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Orazio Attanasio
Nicola Pavoni

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ABSTRACT

We derive testable implications of model in which first best allocations are not achieved because of a moral hazard problem with hidden saving. We show that in this environment agents typically achieve more insurance than that obtained under autarchy via saving, and that consumption allocation gives rise to 'excess smoothness of consumption', as found and defined by Campbell and Deaton (1987). We argue that the evidence on excess smoothness is consistent with a violation of the simple intertemporal budget constraint considered in a Bewley economy (with a single asset) and use techniques proposed by Hansen et al. (1991) to test the intertemporal budget constraint. We also construct closed form examples where the excess smoothness parameter has a structural interpretation in terms of the severity of the moral hazard problem. Evidence from the UK on the dynamic properties of consumption and income in micro data is consistent with the implications of the model.

Orazio Attanasio
Department of Economics
University College London
Gower Street
London WC1E 6BT UNITED KINGDOM
and NBER
o.attanasio@ucl.ac.uk

Nicola Pavoni
Department of Economics
University College London
Gower Street
London WC1E 6BT UNITED KINGDOM
n.pavoni@ucl.ac.uk

1 Introduction

This paper characterizes the relationship between consumption and income variability in a class of private information models with asset accumulation and uses this characterization to derive some of their empirical implications. Interest in these models is partly motivated by the empirical rejection of simpler approaches, including the hypothesis of complete insurance markets and models where the only insurance available to agents is self insurance, such as simple versions of the life cycle/ permanent income hypothesis. Our approach enables us to interpret some of the results in the literature (and those we present below) as providing evidence on the market structure facing economic agents. Interestingly, we are able to do so without having to specify the information set available to economic agents.

The complete insurance hypothesis is soundly rejected by the data (e.g. Attanasio and Davis, 1996). A commonly used alternative is to assume that markets are exogenously incomplete. For example, the Bewley model embeds a version of the permanent income model in a market structure where the only mechanism available to agents to smooth consumption is through personal savings (and borrowing), possibly with a single asset. Intertemporal trades can be further limited by the impossibility of borrowing beyond a certain level, possibly zero.[1] In between the two extremes of complete markets and very limited and exogenously given intertemporal trade opportunities there are other possibilities were individuals have access to some state contingent mechanisms that provide insurance over and above the 'self insurance' considered in the Bewley model. These intermediate cases include models where the intertemporal markets available to agents are exogenously given and models were the market structure arises endogenously from specific imperfections, such as the lack of contracts enforceability or private information. The model we propose belongs to this latter set. In particular, we focus on settings with *private information problems.*

In a life-cycle model (or in a Permanent Income model with infinite horizon), consumption levels are pinned down by two sets of equations. A set of intertemporal Euler equations, which relate expected changes in consumption over time to intertemporal prices, and an intertemporal budget constraint. The former equations are valid under a variety of circumstances: in particular, one does not need to specify the complete set of assets (contingent and not) that are available to a consumer. As long as one considers an asset for which the consumer is not at a corner, then there is an Euler equation holding for that particular asset, regardless of what the consumer is doing in other markets. Moreover, the orthogonality restrictions implied by an Euler equation would also

[1] Whether these constraints are ever binding depends on the properties of the income process and, in particular, on its support.

hold if one were to mis-specify the information set available to agents. As long as the agents know more than the econometrician, by the law of iterated expectations, the Euler equation hold even with coarser information sets.

The robustness of the Euler equation is a big advantage from an empirical point of view. Starting with Hall (1978), many authors have focused on the orthogonality restrictions implied by the Euler equation for consumption that can be derived from a consumer maximization problem. With this approach one can be agnostic about many aspects of the environment in which the consumer operates and even on the information set available to agents. While the level of consumption might depend in an unknown way on expectations of future income and other unobservable quantities, the Euler equation does not need a closed form solution for consumption and exploits the fact that *changes* in marginal utility and, under some functional form assumptions, (log) consumption, should be unpredictable and, in particular, should not be related to predictable changes in income. Many authors reported violations of the orthogonality restrictions that take the form of 'excess sensitivity' of consumption growth to expected changes in income and interpreted this as evidence of restrictions to intertemporal trades or liquidity constraints. Other authors (see Attanasio 2000 for a survey and a discussion), instead, have argued that the so-called excess sensitivity of consumption is not necessary due to *binding* liquidity constraints and can be for explained away by non-separability between leisure and consumption, demographic effects and aggregation problems.

A different approach to the empirical implications of the life cycle model is to focus on the *level* of consumption and on its relationship with income. Flavin (1981), for instance, explored the cross equation restrictions imposed on the VAR representation of consumption and income by the PIH. Campbell (1987), in a related contribution, shows that savings should be predicting subsequent declines in labour income. To derive these restrictions one uses both the Euler equation *and* the intertemporal budget constraint. Campbell and Deaton (1989), using a similar approach, pointed out that consumption seems to be 'excessively smooth' to be consistent with the PIH: that is consumption does not seem to react 'enough' to permanent innovation to income. If one assumes that consumers have only access to an asset with a specified interest rate to borrow and save, then, Campbell and Deaton (1989), show how excess sensitivity and excess smoothness can be related and, *conditional on the intertemporal budget constraint holding*, are essentially equivalent.

However, we can have situations where the Euler equation for a given asset is not violated and yet, an intertemporal budget constraint where that asset is the only one available to the agent is violated as one neglects all the other (possibly state contingent) assets available to the consumer. And this violation of the IBC can be such as to imply 'excess smoothness' of consumption.

In this paper, we show how a model where risk is shared imperfectly because of moral hazard can generate 'excess smoothness' in the absence of 'excess sensitivity'. In particular, consumption will not exhibit excess sensitivity but, because of the additional insurance provided to consumers relative to a Bewley economy, gives rise to 'excess smoothness' in the sense of Campbell and Deaton. We can therefore distinguish sharply between 'excess sensitivity' and 'excess smoothness'.

We start by developing a common theoretical framework that allow us to compare two dynamic asymmetric information models with asset accumulation that virtually exhaust the existing literature on dynamic contracting: the hidden income (adverse selection) model and the action moral hazard framework. Moreover, we crucially assume that agents have secret (or non contractible) access to credit market. Within this framework with hidden asset accumulation, Allen (1985) and Cole and Kocherlakota (2001) (ACK) show that in the pure adverse selection model the optimal allocation of consumption coincides with the one the agents would get by insuring themselves through borrowing and lending at a given interest rate. Abraham and Pavoni (2004) (AP), in contrast show that, in the action moral hazard model, the efficient allocation of consumption generically differs from that arising from self insurance.[2] Our empirical strategy exploits this marked discrepancy between the two allocations under hidden assets to disentangle the nature of the information imperfection most relevant in reality.

The fact that individual consumption satisfies an Euler equation is a key distinguishing feature of models with hidden assets respect to models of asymmetric information where the social planner has information on assets and, effectively, controls intertemporal trades (Rogerson, 1985; and Ligon, 1998). Because of incentive compatibility on saving decisions, both in ACK and AP, the time series of individual consumption satisfies the usual Euler equation. This implies that, in both models, conditional on the past marginal utility of consumption, the current marginal utility of consumption should not react to predictable changes in variables known to the consumer and therefore to predictable changes in income.

The difference between the two models arises in terms of the degree of insurance agents can achieve. In ACK, agents cannot insure more than in a standard PIH model with a single asset. In AP, agents can get some additional insurance. This additional insurance, while still maintaining the Euler equation, can only be achieved by violating the intertemporal budget constraint with a single asset. Another way of saying the same thing, is that the allocations in AP are equivalent to those that would occur if the agents had access to a certain set of state contingent trades, rather

[2] AP also show that, in its general formulation, the action moral hazard model actually nests the ACK model of adverse selection. This is at the essence of the common framework we propose here.

than only to a single asset with a fixed interest rate. Or, if one prefers the metaphor of the social planner, to get the standard PIH results, one should be considering income net of transfers received from the planner, rather than the standard income concept used in the PIH literature.

In the single asset version of the self-insurance/ACK model consumption moves one to one with permanent income. Hence, it should fully react to unexpected shocks to permanent income.[3] In terms of Campbell and Deaton (1989), consumption should not display excess smoothness. Since in the AP model consumers obtain some additional insurance relative to what they get by self insuring with saving, consumption moves only partially to innovations to permanent income, therefore exhibiting excess smoothness. In a situation in which 'excess sensitivity tests' do not reject the martingale hypothesis for the marginal utility of consumption, we can interpret the 'excess smoothness' test as providing evidence on the market available to consumers. A failure to reject the null would constitute evidence in favour of the ACK or PIH model, while evidence of excess smoothness would be consistent with the model we present.

While in what follows we give some general results, we also present two specifications of the model (one with quadratic preferences, another with logarithmic utility) that allow the derivation of closed form solution for consumption. These are useful because the magnitude of the excess smoothness of consumption can be directly related to the degree of control the agent has on public outcomes i.e., to the degree of private information the agent has (as measured by a single parameter). Related to this set of issues, we also discuss how the model can be used to provide a structural interpretation of recent empirical evidence of Blundell et al. (2004).

As our approach stresses the distinction between the orthogonality restrictions of the Euler equation and the intertemporal budget constraint, in the empirical section of this paper, we use the test of intertemporal budget constraints proposed by Hansen, Roberds and Sargent (1991) (HRS from now on). HRS show that when (the marginal utility of) consumption follows a martingale, the intertemporal budget constraint does impose testable restrictions on the time series properties of consumption and income. The type of test HRS derive is related to those derived by West (1988), Deaton and Campbell (1989) and Gali (1991). These papers interpret violations of their test as a rejection of the life cycle/permanent income model, as they always take the intertemporal budget constraint as given. A contribution of this paper, in addition to apply the test to micro data, is to point out that the endogenously determined amount of risk sharing that can be observed in the

[3]When the agent has quadratic utility this result holds exactly for the onsumption and income in levels. If agent's utility is isoelastic the same implication can be derived for consumption and income in logs, using standard approximations (e.g., Deaton, 1992)

presence of moral hazard and hidden assets might imply a process for consumption that is smoother than the one that would arise in a Bewley economy and could manifest itself in a violation of the intertemporal budget constraint with a single asset. In other words, we show how to interpret a test of an intertemporal budget constraint as a test of a market structure, and under what conditions *excess smoothness* of consumption can be interpreted as *risk sharing* across individuals. In this sense, in this paper we also provide a new measure of the risk sharing available if the economy.

An important feature of our test is that it shares with the HRS and West tests the fact that it is robust to some mis-specifications of the agents' information sets. In particular, we only need to assume that the information set the agents have and use is not smaller than that of the econometrician. In other terms, we can allow agents to have an informational advantage over the econometrician. We therefore have a test of market structures that is robust to mis-specification of the agents' information sets. As we will explain more in detail in the main body of the paper, the Euler equation (whose validity is a unique feature of our model among the models of asymmetric information previously proposed in the empirical literature) play a essential role in the identification of some of the information set available to the agent. Following HRS we are then able to test the validity of the intertemporal budget constraint along this dimension.

In addition to the HRS test, we also pursue an alternative approach based on the dynamics of cross sectional variances of consumption and income. This approach is somewhat related to that in Deaton and Paxson (1994), Attanasio and Jappelli (2001), Attanasio and Szekely (2004) and Blundell, Pistaferri and Preston (2004). However, our approach, is derived directly from the equation from consumption *levels* rather than consumption *changes.* Moreover, as in the case of the evidence based on the HRS test, we can give a structural interpretation to the estimated coefficients of our regression, related to the importance of the moral hazard model.

Our tests are based on the identification of certain time series properties of the cross sectional moments (means and variances) of (the marginal utility of) consumption. For this reason, we need long time series. In the absence of sufficiently long longitudinal data on consumption, we have to use synthetic panels derived from long time series of cross section. In such a situation the two approaches we propose are strongly complementary as one focuses on insurance across groups while the other focuses on within group risk sharing.

To perform the empirical test we propose we use synthetic cohort data constructed from the UK Family Expenditure Survey (FES). With this pseudo panel of cohort aggregated data on consumption and income we estimate the parameters of a time series model for individual income and consumption processes that can be used to perform the test proposed by HRS. We also estimate the relationship between the dynamics of consumption and income cross sectional variances. Using both approaches we find evidence that is consistent with the model we describe in what follows.

The rest of the paper is organized as follows. In section 2, we present the building blocks of our model. In section 3, we discuss alternative market structures: complete markets, a Bewley economy and two different forms of endogenously incomplete markets, with observable and unobservable assets. In Section 4 we characterizes the equilibria in the different market environments and present some examples that yield interesting closed form solutions. In Section 5, we discuss the empirical implications of the equilibria we considered in Section 4. In Section 6, after briefly presenting the data we use, we describe our two empirical approaches and report the results we obtain with each of them. Section 7 concludes the paper. The two appendices contain most of the proofs of the results stated in the text.

2 Model: Tastes and technology

Consider an economy consisting of a large number of agents that are ex-ante identical, and who each live $T \leq \infty$ periods. The individual income (neglecting individual indexes for notational ease) follows the process:

$$y_t = x_t + \xi_t$$

where x_t and ξ_t summarize respectively the permanent and temporary components of income shocks. We allow for moral hazard problems to the innovations to income. We assume that each agent is endowed with a private stochastic production technology which takes the following form:

$$x_t = f(\theta_t, e_t). \tag{1}$$

That is, the individual income shock $x_t \in X$ can be affected by the agent's effort level $e_t \in E \subset \Re$ and the shock $\theta_t \in \Theta \subset \Re$ which, consistently with previous empirical studies,[4] is assumed to follow a martingale process of the form

$$\theta_t = \theta_{t-1} + v_t^p.$$

The $i.i.d$ shock v_t^p can be interpreted as a permanent shock on agent's skill level. In each period, the effort e_t is taken after having observed the shock θ_t. The function f is assumed to be continuous, and increasing in both arguments. Both the effort e and the shocks θ (hence ν) will be considered private information, while x_t is publicly observable. Similarly, we assume $\xi_t = g\left(v_t^T, l_t\right)$ where

[4]See, for example, Abowd and Card (1989), and Meghir and Pistaferri (2004).

v_t^T is a *iid* shock[5] only observable by the individual, l_t is private effort (which is taken after the realization of v_t^T), and ξ_t is publicly observable.

Below, we provide a closed form where optimal effort is time constant, delivering an equilibrium individual income process of the form:

$$y_t = y_{t-1} + v_t^p + \Delta v_t^T.$$

This characterization of the income process is reasonably general and in line with the permanent/transitory representation of income often used in permanent income models. For expositional simplicity, below we focus on moral hazard problems in the innovation to permanent income. We hence assume $g \equiv 0$ and normalize l_t to zero. Later on, we will consider the general case.

The history of income up to period t will be denoted by $x^t = (x_1, ..., x_t)$, while the agent's private history of shocks is $\theta^t = (\theta_1, ..., \theta_t)$.

Agents are born with no wealth, have von Neumann-Morgenstern preferences, and rank deterministic sequences according to

$$\sum_{t=1}^{T} \delta^{t-1}\, u\left(c_t, e_t\right),$$

with $c_t \in C$ and $\delta \in (0,1)$. We assume u to be real valued, continuous, strictly concave, and smooth. Moreover, we require u to be strictly increasing in c and decreasing in e. Notice that, given a plan for effort levels there is a deterministic and one-to-one mapping between histories of the private shocks θ^t and x^t, as a consequence we are entitled to use θ^t alone. Denote by μ^t the probability measure on Θ^t and assume that the law of large numbers applies so that $\mu^t(A)$ is also the fraction of agents with histories $\theta^t \in A$ at time t.

Since θ^t are unobservable, we make use of the revelation principle and define a reporting strategy $\sigma = \{\boldsymbol{\sigma}_t\}_{t=1}^T$ as a sequence of θ^t-measurable functions such that $\boldsymbol{\sigma}_t : \Theta^t \to \Theta$ and $\boldsymbol{\sigma}_t(\theta^t) = \hat{\theta}_t$ for some $\hat{\theta}_t \in \Theta$. A truthful reporting strategy σ^* is such that $\boldsymbol{\sigma}_t^*(\theta^t) = \theta_t$ a.s. for all θ_t. Let Σ be the set of all possible reporting strategies. A reporting strategy essentially generates publicly observable histories according to $h^t = \sigma\left(\theta^t\right) = \left(\sigma_1\left(\theta_1\right),\ ...,\ \sigma_t\left(\theta^t\right)\right)$, with $h^t = \theta^t$ when $\sigma = \sigma^*$.

An allocation (α, c, x) consists in a triplet $\{\mathbf{e}_t, \mathbf{c}_t, \mathbf{x}_t\}_{t=1}^T$ of θ^t-measurable functions for effort, consumption and income growths (production) such that they are 'technically' attainable

$$\Omega = \left\{(\alpha, c, x) : \forall\ t \geq 1,\ \theta^t,\ \mathbf{e}_t(\theta^t) \in E,\ \mathbf{c}_t(\theta^t) \in C \text{ and } \mathbf{x}_t(\theta^t) = f\left(\theta_t, \mathbf{e}_t(\theta^t)\right)\right\}.$$

The idea behind this notation is that incentive compatibility will guarantee that the agent announces truthfully his endowments (i.e. uses σ^*) so that in equilibrium private histories are

[5] We could easily allow for v_t^T to follow an $MA(d)$ process.

public information. Resources feasibility implies that by the law of large numbers

$$\mathbf{E}\left[c_t\right] \leq \mathbf{E}\left[y_t\right], \text{ for all } t, \tag{2}$$

where the expectation is taken with respect to the measure μ^t on θ^t. Equation (2) pins down the interest rate in this economy. For simplicity we disregard aggregate shocks, however since we consider a production economy, the aggregate income level may change with t. We could have considered alternative set ups: we could have assumed an exogenously given level of the interest rate (considering a small open economy or a village economy where a money lender has access to an external market) or even a production economy where savings takes the form of capital.[6] For what we do below, how we close the model in this dimension is not particularly important.

3 Market Arrangements

Having specified agents' tastes and the technological environment they face, to characterize intertemporal allocations we need to specify the market arrangements in which they operate. We consider three different environments. The first two are exogenously given, while in the third the type of trades that are feasible is derived from the details of the specific information imperfection we consider. The first environment we consider is that of complete contingent markets. We then move on to the opposite extreme and assume that the only asset available to agents is a bond with certain return. Finally, we consider two types of endogenously incomplete markets where trades satisfy incentive compatibility constraints.

3.1 Full Information

In the complete market model there is no private information problem. In this economy, the representative agent solves

$$\max_{(\alpha,c,x)\in\Omega} \mathbf{E}\left[\sum_{t=1}^{T} \delta^{t-1}\, u\left(c_t, e_t\right)\right],$$

$$\text{s.t. } \sum_t \int_{\Theta^t} \mathbf{p}_t(\theta^t)\left(\mathbf{c}_t(\theta^t) - \mathbf{y}_t(\theta^t)\right) d\theta^t \leq 0, \tag{3}$$

where $\mathbf{p}_t(\theta^t)$ is the (Arrow-Debreu) price of consumption (and income) in state θ^t, and for all $A \subset \Theta^t$, we have $\int_A \mathbf{p}_t(\theta^t) d\theta^t = q_t^0 \mu^t(A)$.[7] This is the price at which the agent both buys rights

[6] For a similar model in a small open economy see Abraham and Pavoni, 2004. For a simple analysis in a closed economy with capital see Golosov *et al.*, 2003.

[7] We have hence guessed the 'fair price' equilibrium. Under standard conditions such equilibrium always exists and it is robust to the presence of asymmetric information (e.g., Bisin and Gottardi, 1999). We will disregard all technical complications associated to the fact that we have allow for a continuum of values for θ, and assume that C and E are such that expectations are always well defined.

to $\mathbf{c}_t(\theta^t)$ units of consumption goods and sells (and commits to supply) $\mathbf{y}_t(\theta^t)$ units of the same good. q_t^0 is the period one price of a bond with maturity t, and $q_1^0 = 1$. The budget constraint faced by agents under complete markets makes it clear that they have available a very wide set of securities whose return is state contingent. This richness in available assets imply that agents' marginal utilities are equated across histories

$$u'\left(c_t(\theta^t), e_t(\theta^t)\right) = u'\left(c_s(\theta^s), e_s(\theta^s)\right) \text{ for all } t, s \text{ and } \theta^t, \theta^s. \tag{4}$$

Since there are no aggregate shocks one can restrict attention to equilibria where q_t^0 are deterministic function of time, and determined so that the resources feasibility condition

$$\int_{\Theta^t} \left(\mathbf{c}_t(\theta^t) - \mathbf{y}_t(\theta^t)\right) d\mu^t = 0$$

holds in each period.

3.2 Permanent Income (Self Insurance)

We call permanent income or self-insurance the allocation derived from autarchy by allowing the agents to participate to a simple credit market. They do not have access to any asset other than a risk free bond. Let $\{q_t\}$ the sequence of one period bonds prices and $b = \{\mathbf{b}_{t+1}\}_{t=1}^T$ the plan of asset holding, where $\mathbf{b}_t$ is a $\theta^{t-1}-$measurable functions. We have

$$\sup_{b,(\alpha,c,x)\in\Omega} \mathbf{E}\left[\sum_{t=1}^{T} \delta^{t-1}\, u\left(c_t, e_t\right)\right] \tag{5}$$

subject to

$$\mathbf{c}_t(\theta^t) + q_t \mathbf{b}_{t+1}(\theta^t) \leq \mathbf{b}_t(\theta^{t-1}) + \mathbf{y}_t(\theta^t), \tag{6}$$

where $b_0 = 0$. As usual we rule out Ponzi games by requiring that $\lim_{t\to T} q_t^0 \mathbf{b}_t(\theta^t) \geq 0$. The constraint (6) is the budget restriction typically used in Permanent Income models when the agent has only access to a risk free bond market. For future reference, notice that this problem can be seen as an extension of the permanent-income model studied by Bewley (1977) which allows for endogenous labor supply and non stationary income. If one wants to consider, as is often done in the consumption literature, the case in which the return on the bond is constant (may be because equation (2) is substituted by an open economy assumption), one can consider equation (6) with a fixed q, and $q_{t+1}^0 = (q)^t$.[8]

It is well known that one of the main implications of the self insurance model can be obtained by considering the following perturbation of the agent's consumption plan: reduce consumption

[8] Recall indeed that by arbitrage $q_{t+1}^0 = \prod_{n=1}^t q_n = q_t^0 q_t$.

infinitesimally at date t (node θ^t after $e_t(\theta^t)$ has been taken), invest $\frac{1}{q_t}$ this amount for one period, then consume the proceeds of the investment at date $t+1$. If the consumption plan is optimal, this perturbation must not affect the agent's utility level. The first-order necessary condition for his utility not to be affected is the Euler equation:

$$u_c'(c_t, e_t) = \frac{\delta}{q_t} \mathbf{E}_t \left[u_c'(c_{t+1}, e_{t+1}) \right], \tag{7}$$

where $\mathbf{E}_t[\cdot]$ is the conditional expectation operator on future histories given θ^t.

Another necessary condition that individual intertemporal allocations have to satisfy in this model can be derived by repeatedly applying the budget constraint (6) starting from any period $t \geq 1$ asset holding level b_t we have that almost surely (a.s.) for all histories $\bar{\theta}^T$ emanating from node $\bar{\theta}^{t-1}$ the following net present value condition (NPVC) must be satisfied

$$\sum_{n=t}^{T} \frac{q_{n-1}^0}{q_{t-1}^0} \left(\mathbf{c}_n(\bar{\theta}^n) - \mathbf{y}_n(\bar{\theta}^n) \right) \leq \mathbf{b}_t(\bar{\theta}^{t-1}). \tag{8}$$

Notice that, given the income process and the sequence $\{q_t\}$ equations (7) and (8) define (even when a closed form solution does not exist), consumption. It is interesting to compare equation (8) for $t = 1$ with equation (3). In the complete market case, the agent has available a wide array of state contingent securities that are linked in an individual budget constraint that sums over time and across histories, as all trades can be made at time 1. In the permanent income model, the agent has a single asset. This restriction on trade requires that the net present value on consumption minus income equals *the same* $b_t = \mathbf{b}_t(\bar{\theta}^{t-1})$ for *all* future histories emanating from node $\bar{\theta}^{t-1}$. In other terms the intertemporal transfer technology does not permit cross-subsidizations of consumption across income histories.

3.3 Endogenously incomplete Markets

We will now consider a series of complete market economies with different assumptions on the degree of private information.

We have in mind an equilibrium concept à la Prescott and Townsend (1984a-b) and Kehoe and Levine (2001). We will therefore use the following definition of equilibrium.

Definition 1 *An equilibrium for economy i is an allocation (α, c, x) and a set of prices $p = \{\mathbf{p}_t(\theta^t)\}_{t=1}^T$ such that - given p - the agent maximizes his expected discounted utility subject to the (Arrow-Debreu) budget constraint, and incentive compatibility constraint IC_i, and all markets clear.*

One might interpret the different degree of asymmetric information as different market arrangements, i.e. as endogenous limitation on the 'set' of available assets. It is however important to notice that all equilibria will be (almost by construction) constrained efficient. The full information model can also be seen as a special case of this model, when the incentive compatibility constraints are not restrictive since effort is fully observable. It is easy to see indeed that in this case the only announcement strategy consistent with (α, c, x) is σ^*. Below we show that the Permanent Income allocation can also be generated as the equilibrium outcome of a special case of the moral hazard model with hidden assets.

We assume that effort is not observable. Within this environment with imperfect information we consider two cases. In the first, private assets are observable. This is equivalent to considering a situation where there are no private assets and all savings are done by the planner. In the second, instead, private assets are hidden.

3.3.1 Moral Hazard with Monitorable Asset Holdings

Consider the case where each agent has private information on his/her effort level e, but there is full information on consumption and asset decisions, and trade contracts can be made conditional on these decisions. We define the expected utility from reporting strategy $\sigma \in \Sigma$, given the allocation $(\alpha, c, x) \in \Omega$ as

$$\mathbf{E}\left[\sum_{t=1}^{T} \delta^{t-1} u\left(c_t, e_t\right) \ \backslash \ (\alpha, c, x)\, ; \sigma\right] = \sum_{t=1}^{T} \delta^{t-1} \int_{\Theta^t} u\left(\mathbf{c}_t\left(\sigma(\theta^t)\right), g\left(\theta_t, \mathbf{x}_t\left(\sigma(\theta^t)\right)\right)\right) d\mu^t(\theta^t)$$

where $g(x, \theta)$ represents the effort level needed to generate x when shock is θ, i.e., g is the inverse of f with respect to e keeping fixed θ. Since x is observable, the mis-reporting agent must adjust his/her effort level so that the lie is not detected.

The equilibrium allocation solves the following problem for the agent

$$\max_{(\alpha,c,x)\in\Omega} \mathbf{E}\left[\sum_{t=1}^{T} \delta^{t-1}\, u\left(c_t, e_t\right)\right],$$

$$\text{s.t. } \sum_{t} \int_{\Theta^t} \mathbf{p}_t(\theta^t) \left(\mathbf{c}_t(\theta^t) - \mathbf{y}_t(\theta^t)\right) d\theta^t \leq 0, \tag{9}$$

together with the incentive compatibility constraint

$$\mathbf{E}\left[\sum_{t=1}^{T} \delta^{t-1}\, u\left(c_t, e_t\right) \ \backslash \ (\alpha, c, x)\, ; \ \sigma^*\right] \geq \mathbf{E}\left[\sum_{t=1}^{T} \delta^{t-1}\, u\left(c_t, e_t\right) \ \backslash \ (\alpha, c, x)\, ; \ \sigma\right] \tag{10}$$

for all $\sigma \in \Sigma$. The key difference between this problem and that of full information is the incentive constraint (10), which defines the set of allocations for which the agent will be induced to tell the truth and supplying the effort plan α.

In the additive separable case, $u(c,e) = u(c) - v(e)$,[9] the key characteristic of the equilibrium allocation is summarized by (e.g., Rogerson, 1985; and Golosov *et al.*, 2003)

$$q_t \mathbf{E}_t \left[\frac{1}{u'(c_{t+1})} \right] = \delta \frac{1}{u'(c_t)}. \tag{11}$$

In order to relate (11) to the Euler equation (7), notice that the inverse $(1/x)$ is a strictly convex transformation. As a consequence, Jensen inequality implies

$$u'(c_t) \leq \frac{\delta}{q_t} \mathbf{E}_t \left[u'(c_{t+1}) \right],$$

with strict inequality if c_{t+1} is not constant with positive probability. That is, the optimality condition (11) is incompatible with the Euler equation (7). The optimal pure moral hazard contract tends to front-load transfers, and agents' consumption process behaves as it the agent were saving constrained. This consideration may play an important role in distinguishing this allocation from permanent income (see Ligon, 1998; and Section 4.5 below).

3.3.2 Moral Hazard with Hidden Asset Accumulation

Assume now that in addition to the moral hazard problem, agents have hidden access to a simple credit market and consumption is not observable (and/or contractable). The agents do not have private access to any other asset market. The equilibrium allocation must solve the following problem

$$\max_{(\alpha,c,x)\in\Omega} \mathbf{E} \left[\sum_{t=1}^{T} \delta^{t-1} \, u\left(c_t, e_t\right) \right],$$

$$\text{s.t.} \ \sum_t \int_{\Theta^t} \mathbf{p}_t(\theta^t) \left(\mathbf{c}_t(\theta^t) - \mathbf{y}_t(\theta^t) \right) d\theta^t \leq 0,$$

and the incentive compatibility constraint:

$$\mathbf{E} \left[\sum_{t=1}^{T} \delta^{t-1} \, u\left(c_t, e_t\right) \ \backslash \ (\alpha, c, x) \, ; \ \sigma^* \right] \geq \mathbf{E} \left[\sum_{t=1}^{T} \delta^{t-1} \, u\left(\hat{c}_t, e_t\right) \ \backslash \ (\alpha, c, x) \, ; \ \sigma \right] \ \text{for all } \sigma \in \Sigma, \tag{12}$$

where the deviation for consumption $\hat{c}$ must be such that the new path of consumption can be replicated by a risk free bond, hence satisfy the self insurance budget constraint (6).[10] It is straightforward to see that the incentive constraint (12) (considered at σ^*) implies that the allocation (α, c, x)

[9]Will explain below that this property is in fact satisfied by set of model specifications larger than the additive separable case.

[10]Formally, for any σ, a deviation $\hat{c}^\sigma$ is admissible if there is a plan of bond holdings $\hat{b}^\sigma$ such that for all t and a.s. for all histories θ^t we have

$$\hat{\mathbf{c}}_t^\sigma \left(\theta^t\right) = \mathbf{c}_t \left(\sigma(\theta^t)\right) + \hat{\mathbf{b}}_t^\sigma(\theta^{t-1}) - q_t \hat{\mathbf{b}}_{t+1}^\sigma(\theta^t), \tag{13}$$

and $\lim_{t\to T} q_t^0 \hat{\mathbf{b}}_{t+1}^\sigma(\theta^t) \geq 0$.

must satisfy

$$u_c'(c_t, e_t) = \frac{\delta}{q_t} \mathbf{E}_t \left[u_c' \left(c_{t+1}, e_{t+1} \right) \right], \tag{14}$$

where the marginal utilities are evaluated at the equilibrium values dictated by (α, c, x).[11] Notice that condition (14) essentially replicates (7). Clearly the Euler equation is consistent with many stochastic processes for consumption. In particular it does not says anything about the variance of c_t. For example, the full information model satisfies this conditions as well, and when preferences are separable c_t has zero variance. The key distinguishing feature between this allocation and the permanent income model is the fact that the former does not satisfy the (NPVC) (8). That is, the intertemporal budget constraint based on that *single asset* is violated because it ignores the state contingent transfers implied by the constrained efficient equilibrium allocation.

4 Characterizing equilibria

In this section we consider the properties of the different market environments we considered and, in particular, that of endogenously incomplete markets. We have mentioned that the equilibrium allocations we consider are constrained Pareto efficient. This means that the equilibrium allocation (α, c, x) can be replicated by an incentive compatible plan of lump sum transfers $\tau = \{\boldsymbol{\tau}_t(\theta^t)\}_{t=1}^T$ that solves the constrained welfare maximization problem of a benevolent social planner who can transfers resources intertemporally at a rate q_t (dictated by the aggregate feasibility constraint). The optimal transfer scheme τ solves

$$\max_{\tau, (\alpha, c, x) \in \Omega} \mathbf{E} \left[\sum_{t=1}^{T} \delta^{t-1} \, u\left(c_t, e_t\right) \right]$$

s.t.

$$\mathbf{c}_t(\theta^t) = \mathbf{y}_t(\theta^t) + \boldsymbol{\tau}_t(\theta^t), \tag{15}$$

the incentive constraint (12), and the planner intertemporal budget constraint

$$\mathbf{E} \left[\sum_{t=1}^{T} q_t^0 \tau_t \right] \leq 0.$$

¿From condition (15) it is easy to see that the optimal allocation implies that the agents do not trade intertemporally ($b_t \equiv 0$). This is just a normalization. Alternatively, τ could be chosen so that the transfer $\tau_t = \boldsymbol{\tau}_t(\theta^t)$ represents the net trade on state contingent assets the agent implements at each date t node θ^t. In this case, resources feasibility would require $\mathbf{E}[\tau_t] = 0$. In Appendix A

[11] This condition is the first order equivalent to the incentive constraint that prevents the controls that the agent is not willing to deviate in assets decisions alone, while contemplating to tell the truth about shock histories θ^t.

we show that under some conditions the optimal allocation (α, c, x) can be 'implemented' using a transfer scheme τ^* which is function of income histories x^t alone.[12] This will simplify the analysis and allow us to describe the consumption allocation in terms of observables. At history x^t an agent with asset level b_t will face the following budget constraint

$$c_t + q_t b_{t+1} = x_t + \boldsymbol{\tau}_t^*(x^t) + b_t.$$

In what follows, we first present a specific economy where we get the 'Allen-Cole-Kocherlakota' (ACK) result. As stressed in AP, the crucial restrictions to obtain the 'self-insurance' result are on the way effort is converted into output. We then move on to relax these restrictions. Within the more general case, we consider a specific parametrization of the income process that allows us to obtain a closed form solution for the optimal transfers. While this example is useful because it gives very sharp predictions, some of the properties of the allocations we discuss generalize to the more general case and inform our empirical specification.

4.1 The ACK economy as a foundation of the Bewley model

It seems intuitive that our model nests that of Allen (1985). In order to clarify this link and introduce our closed forms, we derive the ACK result within our framework. Perhaps the analysis that follows also clarifies further the nature the ACK self-insurance result in terms of a the degree of asymmetric information in the economy. The following model builds on Allen's.

The first specification regards preferences: $u(c, e) = u(c - e)$. Since consumption and effort enter the utility function in a liner fashion, effectively they can be considered as essentially the same good. The second specification, crucial in order to obtains the self-insurance result, is the use of a production function which is linear in effort and separable in the shock (the linearity in θ is obviously irrelevant)

$$x_t = f(e_t, \theta_t) = \theta_t + e_t,$$

with $\Theta = (-\infty, \theta_{\max}]$ and $E = (-\infty, e_{\max}]$. Obviously, in this environment the plan α of effort levels will be indeterminate. We can hence set, without loss of generality, $e_t \equiv 0$. This normalization has two advantages. First, since e_t does not change with θ^t while $f(\theta_t, e_t)$ is strictly increasing in θ_t, all variations in θ_t will induce variations in x_t, automatically guaranteeing the x^t-measurability of c (see Appendix A). Second, an added bonus of the constant effort is that we can focus on the risk sharing dimension of the optimal allocation. This last argument also motivates the modeling choice for our closed form solution below.

[12] In particular, we assume that the optimal plan of consumption c is x^t-measurable.

We now show that, for this specification, incentive compatibility fully characterizes the efficient allocation. Assume $T < \infty$ and consider the last period of the program. Using the budget constraint, given any history x^{T-1} we have

$$u\left(c_T - e_T\right) = u\left(\theta_T + \boldsymbol{\tau}_T^*\left(x^{T-1}, x_T\right) + b_T\right) = u\left(\theta_T + \boldsymbol{\tau}_T^*\left(x^{T-1}, \theta_T + e_T\right) + b_T\right).$$

The key aspect to notice here is that since utility depends on the effort choice only through the transfer, the agent will make to happen the income level x_T delivering the maximal transfer. Since this structure of production allows the agent to obtain any x_T from any θ_T this will always be possible. In addition, the preferences allow him to do it at no cost. In order to be incentive compatible, the transfer scheme must hence be invariant across x_T's.[13]

Now consider the problem in period $T-1$. Taking into account this invariance of τ_T from x_T, the incentive compatibility constraint for e_{T-1} becomes (we set $b_{T-1} = b_T = 0$ for expositional simplicity)

$$\begin{aligned} &u\left(\theta_{T-1} + \boldsymbol{\tau}_{T-1}^*(x^{T-2}, x_{T-1})\right) + \delta \mathbf{E}_{T-1} u\left(\theta_T + \boldsymbol{\tau}_T^*(x^{T-2}, x_{T-1})\right) \\ \geq\, &u\left(\theta_{T-1} + \boldsymbol{\tau}_{T-1}^*\left(x^{T-2}, \hat{x}_{T-1}\right)\right) + \delta \mathbf{E}_{T-1} u\left(\theta_T + \boldsymbol{\tau}_T^*\left(x^{T-2}, \hat{x}_{T-1}\right)\right) \text{ for all } \hat{x}_{T-1} \end{aligned}$$

This constraint says that given x^{T-2} and θ_{T-1}, the planner can only transfer deterministically across time. When the agent can save and borrow he/she will induce the x_{T-1} realization generating the largest $T-1$ net present value of transfers. In order to see it more easily assume the transfer scheme is differentiable and write the agent's first order conditions with respect to e_{T-1} evaluated at $e_T^* = e_{T-1}^* = 0$:

$$\frac{\partial \boldsymbol{\tau}_{T-1}^*\left(x^{T-1}\right)}{\partial x_{T-1}} + \delta \frac{\partial \boldsymbol{\tau}_T^*\left(x^T\right)}{\partial x_{T-1}} \mathbf{E}_{T-1}\left[\frac{u'\left(c_T^*\right)}{u'\left(c_{T-1}^*\right)}\right] = 0.$$

Notice that $\frac{\partial \boldsymbol{\tau}_T^*(x^T)}{\partial x_{T-1}}$ has been taken out from the expectation operator as we saw that $\boldsymbol{\tau}_T$ is constant in x_T shocks. From the Euler equation - i.e. the incentive constraint for bond holding - we get $\mathbf{E}_{T-1}\left[\frac{u'(c_T^*)}{u'\left(c_{T-1}^*\right)}\right] = \frac{q_{T-1}}{\delta}$, which implies

$$\frac{\partial \boldsymbol{\tau}_{T-1}^*(x^{T-1})}{\partial x_{T-1}} + q_{T-1} \frac{\partial \boldsymbol{\tau}_T^*(x^T)}{\partial x_{T-1}} = 0.$$

[13] Recall that the incentive constraint for $e_T^* = 0$ is

$$\begin{aligned} &u\left(\theta_T + \boldsymbol{\tau}_T^*\left(x^{T-1}, \theta_T\right) + b_T\right) \\ \geq\, &u\left(\theta_T + \boldsymbol{\tau}_T^*\left(x^{T-1}, \theta_T + \hat{e}_T\right) + b_T\right) \text{ for all } \hat{e}_T \in E. \end{aligned}$$

The discounted value of transfer must hence be *constant* across x_{T-1} as well, that is, $\boldsymbol{\tau}^*_{T-1}(x^{T-1}) + q_T\boldsymbol{\tau}^*_T(x^T)$ is x^{T-2}-measurable. The fact that the agent faces the same interest rate of the planner implies that for any x^{T-1} incentive compatibility requires that the net present value of the transfers must be the same across x^{T-1} histories, hence as a sole function of x^{T-2}. Going backward, we get that $\sum_{n=1}^{T-t} q^0_{t+n}\boldsymbol{\tau}^*_{t+n}(x^{t+n})$ is x^{t-1} measurable since it is a constant number for all history continuations $x_t, x_{t+1}, ..., x_T$.

Now recall that self insurance has two defining properties: first, it must satisfy the Euler equation. Second, it must satisfy the intertemporal budget constraint with one bond, i.e., the period zero net present value must be zero for all x^T.[14] Since the Euler equation is always satisfied here, the only way of obtaining a different allocation is that the transfers scheme τ^* permits to violate the agent's period zero self insurance intertemporal budget constraint for some history x^T. The previous argument demonstrates that it cannot be the case.[15] This implies that the 'relaxed-optimal' contract obtained by using the first-order-condition version of the incentive constraint corresponds to the bond economy allocation. Since this allocation is obviously incentive compatible, it must be the optimal one.

The intuition for this result is simple. First, as emphasized by ACK, the free access to the credit market implies that the agent only cares about the *net present value* of transfers (i.e., he(she does not care about the exact timing of transfer payments). Second, our definition of the set E of available effort levels imply that at each t the agent has *full control* over the publicly observable outcome x_t. Moreover, the perfect substitutability between consumption and effort in the utility function on one side and between income and effort in production on the other side imply that the agent can substitute effort for income at *no cost*. Agent's preferences over income histories hence only depend on the planner net present value of transfers, which must hence be constant across histories. Making the self insurance allocation the only incentive feasible allocation. The result can be summarized as follows.

[14] Recall conditions (7) and (8) and the fact that there is a one-to-one onto mapping between x^t and θ^t histories in this model.

[15] If we normalize asset holding to zero, we indeed have $\tau^*_T = c^*_T - y^*_T$, $\tau^*_{T-1} = c^*_{T-1} - y^*_{T-1}$, and so on. The above argument hence implies that for all t the quantity

$$NPV_t = \sum_{n=1}^{T-t} q^0_{t+n}\boldsymbol{\tau}^*_{t+n}(x^{t+n}) = \sum_{n=1}^{T-t} q^0_{t+n}\left(\mathbf{c}^*_{t+n}(x^{t+n}) - \mathbf{y}^*_{t+n}(x^{t+n})\right)$$

is x^{t-1} measurable. If we consider the $t = 1$ case, incentive compatibility implies that $NPV_1\left(x^T\right) = \sum_{n=1}^{T} q^0_n\left(c_n - y_n\right)$ is one number a.s. for any history x^T. Hence, in order to satisfy resources feasibility the planner is forced to set this number to zero.

Proposition 1 *Assume $T < \infty$ and that agents have perfect and costless control over publicly observable income histories. Then the efficient allocation coincides with self insurance. However, in more general specifications of the income process, the efficient allocation differs from self insurance.*

The first part of the proposition has been shown above. Notice that we never used the time series properties of θ_t. Indeed this result is pretty general, and as we will see below it also applies to the case with two type of shocks.[16] In order to see an example of what is stated in the second part of the proposition we will now generalize the production function to allow for non-linearities.

4.2 The Case with Some Risk Sharing: Excess Smoothness

Consider now the general case where the function f is left unspecified. Recall that at period t the agent objective function is:

$$u\left(c_t - e_t\right) = u\left(f\left(\theta_t, e_t\right) - e_t + \boldsymbol{\tau}_t\left(x^{t-1}, f\left(\theta_t, e_t\right)\right)\right)$$

To gain the most basic intuition, we start by considering the final period (T) of the model. The incentive constraint (IC) in the last period is

$$\begin{aligned} & u\left(f\left(\theta_T, e_T^*\right) - e_T^* + \boldsymbol{\tau}_T^*\left(x^{T-1}, f\left(\theta_T, e_T^*\right)\right)\right) \\ \geq\, & u\left(f\left(\theta_T, \hat{e}_T\right) - \hat{e}_T + \boldsymbol{\tau}_T^*\left(x^{T-1}, f\left(\theta_T, \hat{e}_T\right)\right)\right) \end{aligned}$$

which, in its first order condition form becomes

$$1 + \frac{\partial \boldsymbol{\tau}_T^*\left(x^{T-1}, x_T\right)}{\partial x_T} = \frac{1}{f_e'}. \tag{16}$$

Recall that in the ACK model we had $f_e' = \frac{\partial f\left(\theta_T, e_T^*\right)}{\partial e_T} = 1$. Since risk sharing requires $\frac{\partial \boldsymbol{\tau}_T^*\left(x^{T-1}, x_T\right)}{\partial x_T} < 0$ no insurance is possible in this environment. However, in general, f_e' might be greater than one, and this is compatible with some risk sharing. We will see that this argument translates into a multi-period setting as well.

What is the intuition for this fact? If the planner's aim is to make agents share risk, the key margin for an optimal scheme is to guarantee that the agent does not shirk. That is that she does not *reduce* effort. The value $\frac{1}{f_e'}$ in the right hand side of (16) represents the return (in terms of consumption) the agent derives by shirking so that to reduce output by one (marginal) unit. The left hand side is the net consumption loss: when the marginal tax/transfer is negative the direct

[16] ACK, they all study the case with *iid* shocks. It is worth noticing, however, that in the presence of liquidity constraints (the case considered by Cole and Kocherlakota), the extension to any persistence of shocks does not apply.

reduction of one unit of consumption is mitigated by the increase in tax revenues. A large f_e' reduces shirking returns making easier for the planner to satisfy the incentive compatibility, hence to provide insurance.[17]

We now consider a model that uses this intuition heavily to deliver a closed form for the transfers, both in the static and dynamic environments. Consider first the last two periods. It is easy to see that the first order version of the effort incentive compatibility becomes

$$1+\frac{\partial \boldsymbol{\tau}_{T-1}^{*}\left(x^{T-1}\right)}{\partial x_{T-1}}+q\frac{\partial \boldsymbol{\tau}_{T}^{*}\left(x^{T}\right)}{\partial x_{T-1}}=\frac{1}{f_e'}.$$

The intuition is the same. The left had side represents the net cost of shirking while in the right hand side we have the agent's return from misbehaving. For the same reasons we explained above, when $f_e' > 1$, the optimal scheme permits the net present value of transfers to decrease with x_T, allowing for some additional risk sharing on top of self insurance.

4.3 Closed Forms

Now consider a related special specification for f. Assume income x_t depends on exogenous shocks θ_t and effort e_t as follows:

$$x_t = f\left(\theta_t, e_t\right) = \theta_t + a\min\left\{e_t, 0\right\} + b\max\left\{e_t, 0\right\}, \tag{17}$$

with $a \geq 1 \geq b$. In Figure 1 we represent graphically the production function f in this case. Notice that when $a = b = 1$, one obtains the linear specification used to obtain the ACK result. Preferences are as in the previous section

$$u\left(c_t, e_t\right) = u\left(c_t - e_t\right).$$

The budget constraint obviously does not change:

$$c_t = y_t + b_t + \tau_t - qb_{t+1},$$

where all t subscript variables are x^t measurable but b_t, which is x^{t-1} measurable.

[17] An equivalent intuition, based on the revelation game, suggests that f_e' may be related to the marginal cost of lying. Recall that in this model x_T is observable by the planner, and notice that when the agent gets the realization θ_T and declares $\hat{\theta}_T$ instead, it must adjust effort so that x_T is consistent with the declaration. If we denote $x_T\left(\hat{\theta}_T\right) - x_T\left(\theta_T\right) = \hat{\Delta}$. The agent obliviously enjoy $\frac{\partial \boldsymbol{\tau}_T^*\left(x^{T-1}, x_T\right)}{\partial x_T}\hat{\Delta}$ but he is forced to reduce 'consumption' $\left(c_T - e_T\right)$ by $\frac{f_e'-1}{f_e'}\hat{\Delta}$ in order to make $x_T\left(\hat{\theta}_T\right)$ to appear in place of $x_T\left(\theta_T\right)$. The quantity $\frac{f_e'-1}{f_e'}$ can hence be seen as the 'net cost' of lying.

In Proposition 2 in Appendix B we show that in the case of a constant q, we get that the following simple closed form for transfers:[18]

$$(\mathbf{E}_{t+1} - \mathbf{E}_t) \left[\sum_{n=0}^{T-t-1} q^n \tau_{t+1+n} \right] = \left(\frac{1}{a} - 1 \right) (\mathbf{E}_{t+1} - \mathbf{E}_t) \left[\sum_{n=0}^{T-t-1} q^n y_{t+1+n} \right] \tag{18}$$

Moreover, given the assumed process for skills, and $e_t \equiv 0$ we have

$$(\mathbf{E}_{t+1} - \mathbf{E}_t) \left[\sum_{n=0}^{T-t-1} q^n y_{t+1+n} \right] = \frac{1 - q^{T-t}}{1 - q} (\theta_{t+1} - \theta_t) = \frac{1 - q^{T-t}}{1 - q} v_{t+1}^p, \tag{19}$$

where v_{t+1}^p is the permanent shock on income. When u is quadratic and $\delta = q$, the Euler equation implies:

$$c_t = \mathbf{E}_t c_{t+s}$$

for all $s \geq 1$. From the budget constraint together with $b_t \equiv 0$, we hence get

$$\Delta c_{t+1} = \frac{(1-q)}{1-q^{T-t}} (\mathbf{E}_{t+1} - \mathbf{E}_t) \left[\sum_{n=0}^{T-t-1} q^n y_{t+n+1} \right] + \frac{(1-q)}{1-q^{T-t}} (\mathbf{E}_{t+1} - \mathbf{E}_t) \left[\sum_{n=0}^{T-t-1} q^n \tau_{t+n+1} \right].$$

Combining (18) and (19), we get

$$\Delta c_{t+1} = \frac{1}{a} \frac{(1-q)}{1-q^{T-t}} (\mathbf{E}_{t+1} - \mathbf{E}_t) \left[\sum_{n=0}^{T-t-1} q^n x_{t+1} \right] = \frac{1}{a} v_{t+1}^p. \tag{20}$$

Hence for $a = 1$ we are back to the PIH, for larger a we get some more risk sharing over and above self-insurance, with full insurance obtainable as a limit case for $a \to \infty$.

In this example, our ability to derive a closed form solution is driven by two factors. The assumption of quadratic utility (which, as is well known, allows one to derive closed form solution in a standard life cycle model) and the simple concavity assumed on the income process that takes the form of a piece-wise linear function. Such a simple function makes zero effort the optimal level the planner is trying to implement so that one can clearly separate the incentive and the risk sharing margins. The amount of risk sharing the planner can give the agent is that amount that does not induce the agent to shirk. If the planner tries to guarantee a bit more risk sharing, the agent will set effort equal to $-\infty$. For values of a close to one, the amount of risk sharing that can be implemented is not much and, in the limit, when a approaches unity, the planner cannot do

[18] This formulation is independent of the relationship between the discount factor δ and the price of the bond q, and of assumptions on the utility function u (as long as $u(c, e) = u(c - e)$). Moreover, one can easily show that the formula can be generalized to the case where q_t evolves deterministically with time. It must however be said that without further specifications the expression (18) represents only a *necessary* condition for optimality. See Appendix B for details.

better than self insurance. When a is very large, the planner can actually approach the allocation obtainable under complete markets.

A couple of remarks regarding our specification are probably needed. First, the fact that zero is the optimal level of effort, can be interpreted as a normalization. Second, at the optimal level of effort, the income process is actually identical to the standard income process used in the permanent income literature, with the innovation to permanent income equal to the random variable θ_t.

4.4 Introducing Temporary Shocks

We now allow for temporary shocks. Recall that $\xi_t = g\left(v_t^T, l_t\right)$. Let specify a production function for g of the form

$$\xi_t = g\left(v_t^T, l_t\right) = v_t^T + a^T \min\{l_t, 0\} + b^T \max\{l_t, 0\} \text{ with } a^T > 1 > b^T,$$

f as in (17), and the following agent's preferences over c_t, l_t and $e_t : u(c_t - l_t - e_t)$. We can now follow a similar line of proof than that adopted for the permanent shock (see Appendix B), and show that the reaction of consumption to the different shocks can be written as

$$\begin{aligned}\Delta c_{t+1} &= \frac{1}{a^p}\Delta x_{t+1} + \frac{1-q}{a^T}\Delta \xi_{t+1} \\ &= \frac{1}{a^p} v_{t+1}^p + \frac{1-q}{a^T} v_{t+1}^T,\end{aligned}$$

where, for consistency, we denoted by a^p the slope of f for $e_t \leq 0$.

Interestingly, our closed form with temporary shocks provides a structural interpretation of recent empirical evidence. Using the evolution of the cross sectional variance and covariance of consumption and income, Blundell et al. (2004) estimate two parameters representing respectively the fraction of permanent and temporary shocks reflected into consumption. Within our model such estimated parameters can be interpreted as the severity of informational problems to income shocks of different persistence.[19]

4.5 The Case with Isoelastic Utility

We now discuss again the simpler version of the model with only permanent shocks, but assume that agents have isoelastic preferences. Full details are to be found in Appendix C. Let us assume that preferences take the following CRRA form:

$$\mathbf{E}_0 \sum_{t=0} \delta^t \frac{\left(C_t \cdot N_t^{-1}\right)^{1-\gamma}}{1-\gamma} \text{ for } \gamma > 1; \quad \text{and} \quad \mathbf{E}_0 \sum_{t=0} \delta^t \left(\ln C_t - \ln N_t\right) \text{ for } \gamma = 1,$$

[19] Blundel et al. use the log formulation of the model, which we will develop below.

and that the production function is represented by a relatively small departure from the Cobb-Douglas:

$$X_t = \Theta_t N_t^a \quad \text{for} \quad N_t \leq 1; \quad \text{and} \quad X_t = \Theta_t N_t^b \quad \text{for} \quad N_t \geq 1, \qquad \text{with} \qquad a > 1 > b,$$

where, abusing in notation:

$$\ln \Theta_t \equiv \theta_t = \theta_{t-1} + v_t^p,$$

and v_t^p is *normally distributed* with zero mean and variance $\sigma_{v^p}^2$. We are able to obtain a very similar closed form for discounted taxes, which lead to the following permanent income formulation for $\delta = q$ and T very large[20]

$$\Delta \ln C_{t+1} := \Delta c_{t+1} = \frac{1}{a} v_{t+1}^p + \frac{\gamma}{2a^2} \sigma_{v^p}^2.$$

The first noticeable difference it of course that all variables are expressed in logs. Moreover, the presence of precautionary saving motive implies that the efficient allocation to displays increasing (log) consumption. Notice interestingly, that in this case $a > 1$ permits both to reduce the cross sectional dispersion of consumption and to mitigates the precautionary saving motives, hence the steepness of consumption (i.e., 'intertemporal dispersion'). The model implies a very tight relationship between these two moments. Finally, notice that for $a = 1$ we obtain the same expression derived *through approximations* in the self insurance literature (e.g., Deaton, 1992, and Banks et al., 2001, and Blundell et al., 2004). It is hence in this approximate sense that we are able to 'test' the self insurance model in the empirical analysis based on our closed form solution in logs.[21]

Observable Assets. The further interesting aspect of this formulation in logs is that, under the same assumptions on preferences and technology, when assets are *monitorable,* for $\delta = q$ we get the

[20] The derivation follows closely that in levels, and uses the log normality of shocks to θ_t in order to get a closed form expression for the Euler equation (in logs) and the discounted value of taxes. For $\delta = q$, when consumption is log normally distributed, we get

$$\mathbf{E}_t \Delta \ln C_{t+1} = \frac{\gamma}{2} \sigma_c^2, \tag{21}$$

where σ_c^2 is the conditional variance of $\Delta \ln C_{t+1}$ and γ is the coefficient of risk aversion.

When considering both termporary and permanent shocks, one obtains

$$\Delta \ln C_{t+1} = \frac{1}{a^p} v_{t+1}^p + \frac{1-q}{a^T} v_{t+1}^T + \frac{\gamma}{2} \left[\left(\frac{1}{a^p} \right)^2 \sigma_{v^p}^2 + \left(\frac{1-q}{a^T} \right)^2 \sigma_{v^p}^2 \right]. \tag{22}$$

[21] Strictly speaking, the assumptions of Proposition 1 are not satisfied in our specification o the income process for the case with isoelastic utility, not even for $a = 1$. Hence the necessity to appeal to approximations.

following expression for expected consumption growth (see equation (11)):

$$\mathbf{E}_t \Delta \ln C_{t+1} = -\frac{\gamma}{2}\sigma_c^2, \tag{23}$$

i.e., the trend in log consumption is now *negatively* affected by consumption dispersion.[22]

Before moving to the empirical specification, we note that even when the income process is more general than the one that yields a closed form solution, the general results that the planner can provide agents with more insurance than the Permanent Income model typically holds in the class of moral hazard models we are considering. This fact is important for our empirical approach.

5 Empirical Implications of the Model

The main implication of the ideas discussed in the previous sections is that in a model with moral hazard and hidden saving, the typical consumer is able to insure more of her idiosyncratic risk than in the standard permanent income model in which individuals can transfer resources over time using a single asset with a given interest rate, even though the presence of the information asymmetries prevents first best allocations to be achievable. We also noticed that with hidden assets the Euler equation for consumption will always hold. In the standard Permanent Income model with a single asset that pays a fixed interest rate, intertemporal allocations are completely pinned down by the Euler equation and the intertemporal budget constraint. Therefore, if the consumer has to get some additional insurance while at the same time the Euler equation holds, it has to be the case that the IBC with a single asset has to be violated.[23] And within the class of models we are considering, it has to be violated in a way that gives additional insurance. This implies having consumption higher than under the PIH when shocks are 'bad' and lower when shocks are 'good'. Notice also that this implies that the consumer reacts less than under the PIH to innovation to permanent income, that is consumption is 'excessively' smooth. In this sense our model can explain a result in the empirical literature on consumption: Campbell and Deaton (1989) and West (1988) had stressed the fact that consumption did not seem to be reactive enough to permanent innovations to income.

¿From these considerations, it also follows that our model has very different implications than one with observable assets, as studied, for instance, by Ligon (1998). In our case, the Euler equation holds while in a model with observable assets it does not, as shown in equation (11). With observable assets, all saving is effectively done by the social planner who then allocates consumption

[22]It is not difficult to derive the expression for the specification of preferences (and income) delivering our closed form. Details are available upon request. Regarding the discrepancies on consumption patterns between our model and that with observable assets see also AP for an analysis based on simulations.

[23]More formally, the equality version of condition (8) is violated for some histories of shocks.

to agents with a system of transfers that guarantees that incentive compatibility (only with respect to effort) is satisfied. When assets are hidden, the set of incentive compatibility constraints must be amended so that to include (at least) the Euler equation with the interest rate provided by the hidden intertemporal transfer technology because such a margin is always available to the agents.

The Euler equation (7) has been used extensively in testing the PIH. It can be derived and holds under a variety of situations. In particular, the Euler equation holds whenever the agent is not at a corner with respect to the decision of holding the asset whose return is being considered. It is important to stress that this result holds regardless of the availability of other assets or of the presence of imperfections in the trading of different assets. Situations where the equation does not hold include the presence of exogenous liquidity constraints (which effectively means that the agent is at a corner as she would like to borrow more than what is allowed on the market) and the model with asymmetric information and observed savings.

An attractive features of the Euler equation is that it implies some strong orthogonality conditions that have been used extensively both to estimate preference parameters and to test the model. According to the model, innovations to marginal utility between time t and $t+1$ have to be uncorrelated with all information available to the consumer at time t. By the law of iterated expectations, this type of test does not require the exact specification of the agents information set, as long as such a set *includes* the variables observed by the econometrician. This logic has been used to derive the so-called excess sensitivity tests of the permanent income model: consumption, conditional on lagged consumption should not be related on any other lagged variable, including those that help to predict income.

Excess sensitivity tests and more generally the Euler equation approach to estimating the parameters and testing the implications of the PIH do not require the existence of a closed form solution for consumption. An implication of this, however, is that the Euler equation is silent about the way in which innovations to (permanent) income are translated into innovations into consumption. When a closed form solution for consumption is available, however, it is possible to derive the implications for the correlation between contemporaneous income and consumption. The standard case that has been extensively studied in the literature is that of quadratic utility case with a fixed interest rate. In such a situation, given a time series process for income, the intertemporal budget constraint and the Euler equation for consumption will induce a set of cross equation restrictions on the joint representation of consumption and income. This is the strategy behind Flavin (1981) and, subsequently, Campbell (1987). As we discussed above, a similar strategy can be followed with more general utility functions if one is willing to rely on approximations. Blundell and Preston

(1998) and Blundell, Pistaferri and Preston (2004) exploit the relationship between income shocks and consumption innovations to study the evolution over time of cross sectional second moments.

In addition to the cross equation restrictions stressed by Flavin (1981), one can consider other restrictions implied by the PIH. In particular, Campbell and Deaton (1989) and West (1988), pointed out that, if current income has a permanent component, consumption should fully adjust to innovations to such component, because they are fully reflected in permanent income. These studies then go on to show that, at least in aggregate data, consumption seems to be excessively smooth, in that it does not react to innovations in permanent income. Campbell and Deaton (1989) also point out that, if one *takes as valid* the intertemporal budget constraint, excess smoothness can be related to excess sensitivity, in that the failure of consumption to react to innovations to permanent income can be recasted, if one imposes the intertemporal budget constraint, in terms of predictability of consumption changes with lagged information.

Our approach is obviously related to these papers. However, in our theoretical model, the intertemporal budget constraint *with a single risk-free asset* does not necessarily hold. Indeed, the additional insurance consumers get in our model relatively to a Bewley model is obtained by violating such intertemporal budget constraint: consumers with positive innovations to permanent income would consume quantities that are below what would be predicted by the PIH, while consumers with negative innovations would consume in excess of what would be predicted by the PIH. Notice that these deviations effectively imply what has been defined as excess smoothness of consumption. At the same time, however, consumption allocations would satisfy the Euler equation for consumption and therefore would not show 'excess sensitivity'. The deviation of consumption allocations that our model have from the predictions of the standard PIH would hence be very different from the directions observed under *binding* liquidity constraints.

A study that explicitly tests the empirical implications of intertemporal budget constraints is the paper by Hansen, Roberds and Sargent (1991) (HRS). HRS show two important results. First, given a consumption and income process and an asset that pays a fixed interest rate, it is not possible to test the implications of the intertemporal budget constraints without imposing additional structure. Second, and more importantly for us, if the income process is exogenous and the consumption process is determined by a linear-quadratic model, so that consumption follows a martingale, the intertemporal budget constraint imposes additional restrictions on the joint process of income and consumption. In this sense the implications of the Euler equation (which informs many of the so-called excess sensitivity tests), and the additional restriction implied by the intertemporal budget constraint are distinct. [24] The test of the intertemporal budget constrained proposed by HRS is equivalent to some of the tests of excess smoothness in the literature. We can use it within our context and interpret as test of market structure: the null considered in this test is the Bewley model, while the specific alternative implied by 'excess smoothness' would be an implication of the market structure that would prevail under moral hazard with hidden assets. We now make these connections precise.

5.1 Explaining 'excess' smoothness

HRS consider an income process y_t which is one element of the information structure available to the consumer and assume that it admits the following representation:[25]

$$(1-L)y_t \equiv \Delta y_t = \rho(L)w_t \tag{24}$$

where w_t is a $n-$dimensional vector of orthogonal covariance stationary random variable that represent the information available to the consumer. $\rho(L)$ is a $1 \times n$ vector of polynomials in the lag operator L. The martingale restriction on the process for consumption implies that consumption can be represented by:

$$\Delta c_t = \pi w_{1t} \tag{25}$$

where (w_t has been chosen so that) w_{1t} is the first element of w_t and π is a scalar different from zero. Notice that equation (25) does not include lags. The coefficient π represents the extent to which income news are reflected into consumption. It is useful to decompose the right hand side of equation (24) into its first component and the remaining ones:

$$\Delta y_t = \rho_1(L)w_{1t} + \rho_2(L)w_{2t} \tag{26}$$

HRS show that, given this structure, the NPV implies some restrictions on the coefficients of equations (25) and (26) In particular, the intertemporal budget constraint implies that:

$$\pi = \rho_1(q) \tag{27}$$

[24]The HRS result is true for the levels of consumption. However, the analysis we perform in Section 4.5 implies that a similar results holds (under different parametric restrictions for preferences) for the same set of varible in logs.

[25]HRS work with a slightly more general framework where the income process is made stationary by the transformation $\gamma(L)$, which we are assuming to be equal to $(1-L)$. In what follows, we adopt only in part the HRS notation and adapt it to ours. In particular, we start from a representation for the income process that has already been rotated so that its first component represents the innovation for the martingale process that generates consumption. At a more trivial level, we use y and c for income and consumption instead of r and p.

$$\rho_2(q) = 0 \tag{28}$$

HRS show that restriction (27) is testable, while restriction (28) is not, in that there exist other representations for income, that are observationally equivalent to (26) for which the restriction holds by construction.

The theoretical structure we have illustrated in the previous section provides a new interpretation to the 'excess smoothness' test and to the HRS test. Notice the similarity of equations (17) and (20) to equations (25) and (26).[26] Our contribution is to point out that the HRS test of the NPV is effectively a test about the market structure. The null considered by HRS corresponds to the Bewley model we considered in Section 3.2, which also corresponds to a special case of our model (see Section 4.1). We also point out that the model with moral hazard generates a specific deviation from the null that is consistent with much of the evidence obtained in this literature. The *alternative hypothesis that* $\pi < \rho_1(q)$ is equivalent to what Campbell and Deaton (1989) and West (1988) define as 'excess smoothness' of consumption.[27] The model we present implies $\rho_1(q)/\pi = a$. The extent of 'excess smoothness' has in our context, at least for this example, a structural interpretation. It represents the severity of the incentive problem. As we discuss above, when a is much larger than 1, one gets a considerable amount of risk sharing, and, in the limit case, one gets full insurance. This is because the return to shirk is very low, the social planner hence finds it relatively easy to motivate agents to work hard, and can provide more insurance. On the other extreme, when a is close to unity, the allocations are similar to those that one would observe under the PIH.

An important feature of the HRS approach is that the test of the NPV restriction does not require the econometrician to identify *all* information available to the consumer. Intuitively, the test uses two facts. First, under the null the intertemporal budget constraint with a single assets must hold whatever is the information set available to the agent. Second, under the assumption that the agent has no coarser information than the econometrician, the validity of the Euler equation implies that consumption innovation reveals part of the information available to the agent. By

[26] In particular, set $\pi = \frac{1}{a}$, $w_{1t} = v_t^p$, $\rho_1(L) = 1$, and $\rho_2(L) = 0$.

[27] Another paper that is related to our empirical strategy is Gali (1991), who imposes the intertemporal budget constraint on top of the Euler equation for consumption so to derive a test that uses only data on consumption. The intertermporal budget constraint is used to derive the relationship between the spectral density at zero frequency of consumption and the variance of innovations to permanent income. While Gali (1991), as Campbell and Deaton (1989), interpret his test as a test of the PIH (in various incarnations) this is because he takes for granted the intertemporal budget constraint with a single asset.

following the HRS strategy we perform a test of the intertemporal budget constraint along the dimensions of information identified via the Euler equation (which is not zero as long as insurance markets are not complete).

While the structural interpretation we have just given only holds for the particular example we have considered, as we explained in Section 4.2 the intuition about the relationship between excess smoothness and the trade-off between incentives and insurance is more general. Unless we are in a situation in which the optimal effort is at higher levels than what would prevail under the PIH (maybe because of some type of complementarity between shocks and effort and negative wealth effects) the intuition will be valid. It should also be noticed that while the specific income process we consider seems special, the income process we get *in equilibrium* from such an example is identical to the income process that is typically used in the PIH literature.

Equation (25) can be solved for w_{1t} and the result substituted in (26) to get:

$$\Delta y_t = \beta(L)\Delta c_t + \varepsilon_t \tag{29}$$

where $\beta(L) = \rho(L)/\pi$ and $\varepsilon_t = \rho_2(L)w_{2t}$. Expressed in terms of the quantities in equation (29) the restriction (27) will be:

$$\beta(q) = 1 \tag{30}$$

where $q = 1/(1+r)$. The test in (30) is very simple to implement and has recently been used by Nalewaik (2004) as a test of the PIH.

The representation in equations (25) and (26) is derived under the assumption that preferences are time separable. When this is not the case, maybe because of the presence of multiple consumption goods some of which are durables and some create habits, HRS (Section 4) show how to generalize equation (25) to the following representation for total consumption expenditure:

$$\Delta c_t = \pi\psi(L)w_{1t} \tag{31}$$

where c in equation (31) represents total consumption expenditure, which enters the budget constraint, while utility is defined over the consumption services which, in turn are a function of current and, possibly, past expenditure. The polynomial in the lag operator $\psi(L)$ reflects these non-separabilities. In this case, the NPV restriction takes the same form as in equation (27). An MA structure in the consumption equation can also be obtained in the presence of i.i.d. taste shocks to the instantaneous utility function, as discussed, for instance, in Attanasio (1999).

5.2 Reformulating the intertemporal budget constraint

The fact that assets are unobservable makes the moral hazard models we are considering very close to the PIH, which is one aspect of the ACK results. Indeed, as we showed, there is a set of parameter values that makes the allocations of the two models equivalent. A useful way to re-write the intertemporal budget constraint faced by a consumer is as a sequence of period to period budget constraints where the consumer receives, in addition to her earnings, the contingent transfers made by the social planner:

$$\tau_t(x^t) + y_t - c_t + b_t \geq q_t b_{t+1} \tag{32}$$

where b_t is the amount held in the hidden saving technology, which pays an exogenously given return r, and $\tau_t(x^t)$ is the state contingent transfer that the consumer receives from the social planner. Such sequence of transfers and the corresponding consumption allocations will satisfy the incentive compatibility constraint (14). The advantage of re-writing the intertemporal budget constraint as (32) are two. First, this formulation makes it clear that the Euler equation

$$u_c'(c_t, e_t) = \frac{\delta}{q_t}\mathbf{E}_t\left[u_c'(c_{t+1}, e_{t+1})\right], \tag{33}$$

must be part of the incentive compatibility constraints. Moreover, many of the standard results one gets for the standard Permanent Income model can be applied here, considering the transfer as a part of the income process.[28] For instance, suppose that utility in consumption is quadratic.[29] Define total income, including the (net) transfer $\tau_t(x^t)$, as $\widetilde{y}_t = \tau_t(x^t) + y_t$. If the return on the (hidden) intertemporal transfer technology is constant and equal to $\frac{1}{q} - 1$, then consumption at time t will be given by:

$$c_t = (1-q)\mathbf{E}_t\left[\sum_{j=1}^{\infty} q^j \widetilde{y}_{t+j}\right]. \tag{34}$$

Equation (34) is derived using the Euler equation for quadratic utility and the intertemporal budget constraint (32) and implies that changes in consumption are equal to the present discounted value of revised expectations about future values of $\widetilde{y}_t$. Equation (34) also makes it clear why the test of the intertemporal budget constraint is informative in this context. If one could observe income *net of all the planner transfers* $\widetilde{y}_t = \tau_t(x^t) + y_t$, and where to formulate the excess smoothness test using

[28] Notice that, as long as the utility function is additively separable in effort and consumption, the fact that income and the transfer are endogenously determined does not prevent us from using the PIH results, as we can obtain them conditioning on the equilibrium level of income and transfers. Of course, the same is true for our closed form specification, since effort is constant in equilibrium.

[29] Moreover assume thet either $u(c, e)$ is additive separable or we are in the case considered in Section 4.3.

this definition of income, one should not get a rejection of the null hypothesis. Using the notation above, one would get that $\pi = \rho_1(q)$. These considerations suggest an empirical strategy based on alternative definitions of income. The net transfers made by the social planners are obviously a metaphor that allow us not to be specific about the particular decentralized instruments agents use in reality. These transfers therefore may include a variety of 'income' sources, ranging from interpersonal transfers, to public transfers to the (net) purchases of state contingent assets. One could then start from definitions of income that do not include any form of shock related transfers, such as gross income to move on to alternative definitions that include explicit or implicit smoothing mechanisms, such as taxes and benefits and interpersonal transfers. One should then find more 'excess' smoothness using definitions of income that do not include smoothing mechanisms.

5.3 Using Cross-Sectional Variances

The empirical implications stressed so far refer to the means of consumption and income. An equation such as (29) relates the time series means of consumption and income. For reasons that will become obvious, it might also be useful to consider the implications of the theory for the cross sectional variances of income and consumption. For this purpose, it is particularly useful the closed form solution derived in Appendix C for the Isoelasic case (see equation (66)):

$$\ln C_t^i := c_t^i = x_t^i + \tau_t(x^{t,i}) = \frac{1}{a}x_t^i + t\left[\frac{\gamma^i}{2a^2}\sigma_{v^P}^2 - \frac{\ln\frac{q}{\delta^i}}{\gamma^i}\right] + \lambda^i + z_t, \tag{35}$$

where we should stress that we do not allow the price of a bond q to change with time, or any sort of heterogeneity in the crucial parameter a. The term λ^i generalizes the expression in the Appendix by adding some form of *ex-ante* heterogeneity, which could capture distributional issues, the initial level of assets of individual i, or unobserved individual variables (all observables can of course be included in the model, and the relative variances identified independently, see Attanasio and Jappelli, 1998). Notice that we allow for possible differences in the taste parameters γ and δ. The term z_t allows for aggregate shocks, which will be assumed to be orthogonal to individual shocks and included in the information set of all agents in the economy.

Let $\Gamma^i = \left[\frac{\gamma^i}{2a^2} - \frac{\ln\frac{q}{\delta^i}}{\gamma^i}\right]$. If we compute the cross-sectional variance at time t of both sides of equation (35), we have

$$Var\left(c_t^i\right) = \left(\frac{1}{a}\right)^2 Var\left(x_t^i\right) + t^2 Var\left(\Gamma^i\right) + Var\left(\lambda^i\right) + \frac{2}{a}Cov\left(x_t^i, \lambda^i\right) + \frac{2t}{a}Cov\left(x_t^i, \Gamma^i\right) + 2tCov\left(\Gamma^i, \lambda^i\right) \tag{36}$$

We start by assuming that both $Cov\left(x_t^i, \lambda^i\right)$ and $Cov\left(x_t^i, \Gamma^i\right)$ are time invariant. These two assumptions are not particularly strong. For instance, if all agents within the group over which the variances are computed have the same γ and δ (risk aversion and discount factors) $Cov\left(x_t^i,\Gamma\right)=0$ for all t. The invariance of $Cov\left(x_t^i,\lambda^i\right)$ can be obtained by assuming constant Pareto weights (and unobservables) across agents in the same group (like in our theoretical model).

If we now take first difference of equation (36), neglecting the individual indexes, we have:

$$\Delta Var\left(c_t\right)=\left(\frac{1}{a}\right)^2\Delta Var\left(x_t\right)+\left(2t+1\right)Var\left(\Gamma\right)+\frac{2}{a}Cov\left(x_t,\Gamma\right)+2Cov\left(\Gamma,\lambda\right) \tag{37}$$

where we used $\left(t+1\right)^2-t^2=2t+1$.

If we assume that both δ and γ are homogeneous across agents, the expression further simplifies to

$$\Delta Var\left(c_t\right)=\frac{1}{a^2}\Delta Var\left(x_t\right). \tag{38}$$

More in general, we could take the second difference and obtain an identification of the degree of cross-sectional heterogeneity in Γ via the intercept of the following regression:[30]

$$\Delta^2 Var\left(c_t\right)=\frac{1}{a^2}\Delta^2 Var\left(x_t\right)+2Var\left(\Gamma\right). \tag{39}$$

Notice that equations (37), (38) and (39) allow the identification of the structural parameter a, which reflects the severity of the moral hazard problem. As noted by Deaton and Paxson (1994), under perfect risk sharing, the cross sectional variance of consumption is constant over time. Under the PIH, as pointed out by Blundell and Preston (1998), the changes in the variance of consumption reflect changes in the variance of (permanent) income. Here, we consider a specific alternative to the perfect insurance hypothesis that implies that consumption variance grows, but less than the increase in the variance of permanent income.

In the presence of transitory shocks, the tests based on equations (37), (38) and (39) remain valid under the assumption that the variance of transitory shocks does not change over time.[31] If that assumption were to be violated, then one would have to identify the fraction of the variance of

[30] A similar expression could be obtained, under the assumption of homogeneous δ and γ while allowing for time-changing variance of individual income $\sigma^2_{v^p}$, as long as the cross-sectional variance of $\bar{\sigma}_t^i\equiv\frac{1}{t}\sum_t\left(\sigma_t^i\right)^2$ is time constant. Details are available upon request.

[31] More precisely, the expression for consumption when income includes both temporary and permanent shocks becomes:

$$c_t^i=\frac{1}{a^p}x_t^i+\frac{(1-q)}{a^T}\xi_t^i+t\frac{\gamma^i}{2}\left[\left(\frac{1}{a^p}\right)^2\sigma^2_{v^p}+\left(\frac{1-q}{a^T}\right)^2\sigma^2_{v^T}\right]-t\frac{\ln\frac{q}{\delta^i}}{\gamma^i}+\lambda^i+z_t,.$$

Under the assumptions that $Cov\left(\xi_t^i,\lambda^i\right)$, $Cov\left(\xi_t^i,\Gamma^i\right)$, $Cov\left(x_t,\xi_t\right)$ and $Var\left(\xi_t\right)$ are time invariant (as it is the case in our model), the analysis we performed in the main text remains valid, with exactly the same interpretations for the coefficients. Of course, since $\Delta Var\left(\xi_t\right)=0$ we would have $\Delta Var\left(x_t\right)=\Delta Var\left(y_t\right)$, hence only the $a^p(=a)$ can be identified.

income accounted for by permanent and transitory income using longitudinal data and then relate each of them to the evolution of consumption inequality. The empirical strategy we follow here is quite different from Blundell, Pistaferri and Preston (2005). They study the evolution of the cross sectional variance of consumption *growth*,[32] while we start from the specification for consumption *levels* in equation (35) to derive equations (38) and (39).

6 Empirical strategy and results

The time series properties for income and consumption we discussed in the previous sections are derived, for the most part, from the maximization problem of an individual consumer. Ideally, therefore, one would like to use individual level data to estimate the relevant parameters and test the restrictions implied by the alternative theories. The main difficulty that arises in pursuing such a strategy, however, is the lack of long longitudinal data on consumption. A long time horizon is clearly crucial to estimate the time series properties of income and consumption without using unduly restrictive assumptions. In the available micro data sources, however, either the longitudinal dimension of the data is very short (like in the data from the UK we use below) or the information on consumption is very limited (like in the PSID, which many people have used). To overcome this difficulty we follow two different strategies. Both of them involve the creation of synthetic cohort data or pseudo-panels, along the lines proposed by Deaton (1985) and Browning, Deaton and Irish (1985). That is, as we cannot follow the same individuals over time, we will form groups and follow moments of the variables of interest for these groups, that will be assumed to have, in the population, constant membership over time. The sample moments, therefore, will approximate the corresponding population moments.

The first strategy will be based on the dynamics of cell *means,* while the second uses cell *variances.* The two tests, while allowing us to estimate the same structural parameter a, have a different focus. The first focuses on insurance across groups, as it exploits variation in group mean for consumption and income. The second, focuses on insurance within a group, as it exploits variation in the cross sectional distributions of consumption (and income) within a group. The two tests, therefore, are complements, rather than substitute. Before describing the two approaches in detail, we discuss briefly the data we use for these exercise.

[32] This has important advantages, but it forces to use the approximation $Var(\Delta c_t) \approx \Delta Var(c_t)$.

6.1 The data

Our main source is the UK Family Expenditure Survey from 1974 to 2002. The FES is a time series of repeated cross sections which is collected for the main purpose of computing the weights for the Consumer Price Index. Each survey consists of about 7,000 contacted over two-week periods throughout the year. We use data on households headed by individuals born in the 1930s, 1940s, 1950s and 1960s to form pseudo panels for 4 year of birth cohorts. As we truncate the samples so to have individuals aged between 25 and 60, the four cohorts form an unbalanced sample. The 1930s cohort is observed over later periods of its life cycle and exits before the end of our sample , while the opposite is true for the 1960s cohort. A part from the year of birth, the other selection criteria we used for this study is marital status. As we want to study relatively homogeneous groups, we excluded from our sample unmarried individuals. We also excluded the self-employed.

These data, which has been used in many studies of consumption (see, for instance, Attanasio and Weber, 1993), contains detailed information on consumption, income and various demographic and economic variables. We report results obtained two different definitions of consumption. The first uses as 'consumption' expenditure on non durable items and services, in real terms and divided by the number of adult equivalents in the households (where for the latter we use the McClemens definition of adult equivalents). The second also includes the expenditure on durables. For income, we also consider different definitions. In particular, we start with gross earnings, to move on to gross earnings plus benefits and finally net earnings plus benefits. As mentioned above, the idea behind using different definitions of income is to gain insights on the role played in terms of providing insurance by different mechanisms, such as the benefit or the tax system.

As households are interviewed every week throughout the year, the FES data are used to construct quarterly time series. This allows us to exploit a relatively long time series horizon.

6.2 The HRS approach

We follow HRS in that we estimate the equations (26) and (25) - or (26) and (31) - by Maximum Likelihood, assuming normality of the relevant residuals and implementing a state space representation of the system. However, some important modifications of the standard procedure are necessary, induced by the fact that we are using micro level data and by the fact that we do not have longitudinal data. To address these two problems, we use the approach recently developed by Attanasio and Borella (2006).

We start by noting that we do not observe the quantities on the left-hand-side of equations (26) and (25), as our data do not have a longitudinal dimension. However, identifying groups of fixed membership, we can define the (population) means (or other moments) for these groups. In particular, given an individual variable z_{gt}^h (where the index h refers to the individual and the index g to the group), we can state, without loss of generality:

$$z_{gt}^h = \overline{z}_{gt} + \eta_{gt}^h$$

where the first term on the right hand side defines the population group mean. We do not observe $\overline{z}_{gt}$, but we can obtain a consistent estimate $\widetilde{z}_{ct}$ of it from our sample. This will differ from the population mean by an error whose variance can be consistently estimated given the within cell variability and cell size (see Deaton, 1985). The presence of this measurement error in the levels will induce an additional MA(1) component in the time series behavior of the changes in the variables of interest. The variability of this component will have to be taken into account when estimating the parameters of the model. We do so by assuming that the information on within cell variability provides an exact measurement of the variance of this component. As discussed in Attanasio and Borella (2006), given the sample size involved, such an assumption is not a very strong one. Given the known values for the variance covariance matrix of the sampling error component, the likelihood function of the MA system in (26) and (25) can be computed using the Kalman filter (for details see Attanasio and Borella (2006)).

As for the specification of the multivariate model, we start by specifying the following latent variable model.

$$\begin{aligned}
\Delta y_{gt} &= \alpha_0^{yy} w_{1gt} + \alpha_1^{yy} w_{1gt-1} + \alpha_2^{yy} w_{1gt-2} + \alpha_0^{yc} w_{2gt} + \alpha_1^{yc} w_{2gt-1} + \alpha_2^{yc} w_{2gt-2} \\
\Delta c_{gt} &= \alpha_0^{cc} w_{1gt} + \alpha_1^{cc} w_{1gt-1} + \alpha_2^{cc} w_{1gt-2} + \alpha_0^{cy} w_{2gt} + \alpha_1^{cy} w_{2gt-1} + \alpha_2^{cy} w_{2gt-2}
\end{aligned} \tag{40}$$

where the two unobserved factors $w_{1,gt}$ and $w_{2,gt}$ are independent over time and with one another. We allow for correlation between the w of different cohorts observed at the same time. In system (40), we only allowed two lags of each of the two factors for the sake of notational simplicity, but larger number of lags can be considered without any problem. As we estimate the variance of $w_{1,gt}$ and $w_{2,gt}$ we normalize the coefficients α_0^{yc} and α_0^{cc} to 1. Moreover, as is standard in multivariate system, for identification we need to restrict one of the coefficients governing the contemporaneous between consumption and income changes to be zero. We follow HRS and set $\alpha_0^{cy} = 0$.[33] This assumption on the one hand imposes a triangular structure on the contemporaneous correlation, and on the other identifies the factor $w_{2,gt}$ as an 'income' shock.

[33] Attanasio and Borella (2006) set to zero the contemporaneous coefficient of w_{2t} in the income equation, choosing the opposite triangular structure.

The martingale property of consumption implies that the coefficient on lagged shocks in the second equation of the system (40) should be zero. However, as discussed in HRS, the presence of durability in some components of consumption could lead to a specification where lagged coefficients on w_1 (α_1^{cc} and α_2^{cc} in our system (40))are different from zero. However, the coefficients on lagged values of w_2 should be zero. This hypothesis can be tested and results of such a test can be interpreted as 'excess sensitivity' tests. This is important both because this test corresponds to a test of the Euler equation and because, as stressed by HRS, if consumption does not satisfy the martingale property implied by the Euler equation, we cannot meaningfully test the intertemporal budget constraint.

Imposing all these restrictions, the system can be written as follows:

$$\Delta y_{gt} = \alpha_0^{yy} w_{1gt} + \alpha_1^{yy} w_{1gt-1} + \alpha_2^{yy} w_{1gt-2} + w_{2gt} + \alpha_1^{yc} w_{2gt-1} + \alpha_2^{yc} w_{2gt-2} \tag{41}$$

$$\Delta c_{gt} = w_{1gt} + \alpha_1^{cc} w_{1gt-1} + \alpha_2^{cc} w_{1gt-2}$$

$$\begin{pmatrix} w_{1,gt} \\ w_{2,gt} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_{1,g}^2 & 0 \\ 0 & \sigma_{2,g}^2 \end{pmatrix} \right), \quad Cov(w_{i,kt}, w_{i,jt}) = \sigma_{i,jk}; \quad i = 1, 2.$$

In section 5, we showed how the equations for consumption and income that make the systems of equation (40) is very similar to the equations of the closed form system we obtain in Section 4 (equations (17) and (20)). Those equations where derived in levels. In Section 4.5, we showed how similar expressions can be derived for the log of income and consumption. We now show both the results in levels and in logs.

6.2.1 Results in levels

In Tables 1 and 2 we report the estimates we obtain estimating the MA system (40) by Maximum Likelihood. As we allow the variance covariance matrix in the system to be cohort specific, we limit the estimation to cohorts that are observed over a long time period. This meant to use a balanced pseudo panels with two cohorts: that born in the 1940s and that born in the 1950s. We experimented with several specifications that differed in terms of the number of lags considered in the system. The most general specification included up to eight lags in both the consumption and income equation. However, no coefficient beyond lag 2 was either individually or jointly significant. In the Table, therefore, we focus on the specification with 2 lags.

In the tables we impose the restriction that the coefficients on the lagged value of w_{2t} are zero. While we do not report the results for the sake of brevity, this restriction is never violated in our data. The estimated coefficients are small in size and never significantly different from zero, either individually or jointly. This is an important result as it correspond to a non-violation of the excess sensitivity test.

Table 1 uses as a definition of consumption the expenditure on non-durables and services. We use three different definitions of income. The first is gross earnings, the second gross earnings plus benefits (such as unemployment insurance and housing benefits) and the third is net earnings plus benefits. For each of the three definitions we report two specifications: one with 2 lags in each of the two equations and one where the insignificant coefficients are restricted to zero.

Several interesting elements come out of the Table. First, the dynamics of income is richer than that of consumption. However, and perhaps surprisingly, the coefficients on the lags of w_{2gt} are not statistically significant and are constrained to zero in columns 2, 4 and 6. In the consumption equation the coefficient on the first lag of w_{1gt} is consistently significant and attracts a negative sign. As discussed above, this could be a sign of intertemporal non-separability of preference, maybe induced by some elements of non durable consumption to have some durability at the quarterly frequency.

The test of the intertemporal budget constraint, which is parametrized as $\pi\psi(q) - \rho_1(q)$ clearly shows the presence of excess smoothness. Interestingly, such evidence is stronger for gross earnings. The value of the test does not change much when we add to gross earnings benefits (as in columns 3 and 4). However, when we consider net earnings plus benefits, the value of the test is greatly reduced in absolute value (moving from -0.49 to -0.26), although still statistically different from zero. Therefore, when we use a definition of income that includes an important smoothing mechanism, we find much less evidence of consumption 'excess smoothness' *relative to that income definition.*

Table 2 mirrors the content of Table 1, with the difference that the definition of consumption we use now includes the expenditure on durables. The results we obtain are, in many ways, similar to those of Table 1. Perhaps surprisingly, the coefficient on lagged w_{1gt} in the consumption equation is smaller in absolute value than in Table 1 and for two of the three income definitions, not statistically different from zero. The most interesting piece of evidence, however, is that the coefficient that measures excess smoothness is now considerably lower, indicating less consumption smoothing relative to the null of the Bewley model. This is suggestive of the fact that durables might be playing an important role in the absorption of shocks, as speculated, for instance by Browning and Crossley (2004). However, when we consider different income definitions, the evidence is consistent with that reported in Table 1, in that relative to net earnings consumption exhibit much less 'excess smoothness' than relative to gross earnings.

6.2.2 Results in logs

When re-estimating the system using the specification in logs, we try to use the same sample used in the specification in levels. However, as we aggregate the non linear relationship (that is we take the group average of logs), we are forced to drop observations that have zero or negative income. A part from this, the sample is the same. We report our estimates in Table 4. Given the evidence on the dynamics of consumption discussed above, we only report the results for total consumption, which includes the expenditure on durables. Results for non durables and services are available upon request from the authors. The evidence is consistent with that of Table 2 in levels in that we do find evidence of excess smoothness. The drop in the size of the excess smoothness parameter when we move to definitions of income that include some smoothing mechanisms is even more dramatic than in Table 2. In the last column, corresponding to net earnings plus benefits, the excess smoothness parameter, while still negative, is insignificantly different from zero.

6.3 The dynamics of consumption variances

The estimation of equation (38) also requires the identification of groups. Here the group implicitly defines the participants in a risk sharing arrangement and the test will identify the amount of risk sharing within that group. As with the estimation of the HRS system, the lack of truly longitudinal data and the use of time series of cross sections, implies that the estimated variances (for income and consumption) will have an error component induced by the variability of the sample. This is particularly important for the changes in the variance of income on the right hand side: the problem induced is effectively a measurement error problem which induces a bias in the estimated coefficient.

In particular, the observable version of equation (38) will be:

$$\Delta\overline{Var\left(c_{gt}\right)} = \frac{1}{a^2}\Delta\overline{Var\left(x_{gt}\right)} + \frac{1}{a^2}\Delta\varepsilon^x_{gt} - \Delta\varepsilon^c_{gt}, \tag{42}$$

where $\varepsilon^x_{gt} = Var(x_{gt}) - \overline{Var\left(x_{gt}\right)}$, and $\varepsilon^c_{gt} = Var(c_{gt}) - \overline{Var\left(c_{gt}\right)}$. Analogous considerations will hold for equations (37) and (39). The variance of the residuals ε will go to zero as the size of the cells in each time period increases. Moreover, information on the within cell variability can be used to correct OLS estimates of the coefficients in equation (42). In particular, a bias correct estimator will be given by the following expression:

$$\widehat{\theta} = A^{-1}[\widetilde{\theta} - B] \tag{43}$$

where $\widetilde{\theta} = (Z'Z)^{-1}Z'w$ is the OLS estimator, B allows for the possibility of correlation between the ε_t^x and ε_t^c and A is determined by

$$B = (Z'Z)^{-1}\,\Gamma = (Z'Z)^{-1}\left\{\tfrac{1}{T-1}\sum_{t=2}^{T}\tfrac{\sigma_{cygt}}{N_{gt}} + \tfrac{\sigma_{cygt-1}}{N_{gt-1}}\right\}$$

and

$$A = \left[I - (Z'Z)^{-1}\,\Omega\right],$$

where

$$\Omega = \frac{1}{T-1}\sum_{t=2}^{T}\left(\frac{\sigma_{xgt}^2}{N_{gt}} + \frac{\sigma_{xgt-1}^2}{N_{gt-1}}\right).$$

In computing the variance covariance matrix of this estimator it will be necessary to take into account the MA structure of the residuals as well as the possibility that observations for different groups observed at the same time will be correlated.

To estimate the parameters in equation (42) we use the same sample we used for the HRS test, with the only difference that we do not limit ourself to the balanced pseudo panel but use four cohorts, although the youngest and oldest are only used for part of the time period. Otherwise the selection criteria used to form our sample are the same as above.

The results are reported in Table 4. There are four columns in the Table, each reporting the slope coefficient of equation (38) and the implied a with the corresponding standard errors. The standard error of a is computed by the delta method. In the first two columns, we use expenditure on non durables and services as our definition of consumption. In the second column, total consumption is divided by the number of adult equivalents. In the third and fourth column we report the results obtained using total expenditure as our definition of consumption. Once more, in the second of these two columns the total is divided by the number of adult equivalents. The three panels correspond to the same three definitions of income we used for the HRS test.

The first thing to note is that all the slope coefficients are positive and statistically different from zero. Moreover, consistently with the theory, they all imply a value of a greater than unity. Finally, the results are affected only minimally by the consideration of adult equivalents.

If we analyze the difference across income definitions we find results that are consistent with the implications of the model and, by and large, with the evidence from the HRS approach. The coefficient on the changes in the variance of gross earnings is much smaller than the one on the other income definitions. This is consistent with the evidence from Tables 1 and 2 which showed more ‘excess smoothness’ for this definition. Unlike in Table 1 and 2, however, the main difference in the size of the coefficient is between the first income definition on one side and the second and third from the other. With the HRS approach, instead, the main difference was between the first and second on one side and the third on the other.

Finally, if we look at the differences between the definitions of consumption that include durables and those that do not, we find that the coefficients are (except for the first income definition) larger for the former than the latter. Again, this is consistent with the evidence from the HRS approach which finds less 'excess smoothness' when one includes durables in the definition of consumption, i.e., some self-insurance mechanisms seems to be at work via durables.

In addition to equation (38) we also estimated versions of equation (37) which include a time trend and of (39) which involves the second differences of the relevant variances. Remarkably, the results we report in Table 3 are barely affected.

The consistency of the results obtained with the variance approach and those obtained with the HRS approach are remarkable because the two tests, as stressed above, focus on different aspects of risk sharing: the latter on insurance across groups and the former on insurance between group. It is remarkable that both yield results that are in line with our model and indicate that the observed amount of risk sharing is in between that predicted by a simple Permanent Income model and that predicted by perfect insurance markets. Comparing the magnitude of the coefficients obtained with the two approaches, we can have a measure of the different degree of risk sharing possibilities that are available within cohorts as opposed to those available across cohorts.

7 Conclusions

In in this paper, we discuss the theoretical and empirical implications of a model where perfect risk sharing is not achieved because of information problems. A specific (and certainly unique in the empirical literature) feature of our model is the hidden access to the credit market. After characterizing the equilibrium of this model, we have shown how it can be useful to interpret individual data on consumption and income.

We have considered a combination of moral hazard and information problems on assets (and therefore consumption). Developing results in Abraham and Pavoni (2004), we have shown that in the constrained efficient equilibrium in our model agents obtain more insurance than in a Bewley set up. Moreover, we are able to construct examples in which we can get closed form solutions for consumption. These results have more than an aesthetic value: in our empirical approach they allow us to give a structural interpretation to some of the empirical results in the literature and to those we obtain. In particular, we show how our model can explain the so-called 'excess smoothness' of consumption and how tests of excess smoothness are distinct from the so-called excess sensitivity tests.

The presence of excess smoothness follows from the fact that, even in the presence of moral hazard and hidden assets, in general, an efficient competitive equilibrium is able to provide some insurance over and above what individuals achieve on their own by self insurance. The constrained efficient allocations that prevail in the environment we consider differ from those one would obtain in a Bewley model with no insurance, in that the agents are able to share some risk and are not forced to rely only on self-insurance. This additional insurance is what generates excess smoothness in consumption, which can then be interpreted as a violation of the intertemporal budget constraint with a single asset. What such set of conditions is neglecting is the set of state contingent Arrow-Debreu securities that the agent can purchase (possibly with some restrictions on trades), in addition to the riskless asset, in a constrained efficient equilibrium. Or, if one prefers the metaphor of the social planner, to get the standard PIH results, one should be considering income net of transfers, rather than the standard concept used in the PIH literature. From an empirical point of view, we show that in our framework, the so called 'excess sensitivity' tests (effectively tests of the martingale property implied by the intertemporal optimization problem solved by the consumers) are distinct from the so called 'excess smoothness' tests, which, following Hansen, Roberds and Sargent (1991), we frame as tests of an intertemporal budget constraint with a single asset. When agents are able to obtain additional insurance relative to the Bewley economy, the IBC with a single asset is violated because it neglects state contingent transfers. We show that in our model we obtain 'excess smoothness' in the sense of Campbell and Deaton (1989), while we do not get excess sensitivity. An important feature of our test, is that it is robust to different hypothesis about the information structure available to economic agents, as long as the econometrician does not have superior information.

While many papers, starting with Deaton and Campbell (1989) have documented the 'excess smoothness' of consumption using aggregate time series data, the evidence based on micro data is recent and limited. Nalewaik (2004) uses a simplified version of the HRS test (effectively estimating equation (29) which, for example, does not permit non-separabilities) and, based on the US CEX, cannot reject the null of the PIH. This result contrasts both with what is obtained by Blundell et al., 2004 in the PSID, and with what we find in this paper for the UK. Neither Nalewaik (2004), nor Attanasio and Borella (2006), who (use a different identification strategy to) estimate a time series model for micro consumption (and other variables) similar to the one we estimate, give their results an interpretation in terms of risk sharing and test of asset markets.

In addition to the HRS test, we also propose a test based on the cross sectional variance of consumption and income. While related to the work of Deaton and Paxson (1994), Blundell and Preston (1994) and Blundell, Pistaferri and Preston (2004), our approach is different in that it focuses on the variance in the *level* rather than *changes* of consumption. Moreover, as is the case of the version of the HRS test we present, we can give the coefficients we estimate a structural interpretation in terms of our moral hazard model.

The approach followed by Blundell et al., (2004) is different from ours. It is more general in some dimensions and less so in others. In particular, they exploit the panel dimension of the PSID and are able to distinguish the effects of consumption of permanent versus temporary income shocks. In our data we cannot identify separately the two parameters. In any case, if one prefers the assumptions made by Blundell et al. (2004) to those we make in deriving our test based on cross sectional variances, we can still use our model to give a structural interpretation to the Blundell et al. (2004) results.

Using our two different approaches we obtain results that are consistent with our model. We forcefully reject both perfect risk sharing and the simple Bewley economy. Moreover, our rejections are consistent with the model with moral hazard and hidden assets we considered. Particularly suggestive is the evidence that when we consider income definitions that include smoothing mechanisms, such as social assistance and net taxes, we find less evidence of 'excess smoothness'.

Our results have obvious policy implications, as one could, in principle, be able to quantify in terms of welfare the insurance role played by the taxation system or UI. Such computations would be immediately feasible from our analysis. We would also be able to perform accurate counterfactuals in order to evaluate the effects of a given policy. All normative issues, however, are left for future research.

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8 Appendix A: Implementing the efficient allocation with income taxes

In this section we show that under some conditions the optimal allocation (α, c, x) can be 'implemented' using a transfer scheme τ which is function of income histories x^t alone. This will simplify the analysis and allow us to describe the consumption allocation in terms of observables.

Notice first, that through $x_t = \mathbf{x}_t(\theta^t)$ the x component of the optimal allocation generates histories of income levels x^t. Let's denote by $x^t\left(\theta^t\right) = \left(\mathbf{x}_1\left(\theta^1\right), ..., \mathbf{x}_t\left(\theta^t\right)\right)$ this mapping. In general $\mathbf{x}_t(\theta^t)$ is not invertible, as it might be the case that for a positive measure of histories of shocks θ^t we get the same history of incomes x^t. A generalization of the argument used in Kocherlakota (2006) however shows that it suffices to *assume that the optimal plan of consumption c alone is x^t-measurable.*[34] This is what we assume thereafter.

Now, notice that $\mathbf{y}_t$ is x^t-measurable by construction. As a consequence, from (15) is easy to see that the x^t-measurability of c implies that τ is x^t-measurable as well. From the transfer scheme τ, we can hence obtain the new x^t-measurable scheme τ^* as follows: $\boldsymbol{\tau}_t^*\left(x^t\left(\theta^t\right)\right) = \boldsymbol{\tau}_t\left(\theta^t\right)$. Given τ^*, let

$$\mathbf{E}\left[\sum_{t=1}^{T} \delta^{t-1}\, u\left(c_t^*, \hat{e}_t\right) \setminus \hat{\alpha}\right] := \sum_{t=1}^{T} \delta^{t-1} \int_{\Theta^t} u\left(\mathbf{c}_t^*\left(\hat{x}^t(\theta^t)\right), \hat{\mathbf{e}}_t\left(\theta^t\right)\right) d\mu^t(\theta^t)$$

[34] That is, that there exists a sequence of x^t-measurable functions c^* such that for all t, θ^t we have $\mathbf{c}_t^*\left(x^t\left(\theta^t\right)\right) = \mathbf{c}_t(\theta^t)$. We will see below that under fairly general conditions the implementation idea of Kocherlakota (2006) extends to the general case with hidden savings. A more extensive proof of our argument is also available upon request.

where $\mathbf{c}_t^*\left(\hat{x}^t(\theta^t)\right) = \boldsymbol{\tau}_t^*\left(\hat{x}^t\left(\theta^t\right)\right) + \mathbf{y}_t^*\left(\hat{x}^t\left(\theta^t\right)\right)$, and the new mapping is induced by $\hat{\alpha}$ as follows: $\hat{\mathbf{x}}_t\left(\theta^t\right) = f\left(\theta_t, \hat{\mathbf{e}}_t\left(\theta^t\right)\right)$ for all t, θ^t. For any history of shocks θ^t, a plan $\hat{\alpha}$ not only entails different effort costs, it also generates a different distribution over income histories x^t hence on transfers and consumption. This justifies our notation for the conditional expectation.

We say that the optimal allocation (α, c, x) can be *implemented* with x^t-measurable transfers if the agent does not have incentive to deviate from c^*, α^* given τ^*. The incentive constraint in this case is as follows:

$$\mathbf{E}\left[\sum_{t=1}^{T} \delta^{t-1}\, u\left(c_t^*, e_t^*\right) \setminus \alpha^*\right] \geq \mathbf{E}\left[\sum_{t=1}^{T} \delta^{t-1}\, u\left(\hat{c}_t, \hat{e}_t\right) \setminus \hat{\alpha}\right], \tag{44}$$

where, as usual, the deviation path of consumption $\hat{c}$ must be replicated by a plan of risk free bonds $\hat{b}$. An important restriction in the deviations $\hat{\alpha}$ contemplated in constraint (44) is that they are required to generate 'attainable' histories of $x's$, i.e. histories of $x's$ that can happen in an optimal allocation. The idea is that any off-the-equilibrium value for x^t will detect a deviation with certainty. One can hence set the planner's transfers to a very low value (perhaps minus infinity) in these cases, so that the agent will never have incentive to generate such off-the-equilibrium histories.

Finally, suppose the agent chooses an effort plan $\hat{\alpha}$ so that the realized history $\hat{x}^t$ is attainable in equilibrium. This means both that there is a reporting strategy $\hat{\sigma}$ so that $\hat{x}^t = \left(\mathbf{x}_1\left(\hat{\sigma}\right), \ \mathbf{x}_2\left(\hat{\sigma}\right), \ ..., \ \mathbf{x}_t\left(\hat{\sigma}\right)\right)$ and that given a consumption plan $\hat{c}$ the utility the agent gets is $\mathbf{E}\left[\sum_{t=1}^{T} \delta^{t-1}\, u\left(\hat{c}_t, e_t\right) \setminus (\alpha, c, x)\, ;\ \hat{\sigma}\right]$, where the notation is that in the main text.[35] This effectively completes the proof since the incentive constraint (12) guarantees that the agent will chose the truth-telling strategy which implies the equilibrium plans α and c as optimal.

[35] More in detail, notice that by definition we have $\hat{\mathbf{e}}_t\left(\theta^t\right) = g\left(\theta_t, \hat{\mathbf{x}}_t\left(\theta^t\right)\right)$. Since $\hat{\mathbf{x}}_t\left(\theta^t\right)$ is 'attainable' it can be induced from x by a 'lie', i.e., there exists a $\hat{\sigma}$ such that $\hat{\mathbf{x}}_t\left(\theta^t\right) = \mathbf{x}_t\left(\hat{\sigma}\left(\theta^t\right)\right)$. But then $\hat{\mathbf{e}}_t\left(\theta^t\right) = g\left(\theta_t, \mathbf{x}_t\left(\hat{\sigma}\left(\theta^t\right)\right)\right)$ and from the definition of τ^* we have $\mathbf{c}_t^*\left(\hat{\mathbf{x}}_t\left(\theta^t\right)\right) = \mathbf{c}_t^*\left(\mathbf{x}_t\left(\hat{\sigma}\left(\theta^t\right)\right)\right) = \mathbf{c}_t\left(\hat{\sigma}\left(\theta^t\right)\right)$, which implies that

$$\mathbf{E}\left[\sum_{t=1}^{T} \delta^{t-1}\, u\left(c_t^*, \hat{e}_t\right) \setminus \hat{\alpha}\right] = \mathbf{E}\left[\sum_{t=1}^{T} \delta^{t-1}\, u\left(c_t, e_t\right) \setminus (\alpha, c, x)\, ;\ \hat{\sigma}\right]$$

for some $\hat{\sigma} \in \Sigma$. Finally, to see that constraint (13) guarantees incentive compatibility of c^* is straightforward.

A final remark. One can easily show that under the same conditions, α must also be x^t-measurable. The intuition is a follows. If α is not x^t-measurable it means that for at least two some θ^t, and $\bar{\theta}^t$ we have $\mathbf{e}_t\left(\theta^t\right) \neq \mathbf{e}_t\left(\bar{\theta}^t\right)$ while $f\left(\theta_t, \mathbf{e}_t\left(\theta^t\right)\right) = f\left(\theta_t, \mathbf{e}_t\left(\bar{\theta}^t\right)\right)$. However, since u is decreasing in e, effort incentive compatibility (at $b_t = 0$) implies that $\boldsymbol{\tau}_s\left(\theta^s\right) \neq \boldsymbol{\tau}_s\left(\bar{\theta}^s\right)$ for some $s \geq t$ with τ not x^s-measurable. A contradiction to the fact that the optimal transfer scheme τ is x^t-measurable.

9 Appendix B: A Closed Form in Levels

The outcome of this section will be a structural interpretation, in terms of the marginal cost/return of effort, of the coefficient ϕ coming from a generalized permanent income equation of the form:

$$\Delta c_t = \phi \Delta y_t^p.$$

We will have

$$\phi = \frac{1}{a}$$

where $a \geq 1$ and $\frac{1}{a}$ is the marginal return to shirking. Since in our model wealth effects are absent, it will deliver a constant effort level in any period, which will be normalized to zero. Zero effort will also be the first best level of effort. So the whole margin in welfare will derive from risk sharing. The incentive compatibility constraint will hence dictate the degree of such insurance as a function of the marginal cost of effort. A lower effort cost/return will allow the planner to insure a lot the agent without inducing him to shirk. And the planner will use the whole available margin to impose transfers and obtain consumption smoothing.

For didactical reasons, we will now solve the model is steps, with increasing degree of complication.

9.1 The Model

Assume that $u(c, e) = u(c - e)$, and consider the following specification of the technology[36]

$$y_t = f(\theta_t, e_t) = \theta_t + a \min\{e_t, 0\} + b \max\{e_t, 0\}, \quad \text{with} \quad a \geq 1 \geq b, \tag{45}$$

and $e \in (-\infty, e_{\max}]$. In other words, the production function has a kink at zero.[37] Interestingly, as we have seen in Section (4.1) for $a = 1$ we are in the standard ACK case (this is true even when $b < 1$) hence there is no room for risk sharing at all (on top of self insurance) and the model replicates the Bewley model.

Finally, notice that as long as $a > 1$ (and $b < 1$) the first best effort level would be zero. However, the first best allocation would also imply a constant consumption. This allocation can only be obtained by imposing a constant tax rate such that $\tau_t' = -1$. Obviously, this allocation is not incentive feasible in a world where effort (and θ) is private information of the agent.

[36]It will be clear soon that the linearity of f for $e > 0$ is not crucial for the closed form, as long as the slope of f is uniformly bounded above by one in this region.

[37]None of the results change if we choose the kink at any other $\bar{e} > -\infty$.

9.2 Static Conditions

We are looking for optimal transfer schemes among that class of continuously differentiable transfer schemes.

Consider first the static problem. When the production function is of the form (45), the incentive compatibility constraint in its differentiable form is as follows

$$\begin{aligned} b\left(1+\tau'\right) &\leq 1 \quad \text{if } e^* \geq 0 \text{ and } e^* < e_{\max} \\ a\left(1+\tau'\right) &\geq 1 \quad \text{if } e^* \leq 0 \ \text{ and } e^* > -\infty, \end{aligned}$$

with equalities if $e \neq 0$. As long as $\tau' \leq 0$ the relevant deviation would always be to reduce effort, i.e. only the second constraint is relevant. The great advantage of this formulation is that, since production efficiency is not an issue,[38] the planner will always use the whole margin to give risk sharing to the agent. Hence the optimal tax rate is exactly $\tau' = \frac{1}{a} - 1$. That is, τ' increases with a, and it approaches -1 (the first best level) for $a \to \infty$. Recall that the intuition is as follows: when a is large the agent does not find very attractive to deviate from the optimal level $e = 0$ and this reduces incentive costs, allowing more risk sharing. Clearly, a simple normalization $z = ae$ induces preferences of the form $u\left(c - \frac{z}{a}\right)$ hence $\frac{1}{a}$ is the marginal cost/return of effort. When the marginal cost/return of effort is low the agent is easier to convince not to shirk.

Notice two important things that are very evident in this static case, but that will be verified in the general case as well. First, since $\tau' = \frac{1}{a} - 1$ must hold for all income levels, the tax schedule must be linear in x_t. Second, that since agent's utility and the production function are concave, when facing a linear tax schedule the agent problem is concave. Hence the incentive compatibility can be substituted by the agent's first order conditions. We conclude that a linear tax scheme is optimal.

9.3 Two Periods

In order to get an idea about the working of the model in a dynamic framework, let's now consider the two period version of this model for the agent. They can obviously be seen as the last two periods of a general $T < \infty$ horizon model. If we normalize to zero the initial level of assets and neglect the notation for previous history x^{T-2}, the agent objective function is

$$u\left(c_{T-1} - e_{T-1}\right) + \delta \mathbf{E}_{T-1} u\left(c_T - e_T\right)$$

[38]We will normalize $e^* = 0$. Notice that as long as as long as $a > 1 > b$, $e^* = 0$ is the unique efficient effort level under full information. With asymmetric information, incentive costs make $e^* > 0$ even less attractive. Moreover, the linearity for $e < 0$ implies that a negaive effort level cannot be optimal since it requires the same consumption dispersion as $e = 0$ and it implies lower net welfare compared to $e^* = 0$.

where for a given tax schedule $\boldsymbol{\tau}$ we have

$$c_{T-1} = y_{T-1} + \tau_{T-1}(x_{T-1}) - qb_T;$$
$$c_T = y_T + \tau_T(x_{T-1}, x_T) + b_T;$$

and, as before,

$$y_t = \theta_t + a\min\{e_t, 0\} + \max\{e_t, 0\}.$$

The Euler's equation is the usual one (as optimality for b_T)

$$\frac{q}{\delta} = \mathbf{E}_{T-1}\frac{u'(c_T - e_T)}{u'(c_{T-1} - e_{T-1})},$$

while the optimal effort choice e_{T-1} solves

$$1 + \frac{\partial\tau_T(x_{T-1})}{\partial x_{T-1}} + \delta\mathbf{E}_{T-1}\frac{\partial\tau_T(x_{T-1}, x_T)}{\partial x_{T-1}}\frac{u'(c_T - e_T)}{u'(c_{T-1} - e_{T-1})} = \frac{1}{a}. \tag{46}$$

The static conditions we derived above imply that

$$1 + \frac{\partial\tau_T(x_{T-1}, x_T)}{\partial x_T} = \frac{1}{a}.$$

We will see more in detail below that since $\frac{\partial\tau_T(x_{T-1},x_T)}{\partial x_T}$ does not depend on x_{T-1}, whenever the scheme is differentiable, we have that $\frac{\partial\tau_T(x_{T-1},x_T)}{\partial x_{T-1}}$ is constant in x_T. Using the Euler equation, the incentive constraint for e_{T-1} hence takes the simpler form

$$\frac{\partial\tau_T(x_{T-1})}{\partial x_{T-1}} + q\frac{\partial\tau_T(x_{T-1}, x_T)}{\partial x_{T-1}} = \frac{1}{a} - 1, \text{ for all } x_{T-1}, x_T.$$

This implies that the discounted sum of the last two taxes is a linear function of x_{T-1}, whose slope does not depend on x_T. We hence get

$$\begin{aligned}\Delta\tau_T \equiv \tau_T - \mathbf{E}_{T-1}\tau_T &= \left(\frac{1}{a} - 1\right)x_T - \mathbf{E}_{t-1}x_T\left(\frac{1}{a} - 1\right)\\ &= \left(\frac{1}{a} - 1\right)\Delta x_T.\end{aligned}$$

And, similarly for $\Delta(\tau_{T-1} + q\tau_T)$, we get

$$\tau_{T-1} + q\tau_T - \mathbf{E}_{T-2}[\tau_{T-1} + q\tau_T] = \left(\frac{1}{a} - 1\right)(x_{T-1} + qx_T - \mathbf{E}_{T-2}[x_{T-1} + qx_T]).$$

This expression does not depend on the process on θ_t, and it provides a very simple (linear) relationship between the innovation on the expected discounted value of taxes and the innovation in the permanent income.

9.4 Generic Time Horizon

We are now ready to derive the results for a generic finite horizon model. The analogous to (46), for all $s, n, t \geq 0$, is

$$\mathbf{E}_{t-s} \sum_{n=0}^{T-t} \delta^n \left[\frac{\partial \tau_{t+n}\left(x^{t+n}\right)}{\partial x_t} \frac{u'\left(c_{t+n} - e_{t+n}\right)}{u'\left(c_t - e_t\right)} \right] = \mathbf{E}_{t-s} \mathbf{E}_t \left[\sum_{n=0}^{T-t} \delta^n \frac{\partial \tau_{t+n}\left(x^{t+n}\right)}{\partial x_t} \frac{u'\left(c_{t+n} - e_{t+n}\right)}{u'\left(c_t - e_t\right)} \right] = \frac{1}{a} - 1.$$

In order to complete the derivation, we first show the following result.

Lemma 1. *Within the class of continuous differentiable transfer schemes the discounted value of marginal taxes* $\sum_{n=0}^{T-t} q^n \frac{\partial \boldsymbol{\tau}_{t+n}\left(x^{t+n}\right)}{\partial x_s}$ *does not depend on* $(x_t, ..., x_T)$ *for all* s. *They are hence linear functions of* x_s *given* x^t.

The proof is by induction. Recall the discussion made for the static case, and notice that the static model corresponds to the last period of a finite period model. We argued above that the optimal tax satisfies $\frac{\partial \tau_T\left(x^{T-1}, x_T\right)}{\partial x_T} = \frac{1}{a} - 1$ for all x^{T-1} and x_T. This implies that the cross derivative $\frac{\partial \tau_T\left(x^{T-1}, x_T\right)}{\partial x_T \partial x_t} = 0$. Since τ_T is continuously differentiable, it must be that $\frac{\partial \tau_T\left(x^T\right)}{\partial x_t}$ is constant in x_T for all $t < T$ as claimed above.[39]

Now consider τ_{T-1}. Since $\frac{\partial \tau_T\left(x^T\right)}{\partial x_{T-1}}$ does not depend on x_T, the effort incentive compatibility can be written as follows:

$$\frac{\partial \boldsymbol{\tau}_{T-1}\left(x^{T-1}\right)}{\partial x_{T-1}} + \delta \frac{\partial \boldsymbol{\tau}_T\left(x^T\right)}{\partial x_{T-1}} \mathbf{E}_{T-1}\left[\frac{u'\left(c_T\right)}{u'\left(c_{T-1}\right)}\right] = \frac{1}{a} - 1, \qquad \text{for all } x^{T-2} \text{ and } x_{T-1}.$$

Since $\mathbf{E}_{T-1}\left[\frac{u'(c_T)}{u'(c_{T-1})}\right] = \frac{q}{\delta}$, we have that $\frac{\partial \boldsymbol{\tau}_{T-1}\left(x^{T-1}\right)}{\partial x_{T-1}} + q \frac{\partial \boldsymbol{\tau}_T\left(x^T\right)}{\partial x_{T-1}}$ is a constant for all all x^{T-2} and x_{T-1}. Since the tax scheme is assumed to be differentiable, this property implies that $\frac{\partial \boldsymbol{\tau}_{T-1}\left(x^{T-1}\right)}{\partial x_t} + q \frac{\partial \boldsymbol{\tau}_T\left(x^T\right)}{\partial x_t}$ is also constant in x_{T-1} (and x_T) for all t.[40] Going backwards, we have our result: $\sum_{n=t}^{T} q^{n-t} \frac{\partial \boldsymbol{\tau}_n\left(x^n\right)}{\partial x_s}$ is constant in $x_t, ...x_T$ for all s. **Q.E.D.**

Given the above results we can apply the law of iterated expectations and get, for a generic δ and q

$$\mathbf{E}_t \left[\sum_{n=0}^{T-t} \delta^n \frac{\partial \tau_{t+n}\left(x^{t+n}\right)}{\partial x_t} \frac{u'\left(c_{t+n} - e_{t+n}\right)}{u'\left(c_t - e_t\right)} \right]$$

[39] The tax/transfer function can hence be written as $\tau_T\left(x^{T-1}, x_T\right) = h\left(x^{T-1}\right) + \left(\frac{1}{a} - 1\right) x_T$ with h differentiable.

[40] Using the result for τ_T we get that the discounted *sum* takes it takes the following from

$$\tau_{T-1}\left(x^{T-1}\right) + q \tau_T\left(x^T\right) = g\left(x^{T-2}\right) + \left(\frac{1}{a} - 1\right) x_{T-1} + q\left(\frac{1}{a} - 1\right) x_T.$$

$$
\begin{aligned}
&= \mathbf{E}_t \left[\sum_{n=0}^{T-t-1} \delta^n \frac{\partial \tau_{t+n}\left(x^{t+n}\right)}{\partial x_t} \frac{u'\left(c_{t+n}-e_{t+n}\right)}{u'\left(c_t-e_t\right)} + \delta^{T-t} \mathbf{E}_{T-1} \frac{\partial \tau_T\left(x^T\right)}{\partial x_t} \frac{u'\left(c_T-e_T\right)}{u'\left(c_t-e_t\right)} \right] \\
&= \mathbf{E}_t \left[\sum_{n=0}^{T-t-1} \delta^n \frac{\partial \tau_{t+n}\left(x^{t+n}\right)}{\partial x_t} \frac{u'\left(c_{t+n}-e_{t+n}\right)}{u'\left(c_t-e_t\right)} + \delta^{T-t} \frac{\partial \tau_T\left(x^T\right)}{\partial x_t} \mathbf{E}_{T-1} \frac{u'\left(c_T-e_T\right)}{u'\left(c_{T-1}-e_{T-1}\right)} \frac{u'\left(c_{T-1}-e_{T-1}\right)}{u'\left(c_t-e_t\right)} \right] \\
&= \mathbf{E}_t \left[\sum_{n=0}^{T-t-1} \delta^n \frac{\partial \tau_{t+n}\left(x^{t+n}\right)}{\partial x_t} \frac{u'\left(c_{t+n}-e_{t+n}\right)}{u'\left(c_t-e_t\right)} + \delta^{T-t-1} q \frac{\partial \tau_T\left(x^T\right)}{\partial x_t} \frac{u'\left(c_{T-1}-e_{T-1}\right)}{u'\left(c_t-e_t\right)} \right] \\
&= \mathbf{E}_t \left[\sum_{n=0}^{T-t-2} \delta^n \frac{\partial \tau_{t+n}\left(x^{t+n}\right)}{\partial x_t} \frac{u'\left(c_{t+n}-e_{t+n}\right)}{u'\left(c_t-e_t\right)} + \delta^{T-t-1} \mathbf{E}_{T-2} \left(\frac{\partial \tau_{T-1}\left(x^{T-1}\right)}{\partial x_t} + q \frac{\partial \tau_T\left(x^T\right)}{\partial x_t} \right) \frac{u'\left(c_{T-1}-e_{T-1}\right)}{u'\left(c_t-e_t\right)} \right] \\
&= \mathbf{E}_t \left[\sum_{n=0}^{T-t-2} \delta^n \frac{\partial \tau_{t+n}\left(x^{t+n}\right)}{\partial x_t} \frac{u'\left(c_{t+n}-e_{t+n}\right)}{u'\left(c_t-e_t\right)} + \delta^{T-t-2} q \left(\frac{\partial \tau_{T-1}\left(x^{T-1}\right)}{\partial x_t} + q \frac{\partial \tau_T\left(x^T\right)}{\partial x_t} \right) \frac{u'\left(c_{T-2}-e_{T-2}\right)}{u'\left(c_t-e_t\right)} \right] \\
&\ldots \\
&= \mathbf{E}_t \left[\sum_{n=0}^{T-t} q^n \frac{\partial \tau_{t+n}\left(x^{t+n}\right)}{\partial x_t} \right].
\end{aligned}
$$

where we repeatedly used the linearity of expectations and the Euler equation. We hence obtain the following expression:

$$
\left(\mathbf{E}_{t+1}-\mathbf{E}_t\right) \left[\sum_{n=0}^{T-t-1} q^n \tau_{t+1+n}\left(x^{t+1+n}\right) \right] = \left(\frac{1}{a}-1\right) \left(\mathbf{E}_{t+1}-\mathbf{E}_t\right) \left[\sum_{n=0}^{T-t-1} q^n x_{t+1+n} \right]. \tag{47}
$$

Note that the above expression for taxes holds true for every concave u and all values for $\delta, q < 1$. We have hence shown the following:

Proposition 2 *Assume τ is an optimal transfer scheme among all differentiable contracts for $u(c,e) = u(c-e)$, with u increasing concave and differentiable, and f as in (45). Then τ solves (47) or all $t < T < \infty$.*

The above proposition simply says that (47) is a *necessary* condition of optimality. Indeed, it is derived by only using a relaxed version of the incentive compatibility constraint. One line of attach is to show conditions under which this scheme is unique. We follow a different approach. We will show the agent's problem is *globally concave* when facing the optimal tax scheme. Intuitively, since taxes are linear taxes, as long as they are non-negative in the decision variables of the agent the result is obtained by the concavity of the utility and the production functions. The formal proof however forces us to restrict the analysis to the case where $\delta \leq q$ and to specify the preferences of the agent to be quadratic. The specification of preferences will allow us to derive an analytical specification for taxes and to link linearly consumption to income.

Since Θ is bounded above, in any finite horizon problem we can choose a quadratic utility specification such that the bliss point is never reached. We will assume

$$u(c-e) = -\frac{1}{2}\left(\bar{B} - (c-e)\right)^2 \qquad \text{with } \bar{B} >> T\theta_{\max}. \tag{48}$$

Lemma 2. *If the agent has quadratic preferences and $\delta \leq q$, within the class of differentiable schemes, taxes are in fact linear with $\frac{\partial \boldsymbol{\tau}_t(x^t)}{\partial x_{t-s}} \geq 0$ for all $t \geq 0$ and $s > 0$. Moreover, $\frac{1}{a} \geq 1 + \frac{\partial \boldsymbol{\tau}_t(x^t)}{\partial x_t} \geq 0$ for all t. If $\delta = q$ then $\frac{\partial \boldsymbol{\tau}_t(x^t)}{\partial x_{t-s}} = 0$ and $1 + \frac{\partial \boldsymbol{\tau}_t(x^t)}{\partial x_t} = \frac{1}{a}$, this for all $t \geq 0$ and $s > 0$.*

When the agent has quadratic preferences, the Euler equation in each period together with the law of iterated expectations imply[41]

$$-\bar{B} + x_t + \boldsymbol{\tau}_t\left(x^t\right) = \left(\frac{\delta}{q}\right)^s \mathbf{E}_t\left[x_{t+s} + \boldsymbol{\tau}_{t+s}\left(x^{t+s}\right)\right] - \left(\frac{\delta}{q}\right)^s \bar{B} \qquad \text{for all } t, s \geq 0. \tag{49}$$

Moreover, the incentive constraint for e_t imply

$$1 + \frac{\partial \boldsymbol{\tau}_t\left(x^t\right)}{\partial x_t} + q\frac{\partial \boldsymbol{\tau}_{t+1}\left(x^{t+1}\right)}{\partial x_t} + ... + q^{T-t}\frac{\partial \boldsymbol{\tau}_T\left(x^T\right)}{\partial x_t} = \frac{1}{a}. \tag{50}$$

We will work backwards.

As seen above, the (constant) slope of $\boldsymbol{\tau}_T$ in x_T is given by the effort incentive constraint (50) for $t = T$:

$$1 + \frac{\partial \boldsymbol{\tau}_T\left(x^T\right)}{\partial x_T} = \frac{1}{a} := R_T. \tag{51}$$

We now derive $\frac{\partial \boldsymbol{\tau}_T(x^T)}{\partial x_{T-1}}$ and show that it is nonnegative.

Consider the Euler equation between periods $T-1$ and T. Using the linearity of $\boldsymbol{\tau}_T$ in x_T, equation (49) for $t = T-1$ and $s = 1$ specifies to

$$-\bar{B} + x_{T-1} + \boldsymbol{\tau}_{T-1}\left(x^{T-1}\right) = \frac{\delta}{q}\mathbf{E}_{T-1}\left[R_T x_T + \tau_T\left(x^{T-1}\right)\right] - \frac{\delta}{q}\bar{B}. \tag{52}$$

In the expression, we use the fact that in equilibrium $\mathbf{E}_t x_T = x_{T-1}$, and we abuse in notation denoting by $\tau_T\left(x^{T-1}\right)$ the x^{T-1} part of $\boldsymbol{\tau}_T$. According to this notation, $\frac{\partial \tau_T(x^{T-1})}{\partial x_{T-1}} = \frac{\partial \boldsymbol{\tau}_T(x^T)}{\partial x_{T-1}}$ by the linearity of $\boldsymbol{\tau}_T$ in x_T.

In order for equation (52) to hold for all x_{T-1} given any x^{T-2}, it must be that

$$1 + \frac{\partial \boldsymbol{\tau}_{T-1}\left(x^{T-1}\right)}{\partial x_{T-1}} = \frac{\delta}{q}\left[R_T + \frac{\partial \boldsymbol{\tau}_T\left(x^T\right)}{\partial x_{T-1}}\right] \tag{53}$$

[41] Recall that we implement $b_t \equiv 0$ - i.e. $c_t = y_t + \tau_t$ - and $e_t \equiv 0$.

for all x_{T-1}, x_T. If we combine this condition with the incentive constraint for e_{T-1} :

$$1+\frac{\partial \boldsymbol{\tau}_{T-1}\left(x^{T-1}\right)}{\partial x_{T-1}}+q\frac{\partial \boldsymbol{\tau}_{T}\left(x^{T}\right)}{\partial x_{T-1}}=\frac{1}{a}, \tag{54}$$

we are able to eliminate $1+\frac{\partial \boldsymbol{\tau}_{T-1}\left(x^{T-1}\right)}{\partial x_{T-1}}$, and obtain

$$\left(\frac{\delta}{q}+q\right)\frac{\partial \boldsymbol{\tau}_{T}\left(x^{T}\right)}{\partial x_{T-1}}=\frac{1}{a}-\frac{\delta}{q}R_T\geq 0. \tag{55}$$

The last inequality - which implies $\frac{\partial \boldsymbol{\tau}_{T}\left(x^{T}\right)}{\partial x_{T-1}}\geq 0$ - is true since $R_T=\frac{1}{a}$ and we assumed $\delta\leq q$.

Next, we obtain the expression for the contemporaneous tax $1+\frac{\partial \boldsymbol{\tau}_{T-1}\left(x^{T-1}\right)}{\partial x_{T-1}}$ from, say, (53):

$$\begin{aligned}1+\frac{\partial \boldsymbol{\tau}_{T-1}\left(x^{T-1}\right)}{\partial x_{T-1}}&=\frac{\delta}{q}\left[R_T+\frac{\partial \boldsymbol{\tau}_{T}\left(x^{T}\right)}{\partial x_{T-1}}\right]\\&=\frac{\delta}{q}\left(\frac{qR_T+\frac{1}{a}}{\frac{\delta}{q}+q}\right):=R_{T-1}\geq 0.\end{aligned} \tag{56}$$

We are now almost ready to start the recursion.

Our next target is to derive the marginal taxes with respect to x_{T-2}, namely $1+\frac{\partial \boldsymbol{\tau}_{T-2}\left(x^{T-2}\right)}{\partial x_{T-2}}$, $\frac{\partial \boldsymbol{\tau}_{T-1}\left(x^{T-1}\right)}{\partial x_{T-2}}$ and $\frac{\partial \boldsymbol{\tau}_{T}\left(x^{T}\right)}{\partial x_{T-2}}$. Those are obtained in five steps.

First, from (52) we get

$$\frac{\partial \boldsymbol{\tau}_{T-1}\left(x^{T-1}\right)}{\partial x_{T-2}}=\frac{\delta}{q}\frac{\partial \boldsymbol{\tau}_{T}\left(x^{T}\right)}{\partial x_{T-2}}. \tag{57}$$

Second, the version of (53) relative to the Euler equation between periods $T-2$ and $T-1$ is

$$1+\frac{\partial \boldsymbol{\tau}_{T-2}\left(x^{T-2}\right)}{\partial x_{T-2}}=\frac{\delta}{q}R_{T-1}+\frac{\delta}{q}\frac{\partial \boldsymbol{\tau}_{T-1}\left(x^{T-1}\right)}{\partial x_{T-2}}. \tag{58}$$

Third, from (50) for $t=T-2$, using (57) we obtain

$$\begin{aligned}&1+\frac{\partial \boldsymbol{\tau}_{T-2}\left(x^{T-2}\right)}{\partial x_{T-2}}+q\frac{\partial \boldsymbol{\tau}_{T-1}\left(x^{T-1}\right)}{\partial x_{T-2}}+q^2\frac{\partial \boldsymbol{\tau}_{T}\left(x^{T}\right)}{\partial x_{T-2}}\\&=1+\frac{\partial \boldsymbol{\tau}_{T-2}\left(x^{T-2}\right)}{\partial x_{T-2}}+\left(q+q^2\frac{q}{\delta}\right)\frac{\partial \boldsymbol{\tau}_{T-1}\left(x^{T-1}\right)}{\partial x_{T-2}}\\&=\frac{1}{a}.\end{aligned} \tag{59}$$

Fourth, rearranging the last two conditions in order to eliminate $1+\frac{\partial \boldsymbol{\tau}_{T-2}\left(x^{T-2}\right)}{\partial x_{T-2}}$, we obtain

$$\left(\frac{\delta}{q}+q+q^2\frac{q}{\delta}\right)\frac{\partial \boldsymbol{\tau}_{T-1}\left(x^{T-1}\right)}{\partial x_{T-2}}=\frac{1}{a}-\frac{\delta}{q}R_{T-1}\geq 0, \tag{60}$$

where the inequality is true since recalling that $R_T = \frac{1}{a}$, (56) can be written as

$$R_{T-1} = \frac{1}{a}\left[\frac{\frac{\delta}{q}+\delta}{\frac{\delta}{q}+q}\right] \leq \frac{1}{a},$$

and the inequality is implied by the assumption $\delta \leq q$.

Finally, we can now use (58) to obtain R_{T-2} as follows

$$\begin{aligned} 1+\frac{\partial \boldsymbol{\tau}_{T-2}\left(x^{T-2}\right)}{\partial x_{T-2}} &= \frac{\delta}{q}R_{T-1}+\frac{\delta}{q}\frac{\partial \boldsymbol{\tau}_{T-1}\left(x^{T-1}\right)}{\partial x_{T-2}} \\ &= \frac{\delta}{q}R_{T-1}+\frac{\delta}{q}\left[\frac{\frac{1}{a}-\frac{\delta}{q}R_{T-1}}{\frac{\delta}{q}+q+q^2\frac{q}{\delta}}\right] \\ &= \frac{\left(\delta+q^2\right)R_{T-1}+\frac{\delta}{q}\frac{1}{a}}{\frac{\delta}{q}+q+q^2\frac{q}{\delta}} \leq \frac{1}{a}. \end{aligned}$$

Again, the last inequality is implied by the fact that as long as $\delta \leq q$ we have both that $R_{T-1} \leq \frac{1}{a}$ and $\delta + q^2 \leq q + q^2\frac{q}{\delta}$. Of course, once $\frac{\partial \boldsymbol{\tau}_{T-1}\left(x^{T-1}\right)}{\partial x_{T-2}}$ is obtained, it is immediate to derive $\frac{\partial \boldsymbol{\tau}_{T}\left(x^{T}\right)}{\partial x_{T-2}}$ from (57), which is nonnegative if and only if $\frac{\partial \boldsymbol{\tau}_{T-1}\left(x^{T-1}\right)}{\partial x_{T-2}} \geq 0$.

The derivation for all other marginal taxes is tedious but straightforward. Once the expression for the contemporaneous taxes $1+\frac{\partial \boldsymbol{\tau}_{t+1}\left(x^{t+1}\right)}{\partial x_{t+1}} := R_{t+1} \geq 0$ is obtained, one can follow the same steps we have describe above to obtain all marginal taxes with respect to x_t. Namely: **First**, one uses the Euler equation (49) to establish the following relationships:

$$\frac{\partial \boldsymbol{\tau}_{t+1}\left(x^{t+1}\right)}{\partial x_t} = \left(\frac{\delta}{q}\right)^s \frac{\partial \boldsymbol{\tau}_{t+1+s}\left(x^{t+1+s}\right)}{\partial x_t} \text{ for all } s \geq 0.$$

Second, from the Euler equation one expression for contemporaneous taxes is obtained:

$$1+\frac{\partial \boldsymbol{\tau}_t\left(x^t\right)}{\partial x_t} = \frac{\delta}{q}R_{t+1}+\frac{\delta}{q}\frac{\partial \boldsymbol{\tau}_{t+1}\left(x^{t+1}\right)}{\partial x_t}.$$

Third, this expression is combined with that obtained from the incentive compatibility constraint for e_t

$$1+\frac{\partial \boldsymbol{\tau}_t\left(x^t\right)}{\partial x_t}+....+q^s\frac{\partial \boldsymbol{\tau}_{t+s}\left(x^{t+s}\right)}{\partial x_t}+....+q^{T-t}\frac{\partial \boldsymbol{\tau}_T\left(x^T\right)}{\partial x_t} = \frac{1}{a},$$

to obtain (**fourth**) the expression for, say, $\frac{\partial \boldsymbol{\tau}_{t+1}\left(x^{t+1}\right)}{\partial x_t}$. Which on one hand will then be used to obtain R_t. On the other hand it will deliver $\frac{\partial \boldsymbol{\tau}_{t+s}\left(x^{t+s}\right)}{\partial x_t}$ from the **first** step. And so on till period $t = 1$. By direct inspection, it is not difficult to realize that they will all satisfy the conditions stated in the proposition.

The specification of our formulae for $\delta = q$ imply that $R_t = \frac{1}{a}$ and, as a consequence, $\frac{\partial \boldsymbol{\tau}_{T-s}\left(x^{T-s}\right)}{\partial x_t} = 0$ for all t and s. **Q.E.D.**

We now use that fact that the tax scheme is linear to show the following Lemma that concludes the proof.

Lemma 3. *If the agent has quadratic preferences, and $\delta \leq q$ when facing the above tax, the agent's problem is concave, so the so derived tax scheme is optimal.*

We have to show that, when facing the optimal tax scheme, the agent's problem is jointly concave in $\{e_t(x_t)\}_{t=0}^{T}$ and $\{b_{t+1}(x_t)\}_{t=0}^{T}$. Consider two contingent plans $\mathbf{e}^1, \mathbf{b}^1, \mathbf{c}^1$ and $\mathbf{e}^2, \mathbf{b}^2, \mathbf{c}^2$. Now consider the plan $\mathbf{e}^\alpha, \mathbf{b}^\alpha, \mathbf{c}^\alpha$ where for all x^t, and $\alpha \in [0,1]$ we have $e_t^\alpha(x^t) := \alpha e_t^1(x^t) + (1-\alpha) e_t^2(x^t)$, and similarly for b_t and c_t. First of all, since both effort and assets enter linearly in the utility function, the concavity of the agent's utility and the additive separability over time and states imply that if we show that $\mathbf{c}^\alpha$ is attainable we are done.

We will show the statement for $\delta = q$ but the proof is similar for the general case with $\delta \leq q$. Of course, what is crucial for the proof is what we have shown above. Namely, that marginal taxes are all nonnegative but the contemporaneous ones $\frac{\partial \boldsymbol{\tau}_t(x^t)}{\partial x_t}$, which are such that $1 + \frac{\partial \boldsymbol{\tau}_t(x^t)}{\partial x_t} \geq 0$. Recall that when $\delta = q$ $\frac{\partial \boldsymbol{\tau}_t(x^t)}{\partial x_{t-s}} \equiv 0$, and $\frac{\partial \boldsymbol{\tau}_t(x^t)}{\partial x_t} \equiv \tau_t^{(t)} = \frac{1}{a} - 1$ for all t. The final part of the proof is hence very simple. If the agent chooses plan $\mathbf{e}^\alpha$ of effort, period t net income available for consumption is given by

$$\begin{aligned}
x_t^\alpha + \tau_t(x_0^\alpha, x_1^\alpha, ..., x_t^\alpha) &= \left(1 + \tau_t^{(t)}\right) x_t^\alpha + k_t \\
&= \frac{1}{a} x_t^\alpha + k_t \\
&= \frac{1}{a} f(\theta_t, e_t^\alpha) + k_t \\
&\geq \frac{1}{a}\left[\alpha f\left(\theta_t, e_t^1\right) + (1-\alpha) f\left(\theta_t, e_t^2\right)\right] + k_t \\
&= \frac{1}{a}\left[\alpha x_t^1 + (1-\alpha) x_t^2\right] + k_t = c_t^\alpha,
\end{aligned}$$

where the inequality in the penultimate row comes from the concavity of f in e and $a \geq 1 > 0$. The last line is definitional. **Q.E.D.**

When $\delta = q$ and u is quadratic, the derivation of the optimal scheme is particularly simple. Since we know that only contemporaneous taxes are positive, from $b_t = 0$ we must have

$$c_t\left(x^t\right) = x_t + \tau_t\left(x^t\right) = x_t + k_t + \left(\frac{1}{a} - 1\right) x_t = k_t + \frac{1}{a} x_t \text{ for all } t \geq 0.$$

This, together with the Euler equation implies $k_{t+1} = k_t$ and

$$\Delta c_{t+1} = c_{t+1} - c_t = \frac{1}{a}(x_{t+1} - x_t) = \frac{1}{a} v_{t+1}^p. \tag{61}$$

The obvious generalization of the previous expression is valid for $\delta < q$ as well. The reader should not be surprised to realize that when the agent is facing only permanent shocks expression (61) for $q = \delta$ holds true independently from the horizon the agent faces.

Infinite Horizon Since for T sufficiently large any bond $\bar{B}$ will be eventually reached for some history of income shocks, we cannot address the infinite horizon cases properly. On the other hand we can interpret the infinite horizon case as the 'limit case' for T very large, but finite. According to this interpretation, we may (heuristically) write the corresponding expression for (47) as

$$(\mathbf{E}_{t+1} - \mathbf{E}_t)\left[\sum_{n=0}^{\infty} q^n \tau_{t+1+n}\left(x^{t+1+n}\right)\right] = \left(\frac{1}{a} - 1\right)(\mathbf{E}_{t+1} - \mathbf{E}_t)\left[\sum_{n=0}^{\infty} q^n y_{t+1+n}\right]. \tag{62}$$

One can obtain the closed form for the single taxes as well. They will be time-invariant and all corresponding to the limiting expression one can derive for R_1 above as T goes to infinity. Details are available upon request. Of course, for $\delta = q$ also in this limit case we have $\frac{\partial \boldsymbol{\tau}_t(x^t)}{\partial x_{t-s}} \equiv 0$, and $\frac{\partial \boldsymbol{\tau}_t(x^t)}{\partial x_t} \equiv \frac{1}{a} - 1$ no matter the time horizon, and $\Delta c_{t+1} = \frac{1}{a} v_{t+1}^p$.

9.5 Temporary shocks

In presence of temporary shocks the planner should obviously condition on $\xi_t = g\left(v_t^T, l_t\right)$ realizations as well. Denote by $h^t = (x^t, \xi^t)$ the combined public history. If we specify a production function of the form

$$\xi_t = g\left(v_t^T, l_t\right) = v_t^T + a^T \min\{l_t, 0\} + b^T \max\{l_t, 0\} \text{ with } a^T > 1 > b^T,$$

and the following agent's preferences over c_t, l_t and $e_t : u(c_t - l_t - e_t)$.

The analysis is now performed separately for the two type of shocks, and we get

$$\mathbf{E}_t \sum_{n=0}^{T-t} \delta^n \left[\frac{\partial \tau_{t+n}\left(h^{t+n}\right)}{\partial \xi_t} \frac{u'\left(c_{t+n} - e_{t+n} - l_{t+n}\right)}{u'\left(c_t - e_t - l_t\right)}\right] = \frac{1}{a^T} - 1$$

and

$$\mathbf{E}_t \sum_{n=0}^{T-t} \delta^n \left[\frac{\partial \tau_{t+n}\left(h^{t+n}\right)}{\partial x_t} \frac{u'\left(c_{t+n} - e_{t+n} - l_{t+n}\right)}{u'\left(c_t - e_t - l_t\right)}\right] = \frac{1}{a^p} - 1,$$

where, for consistency, we denoted by a^p the slope of f for $e_t \leq 0$. By the same reasons we gave in the proof of Lemma 1, as long as we restrict ourself to differentiable schemes, one can uses the Euler equation and derive the following expressions for the sum of taxes:

$$\mathbf{E}_t \sum_{n=0}^{T-t} q^n \frac{\partial \tau_{t+n}\left(h^{t+n}\right)}{\partial \xi_t} = \frac{1}{a^T} - 1$$

and

$$\mathbf{E}_t \sum_{n=0}^{T-t} q^n \frac{\partial \tau_{t+n}\left(h^{t+n}\right)}{\partial x_t} = \frac{1}{a^p} - 1.$$

Assuming *quadratic preferences*, for permanent shocks, the Euler equation implies the same taxes as we derived above, namely

$$\frac{\partial\tau_{t+s}\left(h^{t+s}\right)}{\partial x_t}\left(\frac{\delta}{q}\right)^s=\left(\frac{\delta}{q}\right)^k\frac{\partial\tau_{t+k}\left(h^{t+k}\right)}{\partial x_t}\geq 0 \quad \text{for all} \quad k,s>0.$$

It is easy to see that for temporary shocks we have essentially the same expressions:

$$\frac{\partial\tau_{t+s}\left(h^{t+s}\right)}{\partial \xi_t}\left(\frac{\delta}{q}\right)^s=\left(\frac{\delta}{q}\right)^k\frac{\partial\tau_{t+k}\left(h^{t+k}\right)}{\partial \xi_t}, \text{ for all} \quad k,s>0.$$

Then, one uses the incentive constraint and follows the same five-steps line of derivation we explained above to obtain, working backwards, the expressions for marginal taxes $\frac{\partial\tau_{t+k}\left(h^{t+k}\right)}{\partial \xi_t}$ and $1+\frac{\partial\tau_{t}\left(h^{t}\right)}{\partial \xi_t}$ and show that all are nonnegative. Details are available upon request.

Give our empirical target, we report here below the (analytically simpler) expressions for the 'infinite-horizon' specification of our model. For permanent shocks, we obviously obtain exactly the same conditions as before. In the limit case for T very large the tax scheme tends to:

$$1+\frac{\partial\tau_{t}\left(h^{t}\right)}{\partial \xi_t}=\left(\frac{\delta}{q}\right)^k\frac{\partial\tau_{t+k}\left(h^{t+k}\right)}{\partial \xi_t}=1+\tau_\xi^{(t)}\geq 0. \tag{63}$$

As explained in the proof for Lemma 2, all the above expressions imply optimality when $\delta\leq q$. According to (63) when $\delta<q$ marginal taxes with respect to temporary shocks explode as k increases. The expressions for taxes are then obtained as solutions to difference equations.

It is easy to see by direct inspection of (63) and from the previous analysis that for $\delta=q$ the infinite horizon case delivers the following expressions for taxes:

$$1+\tau_x=\frac{1}{a^p} \quad \text{and} \quad 1+\tau_\xi=\frac{1-q}{a^T},$$

where $1+\tau_x=1+\frac{\partial\tau_{t}\left(h^{t}\right)}{\partial x_t}$ and $1+\tau_\xi=1+\frac{\partial\tau_{t}\left(h^{t}\right)}{\partial \xi_t}=\frac{\partial\tau_{t+k}\left(h^{t+k}\right)}{\partial \xi_t}$ for $k>0$. Hence tax rates are time-invariant, and the agent's consumption reaction to income shocks is given by:

$$\Delta c_{t+1}=\frac{1}{a^p}\Delta x_{t+1}+\frac{1-q}{a^T}\Delta\xi_{t+1}=\frac{1}{a^p}v_{t+1}^p+\frac{1-q}{a^T}v_{t+1}^T.$$

10 Appendix C: Isoelastic Utility: A Closed Form in Logs

We will only consider the infinite horizon case, and start from the case with only permanent shocks. The outcome of this section will be an expression for innovation in *log* consumption of the form

$$\ln C_{t+1}-\ln C_t=\frac{1}{a^p}v_{t+1}^p+\frac{1-q}{a^T}v_{t+1}^T-\frac{\ln\frac{q}{\delta}}{\gamma}+\frac{\gamma}{2}\left[\left(\frac{1}{a^p}\right)^2\sigma_{v^p}^2+\left(\frac{1-q}{a^T}\right)^2\sigma_{v^p}^2\right],$$

where v_{t+1}^p is the innovation to (log) permanent income and $\frac{1}{\gamma}$ is the intertemporal elasticity of substitution of consumption and C is consumption.[42]

[42] In this section, we change a bit notation hoping that it will be all clear since we define any new variable.

The Isoelastic Model Let $y_t = \ln Y_t$, and $e := \ln N$. We assume the log income process follows:

$$y_t = x_t + \xi_t$$

where

$$x_t = f(\theta_t, e_t) = \theta_t + a^p \min\{0, e\} + b^p \max\{0, e\}, \qquad \text{with} \quad a^p \geq 1 > b^p.$$

This is precisely the same formulation as above, but with all variables interpreted in logs. Similarly, we specify $\xi_t = v_t^T + a^T \min\{l_t, 0\} + b^T \max\{l_t, 0\}$. Again, for notational simplicity, we will disregard ξ_t. We hence denote a^p simply with a.

Notice that the production function corresponds to a modification of the standard Cobb-Douglas: $X_t = \Theta_t N_t^a$, and $X_t = \Theta_t N_t^b$ for $N_t \leq 1$ and $N_t \geq 1$ respectively.

We specify the following process for skills:

$$\ln \Theta_t \equiv \theta_t = \theta_{t-1} + v_t^p.$$

An additional assumption, which will be crucial for us to get an exact closed form, is that the shocks v_t^p are normally distributed with mean μ_{v^p} and variance $\sigma_{v^p}^2$.

Moreover, assume that the agent has Cobb-Douglas/CRRA preferences of the following form:

$$\mathbf{E}_0 \sum_{t=0} \delta^t \frac{\left(C_t \cdot N_t^{-1}\right)^{1-\gamma}}{1-\gamma} \text{ for } \gamma > 1; \quad \text{and} \quad \mathbf{E}_0 \sum_{t=0} \delta^t \left(\ln C_t - \ln N_t\right) \text{ for } \gamma = 1.$$

For future use, notice that we can write:

$$\frac{\left(C_t \cdot N_t^{-1}\right)^{1-\gamma}}{1-\gamma} = \frac{1}{1-\gamma} \exp\{(1-\gamma)(c_t - e_t)\}$$

where $c_t := \ln C_t$.

It will be convenient to write the problem in logs so that we can use the analogies to the case in levels. The budget constraint can be written as follows:

$$\exp\{c_t\} + q b_t = \exp\left\{x_t + \tau_t\left(x^t\right)\right\} + b_{t-1}.$$

Since in equilibrium we will have $N_t^* \equiv 1$, the Euler equation is the usual one

$$\mathbf{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right] = \mathbf{E}_t\left[\exp\left(-\gamma \frac{c_{t+1}}{c_t}\right)\right] = \exp\left(-\gamma \mu_t + \gamma^2 \frac{1}{2} \sigma_t^2\right) = \frac{q}{\delta},$$

where we used the fact that C_{t+1} is log normally distributed, with Δc_{t+1} having conditional mean μ_t and conditional variance σ_t^2.

Since we will implement $b_t^* \equiv 0$, $c_t(x^t) = x_t + \tau_t(x^t)$ and the objective function becomes

$$\mathbf{E}_0 \sum_{t=0} \delta^t \frac{1}{1-\gamma} \exp\left\{(1-\gamma)\left(x_t + \tau_t\left(x^t\right) - e_t\right)\right\}.$$

Given our specification for f, the objective function can be all expressed in logs. It is now easy to see the strong analogy to the case in levels considered above. The first order condition for (log) effort e_t is

$$\mathbf{E}_t \sum_{n=0}^{\infty} \delta^n \left(\frac{C_{t+n}}{C_t}\right)^{1-\gamma} \frac{\partial \tau_{t+n}(x^{t+n})}{\partial x_t} = \frac{1}{a} - 1 \tag{64}$$

One can again show that conditional expectations can be decomposed since $\frac{\partial \tau_{t+n}\left(x^{t+n}\right)}{\partial x_t}$ does not depend on x_{t+n}. This is exactly as above for the model in levels. Moreover, since C_t is log normally distributed, we have

$$\mathbf{E}_t\left[\left(\frac{C_{t+n}}{C_t}\right)^{1-\gamma}\right] = \mathbf{E}_t\left[\exp\left((1-\gamma)\frac{c_{t+1}}{c_t}\right)\right] = \exp\left((1-\gamma)\mu_t + \frac{1}{2}(1-\gamma)^2\sigma_t^2\right).$$

¿From the Euler equation we obtain:

$$\begin{aligned}
\exp\left((1-\gamma)\mu_t + \frac{1}{2}(1-\gamma)^2\sigma_t^2\right) &= \exp\left(-\gamma\mu_t + \gamma^2\frac{1}{2}\sigma_t^2\right)\exp\left(\mu_t + \frac{1}{2}(1-2\gamma)\sigma_t^2\right) \\
&= \frac{q}{\delta}\exp\left(\mu_t + \frac{1}{2}\sigma_t^2 - \gamma\sigma_t^2\right) \\
&: = \frac{q}{\delta} d_t.
\end{aligned} \tag{65}$$

Similarly, by the law of iterated expectations, we get

$$\mathbf{E}_t\left[\left(\frac{C_{t+n}}{C_t}\right)^{-\gamma}\right] = \mathbf{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\mathbf{E}_{t+1}\left(\frac{C_{t+2}}{C_{t+1}}\right)^{-\gamma}\cdot ..\cdot \mathbf{E}_{t+n-1}\left(\frac{C_{t+n}}{C_{t+n-1}}\right)^{-\gamma}\right] = \left(\frac{q}{\delta}\right)^n$$

Using the properties of the normality, and assuming that log consumption innovation conditional variance and conditional mean are constant and equal to μ and σ^2 respectively (a property which can be verified below), and denoting $d := \exp\left(\mu + \left(\frac{1}{2} - \gamma\right)\sigma^2\right) > 0$, we have[43]

$$\begin{aligned}
\mathbf{E}_t\left[\exp\left((1-\gamma)\frac{c_{t+n}}{c_t}\right)\right] &= \mathbf{E}_t\left[\exp\left\{-\gamma\frac{c_{t+1}}{c_t}\right\} d\mathbf{E}_{t+1}\exp\left\{-\gamma\frac{c_{t+2}}{c_{t+1}}\right\} d\cdot ..\cdot \mathbf{E}_{t+n-1}\exp\left\{-\gamma\frac{c_{t+n}}{c_{t+n-1}}\right\} d\right] \\
&= \left(\frac{qd}{\delta}\right)^n.
\end{aligned}$$

[43]It can be deduced from (65) the intuitive fact that when utility is logarithmic, we have $d = \frac{\delta}{q}$ and, obviously this is consistent with

$$\mathbf{E}_t\left[\exp\left((1-\gamma)\frac{c_{t+n}}{c_t}\right)\right] = 1.$$

Again by the law of iterated expectations, the incentive constraint (64) hence becomes

$$\mathbf{E}_t \sum_{n=0}^{\infty} (qd)^n \frac{\partial \tau_{t+n}\left(x^{t+n}\right)}{\partial x_t} = \frac{1}{a} - 1$$

¿From the Euler equation, one can easily see that future marginal taxes must be zero, so that

$$\frac{\partial \tau_t\left(x^t\right)}{\partial x_t} = \frac{1}{a} - 1,$$

and

$$\Delta c_{t+1} = c_{t+1} - c_t = \frac{1}{a} v^p_{t+1} + \mu,$$

where $\mu = \mathbf{E}_t \Delta c_{t+1}$. Since, from the above expression, the variance of log consumption is $\sigma_c^2 = \frac{1}{a^2}\sigma^2_{v^p}$, from the Euler equation, we have

$$\mu = -\frac{\ln \frac{q}{\delta}}{\gamma} + \gamma \frac{1}{2}\sigma_c^2 = -\frac{\ln \frac{q}{\delta}}{\gamma} + \frac{\gamma}{2a^2}\sigma^2_{v^p}.$$

In order to get the expression for log taxes (not in levels) notice that the *log* of tax must display a deterministic drift:

$$\tau_t\left(x^t\right) = \left(\frac{1}{a} - 1\right) x_t + t\left[-\frac{\ln \frac{q}{\delta}}{\gamma} + \frac{\gamma}{2a^2}\sigma^2_{v^p}\right]. \tag{66}$$

Now the analogy to the quadratic case is transparent. One can indeed show that the analysis for temporary shocks is again a combination of that just performed and that we have done when we studied the case with temporary shocks in the quadratic case. We hence obtain the expression reported at the beginning of this section.

Why consumption is log normally distributed? By following a similar derivation to that for the quadratic case, from the Euler equation, we have

$$c_t = x_t + \tau_t\left(x^t\right) = -\ln \mathbf{E}_t \exp\left\{-x_{t+1} - \tau_{t+1}\left(x^{t+1}\right)\right\}$$

Going backward we obtain that τ_{t+1} only depends on x_{t+1}, and it is actually linear in x_{t+1} (with constant slope), hence consumption is log normally distributed. Start by the Euler equation for b_{T-1}, since we know that the tax in the last period is linear in x_T, we have

$$\begin{aligned} x_{T-1} + \tau_{T-1}\left(x^{T-1}\right) &= -\frac{1}{\gamma}\ln \mathbf{E}_{T-1}\exp\left\{-\gamma\frac{1}{a}x_T - \gamma\tau_T\left(x^{T-1}\right)\right\} \\ &= -\frac{1}{\gamma}\ln\exp\left\{-\gamma\frac{1}{a}\mathbf{E}_{T-1}x_T - \gamma\tau_T\left(x^{T-1}\right) + \gamma^2\frac{1}{2}\frac{1}{a^2}\sigma^2_{v^p}\right\} \\ &= \frac{1}{a}x_{T-1} - \tau_T\left(x^{T-1}\right) + \frac{\gamma}{2a^2}\sigma^2_{v^p}. \end{aligned}$$

where we used the fact that in equilibrium $\mathbf{E}_t x_T = x_{T-1}$. And so on till period 1.

Figure 1: The Production Function

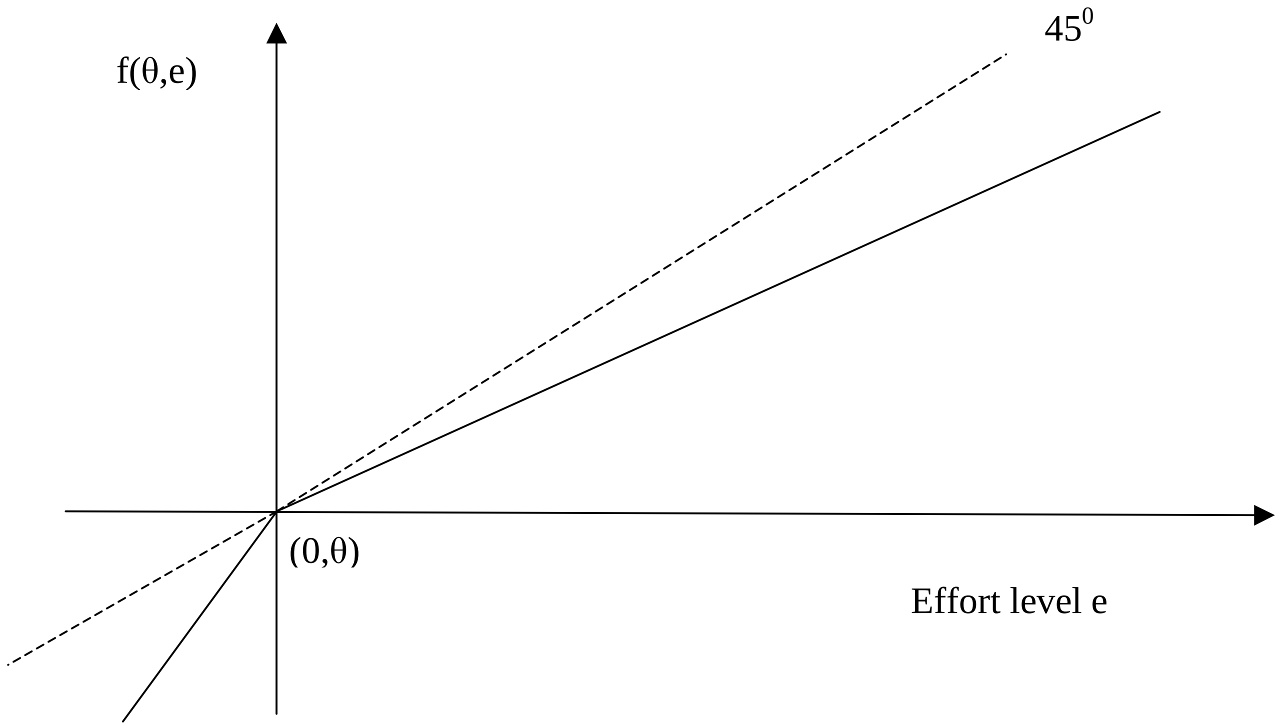

Table 1

Non Durable Consumption

	gross earnings	**gross earnings**	**gross earnings + benefits**	**gross earnings + benefits**	**net earnings + benefits**	**net earnings + benefits**
income equation						
w2	1	1	1	1	1	1
ayc(t)	-	-	-	-	-	-
	0.505	-	0.333	-	0.376	-
ayc(t-1)	*(20.735)*	-	*(25.683)*	-	*(18.240)*	-
	-0.672	-	-0.748	-	-0.507	-
ayc(t-2)	*(12.252)*	-	*(18.183)*	-	*(10.247)*	-
w1						
ayy(t)	1.159	1.171	1.073	1.064	0.771	0.793
	(0.346)	*(0.317)*	*(0.329)*	*(0.317)*	*(0.253)*	*(0.243)*
ayy(t-1)	-1.140	-1.148	-0.887	-0.894	-0.602	-0.587
	(0.454)	*(0.415)*	*(0.492)*	*(0.459)*	*(0.342)*	*(0.323)*
ayy(t-2)	0.992	0.995	0.827	0.844	0.619	0.568
	(0.370)	*(0.331)*	*(0.356)*	*(0.311)*	*(0.257)*	*(0.227)*
consumption equation						
w1						
acc(t)	1	1	1	1	1	1
	-	-	-	-	-	-
acc(t-1)	-0.577	-0.493	-0.604	-0.491	-0.612	-0.499
	(0.196)	*(0.114)*	*(0.193)*	*(0.116)*	*(0.191)*	*(0.111)*
acc(t-2)	0.084	-	0.116	-	0.118	-
	(0.201)	-	*(0.191)*	-	*(0.193)*	-
Log L	-773.3	-773.9	-759.5	-760.6	-685.0	-685.5
excess smoothness	-0.491	-0.499	-0.489	-0.493	-0.273	-0.263
se	*(0.171)*	*(0.165)*	*(0.160)*	*(0.153)*	*(0.132)*	*(0.128)*
Comparison with 4 lags model						
Log L 4lags model	-771.5	-773.6	-756.3	-760.4	-681.7	-685.3
LR	3.7	0.54	6.34	0.42	6.52	0.34
P-value	*0.717*	*0.763*	*0.386*	*0.811*	*0.368*	*0.844*

NOTES:

- all data are in (first diff of) levels
- SE in parentheses
- excess smoothness test computed as sum(acc(t-L))-sum(ayy(t-L))=0, with L=0,…,4
- interest rate =0.01
- *Income/consumption shock* is the shock that enters both the income and the consumption equation
- LR test is the test of current model against previous (to the left) one. In green if restrictions

Table 2
Total Consumption Expenditure

	gross earnings	**gross earnings**	**gross earnings + benefits**	**gross earnings + benefits**	**net earnings + benefits**	**net earnings + benefits**
income equation						
w2	1	1	1	1	1	1
ayc(t)	-	-	-	-	-	-
	51.615	-	0.400	-	0.321	-
ayc(t-1)	*(1535.290)*	-	*(9.124)*	-	*(28.027)*	-
	-36.884	-	-0.968	-	-0.765	-
ayc(t-2)	*(1072.450)*	-	*(7.222)*	-	*(20.287)*	-
w1						
ayy(t)	0.596	0.586	0.730	0.497	0.683	0.662
	(0.427)	*(0.351)*	*(0.383)*	*(0.346)*	*(0.236)*	*(0.205)*
ayy(t-1)	-0.341	-0.673	-0.472	-0.533	-0.277	-0.273
	(0.638)	*(0.473)*	*(0.572)*	*(0.477)*	*(0.358)*	*(0.304)*
ayy(t-2)	0.572	0.981	0.561	0.910	0.330	0.363
	(0.451)	*(0.467)*	*(0.430)*	*(0.451)*	*(0.280)*	*(0.238)*
consumption equation						
w1						
acc(t)	1	1	1	1	1	1
	-	-	-	-	-	-
acc(t-1)	-0.349	-0.346	-0.386	-0.345	-0.372	-0.395
	(0.406)	*(0.208)*	*(0.366)*	*(0.202)*	*(0.367)*	*(0.172)*
acc(t-2)	-0.069	-	-0.022	-	-0.031	-
	(0.358)	-	*(0.330)*	-	*(0.319)*	-
Log L	-885.2	-886.0	-869.4	-870.5	-788.0	-788.9
excess smoothness	-0.233	-0.224	-0.217	-0.203	-0.131	-0.174
se	*(0.128)*	*(0.144)*	*(0.116)*	*(0.113)*	*(0.099)*	*(0.128)*
Comparison with 4 lags model						
Log L 4lags model	-882.3	-885.4	-867.8	-869.6	-786.2	-788.3
LR	5.78	1.24	3.26	1.82	3.54	1.1
P-value	*0.448*	*0.538*	*0.776*	*0.403*	*0.739*	*0.577*

NOTES:

all data are in (first diff of) levels
SE in parentheses
excess smoothness test computed as sum(axx(t-L))-sum(ayx(t-L))=0, with L=0,…,4
interest rate =0.01

Table 3

Total Consumption Expenditure: log specification

	gross earnings	**gross earnings**	**gross earnings + benefits**	**gross earnings + benefits**	**net earnings + benefits**	**net earnings + benefits**
income equation						
Own shock						
ayy(t)	1	1	1	1	1	1
	-	-	-	-	-	-
ayy(t-1)	0.332	-	0.181	-	0.159	-
	(14.452)	-	*(4.936)*	-	*(1.982)*	-
ayy(t-2)	-0.748	-	-0.779	-	-0.595	-
	(10.305)	-	*(3.957)*	-	*(1.426)*	-
Income/Consumption Shock						
ayx(t)	0.889	0.816	0.717	0.748	0.799	0.570
	(0.200)	*(0.155)*	*(0.162)*	*(0.146)*	*(0.191)*	*(0.187)*
ayx(t-1)	-1.102	-1.250	-0.685	-0.836	-0.523	-0.222
	(0.249)	*(0.206)*	*(0.216)*	*(0.177)*	*(0.253)*	*(0.211)*
ayx(t-2)	0.393	0.660	0.094	0.223	-0.120	0.086
	(0.229)	*(0.199)*	*(0.191)*	*(0.139)*	*(0.214)*	*(0.152)*
consumption equation						
Income/Consumption Shock						
axx(t)	1	1	1	1	1	1
	-	-	-	-	-	-
axx(t-1)	-0.601	-1.011	-0.647	-1.011	-0.578	-0.619
	(0.243)	*(0.008)*	*(0.279)*	*(0.008)*	*(0.324)*	*(0.125)*
axx(t-2)	-0.411	-	-0.368	-	-0.444	-
	(0.247)	-	*(0.282)*	-	*(0.330)*	-
Log L	100.9	97.2	153.3	148.6	173.8	168.2
LR		7.4		*9.38*		*11.16*
P-Value		*0.007*		*0.002*		*0.001*
excess smoothness	-0.181	-0.226	-0.133	-0.141	-0.170	-0.048
se	*(0.100)*	*(0.079)*	*(0.050)*	*(0.053)*	*(0.066)*	*(0.106)*

NOTES:

all data are in (first diff of) levels
SE in parentheses
excess smoothness test computed as sum(axx(t-L))-sum(ayx(t-L))=0, with L=0,…,4
interest rate =0.01

Table 4
Variance Based Test

	non durable consumption	non durable consumption per ad.eq.	total consunmption	total consumption per ad.eq.
Ind. Var.				
Gross earnings	0.0709	0.0376	0.0765	0.0547
	0.0133	0.0154	0.0177	0.0196
implied a	3.7556	5.1571	3.6144	4.2746
	0.0484	0.0900	0.0610	0.0865
Gross earnings+	0.2357	0.1476	0.3019	0.2495
benefits	0.0302	0.0355	0.0401	0.0448
implied a	2.0596	2.6032	1.8200	2.0021
	0.0447	0.0747	0.0492	0.0634
Net earnings +	0.2601	0.1466	0.3478	0.2733
benefits	0.0351	0.0413	0.0463	0.0519
implied a	1.9608	2.6121	1.6957	1.9129
	0.0482	0.0871	0.0511	0.0686
Number of observ	505	505	505	505