Non-dual modal operators as a basis for 4-valued accessibility relations in Hybrid logic

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Abstract

The modal operators usually associated with the notions of *possibility* and *necessity* are classically duals. This paper aims to defy that duality in a paraconsistent environment, namely in a Belnapian Hybrid logic where both propositional variables and accessibility relations are four-valued. Hybrid logic, which is an extension of Modal logic, incorporates extra machinery such as nominals – for uniquely naming states – and a satisfaction operator – so that the formula under its scope is evaluated in the state whose name the satisfaction operator indicates.

In classical Hybrid logic the semantics of negation, when it appears before compound formulas, is carried towards subformulas, meaning that eventual inconsistencies can be found at the level of nominals or propositional variables but appear unrelated to the accessibility relations. In this paper we allow inconsistencies in propositional variables and, by breaking the duality between modal operators, inconsistencies at the level of accessibility relations arise. We introduce a sound and complete tableau system and a decision procedure to check if a formula is a consequence of a set of formulas. Tableaux will be used to extract syntactic models for databases, which will then be compared using different inconsistency measures. We conclude with a discussion about bisimulation.

Keywords: Hybrid logic, Four-valued, Paraconsistency, Modal Operators, Tableaux System, Measures of Inconsistency, Bisimulation

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1. Introduction

The introduction of a four-valued logic in the 70s by Nuel Belnap [3] considered an algebraic structure composed of, as the name indicates, four elements $\{\mathrm{t}, \mathrm{f}, \mathrm{b}, \mathrm{n}\}$. These elements intuitively represent the notions of "true", "false", "both true and false" (from a classical point of view, the same as *inconsistent*) and "neither true nor false" (or, in a classical interpretation, *incomplete*). Thus Belnap's logic is not only paraconsistent, as it excludes the Principle of Explosion, but also paracomplete, as it drops also the Principle of the Excluded Middle. Moreover, these four values may be arranged according to two partial orders: the first one, $\leq_{\mathrm{t}}$, reflects the "*quality*" of the information, whereas the second, $\leq_{\mathrm{k}}$, reflects the "*quantity*" of information. The bilattice structure is represented in Figure 1. Four-valued logics have been studied in the context of computer science and artificial intelligence and have been applied in areas such as symbolic model checking [8], semantics of logic programs [11] and inconsistency-tolerant systems.

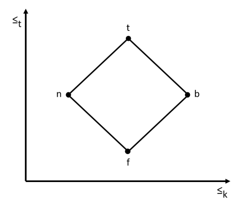


Figure 1: Belnap's bilattice.

In computer programs, relational structures provide a formalism to abstractly depict connections between states; a logic able to formalize these concepts is Modal logic. The notion of satisfiability in Modal logic is local, meaning that formulas are evaluated at a state in a structure. Unfortunately, there is no internal mechanism that allows us to focus on a specific state where we would like to evaluate a formula. It is possible to overcome this limitation if we add to Modal logic a new class of propositional variables, called nominals, which are true at exactly one state, and a satisfaction operator @, that acts as a jump operator. We are in the presence of a more powerful system in terms of expressivity, however still decidable and as complex as standard Modal logic K, called (Basic) Hybrid logic [5]. This extension allows us to refer to a specific state and describe what happens there: the formula $@_i\varphi$ holds at a state if and only if φ holds at the state named by i – actually the state of evaluation of the @-formula is not relevant, it either holds everywhere or nowhere; in particular, we are also able to specify equalities and transitions between (named) states.

Inconsistencies are generally thought of as undesirable and many argue that databases should be inconsistency-free; as such, there are tools designed to eradicate contradictions in order to keep systems consistent. Nonetheless this approach fails to use the benefits of paraconsistency and sometimes precious information is lost, as is the case when contradictions are seen as *mistakes* and one fails to see that their root is a fraudulent operation. Therefore, since contradictory information is everywhere and is actually the norm rather than the exception in the real world, it should be embraced, formalized and used in our favour. One possible use of paraconsistency is that it allows us to compare between different sources and choose the most reliable one based on the information we have in our hands. Observe that this is something that we naturally do in our daily lives: there are situations where we even expect divergences, something as simple as a set of different opinions about a certain subject is an almost guaranteed source of contradictions. Paraconsistent logics are flexible logical systems able to handle heterogeneous and complex data; they accommodate inconsistency in a sensible manner that treats inconsistent information as informative. Four-valued logics are in this category.

The present paper introduces a new four-valued, also known as Belnapian, Hybrid logic where the duality between modal operators is broken. We argue that this is the only way of capturing the real meaning of negation: just because it is not possible that φ, formally represented by $\neg\Diamond\varphi$, it does not mean that the negation of φ is mandatory, represented by $\Box\neg\varphi$. We interpret "*positive*" modal formulas (where negation does not occur immediately before the modal operator) in an almost classical fashion – the subtle difference is the use of positive relations that capture the evidence about the presence of transitions; we interpret "*negative*" modal formulas (where negation appears directly before the modal operator) in a distinct way and by resorting to negative relations that capture the evidence about the absence of transitions. In particular, $@_i\neg\Diamond j$ shall be interpreted as "there is no transition from state i to state j", whereas $@_i\Box\neg j$ is interpreted as "all transitions from state i lead to states different from j". Inconsistencies at the level of the accessibility relation are allowed and correspond to cases when $@_i\Diamond j$ and $@_i\neg\Diamond j$ occur. The logic is called double-Belnapian since it assigns one of four (Belnapian) values to both propositional variables and pairs of states (the accessibility problem). We introduce a tableau system for the logic and a tableau-based procedure in order to check if a formula is a consequence of a set of formulas. The tableau construction algorithm terminates and the system is sound and complete. Another section introduces measures of inconsistency for models and databases. Finally, we talk about bisimulation and how a classical extension does not preserve satisfiability, however a slight change in the definition gives us the desired result.

On contradictions in propositional variables and accessibility relations

Paraconsistent versions of modal logic where both the accessibility relation and the propositional variables are allowed a four-valued behaviour are not a novelty. The works of Wansing and Odintsov with $\mathsf{BK}^{\mathsf{FS}}$ logic [17] and Rivieccio and Jung with Modal bilattice logic MBL [18] are some examples of such logics. For a version of many-valued Modal logic check Fitting's work [12].

Even though proposals of paraconsistent Hybrid logics can be found in [6] and more recently in [9], the work on many-valued Hybrid logic MVHL in [16] seems to be the only version where paraconsistency is present at the level of propositional variables and the accessibility relation. The double-Belnapian

Hybrid logic $\mathsf{DBHL}*$ that we introduce in this paper is neither an extension of pre-existing paraconsistent Modal logics with Hybrid logic features, nor can it be captured by MVHL. The first distinguishing point is the fact that in the semantics for disjunction we resort to the classical notion of disjunctive syllogism. This will force a link between a disjunct and its negation since in case they both hold the other disjunct must hold as well in order to make the whole disjunction hold. Notwithstanding, that is not the main characteristic of $\mathsf{DBHL}*$. The main novelty here is the fact that modal operators $[\pi]$ and $\langle\pi\rangle$ are not considered duals. We argue that this approach is the way to capture the meaning of negation when it appears directly before the modal operator and this is how we will obtain inconsistencies at the level of accessibility relations. If the duality was kept, the usual semantics for modal operators would make it so that saying that in a structure it is possible to make a π-transition between the state named by i and a state where p holds, *i.e.*, the structure satisfies the formula $@_i\langle\pi\rangle p$, and that it is *not* possible to make such transition, *i.e.* $\neg @_i\langle\pi\rangle p$ holds in the structure, results in an explosion created at the level of propositional variable since the latter would be equivalent to $@_i[\pi]\neg p$. It is clear that the focus of negation is not the transition – we want it to be. At this point we would like to mention that $\mathsf{DBHL}*$ as appears in this paper differs from the also double-Belnapian version in [10] in the semantics for $\neg[\pi]\varphi$. The subtle difference is that, as the reader will have the opportunity to check, in $\mathsf{DBHL}*$ we resort to the non-satisfiability of φ, whereas in the older version we resorted to the satisfiability of $\neg\varphi$. Satisfaction coincides for pure formulas, *i.e.* formulas not involving propositional variables, but has a clearly distinct behaviour in other cases.

We propose a paraconsistent and paracomplete version of Hybrid logic such that in a structure both $@_i\langle\pi\rangle j$ and $@_i\neg\langle\pi\rangle j$ may hold or not; they will be interpreted as "there is evidence of a π-transition from the state named by i to the state named by j" and "there is evidence of the *lack* of a π-transition from the state named by i to the state named by j", respectively. The latter is not compatible with the interpretation of $@_i[\pi]\neg j$ which is that "there is evidence that all π-transitions from the state named by i terminate in a state which is not named by j".

The structures underlying this system will incorporate two valuations in order to deal with contradictions at the level of propositional variables, V^+ and V^-, and will, analogously, consider two families of accessibility relations, $(\mathrm{R}^+_\pi)_{\pi\in\mathrm{Mod}}$ and $(\mathrm{R}^-_\pi)_{\pi\in\mathrm{Mod}}$ in order to deal with contradictions at the level of the accessibility relations. The semantics for nominals is the usual: each nominal holds at a unique state.

2. Double-Belnapian Hybrid logic, DBHL∗

Let $\mathcal{L}_\pi = \langle \mathrm{Prop}, \mathrm{Nom}, \mathrm{Mod}\rangle$ be a *hybrid (multimodal) similarity type* where Prop is a countable set of *propositional variables*, Nom is a countable set disjoint from Prop and Mod is a countable set of *modality labels.* We use p, q, r, etc. to refer to the elements in Prop. The elements in Nom are called *nominals* and we typically write them as i, j, k, etc. Modalities are usually represented by π, π', etc.

Definition 1. *The well-formed formulas over* $\mathcal{L}_\pi$*,* $\mathrm{Form}(\mathcal{L}_\pi)$*, are defined by the following recursive definition:*

$$\varphi, \psi := \; i \mid p \mid \bot \mid \neg\varphi \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \varphi \supset \psi \mid \langle\pi\rangle\varphi \mid [\pi]\varphi \mid @_i\varphi$$

where $i \in \mathrm{Nom}, p \in \mathrm{Prop}, \pi \in \mathrm{Mod}$.

For any nominal i *and any formula* φ*,* $@_i\varphi$ *is called a* satisfaction statement*. Both* $@$ *and* $@_i$*, where* $i \in \mathrm{Nom}$*, will be referred to as* satisfaction operators*.* Literals *are formulas of the form* $@_i p, @_i\neg p, @_i\langle\pi\rangle j, @_i\neg\langle\pi\rangle j, @_i j$ *or* $@_i\neg j$ *for* $i, j \in \mathrm{Nom}$*,* $p \in \mathrm{Prop}$*,* $\pi \in \mathrm{Mod}$*.*

A hybrid multistructure is defined as a Kripke frame. Explosion at the level of propositional variables and accessibility relations is avoided and contradictions are allowed by considering two valuations and two families of accessibility relations. By doing so, the interpretation of propositional variables and the interpretation of the negation of propositional variables is independent, as well as the interpretation of positive modal formulas (formulas of the form $\langle\pi\rangle\varphi$ or $[\pi]\varphi$) and the interpretation of negative modal formulas (where negation appears directly before the modal operator).

Definition 2. *A* multistructure $\mathcal{G}$ *is a tuple* $(\mathrm{W}, (\mathrm{R}^+_\pi)_{\pi\in\mathrm{Mod}}, (\mathrm{R}^-_\pi)_{\pi\in\mathrm{Mod}}, \mathrm{N}, \mathrm{V}^+, \mathrm{V}^-)$*, where:*

- $\mathrm{W} \neq \varnothing$ *is the* domain *whose elements are called* states *or* worlds*;*
- *each* R^+_π *and* R^-_π *is a binary relation, called respectively the* positive *and the* negative π-accessibility relation*, such that* $\mathrm{R}^+_\pi, \mathrm{R}^-_\pi \subseteq \mathrm{W} \times \mathrm{W}$*;*
- $\mathrm{N} : \mathrm{Nom} \to \mathrm{W}$ *is a function called* hybrid nomination *that assigns nominals to elements in* W *such that for any nominal* i*,* $\mathrm{N}(i)$ *is the element of* W *named by* i*;*
- V^+ *and* V^- *are* hybrid valuations*, both with domain* Prop *and range* $\mathcal{P}(\mathrm{W})$*, such that* $\mathrm{V}^+(p)$ *is the set of states where the propositional variable* p *holds, and* $\mathrm{V}^-(p)$ *is the set of states where* $\neg p$ *holds.*

Observe that N is not necessarily a bijection. It is possible that some states are not named and that others have multiple names.

Intuitively, each set $\mathrm{V}^+(p)$ consists of evidence that p holds and $\mathrm{V}^-(p)$ consists of evidence that $\neg p$ holds. Analogously, R^+_π relates states between which there is evidence of a π-transition while R^-_π relates states between which there is evidence that the π-transition is missing.

Semantics is formalized as follows:

Definition 3. *A satisfaction relation* $\mathcal{G}, w \Vdash \varphi$ *between a multistructure* $\mathcal{G}$*, a state* w *in the multistructure and a formula* φ *is defined by structural induction on* φ *in Figure 2.*

We say that φ *is* globally satisfied *if* $\mathcal{G} \Vdash \varphi$*,* i.e.*,* $\mathcal{G}, w \Vdash \varphi$ *for all* $w \in \mathrm{W}$.

$$
\begin{array}{rl}
\text{(i)} & \mathcal{G}, w \Vdash p \text{ iff } w \in \mathrm{V}^+(p);\ \mathcal{G}, w \Vdash \neg p \text{ iff } w \in \mathrm{V}^-(p);\\
\text{(ii)} & \mathcal{G}, w \Vdash i \text{ iff } w = \mathrm{N}(i);\ \mathcal{G}, w \Vdash \neg i \text{ iff } w \neq \mathrm{N}(i);\\
\text{(iii)} & \mathcal{G}, w \Vdash \bot \text{ never};\ \mathcal{G}, w \Vdash \neg\bot \text{ always};\\
\text{(iv)} & \mathcal{G}, w \Vdash \neg\neg\varphi \text{ iff } \mathcal{G}, w \Vdash \varphi;\\
\text{(v)} & \mathcal{G}, w \Vdash \varphi \vee \psi \text{ iff } (\mathcal{G}, w \Vdash \varphi \text{ or } \mathcal{G}, w \Vdash \psi)\\
& \qquad\qquad\qquad \text{and } (\mathcal{G}, w \Vdash \neg\varphi \text{ implies } \mathcal{G}, w \Vdash \psi)\\
& \qquad\qquad\qquad \text{and } (\mathcal{G}, w \Vdash \neg\psi \text{ implies } \mathcal{G}, w \Vdash \varphi);\\
& \mathcal{G}, w \Vdash \neg(\varphi \vee \psi) \text{ iff } \mathcal{G}, w \Vdash \neg\varphi \text{ and } \mathcal{G}, w \Vdash \neg\psi;\\
\text{(vi)} & \mathcal{G}, w \Vdash \varphi \wedge \psi \text{ iff } \mathcal{G}, w \Vdash \varphi \text{ and } \mathcal{G}, w \Vdash \psi;\\
& \mathcal{G}, w \Vdash \neg(\varphi \wedge \psi) \text{ iff } \mathcal{G}, w \Vdash \neg\varphi \text{ or } \mathcal{G}, w \Vdash \neg\psi;\\
\text{(vii)} & \mathcal{G}, w \Vdash \varphi \supset \psi \text{ iff } \mathcal{G}, w \Vdash \varphi \text{ implies } \mathcal{G}, w \Vdash \psi;\\
& \mathcal{G}, w \Vdash \neg(\varphi \supset \psi) \text{ iff } \mathcal{G}, w \Vdash \varphi \text{ and } \mathcal{G}, w \Vdash \neg\psi;\\
\text{(viii)} & \mathcal{G}, w \Vdash \langle\pi\rangle\varphi \text{ iff } \exists w'(w\mathrm{R}^+_\pi w' \text{ and } \mathcal{G}, w' \Vdash \varphi);\\
& \mathcal{G}, w \Vdash \neg\langle\pi\rangle\varphi \text{ iff } \forall w'(w(\mathrm{R}^-_\pi)^c w' \text{ implies } \mathcal{G}, w' \nVdash \varphi);\\
\text{(ix)} & \mathcal{G}, w \Vdash [\pi]\varphi \text{ iff } \forall w'(w\mathrm{R}^+_\pi w' \text{ implies } \mathcal{G}, w' \Vdash \varphi);\\
& \mathcal{G}, w \Vdash \neg[\pi]\varphi \text{ iff } \exists w'(w(\mathrm{R}^-_\pi)^c w' \text{ and } \mathcal{G}, w' \nVdash \varphi);\\
\text{(x)} & \mathcal{G}, w \Vdash @_i\varphi \text{ iff } \mathcal{G}, \mathrm{N}(i) \Vdash \varphi;\\
& \mathcal{G}, w \Vdash \neg @_i\varphi \text{ iff } \mathcal{G}, \mathrm{N}(i) \Vdash \neg\varphi.
\end{array}
$$

Figure 2: Definition of the satisfaction relation $\mathcal{G}, w \Vdash \varphi$ for $\mathsf{DBHL}*$.

Notation-wise, $(\mathrm{R}^{\pm}_\pi)^c$ denotes the complement of $\mathrm{R}^{\pm}_\pi$, respectively.

We define models as follows:

Definition 4. *Let Δ be a set of formulas in* $\mathrm{Form}(\mathcal{L}_\pi)$*. A multistructure $\mathcal{G}$ is a* model *of Δ if and only if $\mathcal{G} \Vdash \delta$ for all $\delta \in \Delta$.*

Let us take a closer look at the definition of satisfiability for the disjunction of formulas: for a disjunction to hold, not only at least one of the disjuncts must hold, but also if the negation of one of the disjuncts holds, then the other disjunct must hold as well. A discussion about disjunctive syllogism can be found in [2]. We advocate in its favour by using the same argument as in [4] for Quasi-classical logic; the idea is that this definition links a disjunct and its classical complement and preserves the meaning of the resolution principle.

Remarks on the non-duality of modal operators

We will explore the semantics of modal formulas in detail now. The use of pure formulas, *i.e.*, those which do not contain propositional variables, will play an important role later as a means to represent syntactically the positive and negative transitions in a multistructure.

- $\langle\pi\rangle\varphi$ holds in a multistructure $\mathcal{G}$ at a state w if and only if there is evidence of a π-transition from the state w to a state w' where φ holds. Intuitively, *it is possible φ.*

- $\hookrightarrow$ Thus the formula $@_i\langle\pi\rangle j$ holds if and only if there exists evidence of a π-transition from the state named by the nominal i to the state named by the nominal j, *i.e.*, $\mathrm{N}(i)\mathrm{R}^+_\pi\mathrm{N}(j)$.

- $[\pi]\varphi$ holds in a multistructure $\mathcal{G}$ at a state w if and only if whenever there is evidence of a π-transition from the state w to a state w' then φ holds at w'. Intuitively, *it is necessary* φ.
  - $\hookrightarrow$ Thus the formula $@_i[\pi]\neg j$ holds if and only if every time there is evidence of a π-transition from the state named by the nominal i to a state w', the state w' is not named by j, which is the same as saying that there is not evidence of a π-transition from the state named by the nominal i to the state named by the nominal j, *i.e.*, $\mathrm{N}(i)(\mathrm{R}^+_\pi)^c\mathrm{N}(j)$.

This is when things get interesting: the negated versions.

- $\neg\langle\pi\rangle\varphi$ should then be intuitively thought of as a way of expressing *it is not possible* φ. This is not the same as saying that $\neg\varphi$ is necessary, *i.e.*, that every time there is evidence of a transition, $\neg\varphi$ holds in the accessed state, but rather that there is evidence that the transition is *missing* for all states where φ holds. The formula holds in a multistructure $\mathcal{G}$ at a state w if and only if whenever φ holds at a state w', there is evidence of the lack of a π-transition from the state w to the state w'.
  - $\hookrightarrow$ Thus the formula $@_i\neg\langle\pi\rangle j$ holds if and only if there is evidence of the lack of a π-transition from the state named by the nominal i to the state named by the nominal j, *i.e.*, $\mathrm{N}(i)\mathrm{R}^-_\pi\mathrm{N}(j)$.

- $\neg[\pi]\varphi$ should, by analogy, be intuitively read as *it is not necessary* φ, meaning that there is a state where φ does not hold, and still there is no evidence that a transition to that state is missing. The formula holds in a multistructure $\mathcal{G}$ at a state w if and only if there is a state w' such that there is not a negative π-transition from w to w' and where φ does not hold.
  - $\hookrightarrow$ Thus the formula $@_i\neg[\pi]\neg j$ holds if and only if there is no evidence of the lack of a π-transition from the state named by the nominal i to the state named by the nominal j, *i.e.*, $\mathrm{N}(i)(\mathrm{R}^-_\pi)^c\mathrm{N}(j)$.

Our proposed logic does not only allow local propositional contradictions such as $@_ip, @_i\neg p$, as it also accepts accessibility contradictions such as $@_i\langle\pi\rangle j$, $@_i\neg\langle\pi\rangle j$. In addition to being paraconsistent, the logic is also paracomplete.

Recall that nominals still behave classically and observe that the pairs $@_i\langle\pi\rangle j$, $@_i[\pi]\neg j$, and $@_i\neg\langle\pi\rangle j, @_i\neg[\pi]\neg j$ lead to explosion. For the first pair, by assuming that from i it is *possible* j and that it is *necessary* $\neg j$ we reach an inconsistency at the level of nominals.

A direct four-valued semantics of DBHL∗ *and a comparison with other logics*

In order to provide a semantics of DBHL∗ aesthetically closer to MBL so that a comparison between the two is clearer, we will use an alternative version of the definition of multistructure where, instead of positive and negative accessibility relations and valuations, we consider four-valued accessibility functions and valuations as follows. Recall the four truth-values we are dealing with: true (t), false (f), both true and false (b), neither true nor false (n).

Definition 5. *A* Belnapian structure $\mathcal{B}$ *is a tuple* $(\mathrm{W}, (\mathrm{R}_\pi)_{\pi\in\mathrm{Mod}}, \mathrm{V})$*, where:*

- $\mathrm{W} \neq \varnothing$ *is the domain;*
- *each* R_π *is an accessibility function such that* $\mathrm{R}_\pi : \mathrm{W} \times \mathrm{W} \to \mathbf{4}$*, where* $\mathbf{4}$ *is the usual set of Belnapian truth values:* $\{\mathrm{t}, \mathrm{f}, \mathrm{b}, \mathrm{n}\}$*; and*
- V *is a Belnapian valuation,* i.e.*, a function with domain* $(\mathrm{Prop}\cup\mathrm{Nom})\times\mathrm{W}$ *and range* $\mathbf{4}$ *such that* $\mathrm{V}(i, w) = \mathrm{t}$ *for a unique* $w \in \mathrm{W}$ *and* $\mathrm{V}(i, w') = \mathrm{f}$ *for every other state* w'*.*

Multistructures and Belnapian structures are equivalent when their domains coincide and for all $w, w' \in \mathrm{W}$, $\pi \in \mathrm{Mod}$, $p \in \mathrm{Prop}$, $i \in \mathrm{Nom}$, the equivalences in Figure 3 hold.

$$\begin{array}{lll}
\mathrm{R}_\pi(w, w') \in \{\mathrm{t}, \mathrm{b}\} & \text{iff} & (w, w') \in \mathrm{R}^+_\pi \\
\mathrm{R}_\pi(w, w') \in \{\mathrm{f}, \mathrm{b}\} & \text{iff} & (w, w') \in \mathrm{R}^-_\pi \\
\mathrm{V}(p, w) \in \{\mathrm{t}, \mathrm{b}\} & \text{iff} & w \in \mathrm{V}^+(p) \\
\mathrm{V}(p, w) \in \{\mathrm{f}, \mathrm{b}\} & \text{iff} & w \in \mathrm{V}^-(p) \\
\mathrm{V}(i, w) = \mathrm{t} & \text{iff} & w = \mathrm{N}(i) \\
\mathrm{V}(i, w) = \mathrm{f} & \text{iff} & w \neq \mathrm{N}(i)
\end{array}$$

Figure 3: Equivalences between multistructures and Belnapian structures.

The definition of semantics is simply put as:

Definition 6. *A satisfaction relation* $\mathcal{B}, w \Vdash_{\mathrm{d}} \varphi$ *between a Belnapian structure* $\mathcal{B}$*, a state* w *in the structure and a formula* φ *is defined as follows:*

$$\mathcal{B}, w \Vdash_{\mathrm{d}} \varphi \Leftrightarrow \mathrm{V}(\varphi, w) \in \{\mathrm{t}, \mathrm{b}\}$$

where the valuation V *is extended to all formulas according to Figure 4.*

The implication $\supset$ corresponds to the weak implication connective used, for example, in [18]. Observe also that

$$\mathrm{V}(\varphi \vee \psi, w) = \inf_{\leq_t}\left\{\sup_{\leq_t}\{\mathrm{V}(\varphi, w), \mathrm{V}(\psi, w)\}, \mathrm{V}(\neg\varphi \supset \psi, w), \mathrm{V}(\neg\psi \supset \varphi, w)\right\}.$$

Note that for nominals i and j such that $\mathrm{V}(i, w') = \mathrm{V}(j, w'') = \mathrm{t}$ and for an arbitrary state w, it is the case that:

$$\mathrm{V}(@_i\langle\pi\rangle j, w) = \mathrm{R}_\pi(w', w'').$$

The following result is immediate:

- $\mathrm{V}(\bot, w) = \mathrm{f}$
- $\mathrm{V}(\neg\varphi, w) = \ominus\mathrm{V}(\varphi, w)$, where $\ominus\mathrm{t} = \mathrm{f}$, $\ominus\mathrm{f} = \mathrm{t}$, $\ominus\mathrm{b} = \mathrm{b}$ and $\ominus\mathrm{n} = \mathrm{n}$
- $\mathrm{V}(\varphi \vee \psi, w) = \mathrm{V}(\varphi, w) \circledvee \mathrm{V}(\psi, w)$
- $\mathrm{V}(\varphi \wedge \psi, w) = \inf_{\leq_\mathrm{t}}\{\mathrm{V}(\varphi, w), \mathrm{V}(\psi, w)\}$
- $\mathrm{V}(\varphi \supset \psi, w) = \begin{cases} \mathrm{V}(\psi, w) & \text{if } \mathrm{V}(\varphi, w) \in \{\mathrm{t}, \mathrm{b}\} \\ \mathrm{t} & \text{otherwise} \end{cases}$
- $\mathrm{V}(\langle\pi\rangle\psi, w) = \sup_{\leq_\mathrm{t}}\{\mathrm{R}_\pi(w, w') \circledast \mathrm{V}(\psi, w'),\ w' \in \mathrm{W}\}$
- $\mathrm{V}([\pi]\psi, w) = \inf_{\leq_t}\{\mathrm{R}_\pi(w, w') \looparrowright \mathrm{V}(\psi, w'),\ w' \in \mathrm{W}\}$
- $\mathrm{V}(@_i\varphi, w) = \mathrm{V}(\varphi, w')$, where w' is such that $\mathrm{V}(i, w') = \mathrm{t}$

$\circledvee$, $\circledast$ and $\looparrowright$ are defined by the following matrices:

$\circledvee$	t	f	b	n
t	t	t	t	t
f	t	f	f	n
b	t	f	b	n
n	t	n	n	n

$\circledast$	t	f	b	n
t	t	f	t	f
f	f	f	f	f
b	b	f	b	f
n	n	f	n	f

$\looparrowright$	t	f	b	n
t	t	f	t	f
f	t	t	t	t
b	t	n	t	n
n	t	b	t	b

Figure 4: Extension of **4**-valued V to all formulas.

Lemma 1. *Let $\mathcal{G}$ be a multistructure and $\mathcal{B}$ a Belnapian structure such that $\mathcal{G}$ and $\mathcal{B}$ are equivalent. Then:*

$$\mathcal{G}, w \Vdash \varphi \Leftrightarrow \mathcal{B}, w \Vdash_\mathrm{d} \varphi,\ \textit{for all } \varphi \in \mathrm{Form}(\mathcal{L}_\pi).$$

In what follows we will simply omit the subscript in $\Vdash$.

Let us make a quick **comparison between** DBHL$*$ **and** MBL [18] in what concerns the semantics of modal formulas. We will consider the case with a single modality in what follows; an extension to the multimodal case is straightforward. In MBL a structure is defined as a tuple $\mathcal{K} = \left(\overline{\mathrm{W}}, \overline{\mathrm{R}}, \overline{\mathrm{V}}\right)$ such that $\overline{\mathrm{W}} \neq \varnothing$, $\overline{\mathrm{R}} : \overline{\mathrm{W}} \times \overline{\mathrm{W}} \to \mathbf{4}$, $\overline{\mathrm{V}} : \mathrm{Fm} \times \overline{\mathrm{W}} \to \mathbf{4}$, where Fm is the usual set of formulas in modal logic. The satisfaction relation is defined between a structure $\mathcal{K}$, a state w and a formula φ such that:

$$\mathcal{K}, w \vDash \varphi \Leftrightarrow \overline{\mathrm{V}}(\varphi, w) \in \{\mathrm{t}, \mathrm{b}\}$$

and $\overline{\mathrm{V}}(\Box\psi, w)$ and $\overline{\mathrm{V}}(\Diamond\psi, w)$ are defined as follows:

$$\begin{aligned} \overline{\mathrm{V}}(\Box\psi, w) &:= \inf_{\leq_\mathrm{t}}\{\overline{\mathrm{R}}(w, w') \longrightarrow \overline{\mathrm{V}}(\psi, w'),\ w' \in \mathrm{W}\} \\ \overline{\mathrm{V}}(\Diamond\psi, w) &:= \sup_{\leq_\mathrm{t}}\{\overline{\mathrm{R}}(w, w') * \overline{\mathrm{V}}(\psi, w'),\ w' \in \mathrm{W}\} \end{aligned}$$

where $\longrightarrow$ and $*$ are defined by the following matrices:

$\longrightarrow$	t	f	b	n
t	t	f	f	n
f	t	t	t	t
b	t	f	b	n
n	t	n	n	t

$*$	t	f	b	n
t	t	f	t	n
f	f	f	f	f
b	t	f	b	n
n	n	f	n	f

Compare with the definitions in DBHL$*$ (Figure 4). Observe that in this case $\overline{\mathrm{V}}(\Diamond\psi, w) = \overline{\mathrm{V}}(\neg\Box\neg\psi, w)$.

A curious thing about this system is that when we add nominals to it in order to extend it to hybrid logic, the following happens when ψ is a nominal (which behaves classically):

$*$	t	f
t	t	f
f	f	f
b	t	f
n	n	f

So, even though we can obtain the value n for the valuation of $\Diamond i$, it is never the case that one can obtain the value b. Furthermore, $\overline{\mathrm{V}}(\Diamond j, w) \neq \overline{\mathrm{R}}(w, w')$ where w' is the state such that $\overline{\mathrm{V}}(j, w) = \mathrm{t}$. That is clear when $\overline{\mathrm{R}}(w, w') = \mathrm{b}$ and $\mathrm{V}(\Diamond j, w) = \mathrm{t}$. Therefore an extension of MBL with nominals is not enough to syntactically express the simultaneous information about the presence and absence of transitions.

The second comparison we make is **between** DBHL$*$ **and** $\mathsf{BK}^{\mathsf{FS}}$ [17]. The satisfaction relation $\vDash$ defined between a $\mathsf{BK}^{\mathsf{FS}}$-model $\mathcal{S} = (\overline{\mathrm{W}}, \overline{\mathrm{R}}, \overline{\mathrm{R}'}, \overline{\mathrm{V}^{+}}, \overline{\mathrm{V}^{-}})$, where $\overline{\mathrm{W}} \neq \varnothing$, $\overline{\mathrm{R}}, \overline{\mathrm{R}'} \subseteq \overline{\mathrm{W}} \times \overline{\mathrm{W}}$, $\overline{\mathrm{V}^{+}}, \overline{\mathrm{V}^{-}} : \mathrm{Prop} \to \mathcal{P}(\overline{\mathrm{W}})$, a state w and a formula is such that:

$$
\begin{array}{lll}
\mathcal{S}, w \vDash \neg\neg\varphi & \text{iff} & \mathcal{S}, w \vDash \varphi \\
\mathcal{S}, w \vDash \Diamond\varphi & \text{iff} & \exists u \in \overline{\mathrm{W}}(w\overline{\mathrm{R}}u \text{ and } \mathcal{S}, u \vDash \varphi); \\
\mathcal{S}, w \vDash \neg\Diamond\varphi & \text{iff} & \forall u \in \overline{\mathrm{W}}(w\overline{\mathrm{R}'}u \text{ implies } \mathcal{S}, u \vDash \neg\varphi); \\
\mathcal{S}, w \vDash \Box\varphi & \text{iff} & \mathcal{S}, w \vDash \neg\Diamond\neg\varphi; \text{ and} \\
\mathcal{S}, w \vDash \neg\Box\varphi & \text{iff} & \mathcal{S}, w \vDash \Diamond\neg\varphi.
\end{array}
$$

This definition associates $\Diamond$ with $\overline{\mathrm{R}}$ and $\Box$ with $\overline{\mathrm{R}'}$ which means that the modal operators are not interpreted over the same relation. Thus, contrary to our expectations, $\Diamond\neg p$ and $\Box p$ do not mean that p and $\neg p$ are found simultaneously in a certain state. Take the following example: $\overline{\mathrm{W}} = \{w, w'\}$, $\overline{\mathrm{R}} = \{(w, w)\}$, $\overline{\mathrm{R}'} = \{(w, w')\}$, $\overline{\mathrm{V}^{+}}(p) = \{w'\}$ and $\overline{\mathrm{V}^{-}}(p) = \{w\}$; we can check that $\mathcal{S}, w \vDash \Diamond\neg p$ and $\mathcal{S}, w \vDash \Box p$. In opposition, in DBHL$*$ when in a multistructure $\Diamond\neg p$ and $\Box p$ hold at the same state, it means that there is a state w' such that there is evidence of a transition from w to w', where p and $\neg p$ hold.

Our last comparison is **between** DBHL$*$ **and** MVHL [16]. We restrict MVHL to four values, using Belnap's t-lattice, as follows:

A model is defined as a tuple $\mathcal{M} = \left(\overline{\mathrm{W}}, \overline{\mathrm{R}}, \overline{\mathrm{V}}\right)$ such that $\overline{\mathrm{W}} \neq \varnothing$, $\overline{\mathrm{R}} : \overline{\mathrm{W}} \times \overline{\mathrm{W}} \to \mathbf{4}$, and $\overline{\mathrm{V}} : \mathrm{Prop} \times \overline{\mathrm{W}} \to \mathbf{4}$. $\overline{\mathrm{V}}$ is extended to all formulas in a way such that $\overline{\mathrm{V}}(\Box\psi, w)$ and $\overline{\mathrm{V}}(\Diamond\psi, w)$ are defined as follows:

$$\begin{array}{rcl}\overline{\mathrm{V}}(\Box\psi, w) & := & \inf\limits_{\leq_t}\{\overline{\mathrm{R}}(w, w') \Rightarrow \overline{\mathrm{V}}(\psi, w'),\ w' \in \overline{\mathrm{W}}\} \\ \overline{\mathrm{V}}(\Diamond\psi, w) & := & \sup\limits_{\leq_t}\{\overline{\mathrm{R}}(w, w') \otimes \overline{\mathrm{V}}(\psi, w'),\ w' \in \overline{\mathrm{W}}\}\end{array}$$

where, for values $a, b \in \mathbf{4}$, $a \Rightarrow b$ is the greatest element $x \in \mathbf{4}$ such that $a \otimes x \leq_t b$.

Now consider a model in MVHL where $\overline{\mathrm{W}} = \{w, w'\}$, $\overline{\mathrm{R}}(w, w') = \mathrm{t}$ and $\overline{\mathrm{R}}(w, w) = \overline{\mathrm{R}}(w', w) = \overline{\mathrm{R}}(w', w') = \mathrm{f}$, and $\overline{\mathrm{V}}(p, w') = \mathrm{b}$ and $\overline{\mathrm{V}}(p, w) = \mathrm{n}$. It follows that $\overline{\mathrm{V}}(\Diamond p, w) = \mathrm{b} = \overline{\mathrm{V}}(\neg\Diamond p, w) = \overline{\mathrm{V}}(\Box\neg p, w)$. In DBHL$*$, taking $\mathrm{W} = \overline{\mathrm{W}}, \mathrm{R} = \overline{\mathrm{R}}, \mathrm{V} = \overline{\mathrm{V}}$, implies that $\mathrm{V}(\Diamond p, w) = \mathrm{t}$, $\mathrm{V}(\neg\Diamond p, w) = \mathrm{f}$, and $\mathrm{V}(\Box\neg p, w) = \mathrm{t}$. The interpretations of $\Box$ and $\Diamond$ for MVHL and DBHL$*$ are thus incomparable.

3. A tableau system for DBHL$*$

In this section we will introduce a sound, complete and terminating tableau system for DBHL$*$ and a decision procedure that checks if a formula is a consequence of a set of formulas, called a *database*. In order to do it, we consider an extra-logical operator * that acts on the satisfaction relation in the following sense: for a multistructure $\mathcal{G}$, a state w and a formula $\varphi \in \mathrm{Form}(\mathcal{L}_\pi)$,

$$\mathcal{G}, w \Vdash \varphi^* \Leftrightarrow \mathcal{G}, w \nVdash \varphi$$

and, analogously,

$$\mathcal{G} \Vdash \varphi^* \Leftrightarrow \mathcal{G} \nVdash \varphi.$$

It easy to check that $\mathcal{G} \Vdash \varphi^*$ if and only if it is false that $\forall w \in \mathrm{W}$, $\mathcal{G}, w \Vdash \varphi$ if and only if $\exists w \in \mathrm{W} : \mathcal{G}, w \nVdash \varphi$ if and only if $\exists w \in \mathrm{W} : \mathcal{G}, w \Vdash \varphi^*$. For convenience we will call φ^* a *starred formula*, and the set $\mathrm{Form}^*(\mathcal{L}_\pi) = \mathrm{Form}(\mathcal{L}_\pi) \cup \{\varphi^* \mid \varphi \in \mathrm{Form}(\mathcal{L}_\pi)\}$ the set of all signed formulas over $\mathcal{L}_\pi$.

The tableau system T is composed by the rules in Figures 5 and 6, where the latter deals with the interaction of * with formulas. A tableau in this system will be denoted $\mathcal{T}$.

The rules ($@_\mathrm{I}$), (Id), (Nom), ($[\pi]$), ($\neg\langle\pi\rangle$), ($\langle\pi\rangle^*$) and ($\neg[\pi]^*$) are called *non-destructive rules* and the remaining ones are called *destructive*. This distinction is made so that in the systematic tableau construction algorithm a destructive rule is applied at most once to a formula (a destructive rule has exactly one formula in the premise; the converse is not true). As in [7], the classification of rules as destructive and non-destructive corresponds to a classification of formulas according to their form.

Definition 7. *A* subformula *is defined by the following conditions:*

$$\frac{\varphi}{@_i\varphi}\ (@_\mathrm{I})(\mathrm{i}) \qquad \frac{@_i@_j\varphi}{@_j\varphi}\ (@_\mathrm{E}) \qquad \frac{@_i(\varphi\wedge\psi)}{\begin{array}{c}@_i\varphi\\ @_i\psi\end{array}}\ (\wedge) \qquad \frac{@_i(\varphi\supset\psi)}{(@_i\varphi)^* \mid @_i\psi}\ (\supset)$$

$$\frac{@_i(\varphi\vee\psi)}{\begin{array}{c|c|c|c|c|c|c}@_i\varphi & @_i\varphi & @_i\varphi & @_i\varphi & @_i\psi & @_i\psi & @_i\psi\\ (@_i\neg\varphi)^* & (@_i\neg\varphi)^* & (@_i\neg\psi)^* & @_i\psi & (@_i\neg\varphi)^* & (@_i\neg\psi)^* & (@_i\neg\varphi)^*\\ (@_i\neg\psi)^* & & @_i\psi & & (@_i\neg\psi)^* & & @_i\varphi\end{array}}\ (\vee)$$

$$\frac{@_i[\pi]\varphi, @_i\langle\pi\rangle j}{@_j\varphi}\ ([\pi]) \qquad \frac{@_i\langle\pi\rangle\varphi}{\begin{array}{c}@_i\langle\pi\rangle t\\ @_t\varphi\end{array}}\ (\langle\pi\rangle)(\mathrm{ii}) \qquad \frac{@_i\neg@_j\varphi}{@_j\neg\varphi}\ (\neg@)$$

$$\frac{@_i\neg(\varphi\wedge\psi)}{@_i\neg\varphi \mid @_i\neg\psi}\ (\neg\wedge) \qquad \frac{@_i\neg(\varphi\vee\psi)}{\begin{array}{c}@_i\neg\varphi\\ @_i\neg\psi\end{array}}\ (\neg\vee) \qquad \frac{@_i\neg(\varphi\supset\psi)}{\begin{array}{c}@_i\varphi\\ @_i\neg\psi\end{array}}\ (\neg\supset)$$

$$\frac{@_i\neg[\pi]\varphi}{\begin{array}{c}@_i\neg[\pi]\neg t\\ (@_t\varphi)^*\end{array}}\ (\neg[\pi])(\mathrm{iii}) \qquad \frac{@_i\neg\langle\pi\rangle\varphi, @_i\neg[\pi]\neg j}{(@_j\varphi)^*}\ (\neg\langle\pi\rangle) \qquad \frac{@_i\neg\neg\varphi}{@_i\varphi}\ (\neg\neg)$$

$$\frac{@_ij, @_i\varphi}{@_j\varphi}\ (\mathrm{Nom})(\mathrm{iv}) \qquad \frac{}{@_ii}\ (\mathrm{Id})(\mathrm{v})$$

(i) φ is not a satisfaction statement, i is in the branch;
(ii) $\varphi \notin \mathrm{Nom}$, t is a new nominal;
(iii) $\varphi \neq \neg i$ for all $i \in \mathrm{Nom}$, t is a new nominal;
(iv) for $@_i\varphi$ a literal;
(v) for i in the branch.

Figure 5: Tableau rules for (non-starred) formulas.

- *φ is a subformula of $\varphi \in \mathrm{Form}(\mathcal{L}_\pi)$, and ψ is a subformula of the starred formula ψ^*;*
- *if $\psi\wedge\delta$, $\psi\vee\delta$, or $\psi\supset\delta$ is a subformula of χ (χ is whether a formula or a starred version), then so are ψ and δ;*
- *if $@_i\psi$, $\neg\psi$, $[\pi]\psi$, or $\langle\pi\rangle\psi$ is a subformula of χ, then so is ψ.*

The tableau system T satisfies the following subformula property:

Theorem 1 (Subformula property)**.** *Suppose that $@_i\varphi \in \mathcal{T}$, where φ is not a nominal, $\varphi \neq \langle\pi\rangle j$ and $\varphi \neq \neg[\pi]\neg j$ for $\pi \in \mathrm{Mod}$, $j \in \mathrm{Nom}$ or that $(@_i\varphi)^* \in \mathcal{T}$. If $\varphi = \neg\psi$ then either φ or ψ is a subformula of a root formula. Otherwise, φ is a subformula of a root formula.*

Proof. The proof can be obtained by checking each rule. □

Note the following consequence of Theorem 1:

Lemma 2. *For any tableau $\mathcal{T}$ and nominal i, the following sets are finite:*

$$\frac{\varphi^*}{(@_t\varphi)^*}\;(@^*_{\mathrm{I}})(\text{vi}) \qquad \frac{(@_i@_j\varphi)^*}{(@_j\varphi)^*}\;(@^*_{\mathrm{E}}) \qquad \frac{(@_i(\varphi\wedge\psi))^*}{(@_i\varphi)^* \mid (@_i\psi)^*}\;(\wedge^*)$$

$$\frac{(@_i(\varphi\supset\psi))^*}{\begin{array}{c}@_i\varphi\\(@_i\psi)^*\end{array}}\;(\supset^*) \qquad \frac{(@_i(\varphi\vee\psi))^*}{\begin{array}{c|c|c}(@_i\varphi)^* & @_i\neg\varphi & @_i\neg\psi\\(@_i\psi)^* & (@_i\psi)^* & (@_i\varphi)^*\end{array}}\;(\vee^*) \qquad \frac{(@_i[\pi]\varphi)^*}{\begin{array}{c}@_i\langle\pi\rangle t\\(@_t\varphi)^*\end{array}}\;([\pi]^*)(\text{vii})$$

$$\frac{(@_i\langle\pi\rangle\varphi)^*, @_i\langle\pi\rangle j}{(@_j\varphi)^*}\;(\langle\pi\rangle^*) \qquad \frac{(@_i\neg(@_j\varphi))^*}{(@_j\neg\varphi)^*}\;(\neg @^*) \qquad \frac{(@_i\neg(\varphi\wedge\psi))^*}{\begin{array}{c}(@_i\neg\varphi)^*\\(@_i\neg\psi)^*\end{array}}\;(\neg\wedge^*)$$

$$\frac{(@_i\neg(\varphi\supset\psi))^*}{(@_i\varphi)^* \mid (@_i\neg\psi)^*}\;(\neg\supset^*) \qquad \frac{(@_i\neg(\varphi\vee\psi))^*}{(@_i\neg\varphi)^* \mid (@_i\neg\psi)^*}\;(\neg\vee^*) \qquad \frac{(@_i\neg\neg\varphi)^*}{(@_i\varphi)^*}\;(\neg\neg^*)$$

$$\frac{(@_i\neg[\pi]\varphi)^*, @_i\neg[\pi]\neg j}{@_j\varphi}\;(\neg[\pi]^*) \qquad \frac{(@_i\neg\langle\pi\rangle\varphi)^*}{\begin{array}{c}@_i\neg[\pi]\neg t\\@_t\varphi\end{array}}\;(\neg\langle\pi\rangle^*)(\text{vii}) \qquad \frac{(@_i\varphi)^*}{@_i\neg\varphi}\;(\mathrm{Id}^*)(\text{viii})$$

(vi) φ is not a satisfaction statement, t is a new nominal;
(vii) t is a new nominal;
(viii) $\varphi = j$ or $\varphi = \neg j$, where $j \in \mathrm{Nom}$.

Figure 6: Tableau rules for starred formulas.

$$\begin{array}{rcl}\Gamma_i & = & \{\varphi \mid @_i\varphi \in \mathcal{T},\ \textit{where } \varphi \neq \langle\pi\rangle j, \neg[\pi]\neg j,\ \textit{for } j \in \mathrm{Nom}, \pi \in \mathrm{Mod}\};\\ \Gamma_i^* & = & \{\varphi \mid (@_i\varphi)^* \in \mathcal{T}\}\end{array}$$

We define a binary relation between nominals naming the same states and another binary relation to establish the precedence of nominals as follows:

Definition 8. *Let Θ be a branch of a tableau and let Nom^Θ be the set of nominals occurring in the formulas of Θ. Define a binary relation $\sim_\Theta$ on Nom^Θ by $i \sim_\Theta j$ if and only if the formula $@_i j \in \Theta$.*

Definition 9. *Let i and j be nominals occurring on a branch Θ of a tableau in* T. *The nominal i is* included *in the nominal j with respect to Θ if, for any subformula φ of a root formula, the following holds:*

- *if $@_i\varphi \in \Theta$, then $@_j\varphi \in \Theta$;*
- *if $(@_i\varphi)^* \in \Theta$, then $(@_j\varphi)^* \in \Theta$;*
- *if $@_i\neg\varphi \in \Theta$, then $@_j\neg\varphi \in \Theta$;*
- *if $(@_i\neg\varphi)^* \in \Theta$, then $(@_j\neg\varphi)^* \in \Theta$.*

If i is included in j with respect to Θ, and the first occurrence of j on Θ is before the first occurrence of i, then we write $i \subseteq_\Theta j$.

A tableau is built following this construction:

Definition 10 (Tableau construction)**.** *Let Δ be a finite set of signed formulas in* $\mathrm{Form}^*(\mathcal{L}_\pi)$*. A tableau for Δ is built inductively according to the following rules:*

- *The one branch tableau $\mathcal{T}^0$ composed of the formulas in Δ is a tableau for Δ;*
- *The tableau $\mathcal{T}^{n+1}$ is obtained from the tableau $\mathcal{T}^n$ if it is possible to apply an arbitrary rule to $\mathcal{T}^n$ which obeys the following three restrictions:*
  (1) *If a formula that result from the application of a rule already occurs in the branch, then its addition is simply omitted;*
  (2) *A destructive rule is only applied once to the same formula in each branch;*
  (3) *The existential rules $(\langle\pi\rangle)$, $(\neg[\pi])$, $([\pi]^*)$ and $(\neg\langle\pi\rangle^*)$ are not applied to $@_i\langle\pi\rangle\varphi$, $@_i\neg[\pi]\varphi$, $(@_i[\pi]\varphi)^*$ nor $(@_i\neg\langle\pi\rangle\varphi)^*$ on a branch Θ if there exists a nominal j such that $i \subseteq_\Theta j$.*

Therefore a formula cannot occur more than once on a branch, a destructive rule cannot be applied more than once to the same formula in a branch and the third restriction are loop-check conditions.

Before proving termination of the tableau construction algorithm, let us observe that the only way new satisfaction operators may be introduced in a tableau is by using one of the following rules: $(@^*_{\mathrm{I}}), (\langle\pi\rangle), (\neg[\pi]), ([\pi]^*), (\neg\langle\pi\rangle^*)$ and $(@_{\mathrm{I}})$. The rule $(@^*_{\mathrm{I}})$ introduces a new satisfaction operator $@_t$; whenever a nominal i occurs in the branch but $@_i$ does not, the rule $(@_{\mathrm{I}})$ introduces it. The formulas in the premises of these rules are not satisfaction statements nor starred satisfaction statements. On the other hand, the rules $(\langle\pi\rangle), (\neg[\pi]), ([\pi]^*)$ and $(\neg\langle\pi\rangle^*)$ introduce a new satisfaction operator $@_t$ and the premises in these rules are either a satisfaction statement or a starred satisfaction statement. We distinguish between these two cases as follows:

Definition 11. *Let Θ be a branch of a tableau. If a new satisfaction operator $@_t$ is introduced by applying one of the rules $(\langle\pi\rangle)$, $(\neg[\pi])$, $([\pi]^*)$ or $(\neg\langle\pi\rangle^*)$ on the branch Θ to the formulas $@_i\langle\pi\rangle\varphi$, $@_i\neg[\pi]\varphi$, $(@_i[\pi]\varphi)^*$ or $(@_i\neg\langle\pi\rangle\varphi)^*$, respectively, then we say that t* is generated by i with respect to Θ*; otherwise, if $@_t$ is a new satisfaction operator obtained from $(@_{\mathrm{I}})$ or $(@^*_{\mathrm{I}})$, then we say that the nominal t* is self-generated.

We introduce a (partial) binary relation between nominals to keep track of the introduction of new satisfaction operators:

Definition 12. *Let* Nom^Θ *be the set of nominals occurring in Θ. We define a (partial) binary relation $<_\Theta$ over elements in* $\mathrm{Nom}^\Theta \cup \{\star\}$*, where $\star$ is a new symbol to denote the origin, as follows:*

- *$i <_\Theta j$, with $i, j \in \mathrm{Nom}^\Theta$, if and only if j is generated by i with respect to Θ;*

– $\star <_\Theta j$, *with* $j \in \mathrm{Nom}^\Theta$, *if and only if* $@_j$ *appears in a root formula or the nominal* j *is self-generated;*

– $x \not<_\Theta \star$, *for all* $x \in \mathrm{Nom}^\Theta \cup \{\star\}$.

The following result plays a central role in the proof of termination that will ensue next.

Proposition 1. *Let* Θ *be a branch of a tableau. Let* Nom^Θ *be the set of nominals occurring in* Θ. *The graph* $\left(\mathrm{Nom}^\Theta \cup \{\star\}, <_\Theta\right)$ *is a well-founded (*i.e. *has no infinite descending chain), finitely branching tree.*

Proof. That the graph is well-founded follows from the observation that if $x <_\Theta i$, then either (i) the first occurrence of $@_x$ in Θ is before the first occurrence of $@_i$, if $x \in \mathrm{Nom}^\Theta$, or (ii) if $x = \star$, there is no nominal in Θ which generates an occurrence of a satisfaction statement $@_i\varphi$.

That the graph is a tree follows from the fact that each nominal i in Θ is generated by at most one other nominal, and that all nominals have $\star$ as an ancestor.

That the graph is finitely branching follows from the fact that for any given nominal i, the sets Γ_i, Γ_i^* are finite (Lemma 2) and each of the finitely many formulas in these sets can generate at most one new satisfaction operator $@_t$ (when one of the rules $(\langle\pi\rangle), (\neg[\pi]), ([\pi]^*)$ or $(\neg\langle\pi\rangle^*)$ is applied). $\square$

Termination is proved as follows:

Theorem 2 (Termination)**.** *The systematic tableau construction algorithm terminates.*

Proof. Let us prove this by contradiction, so let us start by assuming that this is not the case. If the algorithm does not terminate, then the tableau must be infinite. Thus it contains an infinite branch, call it Θ. By restriction (1) in Definition 10, all formulas in Θ are distinct. By Theorem 1 and Lemma 2, every satisfaction statement or starred satisfaction statement in the branch of the form $@_i\varphi$ where $\varphi \neq \langle\pi\rangle j, \neg[\pi]\neg j,\ j \in \mathrm{Nom}, \pi \in \mathrm{Mod}$ or $(@_i\varphi)^*$ is such that, if $\varphi = \neg\psi$, then either φ or ψ is a subformula of a root formula; or otherwise φ is a subformula of a root formula. Since the number of subformulas of root formulas is finite and we assumed that the branch is infinite, then it must be the case that there are infinitely many satisfaction operators $@_i$. Therefore, the graph $\left(\mathrm{Nom}^\Theta \cup \{\star\}, <_\Theta\right)$ must be infinite. Since by Proposition 1 the graph is a well-founded, finitely branching tree, it must contain an infinite path $t_1 <_\Theta t_2 <_\Theta t_3 <_\Theta \ldots$.

For each $n > 0$ let Θ_n be the initial segment of Θ up to, but not including, the first satisfaction statement of the form $@_{t_{n+1}}\varphi$.

Also, for each $n > 0$, consider the following sets:

$\Lambda_n \quad = \{\varphi \mid @_{t_n}\varphi \in \Theta_n,\ \text{where } \varphi \neq \langle\pi\rangle j, \neg[\pi]\neg j,\ \text{for } j \in \mathrm{Nom}, \pi \in \mathrm{Mod}\}$; and
$\Lambda_n^* \quad = \{\varphi \mid (@_{t_n}\varphi)^* \in \Theta_n\}$.

All formulas φ in Λ_n and Λ_n^* are such that either φ or $\sim\varphi$ (a formula equivalent to $\neg\varphi$ where the negation symbol appears only directly before propositional variables, nominals or modal operators) are subformulas of a root formula, by Theorem 1. Since there are only finitely many such formulas, not all Λ_n can be distinct and the same happens for Λ_n^*. Thus eventually there exists $l, m \in \mathbb{N}$ with $l < m$ such that $\Lambda_l = \Lambda_m$ and $\Lambda_l^* = \Lambda_m^*$.

We will now prove that t_m is included in t_l with respect to Θ_m:

Let ψ be an arbitrary formula (as long as $\varphi \neq \langle\pi\rangle j, \neg[\pi]\neg j$ for $\pi \in \mathrm{Mod}$, $j \in \mathrm{Nom}$) for which $@_{t_m}\psi$ occurs in Θ_m, *i.e.*, such that $\psi \in \Lambda_m$. Since $\Lambda_l = \Lambda_m$, $\psi \in \Lambda_l$ and thus $@_{t_l}\psi \in \Theta_l$. Since Θ_l is an initial segment of Θ_m, we get that $@_{t_l}\psi \in \Theta_m$. From an analogous reasoning for formulas in Λ_m^*, it is proved that t_m is included in t_l with respect to Θ_m. It follows that $t_m \subseteq_{\Theta_m} t_l$, since the first occurrence of $@_{t_l}$ is before the first occurrence of $@_{t_m}$.

Now consider the first satisfaction statement of the form $@_{t_{m+1}}\delta$. By definition, $@_{t_{m+1}}\delta \notin \Theta_m$. However the nominal t_{m+1} is generated by t_m so it must be introduced by applying one of the rules $(\langle\pi\rangle), (\neg[\pi]), ([\pi]^*)$ or $(\neg\langle\pi\rangle^*)$ to a formula $@_{t_m}\chi$ or $(@_{t_m}\chi)^*$ in Θ_m. But this is in contradiction with restriction (3) in Definition 10 by which none of the rules $(\langle\pi\rangle), (\neg[\pi]), ([\pi]^*), (\neg\langle\pi\rangle^*)$ can be applied to $@_{t_m}\chi$ and $(@_{t_m}\chi)^*$ (for appropriate formulas χ) on the branch Θ_m since $t_m \subseteq_{\Theta_m} t_l$. $\square$

We move on to showing that the tableau system is sound and complete.

Theorem 3 (Soundness)**.** *The tableau rules are sound in the following sense: for any rule* $\dfrac{\Lambda}{\Sigma_1 \mid \cdots \mid \Sigma_n}$*,* $n \geq 1$*, and any multistructure* $\mathcal{G}$*,*

$$\mathcal{G} \Vdash \Lambda \Rightarrow \mathcal{G} \Vdash \Sigma_1 \text{ or } \ldots \text{ or } \mathcal{G} \Vdash \Sigma_n$$

where Λ*,* $\Sigma_1, \ldots, \Sigma_n \subset \mathrm{Form}^*(\mathcal{L}_\pi)$*.*

Proof. The proof can be obtained by checking each rule.

As an example, we will prove soundness for the rules $(\vee)$, $(\langle\pi\rangle)$ and $(\neg[\pi]^*)$:

- $$\dfrac{@_i(\varphi \vee \psi)}{\begin{array}{c|c|c|c|c|c|c} @_i\varphi & @_i\varphi & @_i\varphi & @_i\varphi & @_i\psi & @_i\psi & @_i\psi \\ (@_i\neg\varphi)^* & (@_i\neg\varphi)^* & (@_i\neg\psi)^* & @_i\psi & (@_i\neg\varphi)^* & (@_i\neg\psi)^* & (@_i\neg\varphi)^* \\ (@_i\neg\psi)^* & & @_i\psi & & (@_i\neg\psi)^* & & @_i\varphi \end{array}} \ (\vee)$$

Let $\mathcal{G}$ be a multistructure such that $\mathcal{G} \Vdash @_i(\varphi \vee \psi)$. Then:

$$\begin{array}{ll} & \mathcal{G} \Vdash @_i(\varphi \vee \psi) \\ \Leftrightarrow & \mathcal{G}, w \Vdash @_i(\varphi \vee \psi), \text{ for all } w \in W \\ \Leftrightarrow & \mathcal{G}, N(i) \Vdash (\varphi \vee \psi) \\ \Leftrightarrow & (\mathcal{G}, N(i) \Vdash \varphi \text{ or } \mathcal{G}, N(i) \Vdash \psi) \\ & \text{and } (\mathcal{G}, N(i) \Vdash \neg\varphi \text{ implies } \mathcal{G}, N(i) \Vdash \psi) \\ & \text{and } (\mathcal{G}, N(i) \Vdash \neg\psi \text{ implies } \mathcal{G}, N(i) \Vdash \varphi) \\ \Leftrightarrow & (\mathcal{G}, N(i) \Vdash \varphi \text{ or } \mathcal{G}, N(i) \Vdash \psi) \\ & \text{and } (\mathcal{G}, N(i) \nVdash \neg\varphi \text{ or } \mathcal{G}, N(i) \Vdash \psi) \\ & \text{and } (\mathcal{G}, N(i) \nVdash \neg\psi \text{ or } \mathcal{G}, N(i) \Vdash \varphi) \\ \Leftrightarrow & (\mathcal{G}, N(i) \Vdash \varphi \text{ and } \mathcal{G}, N(i) \nVdash \neg\varphi \text{ and } \mathcal{G}, N(i) \nVdash \neg\psi) \end{array}$$

$$
\begin{array}{ll}
& \text{or } (\mathcal{G}, N(i) \Vdash \varphi \text{ and } \mathcal{G}, N(i) \nVdash \neg\varphi \text{ and } \mathcal{G}, N(i) \Vdash \varphi) \\
& \text{or } (\mathcal{G}, N(i) \Vdash \varphi \text{ and } \mathcal{G}, N(i) \Vdash \psi \text{ and } \mathcal{G}, N(i) \nVdash \neg\psi) \\
& \text{or } (\mathcal{G}, N(i) \Vdash \varphi \text{ and } \mathcal{G}, N(i) \Vdash \psi \text{ and } \mathcal{G}, N(i) \Vdash \varphi) \\
& \text{or } (\mathcal{G}, N(i) \Vdash \psi \text{ and } \mathcal{G}, N(i) \nVdash \neg\varphi \text{ and } \mathcal{G}, N(i) \nVdash \neg\psi) \\
& \text{or } (\mathcal{G}, N(i) \Vdash \psi \text{ and } \mathcal{G}, N(i) \nVdash \neg\varphi \text{ and } \mathcal{G}, N(i) \Vdash \varphi) \\
& \text{or } (\mathcal{G}, N(i) \Vdash \psi \text{ and } \mathcal{G}, N(i) \Vdash \psi \text{ and } \mathcal{G}, N(i) \nVdash \neg\psi) \\
& \text{or } (\mathcal{G}, N(i) \Vdash \psi \text{ and } \mathcal{G}, N(i) \Vdash \psi \text{ and } \mathcal{G}, N(i) \Vdash \varphi) \\
\Leftrightarrow & (\mathcal{G}, N(i) \Vdash \varphi \text{ and } \mathcal{G}, N(i) \Vdash (\neg\varphi)^* \text{ and } \mathcal{G}, N(i) \Vdash (\neg\psi)^*) \\
& \text{or } (\mathcal{G}, N(i) \Vdash \varphi \text{ and } \mathcal{G}, N(i) \Vdash (\neg\varphi)^*) \\
& \text{or } (\mathcal{G}, N(i) \Vdash \varphi \text{ and } \mathcal{G}, N(i) \Vdash \psi \text{ and } \mathcal{G}, N(i) \Vdash (\neg\psi)^*) \\
& \text{or } (\mathcal{G}, N(i) \Vdash \varphi \text{ and } \mathcal{G}, N(i) \Vdash \psi) \\
& \text{or } (\mathcal{G}, N(i) \Vdash \psi \text{ and } \mathcal{G}, N(i) \Vdash (\neg\varphi)^* \text{ and } \mathcal{G}, N(i) \Vdash (\neg\psi)^*) \\
& \text{or } (\mathcal{G}, N(i) \Vdash \psi \text{ and } \mathcal{G}, N(i) \Vdash (\neg\varphi)^* \text{ and } \mathcal{G}, N(i) \Vdash \varphi) \\
& \text{or } (\mathcal{G}, N(i) \Vdash \psi \text{ and } \mathcal{G}, N(i) \Vdash (\neg\psi)^*) \\
\Leftrightarrow & (\mathcal{G} \Vdash @_i\varphi \text{ and } \mathcal{G} \Vdash (@_i\neg\varphi)^* \text{ and } \mathcal{G} \Vdash (@_i\neg\psi)^*) \\
& \text{or } (\mathcal{G} \Vdash @_i\varphi \text{ and } \mathcal{G} \Vdash (@_i\neg\varphi)^*) \\
& \text{or } (\mathcal{G} \Vdash @_i\varphi \text{ and } \mathcal{G} \Vdash @_i\psi \text{ and } \mathcal{G} \Vdash (@_i\neg\psi)^*) \\
& \text{or } (\mathcal{G} \Vdash @_i\varphi \text{ and } \mathcal{G} \Vdash @_i\psi) \\
& \text{or } (\mathcal{G} \Vdash @_i\psi \text{ and } \mathcal{G} \Vdash (@_i\neg\varphi)^* \text{ and } \mathcal{G} \Vdash (@_i\neg\psi)^*) \\
& \text{or } (\mathcal{G} \Vdash @_i\psi \text{ and } \mathcal{G} \Vdash (@_i\neg\varphi)^* \text{ and } \mathcal{G} \Vdash @_i\varphi) \\
& \text{or } (\mathcal{G} \Vdash @_i\psi \text{ and } \mathcal{G} \Vdash (@_i\neg\psi)^*)
\end{array}
$$

- $\dfrac{@_i\langle\pi\rangle\varphi}{\begin{array}{c}@_i\langle\pi\rangle t\\ @_t\varphi\end{array}}$ $(\langle\pi\rangle)$, for t a new nominal, $\varphi \notin \mathrm{Nom}$.

  Let $\mathcal{G}$ be a multistructure such that $\mathcal{G} \Vdash @_i\langle\pi\rangle\varphi$. Then:

$$
\begin{array}{ll}
& \mathcal{G} \Vdash @_i\langle\pi\rangle\varphi \\
\Leftrightarrow & \mathcal{G}, w \Vdash @_i\langle\pi\rangle\varphi, \text{ for all } w \in W \\
\Leftrightarrow & \mathcal{G}, N(i) \Vdash \langle\pi\rangle\varphi \\
\Leftrightarrow & \exists w' \in W\ (N(i) R^+_\pi w' \text{ and } \mathcal{G}, w' \Vdash \varphi) \\
\Leftrightarrow & \mathcal{G} \Vdash @_i\langle\pi\rangle t \text{ and } \mathcal{G}, w' \Vdash \varphi,\ \text{ where } t \text{ is a new nominal such that } N(t) = w' \\
\Leftrightarrow & \mathcal{G} \Vdash @_i\langle\pi\rangle t \text{ and } \mathcal{G} \Vdash @_t\varphi,\ \text{ where } t \text{ is a new nominal such that } N(t) = w' \\
\Rightarrow & \mathcal{G} \Vdash @_i\langle\pi\rangle t \text{ and } \mathcal{G} \Vdash @_t\varphi,\ \text{ for a new nominal } t
\end{array}
$$

- $\dfrac{(@_i\neg[\pi]\varphi)^*, @_i\neg[\pi]\neg j}{@_j\varphi}$ $(\neg[\pi]^*)$

  Let $\mathcal{G}$ be a multistructure such that $\mathcal{G} \Vdash (@_i\neg[\pi]\varphi)^*$ and $\mathcal{G} \Vdash @_i\neg[\pi]\neg j$. Then:

$$
\begin{array}{ll}
& \mathcal{G} \Vdash (@_i\neg[\pi]\varphi)^* \text{ and } \mathcal{G} \Vdash @_i\neg[\pi]\neg j \\
\Leftrightarrow & \mathcal{G} \nVdash @_i\neg[\pi]\varphi \text{ and } \mathcal{G} \Vdash @_i\neg[\pi]\neg j \\
\Leftrightarrow & \mathcal{G}, w \nVdash @_i\neg[\pi]\varphi,\ \text{ for some } w \in W \text{ and } \mathcal{G}, w' \Vdash @_i\neg[\pi]\neg j,\ \text{ for all } w' \in W \\
\Leftrightarrow & \mathcal{G}, N(i) \nVdash \neg[\pi]\varphi \text{ and } \mathcal{G}, N(i) \Vdash \neg[\pi]\neg j \\
\Leftrightarrow & \text{false}(\mathcal{G}, N(i) \Vdash \neg[\pi]\varphi) \text{ and } \mathcal{G}, N(i) \Vdash \neg[\pi]\neg j
\end{array}
$$

$$
\begin{array}{ll}
\Leftrightarrow & \text{false}(\exists w' \in W(N(i)\not R^-_\pi w' \text{ and } \mathcal{G}, w' \not\Vdash \varphi)) \\
& \text{and } \exists w'' \in W(N(i)\not R^-_\pi w'' \text{ and } \mathcal{G}, w'' \not\Vdash \neg j) \\
\Leftrightarrow & \forall w' \in W(N(i) R^-_\pi w' \text{ or } \mathcal{G}, w' \Vdash \varphi) \\
& \text{and } \exists w'' \in W(N(i)\not R^-_\pi w'' \text{ and } w'' = N(j)) \\
\Leftrightarrow & \forall w' \in W(N(i)\not R^-_\pi w' \text{ implies } \mathcal{G}, w' \Vdash \varphi) \\
& \text{and } \exists w'' \in W(N(i)\not R^-_\pi w'' \text{ and } w'' = N(j)) \\
\Rightarrow & \mathcal{G}, w'' \Vdash \varphi \text{ and } w'' = N(j) \\
\Leftrightarrow & \mathcal{G} \Vdash @_j \varphi
\end{array}
$$

The remaining cases are proved analogously. □

A branch is *closed* if and only if there is a formula ψ for which ψ and ψ^* are in that branch or if $@_i\bot$ or $@_i\neg i$ is in the branch for some nominal i. Otherwise the branch is *open*. A tableau is closed if and only if all of its branches are closed; otherwise the tableau is open.

In order to prove completeness, we prove that if a terminal tableau has an open branch Θ, then there exists a model $\mathcal{G}_\Theta$ and a state w where all root formulas are satisfied.

From now on, Θ is a branch of a terminal tableau.

Let U be the subset of Nom^Θ that contains every nominal i for which there is no nominal j such that $i \subseteq_\Theta j$. Let $\approx$ be the restriction of $\sim_\Theta$ (Definition 8) to U. Note that U contains all nominals appearing in the root formulas. Observe also that Θ is closed under the rules (Id) and (Nom), so both $\sim_\Theta$ and $\approx$ are equivalence relations.

Given a nominal i in U, we let $[i]_\approx$ denote the equivalence class of i with respect to $\approx$ and we let $\mathrm{U}/\approx$ denote the set of equivalence classes.

We let R^+_π be the binary relation on U defined by $i\mathrm{R}^+_\pi j$ if and only if there exists a nominal $j' \approx j$ such that one of the following conditions is satisfied:

1. $@_i\langle\pi\rangle j' \in \Theta$; or if
2. there exists a nominal $k \in \mathrm{Nom}^\Theta$ such that $@_i\langle\pi\rangle k \in \Theta$ and $k \subseteq_\Theta j'$.

On the other hand, we let R^-_π be the binary relation on U such that $i\mathrm{R}^-_\pi j$ if and only if $i\not\widehat{\mathrm{R}}^-_\pi j$ (observe that $\widehat{\mathrm{R}}^-_\pi$ is the complement of R^-_π), and $i\widehat{\mathrm{R}}^-_\pi j$ if and only if there exists a nominal $j' \approx j$ such that one of the following conditions is satisfied:

1. $@_i\neg[\pi]\neg j' \in \Theta$; or if
2. there exists a nominal $k \in \mathrm{Nom}^\Theta$ such that $@_i\neg[\pi]\neg k \in \Theta$ and $k \subseteq_\Theta j'$.

Note that the nominal k referred to in the second items is not an element of U. It follows from Θ being closed under the rule (Nom) that R^+_π and R^-_π are compatible with $\approx$ in the first argument and it is trivial that they are compatible with $\approx$ in the second argument. We let $\overline{\mathrm{R}^+_\pi}$, respectively $\overline{\mathrm{R}^-_\pi}$, be the binary relation on $\mathrm{U}/\approx$ defined by $[i]_\approx \overline{\mathrm{R}^+_\pi} [j]_\approx$, respectively $[i]_\approx \overline{\mathrm{R}^-_\pi} [j]_\approx$, if and only if $i\mathrm{R}^+_\pi j$, respectively $i\mathrm{R}^-_\pi j$. Analogously, we let $[i]_\approx \overline{\widehat{\mathrm{R}}^-_\pi} [j]_\approx$ be the binary relation on $\mathrm{U}/\approx$ defined by $[i]_\approx \overline{\widehat{\mathrm{R}}^-_\pi} [j]_\approx$ if and only if $i\widehat{\mathrm{R}}^-_\pi j$.

Let $\overline{\mathrm{N}} : U \to \mathrm{U}/\!\approx$ be defined such that $\overline{\mathrm{N}}(i) = [i]_{\approx}$.

Let V^+ be the function that to each ordinary propositional variable assigns the set of elements of U where that propositional variable occurs, *i.e.*, $i \in \mathrm{V}^+(p)$ if and only if $@_i p \in \Theta$. Conversely $i \notin \mathrm{V}^+(p)$ if and only if $@_i p \notin \Theta$. Analogously, let V^- be the function that to each ordinary propositional variable assigns the set of elements of U where the negation of that propositional variable occurs, *i.e.*, $i \in \mathrm{V}^-(p)$ if and only if $@_i \neg p \in \Theta$. Conversely $i \notin \mathrm{V}^-(p)$ if and only if $@_i \neg p \notin \Theta$. We let $\overline{\mathrm{V}^+}$ be defined by $\overline{\mathrm{V}^+}(p) = \{[i]_{\approx} \mid i \in \mathrm{V}^+(p)\}$. We define $\overline{\mathrm{V}^-}$ analogously: $\overline{\mathrm{V}^-}(p) = \{[i]_{\approx} \mid i \in \mathrm{V}^-(p)\}$.

Given a branch Θ, let $\mathcal{G}_\Theta = \left(U/\!\approx, \left(\overline{\mathrm{R}^+_\pi}\right)_{\pi \in \mathrm{Mod}}, \left(\overline{\mathrm{R}^-_\pi}\right)_{\pi \in \mathrm{Mod}}, \overline{\mathrm{N}}, \overline{\mathrm{V}^+}, \overline{\mathrm{V}^-}\right)$.

We will omit the reference to the branch in $\mathcal{G}_\Theta$ if it is clear from the context.

Theorem 4 (Model Existence)**.** *Let Θ be an open branch of a terminal tableau $\mathcal{T}$. The model extracted from the branch, $\mathcal{G}_\Theta$, is such that the following conditions hold:*

(i) *if $@_i\varphi \in \Theta$, then $\mathcal{G}_\Theta, [i]_{\approx} \Vdash \varphi$;*
(ii) *if $(@_i\varphi)^* \in \Theta$, then $\mathcal{G}_\Theta, [i]_{\approx} \nVdash \varphi$.*

whenever $@_i\varphi$ contains only nominals from U*.*

Proof. The proof is by induction on the complexity of φ:

- The base step for the cases when φ is either a nominal or the negation of a nominal, a propositional variable or the negation of a propositional variable follows from the definition of $\mathcal{G}$.

- The case when φ is $\perp$ is trivial.

Induction Hypothesis (I.H.): the result holds for the formulas $\psi, \delta, \neg\psi, \neg\delta$.

- $\varphi = \neg\neg\psi$
  (i) $@_i\neg\neg\psi \in \Theta$, then, by applying the rule $(\neg\neg)$, $@_i\psi \in \Theta$. By I.H. $\mathcal{G}, [i]_{\approx} \Vdash \psi$ and equivalently $\mathcal{G}, [i]_{\approx} \Vdash \neg\neg\psi$.
  (ii) $(@_i\neg\neg\psi)^* \in \Theta$, then, by applying the rule $(\neg\neg^*)$, $(@_i\psi)^* \in \Theta$. By I.H. $\mathcal{G}, [i]_{\approx} \nVdash \psi$ and equivalently $\mathcal{G}, [i]_{\approx} \nVdash \neg\neg\psi$.

- $\varphi = \psi \wedge \delta$
  (i) $@_i(\psi \wedge \delta) \in \Theta$, then, by applying the rule $(\wedge)$, $@_i\psi, @_i\delta \in \Theta$. By I.H. $\mathcal{G}, [i]_{\approx} \Vdash \psi$ and $\mathcal{G}, [i]_{\approx} \Vdash \delta$. Therefore $\mathcal{G}, [i]_{\approx} \Vdash \psi \wedge \delta$.
  (ii) $(@_i(\psi \wedge \delta))^* \in \Theta$, then, by applying the rule $(\wedge^*)$, $(@_i\psi)^* \in \Theta$ or $(@_i\delta)^* \in \Theta$. Hence, by I.H., $\mathcal{G}, [i]_{\approx} \nVdash \psi$ or $\mathcal{G}, [i]_{\approx} \nVdash \delta$. Therefore $\mathcal{G}, [i]_{\approx} \nVdash \psi \wedge \delta$.

- $\varphi = \neg(\psi \wedge \delta)$
  (i) $@_i\neg(\psi \wedge \delta) \in \Theta$, then, by applying the rule $(\neg\wedge)$, $@_i\neg\psi \in \Theta$ or $@_i\neg\delta \in \Theta$. By I.H. $\mathcal{G}, [i]_{\approx} \Vdash \neg\psi$ or $\mathcal{G}, [i]_{\approx} \Vdash \neg\delta$. Therefore $\mathcal{G}, [i]_{\approx} \Vdash \neg(\psi \wedge \delta)$.

(ii) $(@_i\neg(\psi \wedge \delta))^* \in \Theta$, then, by applying the rule $(\neg\wedge^*)$, $(@_i\neg\psi)^*$, $(@_i\neg\delta)^* \in \Theta$. Hence, by I.H. $\mathcal{G}, [i]_\approx \not\Vdash \neg\psi$ and $\mathcal{G}, [i]_\approx \not\Vdash \neg\delta$. Therefore $\mathcal{G}, [i]_\approx \not\Vdash \neg(\psi \wedge \delta)$.

- $\varphi = \psi \supset \delta$
  (i) $@_i(\psi \supset \delta) \in \Theta$, then, by applying the rule $(\supset)$, $(@_i\psi)^* \in \Theta$ or $@_i\delta \in \Theta$. By I.H. $\mathcal{G}, [i]_\approx \not\Vdash \psi$ or $\mathcal{G}, [i]_\approx \Vdash \delta$. Thus $\mathcal{G}, [i]_\approx \Vdash \psi \supset \delta$.
  (ii) $(@_i(\psi \supset \delta))^* \in \Theta$, then, by applying the rule $(\supset^*)$, $@_i\psi, (@_i\delta)^* \in \Theta$. Hence, by I.H., $\mathcal{G}, [i]_\approx \Vdash \psi$ and $\mathcal{G}, [i]_\approx \not\Vdash \delta$. So $\mathcal{G}, [i]_\approx \not\Vdash \psi \supset \delta$.

- $\varphi = \neg(\psi \supset \delta)$
  (i) $@_i\neg(\psi \supset \delta) \in \Theta$, then, by applying the rule $(\neg \supset)$, $@_i\psi, @_i\neg\delta \in \Theta$. By I.H., $\mathcal{G}, [i]_\approx \Vdash \psi$ and $\mathcal{G}, [i]_\approx \Vdash \neg\delta$. So $\mathcal{G}, [i]_\approx \Vdash \neg(\psi \supset \delta)$.
  (ii) $(@_i\neg(\psi \supset \delta))^* \in \Theta$, then, by applying the rule $(\neg\supset^*)$, $(@_i\psi)^* \in \Theta$ or $(@_i\neg\delta)^* \in \Theta$. Hence, by I.H., $\mathcal{G}, [i]_\approx \not\Vdash \psi$ or $\mathcal{G}, [i]_\approx \not\Vdash \neg\delta$. Therefore $\mathcal{G}, [i]_\approx \not\Vdash \neg(\psi \supset \delta)$.

- $\varphi = @_j\psi$
  (i) $@_i@_j\psi \in \Theta$, then, by applying the rule $(@_{\mathrm{E}})$, $@_j\psi \in \Theta$. By I.H. $\mathcal{G}, [j]_\approx \Vdash \psi$. Thus $\mathcal{G}, [i]_\approx \Vdash @_j\psi$.
  (ii) $(@_i@_j\psi)^* \in \Theta$, then, by applying the rule $(@^*_{\mathrm{E}})$, $(@_j\psi)^* \in \Theta$. By I.H. $\mathcal{G}, [j]_\approx \not\Vdash \psi$ and it follows that $\mathcal{G}, [i]_\approx \not\Vdash @_j\psi$.

- $\varphi = \neg(@_j\psi)$
  (i) $@_i\neg(@_j\psi) \in \Theta$, then, by applying the rule $(\neg@)$, $@_j\neg\psi \in \Theta$. By I.H. $\mathcal{G}, [j]_\approx \Vdash \neg\psi$. Therefore $\mathcal{G}, [i]_\approx \Vdash @_j\neg\psi$, so $\mathcal{G}, [i]_\approx \Vdash \neg@_j\psi$.
  (ii) $(@_i\neg(@_j\psi))^* \in \Theta$, then, by applying the rule $(\neg@^*)$, $(@_j\neg\psi)^* \in \Theta$. By I.H. $\mathcal{G}, [j]_\approx \not\Vdash \neg\psi$ and it follows that $\mathcal{G}, [i]_\approx \not\Vdash @_j\neg\psi$ and so $\mathcal{G}, [i]_\approx \not\Vdash \neg@_j\psi$.

- $\varphi = \psi \vee \delta$
  (i) $@_i(\psi \vee \delta) \in \Theta$, then from applying rule $(\vee)$, one of the following happens:

    1. $@_i\psi, (@_i\neg\psi)^*, (@_i\neg\delta)^* \in \Theta$; or
    2. $@_i\psi, (@_i\neg\psi)^* \in \Theta$; or
    3. $@_i\psi, @_i\delta, (@_i\neg\delta)^* \in \Theta$; or
    4. $@_i\psi, @_i\delta \in \Theta$; or
    5. $@_i\delta, (@_i\neg\psi)^*, (@_i\neg\delta)^* \in \Theta$; or
    6. $@_i\delta, (@_i\neg\delta)^* \in \Theta$; or
    7. $@_i\delta, (@_i\neg\psi)^*, @_i\psi \in \Theta$.

  Recall that

$$\begin{array}{ll} & \mathcal{G}, [i]_\approx \Vdash \psi \vee \delta \\ \Leftrightarrow & (\mathcal{G}, [i]_\approx \Vdash \psi \text{ or } \mathcal{G}, [i]_\approx \Vdash \delta) \\ & \text{and } (\mathcal{G}, [i]_\approx \Vdash \neg\psi \text{ implies } \mathcal{G}, [i]_\approx \Vdash \delta) \\ & \text{and } (\mathcal{G}, [i]_\approx \Vdash \neg\delta \text{ implies } \mathcal{G}, [i]_\approx \Vdash \psi) \end{array}$$

In the first case, by I.H. $\mathcal{G}, [i]_{\approx} \not\Vdash \neg\psi$, $\mathcal{G}, [i]_{\approx} \not\Vdash \neg\delta$, and $\mathcal{G}, [i]_{\approx} \Vdash \psi$. Therefore, $\mathcal{G}, [i]_{\approx} \Vdash \psi \vee \delta$.

Cases 2.–7. follow a similar approach.

Thus if $@_i(\psi \vee \delta) \in \Theta$ then $\mathcal{G}, [i]_{\approx} \Vdash \psi \vee \delta$.

(ii) $(@_i(\psi \vee \delta))^* \in \Theta$, then by applying rule $(\vee^*)$, either:

1. $(@_i\psi)^*, (@_i\delta)^* \in \Theta$; or
2. $@_i\neg\psi, (@_i\delta)^* \in \Theta$; or
3. $@_i\neg\delta, (@_i\psi)^* \in \Theta$.

In case 1. by I.H. $\mathcal{G}, [i]_{\approx} \not\Vdash \psi$ and $\mathcal{G}, [i]_{\approx} \not\Vdash \delta$. Thus, $\mathcal{G}, [i]_{\approx} \not\Vdash \psi \vee \delta$.

In cases 2. and 3. the reasoning is analogous.

In conclusion, if $(@_i(\psi \vee \delta))^* \in \Theta$ then $\mathcal{G}, [i]_{\approx} \not\Vdash \psi \vee \delta$.

- $\varphi = \neg(\psi \vee \delta)$
  (i) $@_i\neg(\psi \vee \delta) \in \Theta$, then from applying rule $(\neg\vee)$, $@_i\neg\psi, @_i\neg\delta \in \Theta$. By I.H. $\mathcal{G}, [i]_{\approx} \Vdash \neg\psi$, and $\mathcal{G}, [i]_{\approx} \Vdash \neg\delta$. Therefore, $\mathcal{G}, [i]_{\approx} \Vdash \neg(\psi \vee \delta)$.
  (ii) $(@_i\neg(\psi \vee \delta))^* \in \Theta$, then by applying rule $(\neg\vee^*)$, it follows that $(@_i\neg\psi)^* \in \Theta$ or $(@_i\neg\delta)^* \in \Theta$. By I.H. $\mathcal{G}, [i]_{\approx} \not\Vdash \neg\psi$ or $\mathcal{G}, [i]_{\approx} \not\Vdash \neg\delta$. It follows that $\mathcal{G}, [i]_{\approx} \not\Vdash \neg(\psi \vee \delta)$.

- $\varphi = \langle\pi\rangle\psi$
  (i) ∗ if $\psi = j$, $j \in \mathrm{Nom}$:
  $@_i\langle\pi\rangle j \in \Theta$, then $[i]_{\approx}\overline{\mathrm{R}_\pi^+}[j]_{\approx}$ and by definition $\mathcal{G}, [i]_{\approx} \Vdash \langle\pi\rangle j$.
  ∗ if ψ is not a nominal:
  $@_i\langle\pi\rangle\psi \in \Theta$, then by the application of the rule $(\langle\pi\rangle)$, $@_i\langle\pi\rangle t$ and $@_t\psi \in \Theta$, for a new nominal t. Then:
    · if $t \in \mathrm{U}$, $[i]_{\approx}\overline{\mathrm{R}_\pi^+}[t]_{\approx}$. By I.H., $\mathcal{G}, [t]_{\approx} \Vdash \psi$, so $\mathcal{G}, [i]_{\approx} \Vdash \langle\pi\rangle\psi$.
    · if $t \notin \mathrm{U}$, $\exists a$ such that $t \subseteq_\Theta a$. Assume that there is no b such that $a \subseteq_\Theta b$, *i.e.*, $a \in \mathrm{U}$. Since $@_t\psi \in \Theta$, from Theorem 1 it follows that if $\psi = \neg\delta$ then either ψ or δ is a subformula of a root formula, otherwise ψ is a subformula of a root formula; Definition 9 implies that $@_a\psi \in \Theta$. By I.H. $\mathcal{G}, [a]_{\approx} \Vdash \psi$ and by definition $[i]_{\approx}\overline{\mathrm{R}_\pi^+}[a]_{\approx}$. So, $\mathcal{G}, [i]_{\approx} \Vdash \langle\pi\rangle\psi$.

  (ii) $(@_i\langle\pi\rangle\psi)^* \in \Theta$. We want to prove that $\mathcal{G}, [i]_{\approx} \not\Vdash \langle\pi\rangle\psi$, *i.e.*, that for all $[k]_{\approx}$ such that $[i]_{\approx}\overline{\mathrm{R}_\pi^+}[k]_{\approx}$, $\mathcal{G}, [k]_{\approx} \not\Vdash \psi$.

  Let k be a nominal such that $[i]_{\approx}\overline{\mathrm{R}_\pi^+}[k]_{\approx}$; by definition exists k' with $k'{\approx}k$ that satisfies one of the following conditions:

  ∗ $@_i\langle\pi\rangle k' \in \Theta$, which then by applying the rule $(\langle\pi\rangle^*)$, implies that $(@_{k'}\psi)^* \in \Theta$. By I.H. $\mathcal{G}, [k']_{\approx} \not\Vdash \psi$. Since $[k']_{\approx} = [k]_{\approx}$, then $\mathcal{G}, [k]_{\approx} \not\Vdash \psi$. Or:
  ∗ $\exists a \in \mathrm{Nom}^\Theta$ such that $@_i\langle\pi\rangle a \in \Theta$ and $a \subseteq_\Theta k'$, then by applying $(\langle\pi\rangle^*)$ it follows that $(@_a\psi)^* \in \Theta$. From Theorem 1, if $\psi = \neg\delta$ either ψ or δ is a subformula of a root formula otherwise ψ is a subformula of a root formula. Since $a \subseteq_\Theta k'$, $(@_{k'}\psi)^* \in \Theta$. By I.H. $\mathcal{G}, [k']_{\approx} \not\Vdash \psi$. Since $[k']_{\approx} = [k]_{\approx}$, then $\mathcal{G}, [k]_{\approx} \not\Vdash \psi$.

It follows that $\mathcal{G}, [i]_\approx \not\Vdash \langle\pi\rangle\psi$.

- $\varphi = \neg\langle\pi\rangle\psi$

  (i) $@_i\neg\langle\pi\rangle\psi \in \Theta$. We want to prove that $\mathcal{G}, [i]_\approx \Vdash \neg\langle\pi\rangle\psi$, *i.e.*, that for all $[k]_\approx$ such that $[i]_\approx \cancel{\overline{\mathrm{R}^-_\pi}} [k]_\approx$, $\mathcal{G}, [k]_\approx \not\Vdash \psi$.

  Let k be a nominal such that $[i]_\approx \cancel{\overline{\mathrm{R}^-_\pi}} [k]_\approx$. That is the case if and only if $[i]_\approx \widehat{\overline{\mathrm{R}^-_\pi}} [k]_\approx$, which by definition implies that exists k' with $k' \approx k$ that satisfies one of the following two conditions:

  - $*$ $@_i\neg[\pi]\neg k' \in \Theta$ which then by applying the rule $(\neg\langle\pi\rangle)$ implies that $(@_{k'}\psi)^* \in \Theta$ and by I.H. $\mathcal{G}, [k']_\approx \not\Vdash \psi$. Since $[k']_\approx = [k]_\approx$, $\mathcal{G}, [k]_\approx \not\Vdash \psi$. Or:
  - $*$ $\exists a \in \mathrm{Nom}^\Theta$ such that $@_i\neg[\pi]\neg a \in \Theta$ and $a \subseteq_\Theta k'$, then by applying the rule $(\neg\langle\pi\rangle)$, $(@_a\psi)^* \in \Theta$. From Theorem 1 if $\psi = \neg\delta$ then either ψ or δ is a subformula of a root formula, otherwise ψ is a subformula of a root formula. Since $a \subseteq_\Theta k'$, $(@_{k'}\psi)^* \in \Theta$. By I.H. $\mathcal{G}, [k']_\approx \not\Vdash \psi$. Since $[k]_\approx = [k']_\approx$, $\mathcal{G}, [k]_\approx \not\Vdash \psi$.

  Therefore $\mathcal{G}, [i]_\approx \Vdash \neg\langle\pi\rangle\psi$

  (ii) $(@_i\neg\langle\pi\rangle\psi)^* \in \Theta$, thus by applying rule $(\neg\langle\pi\rangle^*)$, $@_i\neg[\pi]\neg t, @_t\psi \in \Theta$, for a new nominal t. Then:

  - $*$ if $t \in \mathrm{U}$, $[i]_\approx \widehat{\overline{\mathrm{R}^-_\pi}} [t]_\approx$. By I.H. $\mathcal{G}, [t]_\approx \Vdash \psi$. Thus $\mathcal{G}, [i]_\approx \not\Vdash \neg\langle\pi\rangle\psi$.
  - $*$ if $t \notin \mathrm{U}$, $\exists a$ such that $t \subseteq_\Theta a$. Assume that there is no b such that $a \subseteq_\Theta b$, *i.e.*, $a \in \mathrm{U}$. By Theorem 1 on $@_t\psi \in \Theta$, if $\psi = \neg\delta$ then either ψ or δ is a subformula of a root formula, otherwise ψ is a subformula of a root formula. Since $t \subseteq_\Theta a$, $@_a\psi \in \Theta$. By I.H. $\mathcal{G}, [a]_\approx \Vdash \psi$ and by definition $[i]_\approx \widehat{\overline{\mathrm{R}^-_\pi}} [a]_\approx$. It follows that $\mathcal{G}, [i]_\approx \not\Vdash \neg\langle\pi\rangle\psi$.

- $\varphi = [\pi]\psi$

  (i) $@_i[\pi]\psi \in \Theta$. We want to prove that $\mathcal{G}, [i]_\approx \Vdash [\pi]\psi$, *i.e.*, that for all $[k]_\approx$ such that $[i]_\approx \overline{\mathrm{R}^+_\pi} [k]_\approx$, $\mathcal{G}, [k]_\approx \Vdash \psi$.

  Let k be a nominal such that $[i]_\approx \overline{\mathrm{R}^+_\pi} [k]_\approx$. By definition it implies that exists k' with $k' \approx k$ which satisfies one of the following two conditions:

  - $*$ $@_i\langle\pi\rangle k' \in \Theta$ which then, by applying the rule $([\pi])$, implies that $@_{k'}\psi \in \Theta$ and by I.H. $\mathcal{G}, [k']_\approx \Vdash \psi$. Since $[k']_\approx = [k]_\approx$, then $\mathcal{G}, [k]_\approx \Vdash \psi$. Or:
  - $*$ $\exists a \in \mathrm{Nom}^\Theta$ such that $@_i\langle\pi\rangle a \in \Theta$ and $a \subseteq_\Theta k'$, then by applying the rule $([\pi])$, $@_a\psi \in \Theta$. From Theorem 1 if $\psi = \neg\delta$ either ψ or δ is a subformula of a root formula, otherwise ψ is a subformula of a root formula. Since $a \subseteq_\Theta k'$, $@_{k'}\psi \in \Theta$. By I.H. $\mathcal{G}, [k']_\approx \Vdash \psi$. Since $[k]_\approx = [k']_\approx$, $\mathcal{G}, [k]_\approx \Vdash \psi$.

It follows that $\mathcal{G}, [i]_\approx \Vdash [\pi]\psi$.

(ii) $(@_i[\pi]\psi)^* \in \Theta$, thus by applying the rule $([\pi]^*)$, $@_i\langle\pi\rangle t, (@_t\psi)^* \in \Theta$, for a new nominal t. Then:

- if $t \in \mathrm{U}$, $[i]_\approx \overline{\mathrm{R}^+_\pi}[t]_\approx$. By I.H. $\mathcal{G}, [t]_\approx \nVdash \psi$, thus $\mathcal{G}, [i]_\approx \nVdash [\pi]\psi$.
- if $t \notin \mathrm{U}$, $\exists a$ such that $t \subseteq_\Theta a$. Assume that there is no b such that $a \subseteq_\Theta b$, *i.e.*, $a \in \mathrm{U}$. By Theorem 1 on $(@_t\psi)^* \in \Theta$, if $\psi = \neg\delta$ either ψ or δ is a subformula of a root formula, otherwise ψ is a subformula of a root formula. Since $t \subseteq_\Theta a$, $(@_a\psi)^* \in \Theta$. By I.H. $\mathcal{G}, [a]_\approx \nVdash \psi$ and by definition $[i]_\approx \overline{\mathrm{R}^+_\pi}[a]_\approx$. It follows that $\mathcal{G}, [i]_\approx \nVdash [\pi]\psi$.

- $\varphi = \neg[\pi]\psi$

(i) $@_i\neg[\pi]\psi \in \Theta$, then by applying the rule $(\neg[\pi])$, $@_i\neg[\pi]\neg t, (@_t\psi)^* \in \Theta$, for a new nominal t. Then:

- if $t \in \mathrm{U}$, $[i]_\approx \overline{\widehat{\mathrm{R}}^-_\pi}[t]_\approx$. By I.H. $\mathcal{G}, [t]_\approx \nVdash \psi$. Thus $\mathcal{G}, [i]_\approx \Vdash \neg[\pi]\psi$.
- if $t \notin U$, $\exists a$ such that $t \subseteq_\Theta a$. Assume that there is no b such that $a \subseteq_\Theta b$, *i.e.*, $a \in U$. By Theorem 1 on $(@_t\psi)^* \in \Theta$, if $\psi = \neg\delta$ either ψ or δ is a subformula of a root formula, otherwise ψ is a subformula of a root formula. Since $t \subseteq_\Theta a$, $(@_a\psi)^* \in \Theta$. By I.H., $\mathcal{G}, [a]_\approx \nVdash \psi$ and by definition $[i]_\approx \overline{\widehat{\mathrm{R}}^-_\pi}[a]_\approx$. It follows that $\mathcal{G}, [i]_\approx \Vdash \neg[\pi]\psi$.

(ii) $(@_i\neg[\pi]\psi)^* \in \Theta$. We want to prove that $\mathcal{G}, [i]_\approx \nVdash \neg[\pi]\psi$, *i.e.*, that for all $[k]_\approx$ such that $[i]_\approx \overline{\widehat{R}^-_\pi}[k]_\approx$, $\mathcal{G}, [k]_\approx \Vdash \psi$.

Let k be a nominal such that $[i]_\approx \overline{\widehat{R}^-_\pi}[k]_\approx$. By definition, exists k' with $k' \approx k$ that satisfies one of the following two conditions:

- $@_i\neg[\pi]\neg k' \in \Theta$, which then by applying the rule $(\neg[\pi]^*)$, implies that $@_{k'}\psi \in \Theta$. By I.H. $\mathcal{G}, [k']_\approx \Vdash \psi$. Since $[k']_\approx = [k]_\approx$, then $\mathcal{G}, [k]_\approx \Vdash \psi$. Or:
- $\exists a \in \mathrm{Nom}^\Theta$ such that $@_i\neg[\pi]\neg a \in \Theta$ and $a \subseteq_\Theta k'$, then by applying $(\neg[\pi]^*)$ it follows that $@_a\psi \in \Theta$. From Theorem 1 if $\psi = \neg\delta$ either ψ or δ is a subformula of a root formula, otherwise ψ is a subformula of a root formula. Since $a \subseteq_\Theta k'$, $@_{k'}\psi \in \Theta$. By I.H. $\mathcal{G}, [k']_\approx \Vdash \psi$. Since $[k']_\approx = [k]_\approx$, $\mathcal{G}, [k]_\approx \Vdash \psi$.

Thus $\mathcal{G}, [i]_\approx \nVdash \neg[\pi]\psi$.

$\square$

Observe that root formulas that are satisfaction statements contain only nominals from U, therefore they are captured in this theorem. On the other hand, if a root formula φ (resp. φ^*) is not a satisfaction statement, the application of the rule $(@_\mathrm{I})$ (resp. $(@^*_\mathrm{I})$) turns it into one. Thus, by proving satisfiability of $@_i\varphi$ (resp. $(@_i\varphi)^*$) in a model $\mathcal{G}$, at a state w, where $i \in \mathrm{U}$, we are proving that there exists a model and a state where φ (resp. φ^*) is satisfied. Note also that ($@_\mathrm{I}$) is applied to φ for every i in the branch so there is at least one state where all root formulas are satisfied.

There is a straightforward consequence relation in DBHL$*$, defined as:

Definition 13. *Let Δ be a finite set of signed formulas called* database. *The formula φ is a* consequence *of Δ if and only if for all multistructures $\mathcal{G}$ where all formulas in Δ are globally satisfied, φ is globally satisfied as well.*

Formally,

$$\Delta \Vdash \varphi \Leftrightarrow \forall \mathcal{G}\, (\mathcal{G} \Vdash \Delta \Rightarrow \mathcal{G} \Vdash \varphi)$$

In other words, φ is a consequence of Δ if and only if φ is globally satisfied in all multistructures that are models of Δ. It is clear that the consequence relation $\Vdash$ is non-trivializable. Observe that Δ may include both non-starred as well as starred formulas.

The following result holds:

Proposition 2. *For any finite set of signed formulas $\Delta = \{\delta_1, \ldots, \delta_n\} \subset \mathrm{Form}^*(\mathcal{L}_\pi)$ and any formula $\varphi \in \mathrm{Form}(\mathcal{L}_\pi)$, there is a tableau τ for $\delta_1, \ldots, \delta_n, \varphi^*$ that is closed if and only if there is no multistructure $\mathcal{G}$ such that $\mathcal{G} \Vdash \Delta$ and $\mathcal{G} \Vdash \varphi^*$.*

Example 1. *Let $\Delta = \{@_i\langle\pi\rangle p, @_i[\pi]j, @_i\neg\langle\pi\rangle p\}$. We will check if $\varphi = @_i\neg\langle\pi\rangle j$ is a consequence of Δ using the tableau-based decision procedure described in Proposition 2:*

$@_i\langle\pi\rangle p,\ @_i[\pi]j,\ @_i\neg\langle\pi\rangle p,\ (@_i\neg\langle\pi\rangle j)^*$	*1.*
$@_i\langle\pi\rangle t,\ @_t p$	*2. by* $(\langle\pi\rangle)$ *rule on 1*
$@_t j$	*3. by* $([\pi])$ *rule on 1 and 2*
$@_j p$	*4. by* (Nom) *rule on 3 and 2*
$@_i\neg[\pi]\neg u, @_u j$	*5. by* $(\neg\langle\pi\rangle^*)$ *rule on 1*
$@_i i, @_j j, @_t t, @_u u$	*6. by* (Id) *rule*
$@_j u$	*7. by* (Nom) *rule on 5 and 6*
$@_u p$	*8. by* (Nom) *rule on 7 and 4*
$(@_u p)^*$	*9. by* $(\neg\langle\pi\rangle)$ *rule on 1 and 5*
$\times$	

Since the tableau is closed, φ is a consequence of Δ.

Let us give some intuition behind this result: the multistructure that satisfies the database is such that there is evidence of the presence of a transition from the state named by i to a state where p holds; there is also evidence that the only transition present from the state named by i leads to the state named by j, therefore p holds in that state. We also have evidence of the absence of transitions from the state named by i to states where p holds. Thus, we have evidence about the absence of the transition from the state named by i to the state named by j.

We can also show that, for all models, transitivity of equality between nominals is globally satisfied:

Example 2. *Let* $\Delta = \{\}$ *and* $\varphi = (@_i j \wedge @_j k) \supset @_i k$.

The tableau-based decision procedure described in Proposition 2 yields the following:

$$
\begin{array}{cl}
((@_i j \wedge @_j k) \supset @_i k)^* & \textit{1.} \\
(@_t((@_i j \wedge @_j k) \supset @_i k))^* & \textit{2. by } (@_{\mathrm{I}}^*) \textit{ rule on 1} \\
@_t(@_i j \wedge @_j k), (@_t(@_i k))^* & \textit{3. by } (\supset^*) \textit{ rule on 2} \\
@_t(@_i j), @_t(@_j k) & \textit{4. by } (\wedge) \textit{ rule on 3} \\
@_i j, @_j k, (@_i k)^* & \textit{5. by } (@_{\mathrm{E}}) \textit{ rule on 4, } (@_{\mathrm{E}}^*) \textit{ rule on 3} \\
@_i i & \textit{6. by } (\mathrm{Id}) \textit{ rule} \\
@_j i & \textit{7. by } (\mathrm{Nom}) \textit{ rule on 5 and 6 – } @_i j, @_i i \\
@_i k & \textit{8. by } (\mathrm{Nom}) \textit{ rule on 7 and 5 – } @_j i, @_j k \\
\times &
\end{array}
$$

Since the tableau is closed, φ *is a consequence of* $\Delta = \{\}$ *which means that* φ *is globally satisfied in all models,* i.e., φ *is valid.*

Representation of models via diagrams

From this point on, we will assume that Prop, Nom and Mod are finite sets for any hybrid multimodal similarity type $\mathcal{L}_\pi = \langle \mathrm{Prop}, \mathrm{Nom}, \mathrm{Mod} \rangle$, as is the domain W of any multistructure.

Let $\mathcal{L}_\pi(\mathrm{W})$ denote the expansion of $\mathcal{L}_\pi$ that ensures that all states are named by a nominal and let $\mathcal{G}(\mathrm{W})$ denote the natural expansion of the multistructure $\mathcal{G}$ to the hybrid multimodal similarity type $\mathcal{L}_\pi(\mathrm{W})$.

The diagram of a multistructure will be constituted by all *evidence* of what happens at specific states, all *evidence* about transitions and the lack of transitions, and finally all *evidence* about equalities between states.

We start by introducing the notion of DB-literal that will be used later:

Definition 14. *We define the set of* DB-literals *over a hybrid (multimodal) similarity type* $\mathcal{L}_\pi = \langle \mathrm{Prop}, \mathrm{Nom}, \mathrm{Mod} \rangle$ *as:*

$$
\begin{aligned}
\mathrm{DBLit}(\mathcal{L}_\pi) = \quad & \{@_i p, @_i \neg p, @_i \langle \pi \rangle j, @_i \neg \langle \pi \rangle j, @_i j \mid i, j \in \mathrm{Nom}, \\
& p \in \mathrm{Prop}, \pi \in \mathrm{Mod}\}.
\end{aligned}
$$

Definition 15. *Let* $\mathcal{L}_\pi = \langle \mathrm{Prop}, \mathrm{Nom}, \mathrm{Mod} \rangle$ *be a hybrid (multimodal) similarity type, and* $\mathcal{G} = (\mathrm{W}, (\mathrm{R}_\pi^+)_{\pi \in \mathrm{Mod}}, (\mathrm{R}_\pi^-)_{\pi \in \mathrm{Mod}}, \mathrm{N}, \mathrm{V}^+, \mathrm{V}^-)$ *be a multistructure over* $\mathcal{L}_\pi$*. The* diagram of $\mathcal{G}$*, denoted by* $\mathrm{Diag}(\mathcal{G})$*, is the set of* DB*-literals over* $\mathcal{L}_\pi(\mathrm{W})$ *that hold in* $\mathcal{G}(\mathrm{W})$*,* i.e.*,*

$$
\mathrm{Diag}(\mathcal{G}) = \{\alpha \in \mathrm{DBLit}(\mathcal{L}_\pi(\mathrm{W})) \mid \mathcal{G}(\mathrm{W}) \Vdash \alpha\}.
$$

Two distinct multistructures over $\mathcal{L}_\pi$ with the same domain W induce two distinct diagrams (over $\mathcal{L}_\pi(\mathrm{W})$). Thus the diagram $\mathrm{Diag}(\mathcal{G})$ uniquely defines the multistructure $\mathcal{G}$.

We will use $\mathbb{D}(\Delta, \mathrm{W})$ to denote the set of diagrams of multistructures that are models of Δ, with domain W, over the hybrid (multimodal) similarity type $\mathcal{L}_\pi(\mathrm{W})$, where $\mathcal{L}_\pi$ contains the symbols occurring in Δ.

The following example deals with a single modality which for the sake of simplicity is omitted.

Example 3. *In Figure 7 are represented five different (named) locations, evidence of the presence (full line) and absence (dashed line) of transitions between pairs of locations as well as some local properties.*

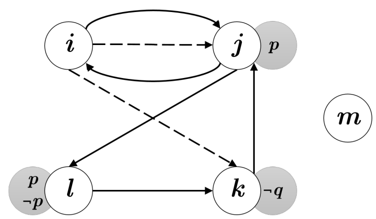


Figure 7: An inconsistent map.

This multistructure is represented by the following diagram:

$$\begin{array}{lr}
\{@_i i, @_j j, @_k k, @_l l, @_m m, & \textit{// nominal equalities} \\
@_j p, @_k \neg q, @_l p, @_l \neg p, & \textit{// local properties} \\
@_i \Diamond j, @_i \neg\Diamond j, @_i \neg\Diamond k, @_j \Diamond i, & \textit{// transitions} \\
@_j \Diamond l, @_k \Diamond j, @_l \Diamond k\} &
\end{array}$$

Note that there is a difference between checking if a formula is satisfied *in* a multistructure and checking if a formula is a *consequence* of the diagram of a multistructure. Recall that a formula is a consequence of a set of formulas Δ if it is globally satisfied in every model of Δ. Observe also that there are multistructures that satisfy all the formulas in the diagram of a particular multistructure $\mathcal{G}$, apart from $\mathcal{G}$ itself; this happens for every multistructure of which $\mathcal{G}$ is a substructure (we consider that $\mathcal{G}$ is a substructure of $\mathcal{G}'$ if the domain of $\mathcal{G}$ is a subset of or equal to the domain of $\mathcal{G}'$, for each modality π, the associated positive and negative accessibility relations R^+_π and R^-_π in $\mathcal{G}$ are a subset or equal to those in $\mathcal{G}'$, each state in $\mathcal{G}'$ is named by at least the same nominal as in $\mathcal{G}$, and finally, for each propositional variable p the positive and negative valuations $\mathrm{V}^+(p)$ and $\mathrm{V}^-(p)$ are a subset or equal to those in $\mathcal{G}'$). In order to check if a formula is globally satisfied in a multistructure, we must check if it is a consequence of its diagram together with the following set of formulas:

$$\{\alpha^* \mid \alpha \in \mathrm{DBLit}(\mathcal{L}_\pi) \backslash \mathrm{Diag}(\mathcal{G})\}$$

where $\mathcal{L}_\pi$ is the hybrid (multimodal) similarity type that contains all symbols appearing in $\mathrm{Diag}(\mathcal{G})$. This construction is always feasible since the multistructures we are considering are finite.

In Example 3, the formula $@_i\Diamond p$ is a consequence of the diagram of the multistructure presented, call it $\mathcal{G}$. However, even though $@_i\neg\Diamond\neg q$ holds in the multistructure, it is not a consequence of its diagram: take a multistructure whose diagram is $\mathrm{Diag}(\mathcal{G}) \cup \{@_l\neg q\}$ – it is a model for the diagram of $\mathcal{G}$ and nonetheless $@_i\neg\Diamond\neg q$ does not hold there.

In order to avoid dealing with unnecessary information, we introduce the notion of minimal model. Minimal models are those where each formula in its diagram is absolutely necessary to keep it a *model*, according to the following definition:

Definition 16. *The set of minimal models with domain* W *for a set of signed formulas* Δ *is the set* $\mathrm{Min}\mathbb{D}(\Delta, \mathrm{W})$ *defined as:*

$$\mathrm{Min}\mathbb{D}(\Delta, \mathrm{W}) = \{\mathbb{M} \in \mathbb{D}(\Delta, \mathrm{W}) \mid \textit{if } \overline{\mathbb{M}} \subset \mathbb{M} \textit{ then } \overline{\mathbb{M}} \notin \mathbb{D}(\Delta, \mathrm{W})\}.$$

Clearly, every model contains a minimal model, *i.e.*, for every model $\mathbb{M}_1$, there is a minimal model $\mathbb{M}_2$ such that $\mathbb{M}_2 \subseteq \mathbb{M}_1$.

No useful information is lost when we use $\mathrm{Min}\mathbb{D}(\Delta, \mathrm{W})$ instead of $\mathbb{D}(\Delta, \mathrm{W})$.

Given a set Δ of signed formulas, there is an algorithm that allows us to extract minimal models for Δ, each of them already represented by its diagram. The algorithm will resort to the tableau system introduced and works as follows:

Algorithm 1. *In order to extract minimal models for* Δ *proceed as follows:*

1. *Build a terminal tableau for* Δ *by applying the tableau rules of system* T*, where condition* (iv) *is restated as follows:*

   (iv) *for* $@_i\varphi$ *a* DB-*literal;*

   together with the following extra rule:

   $$\frac{@_i j, @_k\psi}{@_k(\psi[i/j])}\text{(Bridge)(i)}$$

   (i) $@_k\psi$ *is a* DB-*literal;* $\psi[i/j]$ *is the result of replacing in* ψ *all occurrences of* i *with* j.

   This extra rule is sound and ensures that we have all DB-*literals that are satisfied in our model.*

   Consider only the open branches from now on.

2. *In order to determine minimal models with a certain number of states, introduce formulas of the form* $@_i j$, $@_i\neg j$ *for nominals already occurring; introduce new nominals only if necessary to suit the number of states desired. (If, for example,* i *and* j *are the only nominals occurring in the tableau and we want to determine minimal models with a single state, we*

must add the condition $@_i j$; if we want models with two states, we can either add the condition $@_i \neg j$, or consider the case where $@_i j$ is added and a new nominal k is introduced by adding $@_i \neg k$ as well. The number of combinations grows very rapidly.)

3. *Apply the rules mentioned in step 1., treating the formulas introduced in step 2. as if they were root formulas, until a terminal tableau is reached.*

   Repeat the instructions on step 2. about combining nominals in order to suit the number of states previously set.

   Consider only the open branches.

4. *Finally, with the purpose of defining the positive and negative transitions between states, split each branch into sub-branches such that each sub-branch contains one way of combining the formulas $@_i\langle\pi\rangle j$, $@_i[\pi]\neg j$ and $@_i\neg\langle\pi\rangle j$, $@_i\neg[\pi]\neg j$ for all nominals i and j and modalities π occurring on the branch.*

   Apply the rules indicated in step 1. until a terminal tableau is reached. Each new open branch defines the families of positive and negative accessibility relations $(\mathrm{R}^+_\pi)_{\pi\in\mathrm{Mod}}$ and $(\mathrm{R}^-_\pi)_{\pi\in\mathrm{Mod}}$.

   For each open branch copy the DB*-literals into a set which will be the diagram of a model for Δ. Take the minimal models from amongst those.*

Proposition 3. *There are no minimal models for Δ other than those that are obtained from this algorithm.*

Proof. Suppose that $\mathbb{M}$ is the diagram of a multistructure $\mathcal{G}$ which is a minimal model for Δ and is such that $\mathbb{M} \not\subseteq \Theta$ for all open branches Θ, in the sense that for each branch Θ there exists a literal φ such that $\varphi \in \Theta$ and $\varphi \notin \mathbb{M}$.

Thus, for $\mathbb{M}$ under the conditions described, $\mathcal{G} \nVdash \Theta$ for all Θ.

Recall that by Theorem 3 (Soundness), for each rule $\dfrac{\Delta}{\Sigma_1|\dots|\Sigma_n}$ and any multistructure $\mathcal{G}$, $\mathcal{G} \Vdash \Delta$ implies $\mathcal{G} \Vdash \Sigma_1$ or $\dots$ $\mathcal{G} \Vdash \Sigma_n$. Therefore it follows that $\mathcal{G} \nVdash \Delta$. So $\mathcal{G}$ is not a model for Δ, and therefore $\mathcal{G}$ cannot be a minimal model. Hence there is no such $\mathbb{M}$. □

We present an example as a means to illustrate the algorithm developed:

Example 4. *Let $\Delta = \{@_i\langle\pi\rangle p, @_i\neg\langle\pi\rangle q, @_i q\}$. Let us use the algorithm introduced in order to determine minimal models of Δ with only one state:*

∘ Step 1.

The terminal tableau with root Δ comes as follows:

$$\begin{array}{cl} @_i\langle\pi\rangle p,\ @_i\neg\langle\pi\rangle q,\ @_i q & \textit{1.} \\ @_i\langle\pi\rangle t,\ @_t p & \textit{2. by } (\langle\pi\rangle) \textit{ rule on 1} \\ @_i i,\ @_t t & \textit{3. by } (\mathrm{Id}) \\ \odot & \end{array}$$

∘ Steps 2. and 3.

Now, given that two nominals occur in the tableau, if we want to determine minimal models with a single state, we must add to the tableau the formula $@_i t$, meaning that the nominals i and t name the same state. In this case the tableau is extended as follows:

$$\begin{array}{cl}
@_i t & \textit{4. by step 2. of the algorithm} \\
@_t i,\ @_t q,\ @_t\langle\pi\rangle t & \textit{5. by (Nom) rule on 4 and 3/1/2} \\
@_i p & \textit{6. by (Nom) rule on 5 and 2} \\
@_i\langle\pi\rangle i,\ @_t\langle\pi\rangle i & \textit{7. by (Bridge) rule on 5 and 2/5} \\
\odot &
\end{array}$$

∘ Step 4.

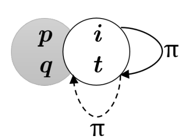
Last step is to determine the positive and negative transitions between states. From the tableau constructed so far, we have that there is a positive transition from the state named by both i and t to itself. Thus the only information missing is if there is or there is not a negative transition from the state named by both i and t to itself. So we split the open branch of the tableau into two in the following way:

$$\begin{array}{c|cl}
@_i\neg\langle\pi\rangle i & @_i\neg[\pi]\neg i & \textit{8. by step 4. of the algorithm} \\
 & (@_t q)^* & \textit{9. by rule } (\neg\langle\pi\rangle) \textit{ on 8 and 1} \\
@_i\neg\langle\pi\rangle t & \times & \textit{10. by rule (Bridge) on 8 and 4} \\
@_t\neg\langle\pi\rangle i & & \textit{11. by rule (Nom) on 8 and 4} \\
@_t\neg\langle\pi\rangle t & & \textit{12. by rule (Bridge) on 11 and 4} \\
\odot & &
\end{array}$$

The minimal model for Δ with one state is:

$$\begin{aligned}
\mathbb{M} = \quad & \{@_i q, @_t q, @_i p, @_t p, @_i i, @_t t, @_i t, @_t i, \\
& @_i\langle\pi\rangle i, @_i\langle\pi\rangle t, @_t\langle\pi\rangle i, @_t\langle\pi\rangle t, \\
& @_i\neg\langle\pi\rangle t, @_i\neg\langle\pi\rangle t, @_t\neg\langle\pi\rangle i, @_t\neg\langle\pi\rangle t\}
\end{aligned}$$

Its graphical representation is as follows:



In the graphical representation the full line represents R_π^+ and the dashed one represents R_π^-.

Note that this algorithm produces very large tableaux – in step 4, when considering n states, we split each open branch into $4^{n\times n}$ branches. However, dealing with each branch separately is as simple as before.

4. Inconsistency measures

The idea of measuring the amount of inconsistent information in paraconsistent structures has been widely addressed in [13], [14] and [15], where a variety of different measures have been proposed. An inconsistency measure is simply a function that assigns a non-negative real value to sets of formulas. Each inconsistency measure is a strategy for analysing inconsistent information by showing how conflicting a set of formulas is. Some measures are more fine-grained than others, but in general what they do is they allow us to compare sets of information.

Inconsistency measures can be classified in various ways and may satisfy certain properties. One distinction is between absolute measures that measure the total amount of contradictions and relative measures that use a ratio to determine how much of the database is inconsistent. A few inconsistency measures for multistructures represented by their diagrams will be presented next, as well as a couple of inconsistency measures for databases.

Absolute measures for multistructures

Given a multistructure $\mathcal{G}$ whose diagram $\mathrm{Diag}(\mathcal{G})$ we will represent by $\mathbb{M}$, an absolute measure that counts how many inconsistencies are in $\mathbb{M}$ is given as follows:

$$\begin{aligned}\mathrm{MInc}_1(\mathbb{M}) = \;& \big|\big\{(p,w) \in \mathrm{Prop} \times \mathrm{W} \;:\; \{@_i p, @_i \neg p\} \subseteq \mathbb{M} \text{ where } \mathrm{N}(i) = w\big\}\big| \\ & + \sum_{\pi \in \mathrm{Mod}} \big|\big\{(w,w') \in \mathrm{W} \times \mathrm{W} \;:\; \{@_i\langle\pi\rangle j, @_i\neg\langle\pi\rangle j\} \subseteq \mathbb{M} \text{ where} \\ & \mathrm{N}(i) = w \text{ and } \mathrm{N}(j) = w'\big\}\big|.\end{aligned}$$

Observe that each component of this measure could be considered as an absolute measure by itself.

MInc_1 is monotonic:

Lemma 3. *Let $\mathcal{G}_1, \mathcal{G}_2$ be two multistructures and $\mathbb{M}_1, \mathbb{M}_2$ be their representations. If $\mathbb{M}_1 \subseteq \mathbb{M}_2$, then $\mathrm{MInc}_1(\mathbb{M}_1) \leq \mathrm{MInc}_1(\mathbb{M}_2)$.*

The next measure counts the number of inconsistencies in a particular connected component of a multistructure, whose definition comes as follows:

$$\begin{aligned}\mathrm{Con}(\mathcal{G}, \mathrm{N}(i)) = \;& \Big\{u \in \mathrm{W} \mid \exists j_1, \ldots, j_n \in \mathrm{Nom} : \\ & ((\exists \pi \in \mathrm{Mod} : @_i\langle\pi\rangle j_1 \in \mathbb{M} \text{ or } @_{j_1}\langle\pi\rangle i \in \mathbb{M}) \\ & \text{and } (\exists \pi \in \mathrm{Mod} : @_{j_1}\langle\pi\rangle j_2 \in \mathbb{M} \text{ or } @_{j_2}\langle\pi\rangle j_1 \in \mathbb{M}), \\ & \ldots, \\ & \text{and } (\exists \pi \in \mathrm{Mod} : @_{j_{n-1}}\langle\pi\rangle j_n \in \mathbb{M} \text{ or } @_{j_n}\langle\pi\rangle j_{n-1} \in \mathbb{M}) \\ & \text{and } \mathrm{N}(j_n) = u)\Big\} \cup \{\mathrm{N}(i)\}.\end{aligned}$$

Clearly $\bigcup_{i \in \mathrm{Nom}} \mathrm{Con}(\mathcal{G}, \mathrm{N}(i)) = \mathrm{W}$.

Then MInc_2 comes as:

$$\begin{aligned}\mathrm{MInc}_2(\mathbb{M}, \mathrm{N}(i)) = \quad & \Big|\Big\{(p,u) \in \mathrm{Prop} \times \mathrm{Con}(\mathcal{G}, \mathrm{N}(i)) : \\ & \{@_t p, @_t \neg p\} \subseteq \mathbb{M} \text{ where } \mathrm{N}(t) = u\Big\}\Big| \\ & + \sum_{\pi \in \mathrm{Mod}} \Big|\Big\{(w,w') \in \mathrm{Con}(\mathcal{G}, \mathrm{N}(i)) \times \mathrm{Con}(\mathcal{G}, \mathrm{N}(i)) : \\ & \{@_i \langle \pi \rangle j, @_i \neg \langle \pi \rangle j\} \subseteq \mathbb{M} \text{ where } \mathrm{N}(i) = w \text{ and } \mathrm{N}(j) = w'\Big\}\Big|.\end{aligned}$$

As MInc_1, MInc_2 is monotonic too.

A third measure counts the number of inconsistencies in a particular path via a modality π, whose definition comes as follows:

$$\begin{aligned}\mathrm{Path}(\mathcal{G}, \mathrm{N}(i)) = \quad & \Big\{u \in \mathrm{W} \mid \exists j_1, \ldots, j_n \in \mathrm{Nom},\ \exists \pi_1, \ldots, \pi_n \in \mathrm{Mod} : \\ & (@_i \langle \pi_1 \rangle j_1, @_{j_1} \langle \pi_2 \rangle j_2, \ldots, @_{j_{n-1}} \langle \pi_n \rangle j_n \in \mathbb{M}) \\ & \text{and } \mathrm{N}(j_n) = u)\Big\} \cup \{\mathrm{N}(i)\}.\end{aligned}$$

The measure is given as:

$$\begin{aligned}\mathrm{MInc}_3(\mathbb{M}, N(i)) = \quad & \Big|\Big\{(p,u) \in \mathrm{Prop} \times \mathrm{Path}(\mathcal{G}, \mathrm{N}(i)) \ : \ \{@_t p, @_t \neg p\} \subseteq \mathbb{M}, \\ & \text{where } N(t) = u\Big\}\Big| \\ & + \sum_{\pi \in \mathrm{Mod}} \Big|\Big\{(w,w') \in \mathrm{Path}(\mathcal{G}, \mathrm{N}(i)) \times \mathrm{Path}(\mathcal{G}, \mathrm{N}(i)) \ : \\ & \{@_i \langle \pi \rangle j, @_i \neg \langle \pi \rangle j\} \subseteq \mathbb{M} \text{ where } \mathrm{N}(i) = w \text{ and } \mathrm{N}(j) = w'\Big\}\Big|.\end{aligned}$$

Once again, this measure is monotonic.

This type of measure may be useful in the future to explore the least inconsistent path in problems that resemble the travelling salesman problem, adapted to deal with inconsistent maps.

Observe that $\mathrm{Path}(\mathcal{G}, \mathrm{N}(i)) \subseteq \mathrm{Con}(\mathcal{G}, \mathrm{N}(i))$, thus the following lemma holds:

Lemma 4. *Let $\mathcal{G}$ be a multistructure represented by $\mathbb{M}$. Then*

$$\mathrm{MInc}_3(\mathbb{M}, N(i)) \leq \mathrm{MInc}_2(\mathbb{M}, N(i)) \leq \mathrm{MInc}_1(\mathbb{M}), \ \textit{for all } i \in \mathrm{Nom}.$$

A weighted measure requires weight vectors for propositional variables and states and a matrix detailing the weight of each transition:

$$\mathrm{weight}_{\mathrm{Prop}} = \Big[\mathrm{weight}_{p_1} \ldots \mathrm{weight}_{p_{|\mathrm{Prop}|}}\Big]$$

and

$$\mathrm{weight}_W = \Big[\mathrm{weight}_{w_1} \ldots \mathrm{weight}_{w_{|W|}}\Big]$$

and

$$\text{weight}_{\mathrm{W}\times\mathrm{W}} = \begin{bmatrix} \text{weight}_{w_1,w_1} & \text{weight}_{w_1,w_2} & \dots & \text{weight}_{w_1,w_n} \\ \text{weight}_{w_2,w_1} & \text{weight}_{w_2,w_2} & \dots & \text{weight}_{w_2,w_n} \\ \vdots & \vdots & & \vdots \\ \text{weight}_{w_n,w_1} & \text{weight}_{w_n,w_2} & \dots & \text{weight}_{w_n,w_n} \end{bmatrix}$$

where all entries are non-negative real numbers.

Let MInc_4 be such that:

$$\begin{aligned} \mathrm{MInc}_4(\mathbb{M}) = & \sum_{\substack{(p,w):\ @_i p, @_i \neg p \in \mathbb{M}, \\ \text{where } \mathrm{N}(i)=w}} \text{weight}_{\mathrm{Prop}_p} \times \text{weight}_{\mathrm{W}w} \\ & + \sum_{\pi \in \mathrm{Mod}} \left(\sum_{\substack{(w,w'):\ @_i\langle\pi\rangle j, @_i \neg\langle\pi\rangle j \in \mathbb{M}, \\ \text{where } \mathrm{N}(i)=w, \mathrm{N}(j)=w'}} \text{weight}_{\mathrm{W}\times\mathrm{W}_{w,w'}} \right). \end{aligned}$$

Relative measures for multistructures

Taking the number of inconsistencies in a multistructure and dividing it by the number of possible inconsistencies gives us a relative measure whose result is a ratio between 0 and 1 that tells us how much of a portion of the multistructure is inconsistent.

Given a multistructure $\mathcal{G}$ represented by $\mathbb{M}$, this inconsistency measure comes in the form:

$$\mathrm{MInc}_5(\mathbb{M}) = \frac{\mathrm{MInc}_1(\mathbb{M})}{|\mathrm{Prop}| \times |\mathrm{W}| + |\mathrm{Mod}| \times |\mathrm{W}| \times |\mathrm{W}|}.$$

This measure is neither monotonic nor anti-monotonic for the following reason:

- if $\mathbb{M}_1 \subseteq \mathbb{M}_2$ and the multistructures that are represented by $\mathbb{M}_1$ and $\mathbb{M}_2$ have the same domain W and are built over the same hybrid (multimodal) similarity type, then

$$\begin{aligned} \mathrm{MInc}_5(\mathbb{M}_1) = & \frac{\mathrm{MInc}_1(\mathbb{M}_1)}{|\mathrm{Prop}| \times |\mathrm{W}| + |\mathrm{Mod}| \times |\mathrm{W}| \times |\mathrm{W}|} \\ \leq & \frac{\mathrm{MInc}_1(\mathbb{M}_2)}{|\mathrm{Prop}| \times |\mathrm{W}| + |\mathrm{Mod}| \times |\mathrm{W}| \times |\mathrm{W}|} = \mathrm{MInc}_5(\mathbb{M}_2) \end{aligned}$$

  since from previous results we already had that $\mathrm{MInc}_1(\mathbb{M}_1) \leq \mathrm{MInc}_1(\mathbb{M}_2)$;
- if, however that is not the case, monotonicity can be broken:

  Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two multistructures over $\mathcal{L}^1_\pi = (\{p\}, \{i\}, \{\pi\})$ and $\mathcal{L}^2_\pi = (\{p\}, \{i, j\}, \{\pi\})$, respectively, such that

– $\mathrm{W}_1 = \{w_1\}$ and $\mathrm{W}_2 = \{w_1, w_2\}$,
– $\mathrm{R}_1^+ = \mathrm{R}_1^- = \mathrm{R}_2^+ = \mathrm{R}_2^- = \varnothing$,
– $\mathrm{N}_1(i) = \mathrm{N}_2(i) = w_1$, $\mathrm{N}_2(j) = w_2$,
– $\mathrm{V}_1^+(p) = \mathrm{V}_1^-(p) = \mathrm{V}_2^+(p) = \mathrm{V}_2^-(p) = \{w_1\}$.

Then $\mathbb{M}_1 = \{@_i p, @_i \neg p, @_i i\}$ and $\mathbb{M}_2 = \{@_i p, @_i \neg p, @_i i, @_j j\}$. Clearly $\mathbb{M}_1 \subset \mathbb{M}_2$ and $\mathrm{MInc}_1(\mathbb{M}_1) = \mathrm{MInc}_1(\mathbb{M}_2) = 1$. Therefore, $\mathrm{MInc}_5(\mathbb{M}_1) = \frac{1}{1} \geq \frac{1}{6} = \mathrm{MInc}_5(\mathbb{M}_2)$.

We can also take the weighted measure MInc_4 and divide it by the sum of the weights of the elements of the multistructure. This inconsistency measure comes as follows:

$$\mathrm{MInc}_6(\mathbb{M}) = \frac{\mathrm{MInc}_4(\mathbb{M})}{\sum\limits_{p \in \mathrm{Prop}, w \in W} \mathrm{weight}_p \times \mathrm{weight}_w \; + \; |\mathrm{Mod}| \times \sum\limits_{w, w' \in W} \mathrm{weight}_{w,w'}}.$$

MInc_6 is also neither monotonic nor anti-monotonic.

Example 5. *Let us calculate some measures of inconsistency for the multistructure in Example 3. We refer to the diagram of the multistructure as* $\mathbb{M}$.
$|\mathrm{W}| = 5$, $|\mathrm{Prop}| = 2$, $|\mathrm{Mod}| = 1$

- $\mathrm{MInc}_1(\mathbb{M}) = 2$.
- $\mathrm{MInc}_2(\mathbb{M}, \mathrm{N}(i)) = 2$.
- $\mathrm{MInc}_3(\mathbb{M}, \mathrm{N}(m)) = 0$.
- $\mathrm{MInc}_5(\mathbb{M}) = \dfrac{2}{10 + 25} = \dfrac{2}{35}$.

Measures for databases

We may want to know whether one source is more inconsistent than another. In particular we would like to determine which is the least inconsistent source of information we have in our hands, which will intuitively be thought of as the least problematical or most reliable source.

First, let us consider minimal models with the least number of inconsistencies, which we will call preferred models.

Definition 17. *The set of preferred models with domain* W *for a set of signed formulas* Δ *is the set* $\mathrm{Pref}\mathbb{D}(\Delta, \mathrm{W})$ *defined as:*

$$\begin{aligned}\mathrm{Pref}\mathbb{D}(\Delta, \mathrm{W}) = \; & \{\mathbb{M} \in \mathrm{Min}\mathbb{D}(\Delta, \mathrm{W}) \mid \textit{for all } \overline{\mathbb{M}} \in \mathrm{Min}\mathbb{D}(\Delta, \mathrm{W}), \\ & \mathrm{MInc}_1(\mathbb{M}) \leq \mathrm{MInc}_1(\overline{\mathbb{M}})\}.\end{aligned}$$

We define $\mathrm{MInc}_7(\Delta, \mathbb{D})$, as a sequence $\langle r_1, \ldots, r_n, \ldots\rangle$ where $r_n = \mathrm{MInc}_5(\mathbb{M})$ if there exists a model $\mathbb{M} \in \mathrm{Pref}\mathbb{D}(\Delta, \mathrm{W}_n)$ with W_n a domain of size n, and $r_n = *$ otherwise. We use $*$ as kind of a null value.

The measure $\mathrm{MInc}_7(\Delta, \mathbb{D})$ captures how the relative measure of inconsistency of preferred models for a database Δ evolves with increasing domain size. There are databases Δ for which all minimal models contain no inconsistencies at all, and others for which all minimal models are inconsistent.

We adopt a lexicographic ordering over the tuples generated by the MInc_7 function as follows:

Definition 18. *Let Δ_1, Δ_2 be two databases. Let $\mathrm{MInc}_7(\Delta_1, \mathbb{D}) = \langle r_1, r_2, \ldots\rangle$ and $\mathrm{MInc}_7(\Delta_2, \mathbb{D}) = \langle s_1, s_2, \ldots\rangle$.*

$$\mathrm{MInc}_7(\Delta_1, \mathbb{D}) \preceq \mathrm{MInc}_7(\Delta_2, \mathbb{D}) \Leftrightarrow \textit{for all } i \geq 1,\ r_i \leq s_i \textit{ or } \ r_i = * \textit{ or } s_i = *.$$

In case $\mathrm{MInc}_7(\Delta_1, \mathbb{D}) \preceq \mathrm{MInc}_7(\Delta_2, \mathbb{D})$, one says that Δ_1 is less or as inconsistent as Δ_2 and denote this by $\Delta_1 \leq_{\mathrm{inc}} \Delta_2$.

Example 6. *Let $\Delta_1 = \{@_i\langle\pi\rangle p, @_i\neg\langle\pi\rangle q, @_j q\}$ and $\Delta_2 = \{@_i p, @_i \neg p\}$. Then:*

$$\mathrm{MInc}_7(\Delta_1, \mathbb{D}) = \left\langle \frac{1}{3}, 0, 0, \ldots, 0, \ldots \right\rangle,$$
$$\mathrm{MInc}_7(\Delta_2, \mathbb{D}) = \left\langle \frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \ldots \frac{1}{n \times (n+1)}, \ldots \right\rangle.$$

Thus $\Delta_1 \leq_{inc} \Delta_2$.

MInc_7 is monotonic: given Δ_1, Δ_2 such that $\Delta_1 \subseteq \Delta_2$, and since additional statements may add but cannot subtract inconsistencies, it follows that $\mathrm{MInc}_7(\Delta_1, \mathbb{D}) \preceq \mathrm{MInc}_7(\Delta_2, \mathbb{D})$.

We define equivalence between databases in the next definition:

Definition 19. *Let Δ_1, Δ_2 be two databases. Δ_1 and Δ_2 are* equivalent *if for all $\mathbb{M}$*

$$\mathbb{M} \textit{ is a model of } \Delta_1 \Leftrightarrow \mathbb{M} \textit{ is a model of } \Delta_2.$$

The following result holds:

Proposition 4. *Let Δ_1, Δ_2 be two databases. If Δ_1 and Δ_2 are equivalent, then*

$$\mathrm{MInc}_7(\Delta_1, \mathbb{D}) = \mathrm{MInc}_7(\Delta_2, \mathbb{D}).$$

Proof. Let $\mathrm{MInc}_7(\Delta_1, \mathbb{D}) = \langle r_1, r_2, \ldots\rangle$ and $\mathrm{MInc}_7(\Delta_2, \mathbb{D}) = \langle s_1, s_2, \ldots\rangle$.

Since Δ_1 and Δ_2 are equivalent, any preferred model $\mathbb{M}$ of Δ_1 with a domain of size n is a model of Δ_2. In fact, $\mathbb{M}$ is a preferred model for Δ_2 as well, for if it were not, then it could not be a preferred model of Δ_1.

Therefore, $r_n = s_n$ for all n and thus $\mathrm{MInc}_7(\Delta_1, \mathbb{D}) = \mathrm{MInc}_7(\Delta_2, \mathbb{D})$. □

5. DB-bisimulation

Bisimulation is the fundamental notion of equivalence between models in Modal logic and extensions to Hybrid logic are not a novelty, [1]. In this section the notion of bisimulation is extended to multistructures. Curiously, the classical construction will not preserve satisfiability of formulas between bisimilar models. However, that will be the case for a posterior definition of DB-bisimulation, which performs a small, yet significant, change in the previous definition.

Definition 20. *Let* $\mathcal{G} = \left(\mathrm{W}, (\mathrm{R}^+_\pi)_{\pi\in\mathrm{Mod}}, (\mathrm{R}^-_\pi)_{\pi\in\mathrm{Mod}}, \mathrm{N}, \mathrm{V}^+, \mathrm{V}^-\right)$ *and* $\overline{\mathcal{G}} = (\overline{\mathrm{W}}, (\overline{\mathrm{R}^+_\pi})_{\pi\in\mathrm{Mod}}, (\overline{\mathrm{R}^-_\pi})_{\pi\in\mathrm{Mod}}, \overline{\mathrm{N}}, \overline{\mathrm{V}^+}, \overline{\mathrm{V}^-})$ *be two hybrid multistructures over the same hybrid (multimodal) similarity type* $\mathcal{L}_\pi = \langle \mathrm{Prop}, \mathrm{Nom}, \mathrm{Mod}\rangle$.

A relation $Z \subseteq \mathrm{W} \times \overline{\mathrm{W}}$ *is a* multi-bisimulation *if* Z *is a bisimulation (in the classical sense) between the hybrid structures* $\mathcal{H} = \left(\mathrm{W}, (\mathrm{R}^*_\pi)_{\pi\in\mathrm{Mod}}, \mathrm{N}, \mathrm{V}^\circ\right)$ *and* $\overline{\mathcal{H}} = \left(\overline{\mathrm{W}}, \left(\overline{\mathrm{R}^*_\pi}\right)_{\pi\in\mathrm{Mod}}, \overline{\mathrm{N}}, \overline{\mathrm{V}^\circ}\right)$, *for each combination* $*, \circ \in \{+,-\}$.

In more detail, Z is a multi-bisimulation if the following conditions are met:

- $(\mathrm{N}(i), \overline{\mathrm{N}}(i)) \in Z$ for all $i \in \mathrm{Nom}$;
- if $(w, \overline{w}) \in Z$, then:
  - atomic conditions:
    - $\rtimes$ $w \in \mathrm{V}^\circ(p)$ iff $\overline{w} \in \overline{\mathrm{V}^\circ}(p)$, for all $\circ \in \{+,-\}$ and $p \in \mathrm{Prop}$;
    - $\rtimes$ $\mathrm{N}(i) = w$ iff $\overline{\mathrm{N}}(i) = w'$, for all $i \in \mathrm{Nom}$;
  - if $w\mathrm{R}^*_\pi u$ for some $u \in \mathrm{W}$, then there is some $\overline{u} \in \overline{\mathrm{W}}$ such that $\overline{w}\overline{\mathrm{R}^*_\pi}\overline{u}$ and $(u, \overline{u}) \in Z$ **(Zig*)**, for $* \in \{+,-\}$;
  - if $\overline{w}\overline{\mathrm{R}^*_\pi}\overline{u}$ for some $\overline{u} \in \overline{\mathrm{W}}$, then there is some $u \in \mathrm{W}$ such that $w\mathrm{R}^*_\pi u$ and $(u, \overline{u}) \in Z$ **(Zag*)**, for $* \in \{+,-\}$.

Two pointed hybrid multistructures $(\mathcal{G}, w)$ and $(\overline{\mathcal{G}}, \overline{w})$ are *multi-bisimilar* if there is a multi-bisimulation Z between $\mathcal{G}$ and $\mathcal{G}'$ such that $(w, \overline{w}) \in Z$.

Theorem 5. DBHL$*$ *is* not *invariant under multi-bisimulation.*

Proof. Take the following pointed hybrid multistructures over $\mathcal{L}_\pi = \langle \{p\}, \varnothing, \{\pi\}\rangle$:

- $(\mathcal{G}, w)$, where $\mathcal{G}$ is such that $\mathrm{W} = \{w\}$, $\mathrm{R}^+_\pi = \varnothing$, $\mathrm{R}^-_\pi = \{(w,w)\}$, N is the empty function, $\mathrm{V}^+(p) = \{w\}$ and $\mathrm{V}^-(p) = \varnothing$; and
- $(\overline{\mathcal{G}}, \overline{w})$, where $\overline{\mathcal{G}}$ is such that $\overline{\mathrm{W}} = \{\overline{w}, \overline{v}\}$, $\overline{\mathrm{R}^+_\pi} = \varnothing$, $\overline{\mathrm{R}^-_\pi} = \{(\overline{w}, \overline{v}), (\overline{v}, \overline{v})\}$, $\overline{\mathrm{N}}$ is the empty function, $\mathrm{V}^+(p) = \{\overline{w}, \overline{v}\}$ and $\mathrm{V}^-(p) = \varnothing$.

Observe that $(\mathcal{G}, w)$ and $(\overline{\mathcal{G}}, \overline{w})$ are multi-bisimilar. The multi-bisimulation $Z = \{(w, \overline{w}), (w, \overline{v})\}$ is represented in the following figure:

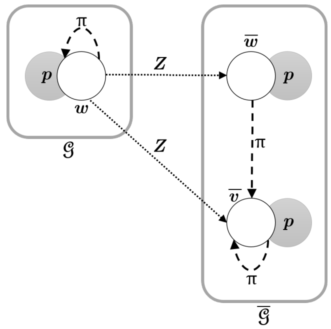


Figure 8: Multi-bisimulation between $\mathcal{G}$ and $\overline{\mathcal{G}}$.

Nonetheless, $\mathcal{G}, w \Vdash \neg\langle\pi\rangle p$ whereas $\overline{\mathcal{G}}, \overline{w} \nVdash \neg\langle\pi\rangle p$. $\square$

So, although this seems like a natural definition for multi-bisimulation, it is clearly not the best since invariance is lost. Note also that a multistructure where all states are named is only multi-bisimilar to itself.

We change Definition 20 and introduce a DB-bisimulation as follows:

Definition 21. *Let* $\mathcal{G} = \big(\mathrm{W}, (\mathrm{R}_\pi^+)_{\pi\in\mathrm{Mod}}, (\mathrm{R}_\pi^-)_{\pi\in\mathrm{Mod}}, \mathrm{N}, \mathrm{V}^+, \mathrm{V}^-\big)$ *and* $\overline{\mathcal{G}} = (\overline{\mathrm{W}}, (\overline{\mathrm{R}_\pi^+})_{\pi\in\mathrm{Mod}}, (\overline{\mathrm{R}_\pi^-})_{\pi\in\mathrm{Mod}}, \overline{\mathrm{N}}, \overline{\mathrm{V}^+}, \overline{\mathrm{V}^-})$ *be two hybrid multistructures over the same hybrid (multimodal) similarity type* $\mathcal{L}_\pi = \langle\mathrm{Prop}, \mathrm{Nom}, \mathrm{Mod}\rangle$.

A relation $Z \subseteq \mathrm{W} \times \overline{\mathrm{W}}$ *is a* DB-bisimulation *if:*

- $(\mathrm{N}(i), \overline{\mathrm{N}}(i)) \in Z$ *for all* $i \in \mathrm{Nom}$;
- *if* $(w, \overline{w}) \in Z$*, then:*
  - *atomic conditions:*
    - $w \in \mathrm{V}^+(p)$ *iff* $\overline{w} \in \overline{\mathrm{V}^+}(p)$*, for all* $p \in \mathrm{Prop}$;
    - $w \in \mathrm{V}^-(p)$ *iff* $\overline{w} \in \overline{\mathrm{V}^-}(p)$*, for all* $p \in \mathrm{Prop}$;
    - $\mathrm{N}(i) = w$ *iff* $\overline{\mathrm{N}}(i) = w'$*, for all* $i \in \mathrm{Nom}$;
  - *if* $w\mathrm{R}_\pi^+ u$ *for some* $u \in \mathrm{W}$*, then there is some* $\overline{u} \in \overline{\mathrm{W}}$ *such that* $\overline{w}\overline{\mathrm{R}_\pi^+}\overline{u}$ *and* $(u, \overline{u}) \in Z$ ***(Zig$^+$)***;
  - *if* $w\cancel{\mathrm{R}_\pi^-}u$ *for some* $u \in \mathrm{W}$*, then there is some* $\overline{u} \in \overline{\mathrm{W}}$ *such that* $\overline{w}\overline{\cancel{\mathrm{R}_\pi^-}}\overline{u}$ *and* $(u, \overline{u}) \in Z$ ***(Zig$^-$)***;
  - *if* $\overline{w}\overline{\mathrm{R}_\pi^+}\overline{u}$ *for some* $\overline{u} \in \overline{\mathrm{W}}$*, then there is some* $u \in \mathrm{W}$ *such that* $w\mathrm{R}_\pi^+ u$ *and* $(u, \overline{u}) \in Z$ ***(Zag$^+$)***;
  - *if* $\overline{w}\overline{\cancel{\mathrm{R}_\pi^-}}\overline{u}$ *for some* $\overline{u} \in \overline{\mathrm{W}}$*, then there is some* $u \in \mathrm{W}$ *such that* $w\cancel{\mathrm{R}_\pi^-}u$ *and* $(u, \overline{u}) \in Z$.

Two pointed hybrid multistructures are DB-bisimilar in an analogous fashion to the definition of multi-bisimilar pointed multistructures.

We conclude with the proof of invariance:

Theorem 6. DBHL$*$ is *invariant under* DB*-bisimulation.*

Proof. Let $(\mathcal{G}, w)$ and $(\overline{\mathcal{G}}, \overline{w})$ be two DB-bisimilar pointed hybrid multistructures over $\mathcal{L}_\pi$. We prove that, for all $\varphi \in \mathrm{Form}(\mathcal{L}_\pi)$, $\mathcal{G}, w \Vdash \varphi \Leftrightarrow \overline{\mathcal{G}}, \overline{w} \Vdash \varphi$.

The proof is by induction on the structure of φ:

- $\varphi = p$:

$$\begin{array}{rcll} \mathcal{G}, w \Vdash p & \Leftrightarrow & w \in \mathrm{V}^+(p) & \\ & \Leftrightarrow & \overline{w} \in \overline{\mathrm{V}^+}(p) & (w, \overline{w}) \in Z: \text{ atomic condition} \\ & \Leftrightarrow & \overline{\mathcal{G}}, \overline{w} \Vdash p & \end{array}$$

- $\varphi = \neg p$:

$$\begin{array}{rcll} \mathcal{G}, w \Vdash \neg p & \Leftrightarrow & w \in \mathrm{V}^-(p) & \\ & \Leftrightarrow & \overline{w} \in \overline{\mathrm{V}^-}(p) & (w, \overline{w}) \in Z: \text{ atomic condition} \\ & \Leftrightarrow & \overline{\mathcal{G}}, \overline{w} \Vdash \neg p & \end{array}$$

- $\varphi = i$:

$$\begin{array}{rcll} \mathcal{G}, w \Vdash i & \Leftrightarrow & w = \mathrm{N}(i) & \\ & \Leftrightarrow & \overline{w} = \overline{\mathrm{N}}(i) & (w, \overline{w}) \in Z: \text{ atomic condition} \\ & \Leftrightarrow & \overline{\mathcal{G}}, \overline{w} \Vdash i & \end{array}$$

- $\varphi = \neg i$:

$$\begin{array}{rcll} \mathcal{G}, w \Vdash \neg i & \Leftrightarrow & w \neq \mathrm{N}(i) & \\ & \Leftrightarrow & \overline{w} \neq \overline{\mathrm{N}}(i) & (w, \overline{w}) \in Z: \text{ atomic condition} \\ & \Leftrightarrow & \overline{\mathcal{G}}, \overline{w} \Vdash \neg i & \end{array}$$

Induction Hypothesis (I.H.): the result holds for subformulas ψ, δ of φ, as well as for $\neg\psi, \neg\delta$.

- $\varphi = \psi \vee \delta$:

$$\begin{array}{rcll} \mathcal{G}, w \Vdash \psi \vee \delta & \Leftrightarrow & (\mathcal{G}, w \Vdash \psi \text{ or } \mathcal{G}, w \Vdash \delta) & \\ & & \text{and } (\mathcal{G}, w \Vdash \neg\psi \text{ implies } \mathcal{G}, w \Vdash \delta) & \\ & & \text{and } (\mathcal{G}, w \Vdash \neg\delta \text{ implies } \mathcal{G}, w \Vdash \psi) & \\ & \Leftrightarrow & \big(\overline{\mathcal{G}}, \overline{w} \Vdash \psi \text{ or } \overline{\mathcal{G}}, \overline{w} \Vdash \delta\big) & \text{(I.H.)} \\ & & \text{and } \big(\overline{\mathcal{G}}, \overline{w} \Vdash \neg\psi \text{ implies } \overline{\mathcal{G}}, \overline{w} \Vdash \delta\big) & \\ & & \text{and } \big(\overline{\mathcal{G}}, \overline{w} \Vdash \neg\delta \text{ implies } \overline{\mathcal{G}}, \overline{w} \Vdash \psi\big) & \\ & \Leftrightarrow & \overline{\mathcal{G}}, \overline{w} \Vdash \psi \vee \delta & \end{array}$$

- $\varphi = \neg(\psi \vee \delta), \psi \wedge \delta, \neg(\psi \wedge \delta), \psi \supset \delta, \neg(\psi \supset \delta)$ follow an analogous reasoning.

- $\varphi = \langle\pi\rangle\psi$:

  In order to give full details, we prove this case in two steps.

$$\begin{array}{rcll} \mathcal{G}, w \Vdash \langle\pi\rangle\psi & \Leftrightarrow & \exists u \in \mathrm{W} : w\mathrm{R}^{+}_{\pi}u \text{ and } \mathcal{G}, u \Vdash \psi & \\ & \Rightarrow & \exists \overline{u} \in \overline{\mathrm{W}} : \overline{w}\overline{\mathrm{R}^{+}_{\pi}}\overline{u} \text{ and } (u, \overline{u}) \in Z & (\mathrm{Zig}^{+}) \\ & \Rightarrow & \exists \overline{u} \in \overline{\mathrm{W}} : \overline{w}\overline{\mathrm{R}^{+}_{\pi}}\overline{u} \text{ and } \overline{\mathcal{G}}, \overline{u} \Vdash \psi & (\text{I.H.}) \\ & \Leftrightarrow & \overline{\mathcal{G}}, \overline{w} \Vdash \langle\pi\rangle\psi & \end{array}$$

$$\begin{array}{rcll} \overline{\mathcal{G}}, \overline{w} \Vdash \langle\pi\rangle\psi & \Leftrightarrow & \exists \overline{u} \in \overline{\mathrm{W}} : \overline{w}\overline{\mathrm{R}^{+}_{\pi}}\overline{u} \text{ and } \overline{\mathcal{G}}, \overline{u} \Vdash \psi & \\ & \Rightarrow & \exists u \in \mathrm{W} : w\mathrm{R}^{+}_{\pi}u \text{ and } (u, \overline{u}) \in Z & (\mathrm{Zag}^{+}) \\ & \Rightarrow & \exists u \in \mathrm{W} : w\mathrm{R}^{+}_{\pi}u \text{ and } \mathcal{G}, u \Vdash \psi & (\text{I.H.}) \\ & \Leftrightarrow & \mathcal{G}, w \Vdash \langle\pi\rangle\psi & \end{array}$$

- $\varphi = [\pi]\psi$:

  Suppose that $\mathcal{G}, w \Vdash [\pi]\psi$ and $\overline{\mathcal{G}}, \overline{w} \nVdash [\pi]\psi$. Then for all $u \in \mathrm{W}, w\mathrm{R}^{+}_{\pi}u$ implies $\mathcal{G}, u \Vdash \psi$. On the other hand, there exists $\overline{u} \in \overline{\mathrm{W}}$ such that $\overline{w}\overline{\mathrm{R}^{+}_{\pi}}\overline{u}$ and $\overline{\mathcal{G}}, \overline{u} \nVdash \psi$. From this, and by (Zag^{+}), there exists $s \in \mathrm{W}$ such that $w\mathrm{R}^{+}_{\pi}s$ and $(s, \overline{u}) \in Z$. Thus, by I.H., $\mathcal{G}, s \nVdash \psi$ and therefore $\mathcal{G}, w \nVdash [\pi]\psi$, which is a contradiction.

  If we assume that $\mathcal{G}, w \nVdash [\pi]\psi$ and $\overline{\mathcal{G}}, \overline{w} \Vdash [\pi]\psi$ we reach a contradiction in an analogous fashion, by using (Zig^{+}) instead. So, $\mathcal{G}, w \Vdash [\pi]\psi$ if and only if $\overline{\mathcal{G}}, \overline{w} \Vdash [\pi]\psi$.

- $\varphi = \neg\langle\pi\rangle\psi$:

  Suppose that $\mathcal{G}, w \Vdash \neg\langle\pi\rangle\psi$ and $\overline{\mathcal{G}}, \overline{w} \nVdash \neg\langle\pi\rangle\psi$. Then for all $u \in \mathrm{W}$, $\mathcal{G}, u \Vdash \psi$ implies $w\mathrm{R}^{-}_{\pi}u$, or equivalently, for all $u \in \mathrm{W}$, $w\cancel{\mathrm{R}^{-}_{\pi}}u$ implies $\mathcal{G}, u \nVdash \psi$. On the other hand, there exists $\overline{u} \in \overline{\mathrm{W}}$ such that $\overline{G}, \overline{u} \Vdash \psi$ and $\overline{w}\cancel{\overline{\mathrm{R}^{-}_{\pi}}}\overline{u}$. From this, and by (Zag^{-}), there exists $s \in \mathrm{W}$ such that $w\cancel{\mathrm{R}^{-}_{\pi}}s$ and $(s, \overline{u}) \in Z$. Thus, by I.H., $\mathcal{G}, s \Vdash \psi$ and therefore $\mathcal{G}, w \nVdash \neg\langle\pi\rangle\psi$, which is a contradiction. So, $\mathcal{G}, w \Vdash \neg\langle\pi\rangle\psi$ if and only if $\overline{\mathcal{G}}, \overline{w} \Vdash \neg\langle\pi\rangle\psi$.

  If we assume that $\mathcal{G}, w \nVdash \neg\langle\pi\rangle\psi$ and $\overline{\mathcal{G}}, \overline{w} \Vdash \neg\langle\pi\rangle\psi$ we reach a contradiction in an analogous fashion, by using (Zig^{-}) instead. So, $\mathcal{G}, w \Vdash \neg\langle\pi\rangle\psi$ if and only if $\overline{\mathcal{G}}, \overline{w} \Vdash \neg\langle\pi\rangle\psi$.

- $\varphi = \neg[\pi]\psi$:

$$\begin{array}{rcll} \mathcal{G}, w \Vdash \neg[\pi]\psi & \Leftrightarrow & \exists u \in \mathrm{W} : w\cancel{\mathrm{R}^{-}_{\pi}}u \text{ and } \mathcal{G}, u \nVdash \psi & \\ & \Leftrightarrow & \exists \overline{u} \in \overline{\mathrm{W}} : \overline{w}\cancel{\overline{\mathrm{R}^{-}_{\pi}}}\overline{u} \text{ and } \overline{\mathcal{G}}, \overline{u} \nVdash \psi & (\mathrm{Zig}^{-}/\mathrm{Zag}^{-}) + (\text{I.H.}) \\ & \Leftrightarrow & \overline{\mathcal{G}}, \overline{w} \Vdash \neg[\pi]\psi & \end{array}$$

$\square$

6. Conclusion

The paper presents a four-valued Hybrid logic where propositional variables and accessibility relations are paraconsistent and paracomplete. The major novelty about this work is the fact that the duality between modal operators is no longer valid. However, the multistructures with which we work are such that they can be described by a set of atomic formulas, a diagram, just like structures can in standard Hybrid logic. We also introduced a sound and complete tableau system, discussed inconsistency measures and notions of bisimulation. This formal system is possibly the key to deal with graph-related problems where inconsistent information regarding local data and transitions is provided. We intend to continue looking for applications of this logic, namely in problems such as the travelling salesman when the underlying map is inconsistent. We also plan on studying fuzzy versions of $\mathsf{DBHL}*$, as well as an extension to dynamic logic in order to check the influence of our four-valued accessibility relations in the behaviour of the composition of actions.

The topic of paraconsistency at the level of nominals should also be addressed in the future. This seemingly easy feature carries a lot of implications and requires a lot of care in many ways. Let us not forget that we cannot simply assign one of four values to pairs (i, w), otherwise nothing would distinguish them from ordinary propositional variables.

Diana Costa acknowledges the support from the UK's Engineering and Physical Sciences Research Council - EPSRC (Grant number EP/R006865/1). Manuel Martins was supported by ERDF, through the COMPETE 2020 Program, and by FCT, within the projects PTDC/CCI-COM/30947/2017 and UIDB/04106/2020.

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