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From a Personal to a Pedagogically Powerful Understanding of School Mathematics

> […] and, since he was scarcely able, unaided, to discern any connection between this task and his university mathematics, he soon fell in with the time honoured way of teaching, and his university studies remained only a more or less pleasant memory which had no influence upon his teaching. (Klein, 1932: 1)

Introduction

What kind of knowledge do teachers need for teaching school mathematics? And how can teachers be supported to tap into such knowledge in ways that empower them pedagogically, with the aim of understanding and supporting their students' thinking and learning of mathematics?

The study of teachers' knowledge of the subject matter they are expected to teach and its relationship to the quality of classroom instruction has been a fruitful area of research since Lee Shulman first launched a call for researching the different components of a professional knowledge base for teaching (Shulman 1986). Ever since, in their efforts to conceptualize mathematics teachers' professional knowledge base for teaching, researchers have put forward various alternatives for conceptualizing teachers' knowledge, each trying to better describe and gain a deeper understanding of the different components (e.g. Kunter and Baumbert 2013; Davis and Simmt 2006; Ma 1999; Rowland, Turner, Thwaites and Huckstep 2009; Schoenfeld and Kilpatrick 2008). Such research was and is still needed in order to understand how to support teachers to build on their own personal understanding of the subject they chose to teach, and develop a pedagogically powerful understanding of the subject with the aim of reaching to students and supporting their learning of the subject.

This chapter contributes to the ongoing discussion on mathematics knowledge for teaching by investigating the case of teachers' knowledge about functions. The claims are substantiated by a report on a professional development workshop, which draws on the analysis of how practising teachers' own understanding of functions becomes more sophisticated and nuanced as they are supported to connect to more advanced knowledge about this mathematics concept. Such new learning also empowers them pedagogically to appreciate better the challenges their students encounter along the way towards developing an understanding of this mathematics idea of high epistemic quality. This mathematics-specific case study contributes thus to the KOSS programme (see Chapter 1), by attempting to characterize the nature of teachers' powerful professional knowledge.

Overview of the chapter

I begin by first considering some of the most influential frameworks describing mathematics teachers' knowledge for teaching. I present an overview of researchers' attempts to describe this body of knowledge, focussing in more depth on Subject Content Knowledge (SCK), Horizon Content Knowledge (HCK), and the more recent and hence less researched Advanced Mathematics Knowledge (AMK).

I then describe how the literature I reviewed informed my design of a professional development workshop aimed at supporting practising teachers connect with their more advanced knowledge of a specific mathematics topic (function). After introducing the empirical study, I analyse the data I collected while the participating teachers engaged with one specific activity in the workshop. In the concluding section, I offer some views, which could serve as a starting point for a more advanced discussion on how teacher education could support teachers develop pedagogically powerful knowledge of the school subject they teach.

Review of the literature

Teachers' knowledge for teaching: some theoretical insights

The study of teachers' knowledge of subject matter and its relationship to the quality of classroom instruction has grown substantially since Lee Shulman launched a call for researching the components of teachers' professional knowledge base for teaching (Shulman 1986). While there is still no easy agreement amongst the mathematics education community about the relationship between these components, research has thrived in efforts to conceptualize related issues for mathematics teachers.

One such successful effort is the mathematics specific "egg" framework advanced by Ball et al. (2008), which builds on and refines Shulman's (1986) initial categorization of types of knowledge of a teacher of any subject, namely subject matter knowledge and pedagogical content knowledge. Their Mathematics Knowledge for Teaching (MKT) framework lays the foundation for a practice-derived theory for mathematical knowledge for teaching. The authors divided Shulman's second category of Pedagogical Content Knowledge (PCK) into two other sub-domains, Knowledge of Content and Students (KCS) and Knowledge of Content and Teaching (KCT), while Shulman's third category of Curricular Knowledge (CK) was also specified under PCK as Knowledge of Content and Curriculum.

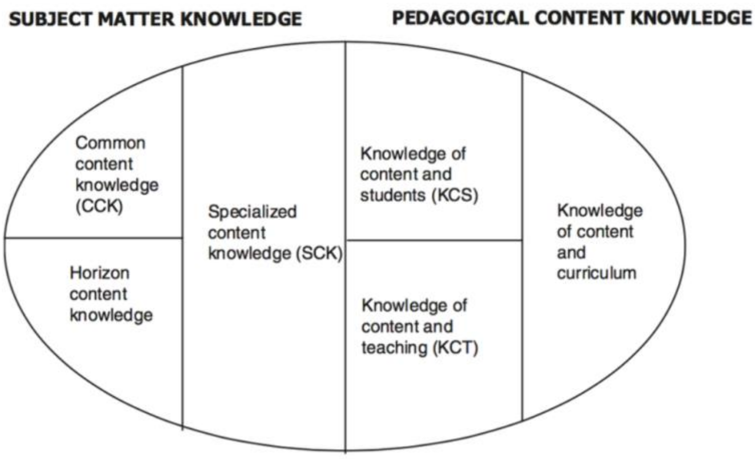


Figure 1 Domains of Mathematical Knowledge for Teaching (Ball, Thames and Phelps, 2008: 403)

The importance of subject knowledge (SK) has been well documented and its deficit linked, for example, to less effective teaching (Bennet and Turner Bisset 1993; Simon and Brown 1996) and over-reliance on commercial schemes (Millett and Johnson 1996). Ball and colleagues (2008) went further and divided Shulman's category of Subject Matter Knowledge into three sub-domains: Common Content Knowledge (CCK), Specialized Content Knowledge (SCK) and Horizon Content Knowledge (HCK), which I briefly describe below, as relevant to this chapter.

Specialized Content Knowledge SCK encompasses knowledge of mathematics needed by teachers, but not necessarily used by others, such as a knowledge of a particular mathematical model or representation useful for teaching a certain concept. For instance, while engineers need to know the "rule" about the product of two negative numbers being a positive number, in their day-to-day jobs they do not need to justify why this rule works. In other words, engineers' knowledge is CCK and used in ways that correspond with how it is used in settings other than teaching; they do not necessarily know the mathematical reasoning behind this rule, nor do they need to know how to explain why it works. Such knowledge is SCK, argued to be an intrinsic part of the foundation for a teacher's everyday classroom teaching.

In my experience as a teacher educator, irrespective of their mathematical background, prospective teachers frequently recall rules (e.g. *minus and minus makes plus*), methods (e.g. the balance method for solving equations: *whatever you do to one side, you should also do to the other side*), acronyms (e.g. *SOHCAHTOA*) that they acquired as learners of mathematics themselves, without always being able to give a mathematically sound justification of why the rule or method works, and just as importantly if it always works. In teacher education courses, recalling such rules is the starting point for prospective teachers' development of their SCK, albeit limited to a few topics. Prospective teachers and practising teachers then continue to broaden their SCK themselves over time, by exploring ways to represent all mathematical ideas, examine alternative representations, provide mathematical explanations of rules and procedures, evaluating unconventional student methods etc, with the aims to reach to students and support their learning (see Zembat, 2013, for a comprehensive list of everyday tasks mathematics teachers need to deal with regularly which require SCK).

Gericke et al. (2018) draw attention to how the "transformation" of knowledge impacts the epistemic quality made available to students in the classroom. What will students learn? How will teachers transform their understanding of mathematics for teaching purposes? Which representations, explanations, instructional resources will they use and how will they analyse and evaluate students' responses and errors? What will be the epistemic values promoted through teachers' SCK?

This raises the question of the quality of teachers' SCK. Ball et al. (2005: 378) describe SCK as a 'bridge' that enables the teacher 'to accurately represent mathematical ideas, provide mathematical explanations of common rules and procedures, and examine and understand unusual solution methods to problems'. Hudson, Henderson and Hudson (2015) argue that there is a need to address the *epistemic quality* of what students come to know, make sense of and be able to do in school mathematics, with the aim of maximizing the chances that all students will have *epistemic access* (Morrow 2008) to school mathematics of high epistemic quality. Hudson (2018) advises that an overemphasis on practice and memorization promotes a fragmented view of the subject and standard procedures reduced simply to rule following result in students learning a mathematics content of *low epistemic quality.* While some memorization of rules, methods, metaphors, rhymes etc, will always happen in mathematics classrooms (as some do help pupils remember "how to do it"), teachers, and students for that matter should also have an awareness of limitations of the validity and applicability of such rules. It is thus important that teachers scrutinize and evaluate instructional materials, and/or design or choose and use appropriate representations, with an awareness of the limitations and potential each such representation brings to the learning process. But how can teachers be supported to develop SCK that supports students' access to school mathematics of *high epistemic quality*, as encapsulated in the National Curriculum (DfE, 2013) aims for mathematics for all pupils, namely: to develop deep conceptual fluency, accompanied by mathematical reasoning and problem-solving?

Bass and Ball (2004) advised that unlike the work of research mathematicians which could be described in terms of 'compressing' information into increasingly concise and powerful formulations, 'the work of teachers is more often just the opposite: teachers must be adept at prying apart concepts, making sense of the analogies, metaphors, images, and logical constructs that give shape to a mathematical construct' (Davis and Simmt 2006: 300). Research evidence has strongly indicated that if teachers' SCK is not built on a conceptually sound understanding of the underlying mathematics, teachers will fall short of providing their students with high epistemic quality mathematics education, consisting of learning experiences that promote conceptual understanding (e.g. see Putnam, Heaton, Prawat and Remillard 1992 for the case for geometry).

To exemplify the above, let us consider the balance method, known as "Whatever you do to one side, you should also do to the other side", which is often heard in mathematics lessons when teachers teach about solving equations. A visual representation of an old-fashioned scale or a seesaw are often invoked to justify why this rule works. And most of the time, the rule works! It works when the four basic operations of addition, subtraction, multiplication and division by a number other than zero are applied to both sides of an equation, but it does not work, for example, if one attempts to balance the given equation by squaring both its sides, as more solutions are yielded than needed, including those of the equation to be solved in the first place. With such awareness, teachers will caution students that the balance method has its own limitations and will not always work. These teachers will be more likely to re-phrase the rule in a more helpful manner, where the "whatever you do" is described precisely in terms of the specific mathematical operations that are permitted if this method were to work, and hence explicitly draw students' attention to the shortcomings of this rule. This type of knowledge is described by Ball and colleagues as Horizon Content Knowledge (HCK).

Horizon Content Knowledge According to Ball et al. (2008: 1, 403), teachers should tap into their Horizon Content Knowledge (HCK), a kind of mathematical '"peripheral vision" needed in teaching, a view of the larger mathematical landscape that teaching requires', including 'the vision useful in seeing connections to much later mathematical ideas'.

Wasserman and Stockton (2013) proposed a helpful division of HCK into: a 'curricular mathematical horizon' (knowing what mathematics is to come in the next few grades) and an 'advanced mathematical horizon' (knowing connections to higher-level mathematical ideas). Advanced mathematical horizon described as such seems to resonate with Jakobsen et al.'s (2012) interpretation of HCK, namely that HCK is about 'being familiar with "advanced" mathematics, but in a way that supports hearing, seeing, sensing, and doing for teaching" (Jakobsen et al. 2012: 4640). Researchers thus argue that HCK relates to the engagement of advanced content in terms of its relevance to teaching and learning, how the content being taught is situated in and connected to the broader mathematical knowledge landscape, going beyond that of school mathematics.

Advanced Mathematics Knowledge The theoretical and empirical work on HCK led researchers' interest to consider a new component of the knowledge base for teaching, namely Advanced Mathematics Knowledge (AMK). While engaging with the MKT framework, Zaskis and Mamolo (2011) proposed to view HCK through the notion of viewing elementary (school) mathematics from an advanced standpoint, thus positioning advanced mathematical knowledge (AMK) as an important aspect of the MKT. The notion of HCK is given by Zazkis and Mamolo in terms of application of the notion of 'advanced mathematical knowledge', which they define as "knowledge of the subject matter acquired during undergraduate studies at colleges or universities" (Zazkis and Leikin 2010: 264).

Wasserman (2016), and later Stockon and Wasserman (2017), narrowed down the description of AMK to knowledge outside the typical scope of what a school mathematics teacher would likely teach, in that AMK is relevant, the advanced mathematical ideas are connected to the content of school mathematics, but also that these forms of knowledge of advanced mathematics are in some way productive for the teaching of school mathematics content. For example, Wasserman (2016) discusses how knowledge of *groups* might influence instruction about solving simple linear equations. As a mathematics topic, *groups* could be classified as an advanced mathematics topic, as it is usually studied at undergraduate, and not school, level. But how about other advanced mathematics topics, such as, for example, non-differentiable geometry? Is this mathematics knowledge relevant or connected to school mathematics or just too distant an area, thus less relevant and less connected to school mathematics and hence not needed by teachers teaching school mathematics? The authors argue that it is this HK which enables teachers to see more, and suggest that AMK is necessary, but not just *any* advanced mathematics knowledge. A provocation was then thrown to the mathematics education researchers: what AMK outside school mathematics has a bearing on school mathematics?

Teachers' perception of how teaching is affected by one's own advanced mathematics knowledge The 52 practising secondary school teachers in Zazkis and Leikin's (2010) study, teaching mathematics in grades 8–12, including Algebra, Geometry and Calculus, agreed that AMK was needed for: personal confidence, for ability to make connections, to respond to pupils' questions, language, aesthetic, precision, proof, elegance of solution, understanding vs. procedural fluency, or connection to history. However, none of the teachers interviewed were able to articulate (clearly or at all) specific examples of advanced mathematics content they had ever used in their teaching or to provide an example of instances where they made explicit connections between university mathematics and concrete pedagogical actions.

Similarly, based on the findings of their survey of future mathematics teachers from Germany, Hong Kong, China (Hangzhou) and South Korea nearing the end of their university studies, Buchholtz et al. (2013) found that the future teachers (including those from top mathematics performing countries or regions) often seem unable to link school and university knowledge systematically. The authors suggested that 'prospective teachers should have adaptable mathematical knowledge: a knowledge that comprises school mathematics, but goes beyond it and relates it to the underlying advanced academic mathematics, which according to Klein (1932) we call the *knowledge of elementary mathematics from an advanced standpoint'* (Buchholtz et al. 2013: 108).

We seem to have come back full circle to the quote at the start of this chapter, when back in 1932 Klein warned that since teachers were unable to see any connection between teaching school mathematics and the more advanced mathematics they studied at university, they "fell in with the time honoured way of teaching", tapping into the "same old" ways of teaching mathematics, thus developing a SCK that held little pedagogic potential for supporting the learning of high epistemic school mathematics knowledge. In his work, Klein refers to the 'double discontinuity' for teachers in their education. The first discontinuity concerns the well-known problems of transition which students face as they enter university, while the second discontinuity is the disconnect for these future teachers in returning back to school mathematics, where university mathematics appeared to be unrelated to the tasks of teaching. While both discontinuities still exist, it is this second discontinuity that is of particular interest in this chapter, namely exploring how knowing advanced mathematics might influence the teaching of school mathematics.

The literature reviewed in this chapter clearly indicates that good teaching requires more than knowledge of the content to be taught. Teachers should understand the processes by which a particular mathematical idea develops as students progress through different levels of school education, and how an elementary school mathematics idea is drawn to completion at advanced levels of mathematics. It is with this understanding that teachers will be better prepared to set students along a powerful mathematical development path (powerful because it helps them develop high epistemic mathematics knowledge, based on a conceptual understanding rather than memorizing facts and rules about the concept), by addressing the obstacles and opportunities that appear most frequently along the way towards an understanding of the idea or concept being taught. To achieve this, Watson and Harel (2013) also propose that teachers should possess personal mathematical knowledge significantly beyond the level at which they are teaching, although the authors themselves admit that the 'question of how formally acquired advanced knowledge becomes tacit and continuously available in teaching remains' (Watson and Harel 2013: 166). This question has remained of interest to researchers, with little progress to date towards understanding the relationship between advanced subject knowledge and subject knowledge for teaching.

The Study

In the following, I present an empirical study intended to gain an insight into this under-developed area of research.

The context of the study

Although prior research has offered some useful descriptions of the advanced mathematics knowledge relevant to teaching school mathematics, and some useful insight into how teachers could be supported in drawing on this knowledge in their teaching, most extant research concerned with advanced mathematics knowledge gives a picture of developments taking place outside England. In contrast to the pre-service teachers in the studies reviewed, most of whom train to become teachers during their undergraduate studies where they study advanced mathematics courses alongside their teacher preparation, the pre-service teachers in England, the UK complete their training immediately, or some years after, they complete their undergraduate studies. Pre-service teachers who enrol in a 1-year postgraduate course would have studied some form of advanced mathematics as part of their undergraduate studies, but usually with no links or reference to school-level mathematics or its teaching.

Moreover, in England, not all pre-service teachers would have studied advanced mathematics in the sense of formal, academic mathematics beyond the school curriculum.

Based on this understanding, I now describe how I approached the design of a professional development course to support teachers in connecting advanced mathematics knowledge/formally acquired advanced knowledge/ academic mathematics knowledge/more advanced and relevant mathematics knowledge/ to their teaching of school mathematics concepts/ideas/topics. Here I have deliberately used the different terminologies encountered in the literature reviewed, as yet another description of this type of knowledge suited to the current study and the England, UK context would further complicate matters. Instead, I come back to such a description in the discussion section of this chapter.

In England, the UK, the requirements of the National curriculum covers what subjects are taught and the standards children should reach in each subject. The New National Curriculum in mathematics (2015) includes *harder* subject material, such as more formulae to learn, set theory, iteration and functions. In the school mathematics curriculum in England, the idea of functions appears in different guises; common ways of representing functions include tables of values, graphs, algebraic representation, words, and problem situations. It is important to note here that the school mathematics curriculum in England has had an informal approach to functions, and as such a formal treatment of functions is not encountered by students unless they choose to study more advanced mathematics courses beyond the age of 17. At the pre-university level, students encounter many more types of functions beyond linear and quadratic functions, and the New National curriculum stipulates that the more formal definition of functions and their features previously taught at pre-university level (17- to 18-year-olds) are now being introduced at lower levels of school education. Students are also expected to learn about more advanced knowledge about functions such as: domain, range, one-to-one function, inverse function, and composition of functions, including the formal definition of a function. The formal definition in school mathematics is consistent with the formal (Dirichlet-Bourbaki) definition, namely: f is a function from one set to another, say **A** to **B**, both sets of real numbers. The main requirement of this modern definition of the function concept is univalence, which requires that for each element in set **A**, called the domain of the function, there is associated only one element of **B**, called the range of the function. The introduction of the New National Curriculum saw a flourishing of professional development courses for teachers to gain familiarity with and confidence with the *harder* topics they were now expected to teach.

The professional development workshop: design consideration

A workshop was thus designed aimed at supporting practising teachers, with experience of teaching mathematics to 11- to 16-year-old students, develop and extend their knowledge for teaching about *functions*. Just as in Stockon and Wasserman (2017), in this study, the designer of the workshop activities is the researcher herself (also the author of this chapter), a mathematician and a mathematics teacher educator with considerable experience of teaching mathematics at undergraduate level, as well as the secondary school level of education in England, UK (non-advanced level: 11- to 16-year-olds, but also advanced level: 17- to 18-year-olds), and also a mathematics educator with considerable experience in initial teacher education.

Wasserman et al.'s (2017) "Building up from and Stepping down" approach to teaching undergraduate-level mathematics to pre-service teachers was adopted and adapted in designing this workshop, in that the teachers did both the *building up* and the *stepping down* *to practice*. Indeed, engaging teachers in mathematical thinking by working on classroom-close mathematics-related tasks that are situated in teaching practice, and reflecting on these experiences, is common to many professional learning programmes (Watson and Mason 2007; Biza, Nardi and Zachariades 2007). As such, the tasks presented to teachers were *building up from practice* by posing mathematics-focused questions, that would lead on the teachers needing to connect with more advanced knowledge about functions, while the *stepping-down to practice* activities would require teachers to react to fictional pupils' scenarios, designed to encourage pedagogical consideration of the new learning.

The participants

The workshop was attended by eight practising mathematics teachers. Data from the initial questionnaire sent to them before the workshop showed that all teachers had gained their qualified teacher status as a result of studying a 1-year teacher training course, after graduating from university. Six teachers (T1, T2, T3, T4 four females and T5, T6 two males) majored in mathematics, one had an engineering background (T7 male), while another teacher had an economics background (T8 male) and introduced himself as a non-specialist mathematics teacher. The participants were practising mathematics teachers, with teaching experiences varying between 1 and 4 years of teaching 11- to 16-year-old students.

When asked about the reasons for choosing to attend this workshop, the teachers mentioned their familiarity with the "usual representations of functions" in school mathematics, such as tables of values, graphs and equations of linear and quadratic functions, but expressed concerns about the need to learn about more advanced knowledge about functions given the requirements of this *harder* topic in the New National curriculum.

Research question

In line with KOSS programme interest in characterizing the nature of teachers' powerful professional knowledge, the research carried out alongside this workshop sought to investigate whether and how participating in a professional development workshop designed to support practising teachers connect with advanced knowledge of function had empowered them pedagogically in ways productive for the teaching and learning of these concepts at all levels of school mathematics education.

Data sources and analysis

This study used a qualitative design. Prior to commencing the programme, participants completed an initial questionnaire. Data from the initial questionnaire provided information about the participants' teaching qualifications, numbers of years of teaching experience and school levels taught.

During delivery of the workshop, textual data were collected through field notes that detailed some of the group interactions, while photos were taken of individual teachers' notes (e.g. their mathematical work and their "reactions" to the pupils' scenarios). Post-session reflective written notes were solicited and collected after the end of the session. The teachers were asked to reflect on and write about the activities in relation to their own learning, their pupils' mathematical learning, and their teaching about functions in the future, as a result of the learning during the workshop.

In this chapter, the data collected following the teachers' involvement with the very first activity of the workshop are analysed. The analysis uses descriptions of the dimensions of the professional knowledge base for teaching as reviewed in this chapter, in particular SCK, HCK, AMK, with a view to allowing for an insight to be gained into how teachers' engagement with the tasks benefitted them both conceptually and pedagogically.

Results

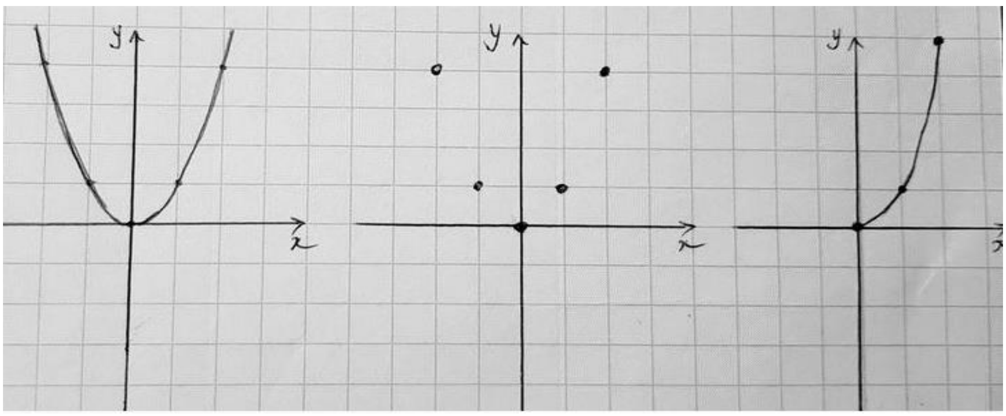
The first activity the participants were given required them to sketch graphs of four functions. The four functions varied in that they had different domains of definition, namely: the whole set of real numbers; an open interval; a union of open and closed intervals; and a discrete set of integer numbers, but they all shared the same function rule (equation), namely $f(x) = x^2$. The eight practising teachers worked in pairs, and each pair was asked to sketch the graph of one of these four functions.

When sharing their graphs and approaches to sketching the graphs, all pairs reported that they straightaway drew *the parabola*, which is in fact the graphical representation of *the quadratic function* $y = x^2$, whose domain of definition is the set of all x real numbers, hence a function distinct from those the teachers had been asked to sketch. In the midst of explaining their approaches to this activity, three teachers turned their attention to the given domains of definitions of the functions assigned to them, raising some confusion about the significance of domains to the graph-sketching activity.

In this activity, the practising teachers displayed the same misconceptions about functions as the well-documented students' misconceptions, namely, that they usually call upon one part-representation of functions, in this case, its equation or function rule (Markovits et al. 1986), namely $(x) = x^2$. The practising teachers also focussed on the function rule, at first ignoring the given domains of definition. This explains why all teachers ended up with the same graphical representation for the four different functions, more precisely, that of a smooth curve in the shape of a parabola.

Teachers reaching for more advanced knowledge about functions

When the graphs produced were shared with the whole group, the teachers became aware of the similarities (the same function rule), but also the differences (the different domains of definition) among their functions. They realized they had been assigned different functions to sketch, and so their graphs could not have looked the same. This realization led to a discussion about the 'full description' or definition of a function, and as a result the teachers were then able to sketch the appropriate graphs to the given functions, as in Figure 2 below:



Domain is 'all real number values of x'; Domain is 'all the integer values of x'; Domain is 'all the positive real values of x'

Figure 2 Graphs of $f(x) = x^2$ for different domains of definitions

In the school mathematics curriculum in England for 11- to 16-year-olds, there is an implicit hidden assumption that the domain of any "school mathematics" function is the whole set of real numbers. This assumption means that most of the "school mathematics" functions are "well behaved", with continuous graphs, and the usual approach to sketching such graphs as encountered by students in textbooks or in their teachers' explanations, is that of choosing a few real number values for the independent variable x in the table of values, usually consisting of "easy integer number values" chosen to be values around zero, followed by calculation of the values of the dependent variable, and lastly plotting and neatly joining up these points either by continuous straight-line segments or line curves.

During this workshop, there was clear evidence that such limited representations of functions were part of the participating teachers' SCK but, most importantly, the teachers themselves became aware of the limitations of their own SCK! Those teachers with a formal mathematics background recalled having encountered the formal definition of a function in their undergraduate studies (HCK), while the others (T7 and T8) did not seem to have any such recollection. In an attempt to define and fully describe a function, suggestions from the teachers were collected, such as domain, range and one-to-one correspondence (T1 to T6), co-domains (T3 and T6), notation conventions for composite functions (T5). Just like the first-year undergraduates in Nardi's (2001) study, these teachers' concept images of functions lacked in understanding the concept as being inextricably connected with its domain, co-domain and relationship expressed algebraically. Such a recollection enabled the practising teachers to assemble and "put together" a definition (the formal definition), with some guidance from the course tutor. A fruitful discussion ensued about the difference between the range and co-domain of a function, and this discussion even led to recollection of other features and properties of functions, such as the relationship between functions and their inverses.

For the two teachers (T7 and T8) who had not studied formal mathematics at undergraduate level, their only recollection about functions was about using the y-notation for functions and solving problems that required differentiation or integration. Their participation in the group discussion about describing fully a function, by asking clarifications from the other teachers about features and properties mentioned meant that they too had familiarized themselves with the formal definition of a function.

Teachers' connecting to more advanced mathematics knowledge

The activity described above provided a stimulus for T1 to T6 to reach for and (partially) recall their advanced knowledge of functions, while for T7 and T8, the activity led to new learning because they had not previously encountered the formal definition of a function.

With gained awareness that functions are uniquely defined by a rule and their domains of definition, the teachers revisited the allocated functions and re-drew their graphs, correctly this time. T2, a mathematics graduate, stated that 'it did not occur to me to relate this activity with the formal definition', while T1 justified the lack of engagement with his advanced mathematics background as 'that was high-level mathematics not much used after the course'.

Teachers' pedagogical learning: understanding curriculum progression

The teachers expressed surprise about the fact that there had never been occasions in their teaching when making explicit these features of a function was ever needed. This led to a fruitful discussion among the teachers about the quality of students' learning experiences of functions as part of the school mathematics curriculum. Indeed, at the secondary school level (in England) the formal definition of a function is not introduced to students until the more advanced levels of education (pre-university and undergraduate levels). The teachers admitted that in their planning they never even thought to consider the formal definition of a function, focusing instead on the common representations of functions across the secondary school curriculum (One-to-one or many-to-one mappings; Input/output machines; Relations between particular x-values and y-values; Expressions to calculate the y-values from given x-values; and Graphs), with the intention to build up to a fuller understanding of functions.

During the workshop, the practising teachers were encouraged to connect their newly acquired/recalled advanced knowledge about functions and relate explicitly each of these representations to the formal definition of functions. In doing so, the practising teachers came to realize that each of these representations explained particular features of the concept itself, but without being able to describe it completely! Teachers realized that overreliance on one representation, and a lack of connections between such representations make way for misconceptions while working with functions. This was strong evidence that the participating teachers had benefitted from engaging with the more advanced, formal definition of a function. Reflecting on their learning, the practising teachers thought they had benefitted conceptually. T3 reckoned that he 'became aware of the stages of building up to the definition of a function', while T1 stated that she had learned about: 'Different representations of functions – I've always seen them as disconnected representations, but they complement each other nicely towards fully understanding functions'.

Teachers' pedagogical learning: supporting pupils' learning

In this workshop, the practising teachers shared the challenges their students encounter while working with functions and graphs. For each challenge shared, the practising teachers were encouraged to find a possible justification in the light of their new learning.

One challenge shared was pupils' perception of the graph of a linear function as a straight-line segment, which neatly joins the plotted points according to their choice of values in the table of values. In fact, this misconception was clearly "demonstrated" by T2 herself while describing her approach to teaching about sketching graphs: 'once the few points in the table of values are plotted, they should always be joined up, with the graph extending between the lowest and highest point plotted […]. The data points are needed in order to draw a smooth graph'. Following a group discussion on this approach, the practising teachers came to realize that lack of explicitness of the convenient choice of a few integer number values of x in the table leads to the students' misconception mentioned above. Lack of explicitness about the domain of a function, usually the whole set of real values, and its connection with drawing a "continuous" graph was also identified by the teachers as a reason why students think of joining plotted points from the table with line segments. There was clear evidence in this episode that teachers' new learning about a function (AMK) had also supported them pedagogically in that not only their SCK was enriched by understanding why students hold specific misconceptions, but they also felt they know how to support students in addressing their misconceptions.

Discussion

This study adds to the evidence that teachers' knowledge of advanced mathematics holds the potential to transform teachers' own understanding of the school mathematics content they teach in ways that are pedagogically powerful.

The analysis of the data collected revealed that the participating teachers had benefitted from engaging with the more advanced, formal definition of a function. While all the participating teachers recognized that such a definition was not appropriate to be taught to secondary school students (unless 17-year-olds choose to study advanced mathematics courses), they all felt they had benefitted conceptually by gaining personal advanced knowledge about functions. The teachers' own understanding of functions became more sophisticated and nuanced as a result.

Moreover, these teachers also believed that they gained a powerful pedagogical understanding by reflecting on how their SCK benefitted from the more advanced knowledge. The pedagogical potential of their SCK was realized, as evidenced in teachers' realization of why students have certain misconceptions and of how they could support students to develop an understanding of this mathematics idea of high epistemic quality. Even if the "methods" and "rules" continued to be used by students, the teachers had become aware of the importance of making explicit to the students the conceptually sound grounding of the underlying mathematics.

Implications for teacher knowledge development

The design of the activities in which the participants were involved in this professional development workshop was important. The mathematics activities created some tension in what the teachers knew about functions and their graphical representations and, in order to address the differences in their solutions, the need to engage with more advanced mathematics knowledge about functions was brought out into the open. With some support and guidance throughout the workshop, the teachers engaged with this knowledge in a way that saw them benefit both conceptually and pedagogically.

These teachers had SCK about various representations of functions, but their SCK was limiting as they all shared common misconceptions about their students' working with functions. 'Today's session helped me understand how I could have addressed the [pupils'] errors and how I can clarify things in the future' (T2), while another teacher shared his learning in the session: 'What I have learnt today? About advanced mathematics knowledge and its place in classroom and planning' (T5).

In responding to the question posed by Furlong and Whitty (2017) as to what may constitute the powerful professional knowledge required for teacher education, this study indicated that with support, when teachers accessed more advanced knowledge, such knowledge then became productive for their teaching and thinking about their students' learning.

This chapter thus helps situate the study of advanced mathematics in relation not only to school mathematics content, but also its teaching, and it proposes that in England such support should be the remit of ITE (teacher training courses, to include CPD opportunities) to look at school mathematics from an advanced standpoint, to examine school mathematics topics by engaging with advanced mathematics knowledge, where guidance is provided in terms of the relevant AMK, and how this could inform the teaching of school mathematics. If left to the teachers themselves, such links might never happen. But if support is provided, and creating links between advanced mathematics and school mathematics is modelled and practised in teacher education programmes, then such habits could be carried forward by teachers when they enter the teaching profession.

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