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Analytic hypoellipticity of Keldysh operators

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Abstract

We consider Keldysh-type operators, $P = x_1 D_{x_1}^2 + a(x)D_{x_1} + Q(x, D_{x'})$, $x = (x_1, x')$ with analytic coefficients, and with $Q(x, D_{x'})$ second order, principally real and elliptic in $D_{x'}$ for x near zero. We show that if $Pu = f$, $u \in C^\infty$, and f is *analytic* in a neighbourhood of 0, then u is *analytic* in a neighbourhood of 0. This is a consequence of a microlocal result valid for operators of any order with Lagrangian radial sets. Our result proves a generalized version of a conjecture made in (Lebeau and Zworski, *Proc. Amer. Math. Soc.* 147 (2019) 145–152; Zworski, *Bull. Math. Sci.* 7 (2017) 1–85) and has applications to scattering theory.

1. *Introduction*

We consider analytic regularity for generalizations of the Keldysh operator [24],

$$P := x_1 D_{x_1}^2 + D_{x_2}^2. \tag{1.1}$$

The operator P has the feature of changing from an elliptic to a hyperbolic operator at $x_1 = 0$. It appears in various places including the study of transsonic flows, see, for instance, Čanić–Keyfitz [8] or population biology — see Epstein–Mazzeo [12]. Our interest in such operators comes from the work of Vasy [31] where the transition at $x_1 = 0$ corresponds to the boundary at infinity for asymptotically hyperbolic manifolds (see [34]), crossing the event horizons of Schwartzschild black holes (see [11, § 5.7]) or the cosmological horizon for de Sitter spaces. The Vasy operator in the asymptotically hyperbolic setting is given by

$$P(\lambda) = 4(x_1 D_{x_1}^2 - (\lambda + i)D_{x_1}) - \Delta_{h(x_1)} + i\gamma(x)\big(2x_1 D_{x_1} - \lambda - i\tfrac{n-1}{2}\big), \tag{1.2}$$

where $h(x_1)$ is a smooth family of Riemannian metrics in x', $x = (x_1, x') \in \mathbb{R}^n$ and $\gamma \in C^\infty(\mathbb{R}^n)$. The resonant states at resonant frequencies λ (see [11, Chapter 5]) are the smooth solutions of $P(\lambda)u = 0$.

For various reasons reviewed in § 1.3, it is interesting to ask if in the case of analytic coefficients the resonant states are real analytic across $x_1 = 0$. That lead to [35, Conjecture 2] which asked if $P(\lambda)u = f$ with u smooth and f analytic near $x_1 = 0$ implies that u is analytic near $x_1 = 0$. For $\gamma(x) \equiv 0$ and h independent of x_1, this was shown by Lebeau–Zworski [25] under the assumption that $\lambda \notin -i\mathbb{N}^*$.

The general case was proved by Zuily [32] under the same restriction on λ. His proof was an elegant adaptation of the work of Baouendi–Goulaouic [1], Bolley–Camus [3] and Bolley–Camus–Hanouzet [4].

In this paper, we prove this result for generalized Keldysh operators with analytic coefficients (1.3). In particular, we do not make any assumptions on lower order terms:

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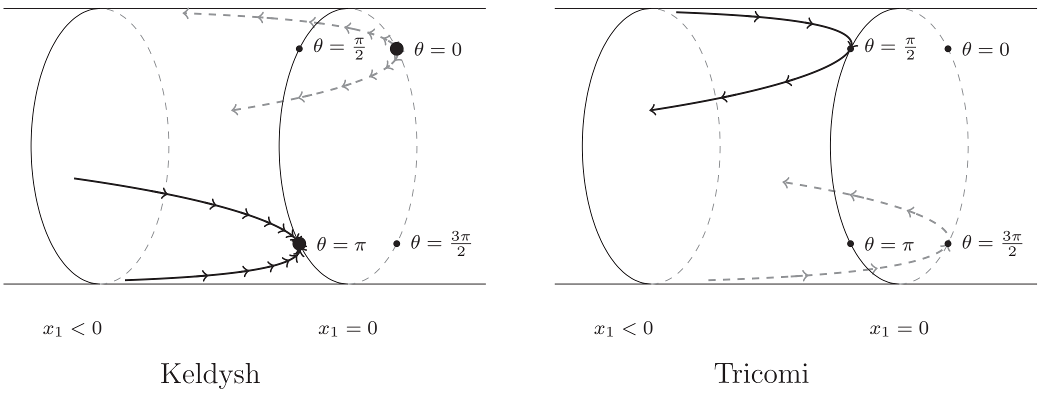

FIGURE 1. *A comparison of the Keldysh operator* (1.1) *and the Tricomi operator* (1.5)*. The figures show the cylinder* $\mathbb{R}_{x_1} \times \mathbb{S}^1_\theta$ *where* $(\xi_1, \xi_2) = |\xi|(\cos\theta, \sin\theta)$ *(this is the boundary of the fibre compactified cotangent bundle* $\overline{T}^*\mathbb{R}^n$ *— see* [**11**, § E.1.3] *— with the* x_2 *variable omitted). The characteristic varieties,* $x_1\cos^2\theta + \sin^2\theta = 0$ *and* $\cos^2\theta + x_1\sin^2\theta = 0$*, respectively, are shown with the direction of the Hamiltonian flow indicated. In the Keldysh case, the two radial Lagrangians,* $\Lambda_\pm$*, correspond to* $\theta = \pi$ *and* $\theta = 0$*, respectively.*

THEOREM 1. *Suppose that* $U \subset \mathbb{R}^n$ *is a neighbourhood of* 0*,*

$$P := x_1 D_{x_1}^2 + a(x)D_{x_1} + Q(x, D_{x'}), \quad x = (x_1, x') \in U, \tag{1.3}$$

has analytic coefficients, $Q(x, D_{x'})$ *is a second-order elliptic operator in* $D_{x'}$ *with a real valued principal symbol. Then there exists a neighbourhood of* 0*,* $U' \subset U$*, such that*

$$Pu \in C^\omega(U), \quad u \in C^\infty(U) \implies u \in C^\omega(U'). \tag{1.4}$$

We will show in § 1.1 that this result follows from a more general microlocal result valid for operators of all orders satisfying a natural geometric condition.

Remarks. (1) In the statement of the theorem 0 can be replaced by any point at which $x_1 \geqslant 0$ and U' can be replaced by U provided we include a bicharacteristic convexity condition. That follows from propagation of analytic singularities — see [**26**, Theorem 4.3.7] or [**22**, Theorem 2.9.1]: since there are no singularities near $x_1 = 0$, there will be no singularities on trajectories hitting $x_1 = 0$ — see Figure 1.

(2) The result is false for the Tricomi operator

$$P := D_{x_1}^2 + x_1 D_{x_2}^2. \tag{1.5}$$

This can be seen using results about propagation of analytic singularities (unlike (1.3) this operator can be microlocally conjugated to D_{y_1} — see Figure 1) but is also easily demonstrated by the following example:

$$u(x) := \int_0^\infty Ai(\tau^{4/3}x_1)e^{i\tau^2 x_2}e^{-\tau}d\tau, \quad Pu = 0, \quad u \in C^\infty(\mathbb{R}^2). \tag{1.6}$$

Here, Ai is the Airy function which satisfies

$$Ai''(t) + tAi(t) = 0, \quad |\partial_t^\ell Ai(t)| \leqslant C_\ell \langle t \rangle^{\frac{\ell}{2} - \frac{1}{4}}, \quad t \in \mathbb{R}, \ \ell \in \mathbb{N}, \quad Ai(0) > 0.$$

We then have

$$D_{x_2}^k u(0) = Ai(0) \int_0^\infty \tau^{2k} e^{-\tau} d\tau = Ai(0)(2k)!$$

and u is not analytic at 0.

(3) Results similar to (1.4) have been obtained in the setting of other operators. In addition to the works [3, 4] cited above, we mention the work of Baouendi–Sjöstrand [2] who considered a class of Fuchsian operators generalizing

$$P = |x|^2\Delta + \mu\langle x, D_x\rangle + \lambda \tag{1.7}$$

In the case of (1.7), (1.4) holds for any $\lambda, \mu \in \mathbb{C}$ and [2] established (1.4) for more general operators satisfying appropriate conditions.

(4) The operators (1.3), (1.5) and (1.7) are not C^∞ hypoelliptic, that is, $Pu \in C^\infty \not\Rightarrow u \in C^\infty$. The study of operators which are C^∞ hypoelliptic but not analytic hypoelliptic has a long tradition with a simple example [23, § 8.6, Example 2] given by

$$P = D_{x_1}^2 + x_1^2 D_{x_2}^2 + D_{x_3}^3.$$

For more complicated cases, references, and connections to several complex variables, see Christ [9] and for some recent progress and additional references, Bove–Mughetti [7].

1.1. *A microlocal result*

We make the following general assumptions. Let P be a differential operator of order m with analytic coefficients:

$$P := \sum_{|\alpha|\leqslant m} a_\alpha(x) D_x^\alpha, \quad a_\alpha \in C^\omega(U), \quad p(x,\xi) := \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha, \tag{1.8}$$

where U is an open neighbourhood of $x_0 \in \mathbb{R}^n$. We make the following assumptions valid in a conic neighbourhood of $(x_0, \xi_0) \in T^*\mathbb{R}^n \setminus 0$: p is *real valued* and there exists a conic Lagrangian submanifold Λ, such that

$$(x_0,\xi_0) \in \Lambda \subset p^{-1}(0), \quad dp|_\Lambda \neq 0, \quad H_p|_\Lambda \parallel \xi\cdot\partial_\xi|_\Lambda. \tag{1.9}$$

Here $\parallel$ means that the two vector fields are *positively* proportional, that is the Lagrangian is *radial* (the positivity assumptions can be achieved by multiplying P by ± 1). Except for the analyticity assumption in (1.8), these are the assumptions made in Haber [19] and Haber–Vasy [20].

Theorem 1 follows from the following microlocal result. We denote by WF the C^∞-wave front set and by WF_a the analytic wave front set — see [23, § 8.1] and [23, § 8.5,9.3], respectively.

THEOREM 2. *Suppose that P and $(x_0,\xi_0) \in T^*\mathbb{R}^n \setminus 0$ satisfy the assumptions (1.8) and (1.9). Then for $u \in \mathscr{D}'(\mathbb{R}^n)$,*

$$(x_0,\xi_0) \notin \mathrm{WF}(u), \quad (x_0,\xi_0) \notin \mathrm{WF}_\mathrm{a}(Pu) \implies (x_0,\xi_0) \notin \mathrm{WF}_\mathrm{a}(u). \tag{1.10}$$

The proof is based on the theory of microlocal symbolic weights developed by Galkowski–Zworski [14] and based on the work of Sjöstrand — see [29, § 2] (and also [21] and [26, § 3.5]). With this theory in place we can use escape functions, G, $H_pG \geqslant 0$, which are logarithmically bounded in ξ (hence the C^∞ wave front set assumption on u allows the use of such weights) and which tend to $\langle\xi\rangle$ in a neighbourhood of (x_0,ξ_0). The normal form for p constructed in [19] (following much earlier work of Guillemin–Schaeffer [18] which was based in turn on Sternberg's linearization theorem [30]) was helpful in the construction of the specific weights needed here. We indicate the method of the proof in § 1.2.

Proof of Theorem 1. Under the assumptions of Theorem 1, the characteristic set of P over $x_1 = 0$ is given by (in $T^*\mathbb{R}^n \setminus 0$)

$$p^{-1}(0) \cap \{x_1 = 0\} = \{(0, x_2, \xi_1, 0) : \xi_1 \in \mathbb{R}\setminus 0; x_2 \in \mathrm{neigh}_{\mathbb{R}^{n-1}}(0)\} = \Lambda_+ \sqcup \Lambda_-,$$

where $\pm\xi_1 > 0$ on $\Lambda_\pm$. These two components are Lagrangian and conic and $H_p|_{\Lambda_\pm} = -\xi_1^2\partial_{\xi_1}|_{\Lambda_\pm}$ is radial. Since $Pu \in C^\omega(U)$, we have $\mathrm{WF_a}(Pu) \cap \{x \in U : x_1 = 0\} = \emptyset$ and hence Theorem 2 shows that $\mathrm{WF_a}(u) \cap \Lambda_\pm = \emptyset$. On the other hand, [**23**, Theorem 8.6.1], $\mathrm{WF_a}(u) \cap \{x_1 = 0\} \subset p^{-1}(0) \cap \{x_1 = 0\} = \Lambda_+ \sqcup \Lambda_-$. Hence $\mathrm{WF_a}(u) \cap \{x_1 = 0\} = \emptyset$ and, since $\mathrm{singsupp_a}\, u = \pi\, \mathrm{WF_a}(u)$, u is analytic near $x_1 = 0$. $\square$

1.2. *A proof in a special case*

To indicate the ideas behind the proof, we consider P given by

$$P = x_1 D_{x_1}^2 + D_{x_2}^2 + aD_{x_1}, \quad a \in \mathbb{C},$$

and a very special u:

$$u = e^{i\tau x_2} v(x_1), \quad v \in \mathscr{S}(\mathbb{R}), \quad Pu = e^{i\tau x_2} f(x_1), \quad e^{|\xi_1|}\widehat{f} \in L^2(\mathbb{R}). \tag{1.11}$$

This assumption is a stronger version of the assumption that f is analytic. We consider a family of smooth functions $G_\epsilon(\xi_1)$ satisfying

$$0 \leqslant G_\epsilon(\xi_1) \leqslant \min(\tfrac{1}{\epsilon}\log(1 + |\xi_1|), |\xi_1|) \tag{1.12}$$

In view of (1.11),

$$\|v_\epsilon\|_{L^2(\mathbb{R})} \leqslant C_\epsilon, \quad \|f_\epsilon\|_{L^2(\mathbb{R})} \leqslant C_0 \quad v_\epsilon := e^{G_\epsilon(D_x)} v, \quad f_\epsilon := e^{G_\epsilon(D_x)} f.$$

where C_0 is independent of ϵ. We then consider

$$P_\epsilon := e^{G_\epsilon(D_x)}(x_1 D_{x_1}^2 + aD_{x_1} + \tau^2)e^{-G_\epsilon(D_x)} = x_1 D_{x_1}^2 + iG_\epsilon'(D_{x_1})D_{x_1}^2 + aD_{x_1} + \tau^2.$$

We have $P_\epsilon v_\epsilon = f_\epsilon$, and

$$\begin{aligned}\operatorname{Im}\langle P_\epsilon v_\epsilon, v_\epsilon\rangle_{L^2(\mathbb{R})} &= \langle G_\epsilon'(D_{x_1})D_{x_1}^2 v_\epsilon, v_\epsilon\rangle_{L^2(\mathbb{R})} + \langle(\operatorname{Im} a + 1)D_{x_1} v_\epsilon, v_\epsilon\rangle_{L^2(\mathbb{R})} \\ &= \langle(\xi_1^2 G_\epsilon'(\xi_1) + (\operatorname{Im} a + 1)\xi_1)\widehat{v}_\epsilon, \widehat{v}_\epsilon\rangle_{L^2(\mathbb{R}_{\xi_1})},\end{aligned}$$

where we took $d\xi_1/(2\pi)$ as the measure on $L^2(\mathbb{R}_{\xi_1})$. Let $\chi \in C^\infty(\mathbb{R};[0,1])$ satisfy $\chi|_{t\leqslant 1} = 1$, $\chi|_{t\geqslant 2} = 0$ and $\chi' \leqslant 0$. We define

$$G_\epsilon(\xi_1) = (1 - \chi(\xi_1))\int_0^{\xi_1}(\chi(\epsilon t) + (1 - \chi(\epsilon t))(\epsilon t)^{-1})dt,$$

which satisfies (1.12) and $G_\epsilon' \geqslant 0$. Moreover, for $\xi_1 \geqslant M \geqslant 2$ and $\epsilon < 1/M$,

$$\xi_1^2 G_\epsilon'(\xi_1) \geqslant \xi_1^2\chi(\epsilon\xi_1) + \epsilon^{-1}\xi_1(1 - \chi(\epsilon\xi_1)) \geqslant M\xi_1.$$

Hence, by taking $M = \max(-\operatorname{Im} a + 1, 2)$, and $\epsilon < 1/M$,

$$\begin{aligned}\|f_\epsilon\|\|\widehat{v}_\epsilon\| &\geqslant \operatorname{Im}\langle P_\epsilon v_\epsilon, v_\epsilon\rangle = \langle(\xi_1^2 G_\epsilon'(\xi_1) + (\operatorname{Im} a + 1)\xi_1)\widehat{v}_\epsilon, \widehat{v}_\epsilon\rangle \\ &\geqslant \|\widehat{v}_\epsilon\|^2 - \|(1 + |\xi_1|(|\operatorname{Im} a| + 1))\widehat{v}_\epsilon|_{\xi_1 \leqslant M}\|\|\widehat{v}_\epsilon\| \geqslant \|\widehat{v}_\epsilon\|^2 - C_1\|\widehat{v}_\epsilon\|,\end{aligned}$$

where $C_1 := (|\operatorname{Im} a| + 1)e^M\|v\|_{H^1}$ is independent of ϵ. This implies

$$\|\widehat{v}_\epsilon\| \leqslant \|f_\epsilon\| + C_1 \leqslant C_0 + C_1.$$

Letting $\epsilon \to 0$ gives $\|e^{\xi_1}\widehat{v}|_{\xi_1 \geqslant 0}\| \leqslant C$. A similar argument applies to $\xi_1 \leqslant 0$ which shows that

$$e^{|\xi_1|}\widehat{v} \in L^2,$$

and consequently that $u(x) = e^{ix_2\tau}v(x_1)$ is analytic.

In the actual proof, the Fourier transform is replaced by the FBI transform (2.1) and its deformation (2.5) defined using a suitably chosen G_ϵ satisfying (1.12) (see Lemma 3.1 which is the heart of the argument). One difficulty not present in the simple one-dimensional case is the localization in other variables. It is here that the C^∞ normal forms of [**18, 19, 30**] are particularly useful. It is essential that no analyticity is needed in the construction of G_ϵ.

1.3. *Applications to scattering theory*

As already indicated in [**32**] analyticity of smooth solution to the Vasy operator (1.2) implies analyticity of resonant states and of their radiation patterns. We review this here and, in Theorem 3, present a slightly stronger result.

For a detailed presentation of scattering on asymptotically hyperbolic manifolds, we refer to [**11**, Chapter 5]. To state Theorem 3, let $\overline{M}$ be a compact $n+1$ dimensional manifold with boundary $\partial M \neq \emptyset$ and let $M := \overline{M} \setminus \partial M$. We assume that $\overline{M}$ is a *real analytic* manifold near ∂M. A metric g on M is called *asymptotically hyperbolic* and *analytic near infinity* if there exist functions $y' \in C^\infty(\overline{M}; \partial M)$ and $y_1 \in C^\infty(\overline{M}; (0,2))$, $y_1|_{\partial M} = 0$, $dy_1|_{\partial M} \neq 0$, such that

$$\overline{M} \supset y_1^{-1}([0,1)) \ni m \mapsto (y_1(m), y'(m)) \in [0,1) \times \partial M \tag{1.13}$$

is a real analytic diffeomorphism, and near ∂M the metric has the form,

$$g|_{y_1 \leqslant \epsilon} = \frac{dy_1^2 + h(y_1)}{y_1^2}, \tag{1.14}$$

where $[0,1) \ni t \mapsto h(t)$, is an analytic family of real analytic Riemannian metrics on ∂M.

Let

$$R_g(\lambda) = (-\Delta_g - \lambda^2 - (n/2)^2)^{-1} : L^2(M, d\,\mathrm{vol}_g) \to H^2(M, d\,\mathrm{vol}_g), \quad \operatorname{Im} \lambda > 0.$$

Mazzeo–Melrose [**27**] and Guillarmou [**17**] proved that

$$R_g(\lambda) : C_c^\infty(M) \to C^\infty(M), \tag{1.15}$$

continues to a meromorphic family of operators for $\lambda \in \mathbb{C} \setminus i(-\frac{1}{2} - \mathbf{N})$. In addition, Guillarmou [**17**] showed that if the metric is *even*, that is,

$$g|_{y_1 \leqslant \epsilon} = \frac{dy_1^2 + h(y_1^2)}{y_1^2}, \tag{1.16}$$

(see [**11**, Theorem 5.6] for an invariant formulation), then $R_g(\lambda)$ is meromorphic in $\mathbb{C}$. In particular, for $\lambda \neq 0$ we have the following Laurent expansion

$$R_g(\zeta) = \sum_{j=1}^{J(\lambda)} \frac{(-\Delta_g - \lambda^2 - (n/2)^2)^{j-1} \Pi(\lambda)}{(\zeta^2 - \lambda^2)^j} + A(\zeta, \lambda), \quad \Pi(\lambda) := \frac{1}{2\pi i} \oint_\lambda R_g(\zeta) 2\zeta d\zeta,$$

where $\zeta \mapsto A(\zeta, \lambda)$ is holomorphic near λ. For $\lambda = 0$, we have a Laurent expansions in powers of ζ^{-j}.

The operator $\Pi(\lambda)$ has finite rank and its range consists of *generalized resonant states*. We then have

THEOREM 3. *Suppose that (M, g) is an even asymptotically hyperbolic manifold (in the sense of (1.16)) analytic near conformal infinity ∂M. Then for $\lambda \in \mathbb{C} \setminus 0$,*

$$u \in \Pi(\lambda) C_c^\infty(M) \implies u = y_1^{-i\lambda + \frac{n}{2}} F, \quad F|_{\partial M} \in C^\omega(\partial M). \tag{1.17}$$

Moreover, in coordinates of (1.16), $F(y) = f(y_1^2, y')$, $y' \in \partial M$ where $f \in C^\omega((-\delta, \delta) \times \partial M)$.

Proof. The metric (1.14) (in the coordinates valid near the boundary) gives the following Laplace operator:

$$\begin{aligned} -\Delta_g &= (y_1 D_{y_1})^2 + i(n + y_1\gamma_0(y_1^2, y'))y_1 D_{y_1} - y_1^2 \Delta_{h(y_1)}, \\ \gamma_0(t, y') &:= -\tfrac{1}{2}\partial_t \bar{h}(t)/\bar{h}(t), \quad \bar{h}(t) := \det h(t), \quad D := \tfrac{1}{i}\partial. \end{aligned} \tag{1.18}$$

Following Vasy [**31**], we change the variables to $x_1 = y_1^2$, $x' = y'$ so that

$$y_1^{i\lambda - \frac{n}{2}}(-\Delta_g - \lambda^2 - (\tfrac{n}{2})^2)y_1^{-i\lambda + \frac{n}{2}} = x_1 P(\lambda), \tag{1.19}$$

where, near ∂M, $P(\lambda)$ is given by (1.2). This operator is considered on $X := ((-\delta, 0]_{x_1} \times \partial M) \sqcup M$. The key fact is that $P(\lambda)$ is a Fredholm family operators on suitable spaces, $P(\lambda)^{-1}$ is meromorphic and its poles can be studied using microlocal methods — see [**31**], [**11**, Chapter 5] and also [**34**, § 2] for a short self-contained presentation.

From meromorphy of $P(\lambda)^{-1}$, we obtain meromorphy of (1.15) using (1.19):

$$R_g(\lambda)f := y_1^{\frac{n}{2} - i\lambda}\Big(P(\lambda)^{-1} y_1^{i\lambda - \frac{n+2}{2}} f\Big)\big|_M \;\in\; C^\infty(M). \tag{1.20}$$

Here we make $y_1^{i\lambda - \frac{n+2}{2}} f$ into an element of $C^\infty_{\rm c}(X)$ by extending it by zero outside of M. Near any λ, $P(\zeta)^{-1} = \sum_{k=1}^{K(\lambda)} Q_j(\lambda)(\zeta-\lambda)^{-j} + Q_0(\zeta, \lambda)$, with $Q_j(\lambda)$ operators of finite rank and $\zeta \mapsto Q_0(\zeta, \lambda)$ is analytic near λ. We then have

$$\Pi(\lambda) = \tfrac{1}{2\lambda} y_1^{\frac{n}{2} - i\lambda} Q_1(\lambda) y_1^{i\lambda - \frac{n+2}{2}}.$$

Hence, the claim about the range of $\Pi(\lambda)$ follows from analyticity of functions in the range of $Q_1(\lambda)$. This follows from Theorem 1. In fact, $P(\zeta) = P(\lambda) + (\zeta - \lambda)V$, $V := -4D_{x_1} + i\gamma(x)$, and hence

$$P(\lambda)Q_k(\lambda) = -VQ_{k+1}(\lambda), \quad Q_{K+1}(\lambda) := 0.$$

Since we already know that the ranges of the operators Q_k are in C^∞ (see [**11**, (5.6.10)]), we inductively conclude that their ranges are in C^ω. $\square$

Remark. Vasy's adaptation of Melrose's radial estimates [**28**] shows that to conclude that $u \in C^\infty$ when $P(\lambda)u \in C^\infty$ (see (1.2)), we only need to assume that $u \in H^{s+1}$ near m_0, where $s + \frac{1}{2} > -\operatorname{Im}\lambda$, see [**34**, § 4, Remark 3].

2. *Preliminaries on FBI transforms and their deformations*

We will use the FBI transform defined in [**14**] in its $\mathbb{R}^n$ (rather than $\mathbb{T}^n$) version. Since the weights we use will be compactly supported in x, the same theory applies. The constructions there are inspired by the works of Boutet de Monvel–Sjöstrand [**6**], Boutet de Monvel–Guillemin [**5**], Helffer–Sjöstrand [**21**] and Sjöstrand [**29**]. An alternative approach to using the classes of weights we need here was developed independently and in greater generality by Guedes Bonthonneau–Jézéquel [**16**].

2.1. *Deformed FBI transforms*

We define

$$Tu(x, \xi) := h^{-\frac{3n}{4}} \int_{\mathbb{R}^n} e^{\frac{i}{h}(\langle x-y, \xi\rangle + \frac{i}{2}\langle\xi\rangle(x-y)^2)} \langle\xi\rangle^{\frac{n}{4}} u(y)dy, \quad u \in C^\infty_{\rm c}(\mathbb{R}^n), \tag{2.1}$$

recalling that the left inverse of T is given by

$$Sv(y) = \frac{2^{\frac{n}{2}} h^{-\frac{3n}{4}}}{(2\pi)^{\frac{3n}{2}}} \int_{\mathbb{R}2n} e^{-\frac{i}{h}(\langle x-y, \xi\rangle - \frac{i}{2}\langle\xi\rangle(x-y)^2)} \langle\xi\rangle^{\frac{n}{4}} (1 + \tfrac{i}{2}\langle x-y, \xi/\langle\xi\rangle\rangle) v(x, \xi) dx d\xi, \tag{2.2}$$

see [**14**, Proposition 2.2].

The first fact we need is the characterization of Sobolev spaces and of the C^∞ wave front set using the FBI transform (2.1). To formulate it we use semiclassical Sobolev spaces H^s_h (see, for instance, [**33**, § 7.1] or [**11**, Definition E.18]) but we should in general think of h as being fixed.

PROPOSITION 2.1. *There exists a constant C such that for $u \in \mathscr{S}'(\mathbb{R}^n)$,*

$$\|u\|_{H^s_h} \leqslant C\|\langle\xi\rangle^s Tu\|_{L^2(T^*\mathbb{R}^n)} \leqslant C^2\|u\|_{H^s_h}. \tag{2.3}$$

Moreover,

$$(x_0,\xi_0) \notin \mathrm{WF}(u) \Leftrightarrow \begin{cases} \exists\, \chi \in S^0(T^*\mathbb{R}^n),\ \chi \equiv 1 \text{ in a conic neighbourhood of } (x_0,\xi_0), \\ \forall\, N\ \exists\, C_N \quad \|\langle\xi\rangle^N \chi Tu\|_{L^2(T^*\mathbb{R}^n)} \leqslant C_N. \end{cases}$$

Proof. This follows from the characterization of the H^s based wave front sets in Gérard [**15**] as stated in [**10**, Theorem 1.2]. Since the arguments are similar to the more involved analytic case presented in Proposition 2.3, we omit the details. □

As in [**29**, § 2] and [**14**, § 3], we introduce a geometric deformation of $\mathbb{R}^{2n}$, $\Lambda = \Lambda_G$:

$$\begin{gathered} \Lambda := \{(x - iG_\xi(x,\xi), \xi + iG_x(x,\xi)) \mid (x,\xi) \in \mathbb{R}^{2n}\} \subset \mathbb{C}^{2n}, \\ \operatorname{supp} G \subset K \times \mathbb{R}^n, \quad K \Subset \mathbb{R}^n, \\ \sup_{|\alpha|+|\beta|\leqslant 2} \langle\xi\rangle^{-1+|\beta|} |\partial_x^\alpha \partial_\xi^\beta G(x,\xi)| \leqslant \epsilon_0, \quad |\partial_x^\alpha\partial_\xi^\beta G(x,\xi)| \leqslant C_{\alpha\beta}\langle\xi\rangle^{1-|\beta|}, \end{gathered} \tag{2.4}$$

where ϵ_0 is small and fixed (so that the constructions below remain valid as in [**14**]). For convenience, we change here the convention from [**14**]: it amounts to replacing G by $-G$ everywhere.

This provides us with the following new objects: the deformed FBI transform (see [**14**, § 4]),

$$\begin{gathered} T_\Lambda u(x,\xi) := Tu(x - iG_\xi(x,\xi), \xi + iG_x(x,\xi)), \quad u \in \mathscr{B}_\delta, \\ \mathscr{B}_\delta := \{u \in \mathscr{S}(\mathbb{R}^n) : \textstyle\int_{\mathbb{R}^n} |\widehat{U}(\xi)|^2 e^{4\delta|\xi|} d\xi < \infty\}, \end{gathered} \tag{2.5}$$

the spaces H^s_Λ, defined as in [**14**, § 4],

$$H^s_\Lambda := \overline{\mathscr{B}_{\delta_0}}^{\|\bullet\|_{H^s_\Lambda}}, \quad \|u\|^2_{H^s_\Lambda} := \int_\Lambda \langle \operatorname{Re}\alpha_\xi\rangle^{2s} |T_\Lambda u(\alpha)|^2 e^{-2H(\alpha)/h} d\alpha, \tag{2.6}$$

and the orthogonal projector

$$\Pi_\Lambda : L_\Lambda := L^2(\Lambda, e^{-2H(\alpha)/h} d\alpha) \to T_\Lambda H_\Lambda, \quad H_\Lambda := H^0_\Lambda,$$

described asymptotically (as $h \to 0$ and as $\xi \to \infty$) in [**14**, § 5]. The weight H appears naturally in this subject and is given by [**14**, (3.3),(3.4)], that is, $H(x,\xi) = \xi \cdot G_\xi(x,\xi) - G(x,\xi)$. The deformed FBI transform T_Λ has an exact left inverse S_Λ obtained by deforming S in (2.2).

We now prove a slightly modified version of [**14**, Proposition 6.2]:

PROPOSITION 2.2. *Suppose that $P = \sum_{|\alpha|\leqslant m} a_\alpha D^\alpha$ is a differential operator with $a_\alpha \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ satisfying,*

$$a_\alpha \in C^\omega(U), \quad K \Subset U,$$

for an open set U and K as in (2.4). Then

$$\Pi_\Lambda T_\Lambda h^m P S_\Lambda = \Pi_\Lambda b_P \Pi_\Lambda + \mathcal{O}(h^\infty)_{H^{-N}_\Lambda \to H^N_\Lambda},$$

where

$$\begin{aligned} b_P(x,\xi) &\sim \sum_{j=0}^{\infty} h^j b_j(x,\xi), \quad b_j \in S^{m-j}(\mathbb{R}^{2n}), \\ b_0 &= p|_\Lambda := p(x - iG_\xi(x,\xi), \xi + iG_x(x,\xi)). \end{aligned} \tag{2.7}$$

We remark that the expansion remains valid when h is fixed. We can use smallness of h to dominate the lower order terms and then keep it fixed.

Proof. The result follows from the analogue of [**14**, Lemma 6.1] where the operator $T_\Lambda h^m P S_\Lambda$ is described in the case where the coefficients of P are globally analytic. Here we point out that the analyticity of the coefficients is only needed in the neighbourhood U of $K \Subset \mathbb{R}^n$ such that in (2.4) $\operatorname{supp} G \subset K \times \mathbb{R}^n$ and ϵ_0 is small enough depending on the size of the complex neighbourhood to which the coefficients extend holomorphically.

In fact, arguing as in the proof of [**14**, Proposition 6.2] all we need is that for $a \in C^\infty_{\rm c}(\mathbb{R}^n)$ and $a \in C^\omega(U)$, the Schwartz kernel of $T_\Lambda M_a S_\Lambda$, $M_a f(x) := a(x) f(x)$, is given by

$$\begin{gathered} K_a(\alpha,\beta) = c_0 h^{-n} e^{\frac{i}{h}\Psi(\alpha,\beta)} A(\alpha,\beta) + r(\alpha,\beta), \quad \alpha,\beta \in \Lambda = \Lambda_G, \\ r(\alpha,\beta) \text{ is the kernel of an operator } R = O(h^\infty) : H_\Lambda^{-N} \to H_\Lambda^N. \end{gathered} \tag{2.8}$$

The phase in (2.8) is given by

$$\Psi(\alpha,\beta) = \frac{i}{2}\frac{(\alpha_\xi - \beta_\xi)^2}{\langle\alpha_\xi\rangle + \langle\beta_\xi\rangle} + \frac{i}{2}\frac{\langle\beta_\xi\rangle\langle\alpha_\xi\rangle(\alpha_x - \beta_x)^2}{\langle\alpha_\xi\rangle + \langle\beta_\xi\rangle} + \frac{\langle\beta_\xi\rangle\alpha_\xi + \langle\alpha_\xi\rangle\beta_\xi}{\langle\alpha_\xi\rangle + \langle\beta_\xi\rangle} \cdot (\alpha_x - \beta_x), \tag{2.9}$$

and the amplitude satisfies

$$A \sim \sum_{j=0}^{\infty} h^j \langle\alpha_\xi\rangle^{-j} A_j, \quad A_0(\alpha,\alpha) = a|_\Lambda(\alpha),$$

and A_j are supported in a small conic neighbourhood of the diagonal in $\Lambda \times \Lambda$. We note that if ϵ_0 is small enough, a extends to some neighbourhood of K in $\mathbb{C}^n$ and hence $a|_\Lambda = a(x - iG_\xi(x,\xi))$ is well defined.

To see (2.8), we use the definitions of T_Λ and S_Λ to write

$$K_a(\alpha,\beta) = c_n \langle\beta_\xi\rangle^{\frac{n}{4}} \langle\alpha_\xi\rangle^{\frac{n}{4}} h^{-\frac{3n}{2}} \int e^{\frac{i}{h}(\varphi_G(\alpha,y) + \varphi_G^*(\beta,y))} a(y)(1 + \langle\beta_x - y, \beta_\xi/\langle\beta_\xi\rangle\rangle) dy, \tag{2.10}$$

where

$$\begin{gathered} \varphi_G(\alpha,y) = \Phi(z,\zeta,y)|_{z=\alpha_x,\zeta=\alpha_\xi}, \quad \varphi_G^*(\alpha,y) = -\bar\Phi(z,\zeta,y)|_{z=\alpha_x,\zeta=\alpha_\xi}, \\ \alpha_x = x - iG_\xi(x,\xi), \quad \alpha_\xi = \xi + iG_x(x,\xi), \\ \Phi(z,\zeta,y) = \langle z - y, \zeta\rangle + \tfrac{i}{2}\langle\zeta\rangle(z-y)^2, \quad \bar\Phi(z,\zeta,y) := \overline{\Phi(\bar z, \bar\zeta, y)}. \end{gathered} \tag{2.11}$$

Let V, V_1 be open such that $K \subset V_1 \Subset V \Subset U$. We start by showing that the contribution to K_a away from the diagonal is negligible. For that let $\chi \in C_c^\infty(\mathbb{R})$ with $\chi \equiv 1$ near 0. Then for all $\delta > 0$ small enough, the operator R_1 with kernel

$$R_1(\alpha,\beta) = K_a(\alpha,\beta)\tilde\chi_\delta(\alpha,\beta),$$

$$\tilde\chi_\delta(\alpha,\beta) := (1 - \chi(\delta^{-1}|\alpha_x - \beta_x|))\Big(1 - \chi\Big(\frac{|\alpha_\xi - \beta_\xi|}{\delta\langle|\alpha_\xi - \beta_\xi|\rangle}\Big)\Big)$$

satisfies $R_1 = O_{H_\Lambda^{-N} \to H_\Lambda^N}(h^\infty)$. This amounts to showing that the operator with kernel $R_1(\alpha,\beta) e^{\frac{1}{h}(H(\beta) - H(\alpha))} \langle\alpha_\xi\rangle^N \langle\beta_\xi\rangle^N$ is bounded on $L^2(\mathbb{R}^{2n})$ with $O(h^\infty)$ norm.

To see this, we first integrate by parts K times in y, using that

$$|\partial_y \Psi| = |\beta_\xi - \alpha_\xi + i(\langle \alpha_\xi\rangle(y-\alpha_x) + \langle\beta_\xi\rangle(y-\beta_x))| \geqslant c(1+|\alpha_\xi|+|\beta_\xi|)$$

on $\operatorname{supp}\tilde{\chi}_\delta$. This reduces the analysis to the case of (2.10) with a replaced by $b(\cdot,\alpha,\beta)\in C^\omega(U)\cap C_c^\infty(\mathbb{R}^n)$ with $|b|\leqslant h^K(\langle|\alpha_\xi|\rangle+\langle|\beta_\xi|\rangle)^{-K}$.

Next, we choose $\psi\in C_c^\infty(\mathbb{R}^n;[0,1])$ with $\psi\equiv 1$ on V and $\operatorname{supp}\psi\subset U$, and $\psi_1\in C_c^\infty(\mathbb{R}^n;[0,1])$ with $\psi_1\equiv 1$ on V_1 and $\operatorname{supp}\psi_1\subset V$. We then deform the contour

$$y\mapsto y+i\epsilon\psi(y)\frac{\overline{\beta_\xi-\alpha_\xi}}{\langle|\beta_\xi-\alpha_\xi|\rangle}.$$

This contour deformation is justified since $a\in C^\omega(U)$. The phase in the integrand of (2.10) becomes

$$\begin{aligned}\Psi =&\langle\alpha_x-y,\alpha_\xi\rangle+\langle y-\beta_x,\beta_\xi\rangle+\frac{i\langle\alpha_\xi\rangle}{2}(\alpha_x-y)^2+\frac{i\langle\beta_\xi\rangle}{2}(\beta_x-y)^2\\ &+i\epsilon\psi(y)\frac{|\beta_\xi-\alpha_\xi|^2}{\langle|\beta_\xi-\alpha_\xi|\rangle}+\frac{i\langle\alpha_\xi\rangle}{2}\left[2\epsilon\psi(y)\left\langle\alpha_x-y,\frac{\overline{\alpha_\xi-\beta_\xi}}{\langle|\beta_\xi-\alpha_\xi|\rangle}\right\rangle-\epsilon^2\psi^2(y)\frac{|\beta_\xi-\alpha_\xi|^2}{\langle|\beta_\xi-\alpha_\xi|\rangle^2}\right]\\ &\frac{i\langle\beta_\xi\rangle}{2}\left[2\epsilon\psi(y)\left\langle\beta_x-y,\frac{\overline{\alpha_\xi-\beta_\xi}}{\langle|\beta_\xi-\alpha_\xi|\rangle}\right\rangle-\epsilon^2\psi^2(y)\frac{|\beta_\xi-\alpha_\xi|^2}{\langle|\beta_\xi-\alpha_\xi|\rangle^2}\right].\end{aligned}$$

In particular, for $y\in V$, and $(\alpha,\beta)\in\operatorname{supp}\tilde{\chi}_\delta$, the integrand is bounded by

$$e^{-c(\langle\alpha_\xi\rangle+\langle\beta_\xi\rangle)\langle\alpha_x-\beta_x\rangle/h}$$

which is negligible (even after multiplication by $e^{\frac{1}{h}(H(\beta)-H(\alpha))}\langle\alpha_\xi\rangle^N\langle\beta_\xi\rangle^N$).

For the integral over $y\notin V$, we consider three cases. First, if both $\operatorname{Re}\alpha_x\in K$ and $\operatorname{Re}\beta_x\in K$, then it is easy to see that the integrand is bounded by

$$e^{-c(\langle\alpha_\xi\rangle+\langle\beta_\xi\rangle)(\langle\alpha_x-\beta_x\rangle+|y|)/h}$$

and hence produces a negligible contribution. Next, if $\operatorname{Re}\alpha_x\notin K$ and $\operatorname{Re}\beta_x\notin K$, then $H(\alpha)=H(\beta)=0$, α,β are real, and integration by parts in y shows that the contribution is negligible.

Finally, we consider the case $\operatorname{Re}\alpha_x\in K$, $\operatorname{Re}\beta_x\notin K$, (the case $\operatorname{Re}\beta_x\in K$ and $\operatorname{Re}\alpha_x\notin K$ being similar). In this case, we have $H(\beta)=0$ and β real. Since $y\notin V$, we have that the integrand is bounded by $e^{-c\langle\alpha_\xi\rangle\langle\alpha_x-y\rangle/h}h^K\langle\beta_\xi\rangle^{-K}$ and hence this term is also negligible.

Since R is negligible, we may assume from now on that

$$|\alpha_x-\beta_x|\ll 1 \quad\text{and}\quad |\alpha_\xi-\beta_\xi|\ll\langle|\alpha_\xi|\rangle+\langle|\beta_\xi|\rangle.$$

In particular, there are three cases: $\operatorname{Re}\alpha_x\in K$ and $\operatorname{Re}\beta_x\in V_1$, $\operatorname{Re}\beta_x\in K$ and $\operatorname{Re}\alpha_x\in V_1$, or $\operatorname{Re}\alpha_x\notin K$ and $\operatorname{Re}\beta_x\notin K$.

The first two cases are similar, so we consider only one of them. Since $\operatorname{Re}\alpha_x\in K$ and $\operatorname{Re}\beta_x\in V_1$, the contribution from $y\notin V$ is negligible. Therefore, we may deform the contour to

$$y\mapsto y+\psi(y)y_c(\alpha,\beta),\qquad y_c(\alpha,\beta)=\frac{i(\beta_\xi-\alpha_\xi)+\langle\alpha_\xi\rangle\alpha_x+\langle\beta_\xi\rangle\beta_x}{\langle\alpha_\xi\rangle+\langle\beta_\xi\rangle}.$$

The proof in this case then follows from the method of complex stationary phase.

When both $\operatorname{Re}\alpha_x\notin K$ and $\operatorname{Re}\beta_x\notin K$, $\alpha=\operatorname{Re}\alpha$, $\beta=\operatorname{Re}\beta$, and $H(\alpha)=H(\beta)=0$. In order to handle this situation, we will Taylor expand $a(y)$ around $y=\alpha_x$. For that we first consider (2.10) with $a=O(|y-\alpha_x|^{2N})$. In that case, we consider the integral

$$\begin{aligned}K_N(\alpha,\beta):=h^{-\frac{3n}{2}}\int e^{\frac{i}{h}(\langle\alpha_x-y,\alpha_\xi\rangle+\frac{i}{2}(\langle\alpha_\xi\rangle(\alpha_x-y)^2+\langle\beta_\xi\rangle(\beta_x-y)^2))}\\ O(|y-\alpha_x|^{2N})\langle\alpha_\xi\rangle^{\frac{n}{4}}\langle\beta_\xi\rangle^{\frac{n}{4}}(1-\tilde{\chi}_\delta(\alpha,\beta))dy.\end{aligned}\tag{2.12}$$

Changing variables $y \mapsto y + \alpha_x$,

$$|K_N(\alpha,\beta)| \leqslant \int \langle\alpha_\xi\rangle^{\frac{n}{4}}\langle\beta_\xi\rangle^{\frac{n}{4}}\frac{h^{N-\frac{3n}{2}}}{\langle\alpha_\xi\rangle^N}e^{-\frac{\langle\beta_\xi\rangle}{2h}(\beta_x-\alpha_x-y)^2}(1-\tilde\chi_\delta)dy$$
$$\leqslant C\frac{h^{N-n}}{(\langle\alpha_\xi\rangle+\langle\beta_\xi\rangle)^N}e^{-c\frac{\langle\alpha_\xi\rangle+\langle\beta_\xi\rangle}{h}(\alpha_x-\beta_x)^2}(1-\tilde\chi_\delta(\alpha,\beta)).$$

Therefore, using the Schur test for boundedness, the operator K_N with kernel $K_N(\alpha,\beta)$ satisfies

$$K_N = O(h^{N-\frac{n}{2}}) : H_\Lambda^{-N+\frac{n}{4}+0} \to H_\Lambda^{N-\frac{n}{4}-0}$$

Now, observe that for any $N > 0$,

$$a(y) = a_N(y) + O(|y-\alpha_x|^{2N}),$$

where $a_N(y)$ is a polynomial of order $2N-1$ in $(y-\alpha_x)$. In particular,

$$K_a(\alpha,\beta) = K_{a_N}(\alpha,\beta) + K_N(\alpha,\beta).$$

Since a_N is analytic and the integrand is exponentially decaying in y, we may deform the contour with $y \mapsto y + y_c(\alpha,\beta)$ in the integral forming the kernel of K_{a_N} and apply complex stationary phase as in the case where $\operatorname{Re}\alpha_x \in K$ or $\operatorname{Re}\beta_x \in K$. This finishes the proof of the proposition after taking N large enough. □

2.2. *Analytic wave front set*

We now relate weighted estimates to analyticity.

Proposition 2.3. *Let T be the FBI transform defined in (2.1) for some fixed h, and let $\psi \in S^1(T^*\mathbb{R}^n)$ satisfy*

$$\psi(x,\xi) \geqslant |\xi|/C, \quad (x,\xi) \in U \times \Gamma, \tag{2.13}$$

where $U \subset \mathbb{R}^n$ and $\Gamma \subset \mathbb{R}^n \setminus 0$ is an open cone. Then, for $u \in H^{-N}(\mathbb{R}^n)$,

$$e^{\psi}\langle\xi\rangle^{-N}Tu \in L^2(T^*\mathbb{R}^n) \implies \operatorname{WF_a}(u) \cap (U\times\Gamma) = \emptyset. \tag{2.14}$$

Conversely, suppose $u \in H^{-N}(\mathbb{R}^n)$, $\Gamma_0 \subset \mathbb{R}^n$ is a conic open set such that $\Gamma_0 \cap \mathbb{S}^{n-1} \Subset \Gamma \cap \mathbb{S}^{n-1}$, $U_0 \Subset U$. Then for any $\psi \in S^1(\mathbb{R}^n \times \mathbb{R}^n)$ with $\operatorname{supp}\psi \subset U_0 \times V_0$,

$$\operatorname{WF_a}(u) \cap (U\times\Gamma) = \emptyset \implies \exists\, \theta > 0 \;\; \langle\xi\rangle^{-N}e^{\theta\psi}Tu \in L^2(T^*\mathbb{R}^n). \tag{2.15}$$

Remark. Here we do not consider uniformity in h in the L^2 bounds. If we demanded that, then we would only need $\psi \in C_c^\infty(T^*\mathbb{R}^n)$, $\psi > 0$ on $U \times (\Gamma \cap \mathbb{S}^{n-1})$.

The proof is based on the following

Lemma 2.4. *Let T and S be given by (2.1) and (2.2), respectively, with h fixed. Suppose that $\chi, \tilde\chi \in S^0(\mathbb{R}^n\times\mathbb{R}^n)$ and $\operatorname{supp}\chi, \operatorname{supp}\chi_1 \subset K \times \mathbb{R}^n$, $K \Subset \mathbb{R}^n$. Then for any $a > 0$, there exists $b > 0$ such that*

$$\chi e^{b\langle\xi\rangle}TS\chi_1 e^{-a\langle\xi\rangle} = \mathcal{O}_N(1) : L^2(\mathbb{R}^{2n}) \to H^N(\mathbb{R}^{2n}), \tag{2.16}$$

for any N.

If in addition $\chi_1 \equiv 1$ on a conic neighbourhood of the support of χ, then there exists $b > 0$ such that

$$\chi e^{b\langle\xi\rangle}TS(1-\chi_1)\langle\xi\rangle^M = \mathcal{O}_{N,M}(1) : L^2(\mathbb{R}^{2n}) \to H^N(\mathbb{R}^{2n}), \tag{2.17}$$

for any N.

Proof. We analyse the Schwartz kernel of the operator in (2.16), $K(x,\xi,y,\eta)$. As in the proofs of [**14**, Lemma 2.1, Proposition 4.5] (the phase of resulting operator can be computed by completion of squares and is given by [**14**, (4.10)] with $\Lambda = T^*\mathbb{R}^n$), we see that

$$\begin{aligned} &|(hD)^\alpha_{x,\xi} K(x,\xi,y,\eta)| \leqslant C_\alpha e^{b\langle\xi\rangle - a\langle\eta\rangle - \psi(x,\xi,y,\eta)}, \\ &\psi := c(\langle\xi\rangle + \langle\eta\rangle)^{-1}\big(|\xi-\eta|^2 + \langle\xi\rangle\langle\eta\rangle|x-y|^2\big). \end{aligned} \tag{2.18}$$

We have

$$b < \tfrac{1}{8}\min(a,c) \ \Rightarrow\ b\langle\xi\rangle - a\langle\eta\rangle - c(\langle\xi\rangle+\langle\eta\rangle)^{-1}|\xi-\eta|^2 \leqslant -\tfrac{1}{2}(b\langle\xi\rangle + a\langle\eta\rangle),$$

if b is sufficiently small. (By taking $b < a/8$, we can assume that $|\eta| \leqslant |\xi|/2$. But then $|\xi-\eta| \geqslant \frac{1}{2}|\xi|$ and $\langle\xi\rangle + \langle\eta\rangle \leqslant 2\langle\eta\rangle$.) This proves (2.16) as we can use the Schur criterion.

To see (2.17), we note that we can now assume that $|\xi/\langle\xi\rangle - \eta/\langle\eta\rangle| > \delta$ or $|x-y| > \delta$. But then if the kernel of the operator in (2.17) is given by $K_M(x,\xi,y,\eta)$ where

$$|(hD_{x,\xi})^\alpha K_N(x,\xi,y,\eta)| \leqslant C_{\alpha,N} e^{b\langle\xi\rangle - M\log\langle\eta\rangle - \psi(x,\xi,y,\eta)}.$$

Now, fix $0 < \delta < 1$ small. Then, when $|\xi/\langle\xi\rangle - \eta/\langle\eta\rangle| > \delta$ or $|x-y| > \delta$,

$$|\xi-\eta|^2 + \langle\xi\rangle\langle\eta\rangle|x-y|^2 \geqslant \frac{\delta^2}{16}(\langle\xi\rangle+\langle\eta\rangle)^2. \tag{2.19}$$

To see this, observe that on

$$\left|\frac{\langle\xi\rangle - \langle\eta\rangle}{\langle\xi\rangle+\langle\eta\rangle}\right| \geqslant \frac{\delta}{4},$$

we have

$$\frac{\delta}{4} \leqslant \left|\frac{\langle\xi\rangle^2 - \langle\eta\rangle^2}{(\langle\xi\rangle+\langle\eta\rangle)^2}\right| \leqslant \frac{|\xi-\eta|}{\langle\xi\rangle+\langle\eta\rangle}.$$

On the other hand, when

$$\left|\frac{\langle\xi\rangle - \langle\eta\rangle}{\langle\xi\rangle+\langle\eta\rangle}\right| \leqslant \frac{\delta}{4},$$

we have

$$\frac{2\langle\xi\rangle\langle\eta\rangle}{\langle\xi\rangle+\langle\eta\rangle} = \frac{\langle\xi\rangle+\langle\eta\rangle}{2}\left(1 - \left[\frac{\langle\eta\rangle-\langle\xi\rangle}{\langle\xi\rangle+\langle\eta\rangle}\right]^2\right) \geqslant \frac{1}{4}(\langle\xi\rangle+\langle\eta\rangle)$$

Therefore, if $|x-y| \geqslant \delta$, (2.19) follows. If instead, $|\xi/\langle\xi\rangle - \eta/\langle\eta\rangle| \geqslant \delta$, then

$$\frac{|\xi-\eta|}{\langle\xi\rangle+\langle\eta\rangle} \geqslant \frac{1}{2}\left[\left|\frac{\xi}{\langle\xi\rangle} - \frac{\eta}{\langle\eta\rangle}\right| - \left(\frac{|\xi|}{\langle\xi\rangle} + \frac{|\eta|}{\langle\eta\rangle}\right)\left|\frac{\langle\xi\rangle-\langle\eta\rangle}{\langle\xi\rangle+\langle\eta\rangle}\right|\right] \geqslant \frac{\delta}{4}$$

and (2.19) follows.

From (2.19), we have that there is $C_{M,\delta} > 0$ such that if $|\xi/\langle\xi\rangle - \eta/\langle\eta\rangle| > \delta$ or $|x-y| > \delta$,

$$\begin{aligned} b\langle\xi\rangle - &c(\langle\xi\rangle+\langle\eta\rangle)^{-1}\big(|\xi-\eta|^2 + \langle\xi\rangle\langle\eta\rangle|x-y|^2\big) + M\log\langle\eta\rangle \\ &\leqslant b\langle\xi\rangle - \tfrac{1}{64}c\delta^2(\langle\xi\rangle+\langle\eta\rangle) - \tfrac{1}{2}c(\langle\xi\rangle+\langle\eta\rangle)^{-1}\big(|\xi-\eta|^2 + \langle\xi\rangle\langle\eta\rangle|x-y|^2\big) + C_{M,\delta}, \end{aligned}$$

and the Schur criterion and gives (2.17) for $b \leqslant \frac{c\delta^2}{64}$. ☐

Proof of Proposition 2.3. We start by recalling the characterization of the analytic wave front set using the standard FBI/Bargmann–Segal transform:

$$\mathscr{T}u(x,\xi;h) := c_n h^{-\frac{3n}{4}} \int_{\mathbb{R}^n} e^{\frac{i}{h}(\langle x-y,\xi\rangle + \frac{i}{2}(x-y)^2)} u(y)dy, \quad u \in \mathscr{S}'(\mathbb{R}^n).$$

Then

$$(x_0,\xi_0)\notin \mathrm{WF}_{\mathrm{a}}(u) \iff \begin{cases} \exists\, \delta,\, U=\mathrm{neigh}((x_0,\xi_0)) \\ |\mathscr{T}u(x,\xi,h)|\leqslant Ce^{-\delta/h}, \quad (x,\xi)\in U, \quad 0<h<h_0. \end{cases} \tag{2.20}$$

see [**23**, Theorem 9.6.3] for a textbook presentation; note the somewhat different convention: $\mathscr{T}u(x,\xi;h)=e^{-\frac{1}{2h}\xi^2}T_{1/h}u(x-i\xi)$.

We first prove (2.14). Hence suppose $(x_0,\xi_0)\in U\times\Gamma$. Let $\chi\in S^0$ be supported in a small conic neighbourhood, $U_0\times\Gamma_0$, of (x_0,ξ_0) and choose $\chi_1\in S^0$ which is supported in $U\times\Gamma$ and is equal to 1 on a conic neighbourhood of the support of χ and $\chi_2\in S^0$ supported in $U\times\Gamma$ and equal to 1 on a conic neighbourhood of the support of χ_1. Our assumptions then show that $e^{a\langle\xi\rangle/h}\chi_2Tu\in L^2(\mathbb{R}^{2n})$ for some $a>0$. We now write

$$\chi e^{b\langle\xi\rangle}Tu=\chi e^{b\langle\xi\rangle}TS\Big(\chi_1e^{-a\langle\xi\rangle}e^{a\langle\xi\rangle}\chi_2Tu+(1-\chi_1)\langle\xi\rangle^N\langle\xi\rangle^{-N}Tu\Big).$$

Since $u\in H^{-N}$, $\langle\xi\rangle^{-N}Tu\in L^2(\mathbb{R}^{2n})$ and (2.16), (2.17), now show that $e^{b\langle\xi\rangle}\chi Tu\in H^K$ for some $b>0$ and any K. By taking $K>n$ and applying [**23**, Corollary 7.9.4], we obtain a uniform bound

$$|Tu(x,\xi)|\leqslant Ce^{-b\langle\xi\rangle}, \quad (x,\xi)\in U_0\times\Gamma_0.$$

Let h_1 be the fixed h in the definition of T. Then,

$$\mathscr{T}(x,\xi/\langle\xi\rangle;h_1/\langle\xi\rangle)=Tu(x,\xi)=\mathcal{O}(e^{-b\langle\xi\rangle}), \quad (x,\xi)\in U_0\times\Gamma_0. \tag{2.21}$$

Putting $\omega_0:=\xi_0/\langle\xi_0\rangle$, it follows that $\mathscr{T}(x,\omega,h)=\mathcal{O}(e^{-\delta/h})$ for (x,ω) in a small neighbourhood of (x_0,ω_0). But then (2.20) shows that $(x_0,\omega_0)\notin\mathrm{WF}_{\mathrm{a}}(u)$. Since $\mathrm{WF}_{\mathrm{a}}(u)$ is a closed conic set, we conclude that $(x_0,\xi_0)\notin\mathrm{WF}_{\mathrm{a}}(u)$.

Now suppose $\mathrm{WF}_{\mathrm{a}}(u)\cap(U\times\Gamma)=\emptyset$. Then for (x,ω) near $U_0\times(\Gamma_0\cap\mathbb{S}^{n-1})$ (with U_0 and Γ_0, as in the statement of the theorem), $\mathscr{T}(x,\omega,h)=\mathcal{O}(e^{-\delta/h})$. Reversing the argument in (2.21), we see that

$$|Tu(x,\xi)|\leqslant Ce^{-b\langle\xi\rangle}, \quad (x,\xi)\in U_0\times\Gamma_0.$$

Now, since $u\in H^{-N}(\mathbb{R}^n)$, $\langle\xi\rangle^{-N}Tu\in L^2(\mathbb{R}^{2n})$. In particular, since $|\psi|\leqslant C\langle\xi\rangle$ and the support of ψ is contained in $U_0\times\Gamma_0$, (2.15) follows. $\square$

The next proposition relates weighted estimates to deformed FBI transform:

PROPOSITION 2.5. *Suppose that H_Λ, $\Lambda=\Lambda_G$, is defined in* [**14**, (4.7)] *with G satisfying* (2.4) *with ϵ_0 chosen as in the definition of H_Λ.*

Then there exists $\psi\in S^1(T^*\mathbb{R}^n)$ such that $T:\mathscr{B}_\delta\to L^2(T^*\mathbb{R}^n,e^{\delta\langle\xi\rangle/Ch}dxd\xi)$ extends to

$$T=\mathcal{O}(1):H_\Lambda\to L^2(T^*\mathbb{R}^n,e^{2\psi(x,\xi)/h}dxd\xi), \tag{2.22}$$

and $S:L^2(T^*\mathbb{R}^n,e^{-C\delta\langle\xi\rangle/h}dxd\xi)\to\mathscr{B}_\delta$, extends to

$$S=\mathcal{O}(1):L^2(T^*\mathbb{R}^n,e^{2\psi(x,\xi)/h}dxd\xi)\to H_\Lambda. \tag{2.23}$$

In addition,

$$\psi(x,\xi)=G(x,\xi)+\mathcal{O}(\epsilon_0^2)_{S^1(T^*\mathbb{R}^n)}. \tag{2.24}$$

For a simpler version of this result in the case of compactly supported weights, see [**13**, § 8].

Proof. The statement (2.22) is equivalent to

$$TS_\Lambda=\mathcal{O}(1):L^2(\Lambda,e^{-2H(\alpha)/h}d\alpha)\to L^2(T^*\mathbb{R}^n,e^{2\psi(\beta)}d\beta)$$

and hence we analyse the kernel of the operator TS_Λ which is given by

$$K(\alpha,\beta)=c_nh^{-\frac{3n}{2}}\int_{\mathbb{R}^n}e^{\frac{i}{h}(\varphi_0(\alpha,y)+\varphi_G^*(\beta,y))}\langle\beta_\xi\rangle^{\frac{n}{4}}\langle\alpha_x\rangle^{\frac{n}{4}}(1+\tfrac{i}{2}\langle\alpha_x-y\rangle)dy,$$

where the notation (and also notation for Φ below) comes from (2.11). The integral in y converges and can be evaluated by a completion of squares as in [**14**, Proposition 4.4]. That gives the phase (2.9) with $\alpha\in T^*\mathbb{R}^n$ and $\beta\in\Lambda$. The critical point in y is given by

$$y_c(\alpha,\beta)=\frac{1}{\langle\alpha_\xi\rangle+\langle\beta_\xi\rangle}(\langle\alpha_\xi\rangle\alpha_x+\langle\beta_\xi\rangle\beta_x+i(\beta_\xi-\alpha_\xi)). \tag{2.25}$$

We then have (2.22) with

$$\psi(\alpha):=\max_{\beta\in\Lambda}(-\operatorname{Im}\Psi(\alpha,\beta)+H(\beta)). \tag{2.26}$$

We have (see [**14**, (3.3),(3.4)])

$$d_\beta(-\operatorname{Im}\Psi(\alpha,\beta)+H(\beta))=\operatorname{Im}(-\partial_{z,\zeta}\Psi(\alpha,(z,\zeta))-\zeta dz|_\Lambda)|_{(z,\zeta)=\beta\in\Lambda}.$$

Now, if $y_c(\alpha,(z,\zeta))$ is the critical point in y, then

$$\begin{aligned}\partial_{z,\zeta}\Psi(\alpha,z)&=\partial_{z,\zeta}(\Phi(\alpha,y_c(\alpha,(z,\zeta)))-\bar\Phi((z,\zeta),y_c(\alpha,(z,\zeta))))=-\partial_{z,\zeta}\bar\Phi\big|_{y=y_c(z,\zeta)}(z,\zeta)\\&=-\zeta\cdot dz+(y_c-z)\cdot d\zeta+i\langle\zeta\rangle(z-y_c)\cdot dz+\tfrac{i}{2}(z-y_c)^2\zeta\cdot d\zeta/\langle\zeta\rangle.\end{aligned}$$

For $G=0$, the critical point (see (2.25)) is given by $\alpha=\beta$. Hence

$$\beta_c=\beta_c(\alpha)=(\alpha_x+\mathcal{O}(\epsilon_0)_{S^0},\alpha_\xi+\mathcal{O}(\epsilon_0)_{S^1}), \tag{2.27}$$

with ϵ_0 as in (2.4).

Hence we obtain ψ by inserting the critical point β_c into the right-hand side of (2.26)

$$\psi(\alpha)=-\operatorname{Im}\Psi(\alpha,\beta_c(\alpha))+H(\beta_c(\alpha))\in S^1(T^*\mathbb{R}^n). \tag{2.28}$$

(We note that for $G=0$ the maximum in (2.26) is non-degenerate and unique and it remains such under small symbolic perturbations.) From (2.9), we see that

$$\operatorname{Im}\Psi(\alpha,\beta_c(\alpha))=\operatorname{Im}\Psi(\alpha,\alpha+\mathcal{O}(\epsilon_0)_{S^0\times S^1})=\alpha_\xi\cdot G_\xi(\alpha)+\mathcal{O}(\epsilon_0^2)_{S^1}.$$

Inserting this into (2.28) and recalling that $H=\xi G_\xi-G$, we obtain (2.24).

To obtain (2.23), we apply the same analysis to $T_\Lambda S$ and we need to show that two weights coincide. That is done as in [**13**, § 8]. ☐

3. *Proof of Theorem* 2

As already indicated in § 1.2, to prove the theorem we construct a family of weights $G_\epsilon\in S^1$, uniformly bounded in S^1, supported in a conic neighbourhood of $\Gamma=\{(0,0,\xi_1,0):\xi_1>M\}$, $M\gg1$, and satisfying $0\leqslant G_\epsilon\leqslant C_\epsilon\log\langle\xi\rangle$. In addition,

$$H_pG_\epsilon\geqslant0,\qquad G_\epsilon\to\xi_1\text{ on }\Gamma\text{ (in }S^{1+}), \tag{3.1}$$

with $H_pG_\epsilon\gg\xi_1^{m-1}$ in a suitable sense (see (3.4)) for $\epsilon\ll1$.

We will then put $\Lambda_\epsilon:=\Lambda_{G_\epsilon}$ so that the assumption $u\in C^\infty$ will give $u\in H_{\Lambda_\epsilon}$. On the other hand, the assumption that $\Gamma\cap\mathrm{WF_a}(Pu)$ shows that $\|Pu\|_{H_{\Lambda_\epsilon}}\leqslant C$ with the constant C independent of ϵ. But then [**14**, Proposition 6.2] and the properties of G_ϵ show that $\|u\|_{H_{\Lambda_\epsilon}}$ is bounded independently of ϵ. Propositions 2.3 and 2.5 then show that $\mathrm{WF_a}(u)\cap\Gamma_0=\emptyset$.

3.1. *Construction of the weight*

We now construct a family of weights, G_ϵ, satisfying (3.1). In fact, we need more precise conditions on G_ϵ given in the following

LEMMA 3.1. *Suppose that p satisfies (1.9) at $\rho_0 = (x_0, \xi_0) \in T^*\mathbb{R}^n \setminus 0$ and Γ is an open conic neighbourhood of ρ_0. Then, there exists $G_\epsilon \in S^1(T^*\mathbb{R}^n)$, $\operatorname{supp} G_\epsilon \subset \Gamma$, such that*

$$\begin{gathered} |\partial_x^\alpha \partial_\xi^\beta G_\epsilon| \leqslant C_{\alpha\beta} \langle \xi \rangle^{1-|\beta|}, \quad 0 \leqslant G_\epsilon \leqslant C\epsilon^{-1} \log\langle \xi \rangle, \\ G_\epsilon(x,\xi)|_{1 \leqslant |\xi| \leqslant 1/\epsilon} = \Phi(x,\xi)|\xi|, \quad \Phi \in S^0_{\mathrm{phg}}(T^*\mathbb{R}^n), \quad \Phi(x_0, t\xi_0) = 1, \ t \gg 1, \end{gathered} \tag{3.2}$$

$$H_p G_\epsilon(x,\xi) \geqslant c_0 \big(\langle \xi \rangle^m |\partial_\xi G_\epsilon(x,\xi)|^2 + \langle \xi \rangle^{m-2} |\partial_x G_\epsilon(x,\xi)|^2 \big), \tag{3.3}$$

$$\forall\, M_1, \gamma \geqslant 0 \ \exists\, M_2, K, \epsilon_0 \,\forall\, 0 < \epsilon < \epsilon_0, \quad H_p G_\epsilon e^{\gamma G_\epsilon} + M_2 \langle \xi \rangle^K \geqslant M_1 \langle \xi \rangle^{m-1} e^{\gamma G_\epsilon}. \tag{3.4}$$

We stress that the constants $C_{\alpha\beta}$ and c_0 are independent of ϵ and M_1.

Proof. We use the normal form for p constructed in [**19**, § 3]. That means that we take $x_0 = 0$ and $\xi_0 = e_1 := (1, 0, \ldots, 0)$ and can assume that $p(x,\xi) = -\xi_1^m x_1$ in a conic neighbourhood of $\rho = (0, e_1)$. For simplicity, we can assume that $m = 1$ as the argument is the same otherwise.

Let $\chi \in C_c^\infty(\mathbb{R}; [0,1])$ satisfy

$$\operatorname{supp} \chi \subset [-2, 2], \quad \chi_{|t| \leqslant 1} = 1, \quad t\chi'(t) \leqslant 0. \tag{3.5}$$

and put $\varphi(t) := \chi(t/\delta)$. Here δ will be fixed depending on Γ. Using this function we define $\Phi = \Phi(x,\xi) := \varphi_1 \varphi_2 \varphi_3 \psi$ where

$$\varphi_1 := \varphi(x_1), \quad \varphi_2 := \varphi(|\xi'|/\xi_1) \quad \varphi_3 = \varphi(|x'|), \quad \psi := (1 - \varphi((\xi_1)_+)). \tag{3.6}$$

We choose δ small enough so that $\operatorname{supp} \Phi \subset \Gamma$.

We define G_ϵ as follows

$$G_\epsilon(x,\xi) = \Phi(x,\xi) q_\epsilon(\xi_1), \quad q_\epsilon(t) := \int_0^t \big(\chi(\epsilon s) + (1 - \chi(\epsilon s))(s\epsilon)^{-1} \big) ds. \tag{3.7}$$

We check that

$$\begin{gathered} \xi_1 \partial_{\xi_1} q_\epsilon \geqslant \min(\xi_1, \epsilon^{-1}), \\ \xi_1 \mathbb{1}_{\xi_1 \leqslant 1/\epsilon} + \epsilon^{-1}(1 + \log(\epsilon\xi_1)) \mathbb{1}_{\xi \geqslant 1/\epsilon} \leqslant q_\epsilon \leqslant \xi_1 \mathbb{1}_{\xi_1 \leqslant 1/\epsilon} + \epsilon^{-1}(2 + \log(\epsilon\xi_1)) \mathbb{1}_{\xi \geqslant 1/\epsilon}. \end{gathered} \tag{3.8}$$

Uniform boundedness of G_ϵ in S^1 means that q_ϵ in (3.7) satisfies $|\partial_{\xi_1}^k q_\epsilon| \leqslant C_k \xi_1^{1-k}$ with functions C_k independent of ϵ. But this is immediate from the definition. We also easily see that G_ϵ converges to $G := \Phi(x,\xi)\xi_1$ in S^{1+} as $\epsilon \to 0$. This proves (3.2).

To see (3.3), we first note that, since $\Phi \geqslant 0$, $\Phi \in S^0$, the standard estimate $f(z) \geqslant 0 \Longrightarrow |df(z)|^2 \leqslant Cf(z)$ gives

$$\Phi(x,\xi) \geqslant c_1 \big(\xi_1^2 |\partial_\xi \Phi(x,\xi)|^2 + |\partial_x \Phi(x,\xi)|^2 \big). \tag{3.9}$$

Note also that we have $H_p = \xi_1 \partial_{\xi_1} - x_1 \partial_{x_1}$ and therefore

$$H_p \Phi = -x_1 \varphi'(x_1) \varphi_2 \varphi_3 \psi - (|\xi'|/\xi_1) \varphi'(|\xi'|/\xi_1) \varphi_1 \varphi_3 \psi - \varphi_1 \varphi_2 \varphi_3 \xi_1 \varphi'((\xi_1)_+) \geqslant 0. \tag{3.10}$$

Since $q_\epsilon \in S^1$, $\xi_1 \partial_{\xi_1} q_\epsilon(\xi_1) \geqslant c_2 \xi_1 (\partial_{\xi_1} q_\epsilon(\xi_1))^2$. We also claim that

$$\xi_1 \partial_{\xi_1} q_\epsilon(\xi_1) \geqslant c_2 \xi_1^{-1} q_\epsilon(\xi_1)^2. \tag{3.11}$$

In fact, using (3.8) we see that to prove (3.11) it is enough to have

$$\min(t, \epsilon^{-1}) \geqslant c_2 t^{-1} \big(t \mathbb{1}_{t \leqslant 1/\epsilon}(t) + \epsilon^{-1}(2 + \log(t\epsilon)) \mathbb{1}_{t \geqslant 1/\epsilon}(t) \big)^2.$$

This clearly holds (with $c_2 = 1$) for $t \leqslant 1/\epsilon$ and for $t \geqslant \epsilon$ is equivalent to $c_2(2+\log s)^2 \leqslant s$, $s = t\epsilon \geqslant 1$, which holds with $c_2 = \frac{1}{4}$. It follows that

$$\xi_1 \partial_{\xi_1} q_\epsilon(\xi_1) \geqslant c_2\big(\xi_1^{-1} q_\epsilon(\xi_1)^2 + \xi_1(\partial_{\xi_1} q_\epsilon(\xi_1))^2\big),$$

which combined with (3.9) and (3.10) gives

$$\begin{aligned} H_p G_\epsilon &= \Phi(\xi_1 \partial_{\xi_1} q_\epsilon) + (H_p \Phi) q_\epsilon \\ &\geqslant \Phi(\xi_1 \partial_{\xi_1} q_\epsilon) \geqslant c_2 \xi_1 \Phi (\partial_{\xi_1} q_\epsilon)^2 + c_3\big(\xi_1^2 |\partial_\xi \Phi|^2 + |\partial_x \Phi|^2\big)\xi_1^{-1} q_\epsilon^2 \\ &\geqslant c_0\big(\xi_1 |\partial_\xi G_\epsilon|^2 + \xi_1^{-1}|\partial_x G_\epsilon|^2\big). \end{aligned}$$

Since $\langle \xi \rangle \sim \xi_1$ on the support of G_ϵ, we obtain (3.3).

Finally we prove (3.4). Since by (3.10) we have $H_p G_\epsilon \geqslant \Phi H_p q_\epsilon$, we see that (3.4) follows from proving that for any M_1 we can find K, M_2 and ϵ_0 such that for $\xi_1 \geqslant 1$,

$$\Phi H_p q_\epsilon e^{\gamma \Phi q_\epsilon} + M_2 \xi_1^K \geqslant M_1 e^{\gamma \Phi q_\epsilon}. \tag{3.12}$$

Using (3.8), we see that for $\xi_1 \leqslant 1/\epsilon$ we need $G_\epsilon e^{\gamma G_\epsilon} + M_2 \xi_1^K \geqslant M_1 e^{\gamma G_\epsilon}$. This holds for

$$K = 0, \quad M_2 = 2\gamma^{-1} e^{\gamma M_1 - 1}$$

since for $\gamma > 0$ and $a \geqslant 0$, $a e^{\gamma a} - M_1 e^{\gamma a} \geqslant -2\gamma^{-1} e^{\gamma M_1 - 1}$.

For $\xi_1 \geqslant 1/\epsilon$, we need to find K and M_2 for which

$$\epsilon^{-1} \Phi e^{\gamma \Phi q_\epsilon} + M_2 \xi_1^K \geqslant M_1 e^{\gamma \Phi q_\epsilon}. \tag{3.13}$$

Using $a e^{ab} + M_1 e^{M_1 b} \geqslant M_1 e^{ab}$ with $a := \epsilon^{-1}\Phi$ and

$$b := \gamma \epsilon q_\epsilon \leqslant \gamma(2 + \log(\epsilon \xi_1)) \leqslant \gamma(2 + \log \xi_1),$$

we obtain (3.13) with $M_2 = M_1 e^{2\gamma M_1}$ and $K = \gamma M_1$. Hence we obtain (3.12) proving (3.4). ☐

3.2. *Microlocal analytic hypoelliticity*

We will have bounds which are uniform in ϵ but not in h. We start with the following

Lemma 3.2. *Suppose that P is of the form (1.8) with real valued principal symbol p and suppose that $\Gamma \subset U \times \mathbb{R}^n \setminus$ is an open cone, $\Gamma \cap \mathbb{S}^{n-1} \Subset U \times \mathbb{S}^{n-1}$ and*

$$\begin{gathered} G \in S^1(\Gamma; \mathbb{R}), \quad |G| \leqslant C \log\langle \xi \rangle, \\ H_p G(x,\xi) \geqslant c_0\big(\langle \xi \rangle^m |\partial_\xi G(x,\xi)|^2 + \langle \xi \rangle^{m-2} |\partial_x G(x,\xi)|^2\big). \end{gathered} \tag{3.14}$$

Then for T_Λ, H_Λ, $\Lambda = \Lambda_{\theta G}$ defined in (2.4) and (2.6), h and θ sufficiently small, and $u \in H_\Lambda^{-N+m}$,

$$\operatorname{Im}\langle h^m P u, u \rangle_{H_\Lambda^{-N}} \geqslant \tfrac{1}{2}\theta \langle H_p G \, \langle \xi \rangle^{-N} T_\Lambda u, \langle \xi \rangle^{-N} T_\Lambda u \rangle_{L^2_\Lambda} - M h \|u\|^2_{H_\Lambda^{\frac{m-1}{2} - N}}, \tag{3.15}$$

where M depends only on P and the semi-norms of G in S^1.

Proof. We use Proposition 2.2 and [**14**, Proposition 6.3] to see that for any $K > 0$,

$$\begin{aligned} \operatorname{Im}\langle h^m P u, u \rangle_{H_\Lambda^{-N}} &= \operatorname{Im}\langle \langle \xi \rangle^{-2N} T_\Lambda h^m P S_\Lambda T_\Lambda u, T_\Lambda u \rangle_{L^2_\Lambda} \\ &= \operatorname{Im}\langle \Pi_\Lambda \langle \xi \rangle^{-2N} \Pi_\Lambda h^m P S_\Lambda \Pi_\Lambda T_\Lambda u, T_\Lambda u \rangle_{L^2_\Lambda} \\ &= \langle (\operatorname{Im} b_{P,N}) T_\Lambda u, T_\Lambda u \rangle_{L^2_\Lambda} + \mathcal{O}(h^\infty) \|u\|_{H_\Lambda^{-K}} \\ &\geqslant \langle (\operatorname{Im} p|_\Lambda) \, \langle \xi \rangle^{-N} T_\Lambda u, \langle \xi \rangle^{-N} T_\Lambda u \rangle_{L^2_\Lambda} - M h \|u\|_{H_\Lambda^{\frac{m-1}{2} - N}}. \end{aligned} \tag{3.16}$$

From (2.7) and (3.14), we obtain

$$\begin{aligned}\operatorname{Im} p|_{\Lambda} &= \operatorname{Im} p(x - i\theta\partial_\xi G(x,\xi), \xi + i\theta\partial_x G(x,\xi))\\ &= \theta H_p G(x,\xi) + \theta^2 \mathcal{O}\big(\langle\xi\rangle^m |\partial_\xi G(x,\xi)|^2 + \langle\xi\rangle^{m-2}|\partial_x G(x,\xi)|^2\big)\\ &\geqslant \tfrac{1}{2}\theta H_p G(x,\xi),\end{aligned}$$

if θ is small enough. $\square$

The next lemma allows us to use smoothness of u to obtain weaker weighted estimates:

Lemma 3.3. *Suppose $U \subset \mathbb{R}^n$ is an open set,*

$$G \in S^1(T^*\mathbb{R}^n), \quad G \geqslant 0, \quad \operatorname{supp} G \subset K \times \mathbb{R}^n, \quad K \Subset U,$$

and T_Λ, H_Λ, $\Lambda = \Lambda_{\theta G}$ are defined in (2.4) and (2.6). Then, there exists $a > 0$ such that for every $\chi, \tilde{\chi} \in S^0$ with $\tilde{\chi} \equiv 1$ in a conic neighbourhood of $\operatorname{supp}\chi$ and every $K, N > 0$, there exists $c, C > 0$ such that for all $u \in H^{-N}(\mathbb{R}^n)$,

$$\|\langle\xi\rangle^K e^{-aG/h}\chi T_\Lambda u\|_{L^2_\Lambda} \leqslant C(\|\langle\xi\rangle^K \tilde{\chi} T u\|_{L^2(T^*\mathbb{R}^n)} + e^{-c/h}\|\langle\xi\rangle^{-N} T u\|_{L^2(T^*\mathbb{R}^n)}). \tag{3.17}$$

In particular, if $\chi \equiv 1$ on $\operatorname{supp} G$, then

$$\begin{aligned}&\|\langle\xi\rangle^K e^{-aG/h}\chi\|_{L^2_\Lambda} + \|\langle\xi\rangle^{-N}(1-\chi)T_\Lambda u\|_{L^2_\Lambda}\\ &\qquad \leqslant C(\|\langle\xi\rangle^K \tilde{\chi} T u\|_{L^2(T^*\mathbb{R}^n)} + \|\langle\xi\rangle^{-N} T u\|_{L^2(T^*\mathbb{R}^n)}).\end{aligned} \tag{3.18}$$

Proof. First, observe that by [**14**, Lemma 4.5], for any $\delta > 0$,

$$T_\Lambda S = K_\delta + O_{N,\delta}(e^{-c_\delta/h})_{\langle\xi\rangle^N L^2(T^*\mathbb{R}^n) \to \langle\xi\rangle^{-N} L^2_\Lambda},$$

and K_δ has kernel, $K_\delta(\alpha,\beta)$, given by

$$h^{-n} e^{\frac{i}{h}\Psi(\alpha,\beta)} k(\alpha,\beta)\psi(\delta^{-1}|\operatorname{Re}\alpha_x - \beta_x|))\psi(\delta^{-1}\min(\langle\operatorname{Re}\alpha_\xi\rangle, \langle\beta_\xi\rangle)^{-1}|\operatorname{Re}\alpha_\xi - \beta_\xi|),$$

where $(\alpha,\beta) \in \Lambda \times T^*\mathbb{R}^n$ and Ψ is as in (2.9), and $\psi \in C_c^\infty(\mathbb{R})$ is identically 1 near 0. Therefore, we need to only consider $K_\delta(\alpha,\beta)$.

To do this, let $\tilde{\chi} \in S^0$ be identically 1 on a conic neighborhood of $\operatorname{supp}\chi$. Then, for $\delta > 0$ small enough,

$$\chi(\operatorname{Re}\alpha)K_\delta(\alpha,\beta)(1-\tilde{\chi})(\beta) \equiv 0.$$

Therefore,

$$\chi e^{-aG/h}\langle\xi\rangle^K T_\Lambda S(1-\tilde{\chi}) = O_N(e^{-c/h})_{\langle\xi\rangle^N L^2(T^*\mathbb{R}^n) \to \langle\xi\rangle^{-N} L^2_\Lambda}.$$

For the mapping properties

$$\chi e^{-aG/h} T_\Lambda S \tilde{\chi} : \langle\xi\rangle^{-K} L^2(T^*\mathbb{R}^n) \to \langle\xi\rangle^{-K} L^2_\Lambda,$$

we consider the operator

$$\chi e^{-aG/h} e^{-H/h}\langle\xi\rangle^K T_\Lambda S \tilde{\chi}\langle\xi\rangle^{-K} : L^2(T^*\mathbb{R}^n) \to L^2(\Lambda; dxd\xi).$$

Modulo negligible terms, the kernel of this operator is given by

$$h^{-n} e^{\frac{i}{h}(\varphi((x,\xi),(y,\eta)))}\tilde{k}((x,\xi),(y,\eta))$$

where $\tilde{k} \in S^0$ has

$$\operatorname{supp}\tilde{k} \subset \{|\xi - \eta| \leqslant C\delta\langle\xi\rangle\} \cap \{|x - y| \leqslant C\delta\}. \tag{3.19}$$

and

$$\varphi = iH(x,\xi) + ia\theta G(x,\xi) + \Psi((x - i\theta G_\xi, \xi + i\theta G_x(x,\xi)), (y,\eta)),$$

with $H(x,\xi) = \theta\langle \xi, G_\xi(x,\xi)\rangle - \theta G(x,\xi)$. Using (3.19), we have

$$\begin{aligned}\operatorname{Im}\varphi &= aG + \theta\xi\cdot G_\xi - \theta G + \frac{\langle\eta\rangle\langle\xi\rangle}{2(\langle\eta\rangle+\langle\xi\rangle)}\big((x-y)^2 - (\theta G_\xi)^2\big) + \frac{(\xi-\eta)^2 - (\theta G_\xi)^2}{2(\langle\eta\rangle+\langle\xi\rangle)}\\ &\quad + \theta\xi\cdot G_\xi + O(\theta(|x-y||G_x| + \langle\xi\rangle^{-1}|\xi-\eta||G_\xi|))\\ &\quad + O(\theta^2(\langle\xi\rangle^{-1}|G_x|^2 + \langle\xi\rangle|G_\xi|^2))\\ &\geqslant (a-\theta)G - C\theta^2(\langle\xi\rangle^{-1}(G_x)^2 + \langle\xi\rangle|G_\xi|^2) + c\langle\xi\rangle(x-y)^2 + c\langle\xi\rangle^{-1}(\xi-\eta)^2.\end{aligned}$$

In particular, taking a large enough and using that $G \geqslant 0$, $G \in S^1$, (see the argument for (3.9)), we have

$$\operatorname{Im}\varphi \geqslant \frac{a}{2}G(x,\xi) + c\langle\xi\rangle(x-y)^2 + c\langle\xi\rangle^{-1}(\xi-\eta)^2.$$

Therefore, applying the Schur test for L^2 boundedness completes the proof that

$$\chi\langle\xi\rangle^K e^{-aG/h} T_\Lambda S\langle\xi\rangle^{-K} = O(1) : L^2(T^*\mathbb{R}^n) \to L^2_\Lambda$$

and the lemma follows. ☐

With these two lemmas in place we can prove the main result:

Proof of Theorem 2. By multiplying u by a C^∞_{c}-function which is 1 in a neighbourhood of x_0, we can assume that $u \in H^{-N+m}$, for some N, is compactly supported in U and $\rho_0 := (x_0,\xi_0) \notin \mathrm{WF}(u)$. By Proposition 2.1, there exists $\tilde\chi \in S^0$ with $\tilde\chi \equiv 1$ in an open conic neighborhood, Γ, of ρ_0 such that for any $K > 0$,

$$\|\langle\xi\rangle^K \tilde\chi T u\|_{L^2} \leqslant C_K. \tag{3.20}$$

Also, since $u \in H^{-N+m}$,

$$\|\langle\xi\rangle^{-N+m} T u\|_{L^2} \leqslant C. \tag{3.21}$$

Let $\Gamma_1 \Subset \Gamma$ be an open conic neighborhood of ρ_0 and $\chi \in S^1$ with $\chi \equiv 1$ on Γ_1 and $\operatorname{supp}\chi \subset \Gamma$.

We choose θ small enough so that (2.4) and (3.16) hold. We then fix $0 < h \leqslant 1$ small enough so that (3.16) holds. From now we neglect the dependence on h which is considered to be a fixed parameter. We choose for $G = G_\epsilon$ constructed in Lemma 3.1 and supported in Γ_1. We recall that the estimates depend only on the S^1 seminorms of G and these are uniform in ϵ. We now claim that

$$u \in H^{-N+m}_{\Lambda_\epsilon}, \qquad \Lambda_\epsilon := \Lambda_{\theta G_\epsilon}.$$

In fact, we can use (3.18) together with (3.20) and (3.21), observing that $\exp(aG_\epsilon/h) = \mathcal{O}_\epsilon(\langle\xi\rangle^{Ca/(h\epsilon)})$ and taking $K = Ca/(h\epsilon) - N + m$.

Next, note that $Pu \in H^{-N}$ is supported in U and $\rho_0 \notin \mathrm{WF}_a(Pu)$. Propositions 2.3 and 2.5 (see (2.15) and (2.23), respectively) then show that for G_ϵ satisfying the assumptions of Lemma 3.2 and θ sufficiently small $\|Pu\|_{H^{-N}_{\Lambda_\epsilon}} \leqslant C_0$, where C_0 depends only on Pu and S^1-seminorms of θG_ϵ.

We now apply (3.15) to obtain with Λ_ϵ as above,

$$\tfrac{1}{2}\|u\|^2_{H^{-N}_{\Lambda_\epsilon}} + 2C_0^2 \geqslant \langle(\theta H_p G_\epsilon - M\langle\xi\rangle^{m-1})\langle\xi\rangle^{-N-m} T_{\Lambda_\epsilon} u, \langle\xi\rangle^{-N} T_{\Lambda_\epsilon} u\rangle_{L^2_{\Lambda_\epsilon}}, \tag{3.22}$$

Let a be given by Lemma 3.3 (so that (3.17) holds). Then by (3.4), there exist M_2 and K such that

$$\theta H_p G_\epsilon + M_2 \langle \xi \rangle^{2K} e^{-2aG_\epsilon/h} \geqslant (M+1)\langle \xi \rangle^{m-1}.$$

From (3.17), we have

$$\begin{aligned} &\|M_2 \chi \langle \xi \rangle^K e^{-aG_\epsilon/h} \langle \xi \rangle^{-N} T_\Lambda u\|^2_{L^2_{\Lambda_\epsilon}} \\ &\leqslant C(\|\langle \xi \rangle^{K-N} \tilde{\chi} T u\|^2_{L^2(T^*\mathbb{R}^n)} + \|\langle \xi \rangle^{-N} T u\|^2_{L^2(T^*\mathbb{R}^n)}) \leqslant C^2{}_1 \end{aligned} \tag{3.23}$$

Therefore, adding (3.23) to (3.22), and using that $\operatorname{supp} G_\epsilon \subset \chi \equiv 1$, we have

$$\begin{aligned} &\tfrac{1}{2}\|u\|^2_{H^{-N}_{\Lambda_\epsilon}} + C_1^2 + 2C_0^2 \\ &\geqslant \langle \chi^2 \langle \xi \rangle^{m-1} \langle \xi \rangle^{-N} T_{\Lambda_\epsilon} u, \langle \xi \rangle^{-N} T_{\Lambda_\epsilon} u \rangle_{L^2_{\Lambda_\epsilon}} \\ &\quad - \langle M(1-\chi^2) \langle \xi \rangle^{m-1} \langle \xi \rangle^{-N} T_{\Lambda_\epsilon} u, \langle \xi \rangle^{-N} T_{\Lambda_\epsilon} u \rangle_{L^2_{\Lambda_\epsilon}} \\ &\geqslant \langle \langle \xi \rangle^{m-1} \langle \xi \rangle^{-N} T_{\Lambda_\epsilon} u, \langle \xi \rangle^{-N} T_{\Lambda_\epsilon} u \rangle_{L^2_{\Lambda_\epsilon}} - (M+1)\|u\|_{H^{-N+\frac{m-1}{2}}}, \end{aligned} \tag{3.24}$$

where in the last line we use that $\chi \equiv 1$ on $\operatorname{supp} G_\epsilon$.

Using $m \geqslant 1$ and rearranging, this yields

$$\|u\|^2_{H^{-N}_{\Lambda_\epsilon}} \leqslant 2C_1^2 + 4C_0^2 + 2(M+1)\|u\|_{H^{-N+\frac{m-1}{2}}}.$$

where C_1, C_0 and M are constants *independent of* ϵ.

Since $\Lambda_\epsilon \cap \{|\xi| < 1/\epsilon\} = \Lambda_0 \cap \{|\xi| < 1/\epsilon\}$ where $G_0 := \Phi|\xi|$, we have that $H_\epsilon|_{|\xi|<1/\epsilon} = H_0|_{|\xi|<1/\epsilon}$, where $H_\epsilon = \theta \xi \partial_\xi G_\epsilon + \theta G$ is the corresponding weight. Therefore, the monotone convergence theorem implies that $u \in H_{\Lambda_0}$. Since $\Phi(x_0, t\xi_0) = 1$, $t \gg 1$, Proposition 2.3 shows that $(x_0, \xi_0) \notin \mathrm{WF_a}(u)$. □

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