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Some Axisymmetric and Asymmetric Stokes Flows Involving Axisymmetric Bodies

by

Abdul Mukid Choudhury

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Department of Mathematics

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Dedicated to

A. Rouf Choudhury

(1929-1996)

&

Shirin Choudhury

– my parents

and to

Abdul Basit Choudhury,

Parveen Nahar Basith

&

Hasneen Choudhury.

Ye shall know the truth and
the truth shall make you angry.

– Isaiah Thompson.

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Abstract

The effect of a layer of an adsorbed monomolecular surfactant film of fluid covering the free surface of a finite or semi-infinite volume of substrate fluid, has been investigated for motion within both surfactant and substrate fluids caused by the slow rotation of a partially submerged solid body of revolution. The resulting boundary value problem is solved for varying depths of partial submersion of the solid body by a method in which the equations governing the motion in the substrate and the surfactant boundary condition are satisfied exactly. The error in satisfying the boundary condition on the solid body surface is minimized according to a least-squares technique. A comparison is made with data available from (a) exact solution and (b) experiment when possible. Illustrations include the sphere, concentric spheres and prolate and oblate ellipsoids.

Methods are presented for obtaining exact solutions in analytic form of the equations of asymmetric Stokes flow when an axisymmetric body is at rest or in motion in homogeneous viscous fluid. One method shows how the difficulty of determining three coupled quasi-harmonic functions simultaneously, which is the general problem encountered in this type of flow, may be overcome by the superposition of solutions for flows involving only two quasi-harmonic functions, with each of these functions determined sequentially. Another method considers a class of asymmetric translation problems which involve only two quasi-harmonic functions and analytical expressions are determined for the drag on the body which are compared with numerical values of the drag already available in the literature for certain body shapes.

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Chapter 1

INTRODUCTION

1.1 Axisymmetric Stokes flow in the presence of a surfactant layer

The fact that a fluid interface is often capable of offering resistance to flow greatly in excess of what may be expected from consideration of bulk phase properties has been known since the time of Plateau (1869). In fact, the interfacial region between two homogeneous phases is composed of matter in a distinct physical state exhibiting properties different from those in the bulkgas, liquid or solid-phase states. Therefore new parameters such as interfacial surface tension enter into the thermodynamic and hydrodynamic description of systems when interfaces are present. In the equilibrium states, the effect of the interfaces often need not be considered explicitly unless the ratio of surface to volume is large, because the contribution from interfacial free energy to the total free energy is usually small. However, the dynamic behaviour of flow systems may be profoundly influenced by interfacial effects even though the material content of the interfacial region may be extremely small. At rest, the interfacial region between two fluids behaves as if it were in a state of uniform tension and it is then usually satisfactory to regard the interface simply as a geometrical surface in tension. This simple view is often sufficient in many flows with free boundaries and indeed forms the basis of classical capillary theory where the effect of surface tension is to produce a discontinuity in the normal stress component across the interface if it has curvature. It is also recognized, in the context of the calming effect of a layer of oil on water waves, that extension and contraction of the surface film produce longitudinal variations in the surface tension, the Plateau-Marangoni-Gibbs effect, and this in turn gives rise to discontinuities in the tangential components of fluid stress at the interface. This departure in the surface tension from its equilibrium state can be attributed to the existence of a surface dilational elasticity or surface-shear viscosity. The surface-shear viscosity is also recognized by physical chemists as playing a significant role in foam stability, as well as in the chemistry and dynamics of insoluble surface films of mono-molecular dimension, known as surfactants, which are often highly viscous.

The first attempt to formally incorporate the concept of a surface viscosity into the equations of motion of a fluid interface was carried out by Boussinesq (1913). This and later work was reviewed by Scriven (1960) who provided a rational theory for the dynamics of a fluid interface and in particular established the equation of motion of a Newtonian fluid surface.

Theoretical and experimental work to measure the coefficient of shear viscosity η was reported in a series of papers by Goodrich *et al.* (1969, 1970, 1971). These authors proposed a viscometer which consisted of a thin circular disk or annulus inserted into the plane interface between the surfactant film and the supporting substrate bulk phase. The disk was slowly rotated, and the torque required to maintain a steady rotation was measured. From a knowledge of this torque and a mathematical formula relating the torque to the shear viscosity, the value of η could then be determined. The theoretical analysis proceeded on the assumption that the Reynolds numbers for the flows of both surfactant and substrate are sufficiently small for the linearized Stokes equations to govern the motions generated in both the surfactant and substrate, and in such motions all fluid particles move in circles with centres along the axis of the disk or annulus perpendicular to its plane. Subject to such assumptions, Scriven's analysis of the motion within of the surfactant layer leads to a boundary condition of the form

$$\mu\frac{\partial v}{\partial z}+\eta\frac{\partial^2 v}{\partial z^2}=0 \quad (z=0), \tag{1.1.1}$$

where v is the rotational fluid velocity, η is the surface viscosity of the surfactant and μ is the dynamic viscosity coefficient in the substrate fluid which occupies the half-space $z < 0$.

The mathematical analysis of Goodrich (1969) was flawed and its shortcomings were exposed and discussed in detail by Shail (1978), who also presented a form of solution using the methods of generalized axially symmetric potential theory to formulate an integral equation for the rotational fluid velocity. Shail's analysis provided both a complete set of numerical data for the torque acting on a disk of radius a, when $\lambda = \eta/\mu a$ takes arbitrary values, and a comprehensive description of the asymptotic structure of the solution in the limits of very small and large values of λ.

The surface viscometer proposed by Goodrich nevertheless provides an interesting mathematical boundary-value problem with the somewhat unusual boundary condition (1.1.1), although there are considerable practical difficulties encountered in using such a viscometer. First, the disk is assumed to have zero thickness so that it lies within the surfactant layer which is assumed to be of zero thickness, but in reality, the thickness of the disk would exceed the mono-molecular dimension of the surfactant layer. Thus the placement of the disk in an experiment so as to minimize errors is crucial but very difficult, particularly since the contribution from the *film* or *ring* torque exerted on the disk by the surfactant layer is a very significant part of the total torque exerted on the disk. The results of Shail's theoretical work further indicated that the rotating disk is not a particularly sensitive device for measuring small coefficients of shear viscosity and consequently errors associated with positioning the disk are further magnified.

A number of studies have sought to minimize the effect of critical positioning by taking the disk out of the surfactant layer. Shail (1979) considered the case when the disk is totally submerged in a semi-infinite substrate fluid and rotates about the normal axis through the centre of the disk which lies in a plane which is parallel to the unbounded surfactant layer. Shail *et al.* (1981, 1982) considered further related problems involving a submerged disk, including the effect of a bounded surfactant layer and when the disk performs torsional oscillations. Exact solutions were also given by Davis and O'Neill (1979) for a sphere totally submerged to any depth below the surfactant layer when it slowly rotates about a diameter perpendicular to the surfactant layer. All of these studies relate to configurations which eliminate the effect of placement error in Goodrich's rotating-disk viscometer and, furthermore, there is no film torque arising from the stress within the surfactant layer. This means that the influence of the presence of the surfactant upon the value of the torque acting on the submerged body enters only in a secondary way through the stress distribution in the substrate fluid. This however has the disadvantage that the effect of the surfactant rapidly decays as the depth of the submerged body below the surfactant layer increases, as is reported in the aforementioned studies. Davis (1984) considered a half-submerged sphere in the limiting cases of very large or very small values of the ratio $\lambda = \eta/\mu a$, with a now denoting the sphere radius. His results provided a greater measurable contribution to the total torque arising from the presence of the surfactant layer in the limiting situations $\lambda << 1$ and $\lambda >> 1$ which in turn gives support to the view that a more effective viscometer would involve a rotating body which straddles the surfactant layer. Such a device clearly avoids the disadvantages of the rotating-disk viscometer proposed by Goodrich while at the same time it provides a mechanism whereby a significant contribution to the total torque due to the surfactant layer can arise from both the film and substrate torques.

O'Neill and Yano (1987) considered a sphere of radius a which straddles the surfactant layer and whose centre may be at any depth h below or above the layer, so that $-a < h < a$. An exact solution to this problem when the surface viscosity η is zero was presented by Schneider, O'Neill and Brenner (1973) assuming negligible effect of the meniscus where the free surface makes contact with the sphere. This enabled the boundary-value problem for the rotational velocity to be solved exactly using toroidal coordinates. A comprehensive set of data was provided for the torque acting on the sphere covering a wide range of values of h/a. A set of experimental results was subsequently published by Kunesh *et al.* (1985); these showed very close agreement between measured and theoretical values of the torque over the range $-1 < h/a < 1$, vindicating the assumption of negligible meniscus effect. The satisfying agreement between theory and experiment for $\eta = 0$ led to O'Neill and Yano

(1987) presenting their theoretical model involving a partially submerged sphere when a surfactant layer of arbitrary shear viscosity is present so as to provide, like the analysis of Schneider *et al.* (1973), a sufficiently accurate theoretical model for use in conjunction with measured data for the torque acting on the sphere and thus enable accurate values of the coefficient of surface-shear viscosity to be determined.

In Chapter 3, a sphere which is partially submerged in the substrate fluid below the surfactant layer, rotates slowly about a diameter perpendicular to the plane of the surfactant layer. This problem, first solved by O'Neill and Yano (1987), is now solved by a different approach which leads to generalization for the solution to other geometries. In O'Neill and Yano's work, neither the boundary condition on the rotating sphere nor the surfactant condition were satisfied exactly and the unknown coefficients in the series representation of the solution were determined by minimizing the combined error in the non-satisfaction of the conditions on the two surfaces. In that work the origin was always located at the centre of the sphere or its reflection in the surfactant layer. In this thesis, we fix the origin within the plane of the surfactant layer. This has the advantage of permitting the surfactant boundary condition to be satisfied exactly, and thereby eliminating one set of unknown coefficients. A feature of the O'Neill and Yano results was that there was a significant deviation between the computed surface velocity when $\lambda = \infty$ and that derived analytically , as indicated in Fig. 10 of their paper. This was unusual since very close agreement between the numerical results and analytical results is reported elsewhere in the paper. We discovered that O'Neill and Yano had left out an eigen-solution in the representation for the velocity, which contributes *only when* $\lambda = \infty$. If this eigen-solution is included, it is shown that excellent agreement between analytical and numerical results is then achieved.

In Chapters *4* and *5* of this thesis, the effect of a layer of an adsorbed mono-molecular surfactant film of fluid covering the free surface of a finite or semi-infinite volume of substrate fluid has been investigated for motion within both surfactant and substrate fluids caused by the slow rotation of a partially submerged solid body. The resulting boundary value problem is considered for varying depths of partial submersion of the solid body by a method in which the equations of motion and continuity in the substrate, and the surfactant boundary condition are satisfied exactly. The boundary condition on the rotating body is satisfied approximately with the error minimized according to a least-squares criterion. A comparison is made with data available from (a) exact solution and (b) experiment where possible. Illustrations of the theory include concentric and eccentric spheres as well as prolate and oblate ellipsoids.

1.2 Asymmetric Stokes flow generated by axisymmetric bodies

A problem of fundamental importance in many engineering applications of the theory of suspensions in sedimentation or aerosols is the determination of the Stokes resistance of a small particle in motion in a fluid which is in general undergoing shear. An example would be the transport of solid particles in a pressure driven flow through a tube or channel. The theoretical problems which model these applications are problems of great mathematical complexity involving in general particle-particle and particle-wall interactions as well as the basic particle-fluid interaction.

For rotation of an axisymmetric body about its axis of symmetry in unbounded fluid the resulting axisymmetric flow problem was investigated by Jeffery (1915) who showed that the pressure field is constant and the fluid velocity consists of one component orthogonal to the azimuthal plane. The solution for this velocity component was found explicitly by Jefferyfor a number of body geometries. Chwang and Wu (1974) approached this problem from a different viewpoint and showed how exact solutions for rotating bodies can be constructed by considering suitably chosen distributions of rotlets along the axis of symmetry. Their work corroborates that of Jeffery for the torque coefficient for a prolate or oblate ellipsoid. Even when the body rotates about an axis of symmetry, there is a scarcity of exact solutions with a limitation being set by the coordinate systems in which Laplace's equation has separable solutions. Slender body theory, applicable when the axial dimension of the body greatly exceeds any transverse dimension, provides approximate solutions for other axisymmetric bodies, as was demonstrated by Batchelor (1970) and Cox

(1970) for example.

An exact solution was determined by Edwards (1892) for slow rotation of a general ellipsoid about a principal axis. The torque on an ellipsoid of revolution rotating about its axis of symmetry is found to agree with that of Jeffery if a numerical factor is replaced by the correct value 16/3. Brenner (1963) examined the limiting case of a circular disk and pointed out that the torque is invariant about any axis of rotation through the centre of the disk. This remarkable property is of course also possessed by the sphere. It was further shown by Brenner to be a property possessed by some other bodies such as a cube, but it is worth noting that no similar drag invariance property exists for the translating disk. Jeffery (1922) obtained an exact solution for a general ellipsoid in a linear shear flow and properties of this solution have been extensively studied by Hinch and Leal (1979).

The asymmetric rotation problem is evidently more complicated analytically because in addition to a non-vanishing pressure field there are three velocity components which must now be determined. As demonstrated for instance by Lamb (1932), the general solution of the Stokes equations involves the evaluation of three quasi-harmonic scalar functions. The purpose of Chapter *6* and *7* of this thesis is to formulate methods for obtaining solutions of the equations of asymmetric Stokes flow for an axisymmetric body which is at rest or in rotation in homogeneous viscous fluid.

In Chapter *6* it is shown how the difficulty of determining the three coupled quasi-harmonic functions *simultaneously* can be overcome by exploiting the linearity of the governing equations. Thus, by superposition, the solutions of various problems may be derived from a set of solutions which involve only *two* quasi-harmonic functions, and each of these pairs of functions may be determined *sequentially.*

In Chapter *7* the ideas of this chapter are further developed by noting that the solution of Oberbeck (1890) for the translation of a *general* ellipsoid involves only two quasi-harmonic scalar functions. It is clear that for the translation of a sphere, circular disk, or a spheroid - prolate or oblate - the solution for the asymmetric translation of these axisymmetric bodies perpendicular to their axis of symmetry involves at most two quasi-harmonic scalar functions. This leads us to conjecture whether there is a general class of axisymmetric bodies for which the solution of the Stokes equations for translation perpendicular to the axis of symmetry involves the determination of only two quasi-harmonic functions. In this chapter we explore the verification of this conjecture for some body shapes for which exact analytic solutions have not been obtained up to now.

Chapter 2

PHYSICAL AND MATHEMATICAL ASPECTS

2.1 Introduction

This chapter is concerned with the mathematical and physical aspects of fluid flow at low Reynolds Number, $\Omega a^2/\nu$, where ν denote the kinematic viscosity of the fluid, a some length scale associated with the body and Ω a constant angular velocity.

2.2 Mathematical aspects

In this section a number of results are established which will be of use in subsequent work.

2.2.1 Legendre functions of the First Kind

For $t = cos\theta$, we find in Morse and Feshbach (1953),

$$P_0^0(t) = 1, \tag{2.2.1}$$

$$P_1^0(t) = t, \tag{2.2.2}$$

$$P_2^0(t) = \frac{1}{2}(3t^2 - 1) \tag{2.2.3}$$

and $P_n^m(t) = (1-t^2)^{m/2}\frac{d^m}{dt^m}P_n^0(t), \ (m, n \geq 0)$, giving

$$P_1^1(t) = (1-t^2)^{1/2}, \tag{2.2.4}$$

$$P_2^1(t) \quad = \quad 3t(1-t^2)^{1/2}. \tag{2.2.5}$$

In general, for integers $n \geq 1$,

$$P_{n+1}^1(t) = \frac{1}{n}\left[(2n+1)tP_n^1(t) - (n+1)P_{n-1}^1(t)\right]. \tag{2.2.6}$$

2.2.2 Integral representations

Consider $\alpha_{m,n}$ defined by

$$\alpha_{m,n} \quad = \quad \int_{-1}^{0} P_{2m-1}^1(t)P_{2n-1}^1(t)dt, \ \ (m,n = 1,2,..) \tag{2.2.7}$$

Morse and Feshbach (1953) show that

$$\int_{-1}^{1} P_\nu^1(t)P_\mu^1(t)dt \quad = \quad \left[\frac{2\nu(\nu+1)}{2\nu+1}\right] \ \delta_{\nu,\mu}. \tag{2.2.8}$$

Since $P_{2m-1}^1(t)$ and $P_{2n-1}^1(t)$ are even functions of t, equation (2.2.7) becomes

$$\begin{aligned} \alpha_{m,n} = \int_{-1}^{0} P_{2m-1}^1(t)P_{2n-1}^1(t)dt \quad &= \quad \frac{1}{2}\int_{-1}^{1} P_{2m-1}^1(t)P_{2n-1}^1(t)dt \\ &= \quad \left[\frac{2m(2m-1)}{(4m-1)}\right] \ \delta_{m,n}. \end{aligned} \tag{2.2.9}$$

Hence

$$\alpha_{m,n} \quad = \quad \left[\frac{2m(2m-1)}{(4m-1)}\right] \ \delta_{m,n}. \tag{2.2.10}$$

Next, consider β_m, defined by

$$\beta_m = \int_{-1}^{0} (t+t^2)P_{2m}'(t)dt. \tag{2.2.11}$$

We first note that from equation (2.2.6)

$$\int_{-1}^{0} tP_m'(t)dt \quad = \quad \left[\frac{m}{(2m+1)}\right]\int_{-1}^{0} P_{m+1}'(t)dt + \left[\frac{(m+1)}{(2m+1)}\right]\int_{-1}^{0} P_{m-1}'(t)dt \tag{2.2.12}$$

and therefore

$$\int_{-1}^{0} t^2P_m'(t)dt \quad = \quad \left[\frac{m}{(2m+1)}\right]\int_{-1}^{0} tP_{m+1}'(t)dt + \left[\frac{(m+1)}{(2m+1)}\right]\int_{-1}^{0} tP_{m-1}'(t)dt. \tag{2.2.13}$$

Changing the suffix, (2.2.12) becomes

$$\begin{aligned}\int_{-1}^{0} tP'_{2m}(t)dt &= \left[\frac{2m}{(4m+1)}\right][P_{2m+1}(t)]_{-1}^{0} + \left[\frac{(2m+1)}{(4m+1)}\right][P_{2m-1}(t)]_{-1}^{0} \\ &= 1 \end{aligned} \qquad (2.2.14)$$

since $P_{2m+1}(0) = P_{2m-1}(0) = 0$ and $P_{2m+1}(-1) = P_{2m-1}(-1) = -1$, and thus (2.2.13) becomes

$$\int_{-1}^{0} t^2 P'_{2m}(t)dt = \left[\frac{2m}{(4m+1)}\right]\int_{-1}^{0} tP'_{2m+1}(t)dt + \left[\frac{(2m+1)}{(4m+1)}\right]\int_{-1}^{0} tP'_{2m-1}(t)dt. \qquad (2.2.15)$$

But (2.2.12) gives

$$\begin{aligned}\int_{-1}^{0} tP'_{2m+1}(t)dt &= \left[\frac{(2m+1)}{(4m+3)}\right][P_{2m+2}(t)]_{-1}^{0} + \left[\frac{(2m+2)}{(4m+3)}\right][P_{2m}(t)]_{-1}^{0} \\ &= -1 + \left[\frac{1}{(4m+3)}\right][(2m+1)P_{2m+2}(0) + (2m+2)P_{2m}(0)] \end{aligned} \qquad (2.2.16)$$

where

$$\begin{aligned}[(2m+1)P_{2m+2}(0) + (2m+2)P_{2m}(0)] &= [P_{2m}(0) - P_{2m+2}(0)] \\ &= P_{2m}(0)\left[1 + \frac{(2m+1)}{(2m+2)}\right] \\ &= P_{2m}(0)\left[\frac{(4m+3)}{(2m+2)}\right] \end{aligned} \qquad (2.2.17)$$

since

$$(2m+2)P_{2m+2}(0) + (2m+1)P_{2m}(0) = 0 \ (m \geq 0)$$

and

$$\left[P_{2m}(0) = (-1)^m (2m-1)!/2^{2m-1} m!(m-1)!\right].$$

Therefore equation (2.2.16) becomes

$$\int_{-1}^{0} tP'_{2m+1}(t)dt = -1 + \left[\frac{1}{(2m+2)}\right] P_{2m}(0). \qquad (2.2.18)$$

Likewise

$$\begin{aligned}\int_{-1}^{0} tP'_{2m-1}(t)dt &= -1 + \left[\frac{1}{(2m)}\right] P_{2m-2}(0) \\ &= -1 - \left[\frac{1}{(2m-1)}\right] P_{2m}(0). \end{aligned} \qquad (2.2.19)$$

Thus, substituting equations (2.2.18) and (2.2.9) into (2.2.14),

$$\begin{aligned}\int_{-1}^{0} t^2 P'_{2m}(t)dt &= \left[\frac{2m}{(4m+1)}\right]\int_{-1}^{0} tP'_{2m+1}(t)dt + \left[\frac{(2m+1)}{(4m+1)}\right]\int_{-1}^{0} tP'_{2m-1}(t)dt \\ &= -1 + P_{2m}(0)\left[\frac{(-8m-2)}{(4m+1)(2m+2)(2m-1)}\right] \\ &= -1 - P_{2m}(0)\left[\frac{2}{(2m+2)(2m-1)}\right]. \qquad (2.2.20)\end{aligned}$$

Therefore

$$\begin{aligned}\beta_m &= \int_{-1}^{0}(t+t^2)P'_{2m}(t)dt \\ &= -P_{2m}(0)\left[\frac{2}{(2m+2)(2m-1)}\right] \qquad (2.2.21)\end{aligned}$$

where

$$\begin{aligned}P_{2m}(0) &= \frac{P_{2m}(0)}{P_{2m-2}(0)}.\frac{P_{2m-2}(0)}{P_{2m-4}}\cdots\frac{P_2(0)}{P_0(0)}.P_0(0) \\ &= (-1)^m\left[\frac{(2m-2)!}{(2^{2m-1})(m)!(m-1)!}\right] \qquad (2.2.22)\end{aligned}$$

for $m \geq 1$. The general expression for the β_m is accordingly

$$\beta_m = (-1)^{m+1}\left[\frac{(2m-2)!}{(2^{2m-1})(m+1)!(m-1)!}\right]. \qquad (2.2.23)$$

Using equations (2.2.10) and (2.2.23),

$$\begin{aligned}\frac{\alpha_{m,m}}{\beta_m} &= (-1)^{m+1}\left[\frac{(2^{2m-1})(m+1)!(m-1)!}{(2m-2)!}\right].\left[\frac{(2m-1)(2m)}{(4m-1)}\right] \\ &= (-1)^{m+1}\left[\frac{(2^{2m})(m+1)!m!}{(2m-2)!}\right].\left[\frac{(2m-1)}{(4m-1)}\right]. \qquad (2.2.24)\end{aligned}$$

Hence

$$\frac{\beta_m}{\alpha_{m,m}} = (-1)^{m+1}\left[\frac{(2m-2)!\,(4m-1)}{(2^{2m})\,(m+1)!\,m!\,(2m-1)}\right] \qquad (2.2.25)$$

for $m \geq 1$.

2.2.3 Legendre functions of the Second Kind

With $s = \cosh\xi$ we have for $s > 1$ from Morse and Feshbach (1953),

$$\begin{aligned}Q_0^0(s) &= \frac{1}{2}\ln\left(\frac{s+1}{s-1}\right), \qquad (2.2.26) \\ Q_1^0(s) &= \frac{1}{2}s\ln\left(\frac{s+1}{s-1}\right) - 1, \qquad (2.2.27) \\ Q_2^0(s) &= \frac{1}{4}(3s^2-1)\ln\left(\frac{s+1}{s-1}\right) - \frac{3}{2}s \qquad (2.2.28)\end{aligned}$$

and

$$Q_n^1(s) = -(s^2-1)^{1/2} Q_n'(s)$$

for $n \geq 0$. In particular

$$Q_1^1(s) = \sqrt{s^2-1}\left[\frac{s}{s^2-1} - \frac{1}{2}\ln\left(\frac{s+1}{s-1}\right)\right], \tag{2.2.29}$$

$$Q_2^1(s) = \sqrt{s^2-1}\left[\frac{3\,s^2-2}{s^2-1} - \frac{3}{2}\,s\ln\left(\frac{s+1}{s-1}\right)\right]. \tag{2.2.30}$$

and in general,

$$Q_{n+1}^1(s) = \frac{1}{n}\left[(2n+1)\,s\,Q_n^1(s) - (n+1)Q_{n-1}^1(s)\right]. \tag{2.2.31}$$

for $n \geq 1$.

2.2.4 Spherical polar coordinates (r, θ, ϕ)

The spherical polar coordinates (r, θ, ϕ) are related to the cylindrical polar coordinates (ρ, ϕ, z) by the relations

$$\rho = r\sin\theta, \quad z = r\cos\theta, \tag{2.2.32}$$

with $r \geq 0$ and $0 \leq \theta \leq \pi$. Thus, the Cartesian coordinates (x, y, z) are expressible as

$$\begin{aligned} x &= r\sin\theta\cos\phi, \\ y &= r\sin\theta\sin\phi, \\ z &= r\cos\theta. \end{aligned} \tag{2.2.33}$$

By restricting the ranges of these coordinates as follows

$$0 \leq r < \infty, \;\; 0 \leq \theta \leq \pi, \;\; 0 \leq \phi < 2\pi \tag{2.2.34}$$

each point in space is represented once and only once, with the exception of the points along the z axis, for which ϕ is undetermined.

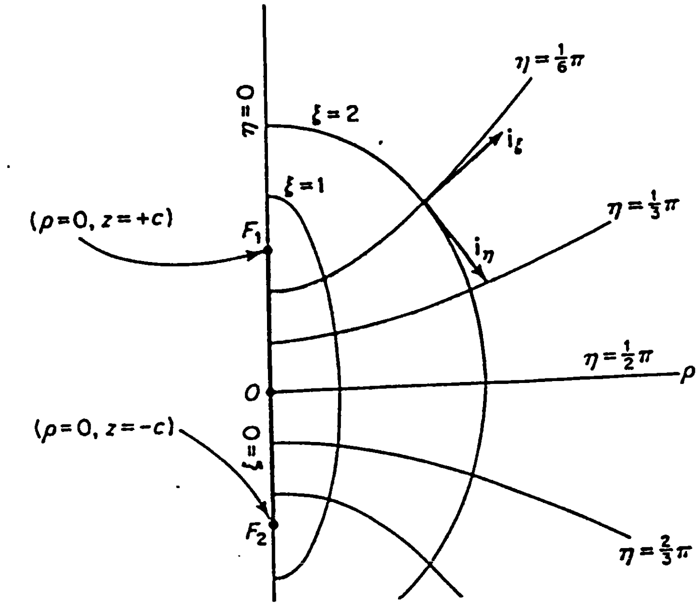

Figure 2.1: Prolate spheroidal coordinates in a meridian plane.

2.2.5 Prolate spheroidal coordinates (ξ, η, ϕ)

The transformation of cylindrical polar coordinates given by

$$z + i\,\rho = c\cosh\,(\xi + i\,\eta) \tag{2.2.35}$$

for $c > 0$, leads to the relations

$$\begin{aligned} z &= c\cosh\xi\,\cos\eta, \\ \rho &= c\sinh\xi\,\sin\eta, \end{aligned} \tag{2.2.36}$$

Each point in space is obtained once and, with minor exceptions, only once by limiting the ranges of the prolate spheroidal coordinates (ξ, η, ϕ) in following manner:

$$\begin{aligned} 0 &\leq \xi < \infty, \\ 0 &\leq \eta \leq \pi, \\ 0 &\leq \phi < 2\,\pi. \end{aligned} \tag{2.2.37}$$

Eliminating η from equation (2.2.36) results in

$$\frac{z^2}{c^2\cosh^2\xi} + \frac{\rho^2}{c^2\sinh^2\xi} = 1. \tag{2.2.38}$$

Since $\cosh\xi > \sinh\,\xi$, the coordinate surface $\xi =$ constant is a member of a family of confocal prolate spheroids having their geometric centre at the origin. Spheroids of this type are generated by the rotation of an ellipse about its major axis – in this instance along the z axis – as indicated in Figures. 2.1 and 2.2.

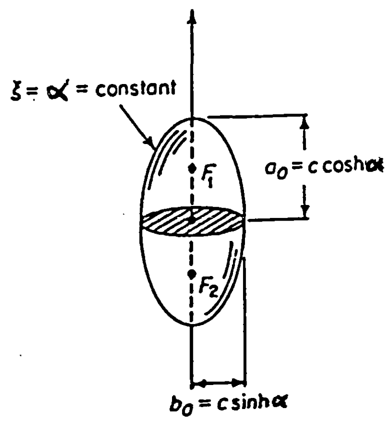

Figure 2.2: Prolate spheroid.

The foci, F_1 and F_2, of the confocal system are located on the z axis at the points $\{\rho = 0, z = \pm c\}$ corresponding to the values $\{\xi = 0, \eta = 0 \text{ and } \pi\}$ respectively. The major and minor semi-axes, lengths a_0 and b_0 respectively, of a typical ellipsoid, $\xi = \alpha =$ constant, lie along the z axis and in the plane $z = 0$, respectively, and the lengths are given by

$$\begin{aligned} a_0 &= c \cosh \alpha, \\ b_0 &= c \sinh \alpha. \end{aligned} \tag{2.2.39}$$

Thus,

$$c^2 = a_0^2 - b_0^2 \tag{2.2.40}$$

and

$$\alpha = \frac{1}{2} \ln \left[\frac{a_0 + b_0}{a_0 - b_0} \right], \tag{2.2.41}$$

which gives the parameters c and α in terms of the lengths of the semi-axes. It should be noted that the eccentricity e_0 of a typical prolate ellipsoid is

$$\begin{aligned} e_0 &= \left[1 - \left(\frac{b_0}{a_0} \right)^2 \right]^{1/2} \\ &= (\cosh \alpha)^{-1} \end{aligned} \tag{2.2.42}$$

The value $\alpha = 0$ is a degenerate ellipsoid which reduces to the line segment $-c \leq z \leq c$ along the z axis, connecting the foci.

2.2.6 Oblate spheroidal coordinates (ξ, η, ϕ)

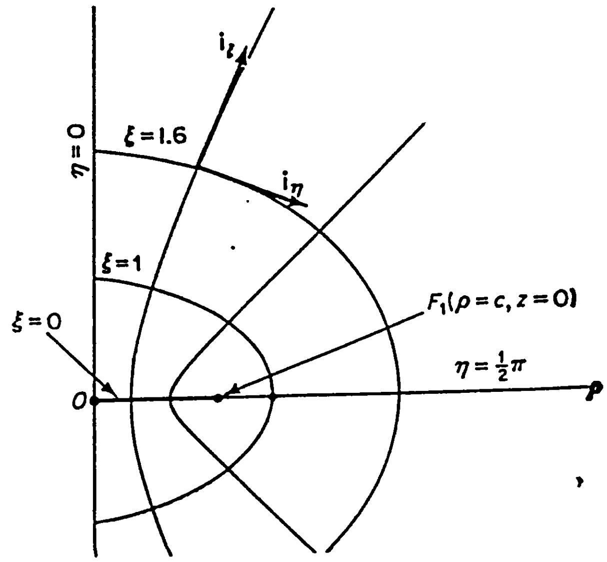

Figure 2.3: Oblate spheroidal coordinates in a meridian plane.

The transformation

$$z + i\,\rho = c \sinh(\xi + i\,\eta) \tag{2.2.43}$$

for $c > 0$, gives rise to the relations

$$\begin{aligned} z &= c \sinh \xi \cos \eta, \\ \rho &= c \cosh \xi \sin \eta \end{aligned} \tag{2.2.44}$$

where again (ρ, ϕ, z) are cylindrical polar coordinates. Every point in space is represented at least once and only once by restricting the ranges of the oblate spheroidal coordinates (ξ, η, ϕ) as follows:

$$\begin{aligned} 0 &\leq \xi < \infty, \\ 0 &\leq \eta \leq \pi, \\ 0 &\leq \phi < 2\,\pi. \end{aligned} \tag{2.2.45}$$

Eliminating η from equation (2.2.46) yields

$$\frac{z^2}{c^2 \sinh^2 \xi} + \frac{\rho^2}{c^2 \cosh^2 \xi} = 1 \tag{2.2.46}$$

from which it is readily established that the coordinate surface ξ = constant is now a member of a family of confocal oblate spheroids having their common centre at the origin. Spheroids of this type are generated by the rotation of an ellipse about its minor axis – in this case along the z axis – as indicated in Figures. 2.3 and 2.4.

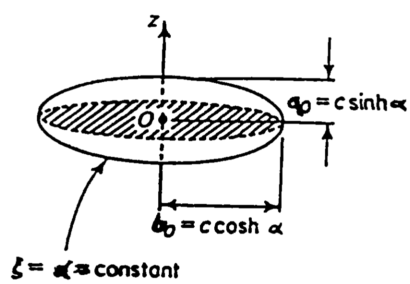

Figure 2.4: Oblate spheroid.

The focal circle of the confocal family lies in the plane $z = 0$ and corresponds to the circle $\rho = c$. The major and minor axes of a typical oblate spheroid, $\xi = \alpha =$ constant, lie in the plane $z = 0$ and along the z axis, respectively. The ellipsoid given by $\alpha = 0$ is degenerate and corresponds to that portion of the plane $z = 0$ inside the focal circle, for which $0 \le \rho \le c$. The lengths of the minor and major semi-axes are $a_0 = c \sinh\, \alpha,\; b_0 = c \cosh\, \alpha$.

2.3 Axisymmetric Stokes flow

2.3.1 A Sphere rotating with a surfactant layer

Consider a partially submerged sphere of radius a slowly rotating with constant angular velocity in a semi-infinite incompressible fluid with dynamic viscosity μ .

The axis of rotation is the diameter of the sphere perpendicular to the surface of the substrate fluid on which there is a layer of surfactant fluid. The depth of the centre C of the sphere *below* the surfactant layer is c, where $-a < c < a$. Thus $c > 0$ or $c < 0$ according as the sphere is more or less than half submerged, respectively.

In order to preserve the symmetry of later analytical work, the system of cylindrical polar coordinates (ρ, ϕ, z) with the origin O lying in the plane of the interface between the surfactant and substrate fluid, will be used. Assuming that the Reynolds number,

$$R_e = \frac{\Omega a^2}{\nu}, \tag{2.3.1}$$

where ν here denotes the kinematic viscosity of the substrate fluid, for the flow induced in the substrate fluid to be sufficiently small to permit the neglect of the inertia terms in the Navier-Stokes equations, then the flow in the substrate fluid is governed by the Stokes equation

$$\mu \nabla^2 \mathbf{v} = \nabla p, \tag{2.3.2}$$

together with the equation of continuity

$$\nabla . \mathbf{v} = 0, \tag{2.3.3}$$

where $\mathbf{v}$ denotes the fluid velocity, p is the fluid pressure and μ is the coefficient of dynamic viscosity of the substrate fluid.

The fluid motion is caused solely by the rotation of the surface, and because of the axisymmetric nature of the problem, it follows that the velocity $\mathbf{v}$ has only one component which is in the azimuthal direction of a system with z - axis along the axis of rotation of the surface and pointing out of the substrate fluid. Thus (2.3.2) and (2.3.3) possess a solution of the form

$$\mathbf{v} = v\,\hat{\phi} \tag{2.3.4}$$

and

$$p = \text{constant} \tag{2.3.5}$$

provided that

$$\nabla^2 v - \frac{v}{\rho^2} = 0. \tag{2.3.6}$$

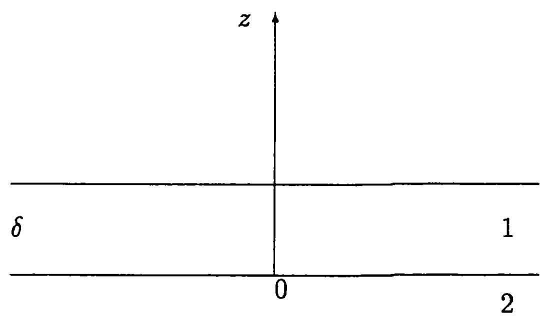

Figure 2.5: Free surface of substrate fluid with a surface film.

2.3.2 Boundary conditions

For the problem of a solid body rotating with a surfactant layer, there are two boundary conditions to be satisfied. One on the surface Γ_1 of the body and the other on the surfactant film Γ_2.

To satisfy the non-slip boundary condition on the body surface Γ_1 requires that

$$v = \rho \tag{2.3.7}$$

where (ρ, ϕ, z) are cylindrical polar coordinates and for rest of this section all physical quantities will be referred to cylindrical polar coordinates.

In the absence of the surfactant layer, when Γ_2 forms the interface between the substrate fluid and a fluid, such as air, imposing negligible shear stress on $z = 0$, then

$$p_{\rho z} = \mu \frac{\partial v}{\partial z} = 0 \ \ at \ \ z = 0. \tag{2.3.8}$$

In order to discuss the effect in the presence of the surfactant layer, a thin fluid layer (1) of thickness δ covering the substrate fluid (2), will be considered. It is assumed that a fluid with negligible shear viscosity (for example air) now bounds the upper surface $z = \delta$ of the surfactant layer, as depicted in Figure 2.5.

Letting suffices 1 and 2 denote quantities pertaining to the fluids (1) and (2) respectively, the boundary conditions to be satisfied are

1. continuity of velocity and stress on $z = 0$,

2. zero tangential stress on $z = \delta$.

Thus

$$\begin{aligned} v_1 &= v_2, \ (z=0) \\ \frac{\mu_1 \partial v_1}{\partial z} &= \frac{\mu_2 \partial v_2}{\partial z} \quad (z=0) \end{aligned} \tag{2.3.9}$$

and

$$\frac{\mu_1 \partial v_1}{\partial z} = 0 \quad (z = \delta). \tag{2.3.10}$$

Equation (2.3.9) and use of the Taylor series expansion

$$\frac{\partial v_1}{\partial z}|_{z=\delta} = \frac{\partial v_1}{\partial z}|_{z=0} + \delta \frac{\partial^2 v_1}{\partial z^2}|_{z=0} + .. \tag{2.3.11}$$

means that equation (2.3.10) implies that

$$\mu_1 \left(\frac{\partial v_1}{\partial z}\right)_{z=0} + \mu_1 \delta \left(\frac{\partial^2 v_1}{\partial z^2}\right)_{z=0} + O\left(\mu_1 \delta^2\right) = 0. \tag{2.3.12}$$

However, assuming that equation (2.3.6) holds fluids 1 and 2, we see that on $z = 0$,

$$\frac{\partial^2 v_1}{\partial z^2} = -\frac{\partial^2 v_1}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial v_1}{\partial \rho} + \frac{v_1}{\rho^2}, \tag{2.3.13}$$

and

$$\frac{\partial^2 v_2}{\partial z^2} = -\frac{\partial^2 v_2}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial v_2}{\partial \rho} + \frac{v_2}{\rho^2}. \tag{2.3.14}$$

Equations (2.3.13) and (2.3.14) together with the first part of equations (2.3.9) give

$$\left(\frac{\partial^2 v_1}{\partial z^2}\right)_{z=0} = \left(\frac{\partial^2 v_2}{\partial z^2}\right)_{z=0}. \tag{2.3.15}$$

Which implies that, as $\delta \to 0$, with the surface viscosity η defined by

$$\eta = lim(\mu_1 \delta) \quad (\delta \to 0, \mu_1 \to \infty), \tag{2.3.16}$$

equation (2.3.10) reduces to

$$\mu \frac{\partial v}{\partial z} + \eta \frac{\partial^2 v}{\partial z^2} = 0 \quad (z = 0). \tag{2.3.17}$$

In equation (2.3.17) the suffix 2 is now being suppressed, and μ denoting the coefficient of dynamic viscosity in the substrate fluid which occupies the region $z < 0$. The equation (2.3.17) is precisely that given by Scriven (1960) when the flow is a swirling motion, but has been obtained here by a more direct approach for this problem. Equation (2.3.17) can be written in dimensionless form as

$$\frac{\partial v}{\partial z} + \lambda \frac{\partial^2 v}{\partial z^2} = 0 \tag{2.3.18}$$

on Γ_2 , where $\lambda = \eta/\mu$.

2.3.3 General solution in spherical polar coordinates for the velocity field

In this section, a systematic method of approach to the problem of solving the differential equation of axisymmetric creeping flow in spherical polar coordinates (r, θ, ϕ) is provided. The Cartesian coordinates which been described in Section *2.2.4* are

$$\begin{aligned} x &= r \sin\theta \cos\phi, \\ y &= r \sin\theta \sin\phi, \\ z &= r\cos\theta, \end{aligned} \tag{2.3.19}$$

where $r \geq 0$, $0 \leq \phi \leq 2\pi$, $0 \leq \theta \leq \pi$.

In spherical polar coordinates, (2.3.6) together with (2.3.4) becomes

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial v}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial v}{\partial \theta}\right) - \frac{v}{\sin^2\theta} = 0. \tag{2.3.20}$$

For brevity, put $t = \cos\theta$, then

$$\begin{aligned} -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta} &= \frac{\partial}{\partial t} \\ \frac{\partial}{\partial \theta} &= -(1-t^2)^{1/2}\frac{\partial}{\partial t}. \end{aligned} \tag{2.3.21}$$

Therefore, (2.3.20) becomes

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial v}{\partial r}\right) + \frac{\partial}{\partial t}\left[(1-t^2)^{1/2}\frac{\partial v}{\partial t}\right] - \frac{v}{(1-t^2)^{1/2}} = 0. \tag{2.3.22}$$

The homogeneous equation (2.3.22) may be solved by separation of variables as follows:

$$v = f(r)g(t) \tag{2.3.23}$$

which, upon substitution in equation (2.3.22), yields

$$\left[\frac{d}{dr}\left(r^2\frac{df(r)}{dr}\right)\right]g(t)+\left[\frac{d}{dt}\left[(1-t^2)^{1/2}\frac{dg(t)}{dt}\right]-\frac{g(t)}{(1-t^2)^{1/2}}\right]f(r)=0. \quad (2.3.24)$$

Dividing through equation (2.3.24), by f(r) and g(t), the resulting expression can only be satisfied if

$$\left[\frac{d}{dr}\left(r^2\frac{df}{dr}\right)\right]-j(j+1)f(r)=0 \quad (2.3.25)$$

and

$$\left[\frac{d}{dt}\left(1-t^2\right)\frac{dg(t)}{dt}\right)-\frac{g(t)}{(1-t^2)}\right]+j(j+1)g(t)=0, \quad (2.3.26)$$

with j an integer or zero. The equation (2.3.25) implies that

$$r^2f''(r)+2rf'(r)-j(j+1)f(r)=0. \quad (2.3.27)$$

Assuming a solution of the form $f(r)=r^\alpha$ exists for the above equation, then

$$\begin{aligned}\alpha(\alpha-1)+2\alpha-j(j+1) &= 0\\ \alpha^2+\alpha-j(j+1) &= 0\end{aligned}$$

giving

$$\alpha = j \ or \ -(j+1). \quad (2.3.28)$$

Thus equation (2.3.23) has as its solution

$$f(r)=A_jr^j+B_jr^{-(j+1)} \quad (2.3.29)$$

with A_j and B_j arbitrary constants, and equation (2.3.26) is Legendre's equation, and has the Legendre functions of the first and second kind, $P_j^1(t)$ and $Q_j^1(t)$, as its independent solutions. The functions $P_j^1(t)$ and $Q_j^1(t)$ can be written as

$$P_j^1(t)=(1-t^2)^{1/2}P_j'(t) \ (j\geq 1) \quad (2.3.30)$$

and

$$Q_j^1(t)=(1-t^2)^{1/2}Q_j'(t) \ (j\geq 0), \quad (2.3.31)$$

with the $'$ denoting differentiation with respect to the argument. The functions $Q_j^1(t)$ are singular at both $t=1$ and $t=-1$., The solutions of (2.3.20), for $j \geq 1$, which are bounded for $-1 \leq t \leq 0$ are therefore

$$r^j P_j^1(t) \ , \ r^{-(j+1)} P_j^1(t), \tag{2.3.32}$$

In the degenerate case $j = 0$, then $f(r) = A + B/r$, where A, B are constants, and equation (2.3.26) becomes

$$\frac{d}{dt}\left[(1-t^2)\frac{dg(t)}{dt}\right] - \frac{g(t)}{(1-t^2)} = 0. \tag{2.3.33}$$

A solution is

$$\frac{1}{(1-t^2)^{1/2}} = Q_0^1(t), \tag{2.3.34}$$

since

$$Q_0(t) = \frac{1}{2}\ \log\left[\frac{(1+t)}{(1-t)}\right]. \tag{2.3.35}$$

To find a second independent solution, let

$$g(t) = \frac{G(t)}{(1-t^2)^{1/2}}. \tag{2.3.36}$$

It then follows that

$$\frac{dg(t)}{dt} = \frac{G'(t)}{(1-t^2)^{1/2}} + \frac{t\,G(t)}{(1-t^2)^{3/2}}. \tag{2.3.37}$$

Multiplying both sides of equation (2.3.37) by $(1-t^2)$ gives

$$(1-t^2)\frac{dg(t)}{dt} = (1-t^2)^{1/2}G'(t) + \frac{t\,G(t)}{(1-t^2)^{1/2}}. \tag{2.3.38}$$

Hence

$$\begin{aligned}\frac{d}{dt}\left[(1-t^2)\frac{dg(t)}{dt}\right] = &\quad (1-t^2)^{1/2}G''(t) \\ + &\quad \left\{\frac{1}{(1-t^2)^{1/2}} + \frac{t^2}{(1-t^2)^{3/2}}\right\}G(t).\end{aligned} \tag{2.3.39}$$

But

$$\left\{\frac{1}{(1-t^2)^{1/2}} + \frac{t^2}{(1-t^2)^{3/2}}\right\} \ = \ \frac{1}{(1-t^2)^{3/2}}. \tag{2.3.40}$$

Hence equation (2.3.39) becomes

$$\frac{d}{dt}\left[(1-t^2)\frac{dg(t)}{dt}\right] = (1-t^2)^{1/2}G''(t) + (1-t^2)^{-3/2}G(t). \tag{2.3.41}$$

Noting that equation (2.3.33) can be satisfied if and only if

$$G''(t) = 0. \tag{2.3.42}$$

Therefore

$$G(t) = C\,t + D, \tag{2.3.43}$$

and it follows that

$$g(t) = \frac{C\,t + D}{(1 - t^2)^{1/2}}, \tag{2.3.44}$$

where C, D are constants. The solution of the equation (2.3.20), such that v is bounded for $-1 \leq t \leq 0$ and $v \to 0$ as $r \to \infty$, has the general solution in spherical coordinates of the form

$$v = B_0 \left[\frac{1}{r}\right] \left[\frac{1+t}{1-t}\right]^{1/2} + \sum_{j=1}^{\infty} B_j \left[\frac{1}{r}\right]^{j+1} P_j^1(t). \tag{2.3.45}$$

If there is an outer boundary, so that r does not extend to infinity, then the appropriate general solution for the velocity field is then

$$\begin{aligned} v = \quad & \left(A_0 + \frac{B_0}{r}\right) \left[\frac{1+t}{1-t}\right]^{1/2} \\ + \quad & \sum_{j=1}^{\infty} \left\{ A_j r^j + B_j \left[\frac{1}{r}\right]^{j+1} P_j^1(t) \right\}. \end{aligned} \tag{2.3.46}$$

2.3.4 General solution in ellipsoidal coordinates for the velocity field

In this section, as in Section *2.3.2*, a systematic method of approach to the problem of solving the differential equation of axisymmetric creeping flow in ellipsoidal coordinates is provided. In order to solve the general problem set out in Section *2.3.1*, it is useful to reintroduce prolate spheroidal coordinates (ξ, η, ϕ) defined by

$$z + i\,\rho = c \cosh(\xi + i\,\eta) \tag{2.3.47}$$

with $c > 0$, giving

$$\begin{aligned} z &= c \cosh\xi \cos\eta, \\ \rho &= c \sinh\xi \sin\eta \end{aligned} \tag{2.3.48}$$

for the cylindrical polar coordinates.

In ellipsoidal coordinates, (2.3.6) together with (2.3.48) becomes

$$\frac{\partial}{\partial \xi}\left(F_1 \frac{\partial v}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(F_1 \frac{\partial v}{\partial \eta}\right) - \left(\frac{F_2}{F_1}\right) v = 0, \tag{2.3.49}$$

where

$$F_1 = \sinh \xi \, \sin \eta \tag{2.3.50}$$

and

$$F_2 = (\cosh^2 \xi - \cos^2 \eta). \tag{2.3.51}$$

The homogeneous equation (2.3.49) may be solved by separation of variables as follows:

$$v = f(s)\, g(t) \tag{2.3.52}$$

For brevity, put

$$t = \cos \eta \tag{2.3.53}$$

and

$$s = \cosh \xi. \tag{2.3.54}$$

Hence

$$\frac{\partial}{\partial \eta} \quad = \quad -(1 - t^2)^{\frac{1}{2}} \frac{\partial}{\partial t} \tag{2.3.55}$$

and

$$\frac{\partial}{\partial \xi} \quad = \quad (s^2 - 1)^{\frac{1}{2}} \frac{\partial}{\partial s}. \tag{2.3.56}$$

Therefore, (2.3.49) becomes

$$\begin{aligned} & \left[\frac{d}{ds}\left((s^2 - 1)\frac{df(s)}{ds}\right) - \frac{f(s)}{(s^2 - 1)}\right] g(t) \\ + & \left[\frac{d}{dt}\left[(1 - t^2)\frac{dg(t)}{dt}\right] - \frac{g(t)}{(1 - t^2)}\right] f(s) \\ = & \ 0. \end{aligned} \tag{2.3.57}$$

Dividing through equation (2.3.57), by $f(r)$ and $g(t)$, the resulting expression can only be satisfied if

$$\left[\frac{d}{ds}\left((s^2-1)\frac{df(s)}{ds}\right)-\frac{f(s)}{(s^2-1)}\right]-n(n+1)f(s)=0 \tag{2.3.58}$$

and

$$\left[\frac{d}{dt}\left[(1-t^2)\frac{dg(t)}{dt}\right]-\frac{g(t)}{(1-t^2)}\right]+n(n+1)g(t)=0, \tag{2.3.59}$$

with n an integer or zero. Then

$$\frac{d}{ds}\left[(s^2-1)\frac{df(s)}{ds}\right]-\frac{f(s)}{(s^2-1)}=-n(n+1)f(s) \tag{2.3.60}$$

and

$$\frac{d}{dt}\left[(1-t^2)\frac{dg(t)}{dt}\right]-\frac{g(t)}{(1-t^2)}=-n(n+1)g(t). \tag{2.3.61}$$

Using the results from the previous section, when $n=0$ the solution of equation (2.3.60) is

$$f(s)=\frac{C_1\,s+D_1}{(s^2-1)^{1/2}}, \tag{2.3.62}$$

where C_1, D_1 are constants. Similarly, when $n=0$ the solution of equation (2.3.61) is

$$g(t)=\frac{C_2\,t+D_2}{(1-t^2)^{1/2}}, \tag{2.3.63}$$

where C_2, D_2 are constants.

The solution of the equation (2.3.49), such that v is bounded when $t=-1$ and $v\to 0$ as $s\to\infty$ requires $C_1=0$ and $C_2=D_2=B_0$, say. Thus the general solution for the velocity in prolate spheroidal coordinates is of the form

$$v=B_0\left[\frac{1}{(s^2-1)^{1/2}}\right]\left[\frac{1+t}{1-t}\right]^{1/2}+\sum_{j=1}^{\infty}B_jQ_j^1(s)P_j^1(t). \tag{2.3.64}$$

In a similar way, it can be shown that the solution corresponding for v in oblate spheroidal coordinates is

$$v=B_0\left[\frac{1}{(s^2+1)^{1/2}}\right]\left[\frac{1+t}{1-t}\right]^{1/2}+\sum_{j=1}^{\infty}B_jq_j^1(s)P_j^1(t), \tag{2.3.65}$$

where

$$q_j^1(s)=Q_j^1(is). \tag{2.3.66}$$

2.4 Non-axisymmetric Stokes flow

2.4.1 The governing equations

The equations governing the motion of an incompressible fluid are the Navier-Stokes equations

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}.\nabla)\mathbf{q} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{q} \tag{2.4.1}$$

and the equation of continuity

$$\nabla.\mathbf{q} = 0 \tag{2.4.2}$$

When the Reynolds number for the flows is very small the terms on the left hand side of equation (2.4.1) are then negligible compared with the terms on the right hand side and the Navier-Stokes equations simplify to the Stokes equation

$$\nabla p = \mu\nabla^2\mathbf{q} \tag{2.4.3}$$

where $\mathbf{q}$ denotes the fluid velocity, p the fluid pressure and μ is the coefficient of dynamic viscosity of the fluid.

If the divergence of both sides of equation (2.4.3) is taken, then by virtue of equation (2.4.2), it follows that

$$\nabla^2 p = 0 \tag{2.4.4}$$

and similarly if the *curl* of both sides of equation (2.4.3) is taken, since $curl\,\nabla p \equiv 0$, it follows that

$$\nabla^2\omega = 0 \tag{2.4.5}$$

with $\omega = curl\,\mathbf{q}$, the vorticity vector. Equations (2.4.4) and (2.4.5) imply that for any Stokes flow the pressure and vorticity are harmonic functions. A consequence of equation (2.4.4) is that the Stokes equation (2.4.3) posses the *particular integral* given by

$$\mathbf{q} = \frac{1}{2\mu}\mathbf{r}p, \tag{2.4.6}$$

and the *general solution* of (2.4.3) is accordingly

$$\mathbf{q} = \frac{1}{\mu}\mathbf{r}p + \mathbf{v}, \tag{2.4.7}$$

where

$$\nabla^2\mathbf{v} = 0. \tag{2.4.8}$$

The form of solution (2.4.7) expresses the velocity in terms of four *scalar* harmonic functions, but to ensure satisfaction of the equation of continuity, the following equation

$$3p + (\mathbf{r}.\nabla)p + 2\mu\nabla.\mathbf{v} = 0 \tag{2.4.9}$$

must be satisfied at all points of the fluid. Equations (2.4.7), (2.4.8) and (2.4.9) therefore imply that for *any* Stokes flow, the velocity is expressible in terms of *no more* than *three* independent scalar harmonic functions of the space variables.

There are various representations of the *general solution* of the Stokes equations and it is usual to combine the three independent harmonic functions so that the equation of continuity is identically satisfied. For instance, Lamb's general solution [see Lamb (1932)] utilizes the fact that the pressure is a harmonic function by expanding it as a series of spherical harmonics in the form

$$p = \sum_{n=-\infty}^{\infty} p_n \tag{2.4.10}$$

where p_n is the spherical harmonic of order n. Lamb further shows that a general solution of (2.4.3) which also satisfies (2.4.2) can be written as

$$\begin{aligned} \mathbf{q} = & \sum_{n=-\infty}^{\infty} [\nabla \times (\mathbf{r}\chi_n) + \nabla(\Phi_n)] \\ & + \sum_{n=-\infty}^{\infty} \left[\frac{(n+3)}{2\mu(n+1)(2n+3)}\right] r^2 \nabla p_n \\ & - \sum_{n=-\infty}^{\infty} \left[\frac{n}{\mu(n+1)(2n+3)}\right] \mathbf{r} p_n \end{aligned} \tag{2.4.11}$$

where χ_n and Φ_n are both spherical harmonics of order n. It is clear that

$$\nabla^2 [\nabla \times (\mathbf{r}\chi_n)] = \nabla^2 [\nabla \Phi_n] = 0, \tag{2.4.12}$$

so that the velocity fields arising from the χ_n and Φ_n functions are each solutions of the Stokes equations for which the pressure is at most a constant, that is, they are *isobaric flows.*

An interesting result which can be derived from Lamb's general solution is set out in Happel and Brenner (1973), where a formula is derived for the force $\mathbf{F}$ acting on a general body in motion within a fluid which in turn has a velocity field $\mathbf{q}_\infty$ far away from the body. The expression for the force is

$$\mathbf{F} = -4\pi\nabla(r^3 p_{-2}) \tag{2.4.13}$$

This result shows that the force can be determined in principle once the pressure in the fluid is known. It is then only necessary to identify the component part of p corresponding to a spherical harmonic of order -2.

A similar expression is derived for the torque $\mathbf{T_0}$ acting on the body when moments of surface stresses are taken about the origin. This formula is given in Happel and Brenner (1973) as

$$\mathbf{T}_0 = -8\pi\mu\nabla(r^3\chi_{-2}) \tag{2.4.14}$$

In practice it is easier to determine the force and torque acting on a body by looking at the far field structure of the velocity and pressure fields, since it is usually extremely difficult to solve a problem using Lamb's general solution. If a body is in motion in a fluid, then the body imposes a force $-\mathbf{F}$ and torque $-\mathbf{T}$ on the fluid. These are equal and opposite to the force and torque exerted by the fluid on the body. Let S be the surface of the body. The force $\mathbf{F}$ and torque $\mathbf{T}$ are then given by

$$\mathbf{F} = \int_S \mathbf{R}_n dS \tag{2.4.15}$$

$$\mathbf{T} = \int_S [\mathbf{r} \times \mathbf{R}_n]\, dS \tag{2.4.16}$$

where $\mathbf{R}_n$ is the stress vector associated with the normal direction to S and in equation (2.4.16) moments of the surface stress vector are taken about the origin. The direction of the unit normal $\hat{\mathrm{n}}$ is that drawn out of the body. If V is the volume of the region bounded by S and any surface Σ enclosing S, it follows from the divergence theorem that

$$\int_\Sigma \mathbf{R}_n dS - \int_S \mathbf{R}_n dS = \int_V \frac{\partial}{\partial x_j}\mathbf{R}_j dV \tag{2.4.17}$$

$$\int_\Sigma [\mathbf{r} \times \mathbf{R}_n]\, dS - \int_S [\mathbf{r} \times \mathbf{R}_n]\, dS = \int_V \frac{\partial}{\partial x_j}[\mathbf{r} \times \mathbf{R}_j]\, dV \tag{2.4.18}$$

since $\mathbf{R}_n = l_j \mathbf{R}_j$ with $\mathbf{R}_j$ the stress vector when $\hat{\mathrm{n}}$ is coincident with $\hat{\mathrm{x}}_j$, the Cartesian unit vector and $\hat{\mathrm{n}} = l_j \hat{\mathrm{x}}_j$ and $\hat{\mathrm{n}}$ is again directed out of the surface Σ. However

$$\frac{\partial \mathbf{R}_j}{\partial x_j} = -\nabla p + \mu\nabla^2 \mathbf{q} = 0, \tag{2.4.19}$$

by virtue of equation (2.4.3). Furthermore since the stress tensor is symmetric

$$\hat{\mathbf{x}}_j \times \mathbf{R}_j = 0, \tag{2.4.20}$$

where the convention of summation over the suffices 1,2,3 is assumed. Thus equation (2.4.17) and equation (2.4.18) yield

$$\mathbf{F} = \int_\Sigma \mathbf{R}_n dS \tag{2.4.21}$$

$$\mathbf{T} = \int_\Sigma [\mathbf{r} \times \mathbf{R}_n]\, dS \tag{2.4.22}$$

The surface Σ may be taken to be a sphere, centre at the origin and radius R arbitrarily large. Thus the force and torque can be determined from the far field asymptotic structure of the velocity and pressure fields. In fact it is the Stokeslet and rotlet contributions to the asymptotic expansions of $\mathbf{q}$ and p for large $|\mathbf{r}|$ which give rise to the force and torque, since at a large distance the body will *appear* to the fluid as if it were a Stokeslet and rotlet located at the centre of mass of the body.

2.4.2 Velocity and pressure fields due to a stokeslet and rotlet

The three-dimensional Stokeslet represents physically an isolated concentration of force acting on the fluid at a point. Let this force be $\mathbf{F} = F\hat{\mathbf{F}}$. If a local system of cylindrical polar coordinates (ρ', ϕ', z'), is chosen with origin at the location of the point force, and the z' axis along the direction of $\mathbf{F}$, then the components of velocity are, according to Lamb (1932),

$$\begin{aligned} q_{\rho'} &= \frac{F}{8\pi\mu}\left[\frac{\rho' z'}{(r')^3}\right], \\ q_{\phi'} &= 0, \\ q_{z'} &= -\frac{F}{8\pi\mu}\left[\frac{(\rho')^2}{(r')^3} - \frac{z'}{r'}\right], \end{aligned} \tag{2.4.23}$$

and the pressure is

$$p = \frac{F}{4\pi}\left[\frac{z'}{(r')^3}\right]. \tag{2.4.24}$$

In particular, with a given Cartesian frame of reference and cylindrical polar coordinates (ρ, ϕ, z) related to (x, y, z) in the usual way, then if $\mathbf{F} = F\hat{\mathbf{k}}$, equation (2.4.23) gives

$$\begin{aligned} q_\rho &= \frac{F}{8\pi\mu}\left[\frac{\rho z}{r^3}\right], \\ q_\phi &= 0, \\ q_z &= -\frac{F}{8\pi\mu}\left[\frac{\rho^2}{r^3}-\frac{z}{r}\right] \end{aligned} \tag{2.4.25}$$

and equation (2.4.24) gives

$$p = \frac{F}{4\pi}\left[\frac{z}{r^3}\right]. \tag{2.4.26}$$

For a point force $\mathbf{F} = F\hat{\mathbf{i}}$, the corresponding velocity components and pressure are

$$\begin{aligned} q_\rho &= -\frac{F}{8\pi\mu}\left[\frac{\rho^2}{r^3}+\frac{1}{r}\right]\cos\phi, \\ q_\phi &= \frac{F}{8\pi\mu}\left[\frac{\sin\phi}{r}\right], \\ q_z &= -\frac{F}{8\pi\mu}\left[\frac{\rho z\cos\phi}{r^3}\right] \end{aligned} \tag{2.4.27}$$

and

$$p = \frac{F}{4\pi}\left[\frac{\rho\cos\phi}{r^3}\right]. \tag{2.4.28}$$

Likewise the three-dimensional rotlet represents physically a point concentration of couple applied to the fluid. If the origin is the point of application of a couple $\mathbf{G} = G\hat{\mathbf{G}}$ then the velocity distribution of the rotlet is

$$\mathbf{q} = \frac{G}{8\pi\mu}\frac{\left[\hat{\mathbf{G}}\times\mathbf{r}\right]}{r^3}. \tag{2.4.29}$$

Since equation (2.4.29) may be written as

$$\mathbf{q} = \frac{G}{8\pi\mu} curl\left[\frac{\hat{\mathbf{G}}}{r^3}\right], \tag{2.4.30}$$

it is evident that $\nabla^2\mathbf{q} = 0$, indicating that the pressure associated with a rotlet is at most a constant. If $\mathbf{G} = G\hat{\mathbf{k}}$, then

$$\begin{aligned} q_\rho &= 0, \\ q_\phi &= \frac{G}{8\pi\mu}\left[\frac{\rho}{r^3}\right], \\ q_z &= 0, \end{aligned} \tag{2.4.31}$$

and if $\mathbf{G} = G\hat{\mathbf{j}} = G(\hat{\rho}\sin\phi + \hat{\phi}\cos\phi)$, then

$$\begin{aligned} q_\rho &= \frac{G}{8\pi\mu}\left[\frac{z\cos\phi}{r^3}\right], \\ q_\phi &= -\frac{G}{8\pi\mu}\left[\frac{z\sin\phi}{r^3}\right], \\ q_z &= -\frac{G}{8\pi\mu}\left[\frac{\rho\cos\phi}{r^3}\right]. \end{aligned} \quad (2.4.32)$$

Using the integral relations (2.4.21) and (2.4.22), it may be verified that the Stokeslet and rotlet gives accordingly **F** for the force and **G** for the torque.

2.4.3 Solution of the Stokes equations

As pointed out above, the solution of any Stokes flow problem involves the determination of up to three independent harmonic functions. For the case of *axisymmetric* flow, the equations for determining the three functions uncouple the problem for translation along the axis of symmetry of the body from that for rotation about the axis of symmetry. The solution of the rotational problem involves only one harmonic function, since there is only one component of velocity – in the azimuthal direction – and the pressure is constant. For the translational problem, the two non-zero velocity components are expressible in terms of a stream function ψ. Thus, with $\mathbf{q} = q_\rho\hat{\rho} + q_\phi\hat{\phi} + q_z\hat{\mathbf{k}}$, it follows that

$$\begin{aligned} q_\rho &= \left[\frac{1}{\rho}\right]\frac{\partial\psi}{\partial z}, \\ q_\phi &= 0, \\ q_z &= -\left[\frac{1}{\rho}\right]\frac{\partial\psi}{\partial\rho}, \end{aligned} \quad (2.4.33)$$

to satisfy the equation of continuity identically. The vorticity ω is given by

$$\begin{aligned} \omega &= \left[\frac{\partial q_\rho}{\partial z} - \frac{\partial q_z}{\partial\rho}\right]\hat{\phi} \\ &= \frac{1}{\rho}\left[\frac{\partial^2\psi}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{\partial^2\psi}{\partial z^2}\right]\hat{\phi} \\ &= \left[\frac{L_{-1}\psi}{\rho}\right]\hat{\phi}, \end{aligned} \quad (2.4.34)$$

where the operator is defined by

$$L_m = \left[\frac{\partial^2}{\partial\rho^2} + \frac{m}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial z^2}\right] \quad (2.4.35)$$

with $m = -1$. It follows that

$$curl\,\omega = -\frac{\hat{\rho}}{\rho}\frac{\partial}{\partial z}(L_{-1}\psi) + \frac{\hat{k}}{\rho}\frac{\partial}{\partial \rho}(L_{-1}\psi) \tag{2.4.36}$$

and

$$curl^2\,\omega = -\left[\frac{L_{-1}^2\psi}{\rho}\right]\hat{\phi}. \tag{2.4.37}$$

Thus in satisfying of equation (2.4.5) and noting that $div\;curl\,\omega \equiv 0$, the equation satisfied by ψ is

$$L_{-1}^2\,\psi = \left[\frac{\partial^2}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2}\right]^2 \psi = 0. \tag{2.4.38}$$

The solution of $L_1 f = 0$ is an *axisymmetric* harmonic and this solution of $L_{-1}f = 0$ will be referred to as a *quasi-harmonic* function. Solutions of $L_m f = 0$ are often referred to as *generalized axisymmetric potential functions.* The stream function ψ may be thought of as a *quasi-biharmonic* function. A harmonic function can be easily constructed from a quasi-harmonic function and vice versa, since if $f(\rho, z)$ satisfies $L_{-1}f = 0$, then

$$\nabla^2\left[\frac{f(\rho, z)}{\rho}\cos\,\phi\right] = 0 \tag{2.4.39}$$

The stream function for axisymmetric flow can be constructed from two generalized axisymmetric potential functions in the form

$$\psi = z\,f^{-1} + g^{-1} \tag{2.4.40}$$

or

$$\psi = \rho^2\,f^1 + g^{-1} \tag{2.4.41}$$

where

$$\begin{aligned} L_1\,(f^1) &= L_{-1}\,(f^{-1}) \\ &= L_{-1}\,(g^{-1}) \\ &= 0. \end{aligned} \tag{2.4.42}$$

Therefore, the determination of the stream function effectively involves the determination of two harmonic functions. The representation of the stream function for axisymmetric flow has been discussed at some length by Payne and Pell (1960).

Chapter 3

THE SINGLE SPHERE PROBLEM

3.1 Introduction

The effect of a layer of an adsorbed monomolecular surfactant film of fluid covering the free surface of a semi-infinite volume of substrate fluid is considered for motion within both surfactant and substrate fluids caused by the slow rotation of a sphere body which is partially submerged in the substrate fluid. The end result of this study will be a theoretical model for determining the surface viscosity of the surfactant. The approach taken involves the use of a variational–least squares criterion for determining the fluid velocity if the motion is considered to be Stokes flow. The theoretical model relates the surface viscosity to the torque acting on the partially submerged sphere in the surfactant and substrate fluids. This work could be appropriate as the basis of a viscometer for measuring surface viscosity with a high degree of accuracy.

In this chapter a sphere, which is partially submerged in the substrate fluid below the surfactant layer, rotates slowly about a diameter perpendicular to the plane of the surfactant layer. It is felt that the choice of a spherical body is particularly advantageous, because this type of geometry ensures that a mathematical formulation of the boundary value problem can be established for all depths of the sphere below the surfactant layer. This has enabled the values of film and substrate torque acting on the partially submerged sphere to be determined for a wide range of values of the depth of the sphere and values of the surface viscosity parameter extending from zero to infinity. Also considered in detail are the limiting cases: (a) when the surface viscosity is zero and the surfactant layer becomes a simple stress free surface, and (b) when the shear viscosity is infinite.

3.2 Sphere rotating with a surfactant layer

Consider a partially submerged sphere of radius a slowly rotating in a semi-infinite incompressible fluid with dynamic viscosity μ . The axis of rotation is the diameter of the sphere perpendicular to the surface of the substrate fluid on which there is a film of an adsorbed monomolecular layer of surfactant fluid possessing surface viscosity η . The depth of the sphere centre C below the surfactant film is c, where c takes values in the range $-a < c < a$ and the sphere rotates with constant angular velocity Ω. Note that the surfactant film is unbounded apart from its intersection with the sphere.

3.2.1 Equations governing the motion

In order to preserve continuity with later analytical work, a system of spherical polar coordinates (r, θ, ϕ) with origin O lying in the plane of the interface, as illustrated in Figure 3.1, will be used. All distances are now regarded as dimensionless relative to the radius of the sphere. Now consider the problem when the centre C of the sphere is below the origin O as shown in Figure 3.1. On the submerged spherical cap

$$1 = r^2 + c^2 + 2rct, \tag{3.2.1}$$

where t = cos θ . Therefore,

$$r = -ct \pm [1 - c^2(1 - t^2)]^{(1/2)}. \tag{3.2.2}$$

The solution with the minus sign can be ignored, since $r > 0$. Hence,

$$r = r_s(t) = [1 - c^2(1 - t^2)]^{(1/2)} - ct, \tag{3.2.3}$$

with $-1 \leq t \leq 0$. Assuming that the Reynolds number, which is defined in equation

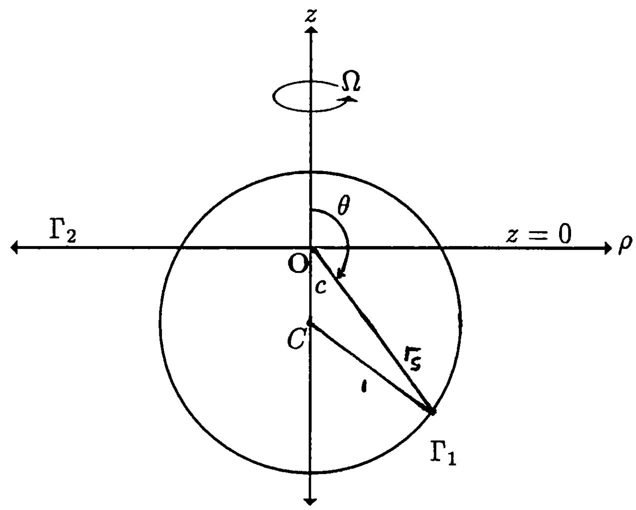

Figure 3.1: The geometry of the single sphere problem.

(2.3.1), for the flows induced in both the substrate fluid and surfactant film are sufficiently small to permit the neglect of the inertia terms in the Navier-Stokes equations, the flows in both the substrate fluid and surfactant film are governed by the Stokes equation (2.3.2) together with the equation of continuity (2.3.3).

The fluid motion is caused solely by the rotation of the sphere, and because of the axisymmetric nature of the problem, it follows that the velocity v has only one component, which is in the azimuthal direction of a system of spherical polar coordinates with $\theta = 0$ along the axis of rotation of the sphere and pointing out of the substrate fluid. The surfactant layer lies in the plane $z = 0$ or $\theta = \pi/2$. Thus (2.3.2) and (2.3.3) possess a solution of the form

$$\mathrm{v} = (0, 0, v(r, \theta)) \tag{3.2.4}$$

and

$$p = constant \tag{3.2.5}$$

provided that

$$\nabla^2 v - \frac{v}{(r^2 \sin^2 \theta)} = 0. \tag{3.2.6}$$

The solution of (3.2.6) which is sought, such that, $v \rightarrow 0$ as $r \rightarrow \infty$ and is bounded for $\pi/2 \leq \theta \leq \pi$. In (r,t) variables equation (3.2.6) can be written as

$$\nabla^2 v - \frac{v}{(r^2(1-t^2))} = 0.$$

3.2.2 Boundary conditions

For this problem there are two boundary conditions to be satisfied, one on the surface Γ_1 of the sphere and the other on of the surfactant film Γ_2, as indicated in Figure 3.1.

To satisfy the non-slip boundary condition on the sphere surface Γ_1 requires that

$$v = r_s(t)(1-t^2)^{1/2} \tag{3.2.7}$$

with $-1 \leq t \leq 0$ and $r_s(t)$ is defined in equation (3.2.3).

In the presence of the surfactant layer, following the analysis of Section *2.3.2*, the boundary condition to be satisfied is

$$\frac{\partial v}{\partial z} + \lambda \frac{\partial^2 v}{\partial z^2} = 0 \tag{3.2.8}$$

on Γ_2 , where

$$\lambda = \eta/\mu. \tag{3.2.9}$$

Here μ denotes the coefficient of the dynamic viscosity in the substrate fluid which occupies the region $z < 0$ and η denotes the surface viscosity.

3.3 Solution of the problem

The general form of solution for v which satisfies (3.2.6) and decays to zero as $r \to \infty$ can be written as

$$v = B_0 \left[\frac{1}{r}\right] \left[\frac{1+t}{1-t}\right] + \sum_{j=1}^{\infty} B_j \left[\frac{1}{r}\right]^{j+1} P_j^1(t) \tag{3.3.1}$$

following the analysis set out in Chapter *2*.

Here r is dimensionless relative to a and v is dimensionless relative to Ωa. In equation (3.3.1), $P_j^1(t)$ is the associated Legendre function of the first kind of order j and degree unity. For a partially submerged sphere with $-1 < c < 1$, the parameter t lies in the range $-1 \leq t \leq 0$, so that, in general, the Legendre functions $P_j^1(t)$ do not form a complete set over this range of values of t. In equation (3.3.1) the unknown coefficients B_j have to be determined, so as to satisfy the boundary conditions on Γ_1 and Γ_2.

The boundary residual ϵ_1 associated with the boundary condition given in equation (3.2.8) is defined as

$$\epsilon_1 = v_s - r_s(t)(1-t^2)^{1/2}, \tag{3.3.2}$$

with v_s the velocity on the body is given by equation (3.3.1), and $r = r_s(t)$ on the body. Thus

$$\epsilon_1 = B_0 \left[\frac{1}{r_s(t)}\right] \left[\frac{1+t}{1-t}\right] + \sum_{j=1}^{\infty} B_j \left[\frac{1}{r_s(t)}\right]^{j+1} P_j^1(t) - r_s(t)(1-t^2)^{1/2}. \tag{3.3.3}$$

Consider now the boundary condition (3.2.8), the derivatives on the right hand side can be expressed in terms of the spherical polar coordinates by

$$\frac{\partial v}{\partial z} = \left[\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right] v, \tag{3.3.4}$$

$$\frac{\partial^2 v}{\partial z^2} = \left[\cos^2\theta \frac{\partial^2}{\partial r^2} + \frac{\sin 2\theta}{r}\left(\frac{1}{r}\frac{\partial}{\partial\theta} - \frac{\partial^2}{\partial r \partial \theta}\right) + \frac{\sin^2\theta}{r}\left(\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial^2}{\partial \theta^2}\right)\right] v. \tag{3.3.5}$$

With the velocity given by equation (3.3.1) and $\theta = \pi/2$, or equivalently $t = 0$, it can be shown that equation (3.2.8) reduces to

$$B_0 \left[\frac{1}{r^2}\right] + \sum_{j=1}^{\infty} B_j \left[\frac{1}{r}\right]^{j+2} \left(\frac{d}{dt}P_j^1(t) - \frac{\lambda}{r}\left((j+1)P_j^1(t) - \frac{d^2}{dt^2}P_j^1(t)\right)\right)_{t=0} = 0, \tag{3.3.6}$$

with $r > r_s(0) = (1-c^2)^{1/2}$. The recurrence formulae relating the Legendre functions, given for instance by Morse and Feshbach (1956), enable the derivatives in (3.3.6) to be written as

$$\frac{dP_j^1(t)}{dt} = (1-t^2)^{-1}\left\{(j+1)tP_j^1(t) - jP_{j+1}^1(t)\right\} \tag{3.3.7}$$

and

$$\begin{aligned}
\frac{d^2P_j^1(t)}{dt^2} &= \frac{d}{dt}\left\{\frac{1}{(1-t^2)}\left((j+1)tP_j^1(t) - jP_{j+1}^1(t)\right)\right\} \\
&= \frac{2t}{(1-t^2)^2}\left[(j+1)tP_j^1(t) - jP_{j+1}^1(t)\right] \\
&+ \frac{1}{(1-t^2)}\left\{(j+1)P_j^1(t) + (j+1)t\frac{d}{dt}P_j^1(t) - j\frac{dP_{j+1}^1(t)}{dt}\right\}. \quad (3.3.8)
\end{aligned}$$

Equation (3.3.7) gives

$$\left(\frac{dP_j^1(t)}{dt}\right)_{t=0} = -jP_{j+1}^1(0), \tag{3.3.9}$$

and equation (3.3.8) gives

$$\left(\frac{d^2P_j^1(t)}{dt^2}\right)_{t=0} = (j+1)P_j^1(0) + j(j+1)P_{j+2}^1(0). \tag{3.3.10}$$

Therefore, the equation (3.3.6) becomes

$$B_0\left[\frac{1}{r^2}\right] + \sum_{j=1}^{\infty} B_j\left[\frac{1}{r}\right]^{j+2}\left(-jP_{j+1}^1(0) + \frac{\lambda}{r}j(j+1)P_{j+2}^1(0)\right)_{t=0} = 0. \tag{3.3.11}$$

Changing the dummy variable of the second term of equation (3.3.11), the following can be obtained

$$B_0\left[\frac{1}{r^2}\right] + \sum_{j=1}^{\infty}\left[\lambda j(j+1)B_j - (j+1)B_{j+1}\right]\frac{P_{j+2}^1(0)}{r^{j+3}} = 0. \tag{3.3.12}$$

If $j = 2m-1$, where $m = 1, 2, ...$, then since $P_{2m}^1(0) = 0$, it follows that (3.3.12) reduces to

$$\begin{aligned} & B_0\left[\frac{1}{r^2}\right] \\ + & \sum_{m=1}^{\infty}\left[\lambda(2m-1)(2m)B_{2m-1} - (2m)B_{2m}\right]P_{2m+1}^1(0)\left(\frac{1}{r}\right)^{2m+2} \\ = & \ 0. \end{aligned} \tag{3.3.13}$$

Since $P_{2m+1}^1(0) \neq 0$ and $r > r_s(0)$ is arbitrary, this implies that

$$B_{2m} = \lambda(2m-1)B_{2m-1}, \quad (m = 1, 2, ..). \tag{3.3.14}$$

If $\lambda \neq \infty$ then

$$\begin{aligned} B_0 &= 0, \\ B_{2m} &= \lambda(2m-1)B_{2m-1}, \quad (m = 1, 2, ..). \end{aligned} \tag{3.3.15}$$

The general solution for $v(r, t)$ which identically satisfies the surfactant condition (3.2.9) is therefore

$$v = \sum_{m=1}^{\infty} B_{2m-1}\left(\left[\frac{1}{[r]^{2m}}\right]P_{2m-1}^1(t) + \lambda\left[\frac{(2m-1)}{[r]^{2m+1}}\right]P_{2m}^1(t)\right) \tag{3.3.16}$$

or

$$v = \sum_{m=1}^{\infty} B_{2m} \left(\left[\frac{1}{[r]^{2m+1}} \right] P_{2m}^1(t) + \frac{1}{\lambda} \left[\frac{1}{(2m-1)[r]^{2m}} \right] P_{2m-1}^1(t) \right) \tag{3.3.17}$$

when $\lambda \neq \infty$, or

$$v = B_0 \left[\frac{1}{r} \right] \left[\frac{(1+t)}{(1-t)} \right]^{1/2} + \sum_{m=1}^{\infty} B_{2m} \left[\frac{1}{[r]^{2m+1}} \right] P_{2m}^1(t), \tag{3.3.18}$$

when $\lambda = \infty$, since $B_{2m-1} = 0$ for $m = 1, 2, ...$

3.4 Determination of the coefficients B_j

3.4.1 Case when $\lambda \neq \infty$

From (3.3.16), the value of v on the partial sphere is

$$v = \sum_{m=1}^{\infty} B_{2m-1} f_m(c, \lambda, t) \tag{3.4.1}$$

where

$$f_m(c, \lambda, t) = \left[\frac{1}{[r_s(t)]^{2m}} \right] P_{2m-1}^1(t) + \lambda \left[\frac{(2m-1)}{[r_s(t)]^{2m+1}} \right] P_{2m}^1(t). \tag{3.4.2}$$

Although the surfactant boundary condition is satisfied identically, there remains the boundary condition on the partial sphere to be satisfied. This requires

$$\sum_{m=1}^{\infty} B_{2m-1} f_m(c, \lambda, t) = r_s(t)(1-t^2)^{1/2} \tag{3.4.3}$$

for $-1 \leq t \leq 0$, where

$$r_s(t) = [1 - c^2(1-t^2)]^{1/2} - ct \tag{3.4.4}$$

with $-1 < c < 1$.

In the particular case when $\lambda = c = 0$, equation (3.4.1) reduces to

$$v = \sum_{m=1}^{\infty} B_{2m-1} P_{2m-1}^1(t). \tag{3.4.5}$$

To satisfy the boundary condition on sphere, requires in this case

$$\sum_{m=1}^{\infty} B_{2m-1} P_{2m-1}^1(t) = (1-t^2)^{1/2} \tag{3.4.6}$$

for $-1 \leq t \leq 0$, since $r_s(t) = 1$. To find the unknown coefficients B_{2m-1}, multiply equation (3.4.6) by $P^1_{2n-1}(t)$ and integrate with respect to t from -1 to 0 to give

$$\sum_{m=1}^{\infty} B_{2m-1} \int_{-1}^{0} P^1_{2m-1}(t) P^1_{2n-1}(t) dt = \int_{-1}^{0} (1-t^2)^{1/2} P^1_{2n-1}(t) dt$$
$$= \int_{-1}^{0} P^1_1(t) P^1_{2n-1}(t) dt \qquad (3.4.7)$$

since

$$P^1_1(t) = (1-t^2)^{1/2}. \qquad (3.4.8)$$

Now, from Section *2.2.2*,

$$\int_{-1}^{0} P^1_{2m-1}(t) P^1_{2n-1}(t) dt = \left[\frac{2n(2n-1)}{(4n-1)}\right] \delta_{m,n} \qquad (3.4.9)$$

and using results from Morse and Fershbach (1953), gives

$$\int_{-1}^{0} P^1_1(t) P^1_{2n-1}(t) dt = \frac{2}{3}\, \delta_{1,n}. \qquad (3.4.10)$$

Hence

$$B_{2n-1} \left[\frac{2n(2n-1)}{(4n-1)}\right] = \frac{2}{3} \delta_{1,n}. \qquad (3.4.11)$$

Therefore

$$B_1 = 1 \qquad (3.4.12)$$

and

$$B_3 = B_5 = .. = 0. \qquad (3.4.13)$$

Since $\lambda = 0$, the even coefficients are accordingly

$$B_2 = B_4 = .. = 0. \qquad (3.4.14)$$

For all other values of c or λ, $r_s(t)$ is no longer a constant, and the orthogonal property of $P^1_{2m-1}(t)$ over $-1 \leq t \leq 0$ cannot be invoked. For this general case, it is necessary to determine the unknown coefficients B_{2m-1} numerically. Consider the function I given by

$$I = \int_{-1}^{0} [v - r_s(t)(1-t)^{1/2}]^2 dt. \qquad (3.4.15)$$

We shall determine B_{2m-1}, so that I is minimized. A necessary set of conditions for minimizing I is

$$\frac{\partial I}{\partial B_{2n-1}} = 0; \; n = 1, 2, ..., \tag{3.4.16}$$

which leads to the infinite system of linear equations,

$$\sum_{m=1}^{\infty} B_{2m-1} S_{m,n} = T_n; \quad (n \geq 1). \tag{3.4.17}$$

The numerical method employed here to solve the boundary value problem for v is one of a general class of least-squares boundary residual methods which was reviewed by Finlayson (1972). The basic idea of this technique was originally applied by Rayleigh (1896) to solve a sound-diffraction problem. In the field of electrical engineering, the technique is known as the mode-matching method.

To solve the equations numerically a finite number J_{max} of equations is fixed and it is assumed that $B_{2m-1} \rightarrow 0$ as $m \rightarrow \infty$. Thus, setting $B_{2m-1} = 0$ for $m > J_{max}$, equation (3.4.17) becomes

$$\sum_{m=1}^{J_{max}} B_{2m-1} S_{m,n} = T_n; \quad 1 \leq n \leq J_{max}, \tag{3.4.18}$$

where

$$S_{m,n} = \int_{-1}^{0} f_m(c, \lambda, t) f_n(c, \lambda, t) dt, \tag{3.4.19}$$

and

$$T_n = \int_{-1}^{0} r_s(t)(1 - t^2)^{1/2} f_n(c, \lambda, t) dt. \tag{3.4.20}$$

Similarly, if we eliminate B_{2m-1} in favour of B_{2m} using equation (3.3.17) instead of equation(3.3.16), equations (3.4.18) to (3.4.20), can be replaced by

$$\sum_{m=1}^{J_{max}} B_{2m} S_{m,n} = T_n; \quad 1 \leq n \leq J_{max} \tag{3.4.21}$$

with

$$S_{m,n} = \int_{-1}^{0} f_m(c, \lambda, t) f_n(c, \lambda, t) dt \tag{3.4.22}$$

and

$$T_n = \int_{-1}^{0} r_s(t)(1 - t^2)^{1/2} f_n(c, \lambda, t) dt \tag{3.4.23}$$

where now, provided that $\lambda \neq 0$,

$$f_n(c, \lambda, t) = \sum_{m=1}^{\infty} B_{2m} \left(\left[\frac{1}{[r_s]^{2m+1}} \right] P_{2m}^1(t) + \frac{1}{\lambda} \left[\frac{1}{(2m-1)[r_s]^{2m}} \right] P_{2m-1}^1(t) \right). \tag{3.4.24}$$

3.4.2 Case when $\lambda = \infty$

For $\lambda = \infty$, to satisfy the surfactant boundary condition, the velocity field is given by

$$v = B_0 \left[\frac{1}{r}\right] \left[\frac{1+t}{1-t}\right]^{1/2} + \sum_{m=1}^{\infty} B_{2m} \left[\frac{1}{r}\right]^{2m+1} P^1_{2m}(t). \tag{3.4.25}$$

There remains the boundary condition on the partial sphere to be satisfied. This requires

$$\sum_{m=0}^{\infty} B_{2m} f_m(c, \infty, t) = r_s(t)(1-t^2)^{1/2} \tag{3.4.26}$$

where

$$f_0(c, \infty, t) = \left[\frac{1}{r_s(t)}\right] \left[\frac{1+t}{1-t}\right]^{1/2} \tag{3.4.27a}$$

and

$$f_m(c, \infty, t) = \left[\frac{1}{r_s(t)}\right]^{2m+1} P^1_{2m}(t); \quad m \geq 1 \tag{3.4.27b}$$

for $-1 \leq t \leq 0$ and $r_s(t)$ is defined by equation (3.4.4). In the case of a half-submerged partial sphere, $c = 0$, $r_s(t) = 1$ and equation (3.4.26) then gives

$$(1-t^2)^{1/2} \quad = \quad B_0 \left[\frac{1+t}{1-t}\right]^{(1/2)} + \sum_{m=1}^{\infty} B_{2m} P^1_{2m}(t). \tag{3.4.28}$$

Since $P^1_{2m}(0) = 0$ for $m \geq 1$ and $t = 0$, equation (3.4.28) gives, on setting $t = 0$,

$$B_0 = 1. \tag{3.4.29}$$

When $t \neq 0$ then equation (3.4.28) implies that

$$(1-t^2)^{1/2} \quad = \quad \left[\frac{1+t}{1-t}\right]^{1/2} + \sum_{m=1}^{\infty} B_{2m} P^1_{2m}(t). \tag{3.4.30}$$

Thus

$$(1-t^2)^{1/2} - \left[\frac{1+t}{1-t}\right]^{1/2} \quad = \quad \sum_{m=1}^{\infty} B_{2m} P^1_{2m}(t), \tag{3.4.31}$$

or

$$(1-t^2)^{1/2} - \left[\frac{1+t}{(1-t^2)^{1/2}}\right] \quad = \quad \sum_{m=1}^{\infty} B_{2m} P^1_{2m}(t). \tag{3.4.32}$$

Hence

$$-\left[\frac{(t+t^2)}{(1-t^2)^{1/2}}\right] = \sum_{m=1}^{\infty} B_{2m} P^1_{2m}(t). \tag{3.4.33}$$

Therefore

$$\beta_m \quad = - \quad \alpha_m \left[B_{2m} \right] \tag{3.4.34}$$

in which, using results from Section*2.2.2*, resulting from the orthogonality of $P_{2m}^1(t)$ over $-1 \leq t \leq 0$,

$$\begin{aligned} \alpha_m &= \int_{-1}^{0} P_{2m}^1(t) P_{2m}^1(t) dt \\ &= \left[\frac{2m(2m-1)}{(4m-1)} \right] \end{aligned} \tag{3.4.35}$$

$$\begin{aligned} \beta_m &= \int_{-1}^{0} (t+t^2) P_{2m}'(t) dt \\ &= (-1)^{m+1} \left[\frac{(2m-2)!}{(2^{2m-1})(m+1)!(m-1)!} \right]. \end{aligned} \tag{3.4.36}$$

Hence

$$B_{2m} = -\frac{\beta_m}{\alpha_m} \quad = \quad (-1)^m \left[\frac{(2m-2)!\,(4m-1)}{(2^{2m})\,(m+1)!\,m!\,(2m-1)} \right] \tag{3.4.37}$$

for $m \geq 1$.

For $c \neq 0$ and $t = 0$, equation (3.4.26) reduces to

$$\sum_{m=0}^{\infty} B_{2m} f_m(c, \infty, 0) = (1-c^2)^{1/2}. \tag{3.4.38}$$

Thus, one can obtain

$$B_0 = (1-c^2), \tag{3.4.39}$$

since $f_0(c, \infty, 0) = (1-c^2)^{-1/2}$ and $f_m(c, \infty, 0) = 0 \ (m \geq 1)$. Unlike the case when $\lambda = 0$ the other coefficients B_{2m} $(m \geq 1)$, cannot now be expressed in a closed form. To determine the unknown coefficients B_{2m} numerically, consider the function I, given by

$$I = \int_{-1}^{0} \left[v_0 + v_1 - r_s(t)(1-t^2)^{1/2} \right]^2 dt, \tag{3.4.40}$$

where

$$v_0 \quad = \quad \left[\frac{(1-c^2)}{r_s(t)} \right] \left[\frac{1+t}{1-t} \right]^{1/2}, \tag{3.4.41}$$

$$v_1 \quad = \quad \sum_{m=1}^{\infty} B_{2m} f_m(c, \infty, t) \tag{3.4.42}$$

and $f_m(c,\infty,t)$ is defined by equations (3.4.27a) and (3.4.27b). We determine B_{2m} so that I is minimized. Accordingly

$$\frac{\partial I}{\partial B_{2m}} = 0 \ , \ j = 0, 1, .. \tag{3.4.43}$$

Again we solve a finite number $J_{max} + 1$ of equations, which are

$$\sum_{m=0}^{Jmax} B_{2m} S_{m,n} = T_n; \quad 0 \leq n \leq J_{max} \tag{3.4.44}$$

where

$$S_{m,n} = \int_{-1}^{0} f_m(c,\infty,t) f_n(c,\infty,t) dt \tag{3.4.45}$$

and

$$T_n = \int_{-1}^{0} \left[v_0 - r_s(t)(1-t^2)^{1/2} \right] f_n(c,\infty,t) dt. \tag{3.4.46}$$

The linear algebraic system, which is described by the sets of equations [(3.4.18) to (3.4.20)], [(3.4.21) to (3.4.23)] and [(3.4.44) to (3.4.46)], can be solved to give the coefficients B_j for $j = 1, 2, ..$ after making the following choices:

1. the depth c of the centre of the sphere body, below the surfactant layer,

2. the number J_{max} in the expression (3.4.18) or (3.4.21) or (3.4.44). This number is determined so that B_m is effectively zero for $m > J_{max}$.

3.4.3 Convergence analysis for the case when $\lambda \neq \infty$

For the convergence of the numerical method, consider the error factor

$$E = \sqrt{I} \tag{3.4.47}$$

where I is defined in equation (3.4.15). Thus, using the representation (3.4.1) for v,

$$\begin{aligned} I &= \int_{-1}^{0} \left[\sum_{m=1}^{\infty} B_{2m-1} f_m(c,\lambda,t) - r_s(t)(1-t^2)^{1/2} \right]^2 dt \\ &= \int_{-1}^{0} \left[\sum_{m=1}^{\infty} B_{2m-1} f_m(c,\lambda,t) \right]^2 dt + \int_{-1}^{0} r_s^2(t)(1-t^2) dt \\ &- 2\int_{-1}^{0} r_s(t)(1-t^2)^{1/2} \left[\sum_{m=1}^{\infty} B_{2m-1} f_m(c,\lambda,t) \right] dt \\ &= \sum_{m=1}^{\infty} B_{2m-1} \sum_{n=1}^{\infty} B_{2n-1} S_{m,n} - 2\sum_{n=1}^{\infty} B_{2n-1} T_n + \int_{-1}^{0} r_s^2(t)(1-t^2) dt, \end{aligned} \tag{3.4.48}$$

where $S_{m,n}$ and T_n are defined in (3.4.19) and (3.4.20) respectively. The value of J_{max} is chosen large enough to ensure that $B_3, B_5, \ldots$ converges to zero. The order of convergence of the numerical method is J_{max} if $B_{2J_{max}}$ is the first non-vanishing constant to a prescribed degree of accuracy. On having solved for the coefficients $B_1, B_3, \ldots$ the value of E represents a measure of how accurately the boundary condition on the partially submerged sphere is satisfied. A similar calculation for E can also be carried out, using the other representation for v given by (3.3.17).

3.4.4 Convergence analysis for the case when $\lambda = \infty$

Similarly, for the convergence of the numerical method, we consider

$$E = \sqrt{I}, \tag{3.4.49}$$

where I is defined by equation (3.4.40). The convergence of the numerical method is achieved by first defining the velocity field, which satisfies the surfactant condition, as

$$v = \sum_{m=0}^{\infty} B_{2m} f_m(c, \infty, t) \tag{3.4.50}$$

where $f_m(c, \infty, t)$ is defined by equations (3.4.27a) and (3.4.27b). Therefore, the boundary condition on $r = r_s(t)$ is satisfied if

$$v_0 + \sum_{m=1}^{\infty} B_{2m} f_m(c, \infty, t) \quad = \quad r_s(t)(1 - c^2)^{1/2} \tag{3.4.51}$$

where $-1 \leq t \leq 0$ and v_0 is defined by (3.4.41). The function I is then given by

$$\begin{aligned} I &= \int_{-1}^{0} \left[v_0 + v_1 - r_s(t)(1 - c^2)^{1/2} \right]^2 dt \\ &= \sum_{m=1}^{\infty} B_{2m} \sum_{n=1}^{\infty} B_{2n} S_{m,n} - 2\sum_{n=1}^{\infty} B_{2n} T_n + \int_{-1}^{0} \left[r_s(t)(1 - c^2)^{1/2} - v_0 \right]^2 dt, \end{aligned} \tag{3.4.52}$$

where $S_{m,n}$ and T_n are defined in (3.4.45) and (3.4.46) respectively and v_0, v_1 are defined by (3.4.41) and (3.4.42), respectively. Again E represents a measure of how accurately the boundary condition on the partially submerged sphere is satisfied.

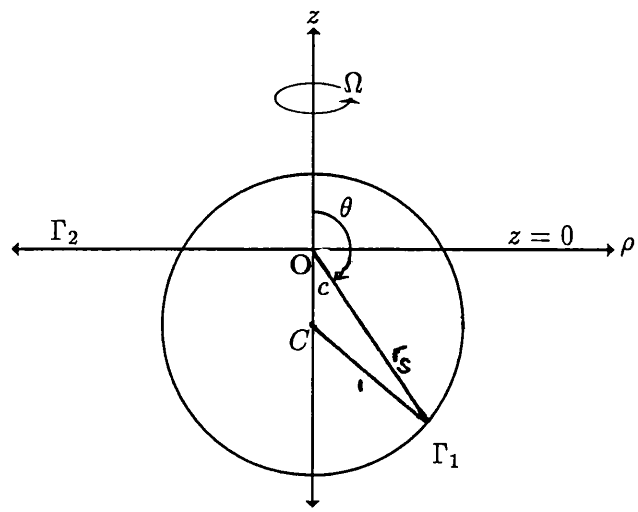

Figure 3.2: The geometry of a partially submerged sphere.

3.5 Expression for the torque acting on a general axi-symmetrical body

There are two contributions to the total torque which acts on the body, the substrate torque T_s and the film torque T_f. The substrate torque arises from the action of the stresses in the substrate fluid and the film torque arises from the action of the stresses in surfactant. The sum of these two torques gives the total torque T_t acting on the body, which is the quantity that would be measured in an experiment.

3.5.1 The substrate torque

A body which has the equation

$$r = r_s(t) \tag{3.5.1}$$

where $-1 \leq t \leq 0$ with $t = \cos\theta$ is considered. Letting $\hat{\mathbf{n}}$ be the general outward drawn unit normal to the surface, the substrate torque $\mathbf{T}_s$ arising from the action of the stresses in the substrate fluid will be

$$\mathbf{T}_s = T_s\hat{\mathbf{k}}, \tag{3.5.2}$$

where

$$-T_s = \hat{\mathbf{k}}.\int_S [\mathbf{r} \times \mathbf{R}_n] dS. \qquad (3.5.3)$$

In equation (3.5.3)

$$\mathbf{R}_n = (\hat{\mathbf{n}}.\hat{\mathbf{r}})\mathbf{R}_r + (\hat{\mathbf{n}}.\hat{\theta})\mathbf{R}_\theta + (\hat{\mathbf{n}}.\hat{\phi})\mathbf{R}_\phi, \qquad (3.5.4)$$

and $\mathbf{r}$ is the position vector of a general point of the surface S of the body, $d\mathbf{S}$ is the areal element of surface orientated in the direction of $\hat{\mathbf{n}}$. In order to simplify equation (3.5.4), we first write

$$\mathbf{R}_r = p_{rr}\hat{\mathbf{r}} + p_{r\theta}\hat{\theta} + p_{r\phi}\hat{\phi}, \qquad (3.5.5)$$

in which p_{ij} is the stress tensor, which for a Newtonian fluid is given by

$$p_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \qquad (3.5.6)$$

Thus

$$[\mathbf{r} \times \mathbf{R}_r] = rp_{r\theta}\hat{\phi} - rp_{r\phi}\hat{\theta}, \qquad (3.5.7)$$

and

$$\hat{\mathbf{k}} = \hat{\mathbf{r}}\cos\theta - \hat{\theta}\sin\theta, \qquad (3.5.8)$$

giving

$$\hat{\mathbf{k}}.[\mathbf{r} \times \mathbf{R}_r] = rp_{r\phi}\sin\theta. \qquad (3.5.9)$$

Similarly,

$$\hat{\mathbf{k}}.[\mathbf{r} \times \mathbf{R}_\theta] = rp_{\theta\phi}\sin\theta \qquad (3.5.10)$$

and

$$\hat{\mathbf{k}}.[\mathbf{r} \times \mathbf{R}_\phi] = 0. \qquad (3.5.11)$$

Now, writing

$$l = (\hat{\mathbf{n}}.\hat{\mathbf{r}}), \quad m = (\hat{\mathbf{n}}.\hat{\theta}), \qquad (3.5.12)$$

equation (3.5.3) becomes

$$T_s = \int_S \frac{r}{l}\left[lp_{r\phi} + mp_{\theta\phi}\right]\sin\theta dS. \qquad (3.5.13)$$

Before proceeding any further, note that the stress components in equation (3.5.13) are

$$p_{r\phi} = \mu\left[\frac{\partial v}{\partial r} - \frac{v}{r}\right] = \mu r\frac{\partial}{\partial r}\left(\frac{v}{r}\right), \qquad (3.5.14)$$

and

$$\begin{aligned} p_{\theta\phi} &= -\mu\left[\frac{v}{r}\cot\theta - \frac{1}{r}\frac{\partial v}{\partial\theta}\right] \\ &= -\mu\left(\frac{v}{r}\frac{t}{(1-t^2)^{1/2}} + \left[\frac{(1-t^2)^{1/2}}{r}\right]\frac{\partial v}{\partial t}\right). \end{aligned} \tag{3.5.15}$$

where μ is the coefficient of viscosity, and using equations (3.5.14) and (3.5.15), it follows that

$$(1-t^2)^{1/2}p_{r\phi} = \mu r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)(1-t^2)^{1/2}, \tag{3.5.16}$$

and

$$(1-t^2)^{1/2}p_{\theta\phi} = \frac{-\mu}{r}\left[tv + (1-t^2)\frac{\partial v}{\partial t}\right]. \tag{3.5.17}$$

To simplify l and m, which are defined in (3.5.12), the equation of the body surface needs to be considered. Suppose the equation of the body surface is $r - r_s(t) = 0$, it then follows that

$$\nabla(r - r_s(t)) = \hat{r} + \frac{\hat{\theta}}{r_s(t)}\frac{dr_s(t)}{dt}. \tag{3.5.18}$$

From equation (3.5.18), the normal vector to the surface can be written as

$$\hat{\mathrm{n}} = \frac{1}{\Upsilon}\left[\hat{r} + \hat{\theta}(1-t^2)^{1/2}\frac{1}{r_s(t)}\frac{dr_s(t)}{dt}\right], \tag{3.5.19}$$

where

$$\Upsilon = \left[1 + \frac{(1-t^2)}{r_s^2(t)}\left(\frac{dr_s(t)}{dt}\right)^2\right]^{1/2}. \tag{3.5.20}$$

Hence

$$l = \hat{\mathrm{n}}.\hat{\mathrm{r}} = \Upsilon^{-1} \tag{3.5.21}$$

and

$$m = \hat{\mathrm{n}}.\hat{\theta} = \Upsilon^{-1}(1-t^2)^{1/2}\frac{1}{r_s(t)}\frac{dr_s(t)}{dt}. \tag{3.5.22}$$

Therefore, equation (3.5.13) can also be written in the form

$$T_s = 2\pi\mu\Omega a^3\int_{-1}^{0}[r_s(t)]^3F(t)\,dt \tag{3.5.23}$$

with $r = r_s(t)$ dimensionless relative to some body dimension a. Hence, it can be shown that

$$T_s = -8\pi\mu\Omega a^3\ \tau_s \tag{3.5.24}$$

with

$$\tau_s = -\frac{1}{4}\int_{-1}^{0} [r_s(t)]^3 F(t)dt, \tag{3.5.25}$$

where

$$\begin{aligned} F(t) &= (1-t^2)^{1/2}\left(r_s(t)\frac{\partial}{\partial r_s(t)}(\frac{v}{r_s(t)}) - \frac{1}{[r_s(t)]^2}\frac{dr_s(t)}{dt}\left[tv + (1-t^2)\frac{\partial v}{\partial t}\right]\right), \\ &= (1-t^2)^{1/2}\left(\left[\frac{\partial v}{\partial r_s(t)} - \frac{v}{r_s(t)}\right] - \frac{1}{[r_s(t)]^2}\frac{dr_s(t)}{dt}\left[tv + (1-t^2)\frac{\partial v}{\partial t}\right]\right). \end{aligned} \tag{3.5.26}$$

In the above Ω is the constant angular velocity of the rotating body, and a a typical length scale for the body.

In order to be able to apply equation (3.5.25) to a partially submerged sphere, we need the equation of the body surface $r = r_s(t)$, with

$$r_s(t) = [1 - c^2(1-t^2)]^{1/2} - ct,$$

where c is the depth of the submerged centre of the sphere. Thus,

$$\begin{aligned} \frac{dr_s(t)}{dt} &= \frac{c^2 t}{[1-c^2(1-t^2)]^{1/2}} - c, \\ &= -\frac{cr_s(t)}{[r_s(t)+ct]}. \end{aligned} \tag{3.5.27}$$

Noting that the general solution for $v = v(r,\theta)$, has the form

$$v = B_0\left[\frac{1}{r_s(t)}\right]\left[\frac{1+t}{1-t}\right]^{1/2} + \sum_{j=1}^{\infty} B_j\left[\frac{1}{r_s(t)}\right]^{j+1} P_j^1(t).$$

Hence,

$$\begin{aligned} \left[\frac{\partial v}{\partial r_s(t)} - \frac{v}{r_s(t)}\right] &= -2B_0\left[\frac{1}{r_s(t)}\right]\left[\frac{1+t}{1-t}\right]^{1/2} \\ &- \sum_{j=1}^{\infty}(j+2)B_j\left[\frac{1}{r_s(t)}\right]^{j+2} P_j^1(t) \end{aligned} \tag{3.5.28}$$

and

$$\begin{aligned} &- \frac{1}{r_s^2(t)}\left[\frac{dr_s(t)}{dt}\right]\left[tv + (1-t^2)\frac{\partial v}{\partial t}\right] = \\ &- \frac{c}{r_s(t)+ct}(1-t)B_0\left[\frac{1}{r_s(t)}\right]^2\left[\frac{1+t}{1-t}\right]^{1/2} \\ &+ \frac{c}{r_s(t)+ct}\sum_{j=1}^{\infty} B_j\left[\frac{1}{r_s(t)}\right]^{j+2}\left[(j+2)tP_j^1(t) - jP_{j+1}^1(t)\right]. \end{aligned} \tag{3.5.29}$$

It therefore follows that

$$\left(\left[\frac{\partial v}{\partial r_s}-\frac{v}{r_s(t)}\right]-\frac{1}{[r_s(t)]^2}\frac{dr_s(t)}{dt}\left[tv+(1-t^2)\frac{\partial v}{\partial t}\right]\right)=$$
$$-\quad\frac{cB_0}{r_s(t)+ct}(2r_s(t)+c+ct)\left[\frac{1}{r}\right]\left[\frac{(1+t)}{(1-t)}\right]^{1/2}$$
$$+\quad\sum_{j=1}^{\infty}\frac{B_j}{(r_s(t)+ct)}\left\{(j+2)\left[\frac{1}{r_s(t)}\right]^{j+1}P_j^1(t)+jc\left[\frac{1}{r(t)}\right]^{j+2}P_{j+1}^1(t)\right\}. \tag{3.5.30}$$

Therefore, the expression for torque coefficient τ_s may be written as

$$\tau_s=\tau_s^{(0)}+\tau_s^{(1)} \tag{3.5.31}$$

where

$$\tau_s^{(0)} = \frac{1}{4}\int_{-1}^{0}\left[\frac{[r_s(t)]^2(1+t)}{(r_s(t)+ct)}\right]\left[\frac{B_0}{r_s(t)}\right](r_s(t)+c)\,dt + \frac{1}{4}\int_{-1}^{0}B_0\,r_s(t)\,(1+t)\,dt \tag{3.5.32}$$

and

$$\tau_s^{(1)}=\frac{1}{4}\sum_{j=1}^{\infty}B_j\left[(j+2)L_j+jcL_{j+1}\right] \tag{3.5.33}$$

with

$$L_j=\int_{-1}^{0}\frac{(1-t^2)^{1/2}\,P_j^1(t)}{[r_s(t)]^{j-2}(r_s(t)+ct)}\,dt. \tag{3.5.34}$$

3.5.2 The film torque

The film torque T_f is applied by the action of surfactant stresses along the ring of intersection with the body with the surfactant layer. The film torque T_f of a general axisymmetric body can be written as

$$T_f=-2\pi\Omega\eta\left[r^3\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right]_{r=r_s(0)} \tag{3.5.35}$$

with the quantity inside the square bracket evaluated on the ring of intersection $r=r_s(0)$, if the body with the surfactant layer . It is convenient to define a dimensionless film torque coefficient $\tau_f=T_f/8\pi\mu\Omega\eta$. Thus

$$\tau_f=-\frac{1}{4}\lambda\left[r^3\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right]_{r=r_s(0)} \tag{3.5.36}$$

where λ is defined in terms of μ and η by equation (3.2.10).

It should be noted that the general solution for v with $r = r_s(t)$ is

$$v = B_0 \left[\frac{1}{r_s(t)}\right] \left[\frac{1+t}{1-t}\right]^{1/2} + \sum_{j=1}^{\infty} B_j \left[\frac{1}{r_s(t)}\right]^{j+1} P_j^1(t),$$

where $t = \cos\theta$. Thus,

$$\left(\frac{v}{r_s(t)}\right) = -B_0 \left[\frac{1}{r_s(t)}\right]^2 \left[\frac{1+t}{1-t}\right]^{1/2} - \sum_{j=1}^{\infty} B_j \left[\frac{1}{r_s(t)}\right]^{j+2} P_j^1(t). \qquad (3.5.37)$$

from which

$$\frac{\partial}{\partial r_s}\left(\frac{v}{r_s(t)}\right) = -2B_0 \left[\frac{1}{r_s(t)}\right]^3 \left[\frac{1+t}{1-t}\right]^{1/2} - \sum_{j=1}^{\infty} (j+2) B_j \left[\frac{1}{r_s(t)}\right]^{j+3} P_j^1(t) \quad (3.5.38)$$

and

$$r_s(t)^3 \frac{\partial}{\partial r_s}\left(\frac{v}{r_s(t)}\right) = -2B_0 \left[\frac{1+t}{1-t}\right]^{1/2} - \sum_{j=1}^{\infty} (j+2) B_j \left[\frac{1}{r}\right]^{j} P_j^1(t). \qquad (3.5.39)$$

Hence

$$\tau_f = \frac{\lambda}{4}\left(2B_0 + \sum_{j=1}^{\infty} (j+2) B_j \left[\frac{1}{r_s(0)}\right]^{j} P_j^1(0)\right). \qquad (3.5.40)$$

Now $P_j^1(0) = 0$ if j is even so the film torque is then, for a partial sphere of radius a, is given by

$$\tau_f = \frac{\lambda}{4}\left(2B_0 + \sum_{m=1}^{\infty} (2m+1) B_{2m-1} \left[(1-c^2)^{-m+1/2}\right]^{2m-1} P_{2m-1}^1(0)\right), \qquad (3.5.41)$$

since $r_s(0) = (1-c^2)^{1/2}$.

In the case when $\lambda = \infty$, the boundary condition given in (3.2.20) reduces to

$$\frac{\partial^2 v}{\partial z^2} = 0, \quad (z = 0) \qquad (3.5.42)$$

which, together with (3.2.8), implies that

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rv)\right] = 0, \quad (z = 0) \qquad (3.5.43)$$

and therefore, the velocity distribution on the free surface is given by

$$v = \frac{(1-c^2)}{r} \qquad (3.5.44)$$

to satisfy both the boundary condition on the sphere and decay to zero at $r = \infty$. Thus for the case in which $\lambda = \infty$,

$$\frac{\tau_f}{\lambda} = \frac{1}{2}B_0 = \frac{1}{2}(1-c^2). \qquad (3.5.45)$$

When c = 0, which corresponds to a half-submerged sphere, the result (3.5.45) agrees with Davis's (1979) asymptotic result when $\lambda = \infty$.

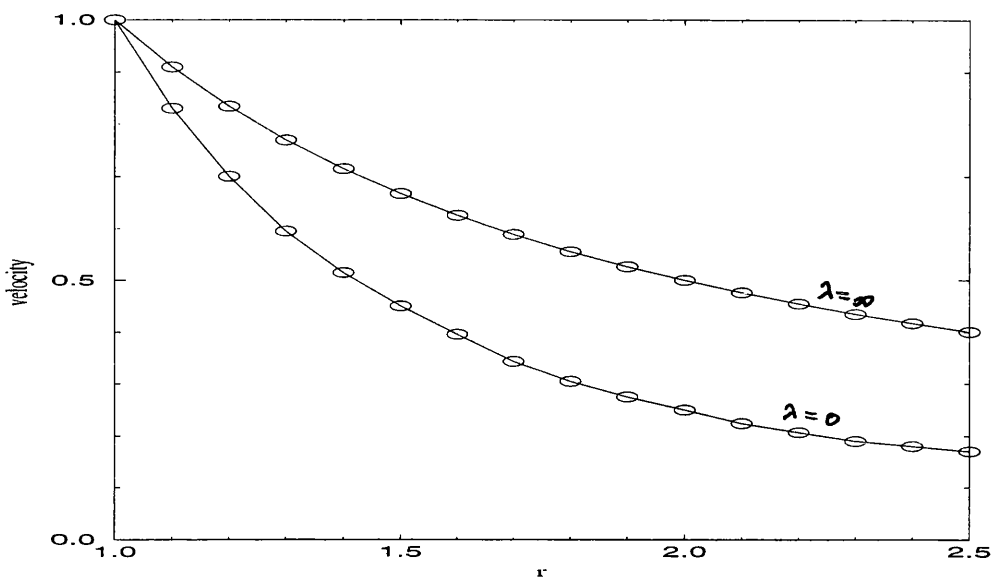

Figure 3.3: The surface velocity distribution when $c = 0$. – exact values and O numerical values.

3.6 Numerical results

3.6.1 The surface velocity distribution

The surface velocity distribution when $c = 0$ is considered for the two cases $\lambda = 0$ and $\lambda = \infty$. The corresponding exact solutions for the velocity profile at $z = 0$, are given by

$$v = \frac{(1-c^2)^{3/2}}{r^2}. \tag{3.6.1}$$

when $\lambda = 0$ and

$$v = \frac{(1-c^2)}{r} \tag{3.6.2}$$

when $\lambda = \infty$. These results are shown with the approximate solution in Figure 3.3. and clearly demonstrate that the approximate solution agrees very well with the exact solution both when $\lambda = 0$ and $\lambda = \infty$. The velocity distribution on the surface for all other values of λ will lie between the two solid curves in Figure 3.3.

c	Exact τ_s	Numerical τ_s
1.00	0.90154	——
0.90	0.87872	0.87872
0.80	0.85185	0.85185
0.70	0.82089	0.82089
0.60	0.78584	0.78584
0.50	0.74683	0.74683
0.40	0.70399	0.70399
0.30	0.65756	0.65756
0.20	0.60783	0.60783
0.10	0.55517	0.55516
0.00	0.50000	0.50000

Table 3.1: Numerical data of τ_s at $\lambda = 0$.

3.6.2 The substrate and film torques

The case when $\lambda = 0$ and the surfactant is effectively absent provides a situation when the exact solution for τ_s is available. The exact solution to the problem was obtained by Schneider et al. (1973). Comparison of the values of τ_s obtained by the exact solution (note that $\tau_f = 0$) provides an opportunity to examine the performance and accuracy of the general numerical method. Practically, this means that the computational parameter J_{max} can be determined to produce acceptable accuracy for the torque acting on the partially submerged sphere for the case of $\lambda = 0$. The actual value for J_{max} was 10, which was found to be satisfactory in this case. It was also found convenient to subdivide Γ_1 into equal subintervals for numerical integration. The computations were effected using, the number of subdivisions $n_{\Gamma_1} = 25$ in the range $0 \leq t \leq -1$ on Γ_1.

The Table 3.1 shows the result for the dimensionless substrate torque τ_s when $\lambda = 0$. The comparative values of τ_s when $\lambda = 0$ given by the exact solution of Schneider et al. (1973) is also included in the Table 3.1.

Figure 3.4 shows the computed values of τ_s plotted against c when $\lambda = 0$ and

c	Exact τ_f/λ	Numerical τ_f/λ
1.00	0.00000	—
0.90	0.09500	0.10000
0.80	0.18000	0.18000
0.70	0.26000	0.26000
0.60	0.32000	0.32000
0.50	0.37000	0.37000
0.40	0.42000	0.42000
0.30	0.46000	0.46000
0.20	0.48000	0.48000
0.10	0.49500	0.49500
0.00	0.50000	0.50000

Table 3.2: Numerical data of τ_f/λ at $\lambda = \infty$.

λ	Numerical E
0.00000	1.1×10^{-8}
0.10000	8.7×10^{-8}
0.20000	8.7×10^{-8}
0.30000	8.5×10^{-8}
0.40000	8.3×10^{-8}
0.50000	8.3×10^{-8}
0.60000	8.2×10^{-8}
0.70000	8.1×10^{-8}
0.80000	8.1×10^{-8}
0.90000	8.1×10^{-8}
1.00000	8.0×10^{-8}

Table 3.3: Numerical data for $Error - factor E$ when $c = 0$.

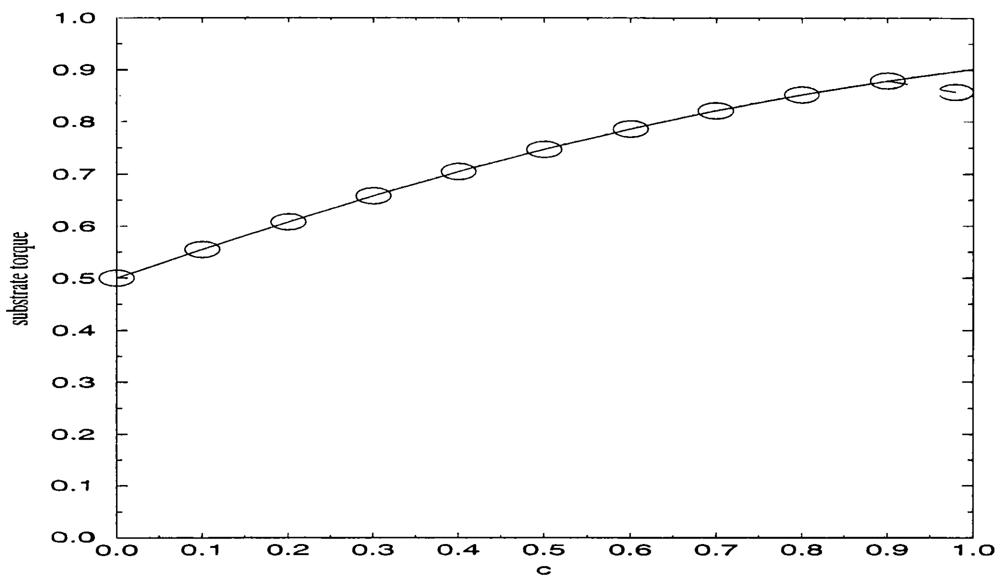

Figure 3.4: The numerical and exact values of substrate torque when $\lambda = 0$. $--$ exact values and O numerical values

compared with the exact value of τ_s by the exact solution of Schneider et al. (1973). The numerical calculations were carried out with $J_{max} = 10$. As demonstrated in Table 3.1 and Figure 3.4, the numerical method gives a high degree of accuracy when compared with the exact solution over all values of c considered (apart from $c = 1.0$). It should be noted that $c = 1.0$ corresponds to the sphere being just fully submerged in the substrate fluid and the origin becomes part of the fluid. Hence the representation for v breaks down.

The Table 3.2 shows the result for the dimensionless film torque τ_f/λ when $\lambda = \infty$.

Figure 3.5 is a graph of the numerical value of τ_s when $\lambda = \infty$, plotted against c. For this case there is a check available against the exact solution $\tau_s = 1.2021$, which was obtained by Davis and O'Neill (1979).

Figure 3.6 shows the computed and exact values of τ_f/λ, which are plotted against c when $\lambda = \infty$. As may be seen from the graph, the computed values of τ_f/λ agree very closely with the exact values determined from (3.5.45) over the entire range of c apart from $c = 1.0$.

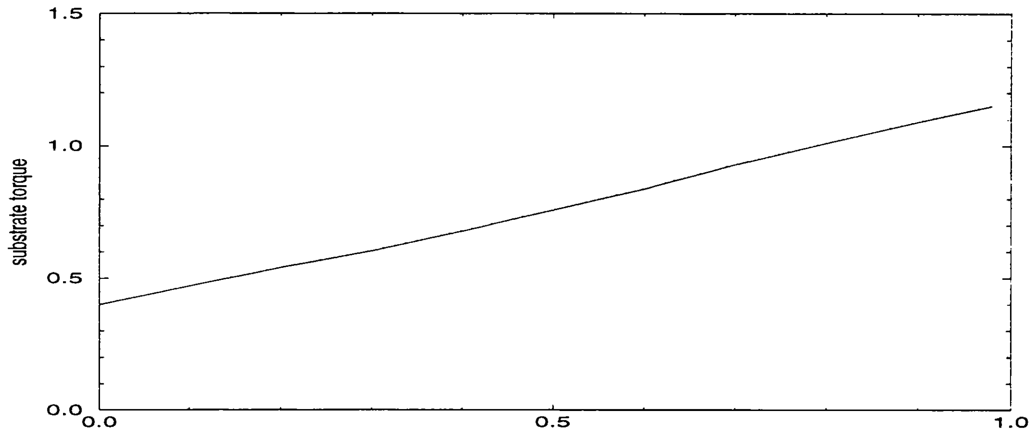

Figure 3.5: The numerical value of substrate torque τ_s when $\lambda = \infty$.

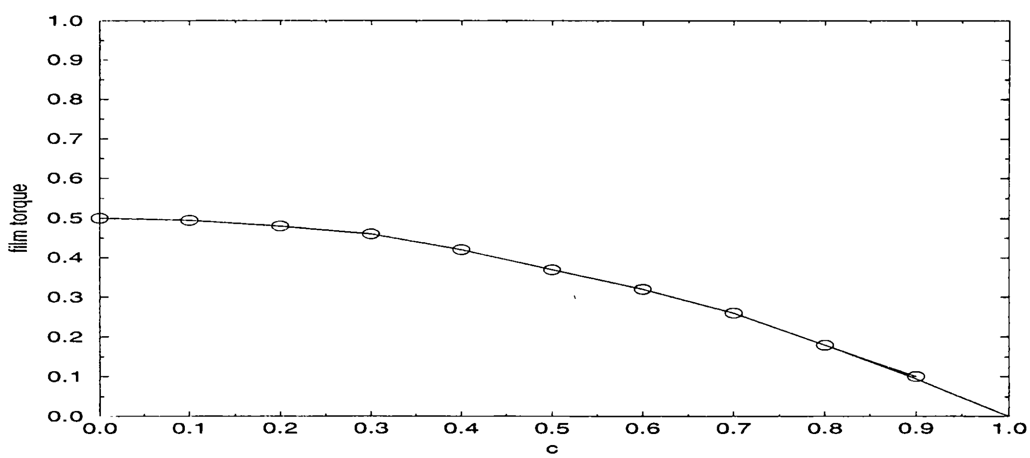

Figure 3.6: The numerical and exact values of film torque τ_f/λ when $\lambda = \infty$.

The formulations and the computational technique presented in this chapter can be generalized, for example, to permit consideration of bounded fluids, all that is necessary is to consider the boundary condition on the appropriate container wall. The problems of a sphere and spherically bounded fluid will be examined in chapter 4.

Chapter 4

CONCENTRIC AND ECCENTRIC SPHERICAL BOUNDARIES

4.1 Introduction

The axisymmetric problem considered in Chapter 3 is now formulated for a slowly rotating solid sphere in a spherical container partially filled with viscous fluid, the plane fluid surface of which is covered with a surfactant film. The geometrical configuration considered now is as follows. A spherical container, contains incompressible viscous fluid on whose plane horizontal surface is a thin layer of immiscible surfactant of typically monomolecular thickness. The wetted surface of the container is denoted by Γ_3 and the surfactant layer by Γ_2. The partially-submerged sphere rotates slowly about a vertical axis through its centre with angular velocity Ω. The wetted surface of the inner sphere is Γ_1, and V is the bulk fluid volume bounded by Γ_1, Γ_2 and Γ_3.

In this chapter, the values of film and substrate torque acting on the partially submerged inner sphere was investigated, and some numerical and graphical results are presented in Section *4.4.3.* These results are computed for varying values of the ratio of the coefficient of surface shear viscosity to the coefficient of viscosity of the

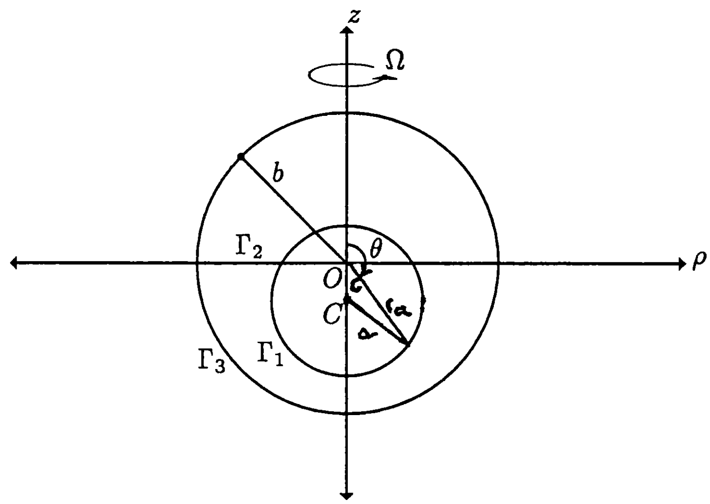

Figure 4.1: A partially submerged inner sphere and half filled outer sphere.

substrate fluid and the depth c of the centre of the inner sphere body below the surfactant film.

4.2 Equations governing the motion

In the treatment presented here, two spherical boundaries are considered. The radii of the inner and the outer boundaries are a and b, respectively, and the outer spherical boundary is assumed to remain at rest and has its centre at the origin O while the inner sphere rotates about the z-axis with constant angular velocity Ω. The inner sphere may or may not be concentric with the outer boundary. The boundary value problem to be solved involves satisfying the creeping motion equations with appropriate boundary conditions. Apart from the presence of the outer boundary condition, and the finiteness of Γ_2 the problem is similar to that of Chapter *3*.

Letting (r, θ, ϕ) be spherical polar coordinates , discussed in as Chapter *3*, then the velocity field v in the substrate fluid satisfies the linearized Navier-Stokes (2.3.2) and equation of continuity (2.3.3).

In the swirling flow under consideration it is clear that equations (2.3.2) and (2.3.3) are satisfied by

$$\mathrm{v} = (0, 0, v(r, \theta)) \tag{4.2.1}$$

and

$$p = \text{constant} \tag{4.2.2}$$

provided that

$$\nabla^2 v - \frac{v}{(r^2 sin^2\theta)} = 0. \tag{4.2.3}$$

The boundary conditions imposed on v are the usual (a) no-slip conditions on Γ_1 and Γ_3, and (b) the condition on the surfactant region Γ_2 that the substrate stresses and the internal film stresses are in balance. Hence, at $r = a$,

$$v = a(1 - t^2)^{1/2} \tag{4.2.4}$$

with $t = \cos\theta$ and $-1 \leq t \leq 0$. At $r = b$ the boundary condition is

$$v = 0. \tag{4.2.5}$$

On the surfactant film, Γ_2,

$$\mu\frac{\partial v}{\partial z} + \eta\frac{\partial^2 v}{\partial z^2} = 0, \tag{4.2.6}$$

where η is the coefficient of surface shear viscosity of the adsorbed film, and μ is the coefficient of viscosity of the substrate fluid. On writing $\lambda = \eta/\mu$, equation (4.2.6) becomes

$$\frac{\partial v}{\partial z} + \lambda\frac{\partial^2 v}{\partial z^2} = 0 \tag{4.2.7}$$

on Γ_2. Thus $\lambda = 0$ corresponds to a uncontaminated surface, and when $\lambda \to \infty$, equation (4.2.7) reduces to $\partial^2 v/\partial z^2 = 0$ on Γ_2.

4.3 Solution of the problem

The non-dimensionalized general form of solution which satisfies (4.2.3) is

$$v = \left(A_0 + \frac{B_0}{r}\right)\left[\frac{(1+t)}{(1-t)}\right]^{1/2} + \sum_{j=1}^{\infty}\left[A_j r^j + B_j\left(\frac{1}{r}\right)^{j+1}\right]P_j^1(t), \tag{4.3.1}$$

following the analysis set out in Chapter 2. In (4.3.1) the radial coordinate r is now dimensionless relative to the radius of the inner sphere, $P_j^1(t)$ is the associated Legendre function of the first kind with $-1 \leq t \leq 0$.

The boundary residual ϵ_1 corresponding to the boundary condition (4.2.4) is defined as

$$\epsilon_1 = v - (1 - t^2)^{1/2}, \tag{4.3.2}$$

with the velocity on the sphere v given by equation (4.3.1) with $r = r_a(t)$. Hence

$$\epsilon_1 = \left(A_0 + \frac{B_0}{r}\right)\left[\frac{(1+t)}{(1-t)}\right]^{1/2} + \sum_{j=1}^{\infty}\left[A_j r^j + B_j\left(\frac{1}{r}\right)^{j+1}\right]P_j^1(t) - (1-t^2)^{1/2}, \tag{4.3.3}$$

with $r = r_a(t)$. Consider now the boundary condition on the surfactant film. The derivatives on the right hand side of equation (4.2.7) can be expressed in terms of spherical polar coordinates as

$$\frac{\partial v}{\partial z} = -\frac{1}{r}\frac{\partial v}{\partial \theta}, \tag{4.3.4}$$

and

$$\frac{\partial^2 v}{\partial z^2} = \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}, \tag{4.3.5}$$

with $z = 0$, or equivalently $\theta = \pi/2$. Hence, using equation (4.3.1), when $\theta = \pi/2$

$$\frac{\partial v}{\partial z} = \left(\frac{A_0}{r} + \frac{B_0}{r^2}\right) + \sum_{j=1}^{\infty}\left[A_j r^{j-1} + B_j\left(\frac{1}{r}\right)^{j+2}\right]P_j''(0), \tag{4.3.6}$$

Furthermore, on $\theta = \pi/2$, or $t = 0$,

$$\frac{\partial^2 v}{\partial z^2} = \frac{v}{r^2} - \frac{\partial^2 v}{\partial r^2} - \frac{1}{r}\frac{\partial v}{\partial r}, \tag{4.3.7}$$

since $\nabla^2 v = v/r^2$, when $t = 0$.

On making use of $P_j''(0) = (j+1)P_{j-1}'(0)$ together with the boundary condition (4.2.7), it follows that substitution of v from equation (4.3.1) gives, when $t = 0$,

$$\begin{aligned}
\frac{\partial v}{\partial z} + \lambda\frac{\partial^2 v}{\partial z^2} = & \\
& \frac{A_0}{r} + \left(\frac{B_0 + \lambda A_0}{r^2}\right) \\
+ & \sum_{j=1}^{\infty}\left[(j+2)A_{j+1}r^j - \lambda(j-1)(j+1)A_j r^{j-2}\right]P_j'(0) \\
+ & \sum_{j=1}^{\infty}\left[(j+2)B_{j+1}\left(\frac{1}{r}\right)^{j+3} - \lambda j(j+2)B_j\left(\frac{1}{r}\right)^{j+3}\right]P_j'(0) \\
= & \quad 0,
\end{aligned} \tag{4.3.8}$$

to satisfy boundary condition (4.2.7).

4.4 Case $\lambda \neq \infty$

If $j = 2m$, where $m = 0, 1, ..,$ then $P'_{2m}(0) = 0$, and it is therefore necessary to consider only $j = 2m+1$ for $m \geq 0$. Thus equation (4.3.8) becomes

$$\begin{aligned}
\frac{\partial v}{\partial z} &+ \lambda\frac{\partial^2 v}{\partial z^2} \\
= & \quad \frac{A_0}{r} + \left(\frac{B_0 + \lambda A_0}{r^2}\right) \\
&+ \sum_{m=0}^{\infty}\left[(2m+3)A_{2m+2}r^{2m+1} - \lambda 2m(2m+2)A_{2m+1}r^{2m-1}\right]P'_{2m+1}(0) \\
&+ \sum_{m=0}^{\infty}\left(\frac{1}{r}\right)^{2m+4}\left[(2m+3)B_{2m+2} - \lambda(2m+1)(2m+3)B_{2m+1}\right]P'_{2m+1}(0) \\
= & \quad 0. \qquad (4.4.1)
\end{aligned}$$

Since $P'_{2m+1}(0) \neq 0$, and $r \geq 1$ is arbitrary, the above boundary condition is then satisfied provided that

$$B_{2m+2} - \lambda(2m+1)B_{2m+1} = 0 \quad (m \geq 0) \qquad (4.4.2)$$

and

$$(2m+3)A_{2m+2}P'_{2m+1}(0) - \lambda(2m+2)(2m+4)A_{2m+3}P'_{2m+3} = 0. \qquad (4.4.3)$$

Now, from Morse and Feshbach (1953),

$$(4m+2)tP_{2m+1}(t) = (2m+2)P_{2m+2}(t) + (2m+1)P_{2m}(t), \qquad (4.4.4)$$

which gives

$$(2m+2)P_{2m+2}(0) + (2m+1)P_{2m}(0) = 0. \qquad (4.4.5)$$

Morse and Feshbach (1953) also gives

$$(1-t^2)P'_{2m+1}(t) = (2m+2)tP_{2m+1}(t) - (2m+2)P_{2m+2}(t). \qquad (4.4.6)$$

Thus

$$\begin{aligned}
P'_{2m+1}(0) = & \; - \; (2m+2)P_{2m+2}(0) \\
= & \quad (2m+1)P_{2m}(0), \qquad (4.4.7)
\end{aligned}$$

and

$$\begin{aligned} P'_{2m+3}(0) &= (2m+3)P_{2m+2}(0) \\ &= -\left[\frac{(2m+3)(2m+1)}{(2m+2)}\right] P_{2m}(0), \end{aligned} \tag{4.4.8}$$

using equation (4.4.5). Hence

$$A_{2m+2} + \lambda(2m+4)A_{2m+3} = 0, \quad (m \geq 0). \tag{4.4.9}$$

Therefore, equation (4.3.1), gives as the velocity field identically satisfying the surfactant boundary condition

$$\begin{aligned} v = & \sum_{m=0}^{\infty} A_{2m+1} \left[r^{2m+1} P^1_{2m+1}(t) - \lambda\,(1-\delta_{m,0})\,(2m+2) r^{2m} P^1_{2m}(t)\right] \\ + & \sum_{m=0}^{\infty} B_{2m+1} \left[\left(\frac{1}{r}\right)^{2m+2} P^1_{2m+1} + \lambda(2m+1)\left(\frac{1}{r}\right)^{2m+3} P^1_{2m+2}(t)\right] \end{aligned} \tag{4.4.10}$$

in which

$$(\delta_{m,0}) = \begin{cases} 1 & \text{if } m{=}0 \\ 0 & \text{if } m \neq 0 \end{cases} . \tag{4.4.11}$$

Equation (4.4.10) may be written as

$$v = \sum_{m=0}^{\infty} \{A_{2m+1} f_m(r,t) + B_{2m+1} g_m(r,t)\} \tag{4.4.12}$$

where

$$f_0(r,t) = r\,P^1_1(t), \tag{4.4.13a}$$

$$f_m(r,t) = r^{2m+1} P^1_{2m+1}(t) - \lambda(2m+2) r^{2m} P^1_{2m}(t) \quad (m \geq 1) \tag{4.4.13b}$$

and

$$g_m(r,t) = \left(\frac{1}{r}\right)^{2m+2} P^1_{2m+1}(t) + \lambda(2m+1)\left(\frac{1}{r}\right)^{2m+3} P^1_{2m+2}(t), \tag{4.4.14}$$

for $m \geq 0$.

4.4.1 Determination of the coefficients A_j and B_j

In (4.4.12), although the surfactant boundary condition is satisfied identically, there remains the boundary conditions on the spherical boundaries to be satisfied. The condition on the inner boundary requires

$$\sum_{m=0}^{\infty} \{A_{2m+1} f_m(r_a,t) + B_{2m+1} g_m(r_a,t)\} = r_a(1-t^2)^{1/2}, \tag{4.4.15}$$

for $-1 \leq t \leq 0$, where $r = r(t)$ takes the value $r_a(t)$ and $r_b(t)$ given by

$$r_a(t) = [1 - c^2(1-t^2)]^{1/2} - ct,$$

with $-1 < c < 1$. The condition on the outer boundary requires

$$\sum_{m=0}^{\infty} \{A_{2m+1}\, f_m(b,t) + B_{2m+1}\, g_m(b,t)\} = 0. \tag{4.4.16}$$

In the particular case when $\lambda = 0$ and $c = 0$, then $r_a(t) = 1$. Using equations (4.4.15) and (4.4.16) and proceeding in the same manner as the analysis in page 51, Chapter 3, the exact solution for the coefficients is obtained as

$$\begin{aligned} A_0 &= \frac{-1}{(b^3-1)} \\ B_0 &= \frac{b^3}{(b^3-1)} \\ A_{2m+1} &= B_{2m+1} = 0 \ (m = 0, 1, ..). \end{aligned} \tag{4.4.17}$$

For all other values of c, $r_a(t)$ is no longer a constant, and the orthogonal property of $P^1_{2m+1}(t)$ over $-1 \leq t \leq 0$ cannot be invoked.

For the general case, it is necessary to determine the unknown coefficients A_{2m+1} and B_{2m+1} numerically. Consider the function I given by

$$I = \int_{-1}^{0} \left[v(r_a,t) - r_a(1-t^2)^{1/2}\right]^2_{r=r_a(t)} dt + \int_{-1}^{0} [v(b,t)]^2 dt. \tag{4.4.18}$$

This is

$$\begin{aligned} I &= \int_{-1}^{0} \left[\sum_{m=0}^{\infty} \{A_{2m+1}\, f_m(r_a,t) + B_{2m+1}\, g_m(r_a,t)\} - r_a(t)(1-t^2)^{1/2}\right]^2 dt \\ &+ \int_{-1}^{0} \left[\sum_{m=0}^{\infty} \{A_{2m+1}\, f_m(b,t) + B_{2m+1}\, g_m(b,t)\}\right]^2 dt \end{aligned} \tag{4.4.19}$$

We shall determine A_{2m+1} and B_{2m+1}, so that I is minimized. A necessary set of conditions for minimizing I is

$$\frac{\partial I}{\partial A_{2n+1}} = \frac{\partial I}{\partial B_{2n+1}} = 0 \ (n = 0, 1, ..). \tag{4.4.20}$$

which lead to the infinite system of linear equations,

$$\begin{aligned}
&\sum_{m=0}^{\infty}\left\{\int_{-1}^{0}[A_{2m+1}\, f_m(r_a,t) + B_{2m+1}\, g_m(r_a,t)]\, f_n(r_a,t)\right\} dt \\
+ &\sum_{m=0}^{\infty}\left\{\int_{-1}^{0}[A_{2m+1}\, f_m(b,t) + B_{2m+1}\, g_m(b,t)]\, f_n(b,t)\right\} dt \\
= &\int_{-1}^{0} r_a(t)(1-t^2)^{1/2} f_n(r_a,t), \ \ n \geq 0
\end{aligned} \tag{4.4.21}$$

and

$$\begin{aligned}
&\sum_{m=0}^{\infty}\left\{\int_{-1}^{0}[A_{2m+1}\, f_m(r_a,t) + B_{2m+1}\, g_m(r_a,t)]\, g_n(r_a,t)\right\} dt \\
+ &\sum_{m=0}^{\infty}\left\{\int_{-1}^{0}[A_{2m+1}\, f_m(b,t) + B_{2m+1}\, g_m(b,t)]\, g_n(b,t)\right\} dt \\
= &\int_{-1}^{0} r_a(t)(1-t^2)^{1/2} g_n(r_a,t), \ \ n \geq 0.
\end{aligned} \tag{4.4.22}$$

The numerical method employed here to solve the boundary value problem for v which is one of a general class of least-squares boundary residual methods.

To solve the equations numerically a finite number J_{max} equations is used and it is assumed that $A_{2m+1}, B_{2m+1} \to 0$ as $m \to \infty$. Thus, setting $A_{2m+1} = B_{2m+1} = 0$ for $m > J_{max}$, equations (4.4.21) and (4.4.22) gives in matrix form

$$\begin{bmatrix} ff(m,n) & \vdots & gf(m,n) \\ \cdots & \cdots & \cdots \\ fg(m,n) & \vdots & gg(m,n) \end{bmatrix} \begin{bmatrix} \mathbf{A} \\ \cdots \\ \mathbf{B} \end{bmatrix} = \begin{bmatrix} tf(n) \\ \cdots \\ tg(n) \end{bmatrix} \tag{4.4.23}$$

where

$$\begin{aligned}
ff(m,n) &= \int_{-1}^{0}[f_m(r_a,t)f_n(r_a,t) + f_m(b,t)f_n(b,t)]\, dt, \\
fg(m,n) &= \int_{-1}^{0}[f_m(r_a,t)g_n(r_a,t) + f_m(b,t)g_n(b,t)]\, dt, \\
gf(m,n) &= \int_{-1}^{0}[f_n(r_a,t)g_m(r_a,t) + f_n(b,t)g_m(b,t)]\, dt, \\
gg(m,n) &= \int_{-1}^{0}[g_m(r_a,t)g_n(r_a,t) + g_m(b,t)g_n(b,t)]\, dt
\end{aligned} \tag{4.4.24}$$

and

$$\begin{aligned} tf(n) &= \int_{-1}^{0} [r_a(t)(1-t^2)^{1/2} f_n(r_a,t)]\, dt, \\ tg(n) &= \int_{-1}^{0} [r_a(t)(1-t^2)^{1/2} g_n(r_a,t)]\, dt. \end{aligned} \tag{4.4.25}$$

and $\mathbf{A} = (A_1, A_3, ..., A_K)'$ and $\mathbf{B} = (B_1, B_3, ..., B_K)'$ with $K = 2J_{max} + 1$. The solutions of these equations determines the values of the coefficients.

4.4.2 Convergence analysis

For the convergence of the numerical method, consider the error factor

$$E = \sqrt{I} \tag{4.4.26}$$

where I is defined in equation (4.4.18), Thus, using the representation (4.3.1) for v,

$$\begin{aligned} I &= \int_{-1}^{0} \left[\sum_{m=0}^{\infty} \{A_{2m+1}\, f_m(r_a,t) + B_{2m+1}\, g_m(r_a,t)\} - r_a(t)(1-t^2)^{1/2} \right]^2 dt \\ &\quad + \int_{-1}^{0} \left[\sum_{m=0}^{\infty} \{A_{2m+1}\, f_m(b,t) + B_{2m+1}\, g_m(b,t)\} \right]^2 dt \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} [A_{2m+1} A_{2n+1}\, ff(m,n)] \\ &\quad + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} [A_{2m+1} B_{2n+1}\, fg(m,n)] \\ &\quad + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} [A_{2n+1} B_{2m+1}\, gf(m,n)] \\ &\quad + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} [B_{2m+1} B_{2n+1}\, gg(m,n)] \\ &\quad -2 \sum_{n=0}^{\infty} [A_{2n+1} tf(n) + B_{2n+1} tg(n)] \\ &\quad + \int_{-1}^{0} \left[[r_a(t)]^2 (1-t^2) \right] dt, \end{aligned} \tag{4.4.27}$$

where the functions ff, fg, gf, gg and tf, tg are defined in equations (4.4.24) and (4.4.25) respectively. The value of J_{max} is chosen large enough to ensure that $A_1, A_3, ..B_1, B_3, ..$ converges to zero. The value of E represents a measure of how accurately the boundary conditions on the spherical boundaries are satisfied.

4.5 Case $\lambda = \infty$

The condition $\lambda = \infty$ implies that the boundary condition (4.2.7) reduces to

$$\partial^2 v / \partial z^2 = 0$$

on Γ_2. It follows that equation (4.3.8) becomes

$$\frac{A_0}{r^2} - \sum_{j=1}^{\infty} \left[(j-1)(j+1) A_j r^{j-2} + j(j+2) \frac{B_j}{r^{j+3}} \right] P_j^1(0) = 0 \tag{4.5.1}$$

Again if $j = 2m$, then $P_{2m}^1(0) = 0$, for $m = 0, 1, 2, \ldots$ Hence

$$\frac{A_0}{r^2}$$

$$- \sum_{m=0}^{\infty} \left[2m(2m+2) A_{2m+1} r^{2m-1} + (2m+1)(2m+3) B_{2m+1} \left(\frac{1}{r} \right)^{2m+4} \right] P_{2m+1}^1(0)$$

$$= 0 \tag{4.5.2}$$

since $P_{2m+1}^1(0) \neq 0$ for $m \geq 0$. Thus, the above equation reduces to

$$\begin{aligned} A_0 &= 0, \\ A_{2m+1} &= 0 \quad (m \geq 1) \end{aligned} \tag{4.5.3}$$

and

$$B_{2m+1} = 0 \quad (m \geq 0). \tag{4.5.4}$$

Therefore

$$\begin{aligned} v &= A_1 r (1-t^2)^{1/2} + \frac{B_0}{r} \left[\frac{(1+t)}{(1-t)} \right]^{1/2} \\ &+ \sum_{m=1}^{\infty} \left[A_{2m} r^{2m} + B_{2m} \frac{1}{r^{2m+1}} \right] P_{2m}^1(t) \end{aligned} \tag{4.5.5}$$

which satisfies the boundary conditions.

4.5.1 Determination of the coefficients A_j and B_j

For $\lambda = \infty$, in satisfying the surfactant boundary condition, the velocity field is given in equation (4.5.5). There remains the boundary condition on the inner partial sphere and spherical boundary to be satisfied. This requires

$$\begin{aligned} r(t)(1-t^2)^{1/2} &= A_1 r(1-t^2)^{1/2} + \frac{B_0}{r}\left[\frac{(1+t)}{(1-t)}\right]^{1/2} \\ &+ \sum_{m=1}^{\infty}\left[A_{2m}r^{2m} + B_{2m}\frac{1}{r^{2m+1}}\right]P_{2m}^1(t), \end{aligned} \quad (4.5.6)$$

for $-1 \le t \le 0$, $r_a(t)$ is defined by equation (4.4.16), and the boundary condition on the outer spherical boundary is

$$v = 0, \quad (4.5.7)$$

on $r = b$, for $-1 \le t \le 0$, and v is defined in equation (4.5.5). In the case of a half-submerged partial inner sphere, $c = 0$, $r_a(t) = 1$ and on the spherical boundary $r = b$, then using equations (4.5.6) and (4.5.7), since $P_{2m}^1(0) = 0$, we obtain

$$1 = B_0 + A_1 \quad (4.5.8)$$

and

$$0 = \frac{B_0}{b} + A_1 b, \quad (4.5.9)$$

giving

$$A_1 = -\frac{1}{b^2-1} \quad (4.5.10)$$

and

$$B_0 = \frac{b^2}{b^2-1}. \quad (4.5.11)$$

When $t \neq 0$, then equation (4.5.6) implies that

$$\begin{aligned} (1-t^2)^{1/2} = &- \left[\frac{1}{(b^2-1)}\right](1-t^2)^{1/2} + \left[\frac{b^2}{(b^2-1)}\right]\left[\frac{1+t}{1-t}\right]^{1/2} \\ &+ \sum_{m=1}^{\infty}\left[A_{2m} + B_{2m}\right]P_{2m}^1(t), \end{aligned} \quad (4.5.12)$$

with $r = 1$, and equation (4.5.7) implies that

$$\begin{aligned} 0 = & - \left[\frac{b}{(b^2-1)}\right](1-t^2)^{1/2} + \left[\frac{b}{(b^2-1)}\right]\left[\frac{1+t}{1-t}\right]^{1/2} \\ & + \sum_{m=1}^{\infty}\left[A_{2m}b^{2m} + B_{2m}\frac{1}{b^{2m+1}}\right]P_{2m}^1(t), \end{aligned} \qquad (4.5.13)$$

with $r = b$. Therefore, equation (4.5.12) gives

$$\left[-\frac{b^2}{(b^2-1)}\right]\beta_m = \alpha_m\left[A_{2m} + B_{2m}\right], \qquad (4.5.14)$$

and equation (4.5.13) gives

$$\left[-\frac{b^2}{(b^2-1)}\right]\beta_m = \alpha_m\left[A_{2m}b^{2m+1} + \frac{B_{2m}}{b^{2m}}\right], \qquad (4.5.15)$$

where

$$\begin{aligned} \alpha_m &= \int_{-1}^{0} P_{2m}^1(t)P_{2m}^1(t)dt \\ &= \left[\frac{2m(2m+1)}{(4m+1)}\right] \end{aligned} \qquad (4.5.16)$$

and

$$\begin{aligned} \beta_m &= \int_{-1}^{0}(t+t^2)P_{2m}'(t)dt \\ &= (-1)^{m+1}\left[\frac{(2m-2)!}{(2^{2m-1})(m+1)!(m-1)!}\right], \end{aligned} \qquad (4.5.17)$$

following the analysis from Chapter 2, and $m \geq 1$. Hence

$$B_{2m} = A_{2m}b^{2m}\left[\frac{(b^{2m+1}-1)}{(b^{2m}-1)}\right] \quad (m \geq 1). \qquad (4.5.18)$$

Thus

$$A_{2m} = \left[-\frac{b^2}{(b^2-1)}\right]\left[\frac{(b^{2m+1}-1)}{(b^{4m+1}-1)}\right]\frac{\beta_m}{\alpha_m} \qquad (4.5.19)$$

and

$$B_{2m} = \left[-\frac{b^2}{(b^2-1)}\right]\left[\frac{b^{2m}(b^{2m+1}-1)}{(b^{4m+1}-1)}\right]\frac{\beta_m}{\alpha_m} \qquad (4.5.20)$$

where $m \geq 1$. It should be noted that as $b \to \infty$ then $B_{2m} \to (-\beta_m/\alpha_m)$, $B_0 \to 1$ and $A_1, A_{2m} \to 0$. Hence, from above,

$$B_{2m} = \frac{\beta_m}{\alpha_m} = (-1)^{m+1}\left[\frac{(2m-2)!\,(4m+1)}{(2^{2m})\,(m+1)!\,m!\,(2m+1)}\right] \qquad (4.5.21)$$

for $m \geq 1$.

4.5.2 Convergence analysis

Similarly, for the convergence of the numerical method, we consider the error factor

$$E = \sqrt{I}, \tag{4.5.22}$$

where the function I is given by

$$\begin{aligned} I &= \int_{-1}^{0} [v_1 - H(t)]^2 \, dt \\ &= \int_{-1}^{0} \left[v_1^2 - 2v_1 H(t) + [H(t)]^2 \right] dt \end{aligned} \tag{4.5.23}$$

and

$$\begin{aligned} v_0 &= A_1 r(1-t^2)^{1/2} + \frac{B_0}{r} \left[\frac{(1+t)}{(1-t)} \right]^{1/2}, \\ v_1 &= \sum_{m=1}^{J_{max}} \left[A_{2m} r^{2m} + B_{2m} \frac{1}{r^{2m+1}} \right] P_{2m}^1(t), \end{aligned} \tag{4.5.24}$$

and

$$H(t) = r_a(t)(1-t^2)^{1/2} - v_0 \tag{4.5.25}$$

with $-1 \le t \le 0$. Again E represents a measure of how accurately the boundary condition on the partially submerged sphere is satisfied.

4.6 Expression for the torque acting on the spheres

4.6.1 The substrate torque

An inner sphere which has surface equation

$$r = r_a(t) \tag{4.6.1}$$

and the outer boundary surface equation

$$r = r_b(t), \tag{4.6.2}$$

where $t = \cos\theta$ and $-1 \le t \le 0$, is considered.

Using the analysis from Chapter *3*, it can be shown that

$$T_s = 2\pi\mu\Omega a^3 \int_{-1}^{0} [r_a(t)]^3 F(t)dt \tag{4.6.3}$$

with $r = r_a(t)$ dimensionless relative to the radius of the inner sphere a, and Ω its angular velocity. Defining the non-dimensional torque coefficient

$$\tau_s = -\frac{T_s}{8\pi\mu\Omega a^3} \tag{4.6.4}$$

it follows that

$$\tau_s = -\frac{1}{4}\int_{-1}^{0} [r_a(t)]^3 F(t)dt \tag{4.6.5}$$

where

$$\begin{aligned} F(t) &= (1-t^2)^{1/2}\left\{\left(r_a(t)\frac{\partial}{\partial r_a(t)}\left(\frac{v}{r_a(t)}\right) - \frac{1}{r_a(t)^2}\frac{dr_a(t)}{dt}\left[tv + (1-t^2)\frac{\partial v}{\partial t}\right]\right)\right\}_{r=r_a(t)} \\ &= (1-t^2)^{1/2}\left\{\left(\left[\frac{\partial v}{\partial r_a(t)} - \frac{v}{r_a(t)}\right] - \frac{1}{r_a(t)^2}\frac{dr_a(t)}{dt}\left[tv + (1-t^2)\frac{\partial v}{\partial t}\right]\right)\right\}_{r=r_a(t)}. \end{aligned} \tag{4.6.6}$$

In order to be able to apply the above equation to a partially submerged inner sphere with spherical outer boundary, we need the equation of the inner sphere surface $r = r_a(t)$, with

$$r_s(t) = [1 - c^2(1-t^2)]^{1/2} - ct, \tag{4.6.7}$$

where c is the depth of the partially submerged inner sphere. Thus,

$$\frac{dr_a(t)}{dt} = \frac{c^2 t}{[1 - c^2(1-t^2)]^{1/2}} - c, \tag{4.6.8}$$

which can also be written as

$$\frac{dr_a(t)}{dt} = -\frac{cr_a(t)}{[r_a(t) + ct]}. \tag{4.6.9}$$

Noting that the general expression for the velocity is

$$v = v_0 + v_1 + v_2 \tag{4.6.10}$$

where

$$\begin{aligned} v_0 &= \left(A_0 + \frac{B_0}{r_a(t)}\right)\left[\frac{(1+t)}{(1-t)}\right]^{1/2}, \\ v_1 &= \sum_{j=1}^{\infty}\left[A_j r_a^j(t)\right] P_j^1(t) \end{aligned} \tag{4.6.11}$$

and

$$v_2 = \sum_{j=1}^{\infty}\left[B_j\left(\frac{1}{r_a(t)}\right)^{j+1}\right]P_j^1(t). \qquad (4.6.12)$$

Hence,

$$\left[\left(\frac{\partial}{\partial r_a(t)}-\frac{1}{r_a(t)}\right)v_0\right] = -\left(A_0+\frac{2B_0}{r_a^2(t)}\right)\left[\frac{(1+t)}{(1-t)}\right]^{1/2}, \qquad (4.6.13)$$

$$\left[\left(\frac{\partial}{\partial r_a(t)}-\frac{1}{r_a(t)}\right)v_1\right] = \sum_{j=1}^{\infty}\left[(j-1)A_j r_a^{j-1}(t)\right]P_j^1(t) \qquad (4.6.14)$$

and

$$\left[\left(\frac{\partial}{\partial r_a(t)}-\frac{1}{r_a(t)}\right)v_2\right] = -\sum_{j=1}^{\infty}\left[(j+2)\frac{B_j}{r_a^{j+2}(t)}\right]P_j^1(t). \qquad (4.6.15)$$

It therefore follows that,

$$\left\{\frac{\partial v_0}{\partial r_a(t)}-\frac{v_0}{r_a(t)}-\left[\frac{1}{r_a^2(t)}\right]\frac{dr_a(t)}{dt}\left[tv_0+(1-t^2)\frac{\partial v_0}{\partial t}\right]\right\}$$
$$= -\left[\frac{(1+t)}{(1-t)}\right]^{1/2}\left[\frac{1}{r_a(t)(r_a(t)+ct)}\right]\left(A_0(r_a(t)+c)+\frac{B_0}{r_a^2(t)}(2r_a(t)+c+ct)\right), \qquad (4.6.16)$$

using the expression for v_0, and

$$\left\{\frac{\partial v_1}{\partial r_a(t)}-\frac{v_1}{r_a(t)}-\left[\frac{1}{r_a^2(t)}\right]\frac{dr_a(t)}{dt}\left[tv_1+(1-t^2)\frac{\partial v_1}{\partial t}\right]\right\}$$
$$= \sum_{j=1}^{\infty}A_j r_a^{j-1}(t)\left[\frac{1}{(r_a(t)+ct)}\right]P_j^1(t)\left[(j+2)ct-(j-1)(r_a(t)+ct)\right]$$
$$-\sum_{j=1}^{\infty}A_j r_a^{j-1}(t)jc\left[\frac{1}{(r_a(t)+ct)}\right]P_{j+1}^1(t)$$
$$= \sum_{j=1}^{\infty}A_j r_a^{j-1}(t)\left[\frac{1}{(r_a(t)+ct)}\right]\left[(2j+1)ctP_j^1(t)-jcP_{j+1}^1(t)+(j-1)r_a(t)P_j^1(t)\right]$$
$$= \sum_{j=1}^{\infty}A_j r_a^{j-1}(t)\left[\frac{1}{(r_a(t)+ct)}\right]\left[(j+1)cP_{j-1}^1(t)+(j-1)r_a(t)P_j^1(t)\right], \qquad (4.6.17)$$

using expression for v_1, and

$$\left\{\frac{\partial v_2}{\partial r_a(t)}-\frac{v_2}{r_a(t)}-\left[\frac{1}{r_a^2(t)}\right]\frac{dr_a(t)}{dt}\left[tv_2+(1-t^2)\frac{\partial v_2}{\partial t}\right\}\right]$$
$$= \sum_{j=1}^{\infty}B_j\left[\frac{1}{r_a^{j+2}(t)}\right]\left[\frac{1}{r_a(t)(r_a(t)+ct)}\right]\left[(j+2)r_a(t)P_j^1(t)+jcP_{j+1}^1(t)\right]. \qquad (4.6.18)$$

Therefore, the expression for the substrate torque coefficient τ_s can be expressed as

$$\tau_s = \tau_s^{(0)} + \tau_s^{(1)} + \tau_s^{(2)} \tag{4.6.19}$$

where

$$\begin{aligned}\tau_s^{(0)} &= \frac{1}{4}\int_{-1}^{0}\left[\frac{[r_a(t)]^2(1+t)}{(r_a(t)+ct)}\right][A_0(r_a(t)+c)+B_0(2r_a(t)+c+ct)]\,dt \\ &= \frac{1}{4}\int_{-1}^{0}\left[\frac{[r_a(t)]^2(1+t)}{(r_a(t)+ct)}\right][(A_0+B_0)(r_a(t)+c)]\,dt \\ &+ \frac{1}{4}\int_{-1}^{0} B_0[r_a(t)]^2(1+t)dt,\end{aligned} \tag{4.6.20}$$

and

$$\tau_s^{(1)} = \frac{1}{4}\sum_{j=1}^{\infty} B_j\left[(j+2)L1_j + jcL1_{j+1}\right], \tag{4.6.21}$$

and

$$\begin{aligned}\tau_s^{(2)} &= -\frac{1}{4}\sum_{j=1}^{\infty}\int_{-1}^{0} A_j[r_a(t)]^{j+2}(1-t^2)^{1/2}\left[\frac{1}{(r_a(t)+ct)}\right]\left[(j+1)cP_{j-1}^1\right]dt \\ &= -\frac{1}{4}\sum_{j=1}^{\infty}\int_{-1}^{0} A_j[r_a(t)]^{j+2}(1-t^2)^{1/2}\left[\frac{1}{(r_a(t)+ct)}\right]\left[(j-1)r_a(t)P_j^1(t)\right]dt \\ &= -\frac{1}{4}\sum_{j=1}^{\infty} A_j\left[(j+1)cL2_{j-1} - (j-1)L2_j\right]\end{aligned} \tag{4.6.22}$$

where

$$L1_j = \int_{-1}^{0}\left[\frac{(1-t^2)^{1/2}}{[r_a(t)]^{j-2}(r_a(t)+ct)}\right]P_j^1(t)dt, \tag{4.6.23}$$

and

$$L2_j = \int_{-1}^{0}\left[\frac{[r_a(t)]^{j+3}(1-t^2)^{1/2}}{(r_a(t)+ct)}\right]P_j^1(t)dt. \tag{4.6.24}$$

It should be noted that as $b \to \infty$ then τ_s agrees with Chapter 3.

4.6.2 The film torque

A film torque T_f is applied to each boundary by the action of surfactant along the ring of intersection with the boundary. The film torque T_f acting on a of a general axisymmetrical body can be written as

$$T_f = -2\pi\Omega\eta \left[r^3 \frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right], \tag{4.6.25}$$

where r is the spherical polar coordinate, the equation of the body is $r = r_a(t)$ and $-1 \leq t \leq 0$. It is convenient to define a dimensionless film torque coefficient $\tau_f = T_f/8\pi\mu\Omega a^3$. Thus

$$\tau_f = -\frac{1}{4}\lambda \left[r^3 \frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right]_{r=r_a(0)}. \tag{4.6.26}$$

The velocity field $v(r,t)$, has the general solution given by

$$v = \left(A_0 + \frac{B_0}{r_a(t)}\right)\left[\frac{1+t}{1-t}\right]^{1/2} + \sum_{j=1}^{\infty}\left[A_j r^j + \frac{B_j}{r_a^{j+1}(t)}\right] P_j^1(t). \tag{4.6.27}$$

Thus

$$\begin{aligned} \frac{\partial}{\partial r}\left(\frac{v}{r_a(t)}\right) &= -\left(\frac{A_0}{r^2} + \frac{2B_0}{r_a(t)^3}\right)\left[\frac{1+t}{1-t}\right]^{1/2} \\ &+ \sum_{j=1}^{\infty}\left[(j-1)A_j r^{j-2} - (j+2)\frac{B_j}{r^{j+3}}\right] P_j^1(t). \end{aligned} \tag{4.6.28}$$

The film torque coefficient is therefore given by

$$\begin{aligned} \tau_f &= \frac{1}{4}\lambda\left(A_0 r + 2B_0\right) \\ &- \frac{1}{4}\lambda \sum_{j=1}^{\infty}\left[(j-1)A_j r_a^{j+1}(0) - (j+2)\frac{B_j}{r_a^j(0)}\right] P_j^1(0). \end{aligned} \tag{4.6.29}$$

When $t = 0$ and j is even then $P_j^1(0) = 0$. Therefore it is necessary only to consider j odd. Hence

$$\begin{aligned} \tau_f &= \frac{1}{4}\lambda\left(A_0 r + 2B_0\right) \\ &- \frac{1}{4}\lambda \sum_{m=1}^{\infty}\left[(2m-2)A_{2m-1} r_a^{2m}(0) - (2m+1)\frac{B_{2m-1}}{r_a^{2m-1}(0)}\right] P_{2m-1}^1(0). \end{aligned} \tag{4.6.30}$$

For a partial inner sphere, the equation of surface when $t = 0$ becomes

$$r_a(0) = (1-c^2)^{1/2} \tag{4.6.31}$$

then

$$\begin{aligned} 4\frac{\tau_f}{\lambda} &= (A_0 r_a(0) + 2B_0) \\ &- \sum_{m=1}^{\infty} \left[(2m-2)A_{2m-1} r_a(0)^{2m} - (2m+1)\frac{B_{2m-1}}{r_a(0)^{2m-1}} \right] P^1_{2m-1}(0) \end{aligned} \tag{4.6.32}$$

with

$$P^1_{2m-1}(0) = \frac{(-1)^{m-1}(2m-1)!}{2^{2m-2}m!(m-2)!}. \tag{4.6.33}$$

In particular case when $r_a(t) = 1$, a half-submerged sphere, the film torque coefficient is

$$\tau_f = -\frac{1}{4}\lambda \left[\frac{\partial}{\partial r} \left(\frac{v}{r} \right) \right]_{r=1}. \tag{4.6.34}$$

Since $P^1_{2m}(0) = 0$, and

$$\frac{\partial}{\partial r} \left(\frac{v}{r} \right)_{r=1} = -2B_0 \tag{4.6.35}$$

Therefore

$$\frac{\tau_f}{\lambda} = \frac{b^2}{2(b^2-1)}, \tag{4.6.36}$$

since

$$B_0 = \left(\frac{b^2}{b^2-1} \right), \tag{4.6.37}$$

from (4.5.11). In the case when $b \to \infty$, equation (4.6.37) reduces to

$$\frac{\tau_f}{\lambda} \to \frac{1}{2}. \tag{4.6.38}$$

4.6.3 Numerical results

The Table 4.1 shows the result for the dimensionless substrate torque when $\lambda = 0$ and outer sphere is half submerged with $b = 100$ and 2, and varying c.

The Table 4.2 shows the result for the dimensionless film torque τ_f/λ when $\lambda = \infty$ with $b = 100$ and 2, and varying c.

The Table 4.3 shows the result for τ_s for various values of c with $\lambda = 1$.

The table 4.4 shows the numerical data for Error-factor E when $\lambda = 1$ with $b = 2$.

c	$b = 100$	$b = 2$
1.00	——	——
0.90	0.8787	1.1879
0.80	0.8519	1.1311
0.70	0.8209	1.0722
0.60	0.7858	0.9998
0.50	0.7468	0.9292
0.40	0.7040	0.8592
0.30	0.6577	0.7887
0.20	0.6078	0.7172
0.10	0.5552	0.6447
0.00	0.5000	0.5714

Table 4.1: The computed values of of τ_s at $\lambda = 0$ with $b = 100$ and 2.

Figure 4.2 shows the computed values of τ_s at $\lambda = 0$ and 1, and Figure 4.3 shows the computed values of τ_f/λ at $\lambda = \infty$.

c	$b = 100$	$b = 2$
1.00	——	——
0.90	0.1000	0.2064
0.80	0.1800	0.2842
0.70	0.2600	0.3537
0.60	0.3200	0.4245
0.50	0.3700	0.4838
0.40	0.4200	0.5322
0.30	0.4600	0.5726
0.20	0.4800	0.6043
0.10	0.4950	0.6254
0.00	0.5000	0.6666

Table 4.2: The computed values of τ_f/λ when $\lambda = \infty$ with $b = 100$ and 2.

c	τ_s
1.00	——
0.90	0.9619
0.80	0.9115
0.70	0.8676
0.60	0.8076
0.50	0.7395
0.40	0.6556
0.30	0.5911
0.20	0.5153
0.10	0.4406
0.00	0.3779

Table 4.3: The computed values of τ_s at $\lambda = 1$ when $b = 2$.

c	Numerical E
0.00000	4.8 $\times 10^{-8}$
0.50000	4.6 $\times 10^{-8}$
0.80000	3.8 $\times 10^{-8}$

Table 4.4: Numerical data for Error – factor E when $\lambda = 1$ with $b = 2$.

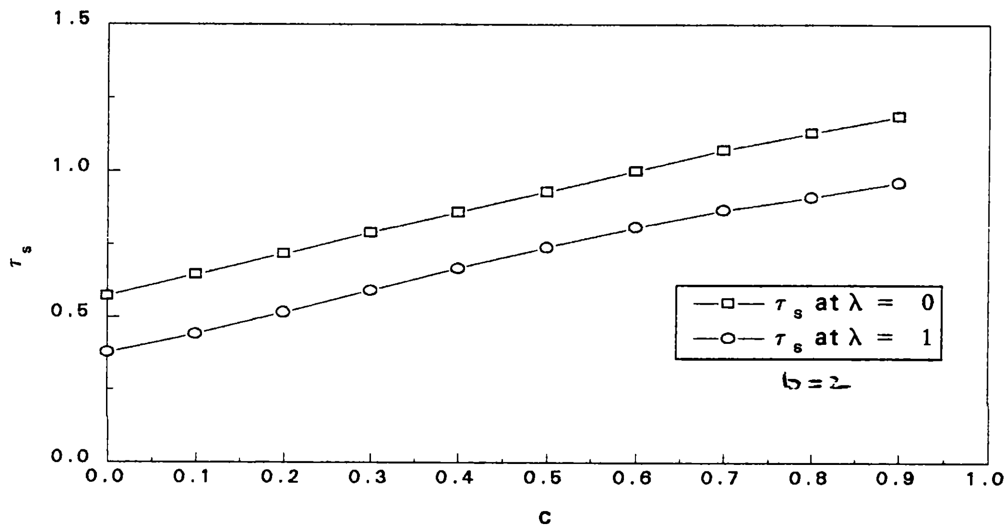

Figure 4.2: The numerical values of substrate torque τ_s at $\lambda = 0$ and 1.

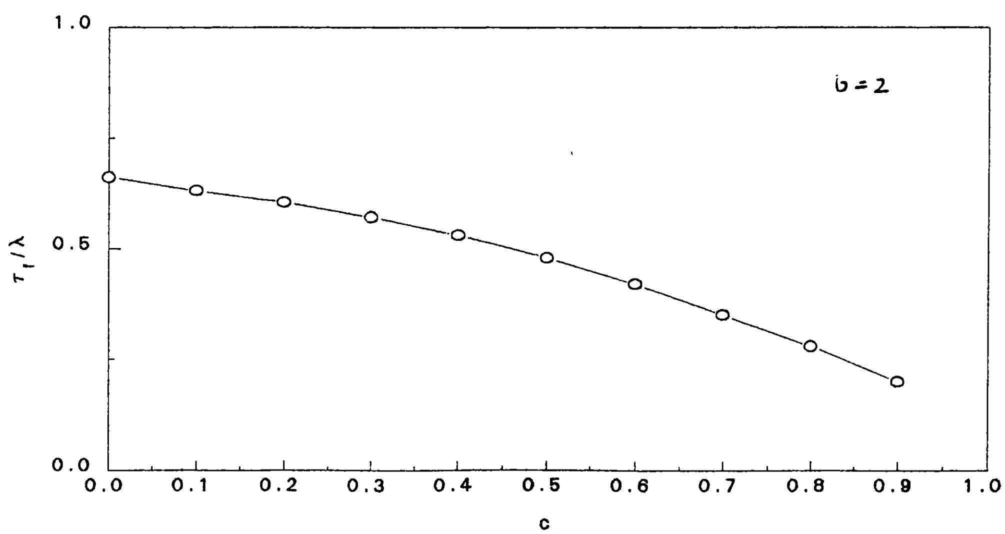

Figure 4.3: The numerical values of film torque τ_f/λ when $\lambda = \infty$.

Chapter 5

THE PROLATE AND OBLATE ELLIPSOIDS

5.1 Introduction

In this chapter the axisymmetric problem of an ellipsoid is considered. The ellipsoid is partially submerged in a substrate fluid below a surfactant layer and rotates slowly about its axis of symmetry which is perpendicular to the plane of the surfactant layer. For an ellipsoidal body the use of ellipsoidal coordinates is particularly advantageous and ensures that a mathematical formulation of the boundary value problem is possible for all depths of the centre ellipsoid below the surfactant layer. This has enabled us to consider in detail the limiting case when the surface viscosity is zero and the surfactant layer becomes a simple stress free surface.

5.2 Prolate ellipsoid

The specific geometry is shown in Figure 5.1. Consider an ellipsoid Γ_1 with the major axis parallel to the z axis. The lengths of the major and minor semi-axes are taken to be a_0 and b_0 respectively. The ellipsoid is partially submerged and slowly rotates with constant angular velocity Ω in a semi-infinite incompressible fluid with dynamic viscosity μ . The axis of rotation is the major axis of the ellipsoid which is perpendicular to the surface Γ_2 of the substrate fluid on which there is a film

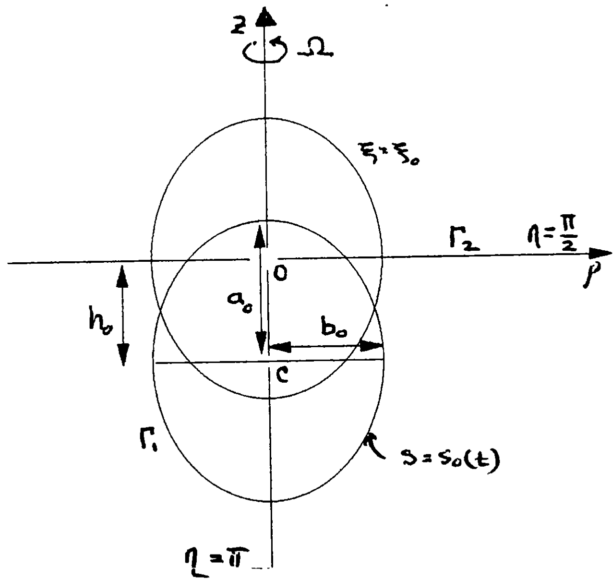

Figure 5.1: The geometry of the prolate ellipsoid.

of an adsorbed monomolecular surfactant fluid possessing surface viscosity η^*. The depth of the ellipsoid centre C *below* the surfactant film is h_0, which takes values in the range $-a_0 < h_0 < a_0$. Note that $h_0 > 0$ or $h_0 < 0$ according as the ellipsoid is more or less than half submerged. The surfactant film is unbounded apart from its intersection with the ellipsoid.

5.2.1 Equations governing the motion

All physical quantities will be referred to the prolate ellipsoidal coordinates (ξ, ϕ, η) related to cylindrical polar coordinates $\{\rho, \phi, z\}$, by the formula

$$z + i\rho = c\cosh(\xi + i\eta) \tag{5.2.1}$$

or equivalently

$$\begin{aligned} z &= c\,\cosh\xi\cos\eta, \\ \rho &= c\,\sinh\xi\sin\eta. \end{aligned} \tag{5.2.2}$$

The origin of coordinates is at the intersection of the major axis of the ellipsoid Γ_1 and the plane containing the surfactant fluid, and c is a constant length which we shall identify with half the distance between the foci of Γ_1.

The surfaces ξ = constant are a set of confocal prolate ellipsoids of revolution, while the surfaces η = constant are a set of confocal hyperboloids of revolution. The foci in each case are at $z = \pm c$, $\rho = 0$.

Now letting the coordinates $\{\rho, \phi, z\}$ be dimensionless relative to c, equations (5.2.2) can be written as

$$z = st, \tag{5.2.3}$$

$$\rho = (s^2 - 1)^{1/2}(1 - t^2)^{1/2}, \tag{5.2.4}$$

with $s = \cosh\xi$ and $t = \cos\eta$. When $h = \frac{h_0}{c} = 0$, the centre of the ellipsoid Γ_1 coincides with the origin O and s = constant = cosh ξ_0, say, on Γ_1. Thus

$$\begin{aligned} a &= \frac{a_0}{c} = \cosh\,\xi_0, \\ b &= \frac{b_0}{c} = \sinh\,\xi_0 = (a^2 - 1)^{1/2}. \end{aligned} \tag{5.2.5}$$

For $h \neq 0$, the parameter s is no longer a constant on Γ_1 and now $s = s_0 = s_0(t)$. The equation of the ellipsoid Γ_1 is accordingly

$$\frac{(s_0 t + h)^2}{a^2} + \frac{(s_0^2 - 1)(1 - t^2)}{(a^2 - 1)} = 1, \tag{5.2.6}$$

where $-1 \leq t \leq 0$ and $-a < h < a$. Thus s_0 satisfies the quadratic equation

$$\alpha s_0^2 + \beta s_0 + \gamma = 0, \tag{5.2.7}$$

where

$$\alpha = (a^2 - t^2), \tag{5.2.8}$$

$$\beta = 2ht(a^2 - 1) \tag{5.2.9}$$

and

$$\gamma = (a^2 - 1)h^2 - (a^2 - t^2)a^2. \tag{5.2.10}$$

The physically meaningful solution of (5.2.7) is

$$s_0(t) = \frac{-\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}. \tag{5.2.11}$$

Assuming that the Reynolds Number for the flow induced in the substrate fluid is sufficiently small to permit the neglect of the inertia terms in the Navier-Stokes equations, the flow produced is governed by the Stokes equation (2.3.2) and the equation of continuity (2.3.3).

The fluid motion is caused solely by the rotation of the ellipsoid, and because of the axisymmetric nature of the flow the velocity v has only one component which is in the azimuthal direction of a system of cylindrical polar coordinates with the z-axis along the axis of rotation of the ellipsoid and pointing out of the fluids. The plane $z = 0$ coincides with that of the surfactant layer.

In this problem, it follows that (2.3.2) and (2.3.3) possess a solution of the form

$$\mathrm{v} = (0, 0, v(s,t)) \tag{5.2.12}$$

with

$$p = \text{constant} \tag{5.2.13}$$

provided that

$$\nabla^2 v - \frac{v}{(s^2-1)(1-t^2)} = 0. \tag{5.2.14}$$

The general solution of (5.2.14), which is bounded in prolate ellipsoidal coordinates, for $-1 \le t \le 0$ and $s > 1$, is of the form

$$v = B_0 \left[\frac{1}{(s^2-1)^{1/2}}\right] \left[\frac{1+t}{1-t}\right]^{1/2} + \sum_{j=1}^{\infty} B_j Q_j^1(s) P_j^1(t), \tag{5.2.15}$$

with $P_j^1(t)$ and $Q_j^1(s)$ the associated Legendre functions of the first kind and the second kind, respectively, of order j and degree unity, as described in Chapter *2*.

5.2.2 Boundary conditions

As for the partially submerged single sphere problem, there are two boundary conditions to be satisfied, one on the wetted surface Γ_1 of the prolate ellipsoid and the other on of the surfactant film Γ_2.

To satisfy the non-slip boundary condition on the surface Γ_1 requires that

$$v = (s_0^2-1)^{1/2}(1-t^2)^{1/2} \tag{5.2.16}$$

with $s = s_0(t)$ on Γ_1 and $-1 \le t \le 0$.

In the presence of the surfactant layer, see Section *2.3.2*, the boundary condition to be satisfied is

$$\frac{\partial v}{\partial z} + \lambda \frac{\partial^2 v}{\partial z^2} = 0 \tag{5.2.17}$$

on Γ_2 , with

$$\lambda = \eta^* / \mu, \tag{5.2.18}$$

where μ denotes the coefficient of dynamic viscosity in the substrate fluid and η^* denotes the surface viscosity of the surfactant layer.

5.2.3 Expressions for $\partial/\partial\rho$ and $\partial/\partial z$

For the following analysis it should be noted that

$$\rho = (s^2-1)^{1/2}\,(1-t^2)^{1/2}, \tag{5.2.19}$$

$$z = st. \tag{5.2.20}$$

Partially differentiating these equations with respect to ρ gives

$$1 = \frac{\partial \rho}{\partial s}\frac{\partial s}{\partial \rho} + \frac{\partial \rho}{\partial t}\frac{\partial t}{\partial \rho}, \tag{5.2.21}$$

$$0 = \frac{\partial z}{\partial s}\frac{\partial s}{\partial \rho} + \frac{\partial z}{\partial t}\frac{\partial t}{\partial \rho}, \tag{5.2.22}$$

giving

$$1 = s\left[\frac{(1-t^2)}{(s^2-1)}\right]^{1/2}\frac{\partial s}{\partial \rho} - t\left[\frac{(s^2-1)}{(1-t^2)}\right]^{1/2}\frac{\partial t}{\partial \rho}, \tag{5.2.23}$$

$$0 = t\frac{\partial s}{\partial \rho} + s\frac{\partial t}{\partial \rho}. \tag{5.2.24}$$

Eliminating $\frac{\partial t}{\partial \rho}$ and $\frac{\partial s}{\partial \rho}$ in turn gives

$$\frac{\partial s}{\partial \rho} = s\left[\frac{(s^2-1)^{1/2}(1-t^2)^{1/2}}{(s^2-t^2)}\right] \tag{5.2.25}$$

and

$$\frac{\partial t}{\partial \rho} = -t\left[\frac{(s^2-1)^{1/2}(1-t^2)^{1/2}}{(s^2-t^2)}\right]. \tag{5.2.26}$$

Therefore

$$\frac{\partial}{\partial \rho} = \left[\frac{(s^2-1)^{1/2}(1-t^2)^{1/2}}{(s^2-t^2)}\right]\left\{s\frac{\partial}{\partial s} - t\frac{\partial}{\partial t}\right\}. \tag{5.2.27}$$

Now, partially differentiating (5.2.20) and (5.2.19) with respect to z gives

$$1 = \frac{\partial z}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial z}{\partial t}\frac{\partial t}{\partial z}, \qquad (5.2.28)$$

$$0 = \frac{\partial \rho}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial \rho}{\partial t}\frac{\partial t}{\partial z}, \qquad (5.2.29)$$

and therefore

$$1 = t\frac{\partial s}{\partial z} + s\frac{\partial t}{\partial z}, \qquad (5.2.30)$$

$$0 = s\left[\frac{(1-t^2)^2}{(s^2-1)^{1/2}}\right]\frac{\partial s}{\partial z} - t\left[\frac{(s^2-1)}{(1-t^2)}\right]^{1/2}\frac{\partial t}{\partial z}. \qquad (5.2.31)$$

Similarly, eliminating $\frac{\partial t}{\partial z}$ and $\frac{\partial s}{\partial z}$ in turn, leads to

$$\frac{\partial s}{\partial z} = t\left[\frac{(s^2-1)}{(s^2-t^2)}\right] \qquad (5.2.32)$$

and

$$\frac{\partial t}{\partial z} = s\left[\frac{(1-t^2)}{(s^2-t^2)}\right]. \qquad (5.2.33)$$

Therefore

$$\frac{\partial}{\partial z} = t\left[\frac{(s^2-1)}{(s^2-t^2)}\right]\frac{\partial}{\partial s} + s\left[\frac{(1-t^2)}{(s^2-t^2)}\frac{\partial}{\partial t}\right], \qquad (5.2.34)$$

and

$$\frac{\partial^2}{\partial z^2} = \left[\frac{1}{(s^2-t^2)}\right]\left\{t(s^2-1)\frac{\partial}{\partial s} + s(1-t^2)\frac{\partial}{\partial t}\right\} \left[\frac{t(s^2-1)}{(s^2-t^2)}\frac{\partial}{\partial s} + \frac{s(1-t^2)}{(s^2-t^2)}\frac{\partial}{\partial t}\right]. \qquad (5.2.35)$$

5.2.4 Expression for $s_0'(t)$

The equation of the body is given by

$$s = s_0(t) = \frac{-\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} \quad (-1 \le t \le 0) \qquad (5.2.36)$$

which satisfies

$$\alpha s_0^2 + \beta s_0 + \gamma = 0, \qquad (5.2.37)$$

and α, β and γ are defined in equations (5.2.8)–(5.2.10), respectively. Hence, differentiating (5.2.37) with respect to t,

$$(2\alpha s_0 + \beta)s_0'(t) - 2ts_0^2 + 2(a^2-1)hs_0 + 2a^2t = 0. \qquad (5.2.38)$$

Therefore

$$\begin{aligned} s_0'(t) &= -\frac{[-ts_0^2 + (a^2-1)hs_0 + a^2 t]}{[(a^2-t^2)s_0 + (a^2-1)ht]} \\ &= \frac{[(s_0^2 - a^2)t - (a^2-1)hs_0]}{[(a^2-t^2)s_0 + (a^2-1)ht]}. \end{aligned} \tag{5.2.39}$$

5.3 Solution of the problem

The form of the general solution which exactly satisfies (5.2.14) can be written as

$$v = v(s,t) = B_0 \left[\frac{1}{(s^2-1)^{1/2}}\right] \left[\frac{1+t}{1-t}\right]^{1/2} + \sum_{j=1}^{\infty} B_j Q_j^1(s) P_j^1(t), \tag{5.3.1}$$

with v dimensionless relative to Ωc. This solution is bounded for $-1 \leq t \leq 0$ and $s > 1$, but in general the Legendre functions $P_j^1(t)$ do not form an orthogonal set over this range of values of t.

On the partially submerged prolate ellipsoid Γ_1, the parameter $s = s_0(t)$ with $-1 \leq t \leq 0$. For $s_0(t) > 1$ for all such t requires $a + h > 1$. This condition ensures exclusion of the line segment $\mid z \mid \leq 1$, $\rho = 0$, on which $s = 1$, from the flow region. The condition is satisfied for all $h \geq 0$, but for $h < 0$ it is necessary for $h > 1 - a$.

The boundary residual ϵ_1 associated with the boundary condition given in equation (5.2.16) is defined as

$$\epsilon_1 = v(s_0, t) - (s_0^2-1)^{1/2}(1-t^2)^{1/2} \tag{5.3.2}$$

$$= B_0 \left[\frac{1}{(s_0^2-1)^{1/2}}\right] \left[\frac{1+t}{1-t}\right]^{1/2} + \sum_{j=1}^{\infty} B_j Q_j^1(s_0) P_j^1(t) - (s_0^2-1)^{1/2}(1-t^2)^{1/2}. \tag{5.3.3}$$

The boundary residual ϵ_2 associated with the boundary (5.2.17) is

$$\epsilon_2 = \left(\frac{\partial v}{\partial z} + \lambda \frac{\partial^2 v}{\partial z^2}\right)_{t=0}, \tag{5.3.4}$$

with the derivatives on the right hand side expressed in terms of the prolate ellipsoidal coordinates as in Section *5.2.3*. Using (5.2.34), (5.2.35), (5.3.1), it can be shown that (5.2.17) reduces to

$$\left\{\left[\frac{1}{s}\right]\frac{\partial v}{\partial t}\right\}_{t=0} + \lambda \left\{\left[\frac{1}{s^3}\right]\left[(s^2-1)\frac{\partial v}{\partial s} + s\frac{\partial^2 v}{\partial t^2}\right]\right\}_{t=0} = 0. \tag{5.3.5}$$

Now

$$\left(\frac{\partial v}{\partial s}\right)_{t=0} = B_0\left[\frac{\partial}{\partial s}(s^2-1)^{-1/2}\right] + \sum_{j=1}^{\infty} B_j\left[\frac{\partial}{\partial s}Q_j^1(s)\right]P_j^1(0), \tag{5.3.6}$$

$$\left(\frac{\partial v}{\partial t}\right)_{t=0} = \sum_{j=1}^{\infty} B_j Q_j^1(s)\left[\frac{d}{dt}P_j^1(t)\right]_{t=0} + \frac{B_0}{(s^2-1)^{1/2}}, \tag{5.3.7}$$

and

$$\left(\frac{\partial^2 v}{\partial t^2}\right)_{t=0} = \sum_{j=1}^{\infty} B_j Q_j^1(s)\left[\frac{d^2}{dt^2}P_j^1(t)\right]_{t=0} + \frac{2B_0}{(s^2-1)^{1/2}}. \tag{5.3.8}$$

Again the recurrence formulae relating the Legendre functions, given for instance by Morse and Feshbach (1953), can be written as

$$(1-t^2)\frac{d}{dt}P_j^1(t) = \left\{(j+1)tP_j^1(t) - jP_{j+1}^1(t)\right\} \tag{5.3.9}$$

for $P_j^1(t)$, and

$$(s^2-1)\frac{d}{ds}Q_j^1(s) = jQ_{j+1}^1(s) - (j+1)sQ_j^1(s) \tag{5.3.10}$$

for $Q_j^1(s)$. Using results from Chapter *3*,

$$\left(\frac{d^2}{dt^2}P_j^1(t)\right)_{t=0} = (j+1)P_j^1(0) + j(j+1)P_{j+2}^1(0). \tag{5.3.11}$$

Thus the equation (5.3.4), with the velocity given by equation (5.3.1) and $t = 0$, becomes

$$\begin{aligned}
\epsilon_2 = \; & - \; \left(\frac{1}{s}\right)\sum_{j=1}^{\infty} B_j Q_j^1(s) j P_{j+1}^1(0) + \frac{B_0}{s(s^2-1)^{1/2}} \\
& + \; \left(\frac{\lambda}{s^2}\right)\sum_{j=1}^{\infty} B_j Q_j^1(s)\left[(j+1)P_j^1(0) + j(j+1)P_{j+2}^1(0)\right] \\
& + \; \left(\frac{\lambda}{s^3}\right)\sum_{j=1}^{\infty} B_j P_j^1(0)\left[jQ_{j+1}^1(s) - (j+1)sQ_j^1(s)\right] \\
& - \; \left(\frac{\lambda}{s^3}\right)B_0\left[\frac{s}{(s^2-1)^{3/2}}\right] + \left(\frac{\lambda}{s^2}\right)\frac{2B_0}{(s^2-1)^{1/2}}.
\end{aligned} \tag{5.3.12}$$

If $j = 2m+1$ where $m = 0, 1, \ldots$ then, since $P_{2m}^1(0) = 0$, it follows that the above equation reduces to

$$\epsilon_2 = \; - \; \left(\frac{1}{s}\right)\sum_{m=1}^{\infty} B_{2m} Q_{2m}^1(s)\left[(2m)P_{2m+1}^1(0)\right] + \frac{B_0}{s(s^2-1)^{1/2}}$$

$$+ \left(\frac{\lambda}{s^2}\right) \sum_{m=0}^{\infty} B_{2m+1} Q^1_{2m+1}(s)$$
$$\left[(2m+2)P^1_{2m+1}(0) + (2m+1)(2m+2)P^1_{2m+3}(0)\right]$$
$$+ \left(\frac{\lambda}{s^3}\right) \sum_{m=0}^{\infty} B_{2m+1} P^1_{2m+1}(0) \left[(2m+1)Q^1_{2m+2}(s) - (2m+2)sQ^1_{2m+1}(s)\right]$$
$$- \left(\frac{\lambda}{s^3}\right) B_0 \left[\frac{s}{(s^2-1)^{3/2}}\right] + \left(\frac{\lambda}{s^2}\right) B_0 \left[\frac{2}{(s^2-1)^{1/2}}\right]. \tag{5.3.13}$$

The surfactant condition is satisfied if $\epsilon_2 = 0$. For $\lambda = 0$, this condition implies that $B_{2m} = 0$, where $m = 0, 1, 2, , ..$, in which case

$$v = \sum_{m=0}^{\infty} B_{2m+1} Q^1_{2m+1}(s) P^1_{2m+1}(t). \tag{5.3.14}$$

For other values of λ, it is not possible to use equation (5.3.13) to express explicitly the odd suffixed coefficients in terms of the even suffixed coefficients or vice versa, as for the partially submerged sphere. We therefore in subsequent analysis consider only the case $\lambda = 0$.

5.4 Determination of the coefficients B_j when $\lambda = 0$

The velocity field is

$$v = \sum_{m=0}^{\infty} B_{2m+1} f_m(s_0, t), \tag{5.4.1}$$

where

$$f_m(s_0, t) = Q^1_{2m+1}(s_0) P^1_{2m+1}(t). \tag{5.4.2}$$

There remains the boundary condition to be satisfied on the ellipsoid, which requires

$$\sum_{m=0}^{\infty} B_{2m+1} f_m(s_0, t) = (s_0^2 - 1)^{1/2}(1-t^2)^{1/2} \tag{5.4.3}$$

for $-1 \le t \le 0$. The unknown coeffients B_{2m+1} are to be determined so that the function I given by

$$I = \int_{-1}^{0} [v - (s_0^2-1)^{1/2}(1-t^2)^{1/2}]^2 dt \tag{5.4.4}$$

is minimized. Clearly a necessary set of conditions for minimizing I is

$$\frac{\partial I}{\partial B_{2m+1}} = 0\,; \;\; m = 0, 1, .. \tag{5.4.5}$$

which leads to the infinite system of linear equations

$$\sum_{m=0}^{\infty} B_{2m+1} S_{m,n} = T_n \; ; \; n \geq 0. \tag{5.4.6}$$

where

$$S_{m,n} = \int_{-1}^{0} Q^1_{2m+1}(s_0) Q^1_{2n+1}(s_0) P^1_{2m+1}(t) P^1_{2n+1}(t) dt \tag{5.4.7}$$

and

$$T_n = \int_{-1}^{0} (s_0^2 - 1)^{1/2} Q^1_{2n+1}(s_0)(1 - t^2)^{1/2} P^1_{2n+1}(t) dt, \tag{5.4.8}$$

with $m \geq 0, n \geq 0$. To solve the equations (5.4.6) numerically, a finite number of equations is used and it is assumed that $B_{2m+1} \rightarrow 0$ as $m \rightarrow \infty$. Consider the $(J_{max}+1)$ equations

$$\sum_{m=0}^{J_{max}} B_{2m+1} S_{m,n} = T_n \; ; \quad 0 \leq n \leq (J_{max}). \tag{5.4.9}$$

A measure of the accuracy of the numerical method, as in chapter *3*, is the error factor E defined as

$$E = \sqrt{I}, \tag{5.4.10}$$

with I given by (5.4.4). Thus, using the representation (5.4.1) for v, this gives

$$\begin{aligned} I &= \int_{-1}^{0} \left[\sum_{m=0}^{\infty} B_{2m+1} f_m(s_0, t) - (s_0^2 - 1)^{1/2} (1 - t^2)^{1/2} \right]^2 dt \\ &= \int_{-1}^{0} \left[\sum_{m=0}^{\infty} B_{2m+1} f_m(s_0, t) \right]^2 dt + \int_{-1}^{0} (s_0^2 - 1)(1 - t^2) dt \\ &\quad -2 \int_{-1}^{0} (s_0^2 - 1)^{1/2} (1 - t^2)^{1/2} \left[\sum_{m=0}^{\infty} B_{2m+1} f_m(s_0, t) \right] dt \\ &= \sum_{m=0}^{\infty} B_{2m+1} \sum_{n=0}^{\infty} B_{2n+1} S_{m,n} - 2 \sum_{n=0}^{\infty} B_{2n+1} T_n + \int_{-1}^{0} (s_0^2 - 1)(1 - t^2) dt, \end{aligned} \tag{5.4.11}$$

where $S_{m,n}$ and T_n are defined in (5.4.7) and (5.4.8) respectively. The value of J_{max} is chosen to be large enough to ensure that B_{2m+1} is effectively *zero* for $m > J_{max}$. Having solved equations (5.4.9) for the coefficients B_{2m+1}, the value of E produces a measure of how accurately the boundary condition on the partially submerged prolate ellipsoid is satisfied.

5.5 Expression for the torque acting on the prolate ellipsoid

There are two types of torque which act on the ellipsoid, as in the sphere problem, namely the substrate torque and film torque. When $\lambda = 0$, the film torque is identically zero.

5.5.1 The substrate torque

The substrate torque T_s arises from the action of the stresses in the substrate fluid. The torque acting on a body, when moments of the surface stresses are taken about the origin, is given by

$$T_s = \hat{\mathbf{k}}.\int_S [\mathbf{r} \times \mathbf{R}_n]\, dS \tag{5.5.1}$$

in which $\mathbf{r}$ is the position vector of a general point of the surface S of the body, and $d\mathbf{S}$=dS $\hat{\mathbf{n}}$ is the areal element of surface orientated in the direction of the outward drawn normal $\hat{\mathbf{n}}$. Since $\mathrm{v} = v(\rho, z)\hat{\phi}$, the only non-zero component of the torque acts along the z-axis. Let the surface of the body have equation

$$\rho = \rho_s(z) \tag{5.5.2}$$

for $-(h + a) \leq z \leq 0$. Now

$$\mathbf{R}_n = (\hat{\mathbf{n}}.\hat{\rho})\mathbf{R}_\rho + (\hat{\mathbf{n}}.\hat{\mathbf{k}})\mathbf{R}_z + (\hat{\mathbf{n}}.\hat{\phi})\mathbf{R}_\phi \tag{5.5.3}$$

with (ρ, ϕ, z) cylindrical polar coordinates.

Letting

$$\begin{aligned} l &= (\hat{\mathbf{n}}.\hat{\rho}), \\ m &= (\hat{\mathbf{n}}.\hat{\mathbf{k}}), \end{aligned} \tag{5.5.4}$$

we then obtain

$$T_s = \int_S \left(\mathbf{R}_n. \left[\hat{\mathbf{k}} \times \mathbf{r}\right]\right) dS. \tag{5.5.5}$$

Now

$$\mathbf{r} = \rho\hat{\rho} + \mathbf{z}\hat{\mathbf{k}} \tag{5.5.6}$$

and thus

$$\left[\hat{\mathbf{k}}\times\mathbf{r}\right] = \rho\hat{\phi}. \tag{5.5.7}$$

Hence

$$T_s = \mu\Omega c^3 \int_S \rho[lP_{\rho\phi} + mP_{\phi z}])dS, \tag{5.5.8}$$

where

$$P_{\rho\phi} = \left[\frac{\partial v}{\partial \rho} - \frac{v}{\rho}\right] \tag{5.5.9}$$

and

$$P_{\phi z} = \frac{\partial v}{\partial z}. \tag{5.5.10}$$

Now

$$l\,dS = \rho_s\,d\phi dz, \tag{5.5.11}$$

where ρ_s is the value of ρ on the body, and

$$\begin{aligned} m\,dS &= -\rho_s\,d\phi\,d\rho_s \\ &= -\rho_s\,d\phi\,\frac{d\rho_s}{dz}\,dz. \end{aligned} \tag{5.5.12}$$

Therefore

$$T_s = 2\pi\mu\Omega c^3 \int_{-(a+h)}^{0} \rho_s^2 \left\{ \left[\frac{\partial v}{\partial \rho} - \frac{v}{\rho}\right] - \frac{\partial v}{\partial z}\frac{d\rho_s}{dz} \right\}_{\rho=\rho_s} dz. \tag{5.5.13}$$

Using the solution for $v(\rho, z)$, or equivalently $v = v(s,t)$, when $\lambda = 0$:

$$\begin{aligned} v &= \sum_{m=1}^{\infty} B_{2m+1} Q^1_{2m+1}(s) P^1_{2m+1}(t) \\ &= -\rho \sum_{m=1}^{\infty} B_{2m+1} Q'_{2m+1}(s) P'_{2m+1}(t), \end{aligned}$$

it follows (5.3.13) that

$$\begin{aligned} \frac{\partial v}{\partial \rho} - \frac{v}{\rho} &= \rho \frac{\partial}{\partial \rho}\left(\frac{v}{\rho}\right) \\ &= -\Phi(s,t)\left[s\frac{\partial}{\partial s} - t\frac{\partial}{\partial t}\right] \sum_{m=0}^{\infty} B_{2m+1} Q'_{2m+1}(s) P'_{2m+1}(t) \\ &= -\Phi(s,t) \sum_{m=0}^{\infty} B_{2m+1} \left\{ sQ''_{2m+1}(s) P'_{2m+1}(t) - tQ'_{2m+1}(s) P''_{2m+1}(t) \right\}, \end{aligned} \tag{5.5.14}$$

where

$$\Phi(s,t) = \frac{(s^2-1)(1-t^2)}{(s^2-t^2)}. \tag{5.5.15}$$

and

$$\begin{aligned}\frac{\partial v}{\partial z} &= \rho\frac{\partial}{\partial z}\left(\frac{v}{\rho}\right)\\ &= -\frac{1}{(s^2-t^2)}\left[t(s^2-1)\frac{\partial}{\partial s}+s(1-t^2)\frac{\partial}{\partial t}\right]\sum_{m=0}^{\infty}B_{2m+1}Q'_{2m+1}(s)P'_j(t)\\ &= -\frac{1}{(s^2-t^2)}\sum_{m=0}^{\infty}B_{2m+1}\\ &\quad \left\{(s^2-1)tQ''_{2m+1}(s)P'_{2m+1}(t)+(1-t^2)sQ'_{2m+1}(s)P''_{2m+1}(t)\right\}\end{aligned} \tag{5.5.16}$$

also

$$\frac{\partial\rho_s}{\partial z} = -\frac{(s_0t+h)(a^2-1)}{(s_0^2-1)^{1/2}(1-t^2)^{1/2}a^2}. \tag{5.5.17}$$

On the body, $s=s_0(t)$ and therefore

$$\begin{aligned}&-\left(\left[\frac{\partial v}{\partial \rho}-\frac{v}{\rho}\right]-\frac{\partial v}{\partial z}\left[\frac{\partial \rho_s}{\partial z}\right]\right)_{\rho=\rho_s}\\ &= \sum_{m=0}^{\infty}\frac{B_{2m+1}}{(s_0^2-t^2)}(s_0^2-1)(1-t^2)\\ &\quad [s_0Q''_{2m+1}(s_0)P'_{2m+1}(t)-tQ'_{2m+1}(s_0)P''_{2m+1}(t)]\\ &+ \sum_{m=0}^{\infty}\frac{B_{2m+1}}{(s_0^2-t^2)}\frac{(a^2-1)}{a^2}(s_0t+h)\\ &\quad \left[(s_0^2-1)tQ''_{2m+1}(s_0)P'_{2m+1}(t)+s_0(1-t^2)Q'_{2m+1}(s_0)P''_{2m+1}(t)\right]\\ &= \sum_{m=0}^{\infty}\frac{B_{2m+1}}{(s_0^2-t^2)}\times\\ &\quad (s_0^2-1)Q''_{2m+1}(s_0)P'_{2m+1}(t)\left[s_0(1-t^2)+\frac{(a^2-1)}{a^2}(s_0t+h)t\right]\\ &+ \sum_{m=0}^{\infty}\frac{B_{2m+1}}{(s_0^2-t^2)}\times\\ &\quad (1-t^2)Q'_{2m+1}(s_0)P''_{2m+1}(t)\left[\frac{(a^2-1)}{a^2}(s_0t+h)s_0-t(s^2-1)\right]\\ &= \sum_{m=0}^{\infty}\frac{B_{2m+1}}{(s_0^2-t^2)}\times\end{aligned}$$

$$
\begin{aligned}
& (s_0^2-1)Q''_{2m+1}(s_0)P_{2m+1}{}'(t)\left[s_0+ht-\frac{1}{a^2}(s_0t^2+ht)\right] \\
+ & \sum_{m=0}^{\infty}\frac{B_{2m+1}}{(s_0^2-t^2)}\times \\
& (1-t^2)Q'_{2m+1}(s_0)P''_{2m+1}(t)\left[s_0h+t-\frac{1}{a^2}(s_0^2t+s_0h)\right] \\
= & \sum_{m=0}^{\infty}\frac{B_{2m+1}}{(s_0^2-t^2)}\{L1(s_0,t)+L2(s_0,t)\}, \qquad (5.5.18)
\end{aligned}
$$

in which

$$
L1(s_0,t) = (s_0^2-1)Q''_{2m+1}(s_0)P'_{2m+1}(t)\left[s_0\left(1-\frac{t^2}{a^2}\right)+\left(1-\frac{1}{a^2}\right)ht\right] \qquad (5.5.19)
$$

$$
L2(s_0,t) = (1-t^2)Q'_{2m+1}(s_0)P''_{2m+1}(t)\left[t\left(1-\frac{s_0^2}{a^2}\right)+\left(1-\frac{1}{a^2}\right)hs_0\right]. \qquad (5.5.20)
$$

Throughout the above $s_0 = s_0(t)$. Defining the non-dimensional torque coefficient τ_s by

$$
\tau_s = -\frac{T_s}{8\pi\mu\Omega c^3},
$$

it follows that for a prolate ellipsoid τ_s is given by

$$
\tau_s = \frac{1}{4}\sum_{m=0}^{\infty}B_{2m+1}\int_{-1}^{0}\frac{(s_0^2-1)(1-t^2)}{(s_0^2-t^2)}(s_0't+s_0)\{L1(s_0,t)+L2(s_0,t)\}\,dt, \quad (5.5.21)
$$

where $L1(s_0,t)$, $L2(s_0,t)$ are as defined in (5.5.19) and (5.5.20) respectively, and $s_0'(t)$ is defined in Section *5.2.4*.

5.6 Oblate ellipsoid

Consider now an oblate ellipsoid Γ_1 with its minor axis parallel to the z axis, as shown in Figure 5.2. The ellipsoid is partially submerged and slowly rotates with constant angular velocity Ω in a semi-infinite incompressible fluid with dynamic viscosity μ . The axis of rotation is the z axis which is perpendicular to the horizontal surface Γ_2 of the substrate fluid. There is a film of an adsorbed monomolecular surfactant fluid possessing surface viscosity η^* on the surface Γ_2. The depth of the ellipsoid centre C *below* the surfactant film is h_0, where h_0 take values in the range $-a_0 < h_0 < a_0$, with a_0 the length of the minor semi-axis. Note that $h_0 > 0$ or $h_0 < 0$ according as

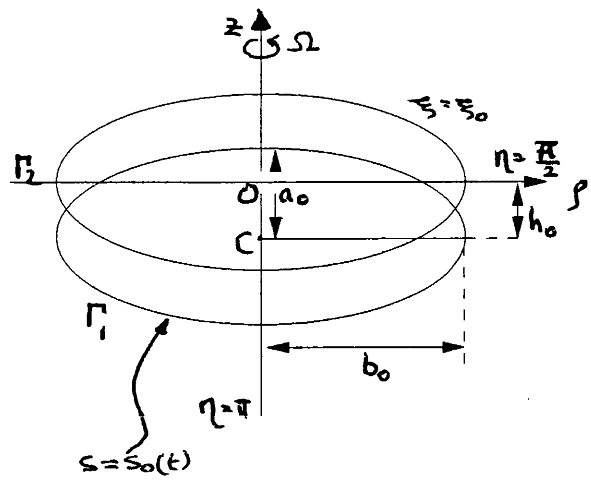

Figure 5.2: The geometry of the oblate ellipsoid.

the ellipsoid is more or less than half submerged. The surfactant film is unbounded apart from its intersection with the ellipsoid.

5.6.1 Equations governing the motion

As before, all physical quantities will be referred to the oblate ellipsoidal coordinates (ξ, ϕ, η), related to cylindrical polar coordinates $\{\rho, \phi, z\}$ by the formula

$$z + i\rho = c \sinh(\xi + i\eta), \tag{5.6.1}$$

or equivalently

$$\begin{aligned} z &= c \sinh \xi \cos \eta, \\ \rho &= c \cosh \xi \sin \eta. \end{aligned} \tag{5.6.2}$$

The surfaces ξ = constant are confocal oblate ellipsoids. The foci in any azimuthal section are such that $\rho = c$, $z = 0$.

Letting coordinates $\{\rho, z\}$ be dimensionless relative to the focal half distance c, equations (5.6.2) can be written as

$$z = st \tag{5.6.3}$$

$$\rho = (s^2 + 1)^{1/2}(1 - t^2)^{1/2}, \tag{5.6.4}$$

with $s = \sinh\xi$ and $t = \cos\eta$. When $h = \frac{h_0}{c} = 0$, the centre of the ellipsoid Γ_1 coincides with the origin O and $s = \text{constant} = \sinh\xi_0$, say, and the dimensionless minor and major semi-axes a and b are

$$\begin{aligned} a &= \frac{a_0}{c} = \sinh\xi_0, \\ b &= \frac{b_0}{c} = \cosh\xi_0 = (a^2+1)^{1/2}. \end{aligned} \tag{5.6.5}$$

The equation of the oblate ellipsoid Γ_1 is

$$\frac{(s_0 t + h)^2}{a^2} + \frac{(s_0^2+1)(1-t^2)}{(a^2+1)} = 1, \tag{5.6.6}$$

with $-1 \le t \le 0$, $s = s_0(t)$ on the body, and $-a < h < a$. Thus (5.6.6) gives

$$\alpha s_0^2 + \beta s_0 + \gamma = 0 \tag{5.6.7}$$

where

$$\alpha = (a^2 + t^2), \tag{5.6.8}$$

$$\beta = 2ht(a^2+1) \tag{5.6.9}$$

and

$$\gamma = (a^2+1)h^2 - (a^2+t^2)a^2. \tag{5.6.10}$$

The physically possible solution is

$$s = s_0(t) = \frac{-\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}. \tag{5.6.11}$$

Also

$$s_0'(t) = -\frac{[(s_0^2 - a^2)t + (a^2+1)hs_0]}{[(a^2+t^2)s_0 + (a^2+1)ht]} \tag{5.6.12}$$

using similar analysis to Section *5.2.4*.

Similarly, in this problem, equations (2.3.2) and (2.3.3) possess a solution of the form

$$\mathbf{v} = (0, 0, v(s,t)) \tag{5.6.13}$$

with

$$p = \text{constant} \tag{5.6.14}$$

provided that

$$\nabla^2 v - \frac{v}{(s^2+1)(1-t^2)} = 0. \tag{5.6.15}$$

5.6.2 Boundary conditions

As for the prolate ellipsoid, in this problem there are two boundary conditions to be satisfied. One on the wetted surface Γ_1 of the oblate ellipsoid and the other on of the surfactant film Γ_2.

To satisfy the non-slip boundary condition on the surface Γ_1 requires that

$$v = (s_0^2+1)^{1/2}(1-t^2)^{1/2} \tag{5.6.16}$$

with $-1 \leq t \leq 0$.

To satisfy the surfactant boundary condition, in dimensionless form as in Section *2.3.2*, requires that on Γ_2

$$\frac{\partial v}{\partial z} + \lambda \frac{\partial^2 v}{\partial z^2} = 0, \tag{5.6.17}$$

where $\lambda = \eta^*/\mu$, with μ denoting the coefficient of dynamic viscosity in the substrate fluid which occupies the region $z < 0$, and η^* denoting the surface viscosity of the surfactant.

5.7 Solution of the problem

The dimensionless general form of solution for v which exactly satisfies (5.6.15) is

$$v = v(s,t) = B_0 \left[\frac{1}{(s^2+1)^{1/2}}\right] \left[\frac{1+t}{1-t}\right]^{1/2} + \sum_{j=1}^{\infty} B_j q_j^1(s) P_j^1(t) \tag{5.7.1}$$

as given in Chapter *2* with $s > 0$ and $-1 \leq t \leq 0$. In general the Legendre functions $P_j^1(t)$ do not form an orthogonal set over this range of values of t.

On the partially submerged oblate ellipsoid Γ_1, the parameter $s = s_0(t)$ with $-1 \leq t \leq 0$. For $s_0(t) > 0$ for all such t and the exclusion of points on the disk $z = 0$, $\rho \leq 1$, on which $s = 0$, from the flow region, it is necessary that $|h| < a^2/(a^2+1)^{1/2}$.

The boundary residual ϵ_1, associated with the boundary condition (5.6.16) is

$$\epsilon_1 = v(s,t) - (s_0^2+1)^{1/2}(1-t^2)^{1/2} \tag{5.7.2}$$

$$= B_0 \left[\frac{1}{(s_0^2+1)^{1/2}}\right] \left[\frac{1+t}{1-t}\right]^{1/2} + \sum_{j=1}^{\infty} B_j q_j^1(s_0) P_j^1(t) - (s_0^2+1)^{1/2}(1-t^2)^{1/2}. \tag{5.7.3}$$

The boundary residual ϵ_2, associated with the surfactant boundary condition is

$$\epsilon_2 = \left(\frac{\partial v}{\partial z} + \lambda \frac{\partial^2 v}{\partial z^2}\right)_{t=0}, \tag{5.7.4}$$

with derivatives on the right hand side expressed in terms of oblate ellipsoidal coordinates as in section 5.2.3. Using similar expressions to (5.2.34) and (5.2.35) together with (5.7.1), when $t = 0$, it can be shown that

$$\begin{aligned}\epsilon_2 = &-\left(\frac{1}{s}\right)\sum_{j=1}^{\infty} B_j q_j^1(s) j P_{j+1}^1(0) + \frac{B_0}{s(s^2+1)^{1/2}} \\ &+ \left(\frac{\lambda}{s^2}\right)\sum_{j=1}^{\infty} B_j q_j^1(s)\left[(j+1)P_j^1(0) + j(j+1)P_{j+2}^1(0)\right] \\ &+ \left(\frac{\lambda}{s^3}\right)\sum_{j=1}^{\infty} B_j P_j^1(0)\left[j q_{j+1}^1(s) - (j+1)s_0 q_j^1(s)\right] \\ &- \left(\frac{\lambda}{s^3}\right) B_0\left[\frac{s}{(s^2+1)^{3/2}}\right] + \left(\frac{\lambda}{s^2}\right)\frac{2B_0}{(s^2+1)^{1/2}}. \end{aligned} \tag{5.7.5}$$

Since $P_{2m}^1(0) = 0$, it follows that the above equation reduces to

$$\begin{aligned}\epsilon_2 = &-\left(\frac{1}{s}\right)\sum_{m=1}^{\infty} B_{2m} q_{2m}^1(s)\left[(2m)P_{2m+1}^1(0)\right] + \frac{B_0}{s(s^2+1)^{1/2}} \\ &+ \left(\frac{\lambda}{s^2}\right)\sum_{m=0}^{\infty} B_{2m+1} q_{2m+1}^1(s). \\ &\qquad \left[(2m+2)P_{2m+1}^1(0) + (2m+1)(2m+2)P_{2m+3}^1(0)\right] \\ &+ \left(\frac{\lambda}{s^3}\right)\sum_{m=0}^{\infty} B_{2m+1} p_{2m+1}^1(0)\left[(2m+1)q_{2m+2}^1(s) - (2m+2)s q_{2m+1}^1(s)\right] \\ &- \left(\frac{\lambda}{s^3}\right) B_0\left[\frac{s}{(s^2+1)^{3/2}}\right] + \left(\frac{\lambda}{s^2}\right)\frac{2B_0}{(s^2+1)^{1/2}}. \end{aligned} \tag{5.7.6}$$

The surfactant condition is satisfied if $\epsilon_2 = 0$. For $\lambda = 0$, this condition implies that $B_{2m} = 0$, where $m = 0, 1, 2, , .., $ in which case

$$v = \sum_{m=0}^{\infty} B_{2m+1} q_{2m+1}^1(s) P_{2m+1}^1(t). \tag{5.7.7}$$

For other values of λ, it is not possible to use equation (5.7.13) to express explicitly the odd suffixed coefficients in terms of the even suffixed coefficients or vice versa, as for the partially submerged sphere. We therefore in subsequent analysis consider only the case $\lambda = 0$.

5.8 Determination of the coefficients B_j when $\lambda = 0$

The velocity field is

$$v = \sum_{m=0}^{\infty} B_{2m+1} f_m(s_0, t) \tag{5.8.1}$$

where

$$f_m(s_0, t) = q^1_{2m+1}(s_0) P^1_{2m+1}(t). \tag{5.8.2}$$

There remains the boundary condition to be satisfied on the body, which requires

$$\sum_{m=0}^{\infty} B_{2m+1} f_m(s_0, t) = (s_0^2 + 1)^{1/2}(1 - t^2)^{1/2} \tag{5.8.3}$$

for $-1 \leq t \leq 0$. The unknown coefficients B_{2m+1} are to be determined so that the function I given by

$$I = \int_{-1}^{0} [v - (s_0^2 + 1)^{1/2}(1 - t^2)^{1/2}]^2 dt \tag{5.8.4}$$

is minimized. Clearly a necessary set of conditions for minimizing I is

$$\frac{\partial I}{\partial B_{2m+1}} = 0\,; \;\; m = 0, 1, .. \tag{5.8.5}$$

which leads to the infinite system of linear equations

$$\sum_{m=0}^{\infty} B_{2m+1} S_{m,n} = T_n\,; \quad n \geq 0, \tag{5.8.6}$$

where

$$S_{m,n} = \int_{-1}^{0} q^1_{2m+1}(s_0) q^1_{2n+1}(s_0) P^1_{2m+1}(t) P^1_{2n+1}(t) dt \tag{5.8.7}$$

and

$$T_n = \int_{-1}^{0} (s_0^2 + 1)^{1/2} q^1_{2n+1}(s_0)(1 - t^2)^{1/2} P^1_{2n+1}(t) dt, \tag{5.8.8}$$

with $m \geq 0, n \geq 0$. To solve the equations numerically a finite number of equations is used and it is assumed that $B_{2m+1} \to 0$ as $m \to \infty$. Consider the equations

$$\sum_{m=0}^{J_{max}} B_{2m+1} S_{m,n} = T_n; \quad (0 \leq n \leq J_{max}) \tag{5.8.9}$$

A measure of the accuracy of the numerical method, as in chapter *3*, is the error factor E defined

$$E = \sqrt{I}, \tag{5.8.10}$$

with I given by (5.8.4). Thus, using the representation (5.8.1) for v, this gives

$$\begin{aligned} I &= \int_{-1}^{0}\left[\sum_{m=0}^{\infty} B_{2m+1}f_m(s_0,t) - (s_0^2+1)^{1/2}(1-t^2)^{1/2}\right]^2 dt \\ &= \int_{-1}^{0}\left[\sum_{m=0}^{\infty} B_{2m+1}f_m(s_0,t)\right]^2 dt + \int_{-1}^{0}(s_0^2+1)(1-t^2)dt \\ &\quad -2\int_{-1}^{0}(s_0^2+1)^{1/2}(1-t^2)^{1/2}\left[\sum_{m=0}^{\infty} B_{2m+1}f_m(s_0,t)\right] dt \\ &= \sum_{m=0}^{\infty} B_{2m+1}\sum_{n=0}^{\infty} B_{2n+1}S_{m,n} - 2\sum_{n=0}^{\infty} B_{2n+1}T_n + \int_{-1}^{0}(s_0^2+1)(1-t^2)dt, \end{aligned} \tag{5.8.11}$$

where $S_{m,n}$ and T_n are defined in (5.8.7) and (5.8.8) respectively. The value of J_{max} is chosen to be large enough to ensure that B_{2m+1} is effectively zero for $m > J_{max}$. Having solved the equations (5.8.9) for the coefficients B_{2m+1}, the value of E provides a measure of how accurately the boundary condition on the partially submerged oblate ellipsoid is satisfied.

5.9 Expression for the torque acting on the oblate ellipsoid

As before, there are two types of torque which act on the oblate ellipsoid, namely, the substrate torque and film torque. The film torque is zero when $\lambda = 0$.

5.9.1 The substrate torque

The substrate torque T_s arising from the action of the stresses in the substrate fluid. The torque acting on the oblate, when moments of the surface stresses are taken about the origin, is given by

$$T_s = 2\pi\mu\Omega c^3 \int_{-(a+h)}^{0} \rho_s^2 \left\{\left[\frac{\partial v}{\partial \rho} - \frac{v}{\rho}\right] - \frac{\partial v}{\partial z}\frac{d\rho_s}{dz}\right\}_{\rho=\rho_s} dz \tag{5.9.1}$$

following the analysis of Section *5.5*. Using the general solution for $v(s,t)$,

$$\begin{aligned} v &= \sum_{m=0}^{\infty} B_{2m+1}q_{2m+1}^1(s)P_{2m+1}^1(t) \\ &= -\rho\sum_{m=0}^{\infty} B_{2m+1}q_{2m+1}'(s)P_{2m+1}'(t) \end{aligned} \tag{5.9.2}$$

gives

$$\begin{aligned}
\frac{\partial v}{\partial \rho} - \frac{v}{\rho} &= \rho \frac{\partial}{\partial \rho}\left(\frac{v}{\rho}\right) \\
&= -\Phi(s,t)\left[s\frac{\partial}{\partial s} - t\frac{\partial}{\partial t}\right]\sum_{m=0}^{\infty} B_{2m+1} q'_{2m+1}(s_0) P'_{2m+1}(t) \\
&= -\Phi(s,t)\sum_{m=0}^{\infty} B_{2m+1}\left\{s q''_{2m+1}(s) P'_{2m+1}(t) - t q'_{2m+1}(s) P''_{2m+1}(t)\right\}
\end{aligned} \tag{5.9.3}$$

where

$$\Phi(s,t) = \frac{(s^2+1)(1-t^2)}{(s^2+t^2)}. \tag{5.9.4}$$

Also

$$\begin{aligned}
\frac{\partial v}{\partial z} &= \rho\frac{\partial}{\partial z}\left(\frac{v}{\rho}\right) \\
&= -\frac{1}{(s^2+t^2)}\left[t(s^2+1)\frac{\partial}{\partial s} + s(1-t^2)\frac{\partial}{\partial t}\right]\sum_{m=0}^{\infty} B_{2m+1} q'_{2m+1}(s) P'_{2m+1}(t) \\
&= -\frac{1}{(s^2+t^2)}\sum_{m=0}^{\infty} B_{2m+1} \times \\
&\quad \left\{(s^2+1) t q''_{2m+1}(s) P'_{2m+1}(t) - (1-t^2) s q_{2m+1}(s) P''_{2m+1}(t)\right\}.
\end{aligned} \tag{5.9.5}$$

and

$$\frac{\partial \rho_s}{\partial z} = -\frac{(s_0 t + h)(a^2+1)}{(s_0^2+1)^{1/2}(1-t^2)^{1/2} a^2}. \tag{5.9.6}$$

On the body, $s = s_0(t)$ and therefore

$$\begin{aligned}
&- \left\{\left[\frac{\partial v}{\partial \rho} - \frac{v}{\rho}\right] - \frac{\partial v}{\partial z}\left(\frac{\partial \rho_s}{\partial z}\right)\right\}_{\rho=\rho_s} \\
&= \sum_{m=0}^{\infty}\frac{B_{2m+1}}{(s_0^2+t^2)}(s_0^2+1)(1-t^2)[s_0 q''_{2m+1}{}''(s_0) P'_{2m+1}(t) - t q'_{2m+1}(s_0) P''_{2m+1}(t)] \\
&+ \sum_{m=0}^{\infty}\frac{B_{2m+1}}{(s_0^2+t^2)}\frac{(a^2+1)}{a^2}(s_0 t + h) \times \\
&\quad \left[(s_0^2+1) t q''_{2m+1}(s_0) P'_{2m+1}{}'(t) + s_0(1-t^2) q'_{2m+1}(s_0) P''_{2m+1}(t)\right] \\
&= \sum_{m=0}^{\infty}\frac{B_{2m+1}}{(s_0^2+t^2)}(s_0^2+1) q''_{2m+1}(s_0) P'_{2m+1}(t)\left[s_0(1-t^2) + \frac{(a^2+1)}{a^2}(s_0 t + h)t\right]
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{m=0}^{\infty} \frac{B_{2m+1}}{(s_0^2+t^2)}(1-t^2)q'_{2m+1}(s_0)P''_{2m+1}(t)\left[\frac{(a^2+1)}{a^2}(s_0t+h)s_0 - t(s_0^2+1)\right] \\
&= \sum_{m=0}^{\infty} \frac{B_{2m+1}}{(s_0^2+t^2)}(s_0^2+1)q''_{2m+1}(s_0)P'_{2m+1}(t)\left[s_0+ht+\frac{1}{a^2}(s_0t^2+ht)\right] \\
&+ \sum_{m=0}^{\infty} \frac{B_{2m+1}}{(s_0^2+t^2)}(1-t^2)q'_{2m+1}(s_0)P''_{2m+1}(t)\left[s_0h-t+\frac{1}{a^2}(s_0^2t+s_0h)\right] \\
&= \sum_{m=0}^{\infty} \frac{B_{2m+1}}{(s_0^2+t^2)}\{L1(s_0,t)+L2(s_0,t)\}
\end{aligned}
\tag{5.9.7}
$$

in which now

$$L1 = (s_0^2+1)q''_{2m+1}(s_0)P'_{2m+1}(t)\left[s_0\left(1+\frac{t^2}{a^2}\right)+\left(1+\frac{1}{a^2}\right)ht\right] \tag{5.9.8}$$

$$L2 = (1-t^2)q'_{2m+1}(s_0)P''_{2m+1}(t)\left[t\left(\frac{s_0^2}{a^2}-1\right)+\left(1+\frac{1}{a^2}\right)hs_0\right]. \tag{5.9.9}$$

Defining the non-dimensional torque coefficient

$$\tau_s = -\frac{T_s}{8\pi\mu\Omega c^3},$$

it follows that τ_s for a partially submerged oblate ellipsoid is

$$\tau_s = \frac{1}{4}\sum_{m=0}^{\infty} B_{2m+1}\int_{-1}^{0}\frac{(s_0^2+1)(1-t^2)}{(s_0^2+t^2)}(s_0't+s_0)\{L1(s_0,t)+L2(s_0,t)\}\,dt, \tag{5.9.10}$$

where $L1(s_0,t)$, $L2(s_0,t)$ are defined in (5.9.8) and (5.9.9), and $s_0'(t)$ is defined in (5.6.12).

5.10 Numerical results

Table 5.1 and Table 5.2 show the results for τ_s when $\lambda = 0$ and $h = 0$ for the prolate and oblate ellipsoids. These are compared with calculations derived from the exact solution of Jeffery (1916) for a rotationally symmetric ellipsoid rotating about its axis of symmetry in infinite fluid. For this problem the velocity v is an even function of z and therefore the boundary condition $\partial v/\partial z = 0$ on $z = 0$ is automatically satisfied. Thus the comparative value of the torque is one half the value of the torque acting on the ellipsoid if rotating in infinite fluid.

We note that k used in Jeffery's analysis is related to a by the formulae: $k = a/(a^2-1)^{1/2}$ for a prolate ellipsoid and $k = a/(a^2+1)^{1/2}$ for an oblate ellipsoid. We further note that in defining the approximate and exact values of τ_s, the dimensionless torque coefficients, c has been used as the length scale.

In calculating τ_s, using the approximate theory, the value of J_{max} was taken to be 20, giving 21 equations to solve for the coefficients. It was found that only the first coefficient was non-zero, in accordance with the exact solution, and the value of the Error-factor E was less than 1.11×10^{-9} for all values of a considered. When $h = 0$, the approximate theory effectively reproduces the exact solution to the problem since there is only one term in either series expression for v. This is further confirmed by Tables 5.1 and 5.2 which indicate that the values of τ_s obtained by both the approximate theory and the exact solutions agree to four decimal places throughout the ranges of values of a considered. In fact, agreement to eight decimal places is achieved over these ranges of values of a.

For the prolate ellipsoid, the exact formula for the dimensionless torque τ_s is, using Jeffery's result, given by

$$\tau_s = \frac{1}{3}\left\{\frac{a}{(a^2-1)} - \frac{1}{2}\ln\left[\frac{a+1}{a-1}\right]\right\}^{-1}. \tag{5.10.1}$$

As $a \to \infty$, the prolate ellipsoid becomes spherical and we find from (5.10.1) that $\tau_s \sim a^3/2$. For $a = 100$ we find that the approximate value of τ_s/a^3 is 0.50002. As $a \to 1$, equation (5.10.1) gives $\tau_s \to 0$.

For the oblate ellipsoid, the exact formula for the dimensionless torque τ_s is found, using Jeffery's result, to be

$$\tau_s = \frac{1}{3}\left\{\tan^{-1}\left(\frac{1}{a}\right) - \left[\frac{a}{a^2+1}\right]\right\}^{-1}. \tag{5.10.2}$$

Again we find that as $a \to \infty$, the ellipsoid becomes a sphere and $\tau_s \sim a^3/2$. As $a \to 0$, the ellipsoid becomes a circular disc of radius c and $\tau_s \to 2/3\pi$, which is one half of the dimensionless torque acting on a circular disc of radius c rotating in infinite fluid about its axis of rotational symmetry. The approximate analysis cannot be used when $a = 0$ but for $a = 0.00001$, we find that $\tau_s = 0.21221$ which agrees with $2/3\pi$ to five decimal places.

Table 5.3 shows results for τ_s and the numerical data for the Error-factor E for a prolate ellipsoid with varying values of h when $\lambda = 0$ and $a = 1.50$. This choice of a corresponds to an ellipsoid with ratio of major to minor semi-axes equal to 1.3416.

The largest value of the Error-factor E is 5.64×10^{-6} which occurs when $h = 1.45$. We are therefore confident that all values of τ_s are correct to four decimal places.

Table 5.4 shows results for τ_s and the numerical data for the Error-factor E for an oblate ellipsoid with varying values of h when $\lambda = 0$ and $a = 0.90$. This choice of a corresponds to an ellipsoid with ratio of minor to major semi-axes equal to 0.6670. The largest value of the Error-factor E is 3.53×10^{-7} which occurs when $h = -0.60$. Again we are confident that all values of τ_s are correct to four decimal places. For the choices of parameters considered for Tables 5.3 and 5.4, the value of J_{max} did not exceed 20 to obtain the stated accuracy. In Table 5.5, we list the values of the coefficients B_{2m+1}, $(m = 0, 1, ..., 9.)$ for the prolate ellipsoid with $a = 1.50$, $h = 1.45$.

The calculations for the prolate and oblate ellipsoids may be repeated for other choices of the parameters a and h for which the theory presented in this chapter is applicable. For those values of the parameters for which the theory cannot be applied, the choice of the length constant c, as the half focal distance of Γ_1, in the spheroidal coordinate systems is not appropriate. In such cases, by choosing, for instance, $c = b_0(a_0^2 - h_0^2)^{1/2}/2a_0$ for the oblate ellipsoid or $c = \frac{1}{2}(a_0 + h_0)$ when $h_0 < 0$ for the prolate ellipsoid, the solutions of the problems may be found by following a procedure similar to that set out in this chapter. The velocity is again expressed by (5.3.14) or (5.7.7) and expressions for τ_s are similar. However, for such choices of c, the parameter s is then generally not a constant on Γ_1 when $h_0 = 0$.

a	τ_s	*Exact*
2.00	2.8403	2.8403
1.90	2.3321	2.3321
1.80	1.8812	1.8812
1.70	1.4847	1.4847
1.60	1.1397	1.1397
1.50	0.8433	0.8433
1.40	0.5926	0.5926
1.30	0.3851	0.3851
1.20	0.2181	0.2181
1.10	0.0897	0.0897

Table 5.1: The approximate and exact values of τ_s when $\lambda = 0$ and $h = 0$ for the prolate ellipsoid.

a	τ_s	*Exact*
2.00	5.2372	5.2372
1.90	4.6085	4.6085
1.80	4.0370	4.0370
1.70	3.5197	3.5197
1.60	3.0536	3.0536
1.50	2.6358	2.6358
1.40	2.2633	2.2633
1.30	1.9332	1.9332
1.20	1.6426	1.6426
1.10	1.3884	1.3884
1.00	1.1680	1.1680
0.90	0.9783	0.9783
0.80	0.8165	0.8165
0.70	0.6799	0.6799
0.60	0.5657	0.5657
0.50	0.4714	0.4714
0.40	0.3943	0.3943
0.30	0.3320	0.3320
0.20	0.2822	0.2822
0.10	0.2429	0.2429
0.00001	0.2122	0.2122

Table 5.2: The approximate and exact values of τ_s when $\lambda = 0$ and $h = 0$ for the oblate ellipsoid.

h	τ_s	E
1.45	1.9994	5.64×10^{-6}
1.40	1.9840	4.52×10^{-7}
1.30	1.7972	3.57×10^{-8}
1.20	1.7155	3.24×10^{-8}
1.10	1.5857	3.13×10^{-8}
1.00	1.5085	2.98×10^{-8}
0.90	1.4774	2.93×10^{-8}
0.80	1.4486	2.81×10^{-8}
0.70	1.4095	2.65×10^{-8}
0.60	1.3638	2.43×10^{-8}
0.50	1.3049	2.25×10^{-8}
0.40	1.2317	2.10×10^{-8}
0.30	1.1479	1.95×10^{-8}
0.20	1.0525	1.74×10^{-8}
0.10	0.9501	1.58×10^{-8}
0.00	0.8433	1.11×10^{-9}
-0.10	0.7345	1.64×10^{-8}
-0.20	0.6106	1.81×10^{-8}
-0.30	0.5253	2.25×10^{-8}
-0.40	0.4271	3.77×10^{-8}

Table 5.3: The computed values of τ_s for the prolate ellipsoid at $\lambda = 0$, when $a = 1.50$ and $a/b = 1.34$.

h	τ_s	E
0.60	3.3143	1.81×10^{-7}
0.50	2.8380	1.60×10^{-8}
0.40	2.2932	1.53×10^{-8}
0.30	1.8703	1.44×10^{-8}
0.20	1.5207	1.38×10^{-8}
0.10	1.2263	1.43×10^{-8}
0.00	0.9783	1.11×10^{-9}
-0.10	0.7709	1.81×10^{-8}
-0.20	0.5993	1.85×10^{-8}
-0.30	0.4616	1.97×10^{-8}
-0.40	0.3566	2.33×10^{-8}
-0.50	0.2854	2.56×10^{-8}
-0.60	0.2777	3.53×10^{-7}

Table 5.4: The computed values of τ_s for the oblate ellipsoid at $\lambda = 0$ when $a = 0.90$ and $a/b = 0.67$.

B_{2m+1}
2.8031120109651
-1.2994279729302
$-2.1588634933057 \times 10^{-2}$
$8.8681696371099 \times 10^{-5}$
$-4.6233949807681 \times 10^{-7}$
$-1.4860945830394 \times 10^{-7}$
$1.1343838320007 \times 10^{-8}$
$-4.3401900630235 \times 10^{-10}$
$8.8711061978190 \times 10^{-12}$
$-7.6583246340209 \times 10^{-14}$

Table 5.5: The computed values of B_{2m+1}, $(m = 0, 1, ..., 9)$ for the prolate ellipsoid with $a = 1.50$ and $h = 1.45$.

Chapter 6

NON-AXISYMMETRIC STOKES FLOW

6.1 Introduction

For non-axisymmetric Stokes flow, as pointed out in Chapter Two, one is forced to accept that the solution will in general involve the determination of three independent harmonic functions. These three functions are inter-related in a complicated way through the boundary conditions and asymptotic conditions, if the fluid is unbounded externally. If there were only two quasi-harmonic functions to be found then there is the possibility of finding these functions *sequentially*. In this chapter it will be shown that it is possible to determine the solutions for some basic Stokes flows by seeking solutions involving two harmonic functions which can be determined sequentially. The solutions of more complicated flow can then be obtained by superposition since the Stokes equations are linear. Although it may seem that the solutions then determined by superposition involve more than three independent quasi-harmonic functions, this cannot of course be true. In like manner, the solutions obtained by O'Neill (1993) to asymmetric Stokes flow problems appeared to involve four harmonic functions, but through the equation of continuity, the linear dependence of these solutions is established, and although it would be a matter of some complexity to identify the three independent harmonic functions within O'Neill's solutions, this must evidently be possible in principle.

6.2 Streaming flow past an axisymmetric body

Consider an axisymmetric body whose axis of symmetry lies along the z-axis and has fore-aft symmetry about the plane $z = 0$, such as a sphere or ellipsoid.

If the body is at rest and the fluid streams past the body with velocity $U\,\hat{\mathbf{i}}$ at infinity, then the fluid motion will not be axisymmetric but will be such that the dependence of the velocity components and pressure on the azimuthal angle ϕ of a system of cylindrical polar coordinates (ρ, ϕ, z) is as follows

$$\begin{aligned} q_\rho &= u \cos \phi, \\ q_\phi &= v \sin \phi, \\ q_z &= w \cos \phi, \\ p &= \bar{p} \cos \phi \end{aligned} \tag{6.2.1}$$

where u, v, w and $\bar{p}$ depend only on the coordinates ρ and z.

A solution of equations (2.4.2) and (2.4.3) is given by

$$\mathbf{q} = -x\nabla F + F\,\hat{\mathbf{i}}, \tag{6.2.2}$$

$$p = -2\mu\frac{\partial F}{\partial x}, \tag{6.2.3}$$

provided that

$$L_1 F = \frac{\partial^2 F}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial F}{\partial \rho} + \frac{\partial^2 F}{\partial z^2} = 0. \tag{6.2.4}$$

This solution is evidently of the correct form to give u, v, w, and $\bar{p}$ as in (6.2.1) and therefore

$$u = -\rho\frac{\partial F}{\partial \rho} + F, \tag{6.2.5}$$

$$v = -F, \tag{6.2.6}$$

$$w = -\rho\frac{\partial F}{\partial z} \tag{6.2.7}$$

and

$$\bar{p} = -2\mu\frac{\partial F}{\partial \rho}. \tag{6.2.8}$$

If we now regard ρ and z as dimensionless with respect to some length scale a associated with the body and u, v, w dimensionless with respect to U, and $\bar{p}$ dimensionless with respect to $\mu U/a$, the boundary conditions require that

$$u = v = w = 0, \quad (\textit{on the body}) \tag{6.2.9}$$

and the asymptotic condition as $r \to \infty$ requires that

$$u = 1, \quad v = -1, \quad w = 0. \quad (r \to \infty) \tag{6.2.10}$$

Let us assume that the equation of the body is given by the conformal mapping

$$z + i\rho = f(\xi + i\eta) \tag{6.2.11}$$

with $f'(\xi + i\eta) \neq 0$ in the flow region and $\xi = \textit{constant} = \alpha$ defines a meridian curve Γ of the body.

A solution of (6.2.5) to (6.2.8) satisfying (6.2.10) is clearly $F = 1$. However to satisfy the boundary condition (6.2.9) require

$$F = 0, \quad \frac{\partial F}{\partial \rho} \quad = \quad \frac{\partial F}{\partial z} = 0 \quad (\xi = \alpha). \tag{6.2.12}$$

The last two equations are equivalent to

$$\frac{\partial F}{\partial \xi} = \frac{\partial F}{\partial \eta} = 0 \quad (\xi = \alpha). \tag{6.2.13}$$

Now, on writing $F = 1 + F_1$, the boundary and asymptotic conditions will be satisfied if

$$\begin{aligned} F_1 &= -1, \quad (\xi = \alpha), \\ \frac{\partial F_1}{\partial \xi} &= 0, \quad (\xi = \alpha), \qquad (6.2.14) \\ F_1 &= o(1), \quad (r \to \infty). \qquad (6.2.15) \end{aligned}$$

Conditions (6.2.14) and (6.2.15) impose three conditions on the axisymmetric harmonic function $F_1(\rho, z)$ which, in general, cannot be satisfied since F_1 will be completely determined either by equations (6.2.14) or one of (6.2.14) together with (6.2.15). This is to be expected since, so far, we have only considered a representation of the solution of the Stokes equations involving *one* harmonic function.

If we choose to solve for F_1 so that the first of (6.2.14) and (6.2.15) are satisfied, then the solution F_1 represents the fundamental axisymmetric harmonic function which has a constant value on $\xi = \alpha$ and decays to zero at infinity. Physically it can be identified with the electrostatic potential field produced by a conducting body $\xi = \alpha$ which has potential equal to -1.

We next find another solution of the Stokes equations which when superposed with that derived from F will satisfy all boundary conditions. We observe that the flow associated with the solution involving F gives rise to a pressure. Thus, in accord with equation (2.4.7) or Lamb's general solution, we may expect that the complementary solution to that involving F will be isobaric, or having at most constant pressure. Consider

$$\mathbf{q} = \nabla \left(\frac{H}{\rho} \cos \phi \right) \tag{6.2.16}$$

This is a solution of the Stokes equations and the equation of continuity which gives rise to at most a constant pressure if

$$L_{-1} H = 0. \tag{6.2.17}$$

From equation (6.2.16), the velocity components are

$$u = \frac{\partial}{\partial \rho}\left(\frac{H}{\rho}\right), \tag{6.2.18}$$

$$v = -\left(\frac{H}{\rho^2}\right), \tag{6.2.19}$$

$$w = \frac{1}{\rho}\frac{\partial H}{\partial z}. \tag{6.2.20}$$

To obtain $u = v = w = 0$ on $\xi = \alpha$ requires

$$H = \frac{\partial H}{\partial z}, \quad (\xi = \alpha) \tag{6.2.21}$$

and the asymptotic condition of zero velocity as $r \to \infty$, if (6.2.18) to (6.2.20) were to be the complementary velocity field to (6.2.5) to (6.2.7) with F already determined as described above, would require

$$H = o(r^2) \quad (r \to \infty) \tag{6.2.22}$$

However, as with the function F, the function H is uniquely determined by both of equations (6.2.21), that is $H \equiv 0$, which is unacceptable, or by one of (6.2.21) taken with (6.2.22). Therefore we proceed as follows: (1) determine $F(\rho, z)$ and (2) sequentially determine $H(\rho, z)$ so as to achieve zero velocity on the body $\xi = \alpha$. The velocity which is produced at infinity cannot then be prescribed. The velocity will be zero on $\xi = \alpha$, when the solutions given by (6.2.5) to (6.2.7) and (6.2.18) to (6.2.20) are combined to give

$$F = H = 0 \quad (\xi = \alpha) \tag{6.2.23}$$

$$\frac{\partial H}{\partial \xi} - \rho^2 \frac{\partial F}{\partial \xi} = 0 \quad (\xi = \alpha) \tag{6.2.24}$$

Furthermore, we note that $L_{-1}(\rho^2) = 0$ and the solution $H = k\rho^2$, where k is a constant, gives rise to the velocity field

$$u = k, \quad v = -k, \quad w = 0, \tag{6.2.25}$$

which is the uniform stream $\mathbf{q} = k\,\hat{\mathbf{i}}$. Thus by writing $H = k(\rho^2 + H_1)$, the problem is solved for a uniform stream $(k+1)\,\hat{\mathbf{i}}$ flowing past the body at rest if

$$F_1 = -1, \quad H_1 = -\rho^2 \quad (\xi = \alpha) \tag{6.2.26}$$

$$k\frac{\partial H_1}{\partial \xi} = \rho^2 \frac{\partial F_1}{\partial \xi} - 2k\rho\frac{\partial \rho}{\partial \xi} \quad (\xi = \alpha) \tag{6.2.27}$$

where

$$\begin{aligned} L_1 F_1 &= 0 \\ L_{-1} H_1 &= 0 \end{aligned} \tag{6.2.28}$$

provided that $H_1 = o(r^2)$ as $r \to \infty$. we note also that if $H_1 = \rho^2 H_3$, then

$$\begin{aligned} L_{-1} H_1 &= \rho^2 \left[\frac{\partial^2 H_3}{\partial \rho^2} + \frac{3}{\rho}\frac{\partial H_3}{\partial \rho} + \frac{\partial^2 H_3}{\partial z^2} \right] \\ &= \rho^2 L_3 H_3 = 0, \end{aligned} \tag{6.2.29}$$

provided that $L_3 H_3 = 0$. For such a solution H_3,

$$H = k\rho^2(1 + H_3), \tag{6.2.30}$$

and equations (6.2.26) and (6.2.27) reduce to

$$F_1 = H_3 = -1, \quad (\xi = \alpha) \tag{6.2.31}$$

$$\frac{\partial F_1}{\partial \xi} - k\frac{\partial H_3}{\partial \xi} = 0, \quad (\xi = \alpha) \tag{6.2.32}$$

and $H_3 = o(1)$ as $r \to \infty$. Thus the boundary and asymptotic condition on F_1 and H_3 are the same.

6.3 Examples of solutions

6.3.1 Sphere

The appropriate solutions are

$$\begin{aligned} F_1 &= -\frac{1}{r}, \\ H_3 &= -\frac{1}{r^3} \end{aligned} \tag{6.3.1}$$

and, to satisfy (6.2.32), require $k = 1/3$. Thus the velocity field derived from

$$F = 1 - \frac{1}{r}, \quad H = \frac{\rho^2}{3}\left(1 - \frac{1}{r^3}\right), \tag{6.3.2}$$

satisfies the boundary conditions on the sphere and is consistent with a uniform stream $(4/3)\hat{\mathbf{i}}$ flowing at infinity. Thus the composite velocity field arising from $(3/4)F$ and $(1/4)H$, with F and H defined by (6.3.2), gives a uniform stream $\hat{\mathbf{i}}$ at infinity.

The pressure in the fluid is accordingly

$$\begin{aligned} p &= -2\mu\frac{\partial F}{\partial \rho}\cos\phi \\ &= -2\mu\frac{\rho}{r^3}\cos\phi \\ &= -2\mu\frac{x}{r^3}. \end{aligned} \tag{6.3.3}$$

6.3.2 Stokeslet

Consider the Stokeslet $k\hat{\mathbf{i}}$ located at the origin. From (2.4.25)

$$\begin{aligned} u &= -\frac{k}{8\pi\mu}\left[\frac{\rho^2}{r^3} + \frac{1}{r}\right] \\ v &= \frac{k}{8\pi\mu}\left[\frac{1}{r}\right] \\ w &= -\frac{k}{8\pi\mu}\left[\frac{\rho^2}{r^3}\right] \end{aligned} \tag{6.3.4}$$

and

$$p = \frac{k}{4\pi}\frac{z}{r^3}. \tag{6.3.5}$$

The combination of flows of types (6.2.2) and (6.2.16) would accordingly be

$$F = -\frac{k}{8\pi\mu}\frac{1}{r}, \quad H = 0, \tag{6.3.6}$$

to produce the velocity and pressure given by (6.3.4) and (6.3.5). The solution given by (6.3.6) is particularly important because it shows that if $H_3 = o(1)$ as $r \to \infty$, then the force exerted on a general body $\xi = \alpha$ will be $6\pi\mu f\hat{\mathbf{i}}$ when placed in a uniform stream $\hat{\mathbf{i}}$ at infinity if the asymptotic structure of F_1 as $r \to \infty$ is

$$(1+k)^{-1}F_1 \sim -\frac{3}{4}f\frac{1}{r} + o\left(\frac{1}{r}\right) \tag{6.3.7}$$

This follows immediately from (2.4.21) when it is remembered that the uniform stream, being a rigid body motion, exerts no net force on the body or a sphere Σ of arbitrary radius enclosing the body.

In the case of the sphere, whose solution is given above, $f = 1$, giving $6\pi\mu$ as the force acting on the sphere. The force may of course be obtained by integration but identification of the Stokeslet term in the far field asymptotic structure of the function F_1 produces the answer far more rapidly. In fact, from equation (6.3.7), it can be seen that

$$\frac{3}{4}f = -\lim_{r\to\infty}\left[\frac{rF_1}{(1+k)}\right] \tag{6.3.8}$$

which is a formula analogous to that derived by Payne and Pell (1960) for *axisymmetric* streaming past an axisymmetric body. Their formula is

$$\frac{3}{4}f = \lim_{r\to\infty}\left[\frac{r\psi_1}{(1+k)}\right]. \tag{6.3.9}$$

where $(1/2)\rho^2 - \psi_1$ is the stream function for a uniform stream $\hat{\mathbf{k}}$ at infinity, and since the stream function for the Stokeslet $6\pi\mu\hat{\mathbf{k}}$ at the origin is

$$\frac{-3}{4}\frac{\rho^2}{r}, \tag{6.3.10}$$

equation (6.3.9) evidently identifies the strength of the Stokeslet term in ψ_1 as $r \to \infty$.

6.3.3 Prolate ellipsoid

Prolate ellipsoidal coordinates are defined by the conformal transformation

$$z + i\rho \quad = \quad c\cosh(\xi + i\eta) \tag{6.3.11}$$

or equivalently

$$\begin{aligned} z &= cst, \\ \rho &= c(s^2-1)^{1/2}(1-t^2)^{1/2} \end{aligned} \qquad (6.3.12)$$

where c is a constant and $s = \cosh\,\xi$, $t = \cos\eta$ with $s \geq 1$, $|t| \leq 1$. The surface $s = \lambda$ corresponds to the ellipsoid

$$\frac{z^2}{c^2\lambda^2} + \frac{\rho^2}{c^2(\lambda^2-1)} = 1, \qquad (6.3.13)$$

with the lengths of the semi-axes being $c\,\lambda$ and $c\,\sqrt{\lambda^2-1}$ respectively, and since $\lambda > \sqrt{\lambda^2-1}$, the ellipsoid is clearly prolate.

The appropriate solutions for F_1 and H_3 in prolate ellipsoidal coordinates satisfying equation (6.2.31) are

$$F_1 = -\left[\frac{Q_0(s)}{Q_0(\lambda)}\right], \qquad (6.3.14)$$

$$H_3 = -\left[\frac{Q_0'(s)}{Q_0'(\lambda)}\right] \qquad (6.3.15)$$

and to satisfy (6.2.32), we find that

$$k = \left[\frac{Q_0'(s)}{Q_0(\lambda)}\right]\left[\frac{Q_1'(s)}{Q_1''(\lambda)}\right] \qquad (6.3.16)$$

As $r \to \infty$ then $s \to \infty$ and

$$\begin{aligned} Q_n(s) &= o\left(s^{-(n+1)}\right) \\ &= o\left(r^{-(n+1)}\right) \end{aligned} \qquad (6.3.17)$$

Thus

$$H_3 = o(1) \qquad (6.3.18)$$

as $r \to \infty$. Accordingly, since $Q_0(s) \sim s^{-1} \sim r^{-1}$ as $r \to \infty$, equation (6.3.8) gives

$$\frac{3}{4}f = \frac{Q_1''(\lambda)}{[Q_0'(\lambda)Q_1'(\lambda) + Q_0(\lambda)Q_1''(\lambda)]} \qquad (6.3.19)$$

However

$$Q_0(\lambda) = \frac{1}{2}\ln\left[\frac{(\lambda+1)}{(\lambda-1)}\right],$$

$$
\begin{aligned}
Q_0'(\lambda) &= -\left[\frac{1}{(\lambda^2-1)}\right], \\
Q_1(\lambda) &= \frac{1}{2}\lambda \ln\left[\frac{(\lambda+1)}{(\lambda-1)}\right] - 1, \\
Q_1'(\lambda) &= \frac{1}{2}\ln\left[\frac{(\lambda+1)}{(\lambda-1)}\right] - \left[\frac{\lambda}{(\lambda^2-1)}\right], \\
Q_1''(\lambda) &= \left[\frac{2}{(\lambda^2-1)^2}\right],
\end{aligned}
\tag{6.3.20}
$$

and it follows that

$$
f = \frac{16}{3}\left\{2\lambda + (3-\lambda^2)\ln\left[\frac{(\lambda+1)}{(\lambda-1)}\right]\right\}^{-1}. \tag{6.3.21}
$$

The force acting on the ellipsoid is accordingly $6\pi\mu a_1 \lambda^{-1} f \hat{\mathrm{i}}$, with a_1 the major semi-axis of the ellipsoid, which agrees with the result of Oberbeck (1890) when the general ellipsoid becomes a prolate ellipsoid of revolution. By letting $\lambda \to \infty$, the result can then be recovered for the sphere since the ratio of major and minor semi-axes tends to unity, in this case equation (6.3.21) gives

$$
\begin{aligned}
\lambda^{-1} f &= \frac{16}{3\lambda}\left\{2\lambda + (3-\lambda^2)\left[\frac{2}{\lambda} + \frac{2}{3\lambda^3} + o\left(\frac{1}{\lambda^5}\right)\right]\right\}^{-1} \\
&= 1 + o(\lambda^{-2})
\end{aligned}
\tag{6.3.22}
$$

giving the correct result for the force in the limit $\lambda \to \infty$.

6.3.4 Oblate ellipsoid

Oblate ellipoidal coordinates are defined by

$$
z + i\rho = c\sinh(\xi + i\eta) \tag{6.3.23}
$$

which gives

$$
\begin{aligned}
z &= cst, \\
\rho &= c(s^2+1)^{1/2}(1-t^2)^{1/2},
\end{aligned}
\tag{6.3.24}
$$

where c is a constant, and $s = \sinh\,\xi$, $t = \cos\eta$ with $s \geq 0$, $|t| \leq \pi$. The surface $s = \lambda$ now corresponds to the ellipsoid

$$
\frac{z^2}{c^2\lambda^2} + \frac{\rho^2}{c^2(\lambda^2+1)} = 1, \tag{6.3.25}
$$

with the lengths of the semi-axes being $c\,\lambda$ and $c\,\sqrt{\lambda^2-1}$ respectively, and since $\lambda > \sqrt{\lambda^2-1}$, the ellipsoid is now oblate.

The appropriate solutions for F_1 and H_3 in oblate ellipsoidal coordinates satisfying equation (6.2.31) are

$$F_1 = -\left[\frac{q_0(s)}{q_0(\lambda)}\right], \tag{6.3.26}$$

$$H_3 = -\left[\frac{q_0'(s)}{q_0'(\lambda)}\right] \tag{6.3.27}$$

with $q_n(s) = i^{(n+1)}Q_n(is)$ and the prime denoting differentiation with respect to s. To satisfy (6.2.32), we find that

$$k = \left[\frac{q_0'(s)}{q_0(\lambda)}\right]\left[\frac{q_1'(s)}{q_1''(\lambda)}\right]. \tag{6.3.28}$$

However

$$3\,q_n'(s) \sim -2s^{-2} \sim -2r^{-2} \tag{6.3.29}$$

as $r \to \infty$, or $s \to \infty$, and it follows that as $r \to \infty$,

$$H_3 = o(1) \tag{6.3.30}$$

and, since $q_0(s) \sim s^{-1} \sim r^{-1}$ as $r \to \infty$, equation (6.3.8) gives

$$\frac{3}{4}f = \frac{q_1''(\lambda)}{[q_0'(\lambda)q_1'(\lambda) + q_0(\lambda)q_1''(\lambda)]}. \tag{6.3.31}$$

However

$$\begin{aligned} q_0(\lambda) &= \tan^{-1}\left[\frac{1}{\lambda}\right], \\ q_0'(\lambda) &= -\left[\frac{1}{(\lambda^2+1)}\right], \\ q_1(\lambda) &= 1 - \lambda\tan^{-1}\left[\frac{1}{\lambda}\right], \\ q_1'(\lambda) &= -\tan^{-1}\left[\frac{1}{\lambda}\right] + \left[\frac{(\lambda)}{(\lambda^2+1)}\right], \\ q_1''(\lambda) &= \left[\frac{2}{(\lambda^2+1)^2}\right], \end{aligned} \tag{6.3.32}$$

so it follows from equation (6.3.31) that

$$f = \frac{16}{3}\left\{3(1+\lambda^2)\tan^{-1}\left[\frac{1}{\lambda}\right] - \lambda\right\}. \tag{6.3.33}$$

The force acting on the oblate ellipsoid is therefore $6\pi\mu a_1(\lambda^2+1)^{-1/2}f\hat{\mathbf{i}}$, with a_1 the major semi-axis of the ellipsoid, which agrees with the result of Oberbeck when the general ellipsoid is an oblate ellipsoid of revolution. By letting $\lambda \to \infty$, the result can again be recovered for the sphere since the ratio of major and minor semi-axes tends to unity. From equation (6.3.33),

$$\begin{aligned}(\lambda^2+1)^{-1/2}f &= \frac{16}{3\lambda}\left\{2\lambda+(3-\lambda^2)\left[\frac{2}{\lambda}+\frac{2}{3\lambda^3}+o\left(\frac{1}{\lambda^5}\right)\right]\right\}^{-1} \\ &= 1+o(\lambda^{-2}) \end{aligned} \tag{6.3.34}$$

Letting $\lambda \to 0$ means the ellipsoid becomes a circular disk of radius a_1. The limiting result for the force is now $(32\mu a_1/3)$ along $\hat{\mathbf{i}}$, which is the well known result.

For the case of a body translating with velocity $-\hat{\mathbf{i}}$ in fluid at rest at infinity, the solution can be found simply by subtracting the solutions for a uniform stream $\hat{\mathbf{i}}$ and $k\hat{\mathbf{i}}$ from the F and H functions found above. Thus for this problem, we have

$$\begin{aligned} F &= F_1, \\ H &= k\,\rho^2\,H_3 \end{aligned} \tag{6.3.35}$$

Chapter 7

Asymmetric Translation of an Axisymmetric Body

7.1 Introduction

In Chapter 6 we investigated how the solution of the problem of the non-axisymmetic Stokes flow about an axisymmetric body, although involving *three* independent quasi-harmonic functions, could be solved by the superposition of solutions determined sequentially and each involving only *two* quasi-harmonic functions. There is of course one axisymmetric body, namely the ellipsoid of revolution, for which it is known that if the body translates perpendicular to its axis of symmetry then the velocity and pressure fields are expressible in terms of just two quasi-harmonic functions. This follows from the work of Oberbeck (1890) who solved the problem of the translation of a general ellipsoid. This remarkable solution uses specific geometrical properties of the general ellipsoid, which has prevented a *generic* type of solution to be found for a general body shape by adaptation and generalization of Oberbeck's analysis.

However, the fact that Oberbeck's solution will give, as special cases, the solution for such specific axisymmetric body shapes as the sphere, spheroid and disk in terms of two quasi-harmonic functions leads one to conjecture whether the problem of the translation of a general body of revolution along an axis perpendicular to its axis of symmetry is soluble in terms of just two quasi-harmonic functions. In this Chapter we consider this conjecture for the class of axisymmetric bodies possessing fore-aft symmetry about a plane perpendicular to the axis of symmetry. Such bodies include the sphere, spheroid, disk and symmetric lens.

7.2 Non-axisymmetric Stokes Flow

The equations governing the flow are

$$\nabla p = \mu \nabla^2 \mathbf{q}, \tag{7.2.1}$$

where $\mathbf{q}$ denotes the fluid velocity, p is the fluid pressure and μ is the coefficient of dynamic viscosity of the fluid; and the equation of continuity is

$$\nabla . \mathbf{q} = 0. \tag{7.2.2}$$

These equations are satisfied identically if

$$\mathbf{q} = -x \nabla F + \nabla \left(\frac{H}{\rho} \cos \phi \right) + F \hat{\mathbf{i}}, \tag{7.2.3}$$

where $L_1 F = 0$, $L_{-1} H = 0$ and the pressure is given by

$$\begin{aligned} p &= -2\mu \frac{\partial F}{\partial x} + \text{constant} \\ &= -2\mu \frac{\partial F}{\partial \rho} \cos \phi + \text{constant}. \end{aligned} \tag{7.2.4}$$

If we consider $\mathbf{q}$ to be of the form

$$\mathbf{q} = u \cos \phi \, \hat{\rho} + v \sin \phi \, \hat{\phi} + w \cos \phi \, \hat{\mathbf{k}}, \tag{7.2.5}$$

with u, v and w independent of ϕ, the solution (7.2.3) is evidently of the correct form to give u, v, w, and therefore the components of (7.2.3) are

$$\begin{aligned} u &= -\rho \frac{\partial F}{\partial \rho} + F + \frac{\partial}{\partial \rho} \left(\frac{H}{\rho} \right) \\ &= -\rho \frac{\partial F}{\partial \rho} + F - \frac{H}{\rho^2} + \frac{1}{\rho} \frac{\partial H}{\partial \rho} \\ &= -\frac{1}{\rho} \frac{\partial}{\partial \rho} \left[\rho^2 F \right] + 3F - \frac{H}{\rho^2} + \frac{1}{\rho} \frac{\partial H}{\partial \rho} \\ &= -\frac{1}{\rho} \frac{\partial}{\partial \rho} \left[\rho^2 F - H \right] + \frac{1}{\rho^2} \left[\rho^2 F - H \right] + 2F, \end{aligned} \tag{7.2.6}$$

$$v = -F - \frac{H}{\rho^2}$$

$$= \frac{1}{\rho^2}\left[\rho^2 F - H\right] - 2F, \qquad (7.2.7)$$

$$w = -\rho\frac{\partial F}{\partial z} + \frac{1}{\rho}\frac{\partial H}{\partial z}$$
$$= -\frac{1}{\rho}\frac{\partial}{\partial z}\left[\rho^2 F - H\right]. \qquad (7.2.8)$$

Since

$$L_1 F = L_{-1} H = 0, \qquad (7.2.9)$$

it follows that

$$L_{-1}^2\left[\rho^2 F - H\right] = 0. \qquad (7.2.10)$$

If we write

$$\Phi = \rho^2 F - H, \qquad (7.2.11)$$

and

$$\psi = 2F, \qquad (7.2.12)$$

then u, v, w and $\bar{p}$ are expressible as

$$u = -\frac{1}{\rho}\frac{\partial \Phi}{\partial \rho} + \frac{\Phi}{\rho^2} + \psi$$
$$= -\frac{\partial}{\partial \rho}\left(\frac{\Phi}{\rho}\right) + \psi, \qquad (7.2.13)$$
$$v = \frac{\Phi}{\rho^2} - \psi, \qquad (7.2.14)$$
$$w = -\frac{\partial}{\partial z}\left(\frac{\Phi}{\rho}\right) \qquad (7.2.15)$$

and

$$p = -\mu\frac{\partial \psi}{\partial \rho}\cos\phi + \text{constant}. \qquad (7.2.16)$$

Thus all three velocity components can be expressed in terms of two scalar functions Φ and ψ which satisfy

$$L_{-1}^2\Phi = L_1\psi = 0. \qquad (7.2.17)$$

7.2.1 Body translating with velocity $\hat{\mathbf{i}}$

The boundary conditions are

$$\begin{aligned} u &= 1, \\ v &= -1, \\ w &= 0 \end{aligned} \tag{7.2.18}$$

on the body and $u, v, w \rightarrow 0$ as $r \rightarrow \infty$. Thus on the body

$$\frac{\partial}{\partial \rho}\left(\frac{\Phi}{\rho}\right) = \psi - 1, \tag{7.2.19}$$

$$\frac{\partial}{\partial z}\left(\frac{\Phi}{\rho}\right) = 0, \tag{7.2.20}$$

$$\frac{\Phi}{\rho^2} = \psi - 1. \tag{7.2.21}$$

If the cylindrical polar coordinates are expressible as

$$z + i\rho = f(\xi + i\eta) \tag{7.2.22}$$

with $\xi = \alpha$ defining the body, equations (7.2.19), (7.2.20) and (7.2.21) are equivalent to

$$\begin{aligned} \frac{\partial}{\partial \xi}\left(\frac{\Phi}{\rho}\right) &= \frac{\Phi}{\rho^2}\frac{\partial \rho}{\partial \xi}, \quad (\xi = \alpha) \\ \frac{\partial}{\partial \eta}\left(\frac{\Phi}{\rho}\right) &= \frac{\Phi}{\rho^2}\frac{\partial \rho}{\partial \eta}, \quad (\xi = \alpha) \\ \frac{\Phi}{\rho^2} &= \psi - 1. \quad (\xi = \alpha) \end{aligned} \tag{7.2.23}$$

We therefore note that the boundary value problem for Φ is uncoupled from that for ψ. Thus if the solution for Φ can be found, then ψ can be determined by solving $L_1\psi = 0$ with the third of the conditions given in equation (7.2.23) the boundary condition on $\xi = \alpha$, since the value of Φ on the boundary would be known.

The force is determined very simply from the coefficient of the Stokeslet term in the velocity field at infinity. For a Stokeslet applying a force $8\pi\mu\nu\hat{\mathbf{i}}$ to the fluid, the velocity field is given by

$$u = \nu\left[\frac{\rho^2}{r^3} + \frac{1}{r}\right],$$

$$\begin{aligned} v &= -\nu\left[\frac{1}{r}\right], \\ w &= \nu\left[\frac{\rho^2}{r^3}\right]. \end{aligned} \tag{7.2.24}$$

This velocity is produced when

$$\begin{aligned} F &= \nu\left[\frac{1}{r}\right], \\ H &= 0. \end{aligned} \tag{7.2.25}$$

Therefore

$$\begin{aligned} \Phi &= \nu\left[\frac{\rho^2}{r}\right], \\ \psi &= 2\nu\left[\frac{1}{r}\right], \\ p &= 2\nu\mu\left[\frac{\rho}{r^3}\right]\cos\phi. \end{aligned} \tag{7.2.26}$$

Note that the Φ function for the Stokeslet $8\pi\mu\nu\hat{\mathbf{i}}$ is the same as the stream function for an axisymmetric Stokeslet $8\pi\mu\nu\hat{\mathbf{k}}$. Thus as $r \to \infty$, we expect the leading terms in Φ and ψ to be such that

$$\begin{aligned} \Phi &\sim \nu\left[\frac{\rho^2}{r}\right], \\ \psi &\sim 2\nu\left[\frac{1}{r}\right]. \end{aligned} \tag{7.2.27}$$

This method of determining the force is a generalisation of the celebrated method of Payne and Pell (1960) for determining the force on a body in an axisymmetric stream.

7.2.2 Solution for Φ

We have to solve

$$L_{-1}^2\Phi = 0 \tag{7.2.28}$$

with the boundary conditions

$$\begin{aligned} \frac{\partial}{\partial\xi}\left(\frac{\Phi}{\rho}\right) &= \frac{\Phi}{\rho^2}\frac{\partial\rho}{\partial\xi}, \quad (\xi = \alpha) \\ \frac{\partial}{\partial\eta}\left(\frac{\Phi}{\rho}\right) &= \frac{\Phi}{\rho^2}\frac{\partial\rho}{\partial\eta}. \quad (\xi = \alpha) \end{aligned} \tag{7.2.29}$$

The second equation gives

$$\frac{\partial}{\partial\eta}\left(\frac{\Phi}{\rho}\right) - \frac{\Phi}{\rho^2}\frac{\partial\rho}{\partial\eta} = \rho\frac{\partial}{\partial\eta}\left(\frac{\Phi}{\rho^2}\right) = 0. \qquad (7.2.30)$$

Therefore

$$\Phi = \mathrm{C}\rho^2 \quad (\xi = \alpha) \qquad (7.2.31)$$

where C is a constant.

The boundary conditions on Φ are therefore

$$\begin{aligned} \Phi &= \mathrm{C}\rho^2, \quad (\xi = \alpha) \\ \frac{\partial\Phi}{\partial\xi} &= 2\mathrm{C}\rho\frac{\partial\rho}{\partial\xi}, \quad (\xi = \alpha) \end{aligned} \qquad (7.2.32)$$

with the asymptotic condition

$$\Phi \sim \nu\left[\frac{\rho^2}{r}\right]. \quad (r \to \infty) \qquad (7.2.33)$$

The solution to this boundary value problem is simply the stream function for *axisymmetric translation* of the body with velocity 2 C $\hat{\mathrm{k}}$ and consequently C is expressible in terms of ν.

7.2.3 Solution for ψ

We now have to solve

$$L_1\psi = 0 \qquad (7.2.34)$$

such that

$$\begin{aligned} \psi &= 1 + \frac{\Phi}{\rho^2} \\ &= 1 + C, \quad (\xi = \alpha) \\ \psi &\sim 2\nu\left[\frac{1}{r}\right]. \quad (r \to \infty) \end{aligned} \qquad (7.2.35)$$

This solution will give rise to a second relation between C and ν which, together with the relation obtained from the solution of (7.2.32), will yield the value of ν and hence the drag on the body.

The solution of (7.2.32) and (7.2.35) provides the *exact* solution of the asymmetric translation problem for an axisymmetric body.

7.3 Examples of asymmetric translation

7.3.1 Prolate ellipsoid

We need to solve

$$L_{-1}^2\Phi = 0 \tag{7.3.1}$$

so that

$$\begin{aligned}\frac{\Phi}{\rho^2} &= C, \quad (s=\lambda)\\ \frac{\partial}{\partial s}\left(\frac{\Phi}{\rho^2}\right) &= 0, \quad (s=\lambda)\end{aligned} \tag{7.3.2}$$

and

$$\Phi \sim \nu\left[\frac{\rho^2}{r}\right]. \quad (r\to\infty) \tag{7.3.3}$$

Here prolate ellipsoidal coordinates

$$\begin{aligned}\rho &= (s^2-1)^{1/2}(1-t^2)^{1/2},\\ z &= st,\end{aligned} \tag{7.3.4}$$

are used, where $1 \le s$ and $-1 \le t \le 1$. A suitable solution is

$$\Phi = C\rho^2\left[AQ_0(s) + BQ_1'(s)\right] \tag{7.3.5}$$

where Q_0 and Q_1' are defined in the Chapter 6. The boundary conditions are satisfied if

$$\begin{aligned}&AQ_0(\lambda) + BQ_1'(\lambda) = 1,\\ &AQ_0'(\lambda) + BQ_1''(\lambda) = 0,\end{aligned} \tag{7.3.6}$$

and noting that $r \to \infty$ corresponds to $s \to \infty$ and

$$\begin{aligned}&Q_0(s) \sim s^{-1} \sim r^{-1},\\ &Q_1(s) \sim s^{-2} \sim r^{-2},\end{aligned} \tag{7.3.7}$$

the far field asymptotic condition requires

$$\nu = CA. \tag{7.3.8}$$

The solution for A accordingly is

$$A = \frac{Q_1''(\lambda)}{[Q_0(\lambda)Q_1''(\lambda) - Q_0'(\lambda)Q_1'(\lambda)]}. \tag{7.3.9}$$

The ψ function must satisfy

$$L_1\psi = 0 \tag{7.3.10}$$

and the conditions

$$\psi = 1 + \mathrm{C}, \quad (s = \lambda) \tag{7.3.11}$$

and

$$\psi \sim 2\nu \left[\frac{1}{r}\right]. \quad (r \to \infty) \tag{7.3.12}$$

The appropriate solution is

$$\psi = (1 + \mathrm{C})Q_0(s)/Q_0(\lambda). \tag{7.3.13}$$

The asymptotic condition gives

$$2Q_0(\lambda)\nu = 1 + \mathrm{C}. \tag{7.3.14}$$

Thus

$$\begin{aligned} \nu &= \left[2Q_0(\lambda) - \frac{1}{A}\right]^{-1} \\ &= \frac{Q_1''(\lambda)}{[Q_0'(\lambda)Q_1'(\lambda) + Q_0(\lambda)Q_1''(\lambda)]}. \end{aligned} \tag{7.3.15}$$

However

$$\begin{aligned} Q_0(\lambda) &= \frac{1}{2}\log\left[\frac{(\lambda+1)}{(\lambda-1)}\right], \\ Q_0'(\lambda) &= -\left[\frac{1}{(\lambda^2-1)}\right], \end{aligned} \tag{7.3.16}$$

$$Q_1(\lambda) = \frac{1}{2}\lambda \log\left[\frac{(\lambda+1)}{(\lambda-1)}\right] - 1, \tag{7.3.17}$$

$$Q_1'(\lambda) = \frac{1}{2}\lambda \log\left[\frac{(\lambda+1)}{(\lambda-1)}\right] - \left[\frac{\lambda}{(\lambda^2-1)}\right], \tag{7.3.18}$$

$$Q_1''(\lambda) = 2\left[\frac{1}{(\lambda^2-1)^2}\right]. \tag{7.3.19}$$

and it follows that

$$\nu = 2\left[\frac{1}{(\lambda^2-1)^2}\right]\left\{\left[\frac{1}{(\lambda^2-1)}\right]\left(\left[\frac{\lambda}{(\lambda^2-1)}\right]-\frac{1}{2}L\right)+\left[\frac{1}{(\lambda^2-1)}L\right]\right\}^{-1}$$

with $L = \ln[(\lambda+1)/(\lambda-1)]$. This expression simplifies to

$$\nu = 4\left\{2\lambda+(3-\lambda^2)\ln\left[\frac{(\lambda+1)}{(\lambda-1)}\right]\right\}^{-1} \tag{7.3.20}$$

which agrees with Oberbeck for the case of a prolate ellipsoid translating perpendicular to its major axis.

7.3.2 Oblate ellipsoid

For an oblate ellipsoid, we need to solve

$$L^2_{-1}\Phi = 0 \tag{7.3.21}$$

so that

$$\begin{aligned}\frac{\Phi}{\rho^2} &= C, \quad (s=\lambda)\\ \frac{\partial}{\partial s}\left(\frac{\Phi}{\rho^2}\right) &= 0, \quad (s=\lambda)\end{aligned} \tag{7.3.22}$$

and

$$\Phi \sim \nu\left[\frac{\rho^2}{r}\right]. \quad (r\to\infty) \tag{7.3.23}$$

Here oblate ellipsoidal coordinates are used

$$\begin{aligned}\rho &= (s^2+1)^{1/2}(1-t^2)^{1/2},\\ z &= st,\end{aligned} \tag{7.3.24}$$

where $1 \le s$ and $-1 \le t \le 1$. A suitable solution is

$$\Phi = C\rho^2\left[Aq_0(s)+Bq_1'(s)\right] \tag{7.3.25}$$

where $q_n(s) = i^{(n+1)}Q_n(is)$. The boundary conditions are satisfied if

$$\begin{aligned}&Aq_0(\lambda)+Bq_1'(\lambda) = 1,\\ &Aq_0'(\lambda)+Bq_1''(\lambda) = 0,\end{aligned} \tag{7.3.26}$$

and noting that $r \to \infty$ corresponds to $s \to \infty$ and

$$q_0(s) \sim s^{-1} \sim r^{-1},$$
$$q_1(s) \sim s^{-2} \sim r^{-2}, \tag{7.3.27}$$

the far field asymptotic condition requires that

$$\nu = \mathrm{C}A. \tag{7.3.28}$$

Again, the solution for A accordingly is

$$A = \frac{q_1''(\lambda)}{[q_0(\lambda)q_1''(\lambda) - q_0'(\lambda)q_1'(\lambda)]}. \tag{7.3.29}$$

The ψ function, must satisfy

$$L_1\psi = 0 \tag{7.3.30}$$

and the conditions

$$\psi = 1 + \mathrm{C}, \quad (s = \lambda) \tag{7.3.31}$$

together with the asymptotic condition

$$\psi \sim 2\nu\left[\frac{1}{r}\right]. \quad (r \to \infty) \tag{7.3.32}$$

The appropriate solution is

$$\psi = (1 + \mathrm{C})q_0(s)/q_0(\lambda). \tag{7.3.33}$$

The asymptotic condition therefore requires that

$$2q_0(\lambda)\nu = 1 + \mathrm{C}. \tag{7.3.34}$$

Thus

$$\begin{aligned}\nu &= \left[2q_0(\lambda) - \frac{1}{A}\right]^{-1} \\ &= \frac{q_1''(\lambda)}{[q_0'(\lambda)q_1'(\lambda) + q_0(\lambda)q_1''(\lambda)]}.\end{aligned} \tag{7.3.35}$$

However,

$$
\begin{aligned}
q_0(\lambda) &= \tan^{-1}\left[\frac{1}{\lambda}\right], &(7.3.36)\\
q_0'(\lambda) &= -\left[\frac{1}{(\lambda^2+1)}\right], &(7.3.37)\\
q_1(\lambda) &= 1-\lambda\,\tan^{-1}\left[\frac{1}{\lambda}\right], &(7.3.38)\\
q_1'(\lambda) &= -\tan^{-1}\left[\frac{1}{\lambda}\right]+\left[\frac{\lambda}{(\lambda^2+1)}\right], &(7.3.39)\\
q_1''(\lambda) &= 2\left[\frac{1}{(\lambda^2+1)^2}\right], &(7.3.40)
\end{aligned}
$$

so it follows from equation (7.3.36) that

$$
\begin{aligned}
\nu &= 2\left[\frac{1}{(\lambda^2+1)^2}\right]\left\{\left[\frac{1}{(\lambda^2+1)}\right]\left(\tan^{-1}\left[\frac{1}{\lambda}\right]-\left[\frac{\lambda}{(\lambda^2+1)}\right]\right)\right\}^{-1}\\
&+ 2\left[\frac{1}{(\lambda^2+1)^2}\right]\left\{\left[\frac{1}{(\lambda^2+1)}\right]\tan^{-1}\left[\frac{1}{\lambda}\right]\right\}^{-1}\\
&= 2\left\{(3+\lambda^2)\,\tan^{-1}\left[\frac{1}{\lambda}\right]-\lambda\right\}^{-1} \qquad (7.3.41)
\end{aligned}
$$

which again agrees with Oberbeck's formula for the drag when an oblate ellipsoid translates along a major axis.

7.3.3 Spherical lens

We assume that the lens has symmetry about the plane $z = 0$ and translates without rotation with velocity $\hat{\mathbf{i}}$ in fluid at rest at infinity. The geometry is depicted in figure 7.1.

We now make use of toroidal coordinates (ξ, ϕ, η), which are related to cylindrical polar coordinates (ρ, ϕ, z), by

$$z = \frac{c\,\sin\,\eta}{\cosh\,\xi - \cos\,\eta} \qquad (7.3.42)$$

and

$$\rho = \frac{c\,\sinh\,\xi}{\cosh\,\xi - \cos\,\eta}, \qquad (7.3.43)$$

where

$$
\begin{aligned}
0 &\le \xi < \infty,\\
-\pi &\le \eta \le \pi,\\
0 &\le \phi < 2\pi. \qquad (7.3.44)
\end{aligned}
$$

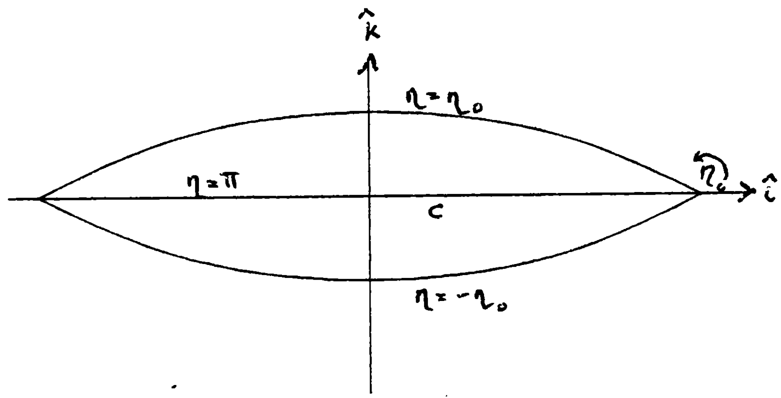

Figure 7.1: Spherical lens.

$\eta_0 = \frac{1}{2}\pi$	lens becomes a sphere of radius a
$\eta_0 = \pi$	lens becomes a disk of radius c
$0 < \eta_0 < \frac{1}{2}\pi$	lens is larger than a sphere of radius c
$\frac{1}{2}\pi < \eta_0 < \pi$	lens is smaller than a sphere of radius c

Table 7.1: Particular lens configurations.

The surface $\eta = \eta_0$ with $0 < \eta_0 < \pi$ is a spherical cap which intersects the plane $z = 0$ in the circle $\rho = c$ and lies above the plane $z = 0$. The surface $\eta = -\eta_0$ is the reflection of the cap $\eta = \eta_0$ in the plane $z = 0$. We note that $c = a \sin \eta_0$, with a the radius of the sphere of which the cap is part.

We further note that $\rho^2 + z^2 >> 1$ or $r >> 1$ corresponds to $\xi = \eta = 0$, and $\rho \to c+$ as $\xi \to +\infty$. Particular lens configurations are set out in Table 7.1.

The solution to the problem

With

$$\begin{aligned} \Phi &= \rho^2 F - H, \\ \psi &= 2\,F, \end{aligned} \tag{7.3.45}$$

then

$$\begin{aligned} u &= -\frac{\partial}{\partial \rho}\left(\frac{\Phi}{\rho}\right) + \psi, \\ v &= \frac{\Phi}{\rho^2} - \psi, & (7.3.46) \\ w &= -\frac{\partial}{\partial z}\left(\frac{\Phi}{\rho}\right), & (7.3.47) \end{aligned}$$

and

$$p = -\mu \frac{\partial \psi}{\partial \rho} \cos \phi + \text{constant}, \quad (7.3.48)$$

where

$$\mathbf{q} = u \cos \phi \, \hat{\rho} + v \sin \phi \, \hat{\phi} + w \cos \phi \, \hat{\mathbf{k}}, \quad (7.3.49)$$

and the functions Φ and ψ satisfy the equations

$$L_{-1}^2 \, \Phi = L_1 \, \psi = 0. \quad (7.3.50)$$

Boundary conditions

The non-slip conditions on the lens requires

$$\begin{aligned} u &= 1, \\ v &= -1, \\ w &= 0 & (7.3.51) \end{aligned}$$

on $\eta = \pm\eta_0$. Therefore

$$\begin{aligned} \frac{\partial}{\partial \rho}\left(\frac{\Phi}{\rho}\right) &= \psi - 1, \quad (\eta = \pm\eta_0) \\ \frac{\partial}{\partial z}\left(\frac{\Phi}{\rho}\right) &= 0, \quad (\eta = \pm\eta_0) \\ \frac{\Phi}{\rho^2} &= \psi - 1. \quad (\eta = \pm\eta_0) & (7.3.52) \end{aligned}$$

Combining the first and third equations gives

$$\rho \frac{\partial}{\partial \rho}\left(\frac{\Phi}{\rho^2}\right) = \rho \frac{\partial}{\partial z}\left(\frac{\Phi}{\rho^2}\right) = 0. \quad (\eta = \pm\eta_0) \quad (7.3.53)$$

The boundary value problem for Φ is uncoupled from that for ψ, so the functions Φ and ψ can be determined sequentially.

The asymptotic conditions on Φ and ψ are such that $u, v, w \to 0$ as $r \to \infty$, thus $\Phi = o(r^2)$, $\psi = o(1)$ as $r \to \infty$.

Equations (7.3.53) are equivalent to

$$\frac{\partial}{\partial \xi}\left(\frac{\Phi}{\rho^2}\right) = \frac{\partial}{\partial \eta}\left(\frac{\Phi}{\rho^2}\right) = 0. \quad (\eta = \pm\eta_0) \tag{7.3.54}$$

Determination of Φ

The velocity components u and v are symmetric about the plane $z = 0$ and the component w is anti-symmetric. Thus Φ and ψ are even functions of z and consequently η. The first of equations (7.3.54) shows that $\Phi = C\rho^2$ on the boundaries $\eta = \pm\eta_0$.

The appropriate solution of $L_{-1}^2\Phi = 0$ which is even in η is

$$\begin{aligned}
\Phi &= C\rho^2\frac{(\cosh\,\xi - \cos\,\eta)^{1/2}}{(\cosh\,\xi + \cos\,\eta)^{1/2}} \\
&+ C\rho^2(\cosh\,\xi - \cos\,\eta)^{1/2}\int_0^\infty [A(s)\cos\,\eta\cosh\,s\eta]K_s'(\cosh\,\xi)ds, \\
&+ C\rho^2(\cosh\,\xi - \cos\,\eta)^{1/2}\int_0^\infty [B(s)\sin\,\eta\sinh\,s\eta]K_s'(\cosh\,\xi)ds, \quad (|\eta| \le \eta_0),
\end{aligned} \tag{7.3.55}$$

where $K_s \equiv P_{-\frac{1}{2}+is}$ is the Mehler conal function. Since

$$\begin{aligned}
\frac{z}{c} &= \frac{\sin\,\eta}{\cosh\,\xi - \cos\,\eta}, \\
\frac{\rho}{c} &= \frac{\sinh\,\xi}{\cosh\,\xi - \cos\,\eta},
\end{aligned}$$

it follows that

$$\begin{aligned}
\frac{(\rho^2 + z^2)}{c^2} &= \frac{\sinh^2\,\xi + \sin^2\,\eta}{(\cosh\,\xi - \cos\,\eta)^2} \\
&= \frac{\cosh^2\,\xi - \cos^2\,\eta}{(\cosh\,\xi - \cos\,\eta)^2} \\
&= \frac{\cosh\,\xi + \cos\,\eta}{(\cosh\,\xi - \cos\,\eta)}.
\end{aligned} \tag{7.3.56}$$

Therefore

$$\begin{aligned}
\frac{(\rho^2 + z^2)^{1/2}}{c} &= \frac{r}{c} \\
&= \frac{(\cosh\,\xi + \cos\,\eta)^{1/2}}{(\cosh\,\xi - \cos\,\eta)^{1/2}}.
\end{aligned} \tag{7.3.57}$$

As $\xi, \eta \to 0$

$$\begin{aligned}\frac{z}{c} &\sim \frac{2\xi}{\xi^2+\eta^2},\\ \frac{\rho}{c} &\sim \frac{2\eta}{\xi^2+\eta^2},\\ \frac{r}{c} &\sim \frac{2}{(\xi^2+\eta^2)^{1/2}},\end{aligned} \qquad (7.3.58)$$

hence

$$\Phi \sim \mathrm{C}\rho^2\left[\frac{c}{r}+\sqrt{2}\frac{c}{r}\int_0^\infty A(s)K_s'(1)ds\right], \qquad (7.3.59)$$

with $K_s'(1) = -\frac{1}{8}(4s^2+1)$. Therefore

$$\Phi \sim \mathrm{C}\frac{\rho^2}{r}a\sin\,\eta_0\left[1-\frac{1}{4\sqrt{2}}\int_0^\infty(4s^2+1)A(s)ds\right]. \qquad (7.3.60)$$

Thus

$$\Phi = o(r^2), \quad (r\to\ \infty) \qquad (7.3.61)$$

with the representation given by equations (7.3.56). The boundary conditions require

$$\begin{aligned}\frac{\Phi}{\rho^2} &= \mathrm{C}, \quad (\eta=\pm\eta_0) &(7.3.62)\\ \frac{\partial}{\partial\eta}\left(\frac{\Phi}{\rho^2}\right) &= 0, \quad (\eta=\pm\eta_0). &(7.3.63)\end{aligned}$$

Equation (7.3.62) gives

$$\begin{aligned}&\frac{1}{(\cosh\,\xi-\cos\,\eta_0)^{1/2}}-\frac{1}{(\cosh\,\xi+\cos\,\eta_0)^{1/2}}\\ =&\int_0^\infty\left[A(s)\cos\,\eta_0\cosh\,s\eta_0+B(s)\sin\,\eta_0\sinh\,s\eta_0\right]K_s'(\cosh\,\xi)ds, \quad (|\eta|\le\eta_0).\end{aligned} \qquad (7.3.64)$$

Now, using formulae given by Schneider *et al.* (1973),

$$(\cosh\,\xi+\cos\,\eta)^{-1/2} = \sqrt{2}\int_0^\infty\left[\frac{\cosh\,s\eta}{\cosh\,s\pi}\right]K_s(\cosh\,\xi)\,ds, \quad (|\eta|<\pi), \qquad (7.3.65)$$

and

$$-\frac{1}{2}(\cosh\,\xi+\cos\,\eta)^{-3/2} = \sqrt{2}\int_0^\infty\left[\frac{\cosh\,s\eta}{\cosh\,s\pi}\right]K_s'(\cosh\,\xi)\,ds, \quad (|\eta|<\pi). \qquad (7.3.66)$$

Hence, multiplying equation (7.3.66) by $\sin\eta$ and integrating with respect to η from η_2 to η_1, we obtain

$$\begin{aligned}
&(\cosh\,\xi + \cos\,\eta_1)^{-1/2} - (\cosh\,\xi + \cos\,\eta_2)^{-1/2} \\
&= \sqrt{2}\int_0^\infty \left[\frac{\cosh\,s\eta}{\cosh\,s\pi}\right]\left[\frac{K_s'(\cosh\xi)}{(s^2+1)}\right] f_1(s,\eta_{1,2})\,ds \\
&\qquad (|\eta_1| < \pi) \text{ and } (|\eta_2| < \pi)
\end{aligned} \tag{7.3.67}$$

with

$$\begin{aligned}
f_1(s,\eta_{1,2}) &= \cos\,\eta_1\,\cosh\,s\eta_1 \\
&- s\,\sin\,\eta_1\sinh\,s\eta_1 \\
&- \cos\,\eta_2\,\cosh\,s\eta_2 \\
&+ s\,\sin\,\eta_2\sinh\,s\eta_2
\end{aligned} \tag{7.3.68}$$

Setting $\eta_1 = \pi - \eta_0$ and $\eta_2 = \eta_0$, we get

$$\begin{aligned}
&(\cosh\,\xi - \cos\,\eta_0)^{-1/2} - (\cosh\,\xi + \cos\,\eta_0)^{-1/2} \\
&= \sqrt{2}\int_0^\infty \left[\frac{\cosh\,s\eta}{\cosh\,s\pi}\right]\left[\frac{K_s'(\cosh\xi)}{(s^2+1)}\right] \\
&\{-\cos\,\eta_0\,\cosh\,s(\pi-\eta_0) - s\,\sin\,\eta_0\sinh\,s(\pi-\eta_0)\} \\
&\{-\cos\,\eta_0\,\cosh\,s\eta_0 + s\,\sin\,\eta_0\sinh\,s\eta_0\}\,ds \\
&= 2\sqrt{2}\int_0^\infty \left[\frac{\cosh\,\frac{1}{2}s\pi}{\cosh\,s\pi}\right]\left[\frac{K_s'\,(\cosh\,\xi)}{(s^2+1)}\right] f_2(s,\eta_0)\,ds
\end{aligned} \tag{7.3.69}$$

where

$$f_2(s,\eta_0) = \left\{s\,\sin\,\eta_0\sinh\,s\,\left(\frac{1}{2}\pi - \eta_0\right) - \cos\,\eta_0\cosh\,s\,\left(\frac{1}{2}\pi - \eta_0\right)\right\} \tag{7.3.70}$$

Equation (7.3.63) gives, using equation (7.3.62),

$$\begin{aligned}
&\left[\int_0^\infty \frac{\partial}{\partial\eta_0}\{A\,\cos\,\eta_0\,\cosh\,s\eta_0 + B\,\sin\,\eta_0\,\sinh\,s\eta_0\}\,K_s'\,(\cosh\,\xi)\,ds\right] \\
&\left[(\cosh\xi - \cos\,\eta_0)^{1/2}\right] \\
+\;& \frac{1}{2}\left[\frac{\sin\,\eta_0}{(\cosh\xi - \cos\,\eta_0)^{1/2}}\right]\left[\frac{1}{(\cosh\xi - \cos\,\eta_0)^{1/2}}\right] \\
+\;& (\cosh\xi - \cos\,\eta_0)^{1/2}\left[\frac{1}{2}\,\frac{\sin\,\eta_0}{(\cosh\xi + \cos\,\eta_0)^{3/2}}\right] \\
=\;& 0,
\end{aligned} \tag{7.3.71}$$

giving

$$\int_0^\infty \frac{\partial}{\partial \eta_0} \{A \cos \eta_0 \cosh s\eta_0 + B \sin \eta_0 \sinh s\eta_0\} K_s' (\cosh \xi)\, ds$$
$$= -\frac{1}{2} \sin \eta_0 \left[\frac{1}{(\cosh \xi - \cos \eta_0)^{3/2}} + \frac{1}{(\cosh \xi + \cos \eta_0)^{3/2}}\right]. \tag{7.3.72}$$

It follows on using equation (7.3.66) that

$$\int_0^\infty \frac{\partial}{\partial \eta_0} \{A \cos \eta_0 \cosh s\eta_0 + B \sin \eta_0 \sinh s\eta_0\} K_s' (\cosh \xi)\, ds$$
$$= \sqrt{2} \sin \eta_0 \int_0^\infty \left[\frac{(\cosh s\eta_0 + \cosh (\pi - \eta_0))}{\cosh s\pi}\right] K_s'(\cosh \xi)\, ds$$
$$= 2\sqrt{2} \sin \eta_0 \int_0^\infty \left[\frac{\cosh \frac{1}{2}s\pi \cosh s(\frac{1}{2}\pi - \eta_0)}{\cosh s\pi}\right] K_s'(\cosh \xi)\, ds. \tag{7.3.73}$$

Equations (7.3.64) and (7.3.73) give

$$\{A \cos \eta_0 \cosh s\eta_0 + B \sin \eta_0 \sinh s\eta_0\}$$
$$= 2\sqrt{2} \left[\frac{\cosh \frac{1}{2}s\pi}{(s^2+1) \cosh s\pi}\right]$$
$$\left[s \sin \eta_0 \sinh s \left(\frac{1}{2}\pi - \eta_0\right) - \cos \eta_0 \cosh s \left(\frac{1}{2}\pi - \eta_0\right)\right] \tag{7.3.74}$$

and

$$A\ [-\sin \eta_0 \cosh s\eta_0 + s \cos \eta_0 \sinh s\eta_0]$$
$$+\quad B\ [\cos \eta_0 \sinh s\eta_0 + s \sin \eta_0 \cosh \eta_0]$$
$$=\quad 2\sqrt{2} \left[\frac{\cosh \frac{1}{2}s\pi}{\cosh s\pi}\right] \left[\sin \eta_0 \cosh s \left(\frac{1}{2}\pi - \eta_0\right)\right] \tag{7.3.75}$$

The solutions of equation (7.3.74) and equation (7.3.75) are

$$A\ [\sinh s\eta_0 \cosh s\eta_0 + s \sin \eta_0 \cos \eta_0]$$
$$=\quad -2\sqrt{2} \left[\frac{\cosh \frac{1}{2}s\pi}{(s^2+1) \cosh s\pi}\right] f_3(s, \eta_0) \tag{7.3.76}$$

with

$$
\begin{aligned}
f_3(s,\eta_0) &= \sinh s\eta_0 \cosh s\left(\frac{1}{2}\pi - \eta_0\right) \\
&+ s \sin \eta_0 \cos \eta_0 \cosh \frac{1}{2}s\pi \\
&+ s^2 \sin^2 \eta_0 \sinh \frac{1}{2}s\pi,
\end{aligned} \tag{7.3.77}
$$

and

$$
\begin{aligned}
B \quad & [\sinh s\eta_0 \cosh s\eta_0 + s \sin \eta_0 \cos \eta_0] \\
= \; & 2\sqrt{2}\left[\frac{\cosh \frac{1}{2}s\pi}{(s^2+1) \cosh s\pi}\right] f_4(s,\eta_0),
\end{aligned} \tag{7.3.78}
$$

where

$$
\begin{aligned}
f_4(s,\eta_0) &= \sinh 2\eta_0 \cosh s\eta_0 \cosh s\left(\frac{1}{2}\pi - \eta_0\right) \\
&+ s \sin^2 \eta_0 \sinh \left(\frac{1}{2}s\pi\right) \\
&- s \sinh s\eta_0 \cosh s\left(\frac{1}{2}\pi - \eta_0\right) \\
&- s^2 \sin \eta_0 \cos \eta_0 \cosh s\left(\frac{1}{2}\pi - \eta_0\right).
\end{aligned} \tag{7.3.79}
$$

Determination of the function ψ

We seek a solution of

$$
L_1\psi = 0 \tag{7.3.80}
$$

which satisfies the boundary conditions

$$
\psi = 1 + \mathrm{C}, \tag{7.3.81}
$$

with $\eta = \pm\eta_0$ and the asymptotic condition $\psi = 0(1)$ as $r \to \infty$. Since ψ must be even in η, the appropriate solution is

$$
\psi = (\cosh \xi - \cosh \eta)^{1/2}(1 + \mathrm{C}) \int_0^\infty \mathrm{E}(s) \cosh s\eta \, K_s(\cosh \xi)\, ds, \quad (|\eta| \le \eta_0) \tag{7.3.82}
$$

where $K_s \equiv P_{-1/2+is}$ is the Mehler conal function. As $\xi, \eta \to 0$ this solution is such that

$$\begin{aligned}\psi &\sim \frac{1}{\sqrt{2}}(\xi^2+\eta^2)^{1/2}\int_0^\infty \mathrm{E}(s)\,ds \\ &\sim \left(\frac{\sqrt{2}}{r}\right) c \int_0^\infty \mathrm{E}(s)\,ds, \end{aligned} \tag{7.3.83}$$

and the asymptotic condition at infinity is satisfied. The condition on $\eta = \pm\eta_0$ requires

$$(\cosh\,\xi - \cosh\,\eta_0)^{-1/2} = \int_0^\infty \mathrm{E}(s)\cosh\,s\eta_0\,K_s(\cosh\,\xi)\,ds, \qquad (|\eta| \le \eta_0) \tag{7.3.84}$$

with

$$(\cosh\,\xi - \cosh\,\eta_0)^{-1/2} = \sqrt{2}\int_0^\infty \frac{\cosh\,s(\pi-\eta_0)}{\cosh\,s\pi} K_s(\cosh\,\xi)\,ds. \qquad (|\eta_0| < \pi) \tag{7.3.85}$$

Equation (7.3.79) is satisfied if

$$\mathrm{E}(s) = \frac{\cosh\,s(\pi-\eta_0)}{\cosh\,s\pi}. \tag{7.3.86}$$

The functions Φ and ψ are thus determined apart from the constant C. As $r \to \infty$, the velocity field and pressure must be that of a Stokeslet of strength $\nu\hat{\mathrm{i}}$ located at the origin with $-8\pi\mu\nu\hat{\mathrm{i}}$ the drag acting on the lens. Thus

$$\begin{aligned} u &\sim \nu\left[\frac{\rho^2}{r^3}+\frac{1}{r}\right], \\ v &\sim -\nu\left[\frac{1}{r}\right], \\ w &\sim \nu\left[\frac{\rho^2}{r^3}\right]. \end{aligned} \tag{7.3.87}$$

Therefore

$$\begin{aligned} -\frac{\partial}{\partial\rho}\left(\frac{\Phi}{\rho}\right)+\psi &\sim \nu\left[\frac{\rho^2}{r^3}+\frac{1}{r}\right], \\ \left(\frac{\Phi}{\rho}\right)-\psi &\sim -\nu\left[\frac{1}{r}\right], \\ -\frac{\partial}{\partial z}\left(\frac{\Phi}{\rho}\right) &\sim \nu\left[\frac{\rho^2}{r^3}\right]. \end{aligned} \tag{7.3.88}$$

Thus

$$\begin{aligned} \Phi &\sim \nu \left[\frac{\rho^2}{r^3}\right], \\ \psi &\sim 2\nu \left[\frac{1}{r}\right]. \end{aligned} \tag{7.3.89}$$

These asymptotic conditions yield the following equations

$$\begin{aligned} \nu &= \mathrm{C}a \sin\,\eta_0 \left[1 - \frac{1}{4\sqrt{2}} \int_0^\infty (4s^2+1) A(s)\, ds\right] \\ &= (1+\mathrm{C}) a \sin\,\eta_0 \int_0^\infty \frac{\cosh\, s(\pi-\eta_0)}{\cosh\, s\pi}\, ds. \end{aligned} \tag{7.3.90}$$

Thus

$$\begin{aligned} \nu &= \mathrm{C}a \sin\,\eta_0 \,[1+\mathrm{J}] \\ &= (1+\mathrm{C}) a \sin\,\eta_0 \mathrm{I}, \end{aligned} \tag{7.3.91}$$

where

$$\mathrm{I} = \int_0^\infty \frac{\cosh\, s(\pi-\eta_0)}{\cosh\, s\pi\, \cosh\, s\eta_0}\, ds \tag{7.3.92}$$

and

$$\mathrm{J} = \int_0^\infty \left[\frac{4s^2+1}{s^2+1}\right] \left[\frac{\cosh\,\frac{1}{2}s\pi}{\cosh\, s\pi}\right] \left[\frac{1}{\sinh\, 2s\eta_0 + s\sin\, 2\eta_0}\right] \mathrm{F}(s,\eta_0) ds, \tag{7.3.93}$$

with

$$\begin{aligned} \mathrm{F}(s,\eta_0) &= \sinh\, s\eta_0 \cosh s \left(\frac{1}{2}\pi - \eta_0\right) \\ &+ s\,\sin\,\eta_0\,\cos\,\eta_0\,\cosh\left(\frac{1}{2}s\pi\right) \\ &+ s^2\,\sin^2\,\eta_0 \sinh\left(\frac{1}{2}s\pi\right). \end{aligned} \tag{7.3.94}$$

Therefore

$$\mathrm{C} = \frac{\mathrm{I}}{(1+\mathrm{J}-\mathrm{I})} \tag{7.3.95}$$

giving

$$\nu = a\,\sin\,\eta_0 \frac{\mathrm{I}(1+\mathrm{J})}{(1+\mathrm{J}-\mathrm{I})} \tag{7.3.96}$$

and

$$f = \frac{4}{3}\left[\frac{\nu}{a}\right] \tag{7.3.97}$$

where $-6\pi\mu a f\hat{\mathbf{i}}$ is the drag on the body.

7.4 Special cases

7.4.1 Sphere $\eta_0 = \frac{1}{2}\pi$:

$$F(s,\eta_0) = (1+s^2)\ \sinh\ \frac{1}{2}s\pi. \tag{7.4.1}$$

Therefore

$$\begin{aligned} I &= \int_0^\infty \text{sech}\ s\pi ds \\ &= \frac{1}{2}, \end{aligned} \tag{7.4.2}$$

and

$$\begin{aligned} J &= \int_0^\infty (4s^2+1)\left[\frac{\cosh\ \frac{1}{2}s\pi}{\cosh\ s\pi}\right]\left[\frac{\sinh\ \frac{1}{2}s\pi}{\sinh\ s\eta_0}\right] ds, \\ &= \frac{1}{2}\int_0^\infty (4s^2+1)\text{sech}\ s\pi\ ds. \end{aligned} \tag{7.4.3}$$

Using result from Appendix I to give

$$\begin{aligned} J &= \frac{1}{2}\left[4.\frac{1}{8}+\frac{1}{2}\right] \\ &= \frac{1}{2}. \end{aligned} \tag{7.4.4}$$

Therefore $C = \frac{1}{2}$, $\nu = \frac{3}{4}a$ and $f = 1$.

7.4.2 Disk $\eta_0 = \pi$:

Since

$$c = a \sin\ \eta_0, \tag{7.4.5}$$

then c is the radius of the disk if $a \to \infty$, $\eta_0 \to \pi$ such that $a \sin\ \eta_0$ remains finite

$$\begin{aligned} I &= \int_0^\infty \text{sech}^2\ s\pi\ ds \\ &= \frac{1}{\pi} = 0.3183. \end{aligned} \tag{7.4.6}$$

and

$$
\begin{aligned}
\mathrm{J} &= \int_0^\infty \left[\frac{4s^2+1}{s^2+1}\right]\left[\frac{\cosh\frac{1}{2}s\pi}{\cosh s\pi}\right]\left[\frac{\sinh s\pi}{\sinh 2s\pi}\right]\cosh\frac{1}{2}s\pi\, ds, \\
&= \frac{1}{4}\int_0^\infty \left[\frac{4s^2+1}{s^2+1}\right]\left[\frac{(\cosh s\pi+1)}{\cosh^2 s\pi}\right] ds, \\
&= \frac{1}{4}\int_0^\infty \left[\frac{4s^2+1}{s^2+1}\right]\left[\operatorname{sech} s\pi + \operatorname{sech}^2 s\pi\right] ds, \\
&= \int_0^\infty \left[\operatorname{sech} s\pi + \operatorname{sech}^2 s\pi\right] ds \\
&\quad -\frac{3}{4}\int_0^\infty \left[\frac{\operatorname{sech}^2 s\pi}{(1+s^2)}\right] ds - \frac{3}{4}\int_0^\infty \left[\frac{\operatorname{sech} s\pi}{(1+s^2)}\right] ds,
\end{aligned}
\tag{7.4.7}
$$

Now $\int_0^\infty \operatorname{sech} s\pi ds = 1/2$, $\int_0^\infty \operatorname{sech}^2 s\pi ds = 1/\pi$, and the other integrals may be evaluated by the residue theorem (as shown in Appendix I) to give

$$
\begin{aligned}
\mathrm{I}_1 &= \int_0^\infty \left[\frac{\operatorname{sech} s\pi}{(1+s^2)}\right] ds \\
&= \left(2 - \frac{\pi}{2}\right) \\
\mathrm{I}_2 &= \int_0^\infty \left[\frac{\operatorname{sech}^2 s\pi}{(1+s^2)}\right] ds \\
&= \left(\frac{\pi}{2} - \frac{4}{\pi}\right)
\end{aligned}
\tag{7.4.8}
$$

Hence

$$
\begin{aligned}
\mathrm{J} &= \frac{1}{2} + \frac{1}{\pi} - \frac{3}{4}\left(\frac{\pi}{2} - \frac{4}{\pi}\right) - \frac{3}{4}\left(2 - \frac{\pi}{2}\right) \\
&= \frac{4}{\pi} - 1 \\
&= 0.2732
\end{aligned}
\tag{7.4.9}
$$

Therefore $\mathrm{C} = \frac{1}{3}$, $\nu = \frac{4c}{3\pi}$. Hence the force acting on the disk is $\frac{-32}{3}\mu c$ along $\hat{\mathbf{i}}$, which is the well known result.

7.4.3 Two equal touching spheres $\eta_0 \to 0$

Now we must evaluate $lim_{\eta_0 \to 0}(\sin\,\eta_0 \mathrm{I})$ and $lim_{\eta_0 \to 0}(\sin\,\eta_0 \mathrm{J})$.

$$
\sin\,\eta_0\,\mathrm{I} = \sin\,\eta_0 \int_0^\infty [1 - \tanh s\pi\, \tanh s\eta_0]\, ds
$$

$$= \frac{\sin\,\eta_0}{\eta_0}\int_0^\infty \left[1 - \tanh\,\left(\frac{x\pi}{\eta_o}\right)\,\tanh\,x\right]\,dx \tag{7.4.10}$$

with $x = \eta_0 s$.

$$\tanh\left(\frac{x\pi}{\eta_0}\right) \to 1 \quad \text{as} \quad \eta_0 \to 0 \quad (x \neq 0) \tag{7.4.11}$$

Therefore $\sin\,\eta_0 I \to \mathrm{I}_0$ implies

$$\begin{aligned} \mathrm{I}_0 &= \int_0^\infty [1 - \tanh\,x]\,dx \\ &= \ln 2. \end{aligned} \tag{7.4.12}$$

$$\mathrm{J} = \left(\frac{\sin\,\eta_0}{\eta_0}\right)\int_0^\infty \left[\frac{(4 + \eta_0^2/x^2)}{(1 + \eta_0^2/x^2)}\right]\left[\frac{\cosh\,\frac{1}{2}(x\pi/\eta_0)}{\cosh\,(x\pi/\eta_0)}\right] \left[\frac{1}{\sinh\,2x\eta_0 + (x\sin\,2\eta_0)/\eta_0}\right]\mathrm{F}(s, \eta_0)ds, \tag{7.4.13}$$

with

$$\begin{aligned} \mathrm{F}(s, \eta_0) &= \sinh\,x\,[\cosh\,(\pi x/2\eta_0)\,\cosh\,x - \sinh(\pi x/2\eta_0)\,\sinh\,x] \\ &+ x\left(\frac{\sin\,\eta_0}{\eta_0}\right)\,\cos\,\eta_0\cosh\,\left(\frac{\pi x}{2\eta_0}\right) \\ &+ x^2\left(\frac{\sin^2\,\eta_0}{\eta_0^2}\right)\,\sinh\,\left(\frac{\pi x}{2\eta_0}\right) \end{aligned} \tag{7.4.14}$$

It follows that

$$\sin\,\eta_0\,\mathrm{J} \to \mathrm{J}_0 = \int_0^\infty \left[\frac{2(x + x^2) + 1 - e^{-2x}}{\sinh\,2x + 2x}\right]\,dx \tag{7.4.15}$$

Therefore

$$\mathrm{C} \to \frac{\mathrm{I}_0}{(\mathrm{J}_0 - \mathrm{I}_0)}, \tag{7.4.16}$$

and

$$\frac{\nu}{a} \to \frac{\mathrm{I}_0\mathrm{J}_0}{(\mathrm{J}_0 - \mathrm{I}_0)}. \tag{7.4.17}$$

Numerical evaluation of the integrals yields $\mathrm{I}_0 = \ln 2 = 0.6931$ and $\mathrm{J}_0 = 1.9354$. These values give $\frac{1}{2}f = 0.7199$ which compares with the calculation of $\frac{1}{2}f = 0.7243$ given by M. E. O'Neill (1969) for the drag on each of two equal touching spheres, solving the problem directly using tangent sphere coordinates. The relative error of the two calculations for f is 0.6%.

k	I	J	f
0	——	——	1.4399
1	2.2191	5.6122	1.3762
2	1.1288	2.4803	1.3093
3	0.7730	1.3227	1.2497
4	0.6002	0.7827	1.1473
5	0.5000	0.5000	1.0000
6	0.4357	0.3539	0.8148
7	0.3917	0.2881	0.6071
8	0.3600	0.2676	0.3940
9	0.3364	0.2677	0.1887
10	0.3183	0.2732	0.0000

Table 7.2: The computed values of I, J and f for $\eta_0 = k\pi/10$, where $k = 0, 1, .., 10$.

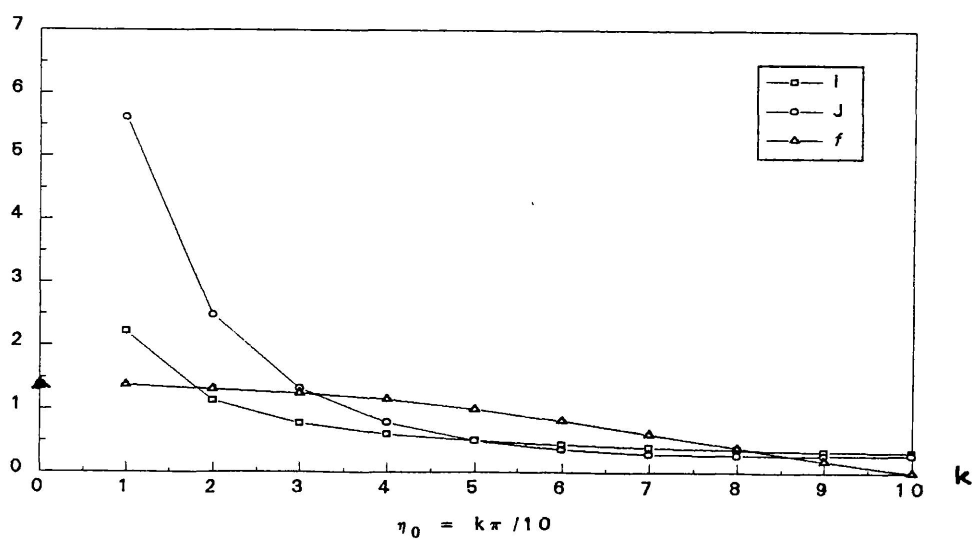

Figure 7.2: The graphs of I, J and f.

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Appendix I

Evaluation of Integrals on page 155

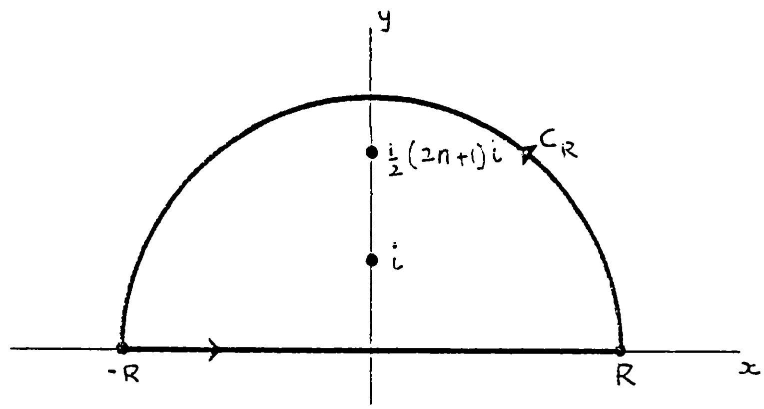

Figure 7.3: Figure I.

Let

$$\begin{aligned} I_1 &= \int_0^\infty \frac{1}{(x^2+1)\cosh \pi x} dx \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{(x^2+1)\cosh \pi x}\, dx \end{aligned} \tag{I.1.1}$$

consider

$$f(z) = \frac{1}{(z^2+1)\cosh \pi z}, \quad (z = x+iy), \tag{I.1.2}$$

$f(z)$ has simple poles at $z = \pm i$ and $z = \pm\frac{1}{2}(2n+1)i$, $(n = 0, 1, 2..)$.

Let $C = \{-R, R\} + C_R$ with C_R a semi-circle radius $R > 1$ not passing through z_n drawn in $y > 0$, that is when $R > 1$, the singular points of f in the upper half plane lie in the interior of the semicircular region bounded by the segment $z = x(-R \leq x \leq R)$ of the real axis and the upper half C_R of the circle $|z| = R$ from $z = R$ to $z = -R$. (Figure I). Integrating f counterclockwise around the boundary of this semicircular region, using residue theorem, we see that

$$\begin{aligned}
\oint_C f(z)dz &= \left[\int_{-R}^{R} + \int_{C_R}\right] f(z)dz, \\
&= \left[\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz\right] \\
&= 2\pi i(B_1 + B_2) \qquad \text{(I.1.3)}
\end{aligned}$$

where B_1 is the residue of f at the point $z = i$ and B_2 is the residue of f at the point $z = \frac{1}{2}(2n+1)i$. It can be shown that the value of the integral

$$\left| \int_{C_R} f(z)dz \right| \to 0$$

as R tends to ∞. Therefore, we need only write

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i(B_1 + B_2) \qquad \text{(I.1.4)}$$

The point $z = i$ is a simple pole of f and that $B_1 = 1/(2i\cos\pi) = -1/(2i)$. The point $z = \frac{1}{2}(2n+1)i = z_n$, where n = 0, 1, 2, .., is also a simple pole, so

$$\begin{aligned}
B_2 &= \frac{1}{[1+z_n^2]}\, lim_{z\to z_n} \frac{(z-z_n)}{\cosh \pi z} \\
&= \frac{4}{[4-(2n+1)^2]} \cdot \frac{1}{\pi \sinh(iz_n\pi)} \\
&= \frac{4}{[4-(2n+1)^2]} \frac{(-1)^n}{\pi i} \\
&= \frac{4i(-1)^n}{\pi(2n-1)(2n+3)}. \qquad \text{(I.1.5)}
\end{aligned}$$

Hence

$$\begin{aligned}
2\pi i(B_1 + B_2) &= -\pi - 8\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n-1)(2n+3)} \\
&= -\pi - 2\sum_{n=0}^{\infty}(-1)^n \left[\frac{1}{(2n-1)} - \frac{1}{(2n+3)}\right] \\
&= -\pi + 4 \qquad \text{(I.1.6)}
\end{aligned}$$

Therefore $2I_1 = -\pi + 4$. It thus follows that

$$\begin{aligned}
I_1 &= \int_0^{\infty} \frac{1}{[(1+x^2)\cosh \pi x]}\, dx \\
&= 2 - \frac{\pi}{2} \qquad \text{(I.1.7)}
\end{aligned}$$

Now consider

$$f(z) = \frac{1}{(1+z^2)\cosh^2 \pi z}.$$

Thus

$$\begin{aligned} \mathrm{I}_2 &= \int_0^\infty \frac{1}{(1+x^2)\cosh^2 \pi x} dx \\ &= \frac{1}{2}\oint_C f(z)dz \end{aligned} \qquad \text{(I.1.8)}$$

As before, $2\mathrm{I}_2 = 2\pi i$ [sum of residues at poles of $f(z)$ in the upper half plane].

Let $2\mathrm{I}_2 = 2\pi i(B_1 + B_2)$, where B_1 is the residue of f at the point $z = i$ and B_2 is the residue of f at $z = z_n$ with $n = 0, 1, \ldots$ The point $z = i$, which lies above the x axis, is a simple pole of f, with residue

$$B_1 = \frac{1}{2i(\cos\pi)^2} = \frac{1}{2i}. \qquad \text{(I.1.9)}$$

Now $f(z)$ has a double pole at $z = z_n$, $(n = 0, 1, 2 \ldots)$. To find the residue of $f(z)$ at $z = z_n$, it is simplest to expand $f(z)$ in a Laurent series about z_n and pick out the coefficient of $(z - z_n)^{-1}$. Thus

$$(z^2+1) = (z_n^2+1) + 2z_n(z-z_n) + O(z-z_n)^2 \qquad \text{(I.1.10)}$$

and

$$\begin{aligned} \cosh \pi z &= \cosh \pi z_n + \pi(z-z_n)\sinh \pi z_n + O(z-z_n)^3 \\ &= i\pi(-1)^n(z-z_n) + O(z-z_n)^3 \end{aligned} \qquad \text{(I.1.11)}$$

and so

$$\cosh^2 \pi z = -\pi^2(z-z_n)^2 + O(z-z_n)^4, \qquad \text{(I.1.12)}$$

and it follows that

$$\begin{aligned} f(z) &= \frac{1}{[(z_n^2+1) + 2z_n(z-z_n) + O(z-z_n)^2][-\pi^2(z-z_n)^2 + O(z-z_n)^4]} \\ &= \frac{1}{(z_n^2+1)} \cdot \frac{(-1)}{\pi^2(z-z_n)^2} Z_1 \cdot [1 + O(z-z_n)^2], \end{aligned} \qquad \text{(I.1.13)}$$

where

$$Z_1 = \left[1 + \frac{2z_n(z-z_n)}{(z_n^2+1)} + O(z-z_n)^2\right]^{-1}. \qquad \text{(I.1.14)}$$

Since coefficient of $(z - z_n)^{-1}$ is

$$\frac{2z_n}{(z_n^2+1)\pi^2} = \frac{16i}{\pi^2}\left[\frac{(2n+1)}{(2n-1)^2(2n+3)^2}\right], \tag{I.1.15}$$

thus f has a double pole at $z = z_n$, with residue

$$B_2 = \frac{16i}{\pi^2}\sum_{n=0}^{\infty}\left[\frac{(2n+1)}{(2n-1)^2(2n+3)^2}\right]. \tag{I.1.16}$$

Hence

$$\begin{aligned} 2\pi i(B_1+B_2) &= 2\pi i\left[\frac{1}{2i}+\frac{16i}{\pi^2}\sum_{n=0}^{\infty}\left[\frac{(2n+1)}{(2n-1)^2(2n+3)^2}\right]\right] \\ &= 2\left[\frac{\pi}{2}-\frac{16}{\pi}\sum_{n=0}^{\infty}\left[\frac{(2n+1)}{(2n-1)^2(2n+3)^2}\right]\right]. \end{aligned} \tag{I.1.17}$$

We know that $2\mathrm{I}_2 = 2\pi i(B_1 + B_2)$, therefore

$$\begin{aligned} \mathrm{I}_2 &= \frac{\pi}{2}-\frac{2}{\pi}\sum_{n=0}^{\infty}\left[\frac{1}{(2n-1)^2}-\frac{1}{(2n+3)^2}\right] \\ &= \frac{\pi}{2}-\frac{2}{\pi}\cdot 2 \\ &= \frac{\pi}{2}-\frac{4}{\pi}, \end{aligned} \tag{I.1.18}$$

and so

$$\begin{aligned} \mathrm{I}_2 &= \int_0^{\infty}\left[\frac{1}{(1+x^2)\cosh^2\pi x}\right]dx \\ &= \frac{\pi}{2}-\frac{4}{\pi}. \end{aligned} \tag{I.1.19}$$