Essays on Uncertainty and Asymmetric information in Financial Markets

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A dissertation submitted in partial fulfillment
of the requirements for the degree of

Doctor of Philosophy
of
University College London

Department of Economics
University College London

November 2020

I, Seungmoon Park, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

Chapter 4 was undertaken as joint work with Marco Cipriani, Antonio Guarino, and Giovanna Di Stefano.

ABSTRACT

This thesis analyses the behaviour of traders in financial markets with various structure of traders, and presents how their actions are affected by uncertainty and information asymmetry. In Chapter 2, a sequential trading model with ambiguity aversion is studied. Traders with and without ambiguity trade an asset in sequence with a market maker updating the expected value of the asset according to a history of actions and their private signal. Ambiguity is only assumed on the precision of the private signal of traders about the value of the asset. The ambiguity averse traders update the set of precision with the recursive multiple prior preference, while the traders without ambiguity and the market maker update the distribution of the precision from uniform distribution as a prior. We find that the behaviour of the ambiguity averse traders differs depending on their updating rule on the set of precision. With the Full Bayesian Updating, herding or contrarian behaviour never occurs while the traders choose no trade in equilibrium even with the informative private signal asymptotically. With the Maximum Likelihood Updating, herding can happen, while no trade is less likely to be chosen by the traders in equilibrium, and they tend to act following their private signals like traders without ambiguity as trade goes on in the limit.

In Chapter 3, a sequential trading model is introduced to estimate ambiguity aversion in the financial market. As with the model in Chapter 2, it has traders with and without ambiguity and they trade an asset with a market maker in sequence. Also, the ambiguity is only on the precision of the private signal received by the traders, and the ambiguity averse traders have recursive multiple prior preference. In addition to that, there is an event uncertainty and the private signal of the traders is continuous. In the model, traders choose not to trade with any signal, or show herding or contrarian behaviour depending on their updating rule on the set of priors about the precision. We estimate the model with trading data on NYSE stocks. The estimation result shows that

strict reevaluation generates more herding or contrarian, while loose reevaluation brings no buying or selling more likely.

In Chapter 4, we consider a sequential trading market with potential manipulator. As in the previous chapters, traders exchange an asset with a market maker. In the market, there are three types of traders: noise traders, informed traders and a potential manipulator. The manipulator receives private information on the asset value like other informed traders, but she has the opportunity to trade twice in a trading day, while the other traders trade only once. We show that, under some conditions, the manipulator does not follow her signal in the first period of action in equilibrium. Instead, she trades against her signal, suffering a loss to distort the expectation of other rational traders and the market maker. The price path is manipulated with the distortion. The manipulator then re-enters the market and makes profits by trading with the price. In this process, the uncertainty is essential as it generates a gap in expectation between the informed traders and the market maker and enables the distortion.

IMPACT STATEMENT

This thesis explores the behaviour of traders in financial markets focusing on the uncertainty and asymmetric information. In Chapter 2 and 3, interesting behaviours such as herding, contrarian or choosing no trade are explained by augmenting ambiguity aversion on traders in the market. Ambiguity aversion is mostly used to explain limited participation in literature. Based on the result of the thesis, ambiguity aversion can also be considered as a possible factor to explain the behaviour of herding and contrarian. The method used to analyse the ambiguity aversion in dynamic setting also can be applied in other areas that are related with agents' decision making based on information asymmetry such as labour market and industrial organisation.

The main idea of the market manipulation in Chapter 4 is that with an event uncertainty, the manipulator has an incentive to distort the beliefs of others even with an initial loss. It also can be applied in various settings with uncertainty to analyse strategic actions of agents.

Outside of academia, the thesis has a potential to affect financial market participants and policy makers. The model with ambiguity aversion explains how the two extreme behaviours of herding and abstaining from trade are generated from the choice of ambiguity averse traders in equilibrium. It brings an insight for the financial market participants to understand the behaviours of traders which are often thought to be irrational, and how to interpret their actions as a meaningful information. Also, it will help policy makers to contain market volatility by refining information structure in the market.

Chapter 4 focuses on the market manipulation in relation with event uncertainty of the market and precision of the private signal. It would help financial regulators to devise policies on restricting market manipulation not only by imposing punishment after it happened, but also by restricting the incentive of manipulation in advance.

ACKNOWLEDGEMENT

First of all, I would like to thank my supervisors Antonio Guarino, Morten Ravn and Wei Cui. Their ceaseless support and encouragement were essential to the completion of this work.

Also, I am really grateful for the faculty members Albert Marcet, Frederic Malherbe and Jihee Yoon who gave me invaluable advice in materialising my crude idea. In addition, I want to give thanks to my fellow students. With them, I could enjoy my study in UCL.

Lastly I would like to thank my wife Juhyun Lee for her support along with patience and understanding.

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Chapter 1

Introduction

In financial markets, traders show intriguing behaviour such as herding, contrarian or abstaining from trading. Also, we can observe the traders who tries to get profits by manipulating the beliefs of others using uncertainty and information asymmetry. This thesis explores the behaviour of traders in sequential trading markets with private information focusing on the uncertainty and information asymmetry and how these are interpreted by the agents in the market. In Chapter 2 and 3, models including traders with ambiguity aversion are presented and the behaviour of the traders are analysed under information asymmetry, and Chapter 4 shows a model with a manipulator who distorts the price of an asset by her action and achieves a profit from it using the event uncertainty and information asymmetry.

Chapter 2 introduces a sequential trading model with ambiguity aversion. In the model only with informed and noise traders, the traders act just following their signal. When the ambiguity averse traders with recursive multiple prior preference are included in the market, the traders can choose no trade or show herding or contrarian behaviour in equilibrium. The ambiguity averse traders can choose no trade in equilibrium since they consider the worst case payoff with a large set of ambiguous parameter using the updating rule such as the Full Bayesian Updating. Also, if the ambiguity averse traders update the set of ambiguous parameter strictly using the updating rule such as the Maximum Likelihood Updating, herding or contrarian behaviour can happen. It is caused by the gap between the precision taken into account by the ambiguity averse traders and the market maker. If all the informed traders in the market are ambiguity averse traders, information cascade can occur with those behaviour in equilibrium.

In Chapter 3, financial market data are estimated by the sequential trading model with ambiguity averse traders. The model is extended from that of Chapter 2 with event uncertainty and continuous signal. It is shown that choosing no trade, or herding or contrarian behaviours also can happen in equilibrium with the extended model. Capturing these behaviours, the share of ambiguity averse traders in the market and their way of updating on the set of the ambiguous parameter are estimated by trading data of NYSE, such as Ashland Inc. and Capital One Financial Corp. The estimation results show that strict updating of the traders with high re-evaluation parameter in the market of Ashland Inc. generates more herding or contrarian, while loose updating of traders in the market of Capital One Financial Corp. makes the traders choose no trade in the equilibrium more likely.

The final chapter of this thesis considers a sequential trading financial market with a manipulator. Similarly to other informed traders, the manipulator receives private information on the asset value. However, she has an opportunity to trade twice in a trading day, whereas other traders trade only once. We show that the manipulator can achieve higher profit by going against her signal in the initial action rather than following the signal. By acting against her signal in the initial period, she can manipulate the other rational traders and the price path. When the manipulator enters the market in the second period, she can make profit from the manipulated price. It is also shown that the manipulation cannot be more profitable than the action following her signal without event uncertainty.

Chapter 2

Ambiguity Averse Traders in Sequential Trading Model

2.1 Introduction

Economic agents' behaviour under ambiguity aversion is studied in many different settings after Ellsberg (1961). Especially in finance literature, the behaviour such as 'limited participation' has been a focus of interest since the ambiguity averse agents avoid to be in a situation with ambiguity by taking part in investments. In this research, a model is introduced to explain not only the behaviour of limited participation but also the concentrated participation such as herding or contrarian with ambiguity aversion.

In the model, traders with and without ambiguity trade an asset with a market maker in sequence. Ambiguity is only assumed in the precision of the private signal the informed traders receive, and only the ambiguity averse traders perceives the ambiguity. Ambiguity aversion is implemented by 'the recursive multiple prior' preference following Epstein and Schneider (2007). The preference is based on the multiple prior preference by Gilboa and Schmeidler (1989), which is also known as 'maxmin' preference, and recursive re-evaluation for the set of ambiguity parameter is augmented on it. With the preference, the agents with ambiguity aversion consider the worst case payoff from the set of ambiguous parameter. As the market maker does not have ambiguity, there could happen a situation that the expected value of the ambiguity averse trader with a good (bad) signal is lower (higher) than the ask (bid) price. It makes the traders abstain from trading in equilibrium even though they have an informative private signal.

It corresponds with limited participation in literature.

The the recursive multiple prior preference also enables herding or contrarian behaviour for the ambiguity averse traders in the model. Those are seemingly the opposite behaviour to limited participation, because it is buying or selling whatever the private signal is. Herding or contrarian occurs when the expected asset value of the traders with a bad (good) signal is higher (lower) than the ask (bid) price. It is caused by the gap between the precision used by the market maker and the ambiguity averse traders. With the recursive multiple prior preference, the set of precision levels considered by the ambiguity averse traders is updated, which is called 're-evaluation' following Epstein and Schneider (2007). As the set is re-evaluated after history of actions, it could happen that the precision considered by the market maker is out of the updated set by the ambiguity averse traders, and herding or contrarian occurs.

The possibility of choosing no trade in equilibrium and herding or contrarian behaviour changes depending on the updating rule of the set of priors of the ambiguous parameter. With the Full Bayesian Updating, which is one extreme way of re-evaluating, the set of priors is sustained as the whole domain of the precision parameter, so the traders are more likely to choose no trade in equilibrium and herding or contrarian is impossible to happen. With other updating rules such as Maximum Likelihood Updating, the set shrinks compared to the Full Bayesian Updating. Hence, herding or contrarian can occur and the traders choose no trade with lower probability.

In addition, there is possibility of informational cascade in the model. Informational cascade is a situation in which agents do not learn from market outcomes in social learning literature. If all the informed traders in the market are ambiguity averse, there could be a situation that the traders choose no trade regardless of their signal, or herding or contrarian behaviour occurs. In those cases, the action of the trader does not reveal any information to the market. Therefore, there is no informational gain after the action, which is informational cascade.

This research is related with literature that study 'limited participation' or 'portfolio inertia' of the agents with ambiguity aversion. Dow and Werlang (1992) presents the non-participation of traders caused by ambiguity aversion using maxmin preference. Epstein and Wang (1994) points out the portfolio inertia for general form of portfolio with multiple assets. In empirical research, Campanale (2009) shows limited participation in life-cycle portfolio allocation. Also, Bossaerts et al. (2010) presents the behaviour of trader not to hold ambiguous asset in experiments.

The other strand of literature is on herding and contrarian behaviour. The seminal papers of Banerjee (2004), Bikhchandani et al. (1992) and Welch (1992) started to explore the theoretical research on herd behaviour and informational cascade in an abstract environment. In their research, it is shown that agents follow the predecessors' action regardless of their private information. The phenomenon is studied in financial market model with sequential trading developed by Glosten and Milgrom (1985). Agents in this type of model decide their action based on the updated beliefs on the state variable. The sequential trading model has an advantage of analysing the behaviour of traders, since it explicitly brings the equilibrium action of traders, which is one of 'buy', 'sell' or 'no trade'. Using these types of models, research such as Avery and Zemsky (1998) and Cipriani and Guarino (2008) show the herding and contrarian behaviour of the traders focused on uncertainty. Also, Dong et al. (2010), Ford et al. (2013) and Boortz (2016) demonstrated those behaviours using ambiguity aversion as a factor to generate them. The difference of this research from those considering ambiguity aversion is that the recursive multiple prior preference is assumed for the ambiguity averse traders rather than Choquet preferences. Choquet preference enables analytic explanation for the behaviour compared to the recursive multiple prior preference. With the recursive multiple prior preference, however, the channel to incur herding or contrarian is explained more explicitly.

This research also shares interest on market participation with other literature. Allen and Gale (1994), Williamson (1994) and Veldkamp (2006) explain that externality of participation generates changes in the agents' participation decision in the financial market. If there is more participation, the agents get external benefits from it, so they would like to participate more. These effects depend on transaction cost or information cost. In this research, the difference in the precision of signal considered by the agents causes changes in participation. If the precision considered by the market maker is within the set of ambiguity averse traders, the traders are not likely to participate; if not, they may participate and sometimes even show herd or contrarian behaviour. In this process, re-evaluation parameter is one of the key factors to govern it.

The paper is organised as follows. After introduction, Section 2 describes the sequential trading model with ambiguity averse traders, and Section 3 presents the behaviour of traders of the model in equilibrium depending on their re-evaluation rule. In Section 4, informational cascade is discussed. Section 5 concludes.

2.2 Model

The model is based on two state Glosten and Milgrom (1985) model with a binary signal.

The market - A sequence of traders trade an asset interacting with a market maker who sets ask and bid prices. At each time $t = 1, 2, 3 \cdots$, a trader is randomly assigned to trade, and she decides whether to buy, sell or not to trade one unit of asset. The action space is, therefore, $\mathcal{A} = \{\textit{buy}, \textit{sell}, \textit{no trade}\}$. The action of a trader at time t is denoted by X_t

The asset - The fundamental value of the asset, V, can be either 1 or 0. It is randomly decided before the beginning of the market and it does not change till the end. True value V is unknown to every agent but has a prior distribution of $Pr(V = 1) = \delta$ as a common knowledge.

The market maker - The market maker sets ask and bid prices at each time taking into account buy or sell action of traders who might have private information. Zero expected profit for the market maker is assumed due to potential competition. The ask price at time t is denoted by a_t, and the bid price by b_t.

$$a_t = E(V|X_t = buy, h_t, a_t, b_t) \quad (2.1)$$

$$b_t = E(V|X_t = sell, h_t, a_t, b_t) \quad (2.2)$$

Note that the ask and bid price are equilibrium products as the strategy of traders is assumed to compute those. a_t and b_t in condition are omitted for notational simplicity afterwards.

The information the market maker has is history of actions $\{h_t\}$ and price quotes $\{a^t, b^t\}$. The market maker is rational and update her belief by Bayesian updating. She does not have ambiguity on the precision of the private signal, although she is not aware of the true precision exactly. The meaning of having ambiguity or not will be discussed in more detail below.

The traders - One trader out of countable number of traders is exogenously assigned to act at time t. Each trader is chosen only once. The traders are categorised by two groups: Informed and noise traders with share of μ and $1-\mu$ respectively with $\mu \in (0,1)$. Informed traders are those who get a private signal S_t. Each informed trader receives one signal $S_t \in \{0,1\}$ about the value of the asset with precision q, $Pr(S_t = 1|V = 1) = Pr(S_t = 0|V = 0) = q$. The traders do not know the true precision of the signal which is q^T, but they know about the domain of q, $\mathcal{Q}_0 = \{q : \underline{q} \leq q \leq \bar{q}\}$, which is a set of possible level of q. It is common knowledge and $0.5 < \underline{q} < \overline{q} < 1$. The noise traders choose their action randomly by the probability of $\epsilon/2$, $\epsilon/2$ and $1-\epsilon$ respectively to buy, sell or not to trade with $0 < \epsilon < 1$.

The informed traders are divided into two groups again. γ share of informed traders who do not have ambiguity in the precision of private signal, and the other $1-\gamma$ who have ambiguity and aversion in it, where $\gamma \in [0,1]$. In this research ambiguity is only assumed to be on the precision of the private signal q. The agents without ambiguity such as informed traders without ambiguity and the market maker also do not know the true level of q. They, however, have proper prior distribution for q to be updated after looking at the history of actions. The prior distribution of q for the agents without ambiguity is assumed to be uniform distribution on $\mathcal{Q}_0$, and it is updated by Bayes updating after trades.

The traders without ambiguity choose their action based on their expectation on payoff. The payoff function is defined as

$$U(V, X_t, a_t, b_t) = \begin{cases} V - a_t & \textit{if } X_t = \textit{buy} \\ 0 & \textit{if } X_t = \textit{no trade} \\ b_t - V & \textit{if } X_t = \textit{sell} \end{cases}$$

X_t is decided to maximize $E(U(V, X_t, a_t, b_t)|S_t, h_t)$. Thus, the informed traders buy when $E(V|S_t, h_t) > a_t$, sell when $E(V|S_t, h_t) < b_t$, and do not trade otherwise.

Ambiguity averse traders, on the other hand, have the recursive multiple prior preference following Epstein and Schneider (2007). It is based on multiple prior preference (also known as 'maxmin' preference) by Gilboa and Schmeidler (1989) and Epstein and Wang (1994) with the feature of re-evaluation on the set of parameters to consider. They have a set of priors on the signal precision to re-evaluate at each time t and their action is decided based on the re-evaluated set. The set of priors is assumed to be the domain of the ambiguity parameter $\mathcal{Q}_0$, and $\mathcal{Q}_t$ is the re-evaluated set at each time t. The process of re-evaluation will be explained in the next section.

The payoff function of the ambiguity averse traders is same as those without ambiguity, but the conditional expectation of the payoff is different because of the ambiguity. Since the agents are averse on ambiguity, they consider the worst case payoff among the elements in the set of ambiguous parameter when they choose their action. It makes the conditional expectation to be $\min_{q \in \mathcal{Q}_t} E(U(V, X_t, a_t, b_t)|S_t, q, h_t)$. Hence, the action of the ambiguity averse traders choose is as follows.

$$\begin{aligned}&\max_{\{\textit{buy,no trade,sell}\}}[\min_{q\in\mathcal{Q}_t}\{E(V|S_t,q,h_t)-a_t\},\,0,\,\min_{q\in\mathcal{Q}_t}\{b_t-E(V|S_t,q,h_t)\}]\\&=\max_{\{\textit{buy,no trade,sell}\}}[\min_{q\in\mathcal{Q}_t}\{E(V|S_t,q,h_t)\}-a_t,\,0,\,b_t-\max_{q\in\mathcal{Q}_t}\{E(V|S_t,q,h_t)\}]\end{aligned} \tag{2.3}$$

The ambiguity averse traders buy if $\min_{q\in\mathcal{Q}_t} E(V|S_t,q,h_t)$ is higher than the ask price, and sell if $\max_{q\in\mathcal{Q}_t} E(V|S_t,q,h_t)$ is lower than the bid. They do not trade otherwise.

2.2.1 Re-evaluating the set of priors of ambiguity averse traders

According to equation (2.3), the maximum or minimum value of the expected value V is constrained by the set of ambiguous parameter q, which is $\mathcal{Q}_t$. Therefore, $\mathcal{Q}_t$ is important for the decision of the ambiguity averse traders.

Since the traders with ambiguity have the set $\mathcal{Q}_0$ as their prior, it is impossible for them to derive a proper distribution on q to be integrated like those without ambiguity. They, however, can re-evaluate[1] the set. It is updating the the set of q to be considered in their decision by eliminating less probable values after observing the history of actions. Following Epstein and Schneider (2007), the re-evaluation is assumed to take the likelihood ratio form as in following equation.

$$\mathcal{Q}_t=\{q:\frac{Pr(h_t|q)}{\max_{\hat{q}\in\mathcal{Q}_0}Pr(h_t|\hat{q})}\geq\rho,\,q\in\mathcal{Q}_0\} \tag{2.4}$$

The parameter ρ, which is $0\leq\rho\leq1$, governs how strictly the agents re-evaluate the set to consider when they make decision. At time t, the set $\mathcal{Q}_t$ includes the q in $\mathcal{Q}_0$ that makes the likelihood of the history h_t to be higher than the maximum likelihood multiplied by ρ. If the level of ρ is low, the threshold for the likelihood becomes low and $\mathcal{Q}_t$ is large as a larger set of q satisfy the condition; if ρ is high, the threshold is high and the $\mathcal{Q}_t$ becomes small.

To understand how the re-evaluation works, let us consider the extreme cases first.

[1]The terminology of re-evaluating environment is from Epstein and Schneider (2007).

When $\rho = 0$, the set $\mathcal{Q}_t$ includes all the possible q in its domain as the likelihood of the history is larger than zero for any q. Hence, $\mathcal{Q}_t$ is constant at $\mathcal{Q}_0$ for any t. It is the loosest re-evaluation of the traders, and the largest set $\mathcal{Q}_t$. It is the *Full Bayesian Updating (FBU)* in Gilboa and Marinacci (2011) and De Filippis et al. (2017). The traders update the expected value of the asset prior by prior in the whole set of priors $\mathcal{Q}_0$ and decide their action based on the maxmin preference.

The strictest case is when $\rho = 1$. The set $\mathcal{Q}_t$ only includes the q that makes the likelihood of the history to be maximum, $\mathcal{Q}_t = \{q : \arg\max_{q \in \mathcal{Q}_0} Pr(h_t|q)\}$. The size of the set is the smallest than any other ρ as $\mathcal{Q}_t$ becomes singleton after a history of actions. When the set $\mathcal{Q}_t$ becomes singleton, the ambiguity disappears and the traders do not need to choose the level q which minimize the payoff. It is like the traders are using the *Maximum Likelihood Updating (MLU)* in Gilboa and Marinacci (2011) and De Filippis et al. (2017).

In the case that ρ is in between zero and one, the re-evaluation process can be interpreted like maximum likelihood ratio test, as it follows the form of the test. The critical value of $\chi^2(1)$ distribution is $-2\log\rho$ for rejecting the hypothesis $q = \hat{q}$. It means that the traders are excluding the level of q out of the confidence interval from $\mathcal{Q}_0$ under the certain significance level determined by ρ. For example, when ρ is 0.26, the significance level for choosing set $\mathcal{Q}_t$ is 10%. The level of q that has probability less than 10% to generate the history h_t are excluded from $\mathcal{Q}_0$ at each time t. Also, if $\rho = 0.79$, the q that makes the probability of the history h_t less than 50% are excluded at each time t. The size of the set $\mathcal{Q}_t$ is larger for $\rho = 0.26$ than $\rho = 0.79$.

2.3 Behaviour of Traders in Equilibrium

The equilibrium concept used here is Perfect Bayesian equilibrium. The equilibrium is defined as the prices $\{a_t, b_t\}$, and the optimal action of traders such that (i) the informed traders without ambiguity maximizes $E(U(V, X_t, a_t, b_t)|S_t, h_t)$, (ii) the traders with ambiguity aversion maximizes their expected utility following equation (2.3), (iii) the market maker sets the ask and the bid price following equation (2.1) and (2.2), and (iv) the beliefs of the agents on the market structure and the optimal action of the traders are correct in equilibrium .

Let us look at the behaviour of the informed traders without ambiguity first.

Proposition 1. *In equilibrium, the informed traders without ambiguity buy with a good signal and sell with a bad signal at any period* t*.*

The formal proofs for proposition 1 to 7 are in the appendix. The traders without ambiguity always trade following their signal as the expected asset value with a good signal($S_t = 1$) is higher than the ask price and that with a bad signal($S_t = 1$) is lower than the bid price even though there exist ambiguity averse traders in the market.

The behaviour of the traders with ambiguity aversion, however, can be different from that of the traders without ambiguity. The other behaviour to be considered is choosing no trade even with an informative signal or buying or selling regardless of their signals, which is herd or contrarian behaviour. To analyse the possibility of these behaviour, let us define herd and contrarian behaviour precisely. It follows Avery and Zemsky (1998) and Cipriani and Guarino (2014).

Definition 1. *A trader with a private signal* S_t *engages in herd behaviour at time* t *if she buys with any signal when* $E(V|h_t) > E(V|h_1)$ *or if she sells with any signal when* $E(V|h_t) < E(V|h_1)$*.*

Also, a trader with private signal S_t *engages in contrarian behaviour at time* t *if she buys with any signal when* $E(V|h_t) < E(V|h_1)$ *or if she sells with any signal when* $E(V|h_t) > E(V|h_1)$*.*

'Herd buying' is a situation that an informed trader buys regardless of her signal and the price itself is higher than the price of initial period since the history is dominated by buy orders. 'Contrarian buying' is that the trader is buying regardless of her signal after a history which generates the price lower than the initial price since it is dominated by sell orders. 'Herd selling' and 'contrarian selling' are symmetric to these.

To begin with, we can consider the condition for the initial period to have the same action with those without ambiguity.

Proposition 2. *There exists a*

$$\tilde{q} = \frac{\mu\bar{q}/2 + (1-\mu)\epsilon/2}{\mu/2 + (1-\mu)\epsilon}$$

such that with $\underline{q} > \tilde{q}$, the ambiguity averse trader at $t = 1$ buys with a good signal and sells with a bad signal in equilibrium.

Note that the condition is about the set $\mathcal{Q}_0$, because it is a relationship between $\underline{q}$ and $\overline{q}$. If the set is too large, the ambiguity averse traders can choose no trade even in the initial period.

Starting from the initial period using proposition 2, the possibility of other behaviour in equilibrium at $t > 1$ is analysed. The behaviour of the ambiguity averse traders after $t = 1$ depends on their re-evaluation rule which is determined by ρ. To look at how the behaviour changes depending on the re-evaluation, we start with two extreme cases of the Full Bayesian Updating with $\rho = 0$ and the Maximum Likelihood Updating with $\rho = 1$.

2.3.1 Case 1: Full Bayesian Updating ($\rho = 0$)

Let us consider the case that ambiguity averse traders are using the Full Bayesian Updating since their re-evaluation parameter $\rho = 0$. In this case, there is positive probability of choosing no trade for the ambiguity averse traders in equilibrium, while herd or contrarian behaviour is not possible.

Proposition 3. *Consider $\tilde{q} < \underline{q} < \overline{q} < 1$. Suppose $\rho = 0$. In equilibrium, herd or contrarian behaviour never occurs.*

Herd or contrarian behaviour, which is buying or selling regardless of the trader's signal, is impossible for the ambiguity averse traders using the Full Bayesian Updating. To generate herd or contrarian behaviour, the traders buy or sell regardless of their signal. It means the traders in equilibrium buy with a bad signal or sell with a good signal. Hence, the expected asset value of the ambiguity averse trader with a bad signal should be higher than the ask price or the expected value with a good signal should be lower than bid price. The re-evaluated set of q for the traders is $\mathcal{Q}_0$ at any t with the Full Bayesian Updating, so the level of q used by the market maker is always within $\mathcal{Q}_0$. It means that the minimum expected value of the traders with a bad signal cannot be higher than the ask price, and the maximum expected asset value of the traders with a good signal cannot be lower than the bid price. Therefore, herd or contrarian behaviour is impossible with the Full Bayesian Updating, which has $\rho = 0$.

The other behaviour we can consider is choosing no trade even with an informative private signal of the ambiguity averse trader.

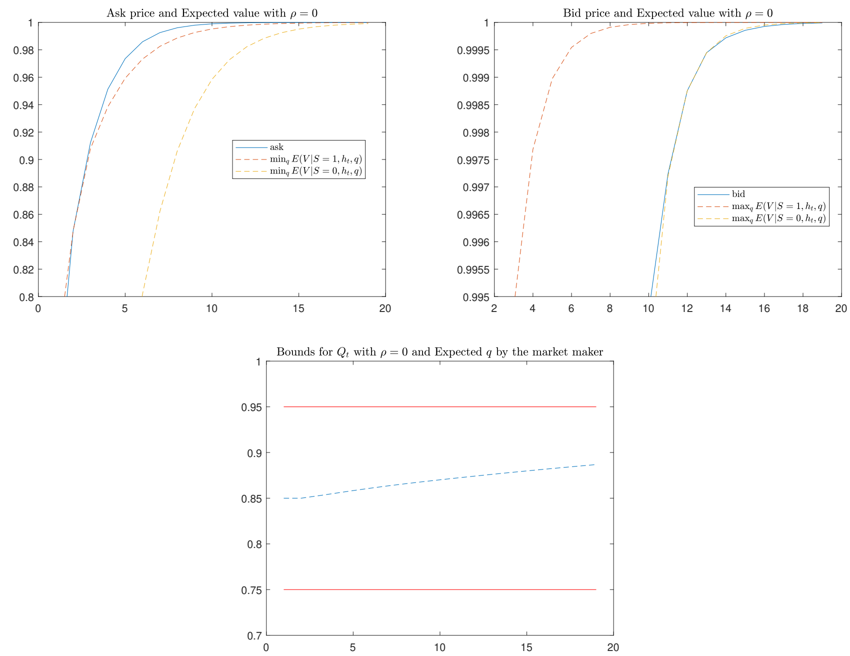
Proposition 4. *Consider $\tilde{q} < \underline{q} < \overline{q} < 1$. Suppose $\rho = 0$. In equilibrium, as t goes to infinity, the probability that the ambiguity averse trader chooses 'no trade' for any signal goes to 1 if $\underline{q} < q^T < \overline{q}$ where q^T is true precision of the private signal.*

If we assume that there is zero probability of choosing no trade regardless of their signal by the traders, the ratio between the orders with a good signal and a bad signal converges to the ratio of $\frac{q^T}{1-q^T}$ where q^T is true level of signal precision, since the private signal is independently and identically distributed. Hence, $E(q|h_t)$ converges to q^T. Suppose $V = 1$. The difference between the number of orders with a good signal and a bad signal increases as the total number of trades increases. It makes the precision of the ambiguity averse traders considering buying become $\underline{q}$ and that of the traders considering selling becomes $\overline{q}$. As the number of trades increases, the relative expected asset value of the traders evaluated at $\underline{q}$ to the ask price which is evaluated at $E(q|h_t)$ decreases, and it eventually makes the traders choose no trade with a good signal. Also, the maximum expected asset value with a bad signal is evaluated at $\overline{q}$ with sufficiently large number of orders. With similar logic of the case with a good signal, the relative expected asset value of the traders evaluated at $\overline{q}$ to the bid price increases. It makes the traders with a bad signal choose no trade in equilibrium and the assumption is violated. Therefore, as t goes to infinity, the ambiguity averse traders choose no trade with any signal in equilibrium with probability 1 under the Full Bayesian Updating. It implies that after sufficiently many periods of trades, the only traders who explicitly trade in the market are those without ambiguity and noise traders.

Example 1. *Choosing no trade with the Full Bayesian Updating*

Under the parameter values of $\gamma = 0.5, \delta = 0.5, \mu = 0.5, \underline{q} = 0.75, \bar{q} = 0.95, \epsilon = 2/3$ and a history of orders such as thirteen consecutive buys $h_{14} = \{buy_1, buy_2, \cdots, buy_{13}\}$, it can be shown that the ambiguity averse traders choose 'no trade' in equilibrium with the Full Bayesian Updating ($\rho = 0$).

Figure 2.1 The expected asset value and $\mathcal{Q}_t$



Note: The upper panels are the ask and bid price and the expected value of the asset by the ambiguity averse traders. The lower panel is bounds of $\mathcal{Q}_t$ and $E(q|h_t)$. It is simulated with a history of 13 buys under the parameter values of $\gamma = 0.5$, $\delta = 0.5$, $\mu = 0.5$, $\underline{q} = 0.75$, $\bar{q} = 0.95$, $\epsilon = 2/3$, and $\rho = 0$.

Looking at the expected asset value of the ambiguity averse traders and the ask price in upper left panel of Figure 2.1, in the case of the Full Bayesian Updating ($\rho = 0$), the ask price is higher than $\min_{q \in \mathcal{Q}_t} E(V|S_t = 1, h_t, q)$ from $t = 3$. It implies that the ambiguity averse traders with a good signal choose no trade in equilibrium. The bid price in the upper right panel becomes lower than $\max_{q \in \mathcal{Q}_t} E(V|S_t = 0, h_t, q)$ from $t = 14$, so the traders do not trade with a bad signal as well. Hence, they choose no trade with any signal. As shown in the lower panel of Figure 2.1, the set $\mathcal{Q}_t$ stays at $\mathcal{Q}_0$ for any t because of $\rho = 0$. With this constant $\mathcal{Q}_t$, the ambiguity averse traders faced with only buy orders evaluate their minimum expected asset value at the lower bound of $\mathcal{Q}_0$, and it keeps their expectation on the asset value relatively lower than the ask price. Also, the maximum expected asset value is evaluated at the upper bound of $\mathcal{Q}_0$ from $t = 12$ after 11 buy orders. It makes the expectation higher than the bid price.

2.3.2 Case 2: Maximum Likelihood Updating ($\rho = 1$)

Herd or contrarian behaviour is impossible with the Full Bayesian Updating because the updated level of q considered by the market maker is always in the set $\mathcal{Q}_t$. If the traders use the Maximum Likelihood Updating, $\mathcal{Q}_t$ shrinks even to a singleton. It opens a possibility of herding or contrarian behaviour.

Proposition 5. *Consider $\tilde{q} < \underline{q} < \overline{q} < 1$. Suppose $\rho = 1$. In equilibrium, the ambiguity averse traders herd with positive probability.*

The logic of the proof for Proposition 5 is as follows. Suppose that there is no herding. In a history only with buy orders, the expected q by the market maker increases with the ask price, while $\mathcal{Q}_t$ shrinks to $\overline{q}$ since there is a positive proportion of the traders without ambiguity who would buy with a good signal at any t, although the ambiguity averse traders sometimes choose no trade with a good signal. After this history, there could happen a situation that the expected asset value of the trader with a bad signal becomes higher than the ask price because the history is dominated with buy orders and the expected asset value of the ambiguity averse traders is evaluated at $\overline{q}$ while the market maker uses $E(q|h_t) < \overline{q}$. Also the price is higher than initial period price. It violates the assumption of no herding. Therefore, herding occurs.

Similar to the Full Bayesian Updating, we also can consider the asymptotic property of the traders' behaviour.

Proposition 6. *Consider* $\tilde{q} < \underline{q} < \overline{q} < 1$. *Suppose* $\rho = 1$. *In equilibrium, as* t *goes to infinity, the probability the ambiguity averse traders buy with a good signal and sell with a bad signal goes to* 1 *if* $\gamma > 0$.

With a positive proportion of the traders without ambiguity, the information about the private signal is always revealed with actions of the traders. It guarantees that both of $E(q|h_t)$ and $\mathcal{Q}_t$ converges to true value q^T as t goes to infinity since the signal is independently and identically distributed. When the level of precision the ambiguity averse traders and the market maker take into account become close enough to each other, the expected asset value of the traders with a good signal is higher than the ask price and that with a bad signal is lower than the bid. Therefore the ambiguity averse traders buy with a good signal and sell with a bad signal.

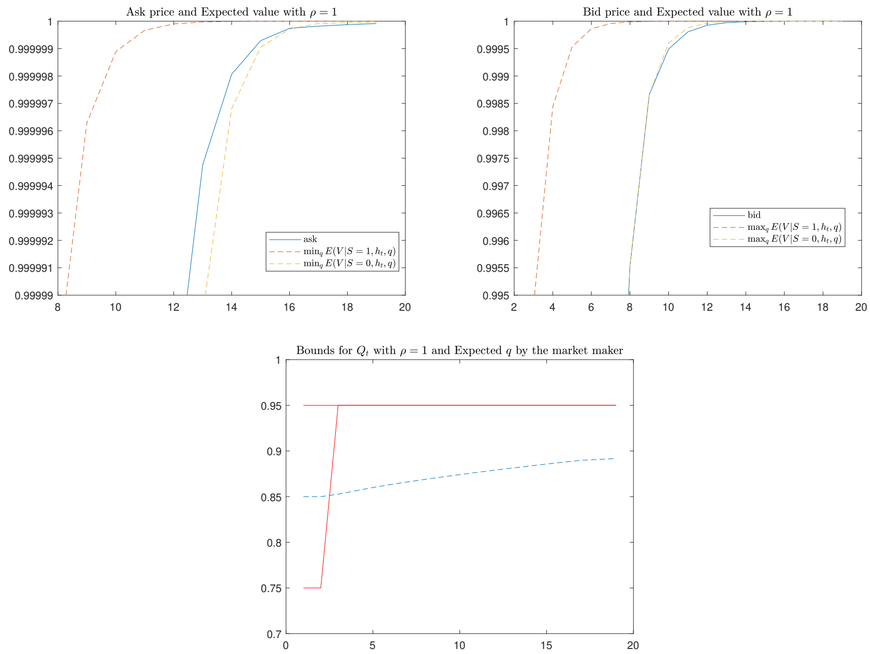
The condition for the contrarian behaviour is not straight forward as herding, but we can check its existence with an example. Followings are examples of herding and contrarian behaviour of the ambiguity averse traders with the Maximum Likelihood Updating.

Example 2. *Herding with the Maximum Likelihood Updating*

Under the parameter values of $\gamma = 0.5, \delta = 0.5, \mu = 0.5, \underline{q} = 0.75, \bar{q} = 0.95, \epsilon = 2/3$ and the history of orders of sixteen consecutive buys $h_{17} = \{buy_1, buy_2, \cdots, buy_{16}\}$, 'herd buying' happens with the Maximum Likelihood Updating ($\rho = 1$), while 'no trade' occurs with the same parameter values and history of the previous example under the Full Bayesian Updating.

In the case of the Maximum Likelihood Updating ($\rho = 1$), $\min_{q \in \mathcal{Q}_t} E(V|S_t = 1, h_t, q)$ is higher than the ask price for the entire periods. Hence, ambiguity averse traders with a good signal choose to buy in equilibrium. They do not choose no trade with a good signal in equilibrium as Example 1 in this case. Moreover, $\min_{q \in \mathcal{Q}_t} E(V|S_t = 0, h_t, q)$ is higher than the ask price from $t = 17$. It means that the ambiguity averse traders even with a bad signal choose to buy. Looking at the bid price, the maximum

Figure 2.2 The expected asset value and $\mathcal{Q}_t$



Note: The upper panels are the ask and bid price and the expected value of the asset by the ambiguity averse traders. The lower panel is bounds of $\mathcal{Q}_t$ and $E(q|h_t)$. It is simulated with a history of 16 buys under the parameter values of $\gamma = 0.5$, $\delta = 0.5$, $\mu = 0.5$, $\underline{q} = 0.75$, $\overline{q} = 0.95$, $\epsilon = 2/3$, and $\rho = 1$.

expected value with a bad signal becomes higher than the bid at $t = 17$, so selling is not chosen by the ambiguity averse traders. Therefore, buy herding happens at $t = 17$.

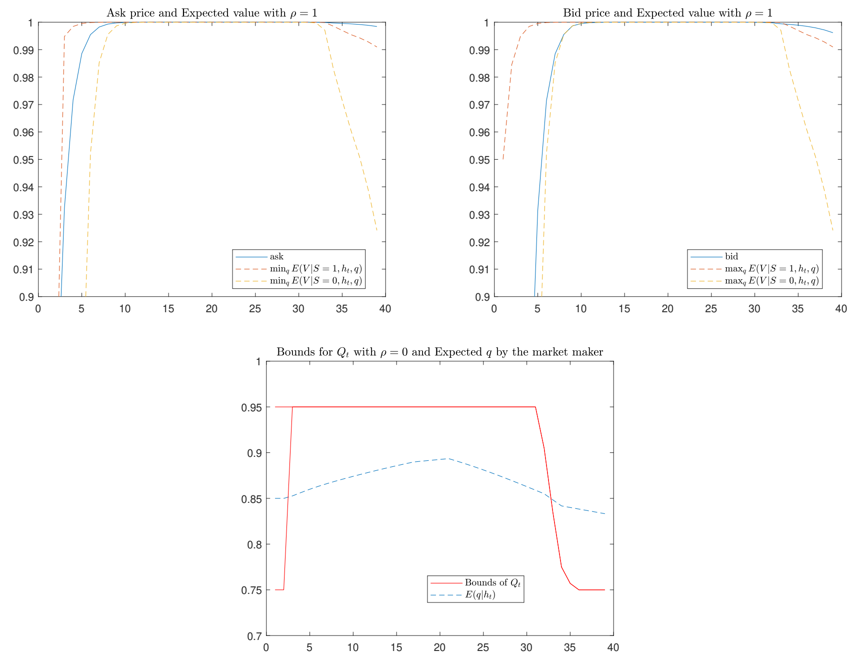
When $\rho = 1$, $\mathcal{Q}_t$ shrinks to a singleton after trades as shown on the lower panel of Figure 2.2. With the history of consecutive buys, $\mathcal{Q}_t$ becomes the upper bound of $\mathcal{Q}_0$ and it makes the expected asset value of ambiguity averse traders increase further than the ask price evaluated at the updated level of q by the market maker.

In the case of the Maximum Likelihood Updating, the contrarian behaviour also occurs. It is caused by the shrunk $\mathcal{Q}_t$ away from the expected q by the market maker, $E(q|h_t)$. Their discrepancy generates the situation that the expected asset value of the ambiguity averse trader with a good signal becomes lower than the bid price after the history dominated by buy orders.

Example 3. *Contrarian behaviour with the Maximum Likelihood Updating*

Under the same parameter values of the previous example and the history of initial 20 buy orders and 14 sell orders after that, we can find that sell contrarian occurs in equilibrium with the Maximum Likelihood Updating ($\rho = 1$).

Figure 2.3 Expected asset value and $\mathcal{Q}_t$



Note: The upper panels are the ask and bid price and expected asset values of the ambiguity averse traders, and the lower panel is $\mathcal{Q}_t$ and $E(q|h_t)$. It is simulated with a history of 20 buys and 14 sells after that under the parameter values of $\gamma = 0.5$, $\delta = 0.5$, $\mu = 0.5$, $\underline{q} = 0.75$, $\overline{q} = 0.95$, $\epsilon = 2/3$, and $\rho = 1$.

At period $t = 34$, bid price is 0.9992 while the maximum expected asset value with a good signal is 0.9985, so the ambiguity averse traders even with a good signal sells when the price is higher than 0.5. The expected q by the market maker when the contrarian happens is 0.8418 and $\mathcal{Q}_{34}$ is 0.7749. As the q used by the ambiguity averse trader is much lower than that of the market maker, the sell contrarian occurs.

We also can check the condition that choosing no trade for any signal is not possible for the ambiguity averse traders with the Maximum Likelihood Updating.

Proposition 7. *Consider $\tilde{q} < \underline{q} < \overline{q} < 1$. Given $\gamma = 0$, in equilibrium the ambiguity averse traders choosing no trade for any signal never occur if $\mathcal{Q}_t$ is a singleton.*

If $\rho = 1$, after some history of orders, $\mathcal{Q}_t$ can become a singleton. When $\mathcal{Q}_t = \{\hat{q}\}$, the minimum expected asset value of the ambiguity averse traders with a good signal is higher than that with a bad signal. If the traders are assumed to choose no trade for any signal in equilibrium and there is only ambiguity averse traders who are informed, the ask and bid price are equivalent. It means that the ask price being higher than the expected value with a good signal and the bid price being lower than that with a bad signal is impossible. Hence, choosing no trade with any signal is impossible if $\mathcal{Q}_t$ is a singleton and there are only the ambiguity averse traders among the informed.

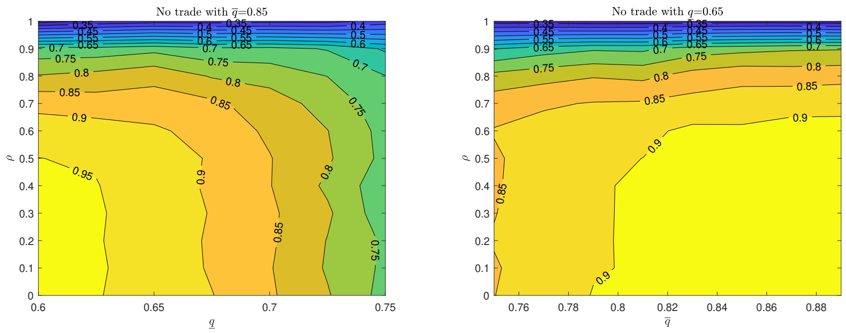
2.3.3 Case between the Full Bayesian Updating and the Maximum Likelihood Updating

In the case between the Full Bayesian Updating and the Maximum Likelihood Updating with $0 < \rho < 1$, the size of the set $\mathcal{Q}_t$ becomes smaller as ρ increases. It means $\min_{q \in \mathcal{Q}_t} E(V|S_t, h_t, q)$ becomes weakly larger and $\max_{q \in \mathcal{Q}_t} E(V|S_t, h_t, q)$ weakly smaller with higher ρ, given $Pr(V = v|h_t, q)$ is the same. Hence, the possibility of choosing no trade of the ambiguity averse traders in equilibrium decreases, while that of herding or contrarian increases with higher ρ. However, the behaviour of the ambiguity averse traders at time before t changes as well if ρ changes. It also affects $Pr(V = v|h_t, q)$. Therefore, the possibility of choosing no trade in equilibrium, or herding or contrarian behaviour depends on both of the effects by the change in ρ.

The complexity of the updated beliefs and the set $\mathcal{Q}_t$ hinders deriving analytic condition for choosing no trade, or the behaviour of herding or contrarian of the ambiguity averse traders in the case between the two extremes. To demonstrate how the behaviour of the trader changes depending on the level of ρ, simulation results are presented. The model is simulated with the true value of asset to be $V = 1$ and the prior probability of the state is $\delta = 0.5$. Half of the traders are informed ($\mu = 0.5$), and half of the informed traders are without ambiguity ($\gamma = 0.5$). True value of the precision q is set to be 0.75 if it is not noted specifically. The results are based on $1,000$ simulations of $T = 100$ for each parameter.

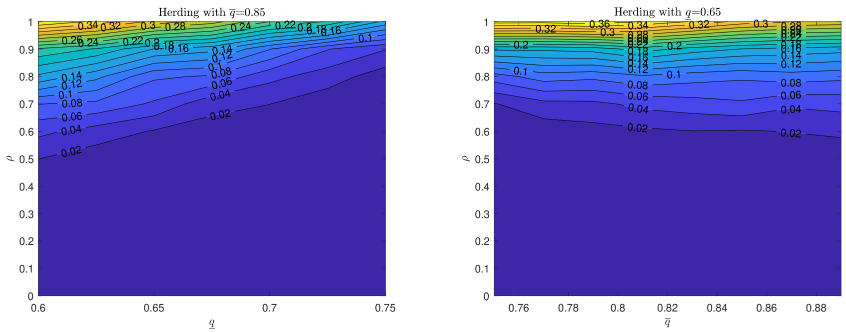
The proportion of choosing no trade, herding and contrarian are presented in figures. Figure 2.4 shows the probability of no trade depending on ρ changing the bounds of $\mathcal{Q}_0$. The left panel presents the proportion depending on ρ by changing the lower bound of

Figure 2.4 Proportion of *no trade* depending on ρ and $\mathcal{Q}_0$



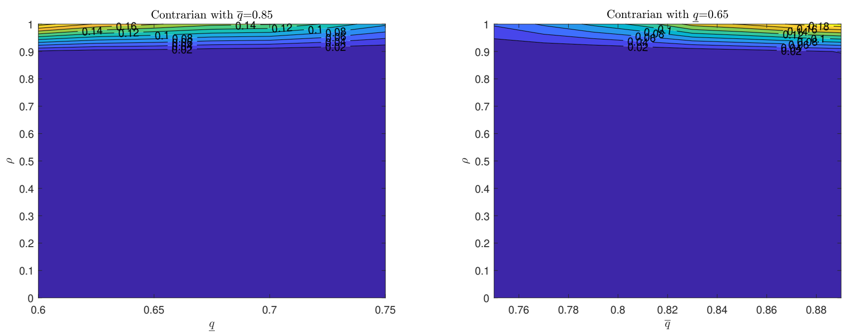
Note: The proportion is computed by the number of periods that the ambiguity averse traders choose no trade in equilibrium over $1,000$ simulations with $T = 100$ at each simulation. The numbers on the figure are the proportion of the action (Higher proportion with yellow, lower proportion with blue color). Choosing no trade includes no trade with any signal, buy with a good signal and no trade with a bad signal, no trade with a good signal and sell with a bad signal, and mixed strategy of them.

Figure 2.5 Proportion of *herding* depending on ρ and $\mathcal{Q}_0$



Note: The proportion is computed by the number of periods that herding happens in equilibrium over $1,000$ simulations with $T = 100$ at each simulation. The numbers on the figure are the proportion of herding (Higher proportion with yellow, lower proportion with blue color). Herding includes buy herding and sell herding although the proportion of sell herding is virtually zero as $V = 1$.

Figure 2.6 Proportion of *Contrarian* depending on ρ and $\mathcal{Q}_0$



Note: The proportion is computed by the number of periods that herding happens in equilibrium over $1,000$ simulations with $T = 100$ at each simulation. The numbers on the figure are the proportion of contrarian behaviour (Higher proportion with yellow, lower proportion with blue color). Contrarian behaviour includes contrarian buying and contrarian selling.

$\mathcal{Q}_0$, while the right panel by changing the upper bound. Both of the graphs display the pattern that the proportion of no trade increases as ρ becomes smaller and the size of

$\mathcal{Q}_0$ becomes larger. As discussed before, we can expect that smaller ρ keeps $\mathcal{Q}_t$ not to shrink a lot. It makes the expected asset value of the ambiguity averse traders with a good private signal lower than the ask price or the expectation with a bad signal higher than the bid price. It brings higher probability of choosing no trade by the traders.

In Figure 2.5, the proportion of herding is presented depending on ρ with changing bounds of $\mathcal{Q}_0$ as in Figure 2.4. The proportion of herding increases with higher level of ρ, smaller $\underline{q}$, and larger $\overline{q}$. It also can be understood by the cause of herding that the private signal precision of the ambiguity averse traders updates rapidly while the market maker updates conservatively. The proportion of the contrarian behaviour in Figure 2.6 shows that it occurs with high ρ, although the proportion itself is not large.

2.4 Informational Cascade

In this section, we show that the market can be in a situation that no more information is flowed into the market and the prices remain at a level away from the fundamental value. It is called informational cascade. The formal definition is from Cipriani and Guarino (2008).

Definition 2. *An informational cascade arises at time t when all informed traders act independently of their own signal.*

When the informational cascade happens, the informed trader in the market choose the same action regardless of her private signal. No information on the signal is revealed in the market with her action. That makes no change in the updated beliefs on the asset value. Therefore, the prices are not updated as well by the market maker.

Since there exists herding or contrarian behaviour, it is shown that the action of the ambiguity averse trader can be the same regardless of her signal. Also, choosing no trade regardless of the signal is possible. If the ambiguity averse traders behave herd or contrarian, or choose no trade regardless of their private signal, the action of the traders does not expose any private information. Therefore, after the action of the ambiguity averse trader, there is no informational gain in the market, which is informational cascade. If there are traders without ambiguity, however, informational cascade is not possible because they always act following their signal according to proposition 1, and it exposes their private information in the market.

If the informational cascade occurs, it lasts forever as in Cipriani and Guarino (2008). The decision problem the agents in the market faces is the same at any time t after the informational cascade happens. Therefore, all of the agents choose the same action and the prices stay the same once the cascade starts.

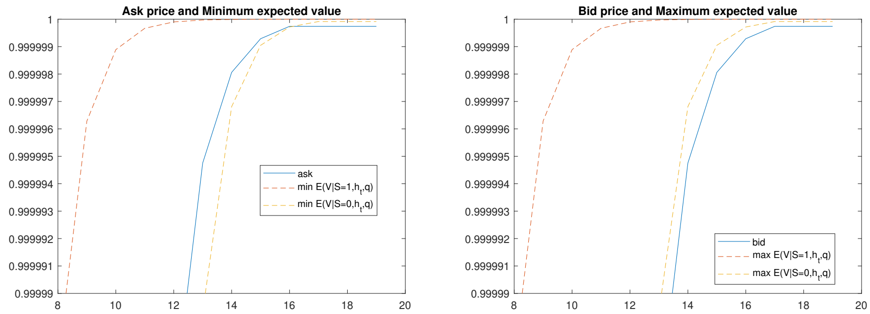
According to Proposition 4, informational cascade occurs with probability one as t goes to infinity if $\gamma = 0$ under the Full Bayesian Updating. The informational cascade is the situation that informed traders choose no trade whatever signal they get, so no one trades in the market when the informational cascade occurs as in Lee (1998), Chari and Kehoe (2004) and Cipriani et al. (2019),.

In this model, however, the informational cascade can also happen with traders keep trading. In the case of herding or contrarian behaviour, the traders continue buying or selling, but the prices do not change as no information flows to the market with the action as in the example below.

Example 4. *Informational Cascade with herding*

With the same parameter values of previous example of the Maximum Likelihood Updating excluding $\gamma = 0$, we can observe informational cascade. After 16 consecutive buys, buy herding occurs from $t = 17$. The minimum expected asset value with a good and bad signal are higher than the ask price at $t = 17$. In this situation, no information flows into the market after any action of traders. All the informed traders, who are ambiguity averse, buy regardless of their signal, and other actions come from noise traders. Hence, the prices and expected asset values stay the same after herding occurs at $t \geq 17$ even though buy order continues.

Figure 2.7 Informational Cascade with buy herding



Note: Simulation with a history of 20 buys under the parameter values of $\gamma = 0$, $\delta = 0.5$, $\mu = 0.5$, $\underline{q} = 0.75$, $\bar{q} = 0.95$, $\epsilon = 2/3$.

2.5 Conclusion

We have analysed how the behaviour of traders changes in the sequential trading model when there are ambiguity averse traders in the market. Given the ambiguity in the precision of private signal, the set of priors on the precision is re-evaluated by the recursive multiple prior preference of the ambiguity averse traders. There are two extreme cases of the re-evaluation: the Full Bayesian Updating and the Maximum Likelihood Updating depending on the re-evaluation parameter ρ.

We present that the behaviour of the ambiguity averse traders in equilibrium differs depending on the updating rule. Herding or contrarian is impossible for the ambiguity averse traders with the Full Bayesian Updating. On the other hand, herding is possible with the Maximum Likelihood Updating. As the period goes to infinity, the ambiguity averse traders with the Full Bayesian Updating tend to choose no trade even with informative signals, and the traders with the Maximum Likelihood Updating trade following their signals. In the case between the two extremes, the probability of no trade decreases with high ρ while that of herding or contrarian increases.

Chapter 3

Estimating a Model of Ambiguity Aversion in Financial Markets

3.1 Introduction

Traders in the financial market show seemingly opposite behaviour such as limited participation and herd or contrarian behaviour. From the empirical results by Mankiw and Zeldes (1991) and Haliassos and Bertaut (1995), it is claimed that there is limited participation of traders in the financial market. Even with the existence of high equity premium, large proportion of households do not hold stock. On the other hand, we often witness heightened volatility in the financial market caused by intensive participation such as herding or contrarian behaviour (Brunnermeier (2001), Chamley (2004)). In this research, it is presented that limited participation (choosing no trade in this context) and herd or contrarian behaviour can occur by ambiguity averse traders and their existence is estimated with trade data in financial market.

The ambiguity aversion is implemented by recursive multiple prior preference in this research following Epstein and Schneider (2007). It is based on multiple prior preference (also known as 'maxmin' preference) of Gilboa and Schmeidler (1989) with re-evaluating[1] set of priors. With the preference, the ambiguity averse traders can choose no trade even with an informative private signal in equilibrium as shown in Dow and Werlang (1992). The ambiguity averse traders, who do not have a proper unique prior distribution for the ambiguous parameter, consider the worst case payoff evaluated prior

[1]The terminology of re-evaluation is from Epstein and Schneider (2007).

by prior in a set of priors, while ask and bid prices are set by a market maker using Bayes updating. Therefore, the traders' expected value of an asset with a good signal can be smaller than the ask price or the expectation with a bad signal can be higher than the bid price. It makes them choose no trade in equilibrium.

In addition, if the traders with ambiguity aversion re-evaluate the possible set of the ambiguous parameter, buying (selling) regardless of any signal can occur, which is herd behaviour if the buying (selling) occurs after a history dominated by buy (sell) orders or contrarian behaviour if the selling (buying) occurs after a history dominated by buy (sell) orders. When the set of the ambiguous parameter is re-evaluated by the traders, it may become different from the expected level of the parameter by the market maker, who does not have ambiguity. If the difference between the parameter values that ambiguity averse traders consider and that of the market maker is large enough, the expected value of the asset even with a bad signal (a good signal) by the ambiguity averse trader can be higher than the ask price (lower than bid price). Then, herding or contrarian behaviour occurs.

We estimate the model by maximum likelihood estimation on trading data of two NYSE stocks, Ashland Inc. and Capital One Financial Corp. in 1995. The methodology is based on Easley et al. (1997) and Cipriani and Guarino (2014) proposed to estimate sequential trading models, and the traders with ambiguity aversion and their re-evaluation on the ambiguous parameter are added on them. Using the method, the proportion of the ambiguity averse traders, the size of the ambiguity in the private signal precision, and the way of re-evaluation are estimated. The proportion of the ambiguity averse traders is estimated to be larger for Capital One Financial Corp., and the precision of signal is re-evaluated more stringently for Ashland Inc. With the estimates, we compute the proportion of not buying or selling with any signal, and herd or contrarian behaviour in each sample. No buying or selling behaviour is occurred more likely by the ambiguity averse traders in the market trading Capital One Financial Corp. and herding or contrarian occurs more likely in the market of Ashland Inc.

Limited participation of economic agents in stock market is documented in various literature such as Mankiw and Zeldes (1991), Guiso et al. (2001) for the US and other countries. The suggested explanation for the phenomena are participation cost (Vissing-Jorgensen (2002)), loss aversion (Dimmock and Kouwenberg (2009), Ang et al. (2005)) and ambiguity aversion (Dow and Werlang (1992), Campanale (2009)). We will focus on the ambiguity aversion as a factor to generate limited participation by considering the situation that the traders do not buy or sell with any signal in equilibrium.

Herd or contrarian behaviour in financial market also has been a popular topic in literature. Banerjee (2004), Bikhchandani et al. (1992) and Welch (1992) study herd behaviour theoretically under abstract settings. Specifically, sequential trading model has an advantage of showing the behaviour of traders directly by their action. Avery and Zemsky (1998) and Cipriani and Guarino (2014) have presented that herd or contrarian behaviour can happen with additional uncertainty such as informational event uncertainty to the standard model of Glosten and Milgrom (1985). Dong et al. (2010), Ford et al. (2013) and Boortz (2016) also show that herd or contrarian behaviour can occur in equilibrium for the ambiguity averse traders within sequential trading model, although they use Choquet preference instead of recursive multiple prior preference used in this research.

The departure of this paper from the previous literature is that both of the behaviour, choosing no buying or selling, and herd or contrarian trades, are explained with the ambiguity aversion. Especially, it uses recursive multiple prior preference following Epstein and Schneider (2007). With the preference, the direct interpretation of ambiguity on the parameter of interest, which is the precision of private signal in this research, is possible. Also, with the re-evaluation embedded in the preference, the reason for the herding is explained explicitly by the gap between the traders and the market maker in the level of the ambiguous parameter they take into account.

Estimation on ambiguity is mostly studied by experiments or surveys. Halevy (2007), Ahn et al. (2014) and Gneezy et al. (2015) estimate ambiguity of individual decision makers in experiments. Dimmock and Kouwenberg (2009) uses survey on US households. On the other hand, Ilut and Schneider (2014) and Jeong et al. (2015) analyses ambiguity with data on macro variables. In this study, the transaction data on individual stock is used for estimating ambiguity. It can show the behaviour of traders more directly than macro data but not in the artificial environment as experiments.

The paper is organised as follows. After introduction, Section 2 explains the model. In Section 3, the behaviour of the informed traders in equilibrium is presented. Section 4 shows empirical results with the presented model, and Section 5 concludes.

3.2 Model

The model is based on Glosten and Milgrom (1985) with continuous signal and event-uncertainty following Cipriani and Guarino (2014). An asset is traded by traders and a market maker for trading days indexed by $d = 1, 2, 3 \cdots$. Within a day, trades occur at discrete time indexed by $t = 1, 2, 3 \cdots$.

3.2.1 The market

The market is composed of countable number of traders and a market maker. One trader is randomly assigned to trade with the market maker at each time t. She decides her action in a space of $\mathcal{A} = \{\textit{buy}, \textit{sell}, \textit{no trade}\}$. The action of the trader at time t of day d is denoted by x_t^d

3.2.2 The asset

There is one risky asset to be traded. It has a fundamental value of V_d in day d. The value stays the same during the day, but it can change in the next day. At the beginning of each day d, every agent in the market knows the value of previous day v_{d-1}. In the next day, the value of the asset can change from the previous day with probability α where $\alpha \in (0, 1]$. Following Cipriani and Guarino (2014), the day when the fundamental value does not change from the previous day, $V_d = v_{d-1}$, is called 'no-event day', which occurs with probability of $1 - \alpha$. With probability of α, the value changes and it is called 'informational event day'. In the informational event day, the value increases to $v_h^d = v_{d-1} + \lambda^h$ with probability δ and it decreases to $v_l^d = v_{d-1} - \lambda^l$ with probability of $1 - \delta$ where $\delta \in (0, 1)$. The former is a 'good event day', and the latter is a 'bad event day'. To guarantee the martingale property of the closing price, $(1 - \delta)\lambda^l = \delta\lambda^h$ is assumed.

3.2.3 The market maker

The asset is traded at ask or bid price that the market maker sets at each time t of day d. The prices are equilibrium products as they are set by the market maker taking into account the strategy of traders. Depending on the action of the traders who have a private information, the probability of buying or selling changes with the ask and bid price. The market maker is assumed to be constrained by zero expected profit condition due to a potential competition. The ask and bid price at time t of day d are denoted by a_t^d and b_t^d respectively.

$$\begin{aligned} a_t^d &= E(V|x_t^d = buy, h_t^d, a_t^d, b_t^d) \\ &= v_{d-1} + \lambda^h Pr(V_d = v_h^d|x_t^d = buy, h_t^d) - \lambda^l Pr(V_d = v_l^d|x_t^d = buy, h_t^d) \end{aligned} \tag{3.1}$$

$$\begin{aligned} b_t^d &= E(V|x_t^d = sell, h_t^d, a_t^d, b_t^d) \\ &= v_{d-1} + \lambda^h Pr(V_d = v_h^d|x_t^d = sell, h_t^d) - \lambda^l Pr(V_d = v_l^d|x_t^d = sell, h_t^d) \end{aligned} \tag{3.2}$$

a_t^d and b_t^d in expectation condition are omitted for notational simplicity afterwards.

In addition, the expected asset value of the market maker without assuming buying or selling at t of day d, $E(V|h_t^d)$ is referred to be the asset price.

3.2.4 The traders

The traders are divided into two groups: Informed traders and noise traders. The informed traders again are grouped by two: Informed with ambiguity aversion and informed without ambiguity. The type of the traders is their private information.

One trader is randomly assigned to trade at time t, and the trader is chosen only once. In the no-event day, there are only noise traders in the market. In the informational event day, there are informed traders with proportion of μ and noise traders with $1-\mu$ where $\mu \in (0,1)$. The traders without ambiguity take γ proportion of informed traders, while the ambiguity averse traders are $1-\gamma$ where $\gamma \in [0,1]$. One trader is chosen out of the three groups at each time t with probability of their proportions.

The informed traders with ambiguity aversion

The informed traders with ambiguity aversion receive a private signal which is distributed by the value-contingent densities below[2] following Cipriani and Guarino (2014).

$$
\begin{aligned}
g^h(s_t^d|v_h^d,\tau) &= 1+\tau(2s_t^d-1) \\
g^l(s_t^d|v_l^d,\tau) &= 1-\tau(2s_t^d-1)
\end{aligned} \tag{3.3}
$$

where $\tau \in [\underline{\tau}, \overline{\tau}]$ with $0<\underline{\tau}<\overline{\tau}<1$ and the support of s_t^d is $[0,1]$.

τ is a parameter represents the precision of the private signal that the informed traders receive. With high level of τ, the traders are more likely to receive high s_t^d if the asset value is high, and vice versa following the conditional density.

In the model, the only parameter considered to be ambiguous is τ. Ambiguity is implemented by *recursive multiple prior preference* following Epstein and Schneider (2007). The recursive multiple prior preference has a feature of updating the set of priors[3], added on the multiple prior preference model suggested by Gilboa and Schmeidler (1989). The agents with ambiguity have multiple prior on the ambiguous parameter τ, and they are open to update the set of priors after a history of actions. The initial set the ambiguity averse traders consider is assumed to be $\mathcal{T}_0^d = \{\tau : \underline{\tau} \leq \tau \leq \overline{\tau}\}$. This set is

[2]For simplicity's sake $V_d = v_i^d$ where $i \in \{h,l\}$ is expressed as v_i^d where $i \in \{h,l\}$ in the conditioning in probabilities and densities afterwards.

[3]It is named as re-evaluation in Epstein and Schneider (2007)

re-evaluated after history of actions following equation (3.4) according to Epstein and Schneider (2007).

$$\mathcal{T}_t^d = \{\tau : \frac{Pr(h_t^d|\tau)}{\max_{\hat{\tau}\in\mathcal{T}_0^d} Pr(h_t^d|\hat{\tau})} \geq \rho,\, \tau \in \mathcal{T}_0^d\} \tag{3.4}$$

The parameter $\rho \in [0,1]$ decides how stringently the agents re-evaluate the set to consider. In the extreme case of $\rho = 0$, the condition in equation (3.4) is satisfied for all the τ in $\mathcal{T}_0^d$. Therefore, $\mathcal{T}_t^d = \mathcal{T}_0^d$ at any time t of day d. It is the Full Bayesian Updating in Gilboa and Marinacci (2011) and De Filippis et al. (2017). The expected value of the asset is updated prior by prior in the whole set of priors $\mathcal{T}_0^d$ and the action is chosen based on the maxmin preference.

In the opposite case of $\rho = 1$, the set would be $\mathcal{T}_t^d = \{\tau : \arg\max_{\tau\in\mathcal{T}_0} Pr(h_t^d|\tau)\}$, only the τ that has maximum probability of generating the history h_t^d becomes the element of the set. It is like the Maximum Likelihood Updating in Gilboa and Marinacci (2011) and De Filippis et al. (2017). In this case, the set $\mathcal{T}_t$ may become a singleton after some history of actions, and ambiguity disappears.

If ρ is in between zero and one, the interpretation of the parameter ρ follows the likelihood ratio test. $-2\log\rho$ is the critical value of $\chi^2(1)$ distribution for rejecting the hypothesis that $\tau = \hat{\tau}$. It means that the traders are excluding the level of τ out of the confidence interval with critical value $-2\log\rho$ from $\mathcal{T}_t^d$. For example, when ρ is 0.26, the significance level for choosing set $\mathcal{T}_t^d$ is 10%, so only the τ which are thought to have a probability more than 10% to generate the history h_t^d survive. If $\rho = 0.79$, the τ that makes the probability of the history h_t^d less than 50% are excluded at each time t. The size of the set $\mathcal{T}_t^d$ is larger for $\rho = 0.26$ than $\rho = 0.79$.

The ambiguity averse traders also update their beliefs on the probability of each state in V_d by Bayes updating. The updated belief itself, however, is a function of τ since it needs to be evaluated at each level of τ in set $\mathcal{T}_t^d$. From the set, the ambiguity averse traders consider the τ which brings the worst outcome when they make their decision, because they are averse on the ambiguity. This is why it is also called maxmin preference. Following this property, their likelihood ratio of a good and bad event day is a function of τ and they consider minimum value of the ratio if they buy, and maximum value if they sell with τ in $\mathcal{T}_t^d \subset \{\tau : \underline{\tau} \leq \tau \leq \overline{\tau}\}$.

$$\begin{cases} \min_{\tau \in \mathcal{T}_t^d} \frac{Pr(v_h^d|s_t^d,h_t^d,\tau)}{Pr(v_l^d|s_t^d,h_t^d,\tau)} = \min_{\tau \in \mathcal{T}_t^d} \frac{g^h(s_t^d|v_h^d,\tau)Pr(v_h^d|h_t^d,\tau)}{g^l(s_t^d|v_l^d,\tau)Pr(v_l^d|h_t^d,\tau)} \text{ for buying} \\ \max_{\tau \in \mathcal{T}_t^d} \frac{Pr(v_h^d|s_t^d,h_t^d,\tau)}{Pr(v_l^d|s_t^d,h_t^d,\tau)} = \max_{\tau \in \mathcal{T}_t^d} \frac{g^h(s_t^d|v_h^d,\tau)Pr(v_h^d|h_t^d,\tau)}{g^l(s_t^d|v_l^d,\tau)Pr(v_l^d|h_t^d,\tau)} \text{ for selling} \end{cases}$$

Monotonic Likelihood Ratio Property (MLRP) satisfies for each case of considering buying or selling. Thus, we can define the signal higher than 0.5 is a good signal and lower than that is a bad signal.

The payoff function of the ambiguity averse traders is as follows.

$$U(V_d, x_t^d, a_t^d, b_t^d) = \begin{cases} V_d - a_t^d & \textit{if } x_t^d = \textit{buy} \\ 0 & \textit{if } x_t^d = \textit{no trade} \\ b_t^d - V_d & \textit{if } x_t^d = \textit{sell} \end{cases}$$

According to maxmin property, the traders are maximizing $\min_{\tau \in \mathcal{T}_t^d} E[U(V_d, X_t^d, a_t^d, b_t^d)|s_t^d, h_t^d, \tau]$. Hence, the ambiguity averse trader chooses her action as follows.

$$\begin{aligned} &\max_{\{\textit{buy,sell,no trade}\}} [\min_{\tau \in \mathcal{T}_t^d} \{E(V_d|h_t^d, s_t^d, \tau) - a_t^d\},\ \min_{\tau \in \mathcal{T}_t^d} \{b_t^d - E(V_d|h_t^d, s_t^d, \tau)\},\ 0] \\ &= \max_{\{\textit{buy,sell,no trade}\}} [\min_{\tau \in \mathcal{T}_t^d} \{E(V_d|h_t^d, s_t^d, \tau)\} - a_t^d,\ b_t^d - \max_{\tau \in \mathcal{T}_t^d} \{E(V_d|h_t^d, s_t^d, \tau)\},\ 0] \end{aligned}$$

From these, the trading decision can be summarized to two thresholds $\beta_{2,t}^d$ and $\sigma_{2,t}^d$.

$$\min_{\tau \in \mathcal{T}_t^d} [E(V_d|h_t^d, \beta_{2,t}^d, \tau)] = a_t^d$$

$$\max_{\tau \in \mathcal{T}_t^d} [E(V_d|h_t^d, \sigma_{2,t}^d, \tau)] = b_t^d$$

The trader with ambiguity aversion buys if she receives a signal higher than $\beta^d_{2,t}$, and sells if the signal is lower than $\sigma^d_{2,t}$. If the signal is between $\beta^d_{2,t}$ and $\sigma^d_{2,t}$, she does not trade.

The informed traders without ambiguity

Informed traders without ambiguity, which comprises $\mu\gamma$ share of all the traders in the informational event day, receive a private signal s^d_t. It follows value-contingent densities as that of the traders with ambiguity in equation (3.3).

The traders, like the market maker, are not assumed to have ambiguity, although they do not know the true value of τ. They are assumed to have a uniform distribution on domain of τ, $\mathcal{T}^d_0$, as a prior for τ, and update their beliefs on τ by Bayesian updating.

The probability of the asset value v_d is updated by Bayesian updating as well. The ratio of a good and bad event day satisfies MLRP, as in the case of the ambiguity averse traders. From this property, the signal $s^d_t > 0.5$ is defined as a good signal and $s^d_t < 0.5$ as a bad signal for the traders without ambiguity as well.

$$\frac{Pr(v^d_h|h^d_t, s^d_t)}{Pr(v^d_l|h^d_t, s^d_t)} = \frac{\int_{\underline{\tau}}^{\overline{\tau}} g^h(s^d_t|v^d_h, \tau) f(\tau|h_t) \mathrm{d}\tau Pr(v^d_h|h^d_t)}{\int_{\underline{\tau}}^{\overline{\tau}} g^l(s^d_t|v^d_l, \tau) f(\tau|h_t) \mathrm{d}\tau Pr(v^d_l|h^d_t)}$$

The payoff function of the informed trader without ambiguity is the same as the ambiguity averse traders. The traders decide their action by maximizing the expected payoff, $E[U(V_d, x^d_t, a^d_t, b^d_t)|s^d_t, h^d_t]$. Hence, the informed traders buy when $E(V_d|s^d_t, h^d_t) > a^d_t$, sell when $E(V_d|s^d_t, h^d_t) < b^d_t$, and do not trade otherwise.

From the decision rule of the informed traders given above, the trading decision is summarised as the thresholds of the signal for buying and selling as $\beta^d_{1,t}$ and $\sigma^d_{1,t}$ respectively.

$$E(V_d|\beta^d_{1,t}, h^d_t) = a^d_t$$

$$E(V_d|\sigma^d_{1,t}, h^d_t) = b^d_t$$

The informed traders without ambiguity will buy if their private signal s_t^d is higher than $\beta_{1,t}^d$ and sell if s_t^d is lower than $\sigma_{1,t}^d$.

The noise traders

The noise traders are assigned randomly by the probability of $\epsilon/2$, $\epsilon/2$ and $1-\epsilon$ respectively to buy, sell and no trade, with $\epsilon \in (0,1)$. As there are only noise traders in no-event day, the buy, sell or no trade occurs according to the probabilities. On the informational event day, $1-\mu$ share of noise traders are in the market. Therefore, the proportion of each order from noise traders is $(1-\mu)\epsilon/2$, $(1-\mu)\epsilon/2$ and $(1-\mu)(1-\epsilon)$.

3.3 Behaviour of Informed Traders in Equilibrium

The action of the informed traders in equilibrium is a fixed point problem about the set of four thresholds $\{\beta_{1,t}^d, \beta_{2,t}^d, \sigma_{1,t}^d, \sigma_{2,t}^d\}$ which satisfies the following equations.

$$E(V_d|\beta_{1,t}^d, h_t^d) = \min_{\tau \in \mathcal{T}_t} E(V|\beta_{2,t}^d, h_t^d, \tau) = a_t^d \tag{3.5}$$

$$E(V_d|\sigma_{1,t}^d, h_t^d) = \max_{\tau \in \mathcal{T}_t} E(V|\sigma_{2,t}^d, h_t^d, \tau) = b_t^d \tag{3.6}$$

$\beta_{1,t}^d$ and $\sigma_{1,t}^d$ are for the traders without ambiguity, and $\beta_{2,t}^d$ and $\sigma_{2,t}^d$ for those with ambiguity aversion. The thresholds are decided by the updated beliefs on the fundamental value V_d by the traders and the market maker.

First of all, the behaviour of the traders without ambiguity is analysed.

Proposition 8. *Suppose* $\alpha = 1$. *In equilibrium,* $0.5 \leq \beta_{1,t}^d < 1$ *and* $0 < \sigma_{1,t}^d \leq 0.5$ *at any period t.*

Proofs for all of the propositions in this chapter are in the appendix. Proposition 8 is proven by showing that $\beta_{1,t}^d$ cannot be smaller than 0.5. It uses the property that $\beta_{1,t}^d$ is increasing function in $\beta_{2,t}^d$ and the lower bound of $\beta_{1,t}$ with $\beta_{2,t}^d = 0$ is 0.5. Proposition 8 is, in other words, that the traders without ambiguity buy with a good signal and sell with a bad signal when there is no event uncertainty ($\alpha = 1$) since the traders buy with the signals higher than $\beta^d_{1,t}$ and sell with a signal lower than $\sigma^d_{1,t}$.

Herd or contrarian behaviour of traders with event-uncertainty is well established in Avery and Zemsky (1998) and Cipriani and Guarino (2014). Without the uncertainty, the traders without ambiguity trade follow their signals. In the case with ambiguity aversion, however, it can be shown that the behaviour of herding or contrarian can occur. Also, the behaviour of not buying or selling with an informative signal.

To analyse the behaviour of the traders, 'herd' and 'contrarian' behaviour are defined following Cipriani and Guarino (2014).

Definition 3. *There is herd behaviour at time t of day d when there is a positive measure of signal realisations for which an informed trader either herd buys or herd sells, that is when*

$$\beta^d_{i,t} < 0.5 \textit{ with } E(V_d|h^d_t) > E(V_d|h^d_1) \textit{ or}$$

$$\sigma^d_{i,t} > 0.5 \textit{ with } E(V_d|h^d_t) < E(V_d|h^d_1) \textit{ for any } i \in \{1,2\}$$

Also, there is contrarian behaviour at time t of day d when there is a positive measure of signal realisations for which an informed trader either contrarian buys or contrarian sells, that is when

$$\beta^d_{i,t} < 0.5 \textit{ with } E(V_d|h^d_t) < E(V_d|h^d_1) \textit{ or}$$

$$\sigma^d_{i,t} > 0.5 \textit{ with } E(V_d|h^d_t) > E(V_d|h^d_1) \textit{ for any } i \in \{1,2\}$$

Herd buy is a behaviour that the informed traders buy even with a bad signal after a history which makes the price higher than the initial price since the buy orders dominate the history. Contrarian buy, on the other hand, is also the behaviour that the informed traders buy even with a bad signal, although the history is dominated by sell orders to reduce the price lower than the initial price.

In addition, we can define 'no buying', 'no selling' or 'no trading' behaviour which is the informed traders do not buy or sell with any signal realisation.

Definition 4. *There is no buying behaviour at time t of day d when*

$$\beta^d_{i,t} = 1 \textit{ with } 0 < \sigma^d_{i,t} \leq 0.5$$

and there is no selling behaviour at time t of day d when

$$\sigma^d_{i,t} = 0 \textit{ with } 0.5 \leq \beta^d_{i,t} < 1 \textit{ for any } i \in \{1,2\}$$

Also, there is no trading behaviour at time t of day d when

$$\beta^d_{i,t} = 1 \textit{ and } \sigma^d_{i,t} = 0 \textit{ for any } i \in \{1,2\}$$

The support for the thresholds $\beta^d_{i,t}$ and $\sigma^d_{i,t}$ and the private signal s^d_t are $[0,1]$. Hence, when $\beta_{i,t}$ is one, there is no probability of getting a signal higher than $\beta^d_{i,t}$. It makes that there is zero probability of an informed trader choosing to buy with $\beta^d_{i,t} = 1$. The case for no selling is symmetric. With $\sigma^d_{i,t} = 0$, no informed trader chooses to sell.

Having the definition of the behaviours, let us start to consider the condition for initial period that the ambiguity averse traders trade following their signal.

Proposition 9. *Suppose $\alpha = 1$. There exists a*

$$\tilde{\tau} = \overline{\tau}\frac{\mu\gamma + 2(1-\mu)\epsilon - 4\sqrt{(1-\mu)\epsilon\{\mu\gamma + (1-\mu)\epsilon/2\}/2}}{\mu\gamma - 4(1-\mu)\epsilon}$$

such that with $\underline{\tau} > \tilde{\tau}$, $0.5 \leq \beta^d_{2,1} < 1$ and $0 < \sigma^d_{2,1} \leq 0.5$ in equilibrium.

The ambiguity averse traders show different behaviour depending on their re-evaluation rule ρ after the initial period. By looking at the two extreme cases of the Full Bayesian Updating and the Maximum Likelihood Updating, the effect of the re-evaluation is analysed.

3.3.1 The Full Bayesian Updating $\rho = 0$

Let us consider the case with the Full Bayesian Updating with $\rho = 0$ first. Under the Full Bayesian Updating, the set $\mathcal{T}_t$ is constant at $\mathcal{T}_0$ for any t. In this case, herd or contrarian behaviour does not occur without the event uncertainty ($\alpha = 1$).

Proposition 10. *Consider $\tilde{\tau} < \underline{\tau} < \overline{\tau} < 1$. Suppose $\alpha = 1$ and $\rho = 0$. In equilibrium, herd or contrarian behaviour never occur.*

If ρ is zero, $\mathcal{T}_t^d$ is always same as $\mathcal{T}_0^d$. It makes the set include the expected τ by the market maker all the time. If the informed trader and the market maker use the same precision τ, the expectation of a trader with a bad signal is always lower than the ask price. It implies that the minimum expected value with a bad signal can never be higher than the ask price as the τ considered by the market maker is always in the set $\mathcal{T}_0^d$. Therefore, herding or contrarian behaviour cannot happen with $\rho = 0$.

Also, we can check the possibility of no buying or no selling behaviour of the ambiguity averse traders with the Full Bayesian Updating.

Proposition 11. *Consider $\tilde{\tau} < \underline{\tau} < \overline{\tau} < 1$. Suppose $\alpha = 1$, $\gamma = 0$ and $\rho = 0$. In equilibrium, no buying or no selling behaviour occurs with positive probability for any $\underline{\tau}$ and $\overline{\tau}$.*

The sketch for the proof is as follows. If we suppose that $\beta_{2,t}^d < 1$ for any t and all the informed traders are ambiguity averse, $\beta_{2,t}^d$ keeps increasing after a consecutive buys. Since the minimum asset value is evaluated at $\underline{\tau}$ by the traders, the expected τ used by the market maker is always higher than that. It makes $\beta_{2,t}^d$ higher to satisfy Equation (3.5). $\beta_{2,t}^d$ increases until $\beta_{2,t}^d = 1$, and this contradicts the assumption of zero possibility of choosing no buying with any signal. Hence, with a positive probability, no buying with any signal occurs. For the case of no selling, it is symmetric to this.

3.3.2 The Maximum Likelihood Updating $\rho = 1$

In the case of the Maximum Likelihood Updating with $\rho = 1$, the probability of herding is positive since $\mathcal{T}_t^d$ shrinks from $\mathcal{T}_0^d$.

Proposition 12. *Consider $\tilde{\tau} < \underline{\tau} < \overline{\tau} < 1$. Suppose $\alpha = 1$, $\gamma = 0$ and $\rho = 1$. In equilibrium, herd behaviour occurs with positive probability.*

After a history of only buys, $\mathcal{T}_t$ becomes a singleton $\{\overline{\tau}\}$, and $E(\tau|h_t^d) < \overline{\tau}$ for any finite t. If there is sufficient number of buy orders, there happens a situation that, even with a signal lower than 0.5, the expected asset value of the ambiguity averse trader becomes higher than the ask price since the value is evaluated at $\overline{\tau}$ by the ambiguity averse traders, while the market maker uses $E(\tau|h_t^d)$. In addition, the price becomes higher than the initial price. Therefore, herding occurs.

When the set $\mathcal{T}_t$ shrinks to a singleton $\{\hat{\tau}\}$ after a history of trades with the Maximum Likelihood Updating, the probability of no trading becomes zero.

Proposition 13. *Consider $\tilde{\tau} < \underline{\tau} < \overline{\tau} < 1$. Suppose $\alpha = 1$, $\gamma = 0$ and $\rho = 1$. In equilibrium, no trading behaviour never occurs if $\mathcal{T}_t$ is a singleton.*

In the case that the only informed traders are the ambiguity averse traders ($\gamma = 0$), suppose that no trading behaviour happen ($\beta_{2,t}^d = 1$, $\sigma_{2,t}^d = 0$), then the ask and bid price are the same since no informed trader will trade. If $\mathcal{T}_t$ is a singleton $\{\hat{\tau}\}$, the expected asset value of the trader is evaluated at $\hat{\tau}$ regardless of the considered action. The thresholds $\beta_{2,t}^d$ and $\sigma_{2,t}^d$, which satisfy Equation (3.5) and (3.6), are 0.5. It is different from the assumed level of thresholds $\beta_{2,t}^d = 1$ and $\sigma_{2,t}^d = 0$. Therefore, no trading behaviour cannot occur in equilibrium.

The main driver of the behaviour such as no buying or no selling, herd or contrarian behaviour is the difference in the level of τ considered by the ambiguity averse traders and the market maker. No buying or no selling behaviour occurs when $E(\tau|h_t^d)$ stays inside the $\mathcal{T}_t$, and herding or contrarian happens when $E(\tau|h_t^d)$ is away from $\mathcal{T}_t$. Hence, in the case between the Full Bayesian Updating and the Maximum Likelihood Updating, the level of ρ determines the probability of these behaviours. If ρ is higher, $\mathcal{T}_t^d$ shrinks with increased lower bound, and the possibility of choosing no trade in equilibrium becomes smaller. As for herding or contrarian, there is more possibility of the set becoming away from the expected τ of the market maker with higher ρ. Therefore, herding or contrarian is more likely to occur.

3.4 Empirical Analysis

Using the trading model with ambiguity aversion, financial market data are estimated structurally by the maximum likelihood method. The estimation results show the proportion of ambiguity averse traders in the market and how they re-evaluate the set of ambiguous parameter.

3.4.1 Likelihood Function

To estimate the model with maximum likelihood method, the likelihood function is specified. The derivation of the function follows Cipriani and Guarino (2014). The likelihood function is written by the history of orders. In this model, there is a one-to-one mapping from history of trades to ask and bid prices, so adding prices in the likelihood function is redundant.

Denoting the history of actions at the end of a day by h^d, the likelihood function is written as

$$\mathcal{L}(\Phi; \{h^d\}_{d=1}^D) = Pr(\{h^d\}_{d=1}^D | \Phi) \tag{3.7}$$

where $\Phi = \{\alpha, \delta, \mu, \gamma, \tau^T, \underline{\tau}, \overline{\tau}, \epsilon, \rho\}$ is a vector of parameters, where τ^T is the true level of precision of the private signal. If we remind that all the agents in the market are aware of the value of the asset on the previous day, v_{d-1}, the sequence of trades on day d only depends on V_d. It makes the likelihood function (3.7) as a product of each day's trading history

$$\mathcal{L}(\Phi; \{h^d\}_{d=1}^D) = Pr(\{h^d\}_{d=1}^D|\Phi) = \prod_{d=1}^{D} Pr(h^d|\Phi) \tag{3.8}$$

Looking at the probability of a day of trading, the probability of an action at time t depends on the path of history. Not only the number of each buy, sell and no trade actions in history h_t^d but also the sequence of them affect the equilibrium, and it decides the probability of actions at time t. Thus, the probability of history of actions is calculated recursively.

$$Pr(h_{t+1}^d|\Phi) = Pr(x_t^d|h_t^d, \Phi)Pr(h_t^d|\Phi) \tag{3.9}$$

The probability of action x_t^d at time t of day d, $Pr(x_t^d|h_t^d, \Phi)$, is decided by the proportion of traders who choose each action $x_t^d \in \{buy_t^d, sell_t^d, no\ trade_t^d\}$ after history of actions h_t^d. It can be expressed as follows using the law of total probability.

$$\begin{aligned} Pr(x_t^d|h_t^d, \Phi) =& Pr(x_t^d|v_h^d, h_t^d, \Phi)Pr(v_h^d|h_t^d, \Phi) \\ &+ Pr(x_t^d|v_l^d, h_t^d, \Phi)Pr(v_l^d|h_t^d, \Phi) \\ &+ Pr(x_t^d|v_{d-1}^d, h_t^d, \Phi)Pr(v_{d-1}^d|h_t^d, \Phi) \end{aligned} \tag{3.10}$$

Let us consider a case of $x_t^d = buy_t^d$ to see how the probabilities in equation (3.10) are calculated. After a history of actions h_t^d, ask price at time t is derived satisfying the following condition from equation (3.5).

$$\begin{aligned} a_t^d &= E(V_d|buy_t^d, h_t^d, a_t^d, b_t^d) \\ &= E(V_d|\beta_{1,t}^d, h_t^d) = \min_{\tau \in \mathcal{T}_t^d} E(V_d|\beta_{2,t}^d, h_t^d, \tau) \end{aligned}$$

The thresholds $\beta_{1,t}^d$ and $\beta_{2,t}^d$ are equilibrium products derived at t. With these at hand, the probability of buying at time t can be calculated as below.

$$\begin{aligned}
Pr(buy_t^d|v_h^d, h_t^d, \Phi) =&\mu\gamma \int_{\beta_{1,t}^d}^1 g^h(s_t^d|v_h, \tau^T)\mathrm{d}s_t^d Pr(v_h^d|h_t, \Phi) \\
&+ \mu(1-\gamma)\int_{\beta_{2,t}^d}^1 g^h(s_t^d|v_h, \tau^T)\mathrm{d}s_t^d Pr(v_h^d|h_t, \Phi) + (1-\mu)\frac{\epsilon}{2} \\
=&\mu\gamma(1-\beta_{1,t}^d)(1-\tau^T\beta_{1,t}^d) + \mu(1-\gamma)(1-\beta_{2,t}^d)(1-\tau^T\beta_{2,t}^d) \\
&+ (1-\mu)\frac{\epsilon}{2}
\end{aligned}$$

Note that true level of τ^T is used to calculate $Pr(buy_t^d|v_h^d, h_t^d, \Phi)$ here instead of $E(\tau|h_t^d)$ used for the market maker to calculate $Pr(buy_t^d|v_h^d, h_t^d)$ in ask price in equation (3.1). The probability of selling is calculated following the same procedure.

$$\begin{aligned}
Pr(sell_t^d|v_h^d, h_t^d, \Phi) =&\mu\gamma \int_0^{\sigma_{1,t}^d} g^h(s_t^d|v_h, \tau^T)\mathrm{d}s_t^d Pr(v_h^d|h_t, \Phi) \\
&+ \mu(1-\gamma)\int_0^{\sigma_{2,t}^d} g^h(s_t^d|v_h, \tau^T)\mathrm{d}s_t^d Pr(v_h^d|h_t, \Phi) + (1-\mu)\frac{\epsilon}{2} \\
=&\mu\gamma\sigma_{1,t}^d\{1-\tau^T(1-\sigma_{1,t}^d)\} + \mu\gamma\sigma_{2,t}^d\{1-\tau^T(1-\sigma_{2,t}^d)\} \\
&+ (1-\mu)\frac{\epsilon}{2}
\end{aligned}$$

Also, the probability of no trade is

$$Pr(no\ trade_t^d|v_h^d, h_t^d, \Phi) = 1 - Pr(buy_t^d|v_h^d, h_t^d, \Phi) - Pr(sell_t^d|v_h^d, h_t^d, \Phi)$$

The case of a bad event day ($V_d = v_l^d$) can be calculated in similar way. In the no informational event day ($V_d = v_{d-1}$), it is simple to compute these probabilities since there are only noise traders in the market. The probability of buy, sell or no trade in the no event day is $\epsilon/2$, $\epsilon/2$ or $1-\epsilon$ respectively. The last term to compute in equation (3.10) is the probability of each state at time t of day d, $Pr(V_d|h_t^d, \Phi)$. It is also computed recursively from the history of actions using Bayes theorem.

From the model presented here, we can think of how each parameter is estimated intuitively, although the likelihood is determined with all parameters combined. The informational uncertainty α decides the ratio between informational event day and no-event day. In no-event day, the traders' action is only governed by ϵ mechanically. In informational event day, the action of traders is different from that of no-event day since only $1-\mu$ share of traders are noise traders. The ratio between a good event day and a bad event day is decided by δ, so if the proportion of a day dominated by buy action is higher, δ is more likely to be high. High level of τ^T, in combination with $\underline{\tau}$ and $\overline{\tau}$, generates more precise signals, so the action of informed traders will be in the right direction. If there are more traders with ambiguity aversion, γ is low, and it brings more no trade or herding and contrarian behaviour relative to the case dominated by traders without ambiguity together with larger gap between $\underline{\tau}$ and $\overline{\tau}$. The re-evaluation parameter ρ determines the proportion between herding or contrarian and no trade. With low level of ρ, no trade is chosen more likely and with high ρ herding or contrarian behaviour happens more likely as explained in the previous section.

3.4.2 Data

Following Easley et al. (1997) and Cipriani and Guarino (2014), TAQ (Trades and Quotes) dataset is used to estimate the model. The data we need to compute the likelihood are history of orders, which are sequences of buy, sell or no trade action. The TAQ dataset is composed of Quotes and Trades. Quotes data provide ask and bid prices with their time to be quoted, and trades data contain records of occurred trades with its transaction prices and time. By matching the quotes and trades, the action at each time can be determined following Lee and Ready (1991) and Cipriani and Guarino (2014)[4].

The trade data itself does not have the sign of trade, such as buying or selling of traders. Therefore, a trade is denoted as buying if the trade price is higher than the mid-point of ask and bid at that time. If the price is lower than the mid-point, it is

[4]As there is a delay in reporting transaction prices, each quote is moved ahead by five seconds following Lee and Ready (1991). Also, the opening trades are excluded as the mechanism of the trade is different from the trades during the day.

denoted to be selling. When the price is just same as the mid-point of ask and bid, it is compared with the previous price. If the price has risen, it is thought to be buy order; if it has fallen, the order is selling. There is no direct information on no trade in the TAQ dataset. Hence, the methodology of inserting no-trade if the time elapsed between explicit trades is longer than a predetermined time interval is used following Easley et al. (1997). The unit of time interval considered to be no trade is calculated by the average time elapsed between the explicit trades.

Using the methods explained above, the data on Ashland Inc. and Capital One Financial Corp. in 1995 are estimated. There were 252 days of trading in 1995. The average number of explicit trades per day on Ashland Inc. is 90 in 1995. It makes the unit of interval to be 259 seconds. Thus, if the interval between two explicit trades is longer than 259 seconds, one 'no trade' is assigned and if it is longer than 518 seconds, two 'no trades' are assigned and so on. Including the no trade, the average trades per day is 149 for Ashland Inc. Following the same methodology, the average trade per day for Capital One Financial Corp. in 1995 is 206 for the same 252 days.

Table 3.1 Trading data

Stock	buys	sells	no trades	total	buy/sell ratio
Ashland Inc.	44.7	45.8	58.8	149.2	1.03
	(42)	(44)	(60)	(146.5)	(0.96)
Capital One Financial Corp.	68.9	56.4	81.2	206.5	1.48
	(60)	(55)	(82)	(200.5)	(1.18)

Note: Average of 252 days in 1995. Numbers in paretheses are median.

The trade data on Ashland Inc. are used by previous literature such as Easley et al. (1997) and Cipriani and Guarino (2014), so the estimation results can be compared with them. Capital One Financial Corp. in 1995 has similar features with Ashland Inc. such as listed on NYSE, similar level of asset value and number of trading, except the year they entered the market. Ashland Inc. was founded in 1924 while Capital One Financial Corp. was in 1994. The gap between them is expected to bring a difference in the ambiguity on signal.

3.4.3 Estimation results

The vector of parameters $\Phi = \{\alpha, \delta, \mu, \gamma, \tau^T, \underline{\tau}, \overline{\tau}, \epsilon, \rho\}$ is estimated by maximum likelihood estimator using the likelihood function explained in the previous section. Looking at the results for Ashland Inc. first, estimates and standard deviations of the nine parameters are in Table 3.2. The informational event probability α is estimated to be 0.28 meaning the fundamental value of the asset changes with a probability a little less than one third. The probability of good event among informational event days, δ, is 0.55 slightly larger than a half. The proportion of informed traders μ is estimated to be 0.46. The estimate for the share of traders without ambiguity γ is 0.79. We can say that around 20% of the informed traders are estimated to be ambiguity averse in this market. It is around 10% among the all traders, including noise traders. The set of $\mathcal{T}_0$ is not small as $\underline{\tau} = 0.22$ and $\overline{\tau} = 0.73$. The re-evaluation parameter ρ, however, is high at 0.99. It means that the ambiguity averse traders only consider the values of τ which have more than 90% of probability to generate the history. Also, it implements that the traders re-evaluate the set $\mathcal{T}_t^d$ so strictly that the τ considered in their decision making is almost close to the Maximum Likelihood Updating. The parameter for the probability of noise trader to buy or sell (ϵ) is estimated to be 0.57, which is, among the noise traders, 57% explicitly buy or sell while 43% choose not to trade.

Table 3.2 Estimation results for Ashland Inc.

Parameter	Ambiguity model		CG model		EKO model	
	Estimate	SD	Estimate	SD	Estimate	SD
α	0.28	0.03	0.28	0.03	0.33	0.04
δ	0.55	0.08	0.62	0.06	0.60	0.06
μ	0.46	0.03	0.42	0.01	0.17	0.01
γ	0.79	0.02	-	-	-	-
τ^T	0.28	0.06	0.45	0.02	-	-
$\underline{\tau}$	0.22	0.02	-	-	-	-
$\overline{\tau}$	0.73	0.02	-	-	-	-
ρ	0.99	0.02	-	-	-	-
ϵ	0.57	0.009	0.57	0.002	0.58	0.002

Note: The estimates for Cipriani and Guarino (2014) (CG) and Easley et al. (1997) (EKO) models are from Cipriani and Guarino (2014)

We can compare the result with literature which used similar sequential trading models, such as Easley et al. (1997) (EKO model) and Cipriani and Guarino (2014) (CG model). The estimates for informational event probability α, good event probability δ, and share of active noise traders ϵ are similar to the previous literature, and δ is insignificantly lower with large standard deviation. The proportion of informed traders μ in the ambiguity model is slightly larger than the other two models. It can be interpreted that the actions considered as coming from noise traders in the previous models are explained by the ambiguity averse traders such as no trade.

With the estimated parameters, the model is simulated again with the trading data. It generates the series of prices and thresholds of each day, and with these the frequency of 'no trading' and 'herd' or 'contrarian' behaviour are computed as in Table 3.3 and 3.4. The frequencies are calculated by the proportion of the periods that satisfy the definitions of each behaviour in equilibrium on the informational event days. The informational event days are defined as the days when $Pr(v_h^d|h_T)$ or $Pr(v_l^d|h_T)$ at last period T is larger than 0.9.

Looking at the frequency of 'no trading' ($\beta_{2,t}^d = 1, \sigma_{2,t}^d = 0$)', 'no buying ($\beta_{2,t}^d = 1, 0 < \sigma_{2,t}^d \leq 0.5$)' and 'no selling ($\sigma_{2,t}^d = 0, 0.5 \leq \beta_{2,t}^d < 1$)' of ambiguity averse traders, they are not common, as 2 or 3%. This can be explained by the high re-evaluation parameter ρ in this case. With high ρ, the set $\mathcal{T}_t^d$ shrinks a lot as time goes by. It makes 'no trading' harder to occur in equilibrium.

Table 3.3 Frequency of 'no trading' for Ashland Inc.

Behaviour	Without ambiguity	Ambiguity averse
No trading	0 (0)	0.021 (0)
No buying	0 (0)	0.035 (0.019)
No selling	0 (0)	0.036 (0.016)

Note: Frequency is the average proportion of periods each behaviour occurs in the informational event days. Numbers in parenthesis are median. No trading as well as no buying and no selling do not happen for the traders without ambiguity in this case.

Looking at the frequency of herd or contrarian behaviour, it occurs with high proportion in Ashland's trading data. Herding happens for the traders without ambiguity more than 15% and for the ambiguity averse traders more than 20% in the informational event days. Herding of the traders without ambiguity mostly comes from the informational asymmetry on event uncertainty between the traders and the market maker as noted in Cipriani and Guarino (2014). For the ambiguity averse traders, herding happens because of the difference between the set $\mathcal{T}_t^d$ and expected τ by the market maker on top of the event uncertainty. With a sizable initial set $\mathcal{T}_0^d$, and high ρ, the high proportion of herding is expected as in the result. The contrarian behaviour does not occur as frequently as herding, but it still takes a larger proportion compared to Capital One Financial Corp. which will be presented later.

Table 3.4 Frequency of 'herd' and 'contrarian' behaviour for Ashland Inc.

Behaviour	Without ambiguity	Ambiguity averse
Herd buy	0.154 (0.105)	0.255 (0.137)
Herd sell	0.189 (0.102)	0.206 (0.068)
Contrarian buy	0.0003 (0)	0.080 (0.007)
Contrarian sell	0 (0)	0.112 (0.048)

Note: Frequency is the average proportion of the periods each behaviour occurs in the informational event days. Numbers in parenthesis are median.

We also can check how likely an informed trader chooses the action of 'no trade' as a part of the behaviour of no buying or no selling (no trading is included as well), and 'buying' or 'selling' as herd or contrarian behaviour. The probability is decided by the signal distributed to the traders at each time. The probability of an informed trader choose no trade as an action of the no buying or no selling behaviour is the probability of getting a signal between $\sigma_{i,t}^d$ and $\beta_{i,t}^d$ with $\beta_{i,t}^d = 1$ or $\sigma_{i,t}^d = 0$. It is computed by $\int_{\sigma_{i,t}^d}^{\beta_{i,t}^d} g^j(s_t^d|v_j^d, \tau)\mathrm{d}s_t^d$ at time t of a day d if $\beta_{i,t}^d = 1$ or $\sigma_{i,t}^d = 0$, where $i \in \{1, 2\}$ and $j \in \{h, l\}$. The probability of the trader to buy as the herd behaviour is a probability of getting a signal between $\beta_{i,t}^d$ and 0.5 when $\beta_{i,t}^d < 0.5$ and $E(V_d|h_t^d) > E(V_d|h_1^d)$. This probability is computed by $\int_{\beta_{i,t}^d}^{0.5} g^h(s_t^d|v_h^d, \tau)\mathrm{d}s_t^d$ if $\beta_{i,t}^d < 0.5$ and $E(V_d|h_t^d) > E(V_d|h_1^d)$. Calculation of the probability of the informed trader to sell in the herd behaviour or buy or sell in the contrarian behaviour are similar to the case of buy herding.

Table 3.5 Probability of the action by informed traders for Ashland Inc.

Action	Without ambiguity	Ambiguity averse
No trade in no buying or no selling behaviour	0 (0)	0.123 (0.126)
Buy in herd behaviour	0.012 (0.005)	0.088 (0.036)
Sell in herd behaviour	0.019 (0.006)	0.074 (0.001)
Buy in contrarian behaviour	0 (0)	0.021 (0)
Sell in contrarian behaviour	0 (0)	0.031 (0.009)

Note: Probabilities are computed by the average proportion of traders getting the signal s_t^d on the informational event days. Numbers in parentheses are median.

Table 3.5 shows the average probability of each action on all the informational event days. No trade would be chosen only by informed traders with ambiguity aversion around 12%. Buy or sell as an action under herd behaviour of the traders without ambiguity occurs less than 2%, while it happens for the ambiguity averse traders around 8%. Considering the proportion of each trader in the market, an informed trader, regardless of the ambiguity, chooses to buy under herding with 2.8% and sell with 3.1% while choosing no trade under no buying or selling behaviour with 2.4% on average. As for the actions in contrarian behaviour, the probability for the ambiguity averse traders is small as 2 or 3%, with zero probability of the traders without ambiguity.

To illustrate how these behaviours occur, one day of trading data for Ashland Inc. is presented in Figure 3.1. The upper panel is the ask and bid prices and accumulated orders which is denoted as 1 for buy, -1 for sell, and 0 for no trade. The middle panel is about the four thresholds $\beta_{1,t}^d, \beta_{2,t}^d, \sigma_{1,t}^d$ and $\sigma_{2,t}^d$. Depending on the thresholds and the prices in the upper panel, the behaviour is decided. The lower panel is about the upper and lower bound for $\mathcal{T}_t^d$ and $E(\tau|h_t^d)$.

The day starts with selling orders that make the ask and bid price decrease with elevated thresholds $\beta_{i,t}^d$ and $\sigma_{i,t}^d$. After a few fluctuations, the trade is eventually dominated by buy orders and the prices subsequently converge to v_h. We can check that $\mathcal{T}_t^d$ shrinks quickly and those two bounds fluctuate a lot with changes in the history of orders, compared to the expected level of τ by the agents without ambiguity, $E(\tau|h_t^d)$. The shrinkage and fluctuations are caused by high level of ρ. The fluctuation in $\mathcal{T}_t^d$ makes the thresholds for the ambiguity averse traders, $\beta_{2,t}^d$ and $\sigma_{2,t}^d$, volatile and generates the herd behaviour.

Figure 3.1 One day of trading: Ashland Inc.

Note: In the upper panel, ask and bid prices are measured on the left axis. Accumulated order, measured on the right axis, is the accumulated sum of history of orders represented by 1 for buying, -1 for selling and 0 for no trade. In the middle panel, herd buying occurs when $\beta_{i,t}$ is lower than 0.5 with price is higher than 0 and herd selling when $\sigma_{i,t}$ is higher than 0.5 and price is lower than 0. In the lower panel, the two red lines are bounds for $\mathcal{T}_t$.

The estimates for the market of the other stock, Capital One Financial Corp., is also presented. The data are about trades in 1995, like Ashland Inc. The company is also listed on NYSE and the level of total asset of the company in 1995 was around 5 billion dollars which is not very different from Ashland of 7 billion. The average number of trading per day for Capital One Financial Corp. is similar to Ashland as well. One thing to note for this stock is that the company was founded in 1994. Compared to Ashland Inc., founded in 1920s, Capital One Financial Corp. is a new company that traders do not seem to have a lot of precise information on it.

The estimates for Capital One Financial Corp. are in Table 3.6. Compared to Ashland data, it shows similar estimates in the probability of good event day ($\delta = 0.53$), and $\mathcal{T}_0^d$ with $\underline{\tau} = 0.22, \overline{\tau} = 0.75$. On the other hand, the probability of informational event ($\alpha = 0.7$), informed traders ($\mu = 0.63$) and the proportion of noise traders to buy or sell ($\epsilon = 0.69$) are higher than those from Ashland Inc. More interesting estimates are the share of traders without ambiguity (γ), and re-evaluation parameter (ρ). γ is relatively low as it is 0.22. It means that there are more share of ambiguity averse traders (78%) in this marekt than in Ashland's. Also, ρ is 0.44, which is low compared to Ashland's 0.99. The ambiguity averse traders in this market do not re-evaluate the set $\mathcal{T}_t^d$ as strictly as those in the market trading Ashland's stock.

Table 3.6 Estimation results for Capital One Financial Corp.

Parameter	Estimate	SD
α	0.70	0.02
δ	0.53	0.02
μ	0.63	0.007
γ	0.22	0.03
τ^T	0.49	0.08
$\underline{\tau}$	0.22	0.03
$\overline{\tau}$	0.75	0.02
ρ	0.44	0.03
ϵ	0.69	0.01

Since the re-evaluation is not conducted strictly with low level of ρ, the set $\mathcal{T}_t^d$ does not shrink as much as the case of Ashland. It makes the behaviour of no buying or no selling occur more likely. In Table 3.7, we can find the frequency of the behaviours of no trading, no buying and no selling. In 20% of periods on the informational event days, 'no trading ($\beta^d_{2,t} = 1, \sigma^d_{2,t} = 0$)' happens for the ambiguity averse traders. 'No buying' or 'no selling' also takes a significant proportion, such as 24 and 27%. Those are much higher relative to Ashland's result at around 3%.

Table 3.7 Frequency of 'no trading' for Capital One Financial Corp.

Behaviour	Without ambiguity	Ambiguity averse
No trading	0 (0)	0.197 (0.154)
No buying	0 (0)	0.243 (0.196)
No selling	0 (0)	0.274 (0.279)

Note: Frequency is the average proportion of periods each behaviour occurs in informational event days. Numbers in parenthesis are median.

The frequency of herding shows the opposite pattern. The proportions of herd buy and herd sell of traders in the informational event days are much lower than those of Ashland data, as those behaviours happen with 11 and 4% for the traders without ambiguity and 8 and 0.2% for those with ambiguity aversion[5]. It can be explained by the high α and low level of ρ. High α means low informational event uncertainty. The low uncertainty makes the frequency of herding lower for all of the traders with and without ambiguity. As for the ambiguity averse traders, low level of ρ affects them as well. Since the set $\mathcal{T}^d_t$ does not shrink much after re-evaluation, the expected τ of the market maker more likely stays within the set. Therefore, herding becomes harder to occur for the ambiguity averse traders. Also, the proportion of contrarian behaviour is almost zero.

Table 3.8 Frequency of 'herd' and 'contrarian' behaviour for Capital One Financial Corp.

Behaviour	Without ambiguity	Ambiguity averse
Herd buy	0.111 (0.058)	0.080 (0)
Herd sell	0.044 (0)	0.002 (0)
Contrarian buy	0.0003 (0)	0.0004 (0)
Contrarian sell	0 (0)	0 (0)

Note: Frequency is the average proportion of periods the behaviour occurs in informational event days. Numbers in parenthesis are median.

[5]There are a few days dominated by buy orders with long period of trading T. It generates the asymmetry between the frequencies of buy herding and sell herding

The probability of the informed traders to choose each action is also computed for Capital One Financial in Table 3.9. The probability of the traders without ambiguity is not significantly different from that of Ashland. The results for the ambiguity averse traders, however, is different from it, in line with the frequency results. No trade under no buying or selling behaviour occurs much more likely of 67% than 12% of Ashland, and buy is chosen under herding around 2% compared to around 8% of Ashland. For all the informed traders considering the proportion γ, no trade under no buying or selling behaviour is selected with probability of more than 53%, and buy and sell under herding is chosen with 2% and 0.2%, respectively. The probability of contrarian behaviour is almost zero here.

Table 3.9 Probability of the action by informed traders for Capital One Financial Corp.

Action	Without ambiguity	Ambiguity averse
No trade in no buying or no selling behaviour	0.002 (0)	0.675 (0.735)
Buy in herd behaviour	0.023 (0.003)	0.023 (0)
Sell in herd behaviour	0.007 (0)	0.0003 (0)

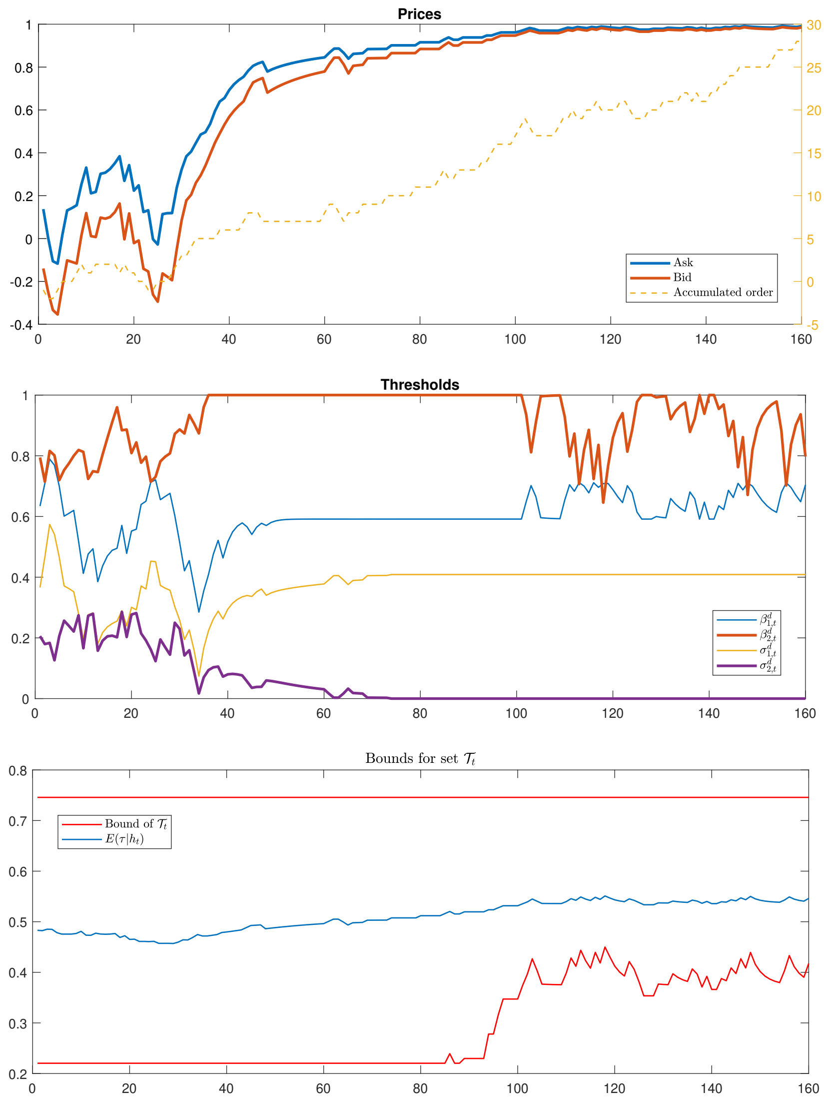
Note: Probabilities are computed by the average proportion of traders getting the signal s_t^d in informational event days. Numbers in parentheses are median. There is zero probability of contrarian behaviour.

One day of trading for Capital One Financial Corp. is also plotted in Figure 3.2. We can see that $\mathcal{T}_t$ stays at $\mathcal{T}_0^d$ up to $t = 86$. It makes 'no buying or selling' behaviour occur for the ambiguity averse traders by generating $\beta_{2,t}^d = 1$ and $\sigma_{2,t}^d = 0$. After that, a series of buy orders finally induce the re-evaluation to shrink $\mathcal{T}_t$ substantially after $t = 94$.

3.5 Conclusion

In this research, it is shown that ambiguity aversion implemented by the recursive multiple prior can generate not only the behaviour of 'no buying' or 'no selling' for any signal but also 'herd' or 'contrarian' behaviour of informed traders. The preference of ambiguity averse traders considering the worst case payoff enables no trade to be chosen even with an informative signal in equilibrium, and the gap between the re-evaluated set

Figure 3.2 One day of trading: Capital One Financial Corp.



Note: In the upper panel, ask and bid prices are measured on the left axis. Accumulated order, measured on the right axis, is the accumulated sum of history of orders represented by 1 for buying, -1 for selling and 0 for no trade. In the middle panel, herd buying occurs when $\beta_{i,t}$ is lower than 0.5 with price is higher than 0 and herd selling when $\sigma_{i,t}$ is higher than 0.5 and price is lower than 0. In the lower panel, the two red lines are bounds for $\mathcal{T}_t$.

of the signal precision τ by ambiguity averse traders and expected level of it by market maker induces herd or contrarian behaviour.

Using this model, trading data of two stocks listed on NYSE are estimated. The market trading Ashland Inc. is estimated to have high re-evaluation parameter ρ, and Capital One Financial Corp. has low ρ. Their results show that, depending on the level of the re-evaluation parameter, frequency of those behaviours differs. High ρ generates herding and contrarian behaviour more likely while low ρ generates the behaviour of no trading more.

Chapter 4

Market Manipulation in Financial Markets

4.1 Introduction

Market manipulation describes a deliberate attempt to interfere with the free and fair operation of the market and create artificial, false or misleading appearances with respect to the price of, or market for, a security, commodity or currency[1]. This phenomenon holds the interest of Financial Institutions, that regulate the behaviour of market participants, in order to guarantee the integrity of financial markets and avoid conducts that undermine the general principle for which all investors must be placed on the same footing. This is the reason why every behaviour that can be identified as market manipulation is banned and, generally, condemned. Notwithstanding, the description of market manipulation is not always clearly specified among regulations about financial markets and, often, examples are provided by laws and Financial Institutions in order to detect cases of market abuse in concrete terms. In the past years also some academic research has been interested in explaining the possibility of "artificially influencing" asset prices and analysing the mechanisms through which it could happen. Among other things, researchers have studied the relationship between this behaviour and specific features of financial markets (e.g., efficiency). A classification of market manipulation schemes can be found in Allen and Gale (1992), who define three types of manipulation:

- action-based manipulation, that is manipulation based on actions that change the

[1]This is the definition provided by a supervisory authority on financial markets: `http://www.asx.com.au/supervision/participants/market_manipulation.htm`

actual or perceived value of the assets;

- information-based manipulation, that is manipulation based on releasing false information or spreading false rumors;
- trade-based manipulation, that "occurs when a trader attempts to manipulate a stock simply by buying and then selling, without taking any publicly observable actions to alter the value of the firm or releasing false information to change the price

Nevertheless, an unambiguous definition of market manipulation cannot be found. More recently, Kyle and Viswanathan (2008) declared that the term manipulation is not always used in a precise manner in economic literature and proposed that a trading strategy is classified as "illegal price manipulation" when the "violator's intent is to pursue a scheme that undermines economic efficiency both by making prices less accurate as signals for efficient resources allocation and by making markets less liquid for risk transfer".

In the present paper, following the Allen and Gale classification scheme, we focus on trade based manipulation and we investigate the mechanisms through which manipulation arises as the optimal behaviour of a rational trader.

We consider a sequential trading financial market with noise traders and two kinds of rational traders: informed traders and a potential manipulator. All types of traders interact and exchange one unit of an asset with a market maker. We show that there are situations in which the potential manipulator disregards her belief in order to manipulate the following rational traders' behaviour. In these cases she is able to affect the pattern of the prices and, through a later trade, she recoups the losses and also gains higher total profits than those taken if she behaved according to her beliefs on both periods. The result is strictly connected to the effects of the manipulator's choice in the first period she trades on the behaviour of the informed traders who enter the market in the next period. In particular, it holds when the behaviour that the rational traders do not follow their signals such as herding arises after the manipulator's action. With the term "herding" we refer to situations in which there is conformity of actions. In these cases, rational traders decide to disregard their private information and to follow the manipulator's behaviour[2].

It is the first time, as far as we know, that a model describes profitable trade based manipulation as the case in which the manipulator, through causing behaviours like herding among the other rational agents, finds it optimal to make the other market participants do what she wants them to do, in order to affect the pattern of the asset prices.

The phenomenon can be identified as "momentum ignition", that, technically, is the entry of orders or a series of orders intended to start or exacerbate a trend, and to encourage other participants to accelerate or extend the trend in order to create an opportunity to unwind or open a position at a favorable price. [3]

Finally, we point out that, in our paper, there are a few similarities with previous research about trade based manipulation studied in a sequential trading financial market's framework. However, it is the first time that the contemporary presence of various classes of traders is analysed and that the interaction among them is evaluated; previous literature, in fact, mainly focuses on the interaction between the manipulator and the market maker.

Review of the literature

Allen and Gale (1992), in the same paper mentioned above, proposed a rational expectations' model in which some equilibria involve manipulation. They defined three types of agents: rational small investors, a large trader, who enters the market only if she has a private information, and a manipulator. Under certain conditions the manipulator can mimic the informed trader's actions and pretend to be informed: therefore investors become uncertain whether the trade is from an informed trader or a manipulator and the

[2] We underline that manipulated traders are rational informed traders, but we study the conditions under which they disregard their private information and imitate the manipulator's action.

[3] The description of "momentum ignition" as a particular type of manipulative behaviour is provided by the European Securities and Markets Authority (ESMA) in one of its official documents: `http://www.esma.europa.eu/system/files/esma_2012_122_en.pdf`.

latter makes profits from this situation.

In a later paper, Aggarwal and Wu (2006) illustrate both a theoretical model and an empirical analysis of the manipulation phenomenon. In the theoretical part they refer to the model of Allen and Gale, adding symmetric information seekers (traders who try to "ferret out information about the firm's prospects"). The latter traders observe what happens in the market and behave according to it and, in general, contribute to improve market efficiency. The authors prove that it does not happen when manipulators are in the market. Other contributions on trade based manipulation are mainly related, as in our case, to one model of market microstructure, either the model by Kyle (1985) or Glosten and Milgrom (1985). Allen and Gale (1992) show that uninformed trade based manipulation can arise in Kyle and Glosten and Milgrom frameworks, when some assumptions are replaced by more natural ones: the authors assert that liquidity traders cannot be treated as symmetric, that is they are equally likely to be buyers as sellers. Moreover, Glosten and Milgrom and Kyle consider sellers and buyers equally likely to be informed, but there are factors such as short sale constraints which imply a different probability of a buyer being informed than a seller. Given the new assumptions, the effects of purchases and sales on prices are "asymmetric" and lead to manipulation opportunities. The authors show that a manipulative strategy "buy-sell" adopted by the trader is, in any case, unprofitable, but they provide an example in which a manipulative strategy "buy-buy-sell-sell" is possible and also profitable. Chakraborty and Yilmaz (2004) use a Glosten and Milgrom setup (but the result is also proved in a Kyle model) to show that, when the market maker is uncertain about the presence of informed traders in the market and the number of trading periods is sufficiently high, informed traders manipulate in every equilibrium. The authors refer to Kyle's result, according to which an informed trader knows that trading reveals her knowledge and chooses a less aggressive strategy in a situation where her trading affects the market price than in a situation where it does not, in order to hide her information as much as possible. Nevertheless, Kyle also demonstrates that there exists a unique equilibrium in the market where the insider trades in the direction of her information. Chakraborty and Yilmaz, on the other hand, show that the informed trader might also find it in her interest to confuse the market maker by trading in the 'wrong' direction with respect to her information, undertaking short-term losses, and then recoup them and make long term profits, if there is a sufficiently large number of periods to do it. In another paper Chakraborty and Yilmaz (2008) again define manipulation as the strategy in which a dynamic informed trader undertakes short term losses to increase noise in the market maker's learning process and makes profits in later trades. In this case the authors consider again rational informed traders and noise traders. Moreover, they introduce rational traders called followers, who do not know the nature of the information of the other traders (who are the 'leaders'), but try to infer it from the first period order flow. As a result, the existence of followers creates incentives for the leader, when she acts in the first period, to trade against her information in order to create noise in the inference problem of the followers (who, otherwise, would compete away all the profits in the second period) and, then, of the market maker.

As already stated, in the papers mentioned above trade based manipulation is never related to herding behaviour. Nevertheless, even if herd behaviour cannot arise in the Glosten and Milgrom set up, the latter is also a basic framework for some models which explain herding in financial markets. In particular, Cipriani and Guarino (2014) develop and estimate a model of informational herding, inspired by the paper by Avery and Zemsky (1998) and based on models by Glosten and Milgrom (1985) and Easley and O'Hara (1987)[4].

Also previous papers about herding in financial markets (among others, Avery and Zemsky (1998), Lee (1998) and Cipriani and Guarino (2008)) analyse a market where informed and uninformed traders sequentially trade a security of unknown value with a market maker. Herding behaviour arises in these cases when there is 'multidimensional

[4]The authors generalize Glosten and Milgrom's model to an economy where trading happens over many days.

uncertainty', that is uncertainty not only about the value of the asset but also about one or more of the other features of the model. For instance, in Avery and Zemsky (1998), as well as in Cipriani and Guarino (2014), there is uncertainty about the occurrence of an information event that affects the value of the asset.

The remainder of this paper is organized as follows: Section 2 presents the model set up; Section 3 describes market manipulation and its properties. In Section 4, the profitable manipulation is examined in various cases. Section 5 shows the conditions for the profitable manipulation; finally, Section 6 offers some conclusions about the model and possible future developments of research about manipulation.

4.2 The Model

The model is based on Glosten and Milgrom (1985). It is modified to have a binary signal on the value of the traded asset, and a potential manipulator is added as a new type of informed trader.

The market

An asset is traded by a sequence of traders who interact with a market maker. Time is represented by a set of trading time indexed by $t = 1, 2, 3, ...$. At each time t, a trader can exchange the asset with the market maker. The trader can buy, sell or decide not to trade. Each trade consists of the exchange of one unit of the asset for cash. We denote the action of the trader at time t byx_t. Moreover, we denote the history of trades and prices until time $t - 1$ by h_t.

At any time t, the market maker sets the prices at which a trader can buy or sell the asset. When posting these prices, she must take into account the possibility of trading with agents who (as we see below) have some private information on the asset value. Therefore, she sets different prices at which she is willing to sell and to buy the asset; that is, there will be a bid-ask spread. We denote the ask price (i.e., the price at which a trader can buy) at time t by a_t and the bid price (i.e., the price at which she can sell) by

b_t.

The asset value

The asset value V is δ with probability $1-\alpha$, which is thought to be the value which stays at the previous day's value, and δ guarantees that the closing price is martingale. With probability α, the asset value changes. In the latter case, since as we will see, there are informed traders in the market, we say that an information event has occurred. If an information event occurs, with probability $1-\delta$ the asset value decreases to 0 ("bad informational event"), and with probability δ it increases to 1 ("good informational event").

The market maker

Unmodeled potential (Bertrand) competition forces the market maker to set prices so as to make zero expected profits in each period t. The market maker observes the history of traders' decisions and prices until time $t-1$, h_t. When setting the prices, the market maker takes into account not only the information conveyed by h_t, but also the information conveyed by the time t decision to buy, to sell or not to trade the asset. Bertrand competition implies that the equilibrium bid (ask) will be the highest (lowest) price satisfying the zero expected profit condition.

Hence, the equilibrium bid and ask prices at time t have to satisfy the following conditions:

$$b_t = E(V|h_t, x_t = sell, a_t, b_t), \tag{4.1}$$

$$a_t = E(V|h_t, x_t = buy, a_t, b_t). \tag{4.2}$$

Finally, we denote the expected value of asset V at time t, *before the trader in t has traded*, by p_t, that is,

$$p_t = E(V|h_t). \tag{4.3}$$

We will refer to p_t as the "price" of the asset.

The traders

Traders act in an exogenously determined sequential order. Each trader is chosen to take an action only once, at time t. Traders are of two types, informed and uninformed (or noise). The trader's type is not known publicly, that is, it is her private information. When there is an informational event, at each time t with probability μ the trader is informed. If there is no event, at any t the trader is noise.

Noise traders

Uninformed (or noise) traders trade for unmodeled (e.g., liquidity) reasons: they buy, sell or do not trade the asset with exogenously given probabilities. We assume that in each period in which they are called to trade, they buy with probability ε_b, sell with probability ε_s, and do not trade with probability $1 - \varepsilon_b - \varepsilon_s$.

Informed traders

Informed traders know their own private component and have private information on the asset value's common component. If at time t an informed trader is chosen to trade, she observes a private signal s_t on the realization of V. s_t is a symmetric binary signal, taking values 0 and 1 with precision $q > \frac{1}{2}$; that is, $\Pr(s_t = 0|V = 0) = \Pr(s_t = 1|V = 1) = q$. In addition to her signal, an informed trader at time t observes the history of trades and prices. Therefore, her expected value of the asset is $E(V|h_t, s_t)$.

The informed traders' payoff function is

$$U(v, x_t, a_t, b_t) = \begin{cases} v - a_t \text{ if } x_t = buy, \\ 0 \text{ if } x_t = no\ trade, \\ b_t - v \text{ if } x_t = sell. \end{cases} \tag{4.4}$$

Informed traders choose x_t to maximize $E(U(\cdot)|h_t, s_t)$, and they are assumed not to trade when they are indifferent between trading and no trade.

4.3 Market Manipulation

To study market manipulation, we need to allow one informed trader to trade twice who is called a manipulator.

Definition 5. *A trade at time t, x_t, is manipulative if the trader suffers an expected loss at that time:* $E(U(V, x_t, a_t, b_t)|h_t, s_t) < 0$.

The trader suffers an immediate loss (in expectation) to gain a higher payoff (in expectation) in the future. Let us assume that other agents (traders and market maker) think manipulation does not occur (e.g., it is a zero probability event). Our aim is to find cases in which in this set up manipulation is optimal. To make our life easier, let us assume that the manipulator has a signal of different precision, q^M. The manipulator trades twice, at times t is τ and τ'.

Let us also explicitly define the standard (non manipulative) trading activity:

Definition 6. *A trade at time t, x_t, is standard if* $x_t \in \arg\max E(U(V, x_t, a_t, b_t)|h_t, s_t)$.

In words, a standard trade maximizes the expected utility obtained at time t only. If the manipulator chooses the standard trade, denoted by x_t^* then her expected profit at time $t = \tau$ is:

$$\Pi^*_{\tau,\tau'} = E(U(V, x^*_\tau, a_\tau, b_\tau)|h_\tau, s_\tau) + E\left(E(U(V, x^*_{\tau'}, a^*_{\tau'}, b^*_{\tau'})|h^*_{\tau'}, s_\tau)|h_\tau, s_\tau\right)$$

If she chooses the manipulative trade x_t^M then her expected profit at time t is:

$$\Pi^M_{\tau,\tau'} = E(U(V, x^M_\tau, a_\tau, b_\tau)|h_\tau, s_\tau) + E\left(E(U(V, x^M_{\tau'}, a^M_{\tau'}, b^M_{\tau'})|h^M_{\tau'}, s_\tau)|h_\tau, s_\tau\right)$$

where the superscript M and $*$ in history $h_{\tau'}$, prices $a_{\tau'}, b_{\tau'}$ and actions $x_{\tau'}$ means those under the manipulative and standard strategy respectively.

To make the manipulative strategy more profitable than the standard strategy, there requires a factor which inflates or deflates the price by the action of the manipulator. The factor we focus in is the herd behaviour. If there is a herd behaviour, the other informed traders ignore their private signal and just follow the dominated action of others. It can boost or plunge the price regardless of the true value. According to Avery and Zemsky (1998) and Cipriani and Guarino (2014), herding behaviour of the informed traders can occur in the sequential trading model with event uncertainty. The profitable manipulation can happen in relation with it. Following Cipriani and Guarino (2014), we define herding as below.

Definition 7. *A trader with private signal s_t engages in 'herding' at time t if she buys with any signal when $E(V|h_t) > p_1$ or if she sells with any signal when $E(V|h_t) < p_1$.*

4.3.1 No event uncertainty

Herding is not possible in this sequential trading model based on Glosten and Milgrom (1985) without event uncertainty according to Avery and Zemsky (1998) and Cipriani and Guarino (2014). In this section we will look at the relationship between the event uncertainty and the manipulative strategy.

Proposition 14. *For a manipulator who can trade at time t is τ and τ', the manipulative strategy is not optimal when there is no event uncertainty.*

When $\alpha = 1$, which implies that there is no event uncertainty, informed traders buy with a signal $s_t = 1$ and sell with a signal $s_t = 0$, and only the number of buys and sells matters for prices and expectations as in the model like the Glosten and Milgrom model without event uncertainty. Suppose the manipulator receives a signal $s^M_\tau = 0$ at $t = \tau$. If she could only act at time τ, she would sell as other informed traders. If she can act twice, at time τ and at a future time $\tau' > \tau$, she finds it optimal to manipulate if the expected sum of the profit at time τ and τ' of the manipulative strategy, $\Pi^M_{\tau,\tau'}$, is larger than that of the standard strategy, $\Pi^*_{\tau,\tau'}$. We know that the profit of the standard strategy at each time is always not negative as the trader can choose not to trade if the other actions would bring a negative profit. The sum of the profit of the manipulative strategy is as follows.

$$
\begin{aligned}
&\Pi^M = E(V|h_\tau, s^M_\tau = 0) - a_\tau \\
&+ E[E(U(V, x^M_{\tau'}, a^M_{\tau'}, b^M_{\tau'})|h^M_{\tau'} \setminus \{buy_\tau\}, s^M_\tau = 0)|h_\tau, s^M_\tau = 0] \\
&= E(V|h_\tau, s^M_\tau = 0) - a_\tau \\
&+ E[E(V|h^M_{\tau'} \setminus \{buy_\tau\}, s^M_\tau = 0) - a^M_{\tau'}|h_\tau, s^M_\tau = 0, E(V|h^M_{\tau'} \setminus \{buy_\tau\}, s_\tau = 0) > a^M_{t'}] \\
&\times Pr(E(V|h^M_{\tau'} \setminus \{buy_\tau\}, s^M_\tau = 0) > a^M_{\tau'}|h_\tau, s^M_\tau = 0) \\
&+ E[b^M_{\tau'} - E(V|h^M_{\tau'} \setminus \{buy_\tau\}, s^M_\tau = 0)|h_\tau, s^M_\tau = 0, E(V|h^M_{\tau'} \setminus \{buy_\tau\}, s^M_\tau = 0) < b^M_{\tau'}] \\
&\times Pr(E(V|h^M_{\tau'} \setminus \{buy_\tau\}, s^M_\tau = 0) < b^M_{\tau'}|h_\tau, s^M_\tau = 0) \\
&+ 0 \times \quad Pr(a^M_{\tau'}|h_\tau, s^M_\tau = 0, a^M_{\tau'} \geq E(V|h^M_{\tau'} \setminus \{buy_\tau\}, s^M_\tau = 0) \geq b^M_{\tau'}|h_\tau, s^M_\tau = 0)
\end{aligned}
$$

where the superscript M and $*$ in history $h_{\tau'}$, prices $a_{\tau'}$, $b_{\tau'}$ and actions $sell_{\tau'}$, $buy_{\tau'}$ means those under the manipulative and standard strategy respectively. The profit at time τ is $E(V|h_\tau, s^M_\tau = 0) - a_\tau$ as the manipulator buys even with the bad signal $s^M_\tau = 0$. The future profit at τ' depends on the history after $t = \tau$. $\{buy_\tau\}$ is excluded from the history as the buy action at $t = \tau$ is chosen by the manipulative motive, so it is not used as an information for the expected asset value. If the expected asset value of the manipulator becomes higher than the ask price at time τ', the manipulator will buy, so she gets the profit $E(V|h^M_{\tau'} \setminus \{buy_\tau\}, s^M_\tau = 0) - a^M_{\tau'}$. If the expectation is lower than the bid price, the profit becomes $b^M_{\tau'} - E(V|h^M_{\tau'} \setminus \{buy_\tau\}, s^M_\tau = 0)$ since she chooses to sell. Otherwise, the profit is zero with no trade. The only possible case to happen is selling at $t = \tau'$. For any history after time $t = \tau$, the expected asset value of the manipulator is

$$
\begin{aligned}
&E(V|h^M_{\tau'} \setminus \{buy_\tau\}, s^M_\tau = 0) \\
&< E(V|h^M_{\tau'} \setminus \{buy_\tau\}) = E(V|h^M_{\tau'}, sell_{\tau'}) = b^M_{\tau'} \\
&< E(V|h^M_{\tau'}) < E(V|h^M_{\tau'}, buy_{\tau'}) = a^M_{\tau'}
\end{aligned}
$$

The bid price at τ' is always higher than the asset value expected by the manipulator. Therefore, with any history after τ, the manipulator sells at τ' and the probability of other actions are zero. The sum of the expected profits can be rewritten as

$$\Pi^M_{\tau,\tau'} = E(V|h_\tau, s^M_\tau = 0) - a_\tau + E[b^M_{\tau'} - E(V|h^M_{\tau'} \setminus \{buy_\tau\}, s^M_\tau = 0)|h_\tau, s^M_\tau = 0]$$
$$= E[b^M_{\tau'}|h_\tau, s^M_\tau = 0] - a_\tau \tag{4.5}$$

Also, the expected bid price at $t = \tau'$ by the manipulator is smaller than ask price at $t = \tau$ because

$$E[b^M_{\tau'}|h_\tau, s^M_\tau = 0] = E[E(V|h^M_{\tau'}, sell_{\tau'})|h_\tau, s^M_\tau = 0]$$
$$< E[E(V|h^M_{\tau'})|h_\tau, s^M_\tau = 0] < E[E(V|h^M_{\tau'})|h_\tau]$$
$$= E(V|h_\tau, buy_\tau) = a_\tau$$

It implies that the sum of expected profit of the manipulative strategy $\Pi^M_{\tau,\tau'} < 0$. Since the sum of profit of the standard strategy $\Pi^*_{\tau,\tau'}$ is not negative, the manipulator does not choose the manipulative strategy. The case with a good signal is symmetric to this.

4.3.2 A trader without signal in the event uncertainty model

We can also think of an environment that the potential manipulator does not have her private signal, although there is event uncertainty. The possibility of manipulation of this case is analysed in this section.

Proposition 15. *The manipulative strategy is not profitable for a trader without a private signal when there is an event uncertainty.*

The market price of the asset is between the ask and bid, $a_t > E(V|h_t) > b_t$ in any state even with the event uncertainty. For a manipulator who can trade at t is τ and τ' but does not have any signal, the optimal action is no-trade at time $t = \tau$, NT_τ, under the standard strategy as $a_t > E(V|h_t) > b_t$ at any t. It means that the expected profit of the standard strategy at time $t = \tau$ is zero. Also, the expected profit of the standard strategy at time τ' would be non-negative as the trader can choose her action among $\{buy_{\tau'}, NT_{\tau'}, sell_{\tau'}\}$ depending on $a_{\tau'}$ and $b_{\tau'}$. Therefore, the manipulative strategy is profitable only if

$$\Pi^M_{\tau,\tau'} = E(V|h_\tau) - a_\tau + E(E(U(V, x^M_{\tau'}, a^M_{\tau'}, b^M_{\tau'})|h^M_{\tau'})|h_\tau) > 0 \tag{4.6}$$

when the manipulator choose to buy at $t = \tau$ as a manipulative strategy. The third term of the equation (4.6) can be re-written as

$$\begin{aligned}
&E(E(U(V, x^M_{\tau'}, a^M_{\tau'}, b^M_{\tau'})|h^M_{\tau'})|h_\tau) \\
&= E[b^M_{\tau'} - E(V|h^M_{\tau'} \setminus \{buy_\tau\})|h_\tau, b^M_{\tau'} > E(V|h^M_{\tau'} \setminus \{buy_\tau\})] \\
&\qquad \times Pr(b^M_{\tau'} > E(V|h^M_{\tau'} \setminus \{buy_\tau\})|h_\tau) \\
&+ E[E(V|h^M_{\tau'} \setminus \{buy_\tau\}) - a^M_{\tau'}|h_\tau, E(V|h^M_{\tau'} \setminus \{buy_\tau\}) > a^M_{\tau'}] \\
&\qquad \times Pr(E(V|h^M_{\tau'} \setminus \{buy_\tau\}) > a^M_{\tau'}|h_\tau)
\end{aligned} \tag{4.7}$$

For any history $h_{\tau'}$, $a^M_{\tau'} = E(V|h^M_{\tau'}, buy_{\tau'}) \geq E(V|h^M_{\tau'}) \geq E(V|h^M_{\tau'} \setminus \{buy_\tau\})$. It ensures the non-positivity of the second term of the equation (4.7). Also, $E(V|h_\tau) < a_\tau$, $E(V|h_\tau) = E[E(V|h^M_{\tau'} \setminus \{buy_\tau\})|h_\tau]$, $a_\tau = E(V|h_\tau, buy_\tau) = E[E(V|h^M_{\tau'})|h_\tau]$, and for any history after $t = \tau$, $E(V|h^M_{\tau'} \setminus \{buy_\tau\}) < E(V|h^M_{\tau'})$. Using these, the expected profit of the manipulative strategy is

$$
\begin{aligned}
\Pi^M_{\tau,\tau'} &= E(V|h_\tau) - a_\tau + E(E(U(V, x^M_{\tau'}, a^M_{\tau'}, b^M_{\tau'})|h^M_{\tau'})|h_\tau) \\
&\leq E(V|h_\tau) - a_\tau + E[b^M_{\tau'} - E(V|h^M_{\tau'} \setminus \{buy_\tau\})|h_\tau, b^M_{\tau'} > E(V|h^M_{\tau'} \setminus \{buy_\tau\})] \\
&\qquad \times Pr(b^M_{\tau'} > E(V|h^M_{\tau'} \setminus \{buy_\tau\})|h_\tau) \\
&\leq \{E(V|h_\tau) - a_\tau + E[b^M_{\tau'} - E(V|h^M_{\tau'} \setminus \{buy_\tau\})|h_\tau, b^M_{\tau'} > E(V|h^M_{\tau'} \setminus \{buy_\tau\})]\} \\
&\qquad \times Pr(b^M_{\tau'} > E(V|h^M_{\tau'} \setminus \{buy_\tau\})|h_\tau) \\
&\leq E[E(V|h^M_{\tau'} \setminus \{buy_\tau\}) - E(V|h^M_{\tau'}) \\
&\qquad + E(V|h^M_{\tau'}, sell_{\tau'}) - E(V|h^M_{\tau'} \setminus \{buy_\tau\})|h_\tau, b^M_{\tau'} > E(V|h^M_{\tau'} \setminus \{buy_\tau\}] \\
&\qquad \times Pr(b^M_{\tau'} > E(V|h^M_{\tau'} \setminus \{buy_\tau\})|h_\tau) \\
&= E[E(V|h^M_{\tau'}, sell_{\tau'}) - E(V|h^M_{\tau'})|h_\tau, b^M_{\tau'} > E(V|h^M_{\tau'} \setminus \{buy_\tau\})] \\
&\qquad \times Pr(b^M_{\tau'} > E(V|h^M_{\tau'} \setminus \{buy_\tau\})|h_\tau) \\
&\leq 0
\end{aligned}
$$

It means that the manipulative strategy is not profitable for the trader without a private signal. In the case of selling at t as a manipulative strategy, the same logic is applicable. Therefore, the manipulative strategy is not profitable without private information in any way.

4.4 Profitable Manipulation

We have shown the conditions that the manipulative strategy cannot be an optimal strategy. Without the event uncertainty or private signal for the manipulator, it is impossible for the manipulative strategy to get a positive profit. In this section, we can show the cases that the manipulative strategy is more profitable than the standard strategy, so the manipulative strategy is optimal.

4.4.1 Case 1: Fixed τ and τ' with $\mu = 1$

The first case to analyse is that the time of the action τ and τ' are fixed. In the example presented, the times are fixed at $\tau = 2$ and $\tau' = 7$. We also focus on the case with $\mu = 1$ in which all traders get a signal on the event day. It helps us restrict the possible histories as there will be no noise traders on the event day. The other parameter values of the example are as follows; $\alpha = 0.01, \delta = 0.5, q = q^M = 0.6$, and $\varepsilon_b = \varepsilon_s = 0.2$. The history of action at $t = \tau = 2$ is buying and the manipulator is assumed to get a bad signal $s_2^M = 0$.

The optimal choice of an ordinary informed trader at $t = 2$ is buy with a good signal and no trade with a bad signal for $a_2 = 0.5118$ and $b_2 = 0.5$[5]. As the precision of the private signal is the same for the ordinary informed trader and the manipulator with $q = q^M$, the optimal action of the manipulator with the standard strategy is not to trade. Therefore, the profit the manipulator would get is zero at $t = \tau = 2$. If the manipulator choose the manipulative strategy, she will buy in order to inflate the price and the profit becomes $E(V|h_2, s_2^M = 0) - a_2 = -0.0118$.

Table 4.1 Expected profit of Case 1

	Expected profit	
	Standard strategy $(\Pi^*_{\tau,\tau'})$	Manipulative strategy $(\Pi^M_{\tau,\tau'})$
$\tau = 2$	0	-0.0118
$\tau' = 3$	0	-0.0118
4	0.0777	-0.0118
5	0.0618	0.0382
6	0.0557	0.0294
7	0.0340	0.0469

Note: the parameter values used in the example are $\alpha = 0.01, \delta = 0.5, \mu = 1, q = q^M = 0.6$, and $\varepsilon_b = \varepsilon_s = 0.2$.

The path after time $t = 2$ differs depending on the strategy. Looking at the standard strategy where the action at $t = \tau = 2$ is no trade, the optimal action at $t = 3$ is buy with a good signal and sell with a bad signal with the ask and bid prices are $a_3 = 0.5048$

[5] Note that the bid $b_t = \delta = 0.5$ if a trader with any signal does not sell and $\mu = 1$ as a selling makes the market maker think that it is a no-event day.

and $b_3 = 0.4952$. If we think about a case that the period of the second action of the manipulator τ' is set to be 3, the expected value of the manipulator is $E(V|h_3^* \setminus \{NT_2\}, s_2^M = 0) = 0.5$. From these, the optimal action of the manipulator is no trade, and the expected profit is $\Pi_{2,3}^* = 0$.

At $t = 4$, there are two possible histories: buying or selling because of the optimal actions depending on the signal at $t = 3$. The probability of each history in the manipulator's point of view is $Pr(buy_3|h_3^* \setminus \{NT_2\}, s_2^M = 0) = 0.5$ and $Pr(sell_3|h_3^* \setminus \{NT_2\}, s_2^M = 0) = 0.5$.

After the history of buying at $t = 3$, the optimal action of the informed traders is buy with a good signal and no trade with a bad signal; the ask and bid prices are $a_4 = 0.5223$ and $b_4 = 0.5$. If $\tau' = 4$, the optimal action of the manipulator is buying because $E(V|h_4^* \setminus \{NT_2\}, s_2^M = 0) = 0.6$. It brings profit of 0.0777.

If the action chosen by the ordinary informed trader is selling at $t = 3$, the optimal action of them becomes no trade with a good signal and sell with a bad signal given $a_4 = 0.5$ and $b_4 = 0.4777$. The expected value of the manipulator after this history is $E(V|h_4^* \setminus \{NT_2\}, sell_3, s_2^M = 0) = 0.4$. From these the optimal action of the manipulator is selling, which brings profit of 0.0777. Therefore, the expected profit of the standard strategy for $\tau' = 4$ is $\Pi_{2,4}^* = 0.0777$.

The expected profit of the standard strategy with $\tau' > 4$ are calculated following the same way. With $\tau' = 7$, there are 14 possible histories[6], which are composed of two buys, five sells, and seven no trades as an optimal action of the manipulator at $t = 7$. As with the optimal action of the manipulator at $t = \tau'$, buys are chosen from the cases with increases in the expectation higher than the ask prices and sells are from cases with decreases in the expectation lower than the bid prices. The total expected profit of the standard strategy when $\tau' = 7$ is $\Pi_{2,7}^* = 0.0340$.

Looking at the manipulative strategy, from $t = 3$ the optimal action of the informed trader is buying with any signal. In other words, herding occurs. The ask and bid price

[6]Details on the histories of the standard and the manipulative strategy are in the appendix

are $a_3 = 0.5475$ and $b_3 = 0.5$. If we consider a case that $\tau' = 3$, the subjective asset value of the manipulator is $E(V|h_3^M \backslash \{buy_2\}, s_2^M = 0) = 0.5$. Hence, the optimal action of the manipulator is not to trade and the expected profit becomes $\Pi_{2,3}^M = -0.0118 + 0 = -0.0118$.

At $t = 4$, there is only one possible history under the manipulative strategy, which is buying because a herd buying occurs at $t = 3$. After observing a buy order, the optimal action of the other traders at $t = 4$ is buying with a good signal and do not trade with a bad signal given $a_4 = 0.6$ and $b_4 = 0.5$. Again, we can think of a case with $\tau' = 4$. The expectation of the manipulator is $E(V|h_4^M \setminus \{buy_2\}, s_2^M = 0) = 0.5$ because buy_3 is not informative as it is the only possible history. The optimal action for the manipulator is not to trade and $\Pi_{2,4}^M = -0.0118 + 0 = -0.0118$.

The expected profits of $\tau' > 4$ under the manipulative strategy is computed in a similar way. With $\tau' = 7$, there are eight possible histories under the manipulative strategy. Among them, the manipulator gets positive profit from four histories and negative profit from the other four histories. In the cases of getting positive profit, the optimal actions of the manipulator at τ' are not to trade. In these cases, there are many buy orders in the history, so the expected values of the manipulator become high at $t = \tau'$. The ask prices after the histories, however, are even higher than those. It makes the optimal action of the manipulator at $t = \tau' = 7$ to be no trade. The manipulator bought the asset at time $t = \tau = 2$ with ask price a_2 under the manipulative strategy. The expected total profit is $E(V|h_7 \setminus \{buy_2\}, s_2^M = 0) - a_2$ and it is higher than the probable profit of executing an additional action of buying or selling at $t = \tau' = 7$. It can be interpreted as keeping the asset which would bring higher value even though it was bought at a slightly high price in the past.

Interesting actions happen in the negative profit cases as well. The optimal action of the manipulator is selling in these cases. Contrary to the positive profit cases, there are fewer buy orders in the history which drives the expected value of the manipulator to decrease. The bid prices, however, do not decline as much as the expected value of the manipulator. Thus, the optimal action of the manipulator is selling although $b_7 < a_2$ and the total profits of these cases are negative. It can be thought to be an action minimizing loss when the bad history occurs. Considering both cases of positive and negative profit, the total expected profit of the manipulative strategy is $\Pi^M_{\tau,\tau'} = 0.0469$ with $\tau' = 7$.

The expected profit of each strategy assuming $\tau' = 3$ to 7 is shown in Table 4.1. We can find that the manipulative strategy gives higher expected profit than the standard one when $\tau' = 7$.

4.4.2 Case 2: Fixed τ and τ' with $\mu < 1$

The second example is more general than the first one. With $\mu < 1$, there are noise traders on the event day. We can find an example that the manipulative strategy is profitable with $\mu = 0.9$ and other parameters are the same.

Time τ and τ' are also fixed at 2 and 7 as before and the manipulator takes action in the asset market at $t = \tau = 2$ with a bad signal ($s^M_2 = 0$) after observing a buy in the initial period. Different from the previous case, the optimal action of the informed traders depending on their signal at $t = 2$ is buying with a good signal and selling with a bad signal. It is because there are not only traders with a good signal but also noise traders who would buy in the previous period, so the bid and ask prices and the expectation of the traders are different from the first case. The optimal action of the manipulator with the standard strategy is selling at $t = \tau = 2$ since $E(V|h_2, s^M_2 = 0) = 0.4956$. The profit from the standard strategy is $b_2 - E(V|h_2, s^M_2 = 0) = 0.0044$. The manipulative strategy in this case is inflating the price by buying even with a negative profit at $t = 2$, which is $E(V|h_2, s^M_2 = 0) - a_2 = -0.01452$.

As before, the histories are different depending on the strategy. After the standard strategy of selling at $t = 2$, the optimal action of the other traders at $t = 3$ is buying with a good signal and selling with a bad signal. The ask and bid price are $a_3 = 0.5107$ and $b_3 = 0.4893$. Given these, the optimal action of the manipulator is no trade since $E(V|h^*_3 \setminus \{sell_2\}, s^M_2 = 0) = 0.4956$. The expected profit is $\Pi^*_{2,3} = 0.0044 + 0 =$

Table 4.2 Expected profit of Case 2

	Expected profit	
	Standard strategy ($\Pi^*_{\tau,\tau'}$)	Manipulative strategy ($\Pi^M_{\tau,\tau'}$)
$\tau = 2$	0.0044	-0.0145
$\tau' = 3$	0.0044	-0.0090
4	0.0538	-0.0057
5	0.0469	0.0304
6	0.0332	0.0274
7	0.0299	0.0404

Note: the parameter values used in the example are $\alpha = 0.01, \delta = 0.5, \mu = 0.9, q = q^M = 0.6$, and $\varepsilon_b = \varepsilon_s = 0.2$.

0.0044.

At $t = 4$, three histories are possible: buy, sell or no trade. The probability of each history from the manipulator's perspective is $Pr(buy_3|h^*_3 \setminus \{sell_2\}, s^M_2 = 0) = 0.4692$, $Pr(NT_3|h^*_3 \setminus \{sell_2\}, s^M_2 = 0) = 0.06$ and $Pr(sell_3|h^*_3 \setminus \{sell_2\}, s^M_2 = 0) = 0.4708$.

The optimal action of the informed traders at $t = 4$ is the same for the three different histories as buying with a good signal and selling with a bad signal. Assuming $\tau' = 4$, the optimal action of the manipulator is buying after observing a buy order at $t = 3$ and it is selling after not to trade or selling at $t = 3$. With the optimal actions of the manipulator, the expected profit becomes $\Pi^*_{2,4} = 0.0538$.

The expected profit of the manipulator under the standard strategy can be calculated with $\tau' > 4$ accordingly. If $\tau' = 7$, there are 81 possible histories and the optimal action of the manipulator is selling in 42 histories, buying in 29 and not to trade in 10 histories. Similar to the first case, buys are chosen from the cases with the expectation of the manipulator increased higher than the ask prices and sells are from the cases with the expectations decreased lower than the bids. From these, the expected total profit is $\Pi^*_{2,7} = 0.0299$.

Looking at the histories under the manipulative strategy, herd buying occurs at $t = 3$ after the manipulative buying at $t = 2$. The optimal action of the informed traders is buying with any signal, and the ask and bid price are $a_3 = 0.5388$ and $b_3 = 0.5011$. If $\tau' = 3$, the expected asset value of the manipulator is $E(V|h^M_3 \setminus \{buy_2\}, s^M_2 = 0) =$

0.4956. Therefore, the optimal action of the manipulator is selling and the expected total profit is $\Pi^M_{2,3} = b_3 - a_2 = -0.0090$.

Even though the previous period is herd buying of the informed traders, there are three possible histories, buy, sell, or no trade, because of the noise traders. Since the action of selling or not to trade comes from the noise traders only, the probability of each history is $Pr(buy_3|h^M_3 \setminus \{buy_2\}, s^M_2 = 0) = 0.92$, $Pr(NT_3|h^M_3 \setminus \{buy_2\}, s^M_2 = 0) = 0.06$ and $Pr(sell_3|h^M_3 \setminus \{buy_2\}, s^M_2 = 0) = 0.02$. The expected value of traders with signals are identical after the three histories as no information is added after the action.

In the history of buy order at $t = 3$, the optimal action of informed traders is buying with a good signal and no trade with a bad signal. After observing no trade or sell order at $t = 3$, herd buying occurs again. If we assume $\tau' = 4$, the expected asset value of the manipulator after the three possible histories are the same as $E(V|h^M_3 \setminus \{buy_2\}, buy_3, s^M_2 = 0) = 0.4956$, and the optimal action of the manipulator is buying for the history of buy order at $t = 3$ and selling for the other histories. The expected total profit of the manipulator is $\Pi^M_{2,4} = -0.0057$.

The expected profit of the manipulator when $\tau' > 4$ can be calculated in a similar way. If $\tau' = 7$, there are 81 possible histories; selling is optimal for the manipulator after 70 histories, not to trade after 8 and buying after 3 histories. As in the first example, the manipulator chooses not to trade when her expected value of the asset is high after histories like 6 buys. This is because keeping the asset is more profitable than closing the position by selling even though it was bought with slightly high price in the past. Also, selling is chosen after histories like 4 sells after initial 2 buys out of the motive to minimize the loss. From these, the expected profit is $\Pi^M_{2,7} = 0.0404$. As shown in Table 4.2, the manipulative strategy brings more profit than the standard strategy with $\tau' = 7$.

4.4.3 Case 3: Without herding

In the previous section, we have shown that the event uncertainty is essential for profitable manipulation. Although the herding is generated by the event uncertainty, it does not guarantee that herding is necessary for the profitable manipulation.

In the previous cases, there are herding behaviours of traders after the manipulative action, and the manipulative strategy becomes profitable. The herding, however, is not necessary to make the manipulative strategy profitable. There exists a case showing this with $\mu = 1$. The other parameter values are $\alpha = 0.04, \delta = 0.5$, $q = q^M = 0.6$, $\varepsilon_b = \varepsilon_s = 0.2$, and the timing is $\tau = 2$ and $\tau' = 6$ with a buy order at $t = 1$ and a bad signal for the manipulator $s_2^M = 0$.

After observing a buy order at $t = 1$, the optimal action of the traders with a good signal is buying; with a bad signal, it is no trade. Under the standard strategy, the manipulator chooses no trade. After the no trade at $t = \tau = 2$, there are 8 possible histories to $t = \tau' = 6$. The expected profit is $\Pi^*_{2,6} = 0.0008$.

The action of the manipulative strategy at $t = \tau = 2$ is buying and it brings a loss of 0.0410. In the previous cases, the herd buying occurs for the other informed traders at $t = 3$ after a manipulative buy. However, the optimal action of ordinary informed traders in this case is buying with a good signal and no trade with a bad signal. In all possible 8 histories up to $t = 6$, the herd buying or herd selling does not appear, but the expected profit of the manipulative strategy is higher than that of the standard strategy, $\Pi^M_{2,6} = 0.0190 > \Pi^*_{2,6} = 0.0008$ with $\tau' = 6$.

From this, we can say that profitable manipulation is possible without herding. Although there is no herding behaviour, the manipulative action which increases the probability of the action by the other informed traders to inflate the price can make the manipulative strategy profitable. It happens by the gap between the expected value of the traders and the market maker caused by the event uncertainty.

Table 4.3 Expected profit of Case 3

	Expected profit	
	Standard strategy $(\Pi^*_{\tau,\tau'})$	Manipulative strategy $(\Pi^M_{\tau,\tau'})$
$t = 2$	0	-0.0410
$t' = 3$	0	-0.0410
4	0.0324	0.0090
5	0.0163	0.0014
6	0.0008	0.0190

Note: the parameter values used in the example are $\alpha = 0.04, \delta = 0.5, \mu = 1, q = q^M = 0.6$, $\varepsilon_b = \varepsilon_s = 0.2$.

4.4.4 Case 4: τ' is not fixed with $\mu = 1$

We also can think about a case in which the time of the second action of the manipulator, τ' is not fixed. The manipulator can choose τ' which makes the expected profit maximum at the point of the first action τ. To make the manipulative strategy profitable the largest expected profit should be higher for the manipulative strategy than the standard one. It can be shown in the example of $\alpha = 0.01$, $\delta = 0.5$, $q = q^M = 0.55$, and $\varepsilon_b = \varepsilon_s = 0.1$. The first point of action is $\tau = 2$ after a buy at $t = 1$, and the manipulator gets a bad signal, $s_2^M = 0$.

The optimal action of an informed trader at $t = 2$ is buying with a good signal and no trade with a bad signal. Since the precision of the signal is the same for the manipulator and other traders, the optimal action of the manipulator under the standard strategy is no trade as well. Considering all the possible histories, the maximum expected profit the manipulator can obtain under the standard strategy is 0.0063 at $\tau' = 4$.[7]

Under the manipulator strategy, the manipulator would buy at $t = \tau = 2$. It brings a negative profit of -0.0201. The expected maximum profit taking into account the loss is obtained with $\tau' = 6$ to be 0.0082.

[7] The expected profits under both of the strategies up to $t' = 8$ are calculated, and those after $t' = 9$ are checked by simulations of 10,000 times.

Table 4.4 Expected profit of Case 4

	Expected profit	
	Standard strategy ($\Pi^*_{\tau,\tau'}$)	Manipulative strategy ($\Pi^M_{\tau,\tau'}$)
$t = 2$	0	-0.0201
$t' = 3$	0	-0.0201
4	0.0063	0.0049
5	0	0.0008
6	0	0.0082
7	0	0.0008

Note: the parameter values used in the example are $\alpha = 0.01$, $\delta = 0.5$, $q = q^M = 0.55$, and $\varepsilon_b = \varepsilon_s = 0.1$.

4.4.5 Case 5: τ' is not fixed with $\mu < 1$

With $\mu < 1$, there exist cases that the maximum profit of the manipulative strategy is more profitable than the standard strategy as well. To analyse this, we rely on simulations of $10,000$ times to calculate the expected profit of each strategy.[8]

In this example, the time of the first action τ is set to be 3, and the manipulator with a bad signal ($s_3^M = 0$) takes part in after observing two buy orders. Other parameters are $\alpha = 0.01$, $\delta = 0.5$, $\mu = 0.3$, $q = q^M = 0.6$ and $\varepsilon_b = \varepsilon_s = 0.2$.

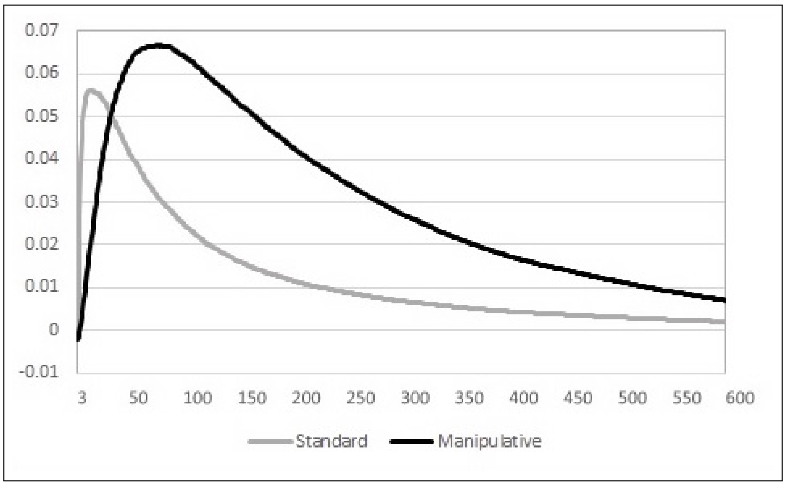
The optimal action of an informed traders at $t = 3$ is buying with a good signal and no trade with a bad signal given the ask and bid price of $a_3 = 0.5046$ and $b_3 = 0.5015$. The expected asset value of the manipulator with a bad signal is $E(V|h_3, s_3^M = 0) = 0.5025$. Hence, the optimal action of the manipulator under the standard strategy is not to trade, and the expected profit is zero. With the manipulative strategy, she buys the asset to inflate the price. It would bring the loss of $E(V|h_3, s_3^M = 0) - a_3 = -0.0022$.

After the action of not to trade under the standard strategy, buy, sell or no trade orders are followed in each period. As shown in Figure 4.1, the maximum profit from the standard strategy can be achieved by $\tau' = 15$ and the profit is $\Pi^*_{3,15} = 0.0563$.

Under the manipulative strategy, the maximum profit can be attained with $\tau' = 80$ and it is $\Pi^M_{3,80} = 0.0665$. Therefore, the manipulative strategy is more profitable than

[8]The differences between the analytic computation and the simulation up to $t = 9$ are less than 0.0005.

Figure 4.1 Expected profit of Case 5



Note: The profits are calculated based on the average of 10,000 simulations of $\alpha = 0.01$, $\delta = 0.5$, $\mu = 0.3$, $q = q^M = 0.6$ and $\varepsilon_b = \varepsilon_s = 0.2$ and $\tau = 3$. The manipulator gets a bad signal ($s_3^M = 0$) and the history of orders before $t = 3$ is two buys.

the standard strategy for the manipulator who can choose τ'.

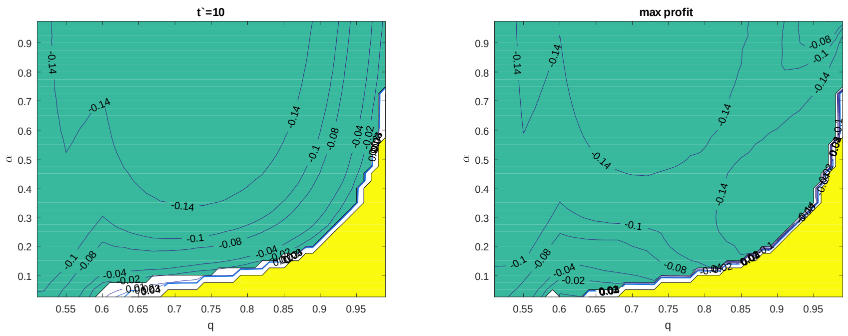
4.5 Conditions for the Profitable Manipulation

We have checked the possibility of the profitable manipulation in different cases. In this section, we analyse the conditions for the profitable manipulation depending on key parameter values. The excess profit of the manipulative strategy is computed by the expected profit of the manipulative strategy deduced by that of the standard strategy. If the excess profit is positive, the manipulative strategy is profitable.

4.5.1 Case with $\mu = 1$

All the examples below are of $\mu = 1$, $\tau = 2$ with $h_2 = \{buy_1\}$, and $\tau' = 10$ or where the expected profit is maximum. The action of noise traders is set to be $\varepsilon_b = \varepsilon_s = 0.2$ if it is not noted specifically. The manipulator has a bad signal ($s_2^M = 0$), and buys at $t = 2$ as a manipulative strategy. The standard strategy at $t = 2$ is decided depending on the parameters. The yellow region is where the optimal action of the standard strategy is buying as well, and the green region is where the manipulative strategy is not profitable (excess profit is negative). The expected profit is calculated based on simulations of 1,000 times.

Figure 4.2 Excess profit with $\mu = 1$ keeping $q^M = 0.6$



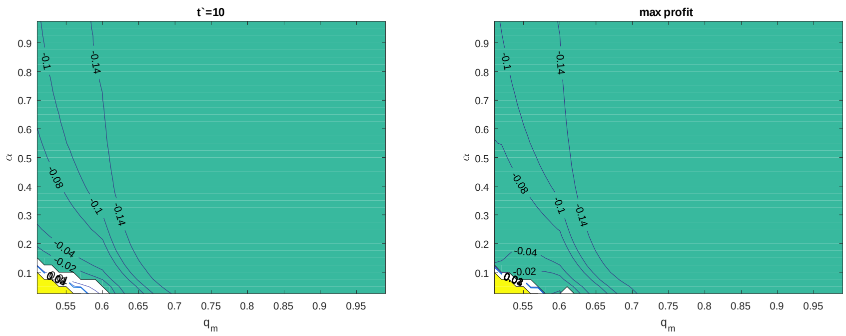
Note: Excess profit is the expected profit of the manipulative strategy minus that of the standard strategy. Numbers on the graph is excess profit of the manipulative strategy with parameters $\mu = 1$, $q^M = 0.6$, $\varepsilon_b = \varepsilon_s = 0.2$, $\tau = 2$ with $h_2 = \{buy_1\}$, and $s_2^M = 0$. The left panel is the case that $\tau' = 10$ and the right panel is that τ' is set to make the excess profit maximum. The green area is where the excess profit is negative; the yellow area is where the optimal action at τ under the standard strategy is buying as well.

Figure 4.2 presents the excess profit of the manipulative strategy when α and q changes keeping $q^M = 0.6$[9]. The left panel shows the results that the second period of action is $\tau' = 10$ and the right panel is that the manipulator can choose τ' at the time of first action τ. Both of the graph show that the profitable manipulation occurs with the combination of small α and small q or large α and large q. It can be interpreted as the influence of the manipulative action to other agents. Small α means high event uncertainty, and it generates herding more likely as the gap in the expected asset value between the informed traders and the market maker is large. Also, with a low q, the informed traders do not rely on their signal that much because of the low precision of their signals. Therefore, with low α and low q, the manipulative action at time t changes the optimal action of the other informed traders and inflate the price to make the manipulation profitable. With large α, there is less possibility of herding, but large q causes the market maker and the other traders to think that the traders' actions are more informative. As the market maker is not aware of the existence of the manipulator, the manipulative action affects prices a lot with high q. Hence, it inflates the price and enables the manipulative strategy to be profitable.

We also can check the excess profit of the manipulative strategy depending on q^M.

[9]The simulation results for the cases changing other parameter values are in appendix such as q and q^M or ε_b and ε_s simultaneously

Figure 4.3 Excess profit with $\mu = 1$ keeping $q = 0.6$



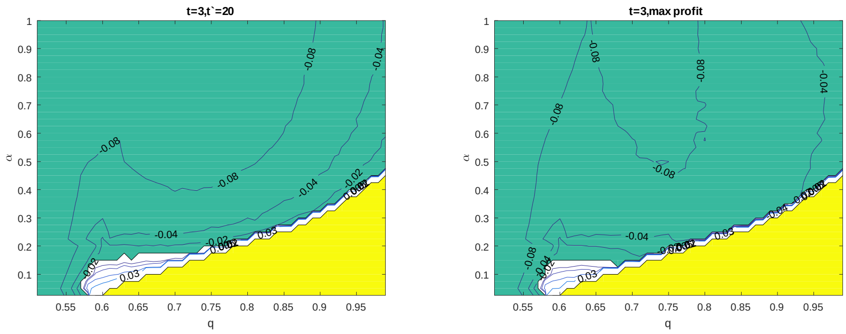
Note: Excess profit is the expected profit of the manipulative strategy minus that of the standard strategy. Numbers on the graph is excess profit of the manipulative strategy with parameters $\mu = 1$, $q = 0.6$, $\varepsilon_b = \varepsilon_s = 0.2$, $\tau = 2$ with $h_2 = \{buy_1\}$, and $s_2^M = 0$. The left panel is the case that $\tau' = 10$ and the right panel is that τ' is set to make the excess profit maximum. The green area is where the excess profit is negative; the yellow area is where the optimal action at τ under the standard strategy is buying as well.

The excess profit of manipulative strategy is presented in Figure 4.3 keeping $q = 0.6$. The first thing to note is that the manipulative strategy is profitable only with low q^M. The manipulative strategy is basically going against the manipulator's private signal. If the precision of the manipulator's private signal is high, she would rely on it and it is hard to get a profit by the action opposite to the signal. As for the event uncertainty, the manipulative strategy is profitable with low α as before. However, the precision of the manipulator's private signal q^M works in the opposite direction to q. As the event uncertainty becomes smaller with higher α, the level of q^M which enables the manipulative strategy to be profitable is smaller. It also can be interpreted that even though the possibility of herding decreases, the manipulative strategy that going against her signal can be profitable if the manipulator's signal becomes less informative.

4.5.2 Case with $\mu = 0.5$

The previous section is about the case that all the traders are informed on the informational event day. In this section, the conditions for the profitable manipulation with $\mu < 1$ are analysed. All the examples below are of $\mu = 0.5$, $\tau = 3$, and $\tau' = 20$ or where the expected profit is maximum after 2 buys in history. The parameter for the noise traders is set to be $\varepsilon_b = \varepsilon_s = 0.2$ if it is not noted specifically. The manipulator has a bad signal ($s_3^M = 0$) as before, and buys at $t = 3$ with a manipulative strategy. The standard strategy at $t = 3$ is decided depending on the parameters. The regions and the number of simulation are the same as before.

Figure 4.4 Excess profit with $\mu = 0.5$ keeping $q^M = 0.6$



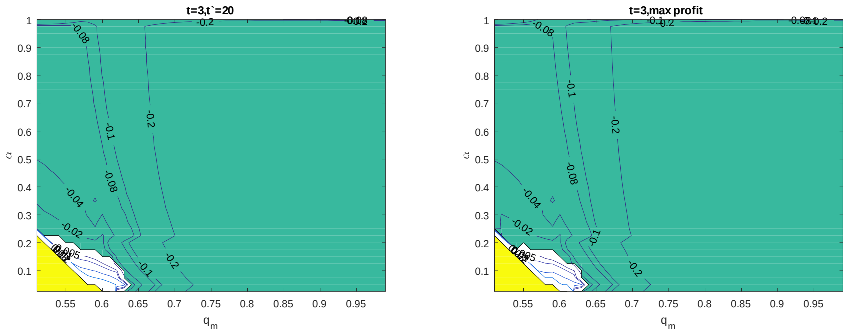
Note: Excess profit is the expected profit of the manipulative strategy minus that of the standard strategy. Numbers on the graph are excess profit of the manipulative strategy with parameters $\mu = 0.5$, $q^M = 0.6$, $\varepsilon_b = \varepsilon_s = 0.2$, $\tau = 3$ with $h_3 = \{buy_1, buy_2\}$, and $s_3^M = 0$. The left panel is the case that $\tau' = 20$ and the right panel is that τ' is set to make the excess profit maximum. The green area is where the excess profit is negative; the yellow area is where the optimal action at τ under the standard strategy is buying as well.

The excess profit of the manipulative strategy keeping q^M is analysed first in Figure 4.4. Similar to the case with $\mu = 1$, the manipulative strategy is profitable with a combination of smaller α and low q, or larger α and high q. As before this property can be explained by the influence of the manipulator's action to other agents.

The difference from the case with $\mu = 1$ is that the profitable region is larger. It can be understood that, with the inclusion of the noise traders, there is more probability of herding because it makes the market maker harder to exclude no event day after looking at the history of orders. Consequently, it causes the profitable manipulation region to become larger.

Figure 4.5 presents the excess profit with $\mu = 0.5$ keeping $q = 0.6$. Again, the profitable region is larger than the case with $\mu = 1$. The level of q^M which enables the manipulation profitable is mostly lower than $q = 0.6$. Also, as α is higher, q^M becomes smaller similar to the case with $\mu = 1$.

Figure 4.5 Excess profit with $\mu = 0.5$ keeping $q = 0.6$



Note: Excess profit is the expected profit of the manipulative strategy minus that of the standard strategy. Numbers on the graph are excess profit of the manipulative strategy with parameters $\mu = 0.5$, $q = 0.6$, $\varepsilon_b = \varepsilon_s = 0.2$, $\tau = 3$ with $h_3 = \{buy_1, buy_2\}$, and $s_3^M = 0$. The left panel is the case that $\tau' = 20$ and the right panel is that τ' is set to make the excess profit maximum. The green area is where the excess profit is negative; the yellow area is where the optimal action under the standard strategy is buying as well.

4.6 Conclusion

Manipulation can be defined as a behaviour according to which one or more market operators try to distort the regular pattern of the prices. Financial institutions and academic research have been studying the phenomenon and trying to provide an exhaustive definition of its features. However, different behaviours, as also detected during the supervision activity of the financial surveillance authorities, can be included in this category.

In this paper, we studied trade based market manipulation and analysed one of the mechanisms through which this phenomenon arises. We considered a sequential trading financial market in which rational informed traders and noise traders exchange an asset with a market maker. Also, we defined a potential manipulator, that is an informed trader who is allowed to trade twice. We showed that there exist cases in which the potential manipulator chooses a manipulative strategy, that is she prefers not to follow her signal in the first period she is allowed to trade.

We also observed that the result is strictly connected to event uncertainty. The behaviour caused by it, such as herding, can arise after the manipulator's first action. In fact, we formally proved that manipulation is never chosen in equilibrium, because it is never profitable, in situations in which there is no event uncertainty. On the other hand, we verified through simulated data that, under given assumptions about the model parameters and for a set of second trades' periods, the potential manipulator prefers the manipulative strategy in equilibrium, because it is more profitable than the strategy in which she always behaved according to her beliefs. Through causing periods like herding, in fact, the manipulator affects the other rational traders' behaviour and the following pattern of the price of the asset and also delays the informational learning process of the market maker. These effects allow her to trade later against the market and take profits from it.

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Appendices

Appendix A

Appendix to Chapter 2

A.1 Expected values and updating beliefs

The expected value of the asset when the trader without ambiguity has a good ($S_t = 1$) and bad ($S_t = 0$) signal, and the ask and bid prices of the market maker are calculated as follows.

$$\begin{aligned} &E(V|S_t = 1, h_t) = Pr(V = 1|S_t = 1, h_t) \\ &= \frac{\int_{\underline{q}}^{\overline{q}} qf(q|h_t)\mathrm{d}qPr(V = 1|h_t)}{\int_{\underline{q}}^{\overline{q}} qf(q|h_t)\mathrm{d}qPr(V = 1|h_t) + \int_{\underline{q}}^{\overline{q}}(1-q)f(q|h_t)\mathrm{d}qPr(V = 0|h_t)} \end{aligned} \tag{A.1}$$

$$\begin{aligned} &E(V|S_t = 0, h_t) = Pr(V = 1|S_t = 0, h_t) \\ &= \frac{\int_{\underline{q}}^{\overline{q}}(1-q)f(q|h_t)\mathrm{d}qPr(V = 1|h_t)}{\int_{\underline{q}}^{\overline{q}}(1-q)f(q|h_t)\mathrm{d}qPr(V = 1|h_t) + \int_{\underline{q}}^{\overline{q}} qf(q|h_t)\mathrm{d}qPr(V = 0|h_t)} \end{aligned}$$

$$\begin{aligned} &a_t = E(V|buy_t, h_t) = Pr(V = 1|buy_t, h_t) \\ &= \frac{\int_{\underline{q}}^{\overline{q}} Pr(buy_t|V = 1, q, h_t)f(q|h_t)\mathrm{d}qPr(V = 1|h_t)}{\sum_{v=0}^{1}\int_{\underline{q}}^{\overline{q}} Pr(buy_t|V = v, q, h_t)f(q|h_t)\mathrm{d}qPr(V = v|h_t)} \end{aligned} \tag{A.2}$$

$$
\begin{aligned}
b_t &= E(V|sell_t, h_t) = Pr(V = 1|sell_t, h_t)\\
&= \frac{\int_{\underline{q}}^{\overline{q}} Pr(sell_t|V = 1, q, h_t) f(q|h_t)\mathrm{d}q Pr(V = 1|h_t)}{\sum_{v=0}^{1} \int_{\underline{q}}^{\overline{q}} Pr(sell_t|V = v, q, h_t) f(q|h_t)\mathrm{d}q Pr(V = v|h_t)}
\end{aligned}
$$

where $X_t = buy$, $X_t = sell$ or $X_t = no\,trade$ are expressed as buy_t, $sell_t$ or NT_t for simplicity in notation.

The expected asset value of the ambiguity averse traders has q as an argument. If the trader has got a good or bad signal, the expected value of V is as follows.

$$
\begin{aligned}
E(V|S_t = 1, h_t, q) &= Pr(V = 1|S_t = 1, h_t, q)\\
&= \frac{qPr(V = 1|h_t, q)}{qPr(V = 1|h_t, q) + (1 - q)Pr(V = 0|h_t, q)}\\
E(V|S_t = 0, h_t, q) &= Pr(V = 1|S_t = 0, h_t, q)\\
&= \frac{(1 - q)Pr(V = 1|h_t, q)}{(1 - q)Pr(V = 1|h_t, q) + qPr(V = 0|h_t, q)}
\end{aligned}
$$

A.2 Proof of Proposition 1

Proposition 1 is proven by comparing the expectation of the trader with a good signal and the ask price. If the expected asset value of the traders without ambiguity with a good signal is higher than the ask price, the trader buys with a good signal; if the expected value with a bad signal is lower than the bid price, the trader sells with a bad signal.

At period t, the expected value of the asset by the trader without ambiguity and the ask price are as in equation (A.1) and (A.2). Comparing these two values can be simplified to be comparing those two followings.

$$\frac{\int_{\underline{q}}^{\overline{q}} q f(q|h_t)\mathrm{d}q}{\int_{\underline{q}}^{\overline{q}}(1-q) f(q|h_t)\mathrm{d}q} \text{ and}$$
$$\frac{\int_{\underline{q}}^{\overline{q}} Pr(buy_t|V=1,q,h_t) f(q|h_t)\mathrm{d}q}{\int_{\underline{q}}^{\overline{q}} Pr(buy_t|V=0,q,h_t) f(q|h_t)\mathrm{d}q}$$

Depending on the assumed action of the traders, $Pr(buy_t|V=v,q,h_t)$ changes. The action of traders that maximise $\frac{\int_{\underline{q}}^{\overline{q}} Pr(buy_t|V=1,q,h_t) f(q|h_t)\mathrm{d}q}{\int_{\underline{q}}^{\overline{q}} Pr(buy_t|V=0,q,h_t) f(q|h_t)\mathrm{d}q}$ for a given $f(q|h_t)$ is that all the informed traders with a good signal buy regardless of their ambiguity.

$$\begin{aligned}&\frac{\int_{\underline{q}}^{\overline{q}} Pr(buy_t|V=1,q,h_t) f(q|h_t)\mathrm{d}q}{\int_{\underline{q}}^{\overline{q}} Pr(buy_t|V=0,q,h_t) f(q|h_t)\mathrm{d}q}\\ &\leq \frac{\int_{\underline{q}}^{\overline{q}}\{\mu q+(1-\mu)\epsilon/2\} f(q|h_t)\mathrm{d}q}{\int_{\underline{q}}^{\overline{q}}\{\mu(1-q)+(1-\mu)\epsilon/2\} f(q|h_t)\mathrm{d}q}\end{aligned} \tag{A.3}$$

Also, $q > 0.5$ implies that

$$\frac{\int_{\underline{q}}^{\overline{q}}\{\mu q+(1-\mu)\epsilon/2\} f(q|h_t)\mathrm{d}q}{\int_{\underline{q}}^{\overline{q}}\{\mu(1-q)+(1-\mu)\epsilon/2\} f(q|h_t)\mathrm{d}q} < \frac{\int_{\underline{q}}^{\overline{q}} q f(q|h_t)\mathrm{d}q}{\int_{\underline{q}}^{\overline{q}}(1-q) f(q|h_t)\mathrm{d}q} \tag{A.4}$$

with $\mu > 0$. From the condition (A.3) and (A.4), $\frac{\int_{\underline{q}}^{\overline{q}} Pr(buy_t|V=1,q,h_t) f(q|h_t)\mathrm{d}q}{\int_{\underline{q}}^{\overline{q}} Pr(buy_t|V=0,q,h_t) f(q|h_t)\mathrm{d}q} < \frac{\int_{\underline{q}}^{\overline{q}} q f(q|h_t)\mathrm{d}q}{\int_{\underline{q}}^{\overline{q}}(1-q) f(q|h_t)\mathrm{d}q}$, so $a_t < E(V|S_t=1,h_t)$. It implies that the traders without ambiguity always buy with a good signal. The case with a bad signal ($S=0$) is symmetric to this.

A.3 Proof of Proposition 2

The condition for the ambiguity averse trader to buy with a good signal and sell with a bad signal at $t=1$ is $\min_{q\in\mathcal{Q}_0} E(V|S_1=1,q) > a_t$ and $\max_{q\in\mathcal{Q}_0} E(V|S_1=0,q) < b_t$.

These are simplified to the followings.

$$\min_{q\in\mathcal{Q}_0}\frac{q}{1-q}\frac{\delta}{1-\delta}>\frac{\mu(\underline{q}+\overline{q})/2+(1-\mu)\epsilon/2}{\mu\{1-(\underline{q}+\overline{q})/2\}+(1-\mu)\epsilon/2}\frac{\delta}{1-\delta}$$
$$\max_{q\in\mathcal{Q}_0}\frac{1-q}{q}\frac{\delta}{1-\delta}>\frac{\mu\{1-(\underline{q}+\overline{q})/2\}+(1-\mu)\epsilon/2}{\mu(\underline{q}+\overline{q})/2+(1-\mu)\epsilon/2}\frac{\delta}{1-\delta}$$

Those conditions are derived to the one below.

$$\underline{q}>\frac{\mu\overline{q}/2+(1-\mu)\epsilon/2}{\mu/2+(1-\mu)\epsilon}=\tilde{q}$$

Therefore, with $\underline{q}>\tilde{q}$ the action of the ambiguity averse traders at $t=1$ becomes 'buy with a good signal and sell with a bad signal'.

A.4 Proof of Proposition 3

Herd buying or contrarian buying is the behaviour of the traders choosing buy regardless of their signals. The condition for the ambiguity averse trader to buy with any signal can be simplified as follows.

$$\begin{aligned}&\min_{q\in\mathcal{Q}_t}\frac{(1-q)Pr(V=1|h_t,q)}{qPr(V=0|h_t,q)}\\&>\frac{\{\mu\gamma E(q|h_t)+\mu(1-\gamma)+(1-\mu)\epsilon/2\}Pr(V=1|h_t)}{\{\mu\gamma(1-E(q|h_t))+\mu(1-\gamma)+(1-\mu)\epsilon/2\}Pr(V=0|h_t)}\end{aligned}\tag{A.5}$$

Since $Pr(V=0|h_t,q)=1-Pr(V=1|h_t,q)$, $\min_{q\in\mathcal{Q}_t}\frac{Pr(V=1|h_t,q)}{Pr(V=0|h_t,q)}=\frac{\min_{q\in\mathcal{Q}_t}Pr(V=1|h_t,q)}{\max_{q\in\mathcal{Q}_t}Pr(V=0|h_t,q)}$. If $\rho=0$, $\mathcal{Q}_t=\mathcal{Q}_0=\{q:\underline{q}\leq q\leq\overline{q}\}$ for any t. Under this $\mathcal{Q}_t$, $\min_{q\in\mathcal{Q}_t}\frac{Pr(V=1|h_t,q)}{Pr(V=0|h_t,q)}\leq\frac{Pr(V=1|h_t)}{Pr(V=0|h_t)}$ by the mean-value theorem. $Pr(V=v|h_t)=\int_{\underline{q}}^{\overline{q}}Pr(V=v|h_t,q)f(q|h_t)\mathrm{d}q$ and $\mathcal{Q}_t$ is support for q. If we define a function $G(x;v,h_t)=\int_{-\infty}^{x}Pr(V=v|h_t,q)f(q|h_t)\mathrm{d}q$, $g(q;v,h_t)=\frac{\mathrm{d}G(q;v,h_t)}{\mathrm{d}q}$, $Pr(V=v|h_t)=\int_{\underline{q}}^{\overline{q}}Pr(V=v|h_t,q)f(q|h_t)\mathrm{d}q=G(\overline{q}|v,h_t)-G(\underline{q}|v,h_t)$. From the mean-value theorem, there exists a $q'\in\{q:\underline{q}\leq q\leq\overline{q}\}$ such that $\frac{G(\overline{q};v,h_t)-G(\underline{q};v,h_t)}{\overline{q}-\underline{q}}=g(q';v,h_t)$. Using this property, $\min_{q\in\mathcal{Q}_t}\frac{Pr(V=1|h_t,q)}{Pr(V=0|h_t,q)}=$

$\frac{\min_{q\in\mathcal{Q}_t} Pr(V=1|h_t,q)}{\max_{q\in\mathcal{Q}_t} Pr(V=0|h_t,q)} \leq \frac{Pr(V=1|h_t)}{Pr(V=0|h_t)}$. It guarantees the following condition.

$$\begin{aligned}
&\min_{q\in\mathcal{Q}_t} \frac{(1-q)Pr(V=1|h_t,q)}{qPr(V=0|h_t,q)} < \min_{q\in\mathcal{Q}_t} \frac{Pr(V=1|h_t,q)}{Pr(V=0|h_t,q)} \leq \frac{Pr(V=1|h_t)}{Pr(V=0|h_t)} \\
&< \frac{\{\mu\gamma E(q|h_t) + \mu(1-\gamma) + (1-\mu)\epsilon/2\}Pr(V=1|h_t)}{\{\mu\gamma(1-E(q|h_t)) + \mu(1-\gamma) + (1-\mu)\epsilon/2\}Pr(V=0|h_t)} \qquad \text{(A.6)}
\end{aligned}$$

Condition (A.6) implies that $\min_{q\in\mathcal{Q}_0} E(V|S_t=0,h_t,q) < a_t$, so it is impossible for the ambiguity averse traders to buy with a bad signal in equilibrium if $\rho = 0$. Therefore, herd buying or contrarian buying never occur. The case of herd selling or contrarian selling is symmetric to this.

A.5 Proof of Proposition 4

Look at the case with $\gamma = 0$ first. With $\rho = 0$, herding or contrarian for the ambiguity averse traders is impossible following Proposition 3. Suppose that there is zero possibility of choosing no trade with any signal for the ambiguity averse traders in equilibrium. Then, after the history of N orders, we can defined n_1 be the number of orders with a good signal, n_2 be the number of orders with a bad signal, and $N - n_1 - n_2$ be the number of orders with pure noise.

Since it is assumed that the possibility of choosing no trade with any signal is zero. As N goes to infinity, $\frac{n_1}{n_2} \to \frac{q^T}{1-q^T}$ where q^T is true precision of the private signal since the private signal S_t is i.i.d. Therefore, $E(q|h_t)$ converges to q^T.

The condition for the ambiguity averse traders with a good signal do not trade in equilibrium can be simplified as follows.

$$\begin{aligned}
&\min_{q\in\mathcal{Q}_t} \frac{q}{1-q}\frac{Pr(V=1|h_N,q)}{1-Pr(V=1|h_N,q)} \\
&< \frac{Pr(buy_t|V=1,h_N)Pr(V=1|h_N)}{Pr(buy_t|V=0,h_N)\{1-Pr(V=1|h_N)\}} \\
&= \frac{Pr(V=1|h_N)}{1-Pr(V=1|h_N)} \qquad \text{(A.7)}
\end{aligned}$$

Suppose $V = 1$, then $m = n_1 - n_2$ increases as N increases. With a sufficiently large m, $arg \min_{q \in \mathcal{Q}_t} \frac{q}{1-q} \frac{Pr(V=1|h_N,q)}{1-Pr(V=1|h_N,q)}$ becomes $\underline{q}$. Then, the previous condition can be log transformed as follows.

$$
\begin{aligned}
&\log \frac{\underline{q}}{1-\underline{q}} \\
&\leq \log \frac{Pr(V=1|h_N)}{1-Pr(V=1|h_N)} - \log \frac{Pr(V=1|h_N,\underline{q})}{1-Pr(V=1|h_N,\underline{q})} \\
&= \log \frac{Pr(h_N|V=1)\delta}{Pr(h_N|V=0)(1-\delta)} - \log \frac{Pr(h_N|V=1,\underline{q})\delta}{Pr(h_N|V=0,\underline{q})(1-\delta)} \\
&= \log \frac{Pr(\hat{a}_N|V=1)Pr(h_{N-1}|V=1)\delta}{Pr(\hat{a}_N|V=0)Pr(h_{N-1}|V=0)(1-\delta)} - \log \frac{Pr(\hat{a}_N|V=1,\underline{q})Pr(h_{N-1}|V=1,\underline{q})\delta}{Pr(\hat{a}_N|V=0,\underline{q})Pr(h_{N-1}|V=0,\underline{q})(1-\delta)}
\end{aligned}
\tag{A.8}
$$

$E(q|h_t) > \underline{q}$. Thus, the last line of condition (A.8) increases as m increases since $\frac{Pr(\hat{a}_N|V=1)}{Pr(\hat{a}_N|V=0)} > \frac{Pr(\hat{a}_N|V=1,\underline{q})}{Pr(\hat{a}_N|V=0,\underline{q})}$ where $\hat{a}_N$ is an action with a good signal at $t = N$ in equilibrium with.

The left hand side of condition (A.8) is set to $\log \frac{\underline{q}}{1-\underline{q}}$ while the last line keeps increasing. Hence, with sufficiently large N, condition (A.8) holds and the ambiguity averse traders with a good signal choose no trade in equilibrium.

Looking at the case with a bad signal, the condition for the ambiguity averse traders choose no trade with a bad signal is as follows.

$$
\begin{aligned}
&\max_{q \in \mathcal{Q}_t} \frac{1-q}{q} \frac{Pr(V=1|h_N,q)}{1-Pr(V=1|h_N,q)} \\
&> \frac{Pr(V=1|h_N)}{1-Pr(V=1|h_N)}
\end{aligned}
\tag{A.9}
$$

As m increases, the influence of the first term in the first line of the previous condition diminishes, so $arg \max_{q \in \mathcal{Q}_t} \frac{1-q}{q} \frac{Pr(V=1|h_N,q)}{1-Pr(V=1|h_N,q)}$ becomes $\overline{q}$. Then, the previous condition can be log transformed as follows.

$$
\begin{aligned}
&\log \frac{1-\overline{q}}{\overline{q}} \\
&\geq \log \frac{Pr(V=1|h_N)}{1-Pr(V=1|h_N)} - \log \frac{Pr(V=1|h_N,\overline{q})}{1-Pr(V=1|h_N,\overline{q})} \\
&= \log \frac{Pr(h_N|V=1)\delta}{Pr(h_N|V=0)(1-\delta)} - \log \frac{Pr(h_N|V=1,\overline{q})\delta}{Pr(h_N|V=0,\overline{q})(1-\delta)} \\
&= \log \frac{Pr(\hat{a}_N|V=1)Pr(h_{N-1}|V=1)\delta}{Pr(\hat{a}_N|V=1)Pr(h_{N-1}|V=0)(1-\delta)} - \log \frac{Pr(\hat{a}_N|V=1,\overline{q})Pr(h_{N-1}|V=1,\overline{q})\delta}{Pr(\hat{a}_N|V=0,\overline{q})Pr(h_{N-1}|V=0,\overline{q})(1-\delta)}
\end{aligned}
\tag{A.10}
$$

The last term of condition (A.10) decreases as m increases since $\frac{Pr(\hat{a}_N|V=1)}{Pr(\hat{a}_N|V=0)} < \frac{Pr(\hat{a}_N|V=1,\overline{q})}{Pr(\hat{a}_N|V=0,\overline{q})}$ where $\hat{a}_N$ an action with a good signal at $t=N$ in equilibrium.

The first line of condition (A.10) is set to $\log \frac{1-\overline{q}}{\overline{q}}$ while the last line keeps decreasing. Hence, with sufficiently large N, condition (A.10) holds and the ambiguity averse traders with a bad signal choose no trade in equilibrium.

Therefore, the ambiguity averse traders with any signal choose no trade. It contradicts that there is zero possibility of the ambiguity averse traders choose no trade with any signal. Once the ambiguity averse traders start to choose no trade with any signal, there is no informational gain as N increases because all the informed traders choose no trade and other trades are just noise traders. The informed traders keep choosing no trade for any signal for any $t \geq N$.

Let us look at the case with $\gamma > 0$. With a positive proportion of the traders without ambiguity, every buy or sell orders in history reveals the information on the private signal. Since herding or contrarian is impossible with $\rho = 0$, for a history of N trades, we can define n_1 be the number of trades with a good signal, n_2 be the number of trades with a bad signal. Then $N - n_1 - n_2$ is the number of trades by noise traders. As N goes to infinity, $\frac{n_1}{n_2} \to \frac{q^T}{1-q^T}$ where q^T is true precision of the private signal since the private signal S_t is i.i.d.

The condition for the ambiguity averse traders with a good signal do not trade in equilibrium can be simplified as follows.

$$\min_{q\in\mathcal{Q}_t} \frac{q}{1-q}\frac{Pr(V=1|h_N,q)}{1-Pr(V=1|h_N,q)} < \frac{\mu\gamma E(q|h_N)+(1-\mu)\epsilon/2}{\mu\gamma(1-E(q|h_N))+(1-\mu)\epsilon/2}\frac{Pr(V=1|h_N)}{1-Pr(V=1|h_N)} \tag{A.11}$$

Suppose $V=1$, then $m=n_1-n_2$ increases as N increases. With a sufficiently large m, $arg\min_{q\in\mathcal{Q}_t}\frac{q}{1-q}\frac{Pr(V=1|h_N,q)}{1-Pr(V=1|h_N,q)}$ becomes $\underline{q}$. Then, the previous condition can be log transformed as follows.

$$\begin{aligned}
&\log\frac{\underline{q}}{1-\underline{q}} - \log\frac{\mu\gamma E(q|h_N)+(1-\mu)\epsilon/2}{\mu\gamma(1-E(q|h_N))+(1-\mu)\epsilon/2} \\
&\le \log\frac{Pr(V=1|h_N)}{1-Pr(V=1|h_N)} - \log\frac{Pr(V=1|h_N,\underline{q})}{1-Pr(V=1|h_N,\underline{q})} \\
&= \log\frac{Pr(h_N|V=1)\delta}{Pr(h_N|V=0)(1-\delta)} - \log\frac{Pr(h_N|V=1,\underline{q})\delta}{Pr(h_N|V=0,\underline{q})(1-\delta)} \\
&= \log\frac{Pr(\hat{a}_N|V=1)Pr(h_{N-1}|V=1)\delta}{Pr(\hat{a}_n|V=0)Pr(h_N|V=0)(1-\delta)} - \log\frac{Pr(\hat{a}_N|V=1,\underline{q})Pr(h_{N-1}|V=1,\underline{q})\delta}{Pr(\hat{a}_n|V=0,\underline{q})Pr(h_{N-1}|V=0,\underline{q})(1-\delta)}
\end{aligned} \tag{A.12}$$

The last line of condition (A.12) increases as m increases since $\frac{Pr(\hat{a}_N|V=1)}{Pr(\hat{a}_N|V=0)} > \frac{Pr(\hat{a}_N|V=1,\underline{q})}{Pr(\hat{a}_N|V=0,\underline{q})}$ where $\hat{a}_N$ an action with a good signal at $t=N$ in equilibrium when m is large enough.

As N goes to infinity the left hand side of condition (A.12) converges to $\log\frac{\underline{q}}{1-\underline{q}} - \log\frac{\mu\gamma q^T+(1-\mu)\epsilon/2}{\mu\gamma(1-q^T)+(1-\mu)\epsilon/2}$ according to Doob (1949) while the last line keeps increasing. Hence, with sufficiently large N, condition (A.12) holds and the ambiguity averse traders with a good signal choose no trade in equilibrium.

Looking at the case with a bad signal, the condition for the ambiguity averse traders choose no trade with a bad signal is as follows.

$$\max_{q\in\mathcal{Q}_t}\frac{1-q}{q}\frac{Pr(V=1|h_N,q)}{1-Pr(V=1|h_N,q)}$$
$$>\frac{\mu\gamma(1-E(q|h_N))+(1-\mu)\epsilon/2}{\mu\gamma E(q|h_N)+(1-\mu)\epsilon/2}\frac{Pr(V=1|h_N)}{1-Pr(V=1|h_N)} \tag{A.13}$$

As m increases, the influence of the first term in the left hand side of the previous condition diminishes, so $arg\max_{q\in\mathcal{Q}_t}\frac{1-q}{q}\frac{Pr(V=1|h_N,q)}{1-Pr(V=1|h_N,q)}$ becomes $\overline{q}$. Then, the previous condition can be log transformed as follows.

$$\log\frac{1-\overline{q}}{\overline{q}}-\log\frac{\mu\gamma(1-E(q|h_N))+(1-\mu)\epsilon/2}{\mu\gamma E(q|h_N)+(1-\mu)\epsilon/2}$$
$$\geq\log\frac{Pr(V=1|h_N)}{1-Pr(V=1|h_N)}-\log\frac{Pr(V=1|h_N,\overline{q})}{1-Pr(V=1|h_N,\overline{q})}$$
$$=\log\frac{Pr(h_N|V=1)\delta}{Pr(h_N|V=0)(1-\delta)}-\log\frac{Pr(h_N|V=1,\overline{q})\delta}{Pr(h_N|V=0,\overline{q})(1-\delta)}$$
$$=\log\frac{Pr(\hat{a}_N|V=1)Pr(h_{N-1}|V=1)\delta}{Pr(\hat{a}_N|V=0)Pr(h_{N-1}|V=0)(1-\delta)}-\log\frac{Pr(\hat{a}_N|V=1,\overline{q})Pr(h_{N-1}|V=1,\overline{q})\delta}{Pr(\hat{a}_N|V=0,\overline{q})Pr(h_{N-1}|V=0,\overline{q})(1-\delta)} \tag{A.14}$$

The last line of condition (A.14) decreases as m increases since $\frac{Pr(\hat{a}_N|V=1)}{Pr(\hat{a}_N|V=0)}<\frac{Pr(\hat{a}_N|V=1,\overline{q})}{Pr(\hat{a}_N|V=0,\overline{q})}$ where $\hat{a}_N$ an action with a good signal at $t=N$ in equilibrium when m is large enough.

As N goes to infinity the left hand side of condition (A.14) converges to $\log\frac{1-\overline{q}}{\overline{q}}-\log\frac{\mu\gamma(1-q^T)+(1-\mu)\epsilon/2}{\mu\gamma q^T+(1-\mu)\epsilon/2}$ while the last line keeps decreasing. Hence, with sufficiently large N, condition (A.14) holds and the ambiguity averse traders with a bad signal choose no trade in equilibrium.

Therefore, the ambiguity averse traders choose no trade with any signal with probability of 1 as t goes to infinity. The case for $V=0$ is symmetric to this.

A.6 Proof of Proposition 5

Suppose that there is probability 0 of herding for the ambiguity averse traders. With a positive probability, there could be a history that there are only n buys. Since there is positive proportion of the traders without ambiguity, the buy orders reveal information on the true value and the signal at each time t, although the ambiguity averse traders sometimes choose no trade with a good signal in equilibrium.

Herding behaviour is composed of two conditions. One is the traders buying (selling) even with a bad (good) signal, and the other is the price is higher (lower) than that of the initial period. The condition for the ambiguity averse traders with a bad signal buy at $t = n + 1$ is as follows.

$$\min_{q \in \mathcal{Q}_{n+1}} E(V|S_{n+1} = 0, q, h_{n+1}) > a_{n+1} = E(V|h_{n+1}, X_{n+1} = buy)$$

The number of buys n is divided into n_1 and n_2 such that n_1 is number of buys under the situation that the informed traders including the traders without ambiguity and the ambiguity averse traders buy with a good signal in equilibrium, and n_2 is number of buys that only the traders without ambiguity buys with a good signal while the ambiguity averse traders would choose no trade with a good signal in equilibrium, and $n = n_1 + n_2$. Then, the condition can be log transformed and simplified as follows.

$$\begin{aligned}
&\min_{q \in \mathcal{Q}_{n+1}} \{\log \frac{1-q}{q} + n_1 \log \frac{\mu q + (1-\mu)\epsilon/2}{\mu(1-q) + (1-\mu)\epsilon/2} + n_2 \log \frac{\mu\gamma q + (1-\mu)\epsilon/2}{\mu\gamma(1-q) + (1-\mu)\epsilon/2}\} \\
&> \log \frac{\mu\gamma E(q|h_{n+1}) + (1-\mu)\epsilon/2}{\mu\gamma\{1 - E(q|h_{n+1})\} + (1-\mu)\epsilon/2} \\
&+ \sum_{t=1}^{n} \log \frac{\mu\{\gamma + (1-\gamma)I(\min_{q \in \mathcal{Q}_t} E(V|S_t = 1, h_t, q) > a_t)\}E(q|h_t) + (1-\mu)\epsilon/2}{\mu[1 - \{\gamma + (1-\gamma)I(\min_{q \in \mathcal{Q}_t} E(V|S_t = 1, h_t, q) > a_t)\}E(q|h_t)] + (1-\mu)\epsilon/2\}}
\end{aligned} \tag{A.15}$$

where $I(c)$ is an indicator function that gives 1 if condition c holds or gives 0 otherwise. Since there are only buys in the history, the $\mathcal{Q}_t$ becomes $\overline{q}$, so the minimum expected asset value is evaluated at $\overline{q}$, while $E(q|h_t) < \overline{q}$ and $E(q|h_t) \geq E(q|h_{t-1})$ for any finite t. We can find n_1 and n_2 such that $\varepsilon_1 = \log \frac{\mu\overline{q}+(1-\mu)\epsilon/2}{\mu(1-\overline{q})+(1-\mu)\epsilon/2} - \log \frac{\mu E(q|h_n)+(1-\mu)\epsilon/2}{\mu\{1-E(q|h_n)\}+(1-\mu)\epsilon/2}$, $\varepsilon_2 = \log \frac{\mu\gamma\overline{q}+(1-\mu)\epsilon/2}{\mu\gamma(1-\overline{q})+(1-\mu)\epsilon/2} - \log \frac{\mu\gamma E(q|h_n)+(1-\mu)\epsilon/2}{\mu\gamma\{1-E(q|h_n)\}+(1-\mu)\epsilon/2}$ and $n_1\varepsilon_1 + n_2\varepsilon_2 > \log \frac{\overline{q}}{1-\overline{q}} + \log \frac{\mu\gamma E(q|h_{n+1})+(1-\mu)\epsilon/2}{\mu\gamma\{1-E(q|h_{n+1})\}+(1-\mu)\epsilon/2}$. Hence, equation (A.15) holds at $t = n$. Also, the price increases from the initial price since there are n buys in history. Therefore, it contradicts the assumption that there is 0 probability of herding. The case for sell herding is symmetric to this.

A.7 Proof of Proposition 6

The condition for the ambiguity averse traders buy with a good signal is as follows.

$$\frac{\hat{q}}{1-\hat{q}}\frac{Pr(V=1|h_t,\hat{q}_t)}{1-Pr(V=1|h_t,\hat{q}_t)} > \frac{\mu E(q|h_t)+(1-\mu)\epsilon/2}{\mu\{1-E(q|h_t)\}+(1-\mu)\epsilon/2}\frac{Pr(V=1|h_t)}{1-Pr(V=1|h_t)} \tag{A.16}$$

where $\hat{q}_t$ is the element of $\mathcal{Q}_t$. We can rewrite this condition as follows.

$$\frac{\frac{\hat{q}}{1-\hat{q}}}{\frac{\mu E(q|h_t)+(1-\mu)\epsilon/2}{\mu\{1-E(q|h_t)\}+(1-\mu)\epsilon/2}} > \frac{\frac{Pr(V=1|h_t)}{1-Pr(V=1|h_t)}}{\frac{Pr(V=1|h_t,\hat{q}_t)}{1-Pr(V=1|h_t,\hat{q}_t)}} \tag{A.17}$$

With $\gamma > 0$, the action of the trader reveals the private information the trader gets since in equilibrium the traders without ambiguity buys with a good signal and sells with a bad signal at any t. Following Doob (1949) and Cramer (1978), $E(q|h_t)$ and Q_t converges to the true value q^T as t goes to infinity since the signal S_t is i.i.d. It implies that the right hand side of equation (A.17) $\frac{Pr(V=1|h_t)}{Pr(V=1|h_t,\hat{q}_t)} \to 1$.

Also the left hand side of equation (A.17) converges to $\frac{\frac{q^T}{1-q^T}}{\frac{\mu q^T+(1-\mu)\epsilon/2}{\mu\{1-q^T\}+(1-\mu)\epsilon/2}} > 1$. Therefore, as t goes to infinity, equation (A.17) satisfies, so the ambiguity averse traders buy with a good signal. The case for selling is symmetric to this.

A.8 Proof of Proposition 7

When $\mathcal{Q}_t = \{\hat{q}\}$ which is a singleton, and there are only ambiguity averse traders who are informed ($\gamma = 0$), the expected asset value with a good signal is higher than that with a bad signal since

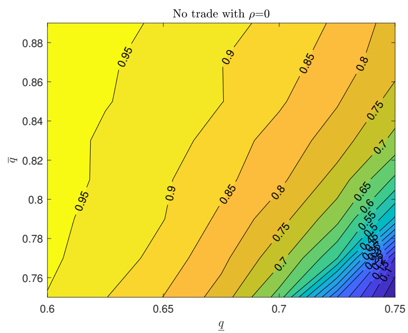
$$\frac{\hat{q}}{1-\hat{q}}\frac{Pr(V=1|h_t,\hat{q})}{1-Pr(V=1|h_t,\hat{q})} > \frac{1-\hat{q}}{\hat{q}}\frac{Pr(V=1|h_t,\hat{q})}{1-Pr(V=1|h_t,\hat{q})}$$

If we assume that the optimal action of the informed traders at t is no trade with any signal, the ask and bid price is the same. Hence, the expected asset value with a good signal is lower than the ask price and that with a bad signal is higher than the bid price is not possible.

A.9 Simulation results with $\gamma = 0.5$

The model is simulated with the true value of asset to be $V = 1$ and the prior probability of the state is $\delta = 0.5$. Half of the traders are informed ($\mu = 0.5$), and half of the informed traders are without ambiguity ($\gamma = 0.5$). True value of the precision q is set to be 0.75 if it is not noted specifically. The results are based on $1,000$ simulations of $T = 100$ for each parameter.

Figure A.1 Proportion of *no trade* with the Full Bayesian Updating ($\rho = 0$)



Note: The proportion is computed by the number of periods that the ambiguity averse traders choose no trade in equilibrium over $1,000$ simulations with $T = 100$ at each simulation. The numbers on the figure are the proportion of the behaviour (Higher proportion with yellow, lower proportion with blue color). Choosing no trade includes no trade with any signal, buy with a good signal and no trade with a bad signal, no trade with a good signal and sell with a bad signal, and mixed strategy of them.

First of all, we start to look at the case of the Full Bayesian Updating with $\rho = 0$. Figure A.1 shows the proportion of choosing no trade for the ambiguity averse traders depending on the lower and upper bound of $\mathcal{Q}_0$, which are $\underline{q}$ and $\overline{q}$. The proportion is calculated by the number of periods that the ambiguity averse traders choose no trade in equilibrium during $1,000$ simulations. Choosing no trade includes three cases such as no trade with any signal, no trade with a good signal and sell with a bad signal, buy with a good signal and no trade with a bad signal, and mixed strategy of them. Note that herding or contrarian behaviour never happens with the Full Bayesian Updating.

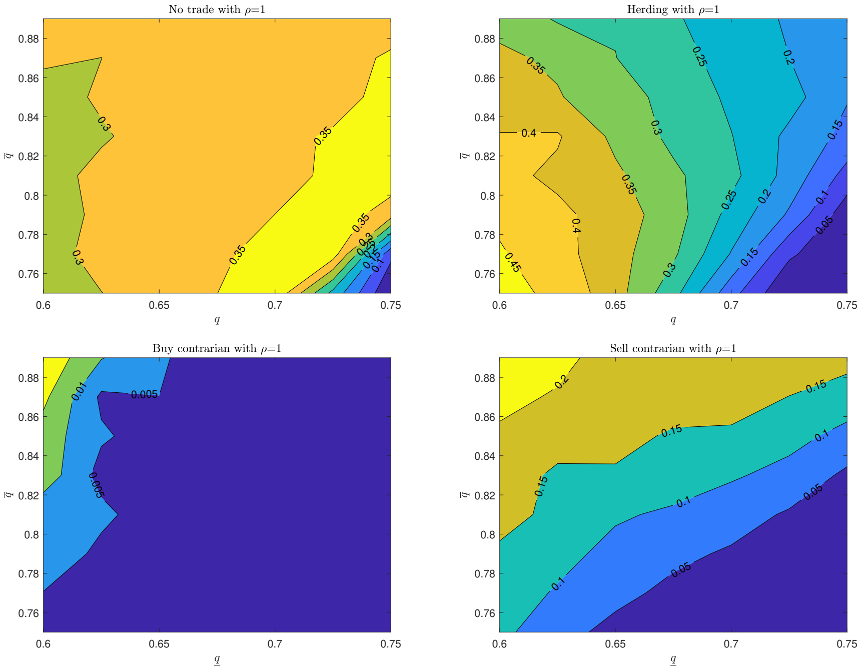
In the case with the Full Bayesian updating, the proportion of no trade increases as the size of $\mathcal{Q}_0$ enlarges by decreasing $\underline{q}$ and increasing $\overline{q}$. As discussed in Chapter 2, we can expect that smaller $\underline{q}$ makes the expected asset value of the ambiguity averse traders with a good private signal lower than the ask price or larger $\overline{q}$ makes the expectation with a bad signal higher than the bid price after buy dominated history. It brings higher probability of choosing no trade by the traders.

Next case is the Maximum Likelihood Updating. The upper panels of Figure A.2 presents the proportion of choosing no trade and herding. As for the proportion of no trade in the upper left panel, we can find the similar trend of increasing the proportion as $\mathcal{Q}_0$ enlarges, although it is only on the $\underline{q}$ between 0.63 and 0.75. When $\underline{q}$ becomes lower than 0.63, the proportion of no trade decreases as the set expands with lower $\underline{q}$. It is because herding and contrarian behaviours take part of the region.

The proportion of herding increases as $\mathcal{Q}_0$ expands although the pattern with low level of $\underline{q}$ is different. Following the explanation about herding in Chapter 2, the distribution of q of the market maker updates marginally if its support $\mathcal{Q}_0$ is large, while $\mathcal{Q}_t$ for the ambiguity averse traders updates rapidly. It brings the trends of higher proportion of herding with larger $\mathcal{Q}_0$.

When $\underline{q}$ is lower than 0.62 the proportion of herding start to decrease as the the parameter decrease. The region is taken by the contrarian behaviour. The contrarian behaviour occurs because the expected value of the trader with a bad signal is higher

Figure A.2 Proportion of *no trade*, *herding* and *contrarian* with the Maximum Likelihood Updating ($\rho = 1$)

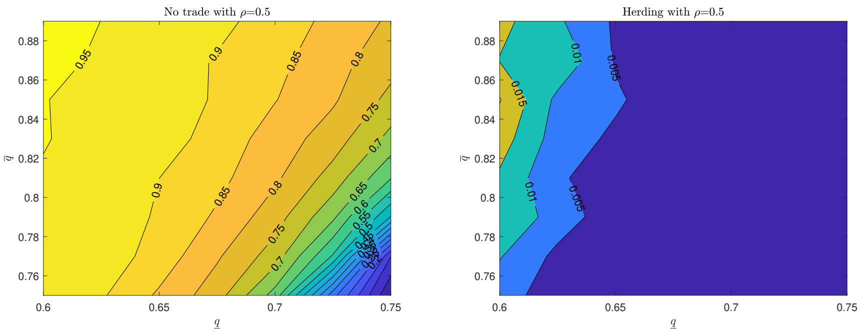


Note: The upper left panel is the proportion of no trade and the upper right panel is of herding. The lower left panel is the proportion of buy contrarian and the lower right is sell contrarian. The proportion is computed by the number of periods that the ambiguity averse traders choose no trade in equilibrium or herding over $1,000$ simulations with $T = 100$ at each simulation. The numbers on the figure are the proportion of the behaviour (Higher proportion with yellow, lower proportion with blue color). Choosing no trade includes no trade with any signal, buy with a good signal and no trade with a bad signal, no trade with a good signal and sell with a bad signal, and mixed strategy of them.

than ask price in a history dominated by selling. We have shown that if $\underline{q}$ is low, the expected asset value of the ambiguity averse trader with a bad signal can be higher than bid price in the analysis on the proportion of no trade. With $\rho = 1$, this situation becomes more extreme that the expectation of the trader with a bad signal is even higher than ask price after a history dominated by selling. We also can check that sell contrarian occurs with a similar pattern.

The case with ρ to be in between these two extreme updating rules is presented in Figure A.3. When $\rho = 0.5$, the patterns for the proportions of no trade and herding are in the middle of those two extremes. No trade increases as $\mathcal{Q}_0$ increases similar to the case with the Full Bayesian Updating. The proportion of herding increases with larger $\mathcal{Q}_0$. The explanation for this trend is same as the Full Bayesian Updating except that

Figure A.3 Proportion of *no trade* and *herding* with $\rho = 0.5$



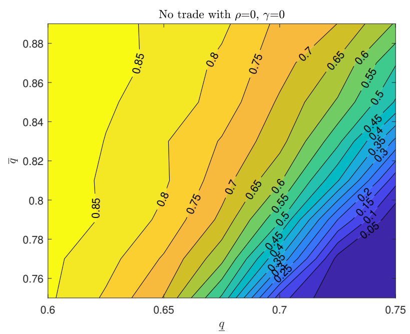
Note: The left panel is the proportion of no trade and the right panel is of herding. The proportion is computed by the number of periods that the ambiguity averse traders choose no trade in equilibrium or herding over $1,000$ simulations with $T = 100$ at each simulation. The numbers on the figure are the proportion of the behaviour (Higher proportion with yellow, lower proportion with blue color). Choosing no trade includes no trade with any signal, buy with a good signal and no trade with a bad signal, no trade with a good signal and sell with a bad signal, and mixed strategy of them.

there is almost no contrarian behaviour in this case.

A.10 Simulation results with $\gamma = 0$

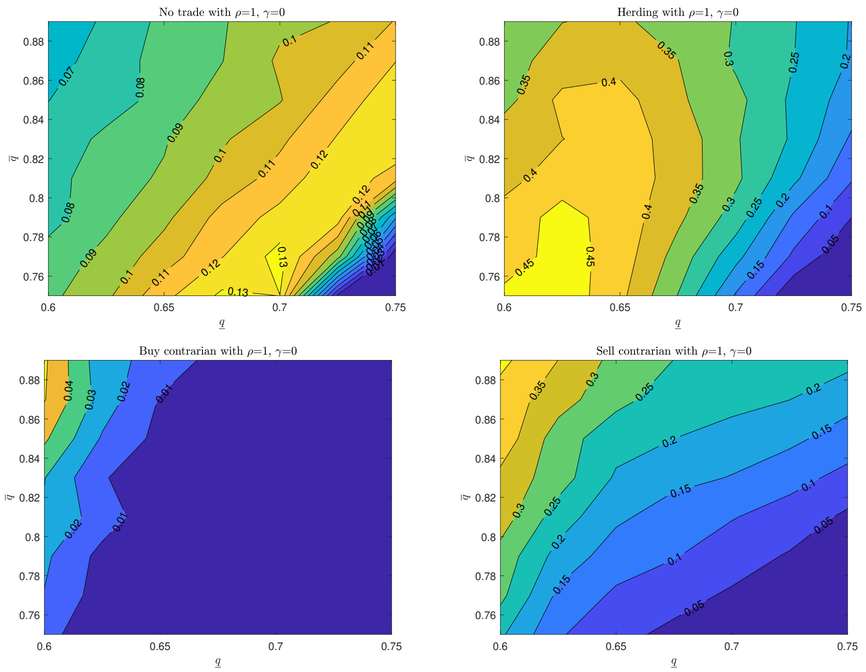
The parameter values used in the simulation is as follows. $V = 1$ and the prior probability of the state is $\delta = 0.5$. Half of the traders are informed ($\mu = 0.5$), and half of the informed traders are without ambiguity ($\gamma = 0$). True value of the precision q is set to be 0.75 if it is not noted specifically. The results are based on $1,000$ simulations of $T = 100$ for each parameter.

Figure A.4 Proportion of *no trade* with the Full Bayesian Updating ($\rho = 0$)



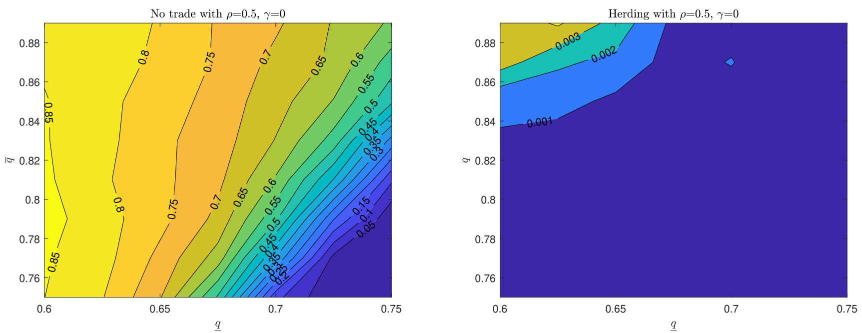
Note: The proportion is computed by the number of periods that the ambiguity averse traders choose no trade in equilibrium over $1,000$ simulations with $T = 100$ at each simulation. The numbers on the figure are the proportion of the behaviour (Higher proportion with yellow, lower proportion with blue color). Choosing no trade includes no trade with any signal, buy with a good signal and no trade with a bad signal, no trade with a good signal and sell with a bad signal, and mixed strategy of them.

Figure A.5 Proportion of *no trade*, *herding* and *contrarian* with the Maximum Likelihood Updating ($\rho = 1$)



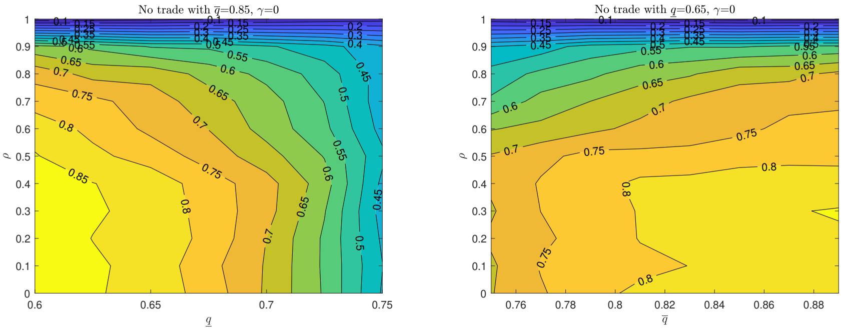
Note: The upper left panel is the proportion of no trade and the upper right panel is of herding. The lower left panel is the proportion of buy contrarian and the lower right is sell contrarian. The proportion is computed by the number of periods that the ambiguity averse traders choose no trade in equilibrium or herding over $1,000$ simulations with $T = 100$ at each simulation. The numbers on the figure are the proportion of the action (Higher proportion with yellow, lower proportion with blue color). Choosing no trade includes no trade with any signal, buy with a good signal and no trade with a bad signal, no trade with a good signal and sell with a bad signal, and mixed strategy of them.

Figure A.6 Proportion of *no trade* and *herding* with $\rho = 0.5$



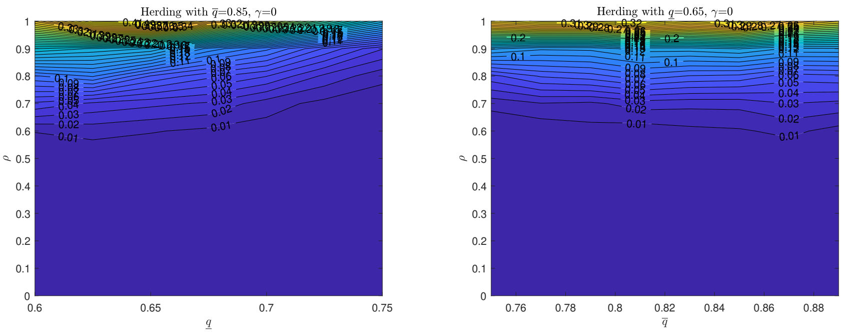
Note: The left panel is the proportion of no trade and the right panel is of herding. The proportion is computed by the number of periods that the ambiguity averse traders choose no trade in equilibrium or herding over $1,000$ simulations with $T = 100$ at each simulation. The numbers on the figure are the proportion of the action (Higher proportion with yellow, lower proportion with blue color). Choosing no trade includes no trade with any signal, buy with a good signal and no trade with a bad signal, no trade with a good signal and sell with a bad signal, and mixed strategy of them.

Figure A.7 Proportion of *no trade* depending on ρ and $\mathcal{Q}_0$



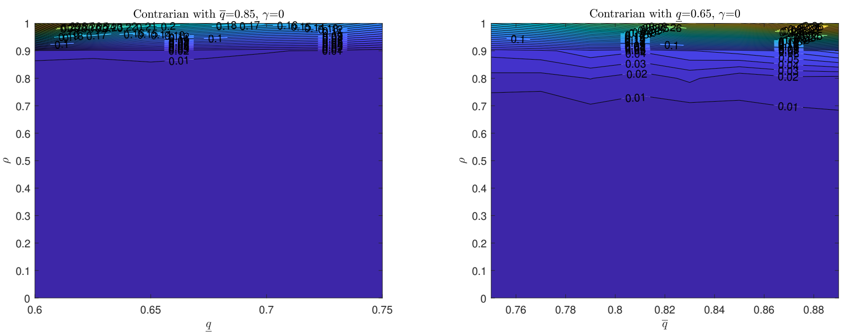
Note: The proportion is computed by the number of periods that the ambiguity averse traders choose no trade in equilibrium over $1,000$ simulations with $T = 100$ at each simulation. The numbers on the figure are the proportion of the action (Higher proportion with yellow, lower proportion with blue color). Choosing no trade includes no trade with any signal, buy with a good signal and no trade with a bad signal, no trade with a good signal and sell with a bad signal, and mixed strategy of them.

Figure A.8 Proportion of *herding* depending on ρ and $\mathcal{Q}_0$



Note: The proportion is computed by the number of periods that herding happens in equilibrium over $1,000$ simulations with $T = 100$ at each simulation. The numbers on the figure are the proportion of herding (Higher proportion with yellow, lower proportion with blue color). Herding includes buy herding and sell herding although the proportion of sell herding is virtually zero as $V = 1$.

Figure A.9 Proportion of *Contrarian* depending on ρ and $\mathcal{Q}_0$



Note: The proportion is computed by the number of periods that herding happens in equilibrium over $1,000$ simulations with $T = 100$ at each simulation. The numbers on the figure are the proportion of contrarian behaviour (Higher proportion with yellow, lower proportion with blue color). Contrarian behaviour includes contrarian buying and contrarian selling.

Appendix B

Appendix to Chapter 3

B.1 Proof of proposition 8

Given $\alpha = 1$, the following two conditions are derived to satisfy equation (3.5).

$$\frac{Pr(v_h^d|\beta_{1,t}^d, h_t^d)}{Pr(v_l^d|\beta_{1,t}^d, h_t^d)} = \min_{\tau \in \mathcal{T}_t} \frac{Pr(v_h^d|\beta_{2,t}^d, h_t^d, \tau)}{Pr(v_l^d|\beta_{2,t}^d, h_t^d, \tau)} \tag{B.1}$$

$$\frac{Pr(v_h^d|\beta_{1,t}^d, h_t^d)}{Pr(v_l^d|\beta_{1,t}^d, h_t^d)} = \frac{Pr(v_h^d|buy_t^d, h_t^d)}{Pr(v_l^d|buy_t^d, h_t^d)} \tag{B.2}$$

From equation (B.1), (B.2) and MLRP of both of the ratios in the equation, $\beta_{1,t}^d$ is increasing in $\beta_{2,t}^d$. Given $\alpha = 1$ and keeping $\beta_{2,t}^d$ at its lowest value 0, meaning that no ambiguity averse traders buy, $\beta_{1,t}^d$ derived from equation (B.2) is as follows.

$$\frac{\mu + (1-\mu)\epsilon/2}{\gamma\mu} - \sqrt{\left\{\frac{\mu + (1-\mu)\epsilon/2}{\gamma\mu}\right\}^2 - \frac{\mu + (1-\mu)\epsilon/2}{\gamma\mu}}$$

It can be transformed to be $h_1(y) = y - \sqrt{y^2 - y}$ where $y = \frac{\mu+(1-\mu)\epsilon/2}{\gamma\mu}$. $y > 1$ with $0 < \gamma \leq 1$ and $\mu > 0$. $h_1(1) = 1$, and $\lim_{y\to\infty} h_1(y) = 0.5$ from $\lim_{y\to\infty} y - \sqrt{y^2 - y} = \lim_{y\to\infty} \frac{1}{1+\sqrt{1-1/y}} = \frac{1}{2}$. Therefore $0.5 \leq \beta_{1,t}^d < 1$ when $\alpha = 1$. $\sigma_{1,t}^d$ is symmetric to $\beta_{1,t}^d$.

B.2 Proof of Proposition 9

Given $\alpha = 1$, equation (B.1) at $t = 1$ can be simplified as follows.

$$\frac{\{1 + (\underline{\tau} + \overline{\tau})/2(2\beta_{1,1}^d - 1)\}\delta}{\{1 - (\underline{\tau} + \overline{\tau})/2(2\beta_{1,1}^d - 1)\}(1 - \delta)} = \min_{\tau} \frac{\{1 + \tau(2\beta_{2,1}^d - 1)\}\delta}{\{1 - \tau(2\beta_{2,1}^d - 1)\}(1 - \delta)} \tag{B.3}$$

From equation (B.2) and (B.3) with $\alpha = 1$, setting $\beta_{2,1}^d = 1$ gives the lower bound for $\underline{\tau}$ as $\underline{\tau}$ is a decreasing function in $\beta_{2,1}^d$. $\tilde{\tau}$ is the solution for $\underline{\tau}$ that satisfies equation (B.2) and (B.3) with $\alpha = 1$ keeping $\beta_{2,1}^d = 1$.

We also can check that $\beta 2, 1^d \geq 0.5$. It is known that $\beta_{1,1}^d \geq 0.5$ from Proposition 8. If we suppose that $\beta_{2,1}^d$ is not lower than 0.5, $\beta_{2,1}^d$ is a decreasing function on $\underline{\tau}$ given $\overline{\tau}$ from equation (B.3). With maximum level of $\underline{\tau}$, which is at $\underline{\tau} = \overline{\tau}$, $\beta_{2,1}^d$ is derived to be

$$1 - [\sqrt{\{(1-\mu)\epsilon/2\}^2 + \mu(1-\mu)\epsilon/2} - (1-\mu)\epsilon/2]/\mu$$

If the function above is defined to be $h_2(\mu, \epsilon)$, it is not lower than 0.5 since $h_2(\mu, \epsilon)$ is a increasing function in μ and $\lim_{\mu \to 0} h_2 = 0.5$. Therefore, $\beta_{2,1}^d \geq 0.5$.

Hence, when $\underline{\tau}$ is higher than $\tilde{\tau}$, $0.5 \leq \beta_{2,1} < 1$. For the case of $\sigma_{2,1}^d$ can be shown in similar way.

B.3 Proof of Proposition 10

One of the condition for herd and contrarian buying of the ambiguity averse traders is $\beta_{2,t}^d < 0.5$. The threshold $\beta_{2,t}^d$ should satisfy the following equation which is simplified from the last two terms of equation (3.5).

$$\begin{aligned}
&\min_{\tau \in \mathcal{T}_t} \frac{1 + \tau(2\beta_{2,t}^d - 1)}{1 - \tau(2\beta_{2,t}^d - 1)} \frac{Pr(v_h^d | h_t^d, \tau)}{Pr(v_l^d | h_t^d, \tau)} \\
&= \frac{\mu\gamma(1 - \beta_{1,t}^d)\{1 + E(\tau|h_t^d)\beta_{1,t}^d\} + \mu(1-\gamma)(1 - \beta_{2,t}^d)\{1 + E(\tau|h_t^d)\beta_{2,t}^d\} + (1-\mu)\epsilon/2}{\mu\gamma(1 - \beta_{1,t}^d)\{1 - E(\tau|h_t^d)\beta_{1,t}^d\} + \mu(1-\gamma)(1 - \beta_{2,t}^d)\{1 - E(\tau|h_t^d)\beta_{2,t}^d\} + (1-\mu)\epsilon/2} \\
&\quad \times \frac{Pr(v_h^d | h_t^d)}{Pr(v_l^d | h_t^d)}
\end{aligned} \tag{B.4}$$

When $\beta^d_{2,t} < 0.5$, $\frac{1+\tau(2\beta^d_{2,t}-1)}{1-\tau(2\beta^d_{2,t}-1)} < 1$ with any τ while

$$\frac{\mu\gamma(1-\beta^d_{1,t})\{1+E(\tau|h^d_t)\beta^d_{1,t}\} + \mu(1-\gamma)(1-\beta^d_{2,t})\{1+E(\tau|h^d_t)\beta^d_{2,t}\} + (1-\mu)\epsilon/2}{\mu\gamma(1-\beta^d_{1,t})\{1-E(\tau|h^d_t)\beta^d_{1,t}\} + \mu(1-\gamma)(1-\beta^d_{2,t})\{1-E(\tau|h^d_t)\beta^d_{2,t}\} + (1-\mu)\epsilon/2}$$
$$> 1$$

Also, $\min_{\tau\in\mathcal{T}_t} \frac{Pr(v^d_h|h^d_t,\tau)}{Pr(v^d_l|h^d_t,\tau)} = \min_{\tau\in\mathcal{T}_t} \frac{Pr(v^d_h|h^d_t,\tau)}{1-Pr(v^d_h|h^d_t,\tau)} = \frac{\min_{\tau\in\mathcal{T}_t} Pr(v^d_h|h^d_t,\tau)}{\max_{\tau\in\mathcal{T}_t}\{1-Pr(v^d_h|h^d_t,\tau)\}} \leq \frac{\int_{\underline{\tau}}^{\overline{\tau}} Pr(v^d_h|h^d_t,\tau)f(\tau|h^d_t)\mathrm{d}\tau}{\int_{\underline{\tau}}^{\overline{\tau}}\{1-Pr(v^d_h|h^d_t,\tau)\}f(\tau|h^d_t)\mathrm{d}\tau} = \frac{Pr(v^d_h|h^d_t)}{Pr(v^d_l|h^d_t)}$ by the mean-value theorem when $\mathcal{T}_t = \{\tau : \underline{\tau} \leq \tau \leq \bar{\tau}\}$ with $\rho = 0$. From these,

$$\min_{\tau\in\mathcal{T}_t} \frac{\{1+\tau(2\beta^d_{2,t}-1)\}Pr(v^d_h|h^d_t,\tau)}{\{1-\tau(2\beta^d_{2,t}-1)\}Pr(v^d_l|h^d_t,\tau)} < \min_{\tau\in\mathcal{T}_t} \frac{Pr(v^d_h|h^d_t,\tau)}{Pr(v^d_l|h^d_t,\tau)} \leq \frac{Pr(v^d_h|h^d_t)}{Pr(v^d_l|h^d_t)}$$
$$< \frac{\mu\gamma(1-\beta^d_{1,t})\{1+E(\tau|h^d_t)\beta^d_{1,t}\} + \mu(1-\gamma)(1-\beta^d_{2,t})\{1+E(\tau|h^d_t)\beta^d_{2,t}\} + (1-\mu)\epsilon/2}{\mu\gamma(1-\beta^d_{1,t})\{1-E(\tau|h^d_t)\beta^d_{1,t}\} + \mu(1-\gamma)(1-\beta^d_{2,t})\{1-E(\tau|h^d_t)\beta^d_{2,t}\} + (1-\mu)\epsilon/2}$$
$$\times \frac{Pr(v^d_h|h^d_t)}{Pr(v^d_l|h^d_t)}$$

Hence, condition (B.4) cannot be satisfied with $\rho = 0$. It means that herd buying or contrarian buying never occurs with $\rho = 0$, given $\alpha = 1$. The case for selling is symmetric to this.

B.4 Proof of Proposition 11

Suppose that $0.5 < \beta^d_{2,t} < 1$. Given $\rho = 0$ and $\gamma = 0$, there could be a history only with buy orders. Equation (3.5) after a history of buys can be simplified as follows.

$$
\begin{aligned}
&\min_{\tau\in[\underline{\tau},\overline{\tau}]}\Big[\frac{\{1+\tau(2\beta^d_{2,t}-1)\}}{\{1-\tau(2\beta^d_{2,t}-1)\}}\\
&\times\prod_{i=1}^{t-1}\frac{\mu(1-\beta^d_{2,i})(1+\tau\beta^d_{2,i})+(1-\mu)\epsilon/2}{\mu(1-\beta^d_{2,i})(1-\tau\beta^d_{2,i})+(1-\mu)\epsilon/2}\frac{\delta}{1-\delta}\Big]\\
&=\frac{\mu(1-\beta^d_{2,t})(1+E(\tau|h^d_t)\beta^d_{2,t})+(1-\mu)\epsilon/2}{\mu(1-\beta^d_{2,t})(1-E(\tau|h^d_t)\beta^d_{2,t})+(1-\mu)\epsilon/2}\\
&\times\prod_{i=1}^{t-1}\frac{\mu(1-\beta^d_{2,i})(1+E(\tau|h^d_i)\beta^d_{2,i})+(1-\mu)\epsilon/2}{\mu(1-\beta^d_{2,i})(1-E(\tau|h^d_i)\beta^d_{2,i})+(1-\mu)\epsilon/2}\frac{\delta}{1-\delta}
\end{aligned}\tag{B.5}
$$

Since there are only buy orders in the history, the minimum value of the left hand side of equation (B.5) is evaluated at $\underline{\tau}$. After a log transformation and some manipulation, equation (B.5) can be transformed as follows.

$$
\begin{aligned}
&\log\frac{\{1+\underline{\tau}(2\beta^d_{2,t}-1)\}}{\{1-\underline{\tau}(2\beta^d_{2,t}-1)\}}\\
&=\log\frac{\mu(1-\beta^d_{2,t})(1+E(\tau|h^d_t)\beta^d_{2,t})+(1-\mu)\epsilon/2}{\mu(1-\beta^d_{2,t})(1-E(\tau|h^d_t)\beta^d_{2,t})+(1-\mu)\epsilon/2}\\
&+\sum_{i=1}^{t-1}\Big\{\log\frac{\mu(1-\beta^d_{2,i})(1+E(\tau|h^d_i)\beta^d_{2,i})+(1-\mu)\epsilon/2}{\mu(1-\beta^d_{2,i})(1-E(\tau|h^d_i)\beta^d_{2,i})+(1-\mu)\epsilon/2}\\
&-\log\frac{\mu(1-\beta^d_{2,i})(1+\underline{\tau}\beta^d_{2,i})+(1-\mu)\epsilon/2}{\mu(1-\beta^d_{2,i})(1-\underline{\tau}\beta^d_{2,i})+(1-\mu)\epsilon/2}\Big\}
\end{aligned}\tag{B.6}
$$

As t increases, the right hand side of equation (B.6) increases since $\beta^d_{2,t}$ is assumed to be lower than one, and $E(\tau|h^d_t)>\underline{\tau}$ for any finite t, and $E(\tau|h^d_t)$ keeps increasing to $\overline{\tau}$ as t increases. To satisfy the equation, $\beta^d_{2,t}$ should increase to one with sufficiently large t. It contradict the assumption of $\beta^d_{2,t}<1$. Hence, $\beta^d_{2,t}$ becomes one with a positive probability for any $\underline{\tau}$ and $\overline{\tau}$. The case with $\sigma^d_{2,t}=0$ is symmetric to this.

B.5 Proof of Proposition 12

Suppose that there is zero probability of $\beta_{2,t}<0.5$. After consecutive n buys, $\mathcal{T}_t$ becomes a singleton $\{\overline{\tau}\}$, so equation (3.5) can be transformed and simplified as follows.

$$
\begin{aligned}
&\log \frac{1+\overline{\tau}(2\beta^d_{2,n+1}-1)}{1-\overline{\tau}(2\beta^d_{2,n+1}-1)} \\
&= \log \frac{\mu(1-\beta^d_{2,n+1})\{1+\beta^d_{2,n+1}E(\tau|h_{n+1})\}+(1-\mu)\epsilon/2}{\mu(1-\beta^d_{2,n+1})\{1-\beta^d_{2,n+1}E(\tau|h_{n+1})\}+(1-\mu)\epsilon/2} \\
&- \sum_{t=1}^{n} \big[\log \frac{\mu(1-\beta^d_{2,t})\{1+\beta^d_{2,t}\overline{\tau}\}+(1-\mu)\epsilon/2}{\mu(1-\beta^d_{2,t})\{1-\beta^d_{2,t}\overline{\tau}\}+(1-\mu)\epsilon/2} \\
&- \log \frac{\mu(1-\beta^d_{2,t})\{1+\beta^d_{2,t}E(\tau|h_t)\}+(1-\mu)\epsilon/2}{\mu(1-\beta^d_{2,t})\{1-\beta^d_{2,t}E(\tau|h_t)\}+(1-\mu)\epsilon/2} \big]
\end{aligned} \tag{B.7}
$$

For any finite t, $E(\tau|h_t) < \overline{\tau}$. Hence, we can find n such that the right hand side of equation (B.7) becomes negative since $\log \frac{\mu(1-\beta^d_{2,t})\{1+\beta^d_{2,t}\overline{\tau}\}+(1-\mu)\epsilon/2}{\mu(1-\beta^d_{2,t})\{1-\beta^d_{2,t}\overline{\tau}\}+(1-\mu)\epsilon/2} > \log \frac{\mu(1-\beta^d_{2,t})\{1+\beta^d_{2,t}E(\tau|h_t)\}+(1-\mu)\epsilon/}{\mu(1-\beta^d_{2,t})\{1-\beta^d_{2,t}E(\tau|h_t)\}+(1-\mu)\epsilon/}$ for any t. To satisfy the equation, $\beta^d_{2,n+1}$ should be lower than 0.5. It contradicts the assumption. Therefore, herd buy can occur with positive probability. The case for herd sell is symmetric to this.

B.6 Proof of Proposition 13

Suppose that no trading behavior occurs at t. It means that $\beta^d_{2,t} = 1$ and $\sigma^d_{2,t} = 0$. Equation (3.5) and (3.6) can be simplified as follows.

$$
\begin{aligned}
&\frac{1+\hat{\tau}(2\beta^d_{2,t}-1)}{1-\hat{\tau}(2\beta^d_{2,t}-1)} \frac{Pr(V_h|h^d_t,\hat{\tau})}{Pr(V_l|h^d_t,\hat{\tau})} \\
&= \frac{\mu(1-\beta^d_{2,t})\{1+\beta^d_{2,t}E(\tau|h^d_t)\}+(1-\mu)\epsilon/2}{\mu(1-\beta^d_{2,t})\{1-\beta^d_{2,t}E(\tau|h^d_t)\}+(1-\mu)\epsilon/2} \frac{Pr(V_h|h^d_t)}{Pr(V_l|h^d_t)}
\end{aligned} \tag{B.8}
$$

$$
\begin{aligned}
&\frac{1+\hat{\tau}(2\sigma^d_{2,t}-1)}{1-\hat{\tau}(2\sigma^d_{2,t}-1)} \frac{Pr(V_h|h^d_t,\hat{\tau})}{Pr(V_l|h^d_t,\hat{\tau})} \\
&= \frac{\mu\{\sigma^d_{2,t}-\sigma^d_{2,t}(1-\sigma^d_{2,t})E(\tau|h^d_t)\}+(1-\mu)\epsilon/2}{\mu\{\sigma^d_{2,t}+\sigma^d_{2,t}(1-\sigma^d_{2,t})E(\tau|h^d_t)\}+(1-\mu)\epsilon/2} \frac{Pr(V_h|h^d_t)}{Pr(V_l|h^d_t)}
\end{aligned} \tag{B.9}
$$

Since $\beta^d_{2,t} = 1$ and $\sigma^d_{2,t} = 0$, the first term of the right hand side of equation (B.8) and (B.9) are 1. It implies that the left hand side of those two equations should be the same. It contradicts $\beta^d_{2,t} = 1$ and $\sigma^d_{2,t} = 0$. Hence, no trading is not possible for the ambiguity averse traders if $\mathcal{T}_t$ is a singleton.

Appendix C

Appendix to Chapter 4

C.1 Example 1

Table C.1 Possible histories at $t = 7$ under the standard strategy

	$h_7 \setminus h_3$	$Pr(h_7 \mid h_2, NT_2^M, s_2^M = 0)$	$E(V \mid h_7, s_2^M = 0)$	a_7	b_7	Action	Profit
	Possible histories under the standard strategy at $t = 7$						
1	$\{buy_3, buy_4, buy_5, buy_6\}$	0.14	0.7714	0.7783	0.5	no trade	0
2	$\{buy_3, buy_4, buy_5, NT_6\}$	0.12	0.6	0.6092	0.5	no trade	0
3	$\{buy_3, NT_4, buy_5, buy_6\}$	0.0624	0.6923	0.6124	0.5	buy	0.0799
4	$\{buy_3, NT_4, buy_5, NT_6\}$	0.0576	0.5	0.5168	0.4832	no trade	0
5	$\{buy_3, NT_4, sell_5, NT_6\}$	0.0576	0.5	0.5168	0.4832	no trade	0
6	$\{buy_3, NT_4, sell_5, sell_6\}$	0.0624	0.3077	0.5	0.3876	sell	0.0799
7	$\{sell_3, NT_4, buy_5, buy_6\}$	0.0624	0.6923	0.6124	0.5168	buy	0.0799
8	$\{sell_3, NT_4, buy_5, NT_6\}$	0.0576	0.5	0.5	0.4832	no trade	0
9	$\{sell_3, NT_4, sell_5, NT_6\}$	0.0576	0.5	0.5168	0.4832	no trade	0
10	$\{sell_3, NT_4, sell_5, sell_6\}$	0.0624	0.3077	0.5	0.3876	sell	0.0799
11	$\{sell_3, sell_4, NT_5, NT_6\}$	0.0576	0.5	0.5168	0.4832	no trade	0
12	$\{sell_3, sell_4, NT_5, sell_6\}$	0.0624	0.3077	0.5	0.3876	sell	0.0799
13	$\{sell_3, sell_4, sell_5, NT_6\}$	0.0624	0.3077	0.5	0.3876	sell	0.0799
14	$\{sell_3, sell_4, sell_5, sell_6\}$	0.0776	0.1695	0.5	0.2179	sell	0.0530
	Total	1					0.0340

Note: the parameter values used in the example are $\alpha = 0.01$, $\delta = 0.5$, $\mu = 1$, $q = q^M = 0.6$, and $\varepsilon_b = \varepsilon_s = 0.2$.

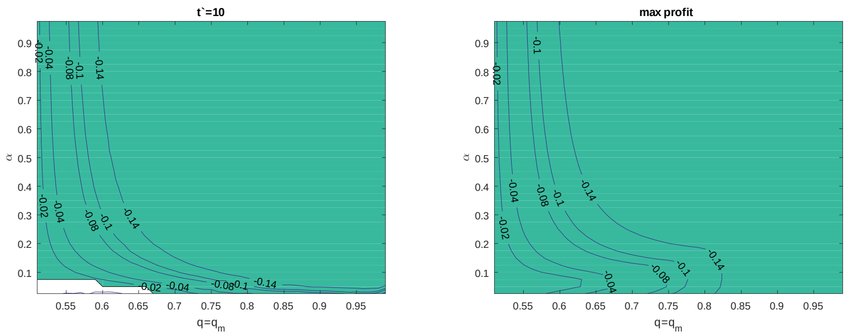
Table C.2 Possible histories at $t = 7$ under the manipulative strategy

	$h_7 \setminus h_3$	$Pr(h_7 \mid h_2, buy_2^M, s_2^M = 0)$	$E(V \mid h_7, s_2^M = 0)$	a_7	b_7	Action	Profit
	Possible histories under the manipulative strategy at $t = 7$						
1	$\{buy_3, buy_4, buy_5, buy_6\}$	0.14	0.7714	0.8994	0.5	no trade	0.2596
2	$\{buy_3, buy_4, buy_5, NT_6\}$	0.12	0.6	0.7783	0.5	no trade	0.0882
3	$\{buy_3, buy_4, NT_5, buy_6\}$	0.12	0.6	0.7783	0.5	no trade	0.0882
4	$\{buy_3, buy_4, NT_5, NT_6\}$	0.12	0.4	0.6092	0.5	sell	-0.0119
5	$\{buy_3, NT_4, buy_5, buy_6\}$	0.12	0.6	0.7783	0.5	no trade	0.0882
6	$\{buy_3, NT_4, buy_5, NT_6\}$	0.12	0.4	0.6092	0.5	sell	-0.0119
7	$\{buy_3, NT_4, NT_5, buy_6\}$	0.12	0.4	0.6092	0.5	sell	-0.0119
8	$\{buy_3, NT_4, NT_5, sell_6\}$	0.14	0.2286	0.5	0.3908	sell	-0.1210
	Total	1					0.0469

Note: the parameter values used in the example are $\alpha = 0.01$, $\delta = 0.5$, $\mu = 1$, $q = q^M = 0.6$, and $\varepsilon_b = \varepsilon_s = 0.2$.

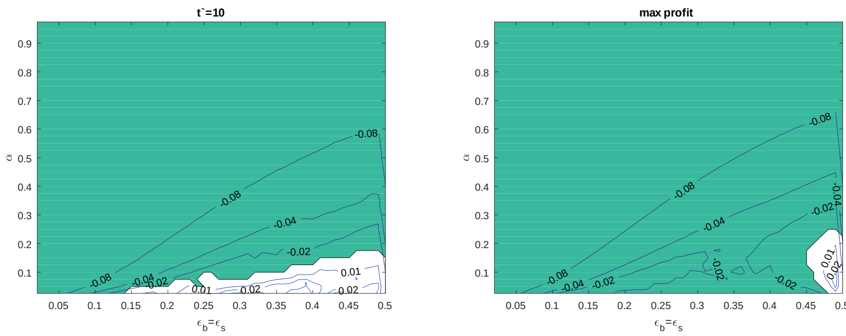
C.2 Conditions for the profitable manipulation

Figure C.1 Excess profit in different α and $q = q^m$ with $\mu = 1$



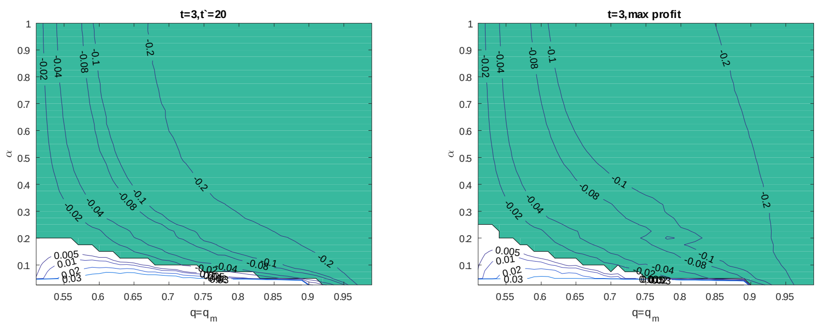
Note: Excess profit is the expected profit of the manipulative strategy minus that of the standard strategy. Numbers on the graph is excess profit of the manipulative strategy with parameters $\mu = 1$, $\varepsilon_b = \varepsilon_s = 0.2$, $\tau = 2$ with $h_2 = \{buy_1\}$, and $s_2^M = 0$. The left panel is the case that $t' = 10$ and the right panel is that τ' is set to make the excess profit maximum. The green area is where the excess profit is negative; the yellow area is where the standard strategy is buying as well.

Figure C.2 Excess profit in different α and $\epsilon_b = \epsilon_s$ with $\mu = 1$



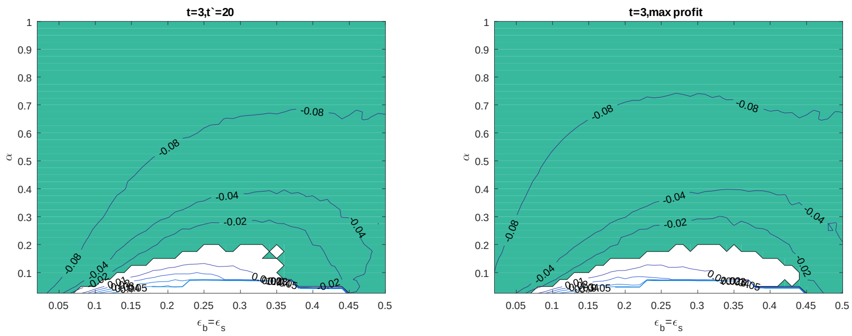
Note: Excess profit is the expected profit of the manipulative strategy minus that of the standard strategy. Numbers on the graph is excess profit of the manipulative strategy with parameters $\mu = 1$, $q = q^M = 0.6$, $\tau = 2$ with $h_2 = \{buy_1\}$, and $s_2^M = 0$. The left panel is the case that $\tau' = 10$ and the right panel is that τ' is set to make the excess profit maximum. The green area is where the excess profit is negative; the yellow area is where the standard strategy is buying as well.

Figure C.3 Excess profit in different α and $q = q^M$ with $\mu = 0.5$



Note: Excess profit is the expected profit of the manipulative strategy minus that of the standard strategy. Numbers on the graph is excess profit of the manipulative strategy with parameters $\mu = 0.5$, $\varepsilon_b = \varepsilon_s = 0.2$, $\tau = 3$ with $h_3 = \{buy_1, buy_2\}$, and $s_3^M = 0$. The left panel is the case that $\tau' = 20$ and the right panel is that τ' is set to make the excess profit maximum. The green area is where the excess profit is negative; the yellow area is where the standard strategy is buying as well.

Figure C.4 Excess profit in different α and $\epsilon_b = \epsilon_s$ with $\mu = 0.5$



Excess profit is the expected profit of the manipulative strategy minus that of the standard strategy. Numbers on the graph is excess profit of the manipulative strategy with parameters $\mu = 0.5$, $q = q^M = 0.6$, $\tau = 3$ with $h_3 = \{buy_1, buy_2\}$, and $s_3^M = 0$. The left panel is the case that $\tau' = 20$ and the right panel is that τ' is set to make the excess profit maximum. The green area is where the excess profit is negative; the yellow area is where the standard strategy is buying as well.