Schwarzian Derivative and Second Order Differential Equations

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Abstract

This thesis is divided into two parts. The first part consisting of the first four chapters. We study mainly the properties of complex function f when some conditions are imposed on the Schwarzian derivative of f.

In Chapter 1, we define the notions of quasiconformal mappings and investigate conditions that allow f to have a quasiconformal extension outside the unit disc Δ to the extended complex plane. We used the method of Ahlfors to obtain and extend the criteria, involving Schwarzian derivatives, obtained earlier by Ahlfors, Krzyż and Lewandowski etc. In Chapter 2 we shall look at the domain constant $\Omega(A)$ introduced by Lehto with the norm of the Schwarzian and logarithmic derivatives.

In Chapter 3, we consider the Schwarzian $S(f,z)$ alone and show that if it is sufficiently small and the second coefficient is also small (depending on $S(f,z)$), then f is a α-strongly starlike function for one such constant, and convex for a smaller constant. Other properties of f when $S(f,z)$ is small are also investigated. The method used depends heavily on the second order differential equations.

Chapter 4 considers the same problems as in Chapter 3, but solved by the use of the Clunie-Jack principle. The advantage of this principle is that it enables us to consider a more restricted class of functions. The results obtained complement that of Chapter 3. With the Clunie-Jack principle, we give alternative proofs of results, in one case with an extension, obtained previously by Miller and Mocanu.

Chapter 5 is our second part. Here we consider the distribution of the zero sequences of the solutions of a second order differential equation, with the given coefficient being an entire transcendental function of finite order. This has been considered by Bank, Laine, Langley and Rossi etc.

'The great end of learning is nothing else

but to seek for the lost mind.'

—Mencins

(An ancient Chinese philosopher.)

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Contents

Preface

This thesis is largely concern, as the title suggests, with the Schwarzian derivative and second order differential equations. It is divided into two parts. The first part consists of the first four chapters dealing with topics in geometric function theory, and the second part, consisting of Chapter 5 dealing with the value distribution theory of differential equations. However the theory and techniques used in the first part are closely related to the second part. In fact many proofs of theorems in part one are based on second order differential equations.

In Chapter 1, we introduce the notion of quasiconformal mappings, a kind of rather general mappings when compared to conformal mappings. We use quasiconformal mappings to construct some univalence criteria and criteria for quasiconformal extensions that involve quantities like f, f' and f'', where f is defined in the unit disc. It is here we first meet the Schwarzian derivative and it together with the logarithmic derivative f''/f' often appears in many such criteria. The modern treatment of the Schwarzian derivative will be discussed in Chapters 2 and 3. We mainly concentrate on a method of obtaining univalence criteria and criteria for quasiconformal extension. This method that we used was due to Ahlfors and it is based on a topological result which asserts that a local homeomorphism of $\bar{\mathbf{C}}$ onto itself is a global homeomorphism. This elegant method allows us to obtain and generalize many well-known criteria due to Lewandowski, Stankiewicz and Krzyż. Several other results are also obtained. We also compare our results with those obtained by Anderson and Hinkkanen.

In Chapter 2, the Schwarzian will be introduced together with some historical remarks and its recent development. However, despite its importence in classical theory of conformal mappings, it has been found recently that it has a relation with Teichmüller theory. We will not discuss anything about the Teichmüller theory here, but this has some close connections with univalent function theory and quasiconformal mappings. In one aspect, Lehto has introduced the notion of Domain Constant of a simply connected domain that depends on the size of the Schwarzian derivative. This seems more or less a modern way of expressing classical ideas, but it has many elegant properties that fit perfectly with the Teichmüller theory. We consider domains that are images of strongly starlike functions defined in the unit disc. An upper bound of the Domain Constant of strongly starlike domains has been found. We also find a corresponding upper bound for the logarithmic derivative for strongly starlike functions. In fact, it is very often the case that when one theorem is true for the Schwarzian derivative then it is also true for the logarithmic derivative. It is known that if the domain constant is small enough then it must be a quasidisc, that is a image of a unit disc under a quasiconformal mapping of $\bar{\mathbf{C}}$. We show that their exist domains whose Domain Constant can be made arbitrary small and they are not starlike. Similar results are also obtained for the logarithmic derivative. We prove these by providing explicit counter examples. The techniques used are based on second order differential equations.

In Chapter 3, we still look at the relation between the size of the Schwarzian derivative and properties of analytic functions defined in the unit disc. However, we shall use a more classical approach. It was Nehari who first obtained important results about the Schwarzian derivatives and univalent functions. Again, the use of second order differential equations is crucial in his proofs. Nehari showed that if the Schwarzian derivative of f, $S(f,z)$ satisfies $|S(f,z)| \le \pi^2/2$ for all z in the unit disc then f is univalent. Gabriel showed if $|S(f,z)| \le c_0$ for all z in the unit disc, where c_0 is a fixed constant, then f is starlike. We define the supremum of the above upper bounds to be Schwarzian radii of univalence and starlikeness respectively. We attempt to find the Schwarzian radius of convexity and some related results are obtained. The main difference between the analysis of Chapter 2 and 3 is to replace the quantity $(1-|z|^2)^2|S(f,z)|$ by $|S(f,z)|$. We find that the latter has some control of the geometrical shape of the image of the unit disc under f whereas the former does not. Some examples are also given at the end of the Chapter.

Chapter 4 was initiated by the private communications with J.G. Clunie and T. Sheil-Small. We continue the study of the problems in Chapter 3. Both Clunie and Sheil-Small have given a method of estimating the Schwarzian radius of convexity for functions defined in the unit disc. Based on the method of the Clunie-Jack principle, we study and obtain several results which were originally obtained by Miller and Mocanu. We then investigate a subclass of strongly gamma starlike functions and show that they belong to strongly starlike functions. Hence they have quasiconformal extensions.

Chapter 5 deals with a different kind of problems. We study the zero distribution of the solutions of second order differential equations of the type $y''+Ay=0$. The basic tool here is the celebrated Nevanlinna theory. A brief introduction of the Nevanlinna theory is also included. The main result that we prove in this chapter is: let f_1, f_2 be linearly independent solutions of equation $y''+Ay=0$ where A is a transcendental entire functions of finite order with $\delta(0,A)=1$. Then the maximum of the exponents of convergence of f_1 and f_2 is infinite. The proof is based on some well-known results of Edrei and Fuchs.

Finally, a word about the references. We try to include all full surnames for each of the literature that we cite. The reference is given at the back of this thesis. Also a '$\Box$' sign is used to indicate the completion of the proof of a theorem.

Chapter One

On Quasiconformal Extensions in the Unit disc and Schwarzian Derivatives

§ *1.1 Notations and Definitions*

We shall first give a definition of *quasiconformal mappings* and some fundamental facts about them.

To begin with, let $\mathbb{C}$ and $\bar{\mathbb{C}}$ denote the complex plane and the extended complex plane respectively. A one to one mapping of a set A onto a set A' is called a ***homeomorphism*** if f and its inverse mapping f^{-1}: $A' \to A$ are both continuous; here we consider A as a subset of $\mathbf{C}$. A *Jordan Curve* C is a set which is homeomorphic to a circle and the Jordan curve theorem states that the complement of the Jordan curve C consists of two disjoint domains, which both have C as their boundary.

We shall also need to clarify the meaning of ***orientation***. The orientation of a Jordan curve can be defined as follows : consider all homeomorphisms that map the unit circle $\partial\Delta = \{e^{i\theta} | \; 0 \leq \theta \leq 2\pi\}$ onto C, and with two such mappings f_1, f_2, the composition $f_2^{-1} \circ f_1(\theta)$ is either increasing or decreasing as θ increases. This divides the homeomorphisms into two classes. If C is a Jordan curve bounding the disjoint domains G_1 and G_2, we can find a linear conformal mapping g which maps G_1 onto a domain $g(G_1)$ containing the origin. Let $f : \partial\Delta \to C$ be a fixed representation (of an equivalent class) of the orientations. As θ increases from 0 to 2π, the argument of $g \circ f(\theta)$ changes by either 2π or -2π . If it is the first case, we say C has positive orientation with respect to G_1, negative otherwise. It is also easy to see that if $g \circ f$ is positive orientated with respect to G_1 then it must be negative orientated with respect to G_2.

Let $w : \bar{D} \to A$ be a homeomorphism, where D is a Jordan domain and $w(D) = D'$. Let f be a representation of the orientation $\partial\Delta \to \partial D$, then $w \circ f : \partial\Delta \to \partial D'$ induces an orientation with respect to $\partial w(D)$. If the orientation is preserved under w(positive or negative), w is called *sense-preserving.* If $w : A \to A'$, where A and A' are general point sets, then w is called sense-preserving homeomorphism if it preserves the orientation of every Jordan domain D such that $\bar{D} \subset A$.

Let $f = u + iv : G \to G'$ be a homeomorphism of plane domains. Let $z_0 \in G$, we define the formal derivatives of f as

$$f_z = \frac{1}{2}(f_x - i\, f_y) \ \text{ and } \ f_{\bar{z}} = \frac{1}{2}(f_x + i\, f_y).$$

where the subscripts x and y are meant to be the partial derivatives of $u(x,y)$ and $v(x,y)$ with respect to x and y. f is said to be *differentiable at* z_0, if we have at the point z_0 the following expression

$$f(z) = f(z_0) + f_z(z_0)(z - z_0) + f_{\bar{z}}(z_0)\overline{(z - z_0)} + o(|z - z_0|). \tag{1.1.1}$$

Let z_0 be an interior point of G, then z_0 is a *regular point* of f if f is differentiable at z_0 and the *Jacobian* $J_f(z_0) \neq 0$. If $J_f(z_0) > 0$ we have the following simple result (Lehto & Virtanen [1] p.10):

If the homeomorphism $f : G \to G'$ possesses a regular point z_0 where $J_f(z_0) > 0$, then f is sense-preserving in G. Conversely, the Jacobian of a sense-preserving homeomorphism is positive at every regular point.

§ 1.2 Quasiconformal Mappings

The linear part of (1.1.1) as $|z - z_0|$ is small is interpreted as the differential

$$df(z_0) = f_z(z_0)\, dz + f_{\bar{z}}(z_0)\, d\bar{z}. \tag{1.2.1}$$

We consider $df(z_0)$ as a mapping from $\mathbb{R}^2 \to \mathbb{R}^2$ defined by $\begin{bmatrix} dx \\ dy \end{bmatrix} \to \begin{bmatrix} du \\ dv \end{bmatrix}$ where $df = du + i\, dv$

$= \begin{bmatrix} du \\ dv \end{bmatrix}$. Then f is differentiable at (x_0,y_0) in the sense of function of two variables, if there exists a matrix A such that $f'(z_0) = A$ and

$$\lim_{|\zeta|\to 0} \frac{|f(z_0+\zeta)-f(z_0)-f'(z_0)\zeta|}{|\zeta|} = 0 .$$

It turns out that the right hand side of (1.2.1) can be written as $f'(z_0)\, d\zeta$ where $f'(z_0) = \begin{bmatrix} u_x\ u_y \\ v_x\ v_y \end{bmatrix}$ evaluated at $z_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$, $\zeta = \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}$. And $df(z_0) = \begin{bmatrix} u_x\ u_y \\ v_x\ v_y \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$. Therefore the differentiability in §1.1 coincides with the differentiability of functions of two variables.

Geometrically (1.2.1) can be interpreted as an affine transformation that maps the circle $|z|=r$ in the plane $dz=(dx,dy)$ locally onto the ellipse in the plane $d\zeta=(du,dv)$. Now at z_0

$$df(z_0)(re^{i\theta}) = f_z(z_0)\,(re^{i\theta}) + f_{\bar z}(z_0)\,(re^{-i\theta}), \quad 0\le\theta\le 2\pi . \tag{1.2.1}$$

As θ increases from 0 to 2π the circle in the dz-plane becomes an ellipse in $d\zeta$-plane with major axis $r(|f_z|+|f_{\bar z}|)e^{\frac{i}{2}arg(f_z f_{\bar z})}$ when $\theta=\frac{1}{2}arg(f_{\bar z}\overline{f_z})$ and minor axis $ir(|f_z|-|f_{\bar z}|)e^{\frac{i}{2}arg(f_z f_{\bar z})}$ when $\theta=\frac{1}{2}arg(f_{\bar z}\overline{f_z})+\frac{\pi}{2}$. Hence the ratio between the length of major and minor axis is defined as

$$D_f(z_0) = \frac{|f_z(z_0)|+|f_{\bar z}(z_0)|}{|f_z(z_0)|-|f_{\bar z}(z_0)|}.$$

It is called the *dilatation quotient* of f at z_0, and it measures the distortion from a conformal mapping locally. Conformal mapping maps a circle onto a circle locally. Thus we have established a measure of *quasiconformality* at a regular point. Under a further assumption on f, apart from a homeomorphism, we shall see f is in fact regular at almost all $z \in G$. Together with the fact that D_f is uniformly bounded by a constant, this gives a initial picture of a quasiconformal mapping.

A function g defined in $\mathbb{I} \subset \mathbb{R}$ is *absolutely continuous on* $\mathbb{I}$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $\sum_i |g(b_i)-g(a_i)|<\epsilon$ for every finite sequence of disjoint intervals (a_i,b_i) whenever $\sum_i |b_i-a_i|<\delta$. The number of intervals (a_i,b_i) can also be infinite. A function f is said to be *absolutely continuous on lines in* G if for every rectangle in G, f is absolutely continuous on almost all horizontal and vertical lines by varying one of the variable and by keeping another variable fixed in each case.

We are now in the position to give the definition:

Definition 1.2.1 (*Analytic definition*)

Let f be a sense preserving homeomorphism in a domain G ($\subseteq \mathbf{C}$) satisfying the following two conditions:

(*i*) *F is absolutely continuous on lines in G (ACL),*

(*ii*) $D_f(z) \leq K$ *for almost all $z \in G$.*

Then f is called a K-quasiconformal mapping of G, or K-qc of G.

Condition (i) asserts that f is ACL on G and any homeomorphism which is ACL on G possesses finite partial derivatives almost everywhere in G (Lehto & Virtanen [1] p.128). It is also a consequence of a theorem due to Lehto and Gehring (see Lehto & Virtanen [1] p.128) that, *if $f : G \to G'$ is a homoemorphism and G, G' are bounded, and if f has a finite partial derivatives a.e. in G, then f must be differentiable almost everywhere in G.* Hence the condition (ii) makes sense and is well defined.

The Definition 1.2.1 is called the analytic definition of a quasiconformal mapping, there is also an equivalent geometric definition, we refer this to the standard reference of Lehto & Virtanen [1]. We shall only use the analytic definition in sequel.

Let us define the function $\mu_f = \frac{f_{\bar{z}}}{f_z}$, which is called the *complex dilatation* of f. It has an obvious relation to the dilatation quotient in the following way, namely

$$D_f(z_0) = \frac{|f_z(z_0)| + |f_{\bar{z}}(z_0)|}{|f_z(z_0)| - |f_{\bar{z}}(z_0)|} = \frac{1+|\mu_f|}{1-|\mu_f|} \leq K, \text{ and this is equivalent to}$$

$$|\mu_f| = \frac{1-D_f}{1+D_f} \leq \frac{1-K}{1+K} = k < 1.$$

A homeomorphism satisfying the Definition 1.2.1 is called a *K-qc mapping*, a *1-qc mapping* is just a conformal mapping. The composition of a *K_1-qc mapping* and a *K_2-qc mapping* is a *K_1K_2-qc mapping.* It can also be shown that the complex dilatation is invariant with respect to any conformal mapping and that the inverse mapping has the same complex dilatation at the corresponding points. A K-qc mapping also satisfies the following fundamental theorem. Although we will not use it, we shall include it here for completeness.

Theorem 1.2.1 (*Lehto & Virtanen* [1] *p.*185) *A homeomorphism f is K-quasiconformal if and only if f is a* L^2*- solution of an equation* $\quad f_{\bar z} = \mu f_z$, (1.2.2)

where μ *satisfies*

$$|\mu| \le \frac{1-K}{1+K} = k < 1,$$

for almost all z.

Equation (1.2.2) is called the *Beltrami equation.*

§ *1.3 Compactness Property*

A family of functions $\mathcal{W}$ which is defined in G is called *normal* if every infinite sequence of elements of $\mathcal{W}$ contains a subsequence which converges uniformly in any compact subset of G. We say that a normal family to be *compact* if the limit of any subsequence converges locally uniformly also belongs to the family. We have the following criterion for normality :

Theorem 1.3.1 (*Lehto & Virtanen* [1] *p.*73) *A family* $\mathcal{W}$ *of K-qc mappings of the domain G is normal if there exists a* $d>0$ *such that, for every mapping* $w \in \mathcal{W}$ *belonging to the family takes values at three different fixed points* z_1, z_2, $z_3 \in G$ *such that the distance* $d(w(z_i), w(z_j)) \ge d > 0$, $i,j=1,2,3$, $i \ne j$.

Quasiconformal mappings also possess the compactness property:

Theorem 1.3.2 (*Lehto & Virtanen* [1] *p.*74) *The limit function w of a sequence* w_n *of K-qc mappings convergent in G is either a constant, or a mapping of G onto two points, or a K-qc mapping of G.*

§ *1.4 Quasiconformal Extensions*

Given a function f which is K-quasiconformal mapping in a region $G(\ne \bar{\mathbb{C}})$, we investigate conditions on f such that f admits a K'*-quasiconformal extension* outside the region G to $\bar{\mathbb{C}}$. In this direction a fundamental result is

Theorem 1.4.1 *(Lehto & Virtanen [1] p.96) Let $f_0 : G \to G'$ be a K-quasiconformal mapping and F a compact subset of G. Then there exists a quasiconformal mapping of the whole plane which coincides with f_0 in F and whose maximal dilatation is bounded by a number depending only on K, G and F.*

For reasons of simplicity, we consider f to be analytic in Δ, and find conditions so that f admits a K-quasiconformal extension to $\bar{\mathbb{C}}$ for some $K \geq 1$. If, in addition, f is locally univalent, that is $f' \neq 0$ in Δ , then the existence of a quasiconformal extension implies that the extension $\tilde{f}$ and f together form a local homeomorphism on the whole Riemann sphere $\bar{\mathbb{C}}$. By a well-known theorem in Topology (see Gordon W.B. [1]) which states that *a local homeomorphism of $\bar{\mathbb{C}}$ onto $\bar{\mathbb{C}}$ is actually a global homeomorphism.* We shall refer this as the *Topology theorem* in this chapter. The fact that has been used by Ahlfors, Anderson-Hinkkanen and many others to conclude that f is univalent in Δ. Hence the quasiconformal extension criteria that we are seeking also give rise to univalency criteria of f. To be more explicit: let f be locally univalent in $\Delta = |z|<1$. We find g in $\overline{\Delta^*} = \{z\colon |z| \geq 1\}$ so that f will have a continuous extension to $|z|=1$, when f satisfies some additional criteria. The extension is given by the following function:

$$F(z) = \begin{cases} f(z) & z \in \Delta \\ g(z) & z \in \overline{\Delta^*} \end{cases}$$

and $f(z)=g(z)$ on $|z|=1$. The modulus of the complex dilatation of F satisfies $|\mu_F| = |F_{\bar{z}}/F_z| \leq k \leq 1$ for almost all $z \in \Delta^*$ and $\mu_F \equiv 0$ for $z \in \Delta$. If F is also locally homeomorphic on $\partial\Delta$, then it is locally homeomorphic and hence globally homeomorphic on $\bar{\mathbb{C}}$. Since $\partial\Delta$ is a removable set of plane measure zero, and F is K-quasiconformal in Δ^* and analytic in Δ, it follows that it is K-qc in $\bar{\mathbb{C}}$ (see Lehto & Virtanen [1]). Clearly f is conformal in Δ by the Topology theorem. Note that the function g chosen must be sufficiently smooth so that it is ACL in Δ. e.g. $g \in C^1(\Delta)$ (the class of functions that have continuous first order partial derivatives in Δ).

There are many methods to produce a quasiconformal extension, for a given locally univalent function in Δ. The whole area of this research was first initiated by the famous paper of Ahlfors and Weill [1], in which they gave the first sufficient criterion for quasiconformal extension, by constructing explicitly the extension:

Theorem 1.4.2 (*Ahlfors and Weill* [1]) *Let f be locally univalent in* Δ, *if*

$$(1-|z|^2)^2|S(f,z)| \leq 2k \quad \forall\ z\in\Delta \tag{1.4.1}$$

where $k < 1$ *and* $S(f, z)$ *is the Schwarzian derivative defined as*

$$S(f,z) = \left(\frac{f''}{f'}\right)'(z) - \frac{1}{2}\left(\frac{f''}{f'}(z)\right)^2,$$

then f has a K-quasiconformal extension to $\bar{\mathbb{C}}$ *with* $K=\frac{1+k}{1-k}$.

The case when $k=1$ was obtained by Z. Nehari in 1949 [1] as a univalence criterion. We shall also consider this case in Chapters 2 and 3, in which we shall look at the Schwarzian derivative more closely.

Other methods, like the Löwner differential equation used by Becker [1, 2] to obtain univalence criteria are extremely powerful. He gave an extension of Theorem 1.4.2 and many others. More recently Gehring and Pommerenke [1] gave a refined version of Theorem 1.4.2. We refer the readers to Becker's survey paper in 1980 [2] which contains an excellent account of many aspects of qc extensions. J.G. Krzyż also obtained many interesting criteria, among them we mention the following:

Definition 1.4.1 Let $f(z)=z+\sum_2^\infty a_n z^n$ *be an analytic function defined in* Δ, *for some* $0<\alpha\leq 1$ *such that f satisfies*

$$\left| \arg \frac{zf'(z)}{f(z)} \right| \leq \frac{\alpha\pi}{2} \quad \forall\ z\in\Delta,$$

then f is said to be a strongly starlike function of order-α. *The family of such functions is denoted by* $S^*(\alpha)$.

Note that when $\alpha=1$, $S^*(1)=S^*$, the class of ordinary starlike functions.

Fait, Krzyż and Zygmunt [1] proved

Theorem 1.4.3 *Let* $f \in S^*(\alpha)$ *for* $0<\alpha<1$, *then the mapping F defined by the formula*

$$F(z)=\begin{cases} f(z) & |z|\leq 1 \\ |f(\zeta)|^2/\bar{f}\left(\frac{1}{\bar{z}}\right) & |z|\geq 1 \end{cases}$$

where ζ *satisfies the conditions:* $|\zeta|=1$, $\arg f(\zeta)=\arg f(1/\bar{z})$, *is K-qc mapping of* $\bar{\mathbb{C}}$ *with* $k\leq \sin(\frac{\alpha\pi}{2})$ *almost everywhere.*

In 1973 Ahlfors [1] published a short paper in which he proved:

Theorem 1.4.4 (*Ahlfors L.V.* [1]) *Let f be locally univalent in* Δ *(i.e.* $f'\neq 0$*) and suppose that f satisfies either*

$$\left|(1-|z|^2)\, z\frac{f''(z)}{f'(z)}-c|z|^2\right| \le k \quad \forall\ z\in\Delta, \tag{1.4.2}$$

or

$$\left|(1-|z|^2)^2 S(f,z)-2c(1-c)\ \bar{z}^2\right| \le 2k|1-c| \quad \forall\ z\in\Delta, \tag{1.4.3}$$

for some constant $c \in \mathbb{C}$ *with* $|c|\le k<1$. *Then there exists a K-qc extension to* $\bar{\mathbb{C}}$.

Note that (1.4.2) is a generalization of an earlier result due to Becker [1] when $c=0$ and (1.4.3) also generalizes Ahlfors' own result (1.3.1). The method of producing criteria (1.4.2) and (1.4.3) were suprisingly simple when compared to, for example, the method of Löwner differential equations used by Becker. Firstly one need to produce an extension and then use a normal family argument. Along this direction, Anderson and Hinkkanen have recently generalized Ahlfors' method completely to the case of domains of the most general nature i.e. a general *K-quasidisc* that is the homeomorphic image of Δ under a K-qc mapping on the plane. If $A=\Delta$, then their results not only give alternative proofs but also generalize earlier results of C.L. Epstein [1] and Ch. Pommerenke [2].

Theorem 1.4.5 (*Anderson and Hinkkanen* [1]) *Let f be locally univalent in* Δ. *Suppose that*

$$\left|\frac{(1-|z|^2)^2\left(g_z(z)-g(z)^2-\frac{1}{2}S(f,z)\right) - 2\bar{z}(1-|z|^2)\ g(z)}{1 + (1-|z|^2)\ g_{\bar{z}}(z)}\right| \le k, \tag{1.4.4}$$

for all $z\in\Delta$, *where* $g \in C^1(\Delta)$ *is a complex-valued function such that it satisfies either*

(*i*) *that* $g_{\bar{z}}$ *is real-valued and* $\qquad \limsup_{|z|\to 1} |g(z)|(1-|z|^2) \le k<1$

$$1+(1-|z|^2)^2 g_{\bar{z}}(z) > 0 \quad \forall\ z\in\Delta,$$

or

(*ii*) $g_{\bar{z}}$ *is complex-valued and* $\qquad \limsup_{|z|\to 1} |g(z)|(1-|z|^2)^2 \le \tau$ *and*

$$\limsup_{|z|\to 1} |g_{\bar{z}}(z)|(1-|z|^2)^2 < \frac{1}{2}\left(1-\frac{4\tau^2(1+k^2)}{(1-k^2)^2}\right) \quad \forall\ z\in\Delta,\ 0\le k+2\tau<1. \tag{1.4.5}$$

Then f is univalent in Δ *and has a K-qc extension* $h(z)$ *to* $\bar{\mathbb{C}}$ *given by*

$$h(z)=f(\tfrac{1}{\bar z}) + \frac{(z-\frac{1}{\bar z})\, f'(\frac{1}{\bar z})}{1+\left(z-\frac{1}{\bar z}\right)\left\{g(\frac{1}{\bar z})-\frac{1}{2}\frac{f''}{f'}(\frac{1}{\bar z})\right\}} \qquad z\in\Delta^*.$$

The above is equivalent to

$$g(z) = \left(z-\frac{1}{\bar z}\right)^{-1} + \frac{1}{2}\frac{f''}{f'}(z) + f'(z)\left(h(\tfrac{1}{\bar z})-f(z)\right)^{-1} \quad \forall\ z\in\Delta. \tag{1.4.6}$$

§ *1.5 Ahlfors' Method*

Let us now discuss Ahlfors' method. We consider f to be locally univalent in Δ, but we shall assume f to be actually locally univalent in the neighbourhood of Δ and remove this extra assumption later. We define an extension of f as follows:

$$F(z) = \begin{cases} f(z) & z\in\overline{\Delta} \\ \overline{g}(1/\overline{z}) & z\in\overline{\Delta^*} \end{cases} \tag{1.5.1}$$

where $g(z)=\overline{f(z)+u(z)}$ is chosen to be sufficiently smooth in Δ. Also $u(z)=0$ on $|z|=1$ so that $f(z)=\overline{g}(z)$ there. F is K-qc in $|z|\geq 1$ if and only if g is a sense preserving K-qc in $|z|\leq 1$. Hence to ensure F is K-qc in $|z|\geq 1$, we need to show, through direct computation and (1.2.2), that

$$|g_{\bar z}(z)| \leq k\ |g_z(z)| \quad \forall\ z\in\Delta\ ,\ k=\frac{K-1}{K+1}. \tag{1.5.2}$$

That is

$$|f'(z)+u_z(z)| \leq k\ |u_{\bar z}(z)| \quad \forall\ z\in\Delta, \tag{1.5.3}$$

so it is necessary that $u_{\bar z}(z)\neq 0\ \ \forall\ z\in\Delta$. Finally, we require F to be locally homeomorphic on $\partial\Delta$ in order to apply the Topology theorem.

Since f is locally univalent and the extension $f+u$ is sufficiently smooth it is continuously differentiable. Hence the weaker condition

$$|g_{\bar z}(z)| < |g_z(z)| \quad \forall\ z\in\overline{\Delta}, \tag{1.5.4}$$

will imply the Jacobian is not equal to zero in $\bar\Delta$. Thus by the *inverse function theorem* (see W. Rudin [1] p.221), F is locally homeomorphic everywhere in $\overline{\mathbf{C}}$ and so homeomorphic in $\overline{\mathbf{C}}$. Hence f is univalent in Δ.

§ *1.6 Application to the Logarithmic Derivative $\frac{f''}{f'}$*

We prove the following:

<u>Theorem</u> <u>1.6.1</u> *Let f be locally univalent in Δ and suppose that $0<k<1$, $1/2\le a\le 1$ and $p(z)$ be analytic in Δ such that it is subordinate to the function $\frac{1+kz}{1-kz}$ in Δ. If f satisfies the inequality*

$$\left| |z|^{2a}\Big(\frac{p(z)-1}{p(z)+1}\Big) - (1-|z|^{2a})\Big\{\frac{1-a}{a}+\frac{1}{a}\Big(\frac{zp'(z)}{1+p(z)}+\frac{zf''(z)}{f'(z)}\Big)\Big\}\right| \le k <1 \quad \forall\ z\in\Delta,$$

then f has a K-qc extension to $\bar{\mathbb{C}}$, $K=\frac{1+k}{1-k}$. If $k=1$ then f is univalent in Δ.

Theorem 1.6.1 is equivalent to the following statement:

<u>Theorem</u> <u>1.6.1'</u> *Let f be locally univalent in Δ and suppose that $0<k<1$ and $1/2\le a\le 1$ and $w(z)$ be analytic in Δ such that $|w(z)|\le k<1$, $w(0)=0$. If f satisfies the inequality*

$$\left| |z|^{2a}w(z) - (1-|z|^{2a})\Big\{\frac{1-a}{a}+\frac{1}{a}\Big(\frac{zw'(z)}{1-w(z)}+\frac{zf''(z)}{f'(z)}\Big)\Big\}\right| \le k <1 \quad \forall\ z\in\Delta, \tag{1.6.1}$$

then f has a K-qc extension to $\bar{\mathbb{C}}$, $K=\frac{1+k}{1-k}$. If $k=1$ and $|w(z)|<1$, then f is univalent in Δ.

The case $k=1$ and $|w(z)|\le 1$, $w\neq 1$ was in fact proved by Z. Lewandowski [2] as a univalency criterion by using the Löwner chain under a more general setting and *with* $a > 1/2$. (see also §1.8). Our theorem does not cover this case and our assumption $|w(z)|<1$ being stronger. The case when $a=1$, that is

$$\left| |z|^{2}w(z) - (1-|z|^{2})\Big(\frac{zw'(z)}{1-w(z)}+\frac{zf''(z)}{f'(z)}\Big)\right| \le k <1 \quad \forall\ z\in\Delta, \tag{1.6.2}$$

has been obtained by J. Miazga and A. Wesołowski [1] using Ahlfors' method. In fact (1.6.2) is also a generalization of an earlier univalency criterion of Z. Lewandowski [1] when $k=1$, $|w(z)|\le 1$, $w\neq 1$ proved by the Löwner chain method, as was (1.6.1). If we now let $w\equiv c\equiv$ constant, then (1.6.1) reduces to (1.4.2), and if $w\equiv 0$ then it reduces to

$$\left| a-1-(1-|z|^{2a})\, z\frac{f''(z)}{f'(z)}\right| \le ka \quad \forall\ z\in\Delta, \tag{1.6.3}$$

with the condition $|a-1|\le ka$. When $k=1$, it can also be found in Lewandowski [2]. (1.6.4)

As pointed out by Lewandowski in [1], the function $f(z)=z+\frac{1}{2}z^2$ does not satisfy (1.4.2) with $c=0$ (Becker's criterion), but it satisfies (1.6.2) with $w(z)=-z$, $k=1$.

Based on the method of Ahlfors and Miazga-Wesołowski, with further modifications, we shall now prove Theorem 1.6.1.

Proof of Theorem 1.6.1' : (a) Following Ahlfors, we assume f to be locally univalent in $\overline{\Delta}$ and we define an extension F as in (1.5.1) and set $g(z)=\overline{f(z)+u(z)}$. Choose $u(z)$ to be

$$u(z)=\frac{1}{a(1-w(z))}(\zeta(z)-z))\, f'(z),$$

where w is analytic in Δ and $|w(z)|\leq k<1$. Here ζ is a reflection in $|z|=1$ and we define the reflection ζ to be $\zeta(z)=z^{1-a}(\bar{z})^{-a}$. Note ζ is a reflection in $|z|=1$ if and only if $a>1/2$. Thus

i.e. $$\bar{g}(z) = f(z) + \frac{1}{a(1-w(z))}(z^{1-a}(\bar{z})^{-a}-z))\, f'(z) = f(z)+\frac{z(1-|z|^{2a})}{a(1-w(z))|z|^{2a}}\, f'(z).$$

We need to show that (1.5.3) is satisfied, and this is equivalent to proving

$$\left| a(1-w(z))+\zeta_z(z)-1+\Big(\frac{f''(z)}{f'(z)}+\frac{w'(z)}{1-w(z)}\Big)(\zeta(z)-z) \right| \leq k\, |\zeta_{\bar{z}}(z)|, \tag{1.6.5}$$

since $$u_z(z)=\frac{\zeta(z)-z}{a(1-w(z))}\Big(f''(z)+\frac{f'(z)w'(z)}{1-w(z)}\Big)+\frac{1}{a(1-w(z))}(\zeta_z(z)-1)f'(z)$$

and $$u_{\bar{z}}(z)=\frac{\zeta_{\bar{z}}(z)f'(z)}{a(1-w(z))}.$$

Now $\zeta_z(z)=\frac{1-a}{|z|^{2a}}$, $\zeta_{\bar{z}}(z)=\frac{-a}{|z|^{2a}}\frac{z}{\bar{z}}$ and hence (1.6.5) becomes, after rearranging the terms,

$$\left| |z|^{2a}w(z) - (1-|z|^{2a})\Big\{\frac{1-a}{a}+\frac{1}{a}\Big(\frac{zw'(z)}{1-w(z)}+\frac{zf''(z)}{f'(z)}\Big)\Big\} \right| \leq k <1 \quad \forall\ z\in\Delta. \tag{1.6.6}$$

The case when $a=1$ is contained in Miazga-Wesołowski [1]. In order to show that F is the required K-qc mapping we need to verify that $u_{\bar{z}}(z)\neq 0$ in (1.5.3). First we consider those points z so that $g(z)\neq\infty$ (this is clear from g defined above except at $z=0$). But

$$u_{\bar{z}}(z)=\frac{1}{a(1-w(z))}\frac{-a}{|z|^{2a}}f'(z) \neq 0 \ \forall\ z\in\Delta,$$

since $f'\neq 0$, so this is satisfied. Next we consider those z such that $g(z)=\infty$ (i.e. when $z=0$). Here we consider $1/g(z)$ instead, since $\mu_g(z)=\mu_{1/g}(z)$ at those z. Now

$$\Big(\frac{1}{g(z)}\Big)_z = \frac{1}{\overline{(f(z)+u(z))_z}} = \frac{-1}{\overline{(f(z)+u(z))^2}}\,\overline{(f(z)+u(z))_{\bar{z}}}$$

$$= \frac{-\overline{u(z)_{\bar{z}}}}{\overline{(f(z)+u(z))^2}} = \frac{-\overline{(u_{\bar{z}}/u^2)}}{\overline{\Big(\frac{f(z)}{u(z)}+1\Big)^2}} = \frac{\overline{(1/u)_{\bar{z}}}}{\overline{\Big(\frac{f(z)}{u(z)}+1\Big)^2}}\,. \tag{1.6.7}$$

Hence it is equivalent to show $(1/u)_{\bar{z}}\neq 0$ at those points.

$$\Big(\frac{1}{u(z)}\Big)_{\bar{z}}=-\frac{u(z)^2}{u_{\bar{z}}(z)}=\frac{f'(z)}{|z|^{2a}(1-w(z))}\frac{z}{\bar{z}}\frac{a^2(1-w(z))^2|z|^{4a}}{z^2(1-|z|^{2a})f'(z)^2}=\frac{a^2(1-w(z))|z|^{2a-2}}{(1-|z|^{2a})^2f'(z)}\neq 0$$

for all $z\in\Delta$, since $f'\neq 0$ and $2a-2\leq 0$. Note that it is precisely here the condition $a\leq 1$ is used.

Hence $u_{\bar{z}} \neq 0 \ \forall \ z \in \Delta$. This shows that $\bar{g}(z)$ is a qc mapping in $\bar{\Delta}$, and hence

$$F(z) = \begin{cases} f(z) & z \in \bar{\Delta} \\ \bar{g}(\frac{1}{\bar{z}}) = f(1/\bar{z}) + u(1/\bar{z}) & z \in \overline{\Delta^*} \end{cases} \tag{1.6.7a}$$

is K-qc in $\bar{\mathbb{C}} \backslash \bar{\Delta}$ and locally univalent in $\bar{\Delta}$.

(b) We need to ensure that F is also locally homeomorphic on $\partial\Delta$. For $e^{i\theta} \in \partial\Delta$ and η sufficiently small,
$$F(e^{i\theta}+\eta) = f(e^{i\theta}) + \eta\, f'(e^{i\theta}) + O(\eta^2)$$
if $e^{i\theta}+\eta \in \bar{\Delta}$; while if $e^{i\theta}+\eta \in \mathbb{C}\backslash\Delta$, we consider $\bar{g}$ at $e^{i\theta}+\delta \in \Delta$ instead, where δ is small. After some calculations we obtain the expansion

$$\bar{g}(e^{i\theta}+\delta) = f(e^{i\theta}) + f'(e^{i\theta})\left\{\frac{-\delta w(e^{i\theta}+\delta) - \bar{\delta}\, e^{i2\theta}}{1 - w(e^{i\theta}+\delta)}\right\} + O(\delta^2). \tag{1.6.7b}$$

Since $f'(e^{i\theta}) \neq 0$ and $|w(e^{i\theta}+\delta)| \leq k < 1$, it follows that F is locally homeomorphic on $\partial\Delta$. By the Topology Theorem F is therefore homeomorphic and hence it is K qc in $\bar{\mathbb{C}} \backslash \bar{\Delta}$ and conformal in $\bar{\Delta}$. Note that J. Miazga and A. Wesołowski [1] did not show that their extension is locally homeomorphic on $\partial\Delta$.

(c) To complete the proof of the Theorem when $0<k<1$, we consider the functions $f(rz)$ and $w(rz)$ where $0<r<1$, now $f(rz)$ and $g(rz)$ are analytic in $\bar{\Delta}$. We aim to show that they also satisfy (1.6.6) and so by what we have proved in the part (a) and (b) we conclude that $f(rz)$ also has a K-qc extension to $\bar{\mathbb{C}}$. Since f is conformal in $\bar{\Delta}$ and so we can choose a constant $d>0$ and three distinct points z_1, z_2 and z_3 belonging to Δ such that the mutual distances between them satisfy $d(f(z_i), f(z_j)) \geq d > 0$, $i,j=1,2,3$, $i \neq j$. Let us choose a sequence $\{r_n\}$ such that $0<r_n<1$, $r_n \to 1$ as $n \to \infty$ and $d(f(r_n z_i), f(r_n z_j)) > d > 0$ $\quad i,j=1,2,3 \quad i \neq j$. Define $f_n(z) = f(r_n z)$, $w_n(z) = w(r_n z)$, and

$$F_n(z) = \begin{cases} f_n(z) & z \in \bar{\Delta} \\ \bar{g}_n(1/\bar{z}) = f(z) + \dfrac{z(1-|z|^{2a})}{a(1-w_n(z))|z|^{2a}}\, f'(z) & z \in \overline{\Delta^*}. \end{cases} \tag{1.6.8}$$

We see that the $\{F_n\}$ is a family of K-qc mapping in $\bar{\mathbb{C}}$ which is conformal in $\bar{\Delta}$ and K-qc in $\bar{\mathbb{C}} \backslash \bar{\Delta}$. Moreover the family is *normal* by the choice of the sequence $\{r_n\}$ and the Theorem 1.3.1. We can thus choose a suitable subsequence such that F_n converges locally uniformly to a limit function, F_0 say. By the compactness Theorem 1.3.2, F_0 is also a K-qc mapping in $\bar{\mathbb{C}}$ since it is not a constant or a mapping of two points. Clearly

$$F_0(z) = \begin{cases} f(z) & z\in\Delta \\ \bar{g}(\frac{1}{\bar{z}}) = f(1/\bar{z}) + u(1/\bar{z}) & z\in\overline{\Delta^*} \end{cases}.$$

Hence f being locally homeomorphic in Δ and has a K-qc extension to $\bar{\mathbf{C}}$. We have completed our proof once we show f_n and w_n also satisfy (1.6.6).

(d) Now let us replace z by $r_n z$ in (1.6.6), and we obtain

$$\left| \frac{1-a}{a} + \frac{1}{a}\left(\frac{r_n z w'(r_n z)}{1-w(r_n z)} + \frac{r_n z f''(r_n z)}{f'(r_n z)}\right) - \frac{|r_n z|^{2a} w(r_n z)}{1-|r_n z|^{2a}} \right| \le \frac{k}{1-|r_n z|^{2a}}. \tag{1.6.9}$$

If f_n and w_n also satisfy (1.6.6), then we have

$$\left| \frac{1-a}{a} + \frac{1}{a}\left(\frac{r_n z w'(r_n z)}{1-w(r_n z)} + \frac{r_n z f''(r_n z)}{f'(r_n z)}\right) - \frac{|z|^{2a} w(r_n z)}{1-|z|^{2a}} \right| \le \frac{k}{1-|z|^{2a}}. \tag{1.6.10}$$

Set
$$A_n(z) = \frac{1-a}{a} + \frac{1}{a}\left(\frac{r_n z w'(r_n z)}{1-w(r_n z)} + \frac{r_n z f''(r_n z)}{f'(r_n z)}\right),$$

and
$$B_n(z) = \frac{|r_n z|^{2a} w(r_n z)}{1-|r_n z|^{2a}}, \qquad C_n(z) = \frac{|z|^{2a} w(r_n z)}{1-|z|^{2a}}.$$

So inequalities (1.6.9) and (1.6.10) can be written as

$$|\, A_n(z) - B_n(z) \,| \le \frac{k}{1-|r_n z|^{2a}}, \tag{1.6.9'}$$

and

$$|\, A_n(z) - C_n(z) \,| \le \frac{k}{1-|z|^{2a}} \tag{1.6.10'}$$

respectively. If we write $\mathfrak{D}(c,r)=\{z\colon\ |z-c|<r\}$ and let $\bar{\mathfrak{D}}(c,r)$ to be the *closure* of $\mathfrak{D}(c,r)$, then (1.6.9)' and (1.6.10)' represent the discs $\bar{\mathfrak{D}}\Big(B_n,\ \frac{k}{1-|r_n z|^{2a}}\Big)$ and $\bar{\mathfrak{D}}\Big(C_n,\ \frac{k}{1-|z|^{2a}}\Big)$. Also f_n and w_n will satisfy (1.6.6) if we can show that (1.6.9)' $\Rightarrow$ (1.6.10)'. i.e. $\bar{\mathfrak{D}}\Big(B_n,\ \frac{k}{1-|r_n z|^{2a}}\Big) \subseteq \bar{\mathfrak{D}}\Big(C_n,\ \frac{k}{1-|z|^{2a}}\Big)$. This will be implied by the fact that if the sum of the distances between the centres of the discs and the smaller radius is *less than or equal to* the larger radius. That is equivalent to proving $|\, B_n(z) - C_n(z) \,| + \frac{k}{1-|r_n z|^{2a}} \le \frac{k}{1-|z|^{2a}}$,

or

$$|\, B_n(z) - C_n(z) \,| \le \frac{k}{1-|z|^{2a}} - \frac{k}{1-|r_n z|^{2a}}. \tag{1.6.11}$$

But

$$B_n(z) - C_n(z) = |z|^{2a} w(r_n z)\Big(\frac{r_n{}^{2a}}{1-|r_n z|^{2a}} - \frac{1}{1-|z|^{2a}}\Big) = w(r_n z)\left(\frac{1}{1-|r_n z|^{2a}} - \frac{1}{1-|z|^{2a}}\right), \tag{1.6.12}$$

since $$|z|^{2a}\Big(\frac{r_n{}^{2a}}{1-|r_n z|^{2a}}-\frac{1}{1-|z|^{2a}}\Big) = \frac{1}{1-|r_n z|^{2a}} - \frac{1}{1-|z|^{2a}},$$

and $|w_n(z)| \leq k$. This proves (1.6.11) and also completes the proof of the theorem when $0<k<1$.

(e) The case when $k=1$ can be proved by modifying the proof of the case when $0<k<1$ above. Let us assume that f satisfies (1.6.1) with $k=1$ and define an extension $g=\overline{f+u}$ as above where f is regular in $\bar{\Delta}$. We note that the extension satisfies (1.6.1) with $k=1$ is precisely the condition
$$|g_{\bar{z}}(z)| \leq |g_z(z)| \quad \forall\ z\in\Delta, \tag{1.6.13}$$
with $g_z(z)\neq 0$ $\forall z\in\Delta$. The proof of the inequality is exactly the same as in part (a) and $|w|<1$ in the expansion (1.6.7a) with $k=1$ in (b). Care must be taken as we allow *equality* to happen in (1.6.13), hence we merely can say that if f satisfies (1.6.1) with $k=1$ and f is regular on $|z|\leq 1$, then it only has a *smooth extension* to $\bar{\mathbb{C}}$, which is *not* necessarily locally homeomorphic everywhere. By defining f_n and w_n as in (c), we may choose the sequence satisfying $0<r_n<1$, $r_n \to 1$ as $n \to \infty$ without further restriction. We show that they satisfy a stronger condition $|(f_n)_{\bar{z}}|<|(f_n)_z|$ in Δ^*(we remember as before that f_n also represents the extension of f_n). Since f satisfies (1.6.6) with $k=1$, hence we have again (1.6.9) or (1.6.9)' with $k=1$. We prove f_n and w_n satisfy
$$|\ A_n(z)\ -\ C_n(z)\ |\ <\ \frac{1}{1-|z|^{2a}}$$
or $\bar{\mathfrak{D}}\Big(B_n,\ \frac{1}{1-|r_n z|^{2a}}\Big) \subset \mathfrak{D}\Big(C_n,\ \frac{1}{1-|z|^{2a}}\Big)$. i.e. $\bar{\mathfrak{D}}\Big(B_n,\ \frac{1}{1-|r_n z|^{2a}}\Big)$ is contained in the *interior* of $\bar{\mathfrak{D}}\Big(C_n,\frac{1}{1-|z|^{2a}}\Big)$. This is indeed the case by applying the *radii argument* above since we have (1.6.12) and $|w|<1$. This shows that each f_n regular in $\bar{\Delta}$ has a locally homeomorphic extension to $\bar{\mathbb{C}}$ (see (1.5.4)) given by (1.6.8), thus each F_n is globally homeomorphic in $\bar{\mathbb{C}}$ and hence univalent in $\bar{\Delta}$, $F_n=f_n$ in Δ. Now $f_n \to f$ as $n \to \infty$, uniformly on any compact subset of $\bar{\Delta}$ and hence on Δ. By Hurwitz's theorem (see Duren [1] p.4), we have that *for any sequence of univalent functions $\{f_n\}$ in a simply open connected set D and $f_n \to f$ as $n \to \infty$, uniformly on any compact subset of D, then f is either univalent or constant in D.* Here we have $D=\Delta$ and since f is not a constant it must be univalent in Δ. This proves the case when $k=1$ and also completes the proof of the theorem. □

§ 1.7 Applications to the Schwarzian Derivative

Following the above ideas, we shall apply the same method to obtain:

Theorem 1.7.1 *Let f be locally univalent in Δ, and suppose $0<k<1$, $\frac{1}{2}\leq\frac{1}{1+k}\leq a\leq 1$ and that $p(z)$ is analytic and subordinate to $\frac{1+\rho z}{1-\rho z}$ in Δ where $\rho=\frac{a-1+ka}{a}\geq 0$. If f satisfies the inequality*

$$\left| a^2\frac{1-p(z)}{1+p(z)}\,|z|^{2a}+a(1-|z|^{2a})\left(1-a+\frac{zp'(z)}{1+p(z)}\right)+\frac{1+p(z)}{4}(1-|z|^{2a})^2S(f,z)|z|^{2-2a}\frac{z}{\bar{z}}\right|\leq ka^2 \tag{1.7.1}$$

$\forall\ z\in\Delta$, then f has a K-qc extension to $\bar{\mathbb{C}}$, where $K=\frac{1+k}{1-k}$. If $k=1$, then f is univalent in Δ.

Note that when $k=1$ and $a=1$, this is a univalency criterion obtained by Z. Lewandowski and J. Stankiewicz [1] using the method of Löwner chains. See also Miazga-Wesołowski [2]. Condition (1.7.1) reduces to :

$$\left|\frac{1-p(z)}{1+p(z)}\,|z|^2+(1-|z|^2)+\frac{zp'(z)}{1+p(z)}+\frac{1+p(z)}{4}\,(1-|z|^2)^2S(f,z)\,\frac{z}{\bar{z}}\right|\leq 1\quad\forall\ z\in\Delta, \tag{1.7.2}$$

where $p(z)$ is analytic and $\Re(p)>0$, $p(0)=0$ in Δ.

We see that by replacing 1 by $k<1$, their univalence criterion becomes a sufficient condition of K-qc extension. Theorem 1.7.1 therefore gives an alternative proof and also generalizes their result. Also by putting $w(z)=\frac{p(z)-1}{p(z)+1}$, (1.7.1) is equivalent to the following criterion:

Theorem 1.7.1' *Let f be locally univalent in Δ and suppose $0<k<1$, $\frac{1}{2}\leq\frac{1}{1+k}\leq a\leq 1$, and let w be analytic with $|w(z)|\leq\frac{a+ak-1}{a}$, $w(0)=0$ in Δ. Suppose f satisfies the inequality*

$$\left| aw(z)|z|^{2a}-a(1-|z|^{2a})\left(1-a+\frac{zw'(z)}{1-w(z)}\right)-\frac{(1-|z|^{2a})^2}{2(1-w(z))}\,S(f,z)|z|^{2-2a}\frac{z}{\bar{z}}\right|\leq ka^2 \tag{1.7.3}$$

$\forall\ z\in\Delta$, then f has a K-qc extension to $\bar{\mathbb{C}}$. If $k=1$ and $|w(z)|<\frac{2a-1}{a}$, then f is univalent in Δ.

It is this form of the theorem that we prove below. Now put $w\equiv c\equiv$constant, then (1.7.3) reduces to

$$\left| a(1-c)c|z|^{2a}-a(1-a)(1-c)(1-|z|^{2a})-\frac{(1-|z|^{2a})^2}{2}\,S(f,z)|z|^{2-2a}\frac{z}{\bar{z}}\right|\leq ka^2|1-c|$$

and $|c|\leq\frac{a+ak-1}{a}<1$. Note that if $a=1$ then we get back to Ahlfors' result (1.4.3). If we put $w\equiv 0$ in (1.7.3) then we obtain

$$\left| a(1-a)(1-|z|^{2a})-\frac{(1-|z|^{2a})^2}{2}\, S(f,z)|z|^{2-2a}\frac{z}{\bar z}\right| \le ka^2 \ \forall\ z\in\Delta.$$

where $1\ge a\ge 1/(1+k)\ge 1/2$.

Proof of Theorem 1.7.1' (a) As before we first assume f is locally univalent in $\overline{\Delta}$ and we define an extension F as in (1.5.1) and *set* $g=\overline{f+u}$. We prove g satisfies (1.5.2). Choose

$$u(z) = \frac{f(z)}{v(z)-\frac{1}{2}\frac{f''(z)}{f'(z)}} \quad \text{and} \quad v(z) = \frac{a(1-w(z))}{\zeta(z)-z},$$

where w satisfies the hypotheses of the theorem and $\zeta(z)=z^{1-a}(\bar z)^{-a}$ again. Clearly $u\equiv 0$ when $|z|=1$. Note that (1.5.3) becomes

$$|\tfrac{1}{2}S(f,z)+v^2-v_z| \le k\,|v_{\bar z}| \quad \forall\ z\in\Delta. \tag{1.7.4}$$

Now
$$v(z) = \frac{a(1-w(z))|z|^{2a}}{z(1-|z|^{2a})}$$
and
$$v(z)_z = \frac{-aw'(z)|z|^{2a}}{z(1-|z|^{2a})} + \frac{a(1-w(z))|z|^{2a}(a-1+|z|^{2a})}{z^2(1-|z|^{2a})^2}\quad,\quad v(z)_{\bar z} = \frac{a^2(1-w(z))|z|^{2a-2}}{(1-|z|^{2a})^2}.$$

Substitute these expressions back into (1.7.4) to obtain (1.7.3). To show that F is the required K-qc mapping we need to verify that $u(z)_{\bar z}\ne 0$ for $z\in\Delta$ such that $u(z)\ne\infty$ in (1.5.3), that is $v_{\bar z}\ne 0$. We also must consider those z such that $v(z)\ne\frac{1}{2}\frac{f''(z)}{f'(z)}$ since $u=\infty$ if and only if $v(z)=\frac{1}{2}\frac{f''(z)}{f'(z)}$.

But
$$v(z)_{\bar z} = \frac{a^2(1-w(z))|z|^{2a-2}}{(1-|z|^{2a})^2} \ne 0 \quad \forall\ z\in\Delta, \text{ since } a\le 1.$$

Next we consider those z such that $u(z)=\infty$. Here we need to consider $1/g$ instead. As shown in the proof of the Theorem 1.6.1, this is equivalent to proving that $(1/u(z))_{\bar z}\ne 0$ at those z . Now

$$\left(\frac{1}{u(z)}\right)_{\bar z} = \left[\frac{v(z)-\frac{1}{2}\frac{f''(z)}{f'(z)}}{f(z)}\right]_{\bar z} = \frac{v(z)_{\bar z}}{f(z)} = \frac{a^2(1-w(z))|z|^{2a-2}}{f(z)(1-|z|^{2a})^2} \ne 0, \text{ since } f\ne 0 \text{ and } a\le 1.$$

This proves that $|\mu_f|\le k$ with the assumption that f is defined in $|z|\le 1$.

(b) To show that F is locally homeomorphic on $\partial\Delta$ we repeat the same argument as in (b) of the proof of Theorem 1.6.1'. We find, after some calculations, that the expansion of F on $\partial\Delta$ is exactly the same as (1.6.7b). Again since $f'(e^{i\theta})\ne 0$ and $|w(e^{i\theta}+\delta)|\le(a+ak-1)/a<1$, it follows that F is locally homeomorphic on $\partial\Delta$. The case when $k=1$ can be dealt with in the same way. Hence F is K qc in $\bar{\mathbb{C}}\backslash\bar{\Delta}$ and conformal in $\bar{\Delta}$.

(c) To complete the proof when $0<k<1$, we use the compactness argument and the normal family argument in the proof of Theorem 1.6.1' (c) again. Firstly, we rewrite (1.7.3) as :

$$\left| A(z) + \frac{a(1-w(z))\Big((1-a)(1-|z|^{2a}-aw(z)|z|^{2a}\Big)}{|z|^{2-2a}(1-|z|^{2a})^2} \right| \le \frac{ka^2|1-w(z)|}{|z|^{2-2a}(1-|z|^{2a})^2}, \qquad (1.7.5)$$

where
$$A(z) = \tfrac{1}{2}S(f,z) + \frac{azw'(z)}{|z|^{2-2a}(1-|z|^{2a})}.$$

Now replace z by $r_n z$ in (1.7.5) and multiply by $r_n{}^2$ on both sides, then with the following notations

$$A_n(z) = \frac{r_n{}^2}{2}S(f,\ r_n z) + \frac{r_n{}^3 a\ zw'(r_n z)}{|r_n z|^{2-2a}(1-|r_n z|^{2a})},$$

$$C(z) = \frac{a\ r_n{}^2(1-w(r_n z))\Big((1-a)(1-|r_n z|^{2a}-aw(r_n z)|r_n z|^{2a}\Big)}{|r_n z|^{2-2a}(1-|r_n z|^{2a})^2},$$

$$\tilde{C}(z) = \frac{r_n{}^2 ka^2|1-w(r_n z)|}{|r_n z|^{2-2a}(1-|r_n z|^{2a})^2}, \qquad D(z) = \frac{a(1-w(r_n z))\Big((1-a)(1-|z|^{2a})-aw(r_n z)|z|^{2a}\Big)}{|z|^{2-2a}(1-|z|^{2a})^2},$$

$$\tilde{D}(z) = \frac{ka^2|1-w(r_n z)|}{|z|^{2-2a}(1-|z|^{2a})^2},$$

the inequality (1.7.5) becomes

$$|\ A_n(z) + C(z)\ | \le \tilde{C}(z). \qquad (1.7.6)$$

However we wish to prove that, with the definitions $f_n(z)=f(r_n z)$ and $w_n(z)=w(r_n z)$, f_n and w_n also satisfiy (1.7.5). Then we deduce the theorem by the compactness property. Substitute f_n and w_n into (1.7.5) to obtain

$$|\ A_n(z) + D(z)\ | \le \tilde{D}(z). \qquad (1.7.7)$$

To prove (1.7.6) $\Rightarrow$ (1.7.7), it is equivalent to proving that, by the radii argument,

$$|\ D(z) - C(z)\ | \le \tilde{D}(z) - \tilde{C}(z) \quad \forall\ z \in \Delta.$$

Now

$$D(z)-C(z) = \frac{a(1-a)|1-w(r_n z)|}{|z|^{2-2a}(1-|z|^{2a})} - \frac{r_n{}^{2a}a(1-a)|1-w(r_n z)|}{|z|^{2-2a}(1-|r_n z|^{2a})} -$$

$$-\left\{\frac{a^2|w(r_n z)(1-w(r_n z))|\ |z|^{2a}}{|z|^{2-2a}(1-|z|^{2a})^2} - \frac{a^2|w(r_n z)(1-w(r_n z))|\ r_n{}^{2a}|r_n z|^{2a}}{|z|^{2-2a}(1-|r_n z|^{2a})^2}\right\}$$

$$=\frac{a|1-w(r_nz)|(1-{r_n}^{2a})}{|z|^{2-2a}}\left\{\frac{1-a}{(1-|z|^{2a})(1-|r_nz|^{2a})}-\frac{a|z|^{2a}w(r_nz)(1-{r_n}^{2a}-2{r_n}^{2a}|z|^{2a})}{(1-|z|^{2a})^2(1-|r_nz|^{2a})^2}\right\}$$

and
$$\tilde{D}(z)-\tilde{C}(z)=\frac{ka^2|1-w(r_nz)|}{|z|^{2-2a}}\left\{\frac{1}{(1-|z|^{2a})^2}-\frac{r^{2a}}{(1-|r_nz|^{2a})^2}\right\}$$

$$=\frac{ka^2|1-w(r_nz)|(1-{r_n}^{2a})(1-{r_n}^{2a}|z|^{4a})}{|z|^{2-2a}(1-|z|^{2a})^2(1-|r_nz|^{2a})^2}.$$

Hence it is sufficient to show

$$\left|(1-a)(1-|z|^{2a})(1-|r_nz|^{2a})-a|z|^{2a}w(r_nz)(1+{r_n}^{2a}-2{r_n}^{2a}|z|^{2a})\right|\le ka(1-{r_n}^{2a}|z|^{4a}). \tag{1.7.9}$$

Note that
$$|z|^{2a}(1+{r_n}^{2a}-2{r_n}^{2a}|z|^{2a})\le 1-{r_n}^{2a}|z|^{4a},$$

so
$$\left|(1-a)(1-|z|^{2a})(1-|r_nz|^{2a})-a|z|^{2a}w(r_nz)(1+{r_n}^{2a}-2{r_n}^{2a}|z|^{2a})\right|$$
$$\le (1-a)(1-|z|^{2a})(1-|r_nz|^{2a})\ +\ a|w(r_nz)|(1-{r_n}^{2a}|z|^{4a})$$
$$\le (1-{r_n}^{2a}|z|^{4a})\Big(1-a\ +\ a|w(r_nz)|\Big)\ \le ka(1-{r_n}^{2a}|z|^{4a}).$$

The last inequality follows since we assumed $|w|\le\frac{a+ak-1}{a}$. This proves (1.7.9). As before we conclude that $\{f_n\}$ form a normal family. Note that $A_n\to A$, $f_n\to f$, $w_n\to w$ through a suitably chosen subsequence. This completes the proof of the theorem when $0<k<1$.

(d) The case when $k=1$ and $|w|<(2a-1)/a$ can be considered as in the proof of the Theorem 1.6.1' (e). We show that if the extension of f satisfies (1.7.3) with $k=1$ and $|w|<(2a-1)/a$ in Δ, then f_n and w_n defined above will also satisfy (1.7.3) with *strict* inequality and $k=1$. i.e.

$$\bar{\mathfrak{D}}(C,\tilde{C})\subset\mathfrak{D}(D,\tilde{D}).$$

This is true since (1.7.10) and $|w|<(2a-1)/a$ and so f_n satisfy $|(f_n)_{\bar{z}}/(f_n)_z|<1$ in $\bar{\mathbf{C}}$. Hence $\{f_n\}$ form a normal family of local homeomorphisms in $\bar{\mathbf{C}}$ and hence univalent in $\bar{\Delta}$. Its limit function f must be univalent in Δ. (for details see part (e) of the proof of Theorem 1.6.1') □

§ *1.8 Some Remarks*

We shall discuss here some fine points about the proofs of the Theorems 1.6.1' and 1.7.1'.

(*i*) In (1.7.2), the univalence criterion of Lewandowski and Stankiewicz, as Lewandowski mentioned (private communication), the hypotheses $p>0$ and $p(0)=1$ can be replaced by $p\geq 0$ in Δ. This means that the corresponding criterion of (1.7.3) when $k=1$ and $a=1$ can be taken as $|w(z)|\leq 1$ and $w\neq 1$ in Δ. Clearly our proof of (1.7.3) does not handle this case and we must have $|w(z)|<1$ for the approximations of the triangle inequalities to go through. In fact, in Theorems 1.6.1' and 1.7.1', the corresponding analytic functions w were assumed to satisfy $w(0)=0$ (hence $p(0)=1$). However, we have not made use of these assumptions.

(*ii*) In Theorem 1.6.1', the assumption $a>\frac{1}{2}$ is necessary in Lewandowski's proof, and our Theorem 1.6.1' also has this inequality. However, although $\zeta(z)=z^{1-a}(\bar{z})^{-a}$ is a reflection on $|z|=1$ only when $a>\frac{1}{2}$, we have not made use of this assumption in our proof. All we required was $a\leq 1$, hence we may relax our assumption to be $0<a\leq 1$. Recently Lewandowski [3] has extended the range of a to $a\geq\frac{1}{2}$ in a more general criterion. This shows that the proof of Lewandowski, which based on Pommerenke's subordinate chains and our proof based on Ahlfors' method give rise to two ranges of values of a, which overlap each other. It should also be noted that, by choosing a suitable branch, a could be assumed to be complex-valued. This is seen by choosing $c=0$ in (1.6.4), where the boundary condition is $|1-a| \leq k|a|$. This inequality implies that a lies in a region which is the intersection of the disc with centre $\left(\frac{1}{1-k^2}, 0\right)$, radius $\frac{k}{1-k^2}$ and the half-plane $\Re(z) \leq 1$. Similar conditions in the other cases also exist, but we choose not to pursue this.

(*iii*) We consider Theorems 1.6.1' and 1.7.1' again; let $a=1$ in both cases. i.e. we consider the criteria

$$\left| |z|^2 w(z) - (1-|z|^2)\left(\frac{zw'(z)}{1-w(z)}+\frac{zf''(z)}{f'(z)}\right)\right| \leq k <1 \quad \forall\ z\in\Delta,\ |w|\leq k, \tag{1.6.2}$$

and $$\left| w(z)|z|^2-(1-|z|^2)\,\frac{zw'(z)}{1-w(z)}-\frac{(1-|z|^2)^2}{2(1-w(z))}\,S(f,z)\frac{z}{\bar{z}}\right| \leq k\ ,\ \forall\ z\in\Delta,\ |w|\leq k.$$

The corresponding extensions g appear in (1.4.6), of the Theorem 1.4.5 of Anderson and Hinkkanen [1], are in fact given by

$$g_1(z) = \frac{-w(z)\,\bar{z}}{1-|z|^2} + \frac{1}{2}\frac{f''(z)}{f'(z)} \quad \text{and} \quad g_2(z) = \frac{-w(z)\,\bar{z}}{1-|z|^2}$$

respectively.

Now
$$(g_1)_{\bar{z}} = \frac{w(z)}{(1-|z|^2)^2} \quad \text{and} \quad (g_2)_{\bar{z}} = \frac{w(z)}{(1-|z|^2)^2},$$
and they both satisfy (1.4.5) with $\tau=0$ since w is generally a complex-valued function. But

$$\limsup_{|z|\to 1} |g_{\bar{z}}(z)|(1-|z|^2)^2 < \frac{1}{2}$$

if and only if $|w(z)|<\frac{1}{2}$. Hence Theorem 1.4.5 does not apply in these cases.

(*iv*) The proof of (c) of Theorem 1.6.1' can be made easier if we assume $f(0)=0$ and $f'(0)=1$. So that, if $\{f_n\}$ satisfy (1.6.6) they have K-qc extension. Also the class of normalized univalent functions in Δ having K-qc extension is a normal compact family with respect to the metric of locally uniform convergence. Hence we can extract a subsequence so that $f_n \to f$ and f also has a K-qc extension (see Schober [1]).

(v) We finally mention that although Theorems 1.6.1' and 1.7.1' were proved under the assumption that $w(z)$ is analytic and bounded in Δ, it seems that all we require is $w(z) \in C^1(\Delta)$ (i.e. f has first order continuous derivatives in Δ) and $|w|\leq k$. This will of course lead to some more general criteria, and we leave this for future work.

§ *1.9 Applications to f'*

Theorem 1.9.1 *Let $f(z)= z+\sum_{n=m}^{\infty} a_n z^{-n}$ $m\geq 0$ be a regular function defined in $|z|>1$. Suppose f also satisfies*

$$\left| f'(z)-1 + \frac{1-a}{|z|^{2a}} \right| \leq \frac{ka}{|z|^{2a}} \qquad |z|>1, \tag{1.9.0}$$

where $m+1 \geq a \geq \frac{1}{1+k} \geq \frac{1}{2}$ and $k<1$. Then f has a K-qc extension to $\bar{\mathbb{C}}$ with $K=\frac{1+k}{1-k}$.

The case when $a=1$ was obtained by Krzyż in 1976 [1] through direct computation. We simply apply the Ahlfors' method again with slight alterations.

Proof We first note that (1.9.0) implies that $f'(z)\neq 0$ for $|z|>1$. For suppose $f'(z_1)=0$, then (1.9.0) becomes $|1-a-|z_1|^{2a}|\leq ka$. Rewrite this as

$$-ka \le 1-a-|z_1|^{2a} \le ka.$$

From the first inequality, we obtain $a+|z_1|^{2a} \le 1+ka$. But $a+|z_1|^{2a} > 1+ka$, hence $f'(z) \ne 0$ for $|z|>1$ and f is locally univalent in $|z|>1$. Now we assume that f is locally univalent in $|z|\ge 1$, and define the extension F (similar to (1.6.7a)) in Δ where $g(z)=\overline{f(z)+(\zeta(z)-z)}$ and $\zeta(z)=z^{1-a}(\bar{z})^{-a}$. Thus

$$g(z)_z = \overline{-a\frac{1}{|z|^{2a}}\frac{z}{\bar{z}}} \quad \text{and} \quad g(z)_{\bar{z}} = \overline{f'(z)+\left((1-a)\frac{1}{|z|^{2a}}-1\right)}\,.$$

Hence from (1.5.3), we show

$$\left| f'(z)-1+\frac{1-a}{|z|^{2a}} \right| \le \frac{ka}{|z|^{2a}} \qquad |z|\ge 1. \tag{1.9.1}$$

Here we have $u(z)=z^{1-a}(\bar{z})^{-a}-z$ as before. Now $u(z)_{\bar{z}} = \frac{-a}{|z|^{2a}}\frac{z}{\bar{z}} = \frac{-ae^{i2\theta}}{r^{2a}} \ne 0$, $|z|\ge 1$ and $z\ne\infty$. When $z=\infty$, we consider $\left(g(\frac{1}{z})^{-1}\right)_z$ at 0 instead. That is to show $1/g(1/z)$ is locally univalent at the origin.

$$\left(g(\tfrac{1}{z})^{-1}\right)_z = \left(\overline{f(1/z)+u(1/z)}\right)^{-1}_z = \left(\overline{f(1/z)+u(1/z)}\right)^{-2}\overline{u(\eta)_{\bar{\eta}}\frac{\partial\bar{\eta}}{\partial\bar{z}}} \quad , \ \eta=\tfrac{1}{z}$$

$$= \frac{\overline{\left(-a|z|^{2a}\frac{z}{\bar{z}}(\frac{-1}{\bar{z}^2})\right)}}{\left(\overline{f(1/z)+u(1/z)}\right)^2} = \frac{\overline{(a|z|^{2a-2})}}{\left(\overline{f(1/z)+u(1/z)}\right)^2}$$

$$= \frac{ar^{2a-2}}{(r^{2a-1}e^{-i\theta}+a_m r^m e^{im\theta}+\cdots)^2} \qquad \text{where } z=re^{i\theta}$$

$$= \frac{a}{(r^a e^{-i\theta}+a_m r^{m+1-a}e^{im\theta}+\cdots)^2} \ne 0 \ \text{ if } \ z=0 \text{ and } m+1\ge a.$$

We then show the extension F is locally homeomorphic on $|z|=1$. It suffice to show that g is locally homeomorphic in the disc $\mathfrak{D}(e^{i\theta},\ \delta)$ for some small δ. If $e^{i\theta}+\delta\in\bar{\mathbb{C}}\backslash\Delta$, then g is locally homeomorphic, while if $e^{i\theta}+\delta\in\Delta$ we consider $g(e^{i\theta}+\delta)$ where $e^{i\theta}+\delta\in\Delta^*$. Now

$$\bar{g}(e^{i\theta}+\delta) = f(e^{i\theta}+\delta)+\left((e^{i\theta}+\delta)^{1-a}(e^{-i\theta}+\bar{\delta})^{-a}-(e^{i\theta}+\delta)\right)$$
$$= f(e^{i\theta})+\delta\left(f'(e^{i\theta})-a\right)-a\bar{\delta}e^{i2\theta}+O(\delta^2).$$

From (1.9.1), we have $|f'(e^{i\theta})-a|\le ka$ and this implies that $\delta\left(f'(e^{i\theta})-a\right)-a\bar{\delta}e^{i2\theta}\ne 0$. Hence F is locally homeomorphic on $|z|=1$.

To remove the extra assumption that f is regular on $|z|=1$, we approximate f by $f_n(z)=\frac{1}{R_n}f(R_n z)$ in (1.9.1) which becomes

$$\left| f'(R_n z)-1+\frac{1-a}{|R_n z|^{2a}} \right| \le \frac{ka}{|R_n z|^{2a}} \qquad |z|\ge 1. \tag{1.9.2}$$

As before we want to show that f_n also satisfies (1.9.1), that is

$$\left| f'(R_n z)-1+\frac{1-a}{|z|^{2a}}\right| \le \frac{ka}{|z|^{2a}} \quad |z|\ge 1. \tag{1.9.3}$$

We have omitted some details. It is now easy to show that (1.9.2) $\Rightarrow$ (1.9.3). We apply the radii argument to the closed discs (1.9.2) and (1.9.3):

$$\frac{1-a}{|z|^{2a}}-\frac{1-a}{|R_n z|^{2a}}=\frac{1-a}{|z|^{2a}}\Big(1-\frac{1}{R_n{}^{2a}}\Big) \quad \text{and} \quad \frac{ka}{|z|^{2a}}-\frac{ka}{|R_n z|^{2a}}=\frac{ka}{|z|^{2a}}\Big(1-\frac{1}{R_n{}^{2a}}\Big).$$

So (1.9.3) is true, since we have assumed that $1-a\le ka$ or $a\ge 1/(1+k)\ge 1/2$. This completes the proof of the Theorem. □

As mentioned before the above theorem, when $a=1$, is a special case of Theorem 1 in Krzyż [1]. We shall alter Krzyż's theorem slightly to obtain the following:

Lemma 1.9.2 *Suppose $w(z)$ is analytic in Δ and such that $|w'(z)|\le 1$ in Δ. Then $f(z)=\frac{1}{z}+w(z)$ is meromorphic and univalent in Δ. Moreover, if $|w'(z)|\le k<1$, then f can have a K-qc extension to $\bar{\mathbb{C}}$ with $K=\frac{1+k}{1-k}$. The extension to $\bar{\mathbb{C}}$ has the form $\tilde f(z)=z+w(1/\bar z)$ $|z|>1$.*

We omit the proof.

Let $f(z)=z+a_2z^2+a_3z^3+\cdots$ be analytic in Δ. We define:

$$g(z)=\frac{f'(\zeta)(1-|\zeta|^2)}{f\Big(\frac{z+\zeta}{1+\bar\zeta z}\Big)-f(\zeta)}=\frac{1}{z}+\Big(\bar\zeta-\frac{1}{2}(1-|\zeta|^2)\frac{f''}{f'}(\zeta)\Big)-\frac{1}{6}(1-|\zeta|^2)^2\Big(\Big(\frac{f''}{f'}\Big)'(\zeta)-\frac{1}{2}\Big(\frac{f''}{f'}(\zeta)\Big)^2\Big)z+\cdots$$

$$:=\frac{1}{z}+h(z,\zeta) \tag{1.9.4}$$

i.e. $h(z,\zeta)$ is equal to the right hand side of the above expression. We prove the following result:

Theorem 1.9.3 *Let $f(z)$, $g(z)$ and $h(z,\zeta)$ be defined as above. If*

$$|h_z(z,\zeta)|\le k<1 \quad \forall z\in\Delta \text{ and } \zeta\in\Delta, \tag{1.9.5}$$

then f has a K-qc extension to $\bar{\mathbb{C}}$. $K=\frac{1+k}{1-k}$.

The case when $k=1$ was a univalency criterion due to Ozaki and Nunokawa [1]. Thus Theorem 1.9.3 shows that by replacing 1 by $k<1$ we actually obtain quasiconformal extension. Also note that $|h_z(0,\zeta)|=\frac{1}{6}(1-|\zeta|^2)^2|S(f,\zeta)|\le k \quad \forall\zeta\in\Delta$,

is a necessary condition if f has a K-qc extension, and

$$|h_z(0,\zeta)| \le \frac{1}{3}k \ \forall \zeta \in \Delta$$

is a sufficient condition for f to have K-qc extension (Nehari's and Ahlfors' conditions).

Proof: Let f, g and h be defined as above. Since $h(z,\zeta)$ is analytic in $z\in\Delta$ and satisfies (1.9.5), it also satisfies Lemma 1.9.2 with $w(z)=h(z,\zeta)$ for a fixed ζ, $|w(z)|\le k<1$. This shows that g has a K-qc extension which is given by

$$\tilde{g}(z) = \begin{cases} \frac{1}{z}+w(z) & |z|<1 \\ z+w(1/\bar{z}) & |z|\ge 1 \end{cases}.$$

Now
$$f\left(\frac{z+\zeta}{1+\bar{\zeta}z}\right) = \frac{f'(\zeta)(1-|\zeta|^2)}{g(z)} + f(\zeta).$$

This is equivalent to
$$f(z) = \frac{f'(\zeta)(1-|\zeta|^2)}{g\left(\frac{z-\zeta}{1-\bar{\zeta}z}\right)} + f(\zeta), \quad z\in\Delta.$$

We have already shown that g has a K-qc extension. It is therefore straightforward to verify f too has the corresponding extension. According to $\tilde{g}$,

$$\tilde{f}(z) = \frac{f'(\zeta)(1-|\zeta|^2)}{\tilde{g}\left(\frac{z-\zeta}{1-\bar{\zeta}z}\right)} + f(\zeta), \quad |z|\ge 1$$

$$= \frac{f'(\zeta)(1-|\zeta|^2)}{\left(\frac{z-\zeta}{1-\bar{\zeta}z}\right)+h\left(1/\overline{\left(\frac{z-\zeta}{1-\bar{\zeta}z}\right)}\right)} + f(\zeta) = \frac{f'(\zeta)(1-|\zeta|^2)}{\left(\frac{z-\zeta}{1-\bar{\zeta}z}\right)+h\left(\overline{\left(\frac{1-\bar{\zeta}z}{z-\zeta}\right)}\right)} + f(\zeta)$$

Let $\eta(z) = \dfrac{1-\bar{\zeta}z}{z-\zeta}$, now

$$\tilde{f}_{\bar{z}}(z) = \frac{f'(\zeta)(1-|\zeta|^2)}{\left\{\left(\frac{z-\zeta}{1-\bar{\zeta}z}\right)+h(\bar{\eta})\right\}^2}\ h_{\bar{\eta}}(\bar{\eta})\ \frac{1-|\zeta|^2}{(\bar{z}-\bar{\zeta})^2}$$

and
$$\tilde{f}_z(z) = \frac{f'(\zeta)(1-|\zeta|^2)}{\left\{\left(\frac{z-\zeta}{1-\bar{\zeta}z}\right)+h(\bar{\eta})\right\}^2}\ \frac{1-|\zeta|^2}{(1-\bar{\zeta}z)^2}.$$

So $\left|\dfrac{\tilde{f}_{\bar{z}}}{\tilde{f}_z}\right| = \left|h_{\bar{\eta}}(\bar{\eta})\left(\dfrac{1-\bar{\zeta}z}{\bar{z}-\bar{\zeta}}\right)^2\right| = |h_{\bar{\eta}}(\bar{\eta})|\left|\dfrac{1-\bar{\zeta}z}{z-\zeta}\right|^2 \le |h_{\bar{\eta}}(\bar{\eta})| \le k < 1$ for $|z|>1$.

This completes the proof of the theorem. □

§ 1.10 A Counter Example

Although the Ahlfors' approximation method works well for finding some of the sufficient conditions for qc-extension, including those obtainable through Löwner chains, we provide here an example criterion that has been obtained by Löwner chain but does not seem obtainable by the radii argument.

<u>Theorem</u> <u>1.10.1</u> *(Becker [2]) Let* $f(z)=z+\frac{b_1}{z}+\cdots$ *be analytic in* $|z|>1$. *If* $|c|\le 1$, $c\ne 1$ *and* $k<1$, *then if*

$$\left|\frac{(1-c|z|^{-2})^2}{1-c}|z|^2(f'(z)-1)-c|z|^{-2}\right|\le k\,,\quad |z|>1 \tag{1.10.1}$$

implies f *has* $\frac{1+k}{1-k}$*-qc extension to* $\bar{\mathbb{C}}$.

We now try to prove this criterion by Ahlfors' method with the extra assumptions that c is real and that $c<(1-k)/2$. Note that f is locally univalent in $|z|>1$. For suppose $f'(z)=0$ for some z, then (1.10.1) and $c<(1-k)/2$ will yield a contradiction. So let f be defined as above but also analytic on $|z|=1$ and define the extension to be $g=\overline{f+(\zeta-z)}$, where $\zeta(z)=\frac{1-c}{1-c/|z|^2}\frac{1}{\bar z}$ is an anti-quasiconformal mapping which maps the Δ onto $|z|>1$ and fixes the $|z|=1$ with $|\mu_\zeta|=|c|/|z|^2\le|c|$ (this anti-qc mapping is also due to Becker). The quasiconformal condition requires $|g_{\bar z}|\le k|g_z|$ $|z|\ge 1$. We obtain

$$\overline{g_z(z)} = (1-c)\left\{\frac{-c}{(1-c/|z|^2)^2}\frac{c}{z\bar z^3}+\frac{1}{1-c/|z|^2}\left(-\frac{1}{\bar z^2}\right)\right\}=\frac{-(1-c)}{\bar z^2(1-c/|z|^2)^2}$$

and

$$\overline{g_{\bar z}(z)} = f'(z)-1-\frac{c(1-c)}{(1-c/|z|^2)^2}\frac{1}{|z|^4}.$$

The quasiconformal condition just becomes (1.10.1). We note that if $e^{i\theta}+\delta\in\Delta$ then the extension

$$F(e^{i\theta}+\delta)=\bar g\Big(1/(e^{-i\theta}+\bar\delta)\Big)=f(e^{i\theta})+\frac{1}{1-c}\Big(\delta+\bar\delta(1-c)(1-f'(e^{i\theta}))+c\Big)e^{i2\theta}+O(\delta^2)$$

and (1.10.1) shows that F is locally homeomorphic on $\partial\Delta$.

Let

$$A(z)=\frac{c(1-c)}{(|R_n z|^2-c)^2}\quad,\ \tilde A(z)=\frac{k|1-c|}{||R_n z|^2-c|^2}$$

and

$$B(z)=\frac{c(1-c)}{(|z|^2-c)^2}\quad,\ \tilde B(z)=\frac{k|1-c|}{||z|^2-c|^2}. \tag{1.10.2}$$

Replace z by $R_n z$ and $R_n\ge 1$ in (1.10.1). This becomes

$$|f'(R_n z)-1-A(z)| \leq \tilde{A}(z). \tag{1.10.3}$$

Consider the family $f_n(z)=\frac{1}{R_n}f(R_n z)$ where $R_n \to 1$ as $n \to \infty$. We want to show $f_n(z)$ also satisfies (1.10.1). i.e.

$$|f'(R_n z)-1-B(z)| \leq \tilde{B}(z). \tag{1.10.4}$$

As we have done before, we want to deduce (1.10.3)⇒(1.10.4). It is easy to obtain that

$$\tilde{B}(z)-\tilde{A}(z)= \frac{k|1-c|(R_n{}^2-1)|z|^2}{||z|^2-c|^2||R_n z|^2-c|^2}\Big\{|z|^2(R_n{}^2+1)-(c+\bar{c})\Big\},$$

and

$$B(z)-A(z)= \frac{c(1-c)(R_n{}^2-1)|z|^2}{(|z|^2-c)^2(|R_n z|^2-c)^2}\Big\{|z|^2(R_n{}^2+1)-2c\Big\}.$$

Hence

$$|A(z)-B(z)| \leq \tilde{B}(z)-\tilde{A}(z),$$

if and only if

$$\left|\frac{|z|^2(R_n{}^2+1)}{2} - c\right| \leq \left|\frac{|z|^2(R_n{}^2+1)}{2} - \Re(c)\right|.$$

The above inequality is true if and only if c is a real number. So we conclude by the Ahlfors' method and the radii argument that we have Theorem 1.10.1 only if c is real.

§ *1.11 A Univalence Criterion involving an Area Integral*

We have already seen many univalence criteria or criteria for quasiconformal extension, most of which were given in the form of inequalities. They can also be given in the form of an area integral. D. London was the first one who gave such a result.

Theorem 1.11.1 (*London D.* [1]) *Let $f(z)=z+\sum_2^\infty a_n z^n$ be an analytic function defined in Δ and if*

$$\iint_\Delta |S(f,z)|\ d\sigma_z \leq 2\pi$$

then f is univalent, and σ_z denotes the area element of z-plane.

Following London's method, we have

Theorem 1.11.2 *Let $f(z)=z+\sum_2^\infty a_n z^n$ be an analytic function defined in Δ and if*

$$\iint_\Delta |T(f,z)|^2\ d\sigma_z \leq \pi, \tag{1.11.1}$$

where $T(f,z)=f''(z)/f'(z)$, *then* f *is univalent.*

We remark that this result has been found independently by V.D. Golovan′ [1] and he also proved that if the inequality in (1.11.1) is strict then $\partial f(\Delta)$ will be rectifiable. The proof of Theorem 1.11.2 is similar to that of Golovan′, but our considerations are quite different and this leads to new problems.

Let $z=z(\zeta)$ be conformal in Δ, then it is easy to verify the following.

$$T(f\circ z,\zeta)=T(f,z)z'(\zeta)+T(z,\zeta) \tag{1.11.2}$$

Also $T(f,z)\equiv 0$ if and only if $f=az+b$ where a, b are constants. The Schwarzian derivatives also have a similar identity which we will see in the next chapter.

We need two lemmas.

<u>Lemma</u> <u>1.11.3</u> (*D. London* [1]) *Let* $g(z)=\sum_0^\infty b_n z^n$ *be an analytic function.*

Then we have
$$\pi|g(0)|\le \iint_\Delta |g(z)|\, d\sigma_z.$$

<u>Lemma</u> <u>1.11.4</u> *Let* $f(z)=z+\sum_2^\infty a_n z^n$ *be defined in* Δ, *then*

$$|T(f,z)|\le \frac{\left(\iint_\Delta |T(f,z)|^2\, d\sigma_z\right)^{\frac{1}{2}}}{\sqrt{\pi}(1-|z|^2)}. \tag{1.11.3}$$

Proof Let $z=z(\zeta)$ be a conformal mapping which maps ζ-plane to z-plane. Consider

$$\iint_{|z|<1} |T(f,z)|^2\, d\sigma_z = \iint_{|\zeta|<1} |T\big(f,z(\zeta)\big)z'(\zeta)|^2\, d\sigma_\zeta$$

$$\ge \pi|T\big(f,z(0)\big)z'(0)|^2 \qquad \text{from Lemma 1.11.3.}$$

We now choose $z(\zeta)$ to be an automorphism of the unit disc and set $z(\zeta)=\dfrac{\zeta+t}{1+\bar{t}\zeta}$.

Hence
$$\iint_{|z|<1} |T(f,z)|^2\, d\sigma_z \ge \pi\, |T(f,t)|^2(1-|t|^2)^2.$$ □

Proof of the theorem Suppose f satisfies the hypotheses of Theorem 1.11.2. Then from (1.11.3)

$$(1-|z|^2)|T(f,z)| \le \frac{1}{\sqrt{\pi}}\left(\iint_\Delta |T(f,z)|^2\, d\sigma_z\right)^{\frac{1}{2}} \le 1.$$

The univalence of f follows immediately from (1.4.2), the theorem of Becker. □

It is also clear we have the following corollary.

<u>Corollary 1.11.5</u> *Let* $f(z)=z+\sum_{2}^{\infty}a_n z^n$ *be an analytic function defined in* Δ. *If*

$$\iint_{\Delta}|T(f,z)|^2\, d\sigma_z \le k\pi,$$

where $0<k<1$, *then* f *has a* K*-qc extension to* $\bar{\mathbb{C}}$, *where* $K=\frac{1+k}{1-k}$.

Remark Theorem 1.11.2 was also obtained by Krzyż [2] in a very different way, involving the Green's function and the Schwarz symmetrization. Krzyż considered the geometrical shape of the set

$$\Omega_f=\{\log f'(z):\ z\in\Delta\}\quad f'\neq 0,$$

and asked under what conditions on Ω_f that implies f is univalent. He proved that if the area $|\Omega_f|\le\pi$, then f is univalent. The proof of Theorem 1.11.2 serves as a simple alternative method of Krzyż.

The correct analogue of Theorem 1.11.1 of London would involve $\iint|T(f,z)|\,d\sigma_z$ instead of the L^2-norm and the question appears to be open.

Chapter Two

Schwarzian Derivatives and the Domain Constants

§ *2.1 Introduction*

Let us recall the definition of the Schwarzian derivative of an analytic function defined in a domain A. We have $S(f,z)=\left(\frac{f''}{f'}\right)'(z)-\frac{1}{2}\left(\frac{f''}{f'}(z)\right)^2$. We shall also write $S(f,z)$ as S_f if we do not want to emphasize z.

Historically, the differential operator was first known to Riemann as early as 1857, but the first person who actually studied it extensively was H.A. Schwarz. He investigated differential operator invariant with respect to Möbius transformations; this later became known as the Schwarzian. Much later M. Lavie [1] showed that under the assumption that $f'\neq 0$ all the differential operators of order n on f (i.e. operator involving f, f', f'',$\cdots$, $f^{(n)}$) invariant with respect to Möbius transformations can be written as rational functions of $S(f,z)$ and its derivatives of order up to $n-3$. The Schwarzian also plays an important role in several branches of complex function theory. We have seen that it is closely related to the theory of quasiconformal extensions in the last chapter and univalent function theory later in this chapter.

Let f be an analytic function defined in a domain A and $z : B \to A$ be analytic, then we have

$$S(f\circ z,\zeta) = S(f,z)\ z'(\zeta)^2 + S(z,\zeta) \tag{2.1.1}$$

It is not difficult to check that $S(z,\zeta)\equiv 0$ if and only if z is a Möbius transformation. Hence if $A=\Delta$ and z is a Möbius transformation mapping the unit disc onto the unit disc then (2.1.1)

becomes
$$S(f\circ z,\zeta) = S(f,z)\, z'(\zeta)^2, \tag{2.1.2}$$
since the Schwarzian of a Möbius transformation is identically equal to zero.

We also recall that the Poincaré density function of a simply connected domain A is defined as

$$\eta_A(z) = \frac{|f'(z)|}{1-|f(z)|^2},$$

where f is any conformal mapping which maps A onto a unit disc. Let f and g both be conformal mappings defined in A, then it follows from (2.1.1) that

$$S(f\circ z,\zeta) - S(g\circ z,\zeta) = \left\{S(f,z) - S(g,z)\right\} z'(\zeta)^2. \tag{2.1.3}$$

Now let $I : B \rightarrow \Delta$ and $J : A \rightarrow \Delta$ be conformal such that $J o z = I$. Since the Poincaré density of B is independent of the conformal mapping, we have

$$\eta_B(\zeta) = \frac{|I'(\zeta)|}{1-|I(\zeta)|^2} = \frac{|J'(z)\, z'(\zeta)|}{1-|J(z)|^2} = \eta_A(z)\, |z'(\zeta)|$$

and (2.1.3) becomes
$$\frac{S(f\circ z,\zeta) - S(g\circ z,\zeta)}{\eta_B(\zeta)^2} = \frac{S(f,z) - S(g,z)}{\eta_A(z)^2}.$$

Now define the *norm of the Schwarzian derivative* to be
$$\|S_f\| = sup\ \{\ |S(f,z)|\ \eta_A(z)^{-2}\!: z\in\Delta\}.$$

Hence
$$\|S_f - S_g\|_A = \|S_{f\circ z} - S_{g\circ z}\|_B. \tag{2.1.4}$$

If we now put $z = g^{-1}$, we obtain
$$\|S_f - S_g\|_A = \|S_{f\circ g^{-1}}\|_{g(A)}. \tag{2.1.5}$$

If f is the identity mapping of A, then (2.1.5) becomes

$$\|S_g\|_A = \|S_{g^{-1}}\|_{g(A)}. \tag{2.1.6}$$

Suppose g is a Möbius transformation in (2.1.5). Then the equation becomes

$$\|S_f\|_A = \|S_{f\circ g^{-1}}\|_{g(A)},$$

and if we now let m be a Möbius transformation, then we have

$$\|S_f\|_A = \|S_{m\circ f\circ g^{-1}}\|_{g(A)}. \tag{2.1.7}$$

i.e. the norm of the Schwarzian is *completely invariant* with respect to Möbius transformations, since when comparing to (2.1.2) the last term disappeared.

§ 2.2 *The Domain Constant* $\Omega(A)$

We now introduce the concept of *domain constant* of a simply connected domain. Let f be a conformal mapping which maps A onto Δ . The domain constant $\Omega(A)$ is defined by

$$\Omega(A) = \|S_f\| = sup \{ |S(f,z)| \ \eta_A(z)^{-2}: z\in A, f: A \to \Delta \ \ conformal\}.$$

This definition is well defined, for if $g : A \to \Delta$ is also conformal, then $f\circ g^{-1}$ is an automorphism of the unit disc and so it must be a Möbius transformation, hence $g = M\circ f$ where M is a Möbius transformation. Thus by (2.1.7), $\Omega(A)$ is well defined. Also from (2.1.6), since

$$\|S_f\|_A = \|S_{f^{-1}}\|_{f(A)},$$

we have the equivalent definition:

$$\Omega(A) = \|S_f\| = sup \{ |S(f,z)| \ \eta_\Delta(z)^{-2}: z\in\Delta, f: \Delta \to A \ \ conformal\}. \qquad (2.2.1)$$

$$= sup \{ (1-|z|^2)^2|S(f,z)| : \ z\in\Delta, f: \Delta \to A \ \ conformal\}.$$

One of the main problems is to determine what $\Omega(A)$ is when given a domain $A\subset\bar{\mathbf{C}}$. Since the Schwarzian derivative of a Möbius transformation is identically equal to zero, so $\Omega(A)$ can be regarded as a measure of the deviation of A from Δ or f from the Möbius transformations. We have an upper bound of $\Omega(A)$ whenever f is conformal. This has been found by Kraus [1] in 1922 but was forgotten and rediscovered by Z. Nehari [1] in 1949. Let us proceed to the proof now. According to (2.2.1) we may assume $f: \Delta \to A$ to be conformal and since for any $z_0\in\Delta$, there exists an automorphism g of Δ such that $g(0)=z_0$, we have by (2.1.7) that

$$(1-|z_0|^2)^2|S(f,z_0)| = (1-|0|^2)^2|S(f\circ g,0)| = |S(f\circ g,0)|.$$

So we have another characterization of $\Omega(A)$ that:

$$\Omega(A) = sup \{ |S_f(0)| : f: \Delta \to A \ conformal \}.$$

Now by (2.1.7) again, we may further assume that $f\in S$ and since $S(f,0) = 6(a_3-a_2^2)$, we finally have

$$\Omega(A) = sup \{ |6(a_3-a_2^2)| : f\in S, f: \Delta \to B \ conformal \ and \ B \ is \ Möbius \ equivalent \ to \ A \}.$$

This new characterization gives a relation between the domain constant $\Omega(A)$ and the coefficients of f which is very useful. Let $f\in S$, then $1/f(1/z)=z+b_0+b_1/z+ \cdots$ is meromorphic in $|z|>1$ and the class is called Σ. It is well known that $|b_1|\leq 1$ by the *area theorem* (see Duren [1] p.29). But $b_1=a_2^2-a_3$, hence $|a_2^2-a_3|\leq 1$. Thus we eventually arrive at the sharp estimate $\Omega(A) \leq 6$ (see also Lehto [2] p.61).

Let $f \in S$, then the renormalization of f is given by

$$g(z)=\frac{f\circ w(z)-f\circ w(0)}{(f\circ w)'(0)},$$

where $w(z)=e^{i\theta}\Big(\frac{z+\zeta}{1+\bar{\zeta}z}\Big)$, $\zeta\in\Delta$, is an automorphism of Δ. Clearly g also belongs to S. We call any family of functions that has the similar property as S above to be *linearly invariant.* The function g is called *Koebe transform* of f; the ranges of f and g are therefore similar. It is clear that the above new characterization for the domain constant depends on the fact that S is linearly invariant.

Recently there has been growing interest in finding the domain constant of different domains A. Nehari [3] himself had found that if A is a convex domain, then $\Omega(A) \le 2$ and the bound is sharp.

We recall that the domain A has *bounded boundary rotation* $k\pi$ if there exists a conformal mapping f of the Δ onto A such that

$$\lim_{r\to 1}\int_0^{2\pi} |u(re^{i\theta})| \; d\theta \le k\pi,$$

where $u(z)=\Re\Big(1+z\frac{f''}{f'}\Big)$. It is therefore easy to see that if the boundary rotation is exactly 2π then f is convex or A is a convex domain. For details about the bounded boundary rotation functions we refer to the book of Duren [1] p.269. Using the techniques described above Lehto and Tammi [1] (see also Lehto [2] p.64) proved that :

<u>Theorem</u> <u>2.2.1</u> *Let A be Möbius equivalent to a domain with bounded boundary rotation less than $k\pi$. If $k<4$ then*

$$\Omega(A) \le \frac{2k+4}{6-k},$$

the bound is sharp.

Similarly we have the following result if the function is *close to convex. A* conformal mapping $f(0)=f'(0)-1=0$ is said to be *close-to-convex of order* β if $\left| \arg\frac{f'(z)}{g'(z)}\right| \le \frac{\beta\pi}{2}$ where g is a convex conformal mapping defined in Δ and $\beta \ge 0$, we denote this class by C_β. A domain is called *close-to-convex of order* β if it is a image of a close-to-convex of order β function in Δ.

<u>Theorem</u> <u>2.2.2</u> (*Koepf W.* [1]) *Let A be Möbius equivalent to a domain close-to-convex of order*

β, then we have $$\Omega(A) = \begin{cases} 2+4\beta & \beta\le 1 \\ 2\beta^2+4\beta & \beta\ge 1 \end{cases}.$$

This result is sharp.

§ 2.3 The Domain Constant of a Strongly Starlike Domain

Let us recall from the Definition 1.4.1 that a function $f(z)=z+a_2z^2+\cdots\in S$ is strongly starlike of order α where $0<\alpha\le 1$, if $|arg\frac{zf'}{f}|\le\frac{\alpha\pi}{2}$. We shall now define a domain A to be *strongly starlike of order* α if it is a image of a function which is α-strongly starlike.

When considering $\Omega(A)$ for different classes of domain A, we found that the method described above in § 2.2 does not seem to work for strongly starlike domains. Both the proofs of Theorems 2.2.1 and 2.2.2 were based on the fact that the classes of functions with bounded boundary rotation and close-to-convex functions are in fact linearly invariant. Hence one can transfer the problem to the origin and find a least upper bound of $|S_f(0)|$ over the corresponding functions and then transfer it to other points in the unit disc. This is exactly what we have done for the class S. But strongly starlike functions are *not* linearly invariant. However we still find that:

Theorem 2.3.1 *Let A be a domain which is Möbius equivalent to a strongly starlike domain of order α, where $0<\alpha\le 1$, then* $$\Omega(A)\le 6\ sin(\tfrac{\alpha\pi}{2}).$$

Let us say that $f\in S_K$ if $f\in S$ and f has a K-qc extension to $\bar{\mathbf{C}}$, where its complex dilitation μ_f satisfies $|\mu_f|\le k=\frac{K-1}{K+1}$. The extension is written as $\tilde f(z)$ when $|z|\ge 1$ and $\tilde f(z)=f(z)$ when $|z|<1$. We also define $S_K(\infty)$ to be the subclass of S_K such that $\tilde f(\infty)=\infty$. We quote the following lemma which is due to R. Kühnau [1] and Lehto [1] :

Lemma 2.3.2 *Suppose $f\in S_K$, then $|a_2{}^2-a_3|\le k=\frac{K-1}{K+1}$. If in addition $\tilde f(\infty)=\infty$ i.e. $f\in S_K(\infty)$, then* $$|a_2|\le 2k$$

and with equality if and only if

$$\tilde f(z) = \begin{cases} z/(1+ke^{i\theta}z)^2 & |z|\le 1 \\ z\bar z/(\sqrt{\bar z}+ke^{i\theta}\sqrt{z})^2 & |z|\ge 1 \end{cases}.$$

Proof of Theorem 2.3.1 By Theorem 1.4.3, $f \in S^*(\alpha)$ then $f \in S_K$ with $k \leq sin(\alpha\pi/2)$. So by Lemma 2.3.2, $|a_2{}^2 - a_3| \leq k$. Since the class S_K is linearly invariant, then according to § 2.2 we deduce $\Omega(A) \leq 6\ sin(\frac{\alpha\pi}{2})$. It is however not known that if this estimate is sharp. $\square$

§ 2.4 *Estimations of the Logarithmic Derivative* $\frac{f''}{f'}$.

Just like estimating the $\|S_f\|$ we can also estimate $(1-|z|^2)|z\frac{f''}{f'}|$ by similar techniques. However the logarithmic derivative does not share the same invariance properties as the Schwarzian derivative. It is well known that if $f \in S$, then $|a_2| \leq 2$ (see Duren [1] p.30) and by the Koebe transform

$$\frac{f\left(\frac{z+\zeta}{1+\bar{\zeta}z}\right) - f(\zeta)}{f'(\zeta)(1-|\zeta|^2)} = z + \left(\tfrac{1}{2}(1-|\zeta|^2)\frac{\zeta f''(\zeta)}{f'(\zeta)} - \bar{\zeta}\right) z^2 + \cdots \tag{2.4.1}$$

still belongs to S. Hence we have $\left|(1-|z|^2)z\frac{f''(z)}{f'(z)} - 2|z|^2\right| \leq 4$

or
$$(1-|z|^2)\left|z\frac{f''(z)}{f'(z)}\right| \leq 6.$$

We have the following analogue when f is convex:

<u>Theorem</u> <u>2.4.1</u> (*Hayman W.K.; see Ahlfors* [2] p.5) *Let* $f(z) = z + \sum_2^{\infty} a_n z^n$ *be a convex function defined in* Δ, *then*

$$(1-|z|^2)\left|z\frac{f''(z)}{f'(z)}\right| \leq 4.$$

We shall be interested in finding the constant for strongly starlike functions and close-to-convex functions. Let us quote the following result:

<u>Theorem</u> <u>2.4.2</u> (*Schiffer and Schober* [1]) *Let* $f \in S_K$, *then*

$$|a_2| \leq 2 - 4\kappa^2, \tag{2.4.2}$$

where $\kappa = \frac{1}{\pi}$ *arccos* $k \in (0, \frac{1}{2}]$. *This estimate is sharp (the extremal function also exists).*

Since f is linearly invariant with respect to S_K, we can use (2.4.1) and (2.4.2) together to obtain

$$(1-|z|^2)\left|z\frac{f''(z)}{f'(z)} - 2|z|^2\right| \le 2(2-4\kappa^2)$$

or
$$(1-|z|^2)\left|z\frac{f''(z)}{f'(z)}\right| \le 6-8\kappa^2.$$

However if we consider $f\in S^*(\alpha)$ we can expect a better estimate. By a closer inspection of the qc-extension of $S^*(\alpha)$ in Theorem 1.4.3, we see that $\tilde{f}(\infty)=\infty$. Hence by Lemma 2.3.2 we deduce that

$$(1-|z|^2)\left|z\frac{f''(z)}{f'(z)}\right| \le 6\ sin(\frac{\alpha\pi}{2}).$$

But this is still not the best possible estimate, we shall now derive the sharp result for the logarithmic derivative for $S^*(\alpha)$.

Theorem 2.4.3 *Let* $f\in S^*(\alpha)$ $0<\alpha\le 1$, *then*

$$(1-|z|^2)\left|z\frac{f''(z)}{f'(z)}\right| \le 6\alpha,$$

where equality can occur if and only if

$$\frac{zf'(z)}{f(z)} = \left(\frac{1+\epsilon z}{1-\epsilon z}\right)^{\alpha},\ |\epsilon|=1 \ and\ \alpha<1. \tag{2.4.3}$$

When $\alpha=1$ *f is just a starlike function and the estimate is also sharp.*

We require the following definition and lemmas. Let f and g be analytic functions in Δ. We say that f is *subordinate* to g in Δ if there exists an analytic function w defined in Δ such that $w(0)=0$, $|w(z)|<1$ when $z\in\Delta$ and $f(z)=g(w(z))$ for $z\in\Delta$. We denote this relation by $f\prec g$. It is well known that if g is univalent in Δ, then $f\prec g$ in Δ if $f(0)=g(0)$ and $f(\Delta)\subset g(\Delta)$.

Lemma 2.4.4 (*Pick's lemma; see Ahlfors* [1] *p.3*) *Let P be an analytic function defined in* Δ *and if* $|P|<1$ *in* Δ, *then*

$$|P'(z)| \le \frac{1-|P(z)|^2}{1-|z|^2} \quad \forall\ z\in\Delta,$$

with equality if and only if P is an automorphism of the unit disc i.e. $P(z) = e^{i\theta}\frac{z-z_1}{1-\bar{z}_1 z}$.

Lemma 2.4.5 (*Rogosinki W.* [1] *p.70*) *Let* $f(z)=\sum_1^\infty a_n z^n$ *be subordinate to* $F(z)=\sum_1^\infty A_n z^n$ *in* Δ. *If f is univalent in* Δ *and* $F(\Delta)$ *is convex univalent, then* $|a_n| \le |A_1|$ $\forall n$. *If* $F(\Delta)$ *is not a half-plane, then the equality can hold for a given n only if* $f(z)=F(\epsilon z^n)$, $|\epsilon|=1$. *If* $F(\Delta)$ *is a half-plane then equality occurs only if*

$$f(z)=\frac{1}{\epsilon}\sum_{j=0}^{n}\mu_j\frac{\epsilon_j z}{1-\epsilon_j z},\ \mu_j\geq 0,\ \sum_1^n\mu_j=A_1,\ \ \epsilon_j=\frac{1}{\epsilon}exp\Big(\frac{j\,2\pi i}{n}\Big).$$

Proof of Theorem 2.4.3 Let $f\in S^*(\alpha)$ and $P(z)=\frac{zf'(z)}{f(z)}$. Then P is subordinate to the function $\left(\frac{1+z}{1-z}\right)^{\alpha}$.

i.e.

$$P(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha} \quad z\in\Delta.$$

Hence there exist an analytic function $w(z)$ such that $w(0)=0$ and $|w|<1$ and

$$P(z) = \left(\frac{1+w(z)}{1-w(z)}\right)^{\alpha} \quad z\in\Delta.$$

Differentiate both sides

$$\frac{zf''(z)}{f'(z)} = P(z)-1+z\frac{P'(z)}{P(z)}. \tag{2.4.4}$$

Now

$$P(z)-1 = \left(\frac{1+w(z)}{1-w(z)}\right)^{\alpha}-1 \prec \left(\frac{1+z}{1-z}\right)^{\alpha}-1 \quad z\in\Delta.$$

Since $\left(\frac{1+z}{1-z}\right)^{\alpha}-1$ is a convex conformal mapping, by Lemma 2.4.5 the coefficients of $P-1$ in the series expansion are dominated by the first coefficient of $\left(\frac{1+z}{1-z}\right)^{\alpha}-1$ which is bounded by 2α. So suppose $\left(\frac{1+w(z)}{1-w(z)}\right)^{\alpha}-1 = \sum_1^{\infty}b_k z^k$, then

$$\left|\left(\frac{1+w(z)}{1-w(z)}\right)^{\alpha}-1\right| \leq \sum_1^{\infty}|b_k z^k| \leq 2\alpha\sum_1^{\infty}|z|^k = \frac{2\alpha|z|}{1-|z|}. \quad |z|<1. \tag{2.4.5}$$

Also since

$$P(z)^{1/\alpha}= \frac{1+w(z)}{1-w(z)},$$

We have

$$\frac{1}{\alpha}\frac{P'(z)}{P(z)} = \frac{w'(z)}{1+w(z)}+\frac{w'(z)}{1-w(z)} = \frac{2w'(z)}{1-w(z)^2}.$$

By Pick's lemma

$$(1-|z|^2)\frac{1}{\alpha}\left|\frac{zP'(z)}{P(z)}\right| = (1-|z|^2)\,\frac{2|zw'(z)|}{|1-w(z)^2|} \leq \frac{2|z|(1-|w(z)|^2)}{|1-w(z)^2|} \leq 2. \tag{2.4.6}$$

From (2.4.4) we deduce that

$$\begin{aligned}(1-|z|^2)\left|z\frac{f''(z)}{f'(z)}\right| &\leq (1-|z|^2)\left|\left(\frac{1+w(z)}{1-w(z)}\right)^{\alpha}-1\right| +(1-|z|^2)\left|\frac{zP'(z)}{P(z)}\right| \\ &\leq (1-|z|^2)\frac{2\alpha|z|}{1-|z|} + 2\alpha \\ &\leq (1+|z|)\,2\alpha + 2\alpha \leq 6\alpha. \end{aligned}\tag{2.4.7}$$

Equality in (2.4.7) can occur if and only if both (2.4.5) and (2.4.6) hold with equalities. From (2.4.5) and Lemma 2.4.5,

$$\frac{zf'(z)}{f(z)} = \left(\frac{1+w(z)}{1-w(z)}\right)^{\alpha}, \ w(z)=\epsilon z^n \ |\epsilon|=1 \ \textit{and} \ \alpha<1.$$

Also from (2.4.5) and Lemma 2.4.4 the only automorphism $w(z)=\epsilon z^n$ of the unit disc such that $w(0)=0$ must be $w(z)=\epsilon z$ where $|\epsilon|=1$. Hence f has the representation (2.4.3). When $\alpha=1$ f is just an ordinary starlike function and it has a standard representation (see Pommerenke [1]). □

Remark 1 The estimate 6α in the above theorem is better than 6 $sin(\frac{\alpha\pi}{2})$.

Remark 2 Let $f\in S^*(\alpha)$ then since $zf'/f \prec \left(\frac{1+z}{1-z}\right)^{\alpha}$, it is reasonable to believe that if f satisfies the equation (2.4.3) then f is the extremal function, just as the Koebe function is the extremal function of many problems in Geometric function theory. Theorem 2.4.3 shows that f is indeed the extremal function for that problem. However Brannan, Clunie and Kirwan [1] found that the f satisfying the (2.4.3) is *not* the extremal function for the coefficient problems, and it is extremal only if α is near to 0 or 1. The situation is more complicated, in fact they proved:

(a) if $|a_2| \le 2\alpha$ and (2.4.3) is the extremal function;

(b) if $0<\alpha<1/3$, then $|a_3| \le \alpha$ and the extremal function satisfies $\frac{zf'}{f}= \left(\frac{1+\epsilon z^2}{1-\epsilon z^2}\right)^{\alpha} |\epsilon|=1$;

(c) if $1/3<\alpha\le 1$, then $|a_3| \le 3\alpha^2$ and the extremal function satisfies (2.4.3);

(d) if $\alpha=1/3$, then $|a_3| \le 1/3$ and the extremal function satisfies

$$\frac{zf'(z)}{f(z)} = \left(\lambda\left(\frac{1+\epsilon z}{1-\epsilon z}\right)+(1-\lambda)\left(\frac{1+\epsilon^2 z^2}{1-\epsilon^2 z^2}\right)\right)^{1/3}$$

where $|\epsilon|=1$ and $0\le\lambda\le 1$.

Similarly, we have the following estimate for the functions which are close-to-convex of order β.

Theorem 2.4.6 (*cf. Theorem* 2.2.2) *Let* $f\in C_\beta$, $\beta\ge 1$, *then*

$$(1-|z|^2)\left|z\frac{f''(z)}{f'(z)}\right| \le 2(2+\beta) \quad \forall z\in\Delta.$$

This estimate is sharp.

Proof Since $f\in C_\beta$, we can find a convex function ϕ such that

$$\left| arg \frac{f'(z)}{\phi'(z)} \right| \le \frac{\beta\pi}{2}.$$

By the subordination principle, there exists an analytic function $w(z)$ such that $w(0)=0$ and $|w|<1$, with

$$\frac{f'(z)}{\phi'(z)} = \left(\frac{1+w(z)}{1-w(z)}\right)^{\beta}.$$

Taking logarithms and differentiating both sides, we obtain

$$\frac{zf''(z)}{f'(z)} = z\left(\frac{\phi''(z)}{\phi'(z)} + \beta\left(\frac{w'(z)}{1+w(z)} + \frac{w'(z)}{1-w(z)}\right)\right) = z\frac{\phi''(z)}{\phi'(z)} + \beta\frac{2zw'(z)}{1-w(z)^{2}}. \tag{2.4.8}$$

We claim that $(1-|z|^2)|z\phi''(z)/\phi'(z)| \le 4$. This of course is the result of Hayman mentioned above. However, we present here a simple proof. Since $\Re\left(1+\frac{z\phi''}{\phi'}\right)>0$ in Δ, by the subordination principle we have $1+\frac{z\phi''(z)}{\phi'(z)} = \frac{1+w_1(z)}{1-w_1(z)}$ for some w_1 *analytic*, $w_1(0)=0$, $|w_1|<1$. As w_1 satisfies the hypotheses of the Schwarz's lemma, we have $|w_1| \le |z|$ and so $|1-w_1| \ge 1-|w_1| \ge 1-|z|$.

Now $\quad (1-|z|^2)\left| z\frac{\phi''(z)}{\phi'(z)}\right| = (1-|z|^2)\frac{2|w_1(z)|}{|1-w_1(z)|} \le (1-|z|^2)\frac{2|w_1(z)|}{1-|z|} \le 2|z|(1+|z|) \le 4.$

Using Pick's lemma again, it follows from (2.4.8) that

$$\begin{aligned}(1-|z|^2)\left| z\frac{f''(z)}{f'(z)}\right| &\le (1-|z|^2)\left| z\frac{\phi''(z)}{\phi'(z)}\right| + (1-|z|^2)\frac{2\beta|zw'(z)|}{|1-w(z)^2|} \\ &\le 4 + 2\beta\frac{1-|w(z)|^2}{|1-w(z)^2|} \le 4+2\beta = 2(2+\beta).\end{aligned}$$ □

§ *2.5 The Geometry and the Domain Constant of A*

Recall the definition of $\Omega(A)$ from § 2.2. We have already seen that different geometrical shapes of A lead to different Domain Constants. We shall now discuss a more general class of domains i.e. *quasidiscs*. A domain A is a *K-quasidisc* if it is an image of the unit disc Δ under a K-qc mapping of the plane. Let $\mathfrak{C}$ be a Jordan curve and z_1, $z_2 \in \mathfrak{C}$ divide it into two arcs $\mathfrak{C}_1$ and $\mathfrak{C}_2$, $\mathfrak{C}$ is said to satisfy the *arc condition* if there exists a constant c such that

$$\min_i\ (diam\ \mathfrak{C}_i) \le c|z_1-z_2| \tag{2.5.0}$$

for all pairs of z_1 and z_2 on $\mathbb{C}$. Ahlfors gave a geometric characterization of quasidiscs (see Lehto [2] p.45): *a Jordan domain A has boundary satisfying the arc condition if and only if A is a quasidisc.* It is easy to see that if $\Omega(A) \le 2k$, $k<1$, then A is a $\frac{1+k}{1-k}$-quasidisc. For, by Theorem 1.4.1, $f : \Delta \to A$ admits a K-qc extension to $\bar{\mathbb{C}}$ and in fact f is conformal in Δ and K-quasiconformal in $\bar{\mathbb{C}}\backslash\Delta$. By the definition, A is therefore a K-quasidisc. On the other hand we have:

Theorem 2.5.1 (*Lehto* [2] *p.73*) *If A is a K-quasidisc, then* $\Omega(A) \le 6\,\frac{K^2-1}{K^2+1}$.

It is not known whether the estimate of the above theorem is best possible. It is therefore natural to ask if $\Omega(A) < \epsilon$ where ϵ is a sufficiently small positive constant, implies that A is a starlike or convex domain. The answer turns out to be *negative.*

Theorem 2.5.2 *Given any $\epsilon>0$, there exists a domain $A(\epsilon)$ containing the origin such that*

$$\Omega(A) < \epsilon,$$

and A is not a starlike domain (let alone a convex domain).

The proof of the above theorem depends upon the theory of the second order differential equation

$$y'' + A\,y = 0, \tag{2.5.1}$$

and its linearly independent solutions. We require some lemmas here which will also be used again in Chapters 3 and 5. Similar to the Schwarzian derivative we introduce the following notation:

$$<E,\ c> = \frac{1}{4}\Big\{\Big(\frac{E'(z)}{E(z)}\Big)^2 - 2\frac{E''(z)}{E(z)} - \Big(\frac{c}{E(z)}\Big)^2\Big\},$$

where c is a constant.

Lemma 2.5.3 (*Bank and Laine* [2]) (*a*) *Let A be meromorphic in a region D, and assume that f_1, f_2 are linearly independent solutions of* (2.5.1). *Then $g=\frac{f_1}{f_2}$ has the following properties;*

(*i*) *All zeros of g' in D are of even multiplicity;*

(*ii*) *All poles of g in D are of odd order;*

(*iii*) $A \equiv \frac{1}{2}S(g,z)$.

(*b*) *Conversely, let g be an non-constant meromorphic function in a simply connected domain D, which possesses properties* (*i*) *and* (*ii*), *and define A by* (*iii*). *Then the equation* (2.5.1) *possesses two linearly independent solutions f_1, f_2 in D such that $g=\frac{f_1}{f_2}$.*

We note that most of the Lemma 2.5.3 is well-known, see Hille [1] or Fuchs [2].

Lemma 2.5.4 (*Bank and Laine* [2]) (*a*) *Let A be meromorphic in a region D, and assume that* (2.5.1) *possesses two linearly independent meromorphic solutions f_1, f_2 in D. Set $E=f_1f_2$ and $c=W(f_1,f_2)$ (the Wronskian of f_1 and f_2). Then,*

(*i*) *All zeros of $E(z)$ in D are simple;*

(*ii*) *All poles of $E(z)$ in D are of even order;*

(*iii*) *At any zero z_1 of E in D, the number $\frac{c}{E'(z_1)}$ is an odd integer;*

(*iv*) $A\equiv<E,c>$.

(*b*) *Conversely, let $E(z)\not\equiv0$ be a meromorphic function in a simply-connected region D, and let c be a non-zero constant such that* (*i*), (*ii*) *and* (*iii*) *above hold. Then, if $A(z)$ is defined by* (*iv*), *the equation* (2.5.1) *possesses two linearly independent meromorphic solutions f_1, f_2 in D such that*

(*v*) $E\equiv f_1f_2$ *and* $c=W(f_1,f_2)$.

Proof of the Theorem 2.5.2 We prove the theorem by providing an explicit counter-example. It is sufficient to show that given $\epsilon>0$ there exists a conformal mapping f (depending on ϵ) in Δ such that $(1-|z|^2)^2|S(f,z)|<\epsilon$ $\forall z\in\Delta$ and f is not a starlike function.

Given $\epsilon>0$, let $\quad E(z)=\frac{z}{(1-z^2)^\lambda}\quad |z|<1$ where $\lambda=i\mu$, $\mu<(2/7)\epsilon$.

Since $E(0)=0$ and $E'(z)=(1+(2\lambda-1)z^2)/(1-z^2)^{1+\lambda}$ we have $E'(0)=1$. Moreover the origin is the only zero of E in Δ and it has no pole in Δ. Therefore E clearly satisfies (*i*), (*ii*) and (*iii*) of Lemma 2.5.4 with $c=-1$ i.e. c is odd. Let $A\equiv<E,c>$, then the differential equation (2.5.1) has two linearly independent analytic solutions f_1, f_2 such that $E\equiv f_1f_2$ and $W(f_1,f_2)\equiv-1=f_1(0)f_2'(0)-f_2(0)f_1'(0)$, since $E(0)=0$ we may assume $f_1(0)=0$. Now

$$A=<E,-1>=<E,1>=\frac{1}{4}\left\{\left(\frac{E'(z)}{E(z)}\right)^2-2\frac{E''(z)}{E(z)}-\frac{1}{E(z)^2}\right\}=\frac{-1}{4}\left\{2\left(\frac{E'(z)}{E(z)}\right)'+\left(\frac{E'(z)}{E(z)}\right)^2+\frac{1}{E(z)^2}\right\}$$

We calculate A. Differentiate *log E*:

$$\frac{E'(z)}{E(z)}=\frac{1}{z}+\frac{2\lambda z}{(1-z^2)}\quad,\quad\left(\frac{E'(z)}{E(z)}\right)'=-\frac{1}{z^2}+\frac{2\lambda+2\lambda z^2}{(1-z^2)^2}.$$

Therefore
$$4A(z)=-\left\{2\left(-\frac{1}{z^2}+\frac{2\lambda+2\lambda z^2}{(1-z^2)^2}\right)+\left(\frac{1}{z}+\frac{2\lambda z}{(1-z^2)}\right)^2+\frac{(1-z^2)^{2\lambda}}{z^2}\right\}$$

$$=\frac{2}{z^2}-\frac{2(2\lambda+2\lambda z^2)}{(1-z^2)^2}-\frac{1}{z^2}-\frac{4\lambda}{(1-z^2)}-\frac{4\lambda^2 z^2}{(1-z^2)^2}-\frac{(1-z^2)^{2\lambda}}{z^2}$$

$$=\frac{1}{z^2}+\frac{-4\lambda-4\lambda z^2-4\lambda(1-z^2)-4\lambda^2 z^2}{(1-z^2)^2}-\frac{(1-z^2)^{2\lambda}}{z^2}$$

$$=\frac{1}{z^2}+\frac{-8\lambda-4\lambda^2 z^2}{(1-z^2)^2}-\frac{1}{z^2}\left(1-2\lambda z^2+\frac{2\lambda(2\lambda-1)}{2!}z^4-\cdots\right)$$

$$=\frac{-8\lambda-4\lambda^2 z^2}{(1-z^2)^2}+\left(2\lambda-\frac{2\lambda(2\lambda-1)}{2!}z^2+\cdots\right).$$

Let
$$P(z)=2\lambda-\frac{2\lambda(2\lambda-1)}{2!}z^2+\cdots$$

be the term in brackets above. The nth coefficient of P is equal to

$$\frac{2\lambda(2\lambda-1)(2\lambda-2)\cdots(2\lambda-n+1)}{n!}=2\lambda\left(\frac{2\lambda-1}{2}\right)\left(\frac{2\lambda-2}{3}\right)\left(\frac{2\lambda-3}{4}\right)\cdots\left(\frac{2\lambda-(n-1)}{n}\right).$$

It is not difficult to see that the modulus of each factor of the right hand side of the above equality is strictly less than 1 as long as $|\lambda|$ is chosen to be sufficiently small (less than 1, say). And so all the coefficients of P are bounded by $2|\lambda|=2\mu$. So we deduce

$$(1-|z|^2)^2|4A|\le(1-|z|^2)^2\left|\frac{-8\lambda-4\lambda^2z^2}{(1-z^2)^2}\right|+(1-|z|^2)^2\left|2\lambda-\frac{2\lambda(2\lambda-1)}{2!}z^2+\cdots\right|$$

$$\le(1-|z|^2)^2\ \frac{8|\lambda|+4|\lambda^2|}{|1-z^2|^2}+(1-|z|^2)^2\left(2|\lambda|+2|\lambda||z|^2+2|\lambda||z|^4+\cdots\right)$$

$$\le 12|\lambda|+(1-|z|^2)^2 2|\lambda|\frac{1}{1-|z|^2}$$

$$=12|\lambda|+2|\lambda|(1-|z|^2)\le 14|\lambda|.$$

i.e.
$$(1-|z|^2)^2|A|\le\frac{14}{4}|\lambda|=\frac{7}{2}\mu<\epsilon. \tag{2.5.2}$$

Since A is analytic in Δ, and f_1, f_2 considered above are linearly independent solutions of (2.5.1), by Lemma 2.5.3 (a) the function defined by

$$g(z)=\frac{f_1(z)}{f_2(z)} \tag{2.5.3}$$

satisfies the identity $A(z)\equiv\frac{1}{2}S(g,z)$. According to the equation (2.5.1) f_1, f_2 are analytic and so is their product $E=f_1f_2$ and since the only zero of $E(z)$ is when $z=0$ which is the zero of f_1. We conclude that f_1 has only one zero and f_2 has no zero in Δ. Hence we deduce that g must be analytic in Δ (in fact, since $g' = \frac{-W(f_1,f_2)}{f_2^{\,2}} \neq 0$ so g is locally univalent in Δ). Now $g(0)=0$ and g is analytic, by (2.5.2) $(1-|z|^2)^2|S(g,z)|$ can be made arbitrary small so g satisfies the Ahlfors-Weill's criterion (Theorem 1.4.1) and it must be conformal in Δ.

The well known criterion for a normalized function to be starlike is that $\Re\left(\frac{zf'}{f}\right)>0$ $\forall z \in \Delta$ (see Duren [1] p.31). However

$$\begin{aligned}
\frac{zg'(z)}{g(z)} &= \frac{z}{f_1 f_2} = \frac{z}{E(z)} \\
&= (1-z^2)^{\lambda} \\
&= exp\Big(\lambda \; log(1-z^2)\Big) \\
&= exp\Big\{i\mu\Big(log\;|1-z^2| + \; iarg(1-z^2)\Big\} \\
&= exp\Big\{-\mu \; arg(1-z^2) + \; i\mu \; log\;|1-z^2|\Big\} \\
&= exp\Big\{-\mu \; arg(1-z^2)\Big\} exp\Big\{i\mu \; log\;|1-z^2|\Big\}.
\end{aligned}$$

Now the argument of $\frac{zg'}{g}$ is $\mu log|1-z^2|$ which tends to negative infinity as $z \rightarrow 1$. Therefore, there exists infinitely many $z \in \Delta$ such that $\Re\left(\frac{zg'}{g}\right)<0$. This shows g cannot be starlike and also completes the proof. $\square$

§ *2.6 An analogue for the Logarithmic Derivative*

The logarithmic derivative does not share the same properties as the Schwarzian derivative, for it is invariant only with respect to linear mappings. Thus it is probably not very useful to define another domain constant analogue to $\Omega(A)$. Theorems 2.4.3 and 2.4.6 show that for different domains, we can have different estimates for the logarithmic derivative $(1-|z|^2)|z\frac{f''}{f'}|$. Since Becker's result (put $c=0$ in (1.4.2) of Theorem 1.4.4) shows that when $(1-|z|^2)|z\frac{f''}{f'}|<1$ then $f(\Delta)$ is a quasidisc, we can thus ask the same question as we did in § 2.4; namely if $(1-|z|^2)|z\frac{f''}{f'}|$ is small, is $f(\Delta)$ necessarily starlike or convex ? The answer can easily be deduced and turns out to be negative as expected from § 2.4 .

Given $\epsilon>0$, we consider the same g as constructed in (2.5.3) of the proof of the Theorem 2.5.2, then $E=f_1f_2=z/(1-z^2)^{\lambda}$ and $|\lambda|<(1/3)\epsilon$ is sufficiently small. It has been shown that

$(1-|z|^2)^2|S(g,z)|$ in § 2.5 can be made arbitrary small and it is not difficult to show $(1-|z|^2)|z\frac{g''}{g'}|$ can also be made arbitrary small. Recall that

$$\frac{zg'(z)}{g(z)} = (1-z^2)^\lambda \text{ and let } P(z) = \frac{zg'(z)}{g(z)}.$$

From (2.4.3) $$\frac{zg''(z)}{g'(z)} = P(z)-1+\frac{zP'(z)}{P(z)} = (1-z^2)^\lambda-1-\frac{2\lambda z^2}{1-z}.$$

Now

$$\begin{aligned}(1-|z|^2)\left|z\frac{g''}{g'}\right| &\le (1-|z|^2)|(1-z^2)^\lambda-1|+(1-|z|^2)\frac{2|\lambda||z|^2}{|1-z^2|}\\ &= (1-|z|^2)|\,(1-\lambda z^2+\frac{\lambda(\lambda-1)}{2!}z^4-\cdots)-1|+2\lambda|z|^2\,\frac{1-|z|^2}{|1-z^2|}\\ &\le(1-|z|^2)|\lambda|(|z|^2+|z|^4+\cdots)+2|\lambda| \qquad (2.5.3)\\ &=(1-|z|^2)\frac{|\lambda||z|^2}{1-|z|^2}+2|\lambda|=|\lambda z^2|+2|\lambda|\le 3|\lambda| = 3\mu<\epsilon.\end{aligned}$$

Note that the inequality (2.5.3) follows since we can choose $|\lambda|$ so small that the coefficients in the series expansion have modulus less than $|\lambda|$ as in the proof of the Theorem 2.5.2. Hence by making μ small the same g satisfies Becker's criterion and so it must be conformal yet it fails to be a starlike function.

We summarize the above results:

Theorem 2.6.1 *Given* $0<\epsilon<1$, *there exists a conformal mapping* g (*depending on* ϵ), $g(0)=0$, *such that*

$$(1-|z|^2)\left|z\frac{g''}{g'}\right| < \epsilon \quad \forall z\in\Delta,$$

and $g(\Delta)$ *is not a starlike domain* (*let alone a convex domain*). □

Remark 1 Note that $E=\frac{z}{(1-z)^\lambda}$ can be choosen instead of $E=\frac{z}{(1-z^2)^\lambda}$ in Theorem 2.6.1 and it is still sufficient to construct, by the same argument as in the proof of the Theorem 2.5.2, a counter example g for Theorem 2.6.1 . But it fails to be a counter example in Theorem 2.5.2.

Remark 2 We shall see in Chapter 4 that, when $|f''/f'|$ is small, f can indeed be starlike.

§ 2.7 *Some more General Problems*

In this section, we would like to discuss the results obtained in the previous sections and to explore more deeply into the relationship between the Schwarzian derivatives and the quasidiscs.

It is obvious that not all starlike domains with respect to the origin are K-quasidiscs. Since starlike domains can have cusps. In Theorem 1.4.1, Krzyż actually shows that for strongly starlike functions of order $\alpha < 1$ can have $K(\alpha)$-qc extension i.e. $f(\Delta)$ is a $K(\alpha)$-quasidisc. Hence α-strongly starlike functions must characterize those starlike domains *without* cusps. On the other hand, Gehring and Pommerenke [1] generalized the original Nehari's result:

<u>Theorem</u> <u>2.7.1</u> *Let f be meromorphic in Δ and let*

$$\|S_f\|_\Delta \leq 2.$$

Then f has a spherically continuous extension to $\bar{\Delta}$ and $f(\Delta)$ is a Jordan domain or the image of the parallel slit $T=\{w\colon |arg\ w|<\pi/2\}$ under a Möbius transformation. Moreover if $z_1 \in \partial\Delta$ and $f(z_1)\neq\infty$, then

$$|f(rz_1)-f(z_1)| = O\Big(dist(f(rz_1),\ \partial f(\Delta))^{1/2}\Big) \quad as\ r \to 1-.$$

This shows that the boundary $\partial f(\Delta)$ can allow certain cusps. On the other hand:

<u>Theorem</u> <u>2.7.2</u> (*Gehring and Pommerenke* [1]): *If f is meromorphic in Δ and if*

$$\|S_f\|_\Delta \leq b < 2,$$

then $f(\Delta)$ is a quasidisc with the constant $c \leq 8(1-b/2)^{-1/2}$, where c is defined in (2.5.0) §2.5. *And the order of the bound c is best possible as $b \to 2$ (see Gehring and Pommerenke* [1] *p229).*

These two theorems indicate that if $\|S_f\|_\Delta$ is small then the boundary of the image $f(\Delta)$ is a quasidisc and is smooth up to a certain degree. However, Theorem 2.5.2 shows that it has little control on the overall geometrical shape of the image $f(\Delta)$ and it certainly cannot guarantee $f(\Delta)$ to be starlike. Little is known just how much $f(\Delta)$ looks like when $\|S_f\|_\Delta$ small, and there is a gap between these results (see also Chapter 3). Of course $f(\Delta)$ is a disc when $\|S_f\|_\Delta=0$. Actually Theorems 2.3.1 and 2.4.3 clearly show that if α is small then the strongly starlike functions satisfy both the Nehari's and Becker's criteria and the opposites is not true. Obviously, an analogue can also occur if we consider the logarithmic derivatives.

Finally we have the following fundamental result due to Ahlfors.

<u>Theorem</u> <u>2.7.3</u> (*see Lehto* [2] *p.81*) *Let A be a K-quasidisc. Then there is a constant $\epsilon(K)>0$,*

depending only on K, such that every function f meromorphic in A with the property

$$\|S_f\|_A < \epsilon(K)$$

is univalent in A and can be extended to a quasiconformal mapping of the plane whose complex dilatation $\mu(z)$ satisfies the inequality

$$\|\mu\|_\infty \leq \|S_f\|_A / \epsilon(K).$$

Note that both Nehari's and Gehring-Pommerenke's results are special cases of this theorem when $A = \Delta$. In view of the Theorem 2.5.2, we may ask the following: *What conclusion can we make other than f is univalent in A, if A is a K-quasidisc and f meromorphic in A such that*

$$\|S_f\|_A < \epsilon(K),$$

where $\epsilon > 0$ is sufficiently small ?

Chapter Three

Properties of Analytic Functions with Small Schwarzian Derivatives

§ *3.1 Introduction*

We have already introduced some univalence criteria obtained by the methods of Ahlfors and Löwner differential equations. We have also seen how to use the second order differential equations techniques to construct counter examples that no matter how small that $\|S_f\|$ is f need not be starlike or convex etc. In this chapter we shall continue to use differential equations techniques to show that, replacing the hypothesis $\|S_f\| < \delta$ by $|S(f,z)| < \delta$ for some sufficiently small δ which also depends on the second coefficient a_2 of f, and $|a_2|$ is also small, then f is indeed starlike for one δ and convex for another δ. We then investigate some other consequences for f, when the Schwarzian derivative is 'small'. We shall introduce a fundamental theorem of Gronwall in differential equations which is of central importance and will be used in the proofs and later in the thesis.

Let us recall the definitions of the Schwarzian derivative of an analytic function defined in the unit disc Δ with $f' \neq 0$ in Δ. We have

$$S(f,z) = \left(\frac{f''}{f'}\right)'(z) - \frac{1}{2}\left(\frac{f''}{f'}(z)\right)^2.$$

By using the method of differential equations, we can study the relations between the Schwarzian and Univalent Function Theory. This line of research was first initiated in the works of Z. Nehari. In his famous paper of 1949, he obtained the necessary and sufficient conditions for an analytic function defined in the unit disc to be univalent:

<u>Theorem</u> <u>3.1.1</u> (*Z. Nehari* [1]) *Suppose f is analytic in Δ and $f' \neq 0$ in Δ. In order that f be univalent in Δ, it is necessary that*

$$(1-|z^2|)^2|S(f,z)| \leq 6 \quad \forall z \in \Delta,$$

and sufficient that

$$(1-|z^2|)^2|S(f,z)| \leq 2 \quad \forall z \in \Delta. \tag{3.1.1}$$

Both constants are best possible.

We recall that in Chapter 1, replacing 2 by $2k$, $k<1$ on the right hand side of (3.1.1), it becomes a sufficient condition for K-quasiconformal extension.

In [1] Nehari considered the second order differential equation

$$y'' + \frac{1}{2}S(f,z) = 0, \tag{3.1.2}$$

and the stability of its solutions. He called (3.1.2) *disconjugate* if and only if no solution can vanish more than once in Δ. He showed by applying *Green's transformation* to (3.1.2), that if $S(f,z)$ satisfies (3.1.1), then it is disconjugate and this implies that f is univalent. The Green's transformation has its origin dating back to the beginning of this century, see Hille [1],[2]. Now f can be written as :

$$f(z) = \frac{f_1(z)}{f_2(z)},$$

where f_1, f_2 are linearly independent solutions of (3.1.2) (see Lemma 2.5.3). Suppose that

$$\frac{f_1(z_1)}{f_2(z_1)} = f(z_1) = f(z_2) = \frac{f_1(z_2)}{f_2(z_2)} = \alpha, \qquad z_1 \neq z_2.$$

Then $f_2(z_1)-\alpha f_1(z_1)=0$, $f_2(z_2)-\alpha f_1(z_2)=0$. This shows that (3.1.2) is not disconjugate, hence f must be univalent. It is therefore easy to see that (3.1.2) is disconjugate if and only if no solution of (3.1.2) can vanish more than once in Δ, hence if and only if f is univalent in Δ.

Later Nehari also published a series of papers in 1954 [2], 1979 [4] and Friedland & Nehari 1970 [1] with more general univalence criteria. Along the same line F. G. Avkhadiev (see Avkhadiev & Aksent'ev [1]) gave a complete generalization of (3.1.1). Many different methods have been developed to obtain these kind of criteria by others (e.g. Löwner differential equations). For a complete review of this area, we refer the readers to a survey paper of F.G. Avkhadiev and L.A. Aksent'ev [1].

We note that Theorem 3.1.1 is irrespective of the normalization of f. However we shall consider the normalized class $N=\{f|\ f$ *analytic in* Δ *and* $f(0)=0,\ f'(0)=1\}$ and the subclass $S \subset N$ such that f is univalent. Such an f has the expansion

$$f(z) = z + a_2 z^2 + \cdots = z + \sum_{2}^{\infty} a_n z^n.$$

We consider also the class $\tilde{N}=\{g|\ g$ *analytic in* $0<|z|<1$, *with a simple pole at the origin of residue* $= 1$ $\}$, that is

$$g(z) = \tfrac{1}{z} + b_0 + b_1 z + b_2 z^2 + \cdots .$$

Also $\tilde{N}_0$ is the subclass $f \in \tilde{N}$ such that $b_0=0$.

§ *3.2 The Problems*

Schwarzian derivative is invariant with respect to the Möbius transformations M and $S(M,z)\equiv 0$. A Möbius transformation is a one to one conformal mapping of $\bar{\mathbf{C}}$. We normalize M such that for each $f \in M$, $f(0)=0$ and $f'(0)=1$. This class is denoted by M_0. For example consider the normalized mapping $z/(1-z)$ which maps the unit disc Δ onto $\Re(z) > -\frac{1}{2}$. It is reasonable to believe that when $S(f,z)$ is small in absolute value for any f, then f is close to M_0. That is f is univalent, starlike or even convex. The following discussions and results show that this is indeed the case and we also give some quantitive estimates. Our problems although similar to those considered in Chapter 2 § 2.1, in which the quantity $\|S_f\|$ the *norm of the Schwarzian derivative* was used to investigate the univalence and relation with M, our assumptions are much stronger and we fixed our normalisation.

Our starting point is from a less well-known sufficient univalence criterion of Nehari also published in [Nehari 1]. He proved:

<u>Theorem</u> <u>3.2.1</u> (*Nehari* [1]) *If f belongs to either* N or $\tilde{N}$, *and satisfies*

$$|S(f,z)| \le \frac{\pi^2}{2}, \ \forall z \in \Delta, \tag{3.2.1}$$

then f is univalent. The results are sharp.

The constant $\pi^2/2$ is best possible. This is shown by the examples $\left(exp(i\pi z)-1\right)/(i\pi)$ and $i\pi/\left(exp(i\pi z)-1\right)$ respectively. Both have Schwarzian derivatives equal to $\pi^2/2$.

Later in 1954, R.F. Gabriel [1] used *Green's transformation* and similar techniques to obtain the following :

Theorem 3.2.2 *Let $g \in \tilde{N}_0$, and suppose that*

$$|S(g,z)| \leq 2c_0 \approx 2.73 \qquad |z|<1,$$

where c_0 is the smallest positive root of the equation

$$2\sqrt{x} - tan\sqrt{x} = 0 ,$$

then g is univalent in $0<|z|<1$ and maps the interior of each circle $|z|=r<1$ onto the exterior of a convex region. The constant c_0 is best possible.

The above result is still true even if $b_0 \neq 0$ i.e. if $f \in \tilde{N}$, since a convex domain remains convex after a translation. Also if $f \in N$ then $1/f \in \tilde{N}$; but since $S(f,z) \equiv S(1/f,z)$, it is easy to obtain the following:

Corollary 3.2.3 (*Gabriel R.F.* [1]) *Let $f \in N$, and suppose that*

$$|S(f,z)| \leq 2c_0 \quad for \ |z| < 1,$$

where the constant c_0 is the same as defined in the Theorem 3.2.2, then f maps Δ onto a starlike domain.

We note that although $2c_0$ is sharp for the Theorem 3.2.2, it does not follow that it is again sharp for Corollary 3.2.3. There are many results of a similar nature about the *solutions* of the equation (3.1.2). Among them we mention the following:

Theorem 3.2.2 (*Robertson M.S.* [1]) *Let $zA(z)$ be analytic in Δ with*

$$\Re(z^2A(z)) \leq \frac{\pi^2}{2}|z|^2, \quad \forall z \in \Delta. \tag{3.2.2}$$

Then the unique solution u, satisfying $u(0)=0$, $u'(0)=1$, of the differential equation

$$y'' + Ay = 0 \tag{3.2.3}$$

is univalent and starlike in Δ. The constant $\pi^2/2$ is best possible.

We note that by putting $A=\frac{1}{2}S(f,z)$ in (3.2.2), then (3.2.1) of Theorem 3.2.1 implies (3.2.2). Hence Nehari's theorem has a stronger hypothesis. Thus Robertson proved that the unique solution of the equation (3.2.3) is starlike whereas Nehari proved that the quotient of the linearly independent solutions of (3.2.3) is univalent.

Let us compare Nehari's and Gabriel's results for the class $\tilde{N}$; we have the following relation:

$$2c_0 < \frac{\pi^2}{2},$$

where $\pi^2/2$ and $2c_0$ are the best possible constants for f to be univalent and convex respectively. This leads naturally to the following problem : what is the best constant so that when $|S(g,z)| \le 2\delta\ \forall\ z \in \Delta$, then g is starlike ? Clearly this constant must lie between $2c_0$ and $\pi^2/2$, otherwise this is meaningless. More precisely we define

$$\frac{\pi^2}{2} = 2\delta(g \in N, \tilde{N}\ ;\ univalence) = \sup\ \{\ 2\tilde{\delta}:\ g \in N, \tilde{N}\ ;\ |S(g,z)| \le 2\tilde{\delta} \Rightarrow g\ univalent\}$$

to be the *Schwarzian radii of univalence* of the classes N and $\tilde{N}$. Let

$$2c_0 = 2\delta(g \in \tilde{N};\ convex) = \sup\ \{\ 2\tilde{\delta}:\ g \in \tilde{N};\ |S(g,z)| \le 2\tilde{\delta} \Rightarrow g\ convex\}$$

to be the *Schwarzian radius of convexity of the class* $\tilde{N}$.

We can therefore put our questions as follows, that is to find

$$2\delta(g \in \tilde{N};\ starlike) = \sup\ \{\ 2\tilde{\delta}:\ g \in \tilde{N}\ ;\ |S(g,z)| \le 2\tilde{\delta} \Rightarrow g\ starlike\},$$

and

$$2\delta(f \in N;\ convex) = \sup\ \{\ 2\tilde{\delta}:\ f \in N;\ |S(f,z)| \le 2\tilde{\delta} \Rightarrow f\ convex\},$$

the *Schwarzian radius of starlikeness of the class* $\tilde{N}$ and convexity of N.

The method of proof is to consider the equation (3.2.3) and its linearly independent solutions. First we shall give a different version of Corollary 3.2.3, under the assumption that a_2, the second coefficient of f, is small. This, of couse, fails to recover the result of the corollary, but we actually obtain a stronger conclusion that f is strongly starlike of order-α: $S^*(\alpha)$, as defined in chapter one. We will also give an example to show the requirement that a_2 being small is necessary. However when $\alpha=1$ and $a_2=0$, Theorem 3.3.2 gives a poor estimate for δ for starlike functions when compared to Corollary 3.2.3.. In fact when $a_2=0$, Theorem 3.3.2 shows that $|S(f,z)|<1.8$ then f is starlike. We have not been able to find a useful estimate for $2\delta(g\in\tilde{N}$: starlike). Corollary 3.2.3 shows that $2c_0<2\delta(f\in N;$ *starlike*). The next obvious question is to find $2\delta(f\in N;$ *convex*). Our methods, similar to that of Theorem 3.3.2, allow us to obtain a lower bound for $2\delta(f\in N$: *convex*) provided a_2 is small. In § 5, by using an earlier result of Clunie and Keogh [1], we find a rough estimate on the coefficients of f in terms of $|S(f,z)|$. In § 6 we show that it is possible to drop the assumption of a_2 being small if f has a quasiconformal extension $\tilde{f}$

to $\overline{\mathbb{C}}$ such that $\tilde{f}(\infty)=\infty$. With the applications of quasiconformal extension we obtain a distortion theorem analogous to Koebe-1/4 theorem in § 7. We shall give two explicit examples of functions which in turn estimate the lower bounds of Schwarzian radii of starlikeness and convexity.

Remark Under the assumption that $|S(f,z)|$ is bounded by a constant in Δ, we deduce that f' cannot have any zero in Δ. For if f' has a zero of order n at z_0 say, then $S(f,z)$ will have a pole of order 2 at z_0. This will contradict the assumption that it is uniformly bounded in Δ. Hence we can drop the assumption that $f'\neq 0$ in the theorems which we are going to state and prove below.

§ *3.3 Main Results and Proofs*

Let us recall the definition of the strongly starlike functions of order α and starlike functions in Chapter 1. Let $f\in S$, then f is called $S^*(\alpha)$ strongly starlike function of order α if and only if $|arg\ zf'/f|\leq\alpha\pi/2$, $\forall z\in\Delta$, $0<\alpha\leq 1$. $S^*(1)= S^*$, the class of starlike functions. We also need the following well known-result for differential equations known as *Gronwall's lemma.* It is a fundamental result to estimate the growth of solutions of a given second order differential equation. We shall also present the proof, for it will be used again later. It is crucial in many of our proofs.

<u>Lemma</u> <u>3.3.1</u> (*Gronwall T.H.; see Hille* [1] *p.*19) *Suppose that $A(t)$ and $g(t)$ are non-negative continuous real functions for $t\geq 0$. Let $k>0$ be a constant. Then the inequality*

$$g(t) \leq k+\int_0^t g(s)A(s)\ ds,$$

implies for all $t>0$ that

$$g(t) \leq k\ exp\Big(\int_0^t A(s)\ ds\Big).$$

Proof Divide through the first inequality by its right hand side and then multiply by A on both sides. We obtain

$$\frac{g(t)A(t)}{k+\int_0^t g(s)A(s)\,ds} \le A(t).$$

Notice that the numerator is the derivative of the denominator. We integrate both sides from 0 to t to obtain :

$$\left[log\left(k+\int_0^{t'} g(s)A(s)\,ds\right)\right]_0^t \le \int_0^t A(s)\,ds.$$

Thus

$$log\left(k+\int_0^t g(s)A(s)\,ds\right)-log\,k \le \int_0^t A(s)\,ds$$

$$g(t) \le k+\int_0^t g(s)A(s)\,ds \le k\,exp\left(\int_0^t A(s)\,ds\right).$$ □

<u>Theorem</u> <u>3.3.2</u> *Let* $f\in N$, *and suppose* $0<\alpha\le 1$ *and* $|a_2|=\eta< sin(\alpha\pi/2)$. *Let*

$$\sup_{z\in\Delta} |S(f,z)| = 2\delta(\eta)\,,$$

where $\delta=\delta(\eta)$ *satisfies the inequality*

$$sin^{-1}\left[\frac{\delta(\eta)exp\delta(\eta)}{2}\right] + sin^{-1}\left[\eta+\frac{(1+\eta)exp(\delta(\eta)/2)\delta(\eta)}{2}\right] \le \frac{\alpha\pi}{2}. \tag{3.3.1}$$

Then $f\in S^*(\alpha)$.

Remark The inequality (3.3.1) enables us to have such a $\delta=\delta(\eta)$ since we have assumed $sin^{-1}\eta < \alpha\pi/2$.

Before we go on to prove this theorem, let us pause for a moment to look at an extremal example when $S(f,z)\equiv 0$. It is well known that $S(g,z)\equiv 0$ if and only if g is a Möbius transformation, we normalize g so that $g\in N$ (in fact $g\in S$), hence

$$g(z)=\frac{z}{1+cz}\ ,\ |c|<1.$$

We require $|c|<1$, since g is analytic in Δ. Note that the series expansion of g is

$$g(z)= z-cz^2+c^2z^3- \cdots. \tag{3.3.2}$$

If $|c|<sin(\alpha\pi/2)$ for some α, $0<\alpha\le 1$, then $g\in S^*(\alpha)$ since f satisfies the hypotheses of Theorem

3.3.2 where δ is identically equal to zero. In fact

$$z\frac{g'(z)}{g(z)} = \frac{1}{1+cz} .$$

So
$$\left| arg\ z\frac{g'(z)}{g(z)} \right| = \left| arg\frac{1}{1+cz} \right| = |arg(1+cz)| \leq sin^{-1}|c|.$$

Hence $\left| arg\ z\frac{g'(z)}{g(z)} \right| \leq \frac{\alpha\pi}{2}$ if and only if $|c| \leq sin(\alpha\pi/2)$, since this inequality is sharp for equality can be attained. Therefore $g \in S^*(\alpha)$ if and only if $|c| \leq sin(\alpha\pi/2)$. This shows, at least in this case, when g is defined as above that it is necessary for $|a_2| < sin\frac{\alpha\pi}{2}$ for g to be strongly starlike of order-α. So the conditions in Theorem 3.3.2 is nearly the best possible. However, when $|c| \geq 1$ the function g does not have the *Taylor expansion* (3.3.2), so it does not serve as a counter example which shows that the Corollary 3.2.3 is false, since $\Re(z\frac{g'}{g}) < 0$ for some z.

Proof of the Theorem 3.3.2 Suppose $u(z)$, $v(z)$ are linearly independent solutions of the differential equation

$$y'' + \tfrac{1}{2}S(f,z)y = 0 \tag{3.3.3}$$

with the normalization $u(0) = u'(0) - 1 = 0$, $v(0) - 1 = v'(0) = 0$. This is always possible since the Wronskian $W(u,v)$ of $u(z)$ and $v(z)$ of a second order differential equation is identically equal to a constant which we may take to be -1. Thus we have $u(z) = z + \cdots$ and $v(z) = 1 + \cdots$.

By Lemma 2.5.3 (b) in Chapter 2, we can find two linearly independent solutions y_1, y_2 of (3.3.3) such that

$$f(z) = \frac{y_1(z)}{y_2(z)} = \frac{au(z) + bv(z)}{cu(z) + dv(z)} , \quad ad - bc \neq 0. \tag{3.3.4}$$

The representation depends on three arbitrary constants only, but $A = \frac{1}{2}S(f,z)$ is a third order differential equation, hence they can be determined uniquely and any solution can be obtained from it by a suitable choice of these constants.

We deduce that $b = 0$ since $f(z) = z + a_2z^2 + \cdots$. We can divide through by a on both sides of (3.3.4) and therefore we may assume $a = 1$. Also $d = 1$ since $f'(0) = 1$, note also that $c = -a_2$, since $f''(0) = 2a_2$. Hence

$$f(z) = \frac{u(z)}{cu(z) + v(z)} .$$

Differentiate f, to obtain $$f'(z)=\frac{(u'(z)v(z)-u(z)v'(z))}{(cu(z)+v(z))^2}$$

$$=\frac{-W(u,v)}{(cu(z)+v(z))^2} = \frac{1}{(cu(z)+v(z))^2}.$$

Hence $$z\frac{f'(z)}{f(z)} = \frac{z}{u(z)(cu(z)+v(z))}.$$

We will show that $\left| arg\ z\frac{f'(z)}{f(z)}\right| \le \frac{\alpha\pi}{2}$. Integrating (3.3.3) by parts, we can write $u(z)$ in the following form:

$$u(z) = z+\int_0^z (\zeta-z)\ A(\zeta)\ u(\zeta)\ d\zeta. \tag{3.3.5}$$

The path of integration is taken along the radius $\zeta(t)=te^{i\theta}$ $t\in[0,r]$, $z=re^{i\theta}$. We have

$$|u(z)| \le\ r+\int_0^r |te^{i\theta}-re^{i\theta}|\ |A(te^{i\theta})|\ |u(te^{i\theta})|\ dt$$

$$< 1+\int_0^r (r-t)\ |A(te^{i\theta})|\ |u(te^{i\theta})|\ dt.$$

Now $|A(z)|<\delta=\delta(\eta)$, where δ satisfies (3.3.1). Thus applying lemma 3.3.1 we deduce:

$$|\mathrm{u(z)}| <\ exp\left[\int_0^r (r-t)\ |A(te^{i\theta})|\ dt\right]$$

$$\le\ exp\left[\delta(\eta)\int_0^r (r-t)\ dt\right]$$

$$= exp\left[\frac{\delta r^2}{2}\right]. \tag{3.3.6}$$

Now substitute back into (3.3.5), to obtain

$$|u(z)-z| \le \int_0^r (r-t)\ |A(te^{i\theta})|\ |u(te^{i\theta})|\ dt$$

$$\le \int_0^r (r-t)\ |A(te^{i\theta})|\ exp(\frac{\delta t^2}{2})\ dt$$

$$\le \delta\ exp(\frac{\delta}{2}) \int_0^r (r-t)\ \ dt$$

$$= \frac{\delta\ exp(\delta/2)\ r^2}{2}.$$

Hence $$\left|\frac{u(z)}{z}-1\right| \le \frac{\delta\, exp(\delta/2)\; r}{2} \le \frac{\delta\, exp(\delta/2)}{2}. \tag{3.3.7}$$

Similarly, $v(z)$ can also be written in the form

$$v(z) = 1+\int_0^z (\zeta-z)\; A(\zeta)\; v(\zeta)\; d\zeta.$$

Combining this with (3.3.5) we obtain

$$cu(z)+v(z) = 1+cz+\int_0^z (\zeta-z)\; A(\zeta)\left(cu(\zeta)+v(\zeta)\right) d\zeta. \tag{3.3.8}$$

The path of integration is chosen as above. So we can estimate $cu+v$ as before

$$|cu(z)+v(z)| \le (1+|c|r)+\int_0^r (r-t)\; |A(te^{i\theta})|\; |cu(te^{i\theta})+v(te^{i\theta})|\; dt.$$

Since $|A| < \delta(\eta)$ where δ satisfies (3.3.1) by the hypotheses, we obtain, by applying Lemma 3.3.1 again that

$$\begin{aligned} |cu(z)+v(z)| &< (1+|c|)\; exp\left[\int_0^r (r-t)\; |A(te^{i\theta})|\; dt\right] \\ &\le (1+|c|)\; exp\left[\frac{\delta r^2}{2}\right] \\ &\le (1+|c|)\; exp\left[\delta/2\right]. \end{aligned} \tag{3.3.9}$$

Substitute this back into (3.3.8) and note that $|c|=\eta<sin(\alpha\pi/2)$. We obtain

$$\begin{aligned} |cu(z)+v(z)-1| &\le |c|r + \int_0^r |\zeta-z|\; |A(\zeta)|\; |cu(\zeta)+v(\zeta)|\; dt \\ &\le \eta + \int_0^r (r-t)\; |A(te^{i\theta})|\; |cu(te^{i\theta})+v(te^{i\theta})|\; dt \\ &\le \eta + (1+\eta)\delta\; exp(\delta/2)\int_0^r (r-t)\; dt \\ &\le \eta + (1+\eta)\delta\; \frac{exp(\delta/2)}{2}\, . \end{aligned} \tag{3.3.10}$$

It follows from (3.3.7) and (3.3.9) that

$$\left|arg\; \frac{zf'(z)}{f(z)}\right| = \left|arg\; \frac{f(z)}{zf'(z)}\right|$$

$$= \left| arg \, \frac{u(z)\{cu(z)+v(z)\}}{z} \right|$$

$$\leq \left| arg \, \frac{u(z)}{z} \right| + \left| arg \, \{u(z)+v(z)\} \right|$$

$$\leq sin^{-1}\Big(\frac{\delta \, exp(\delta/2)}{2}\Big) + sin^{-1}\Big(\eta+(1+\eta)\,\frac{\delta \, exp(\delta/2)}{2}\Big)$$

$$\leq \frac{\alpha\pi}{2}.$$

The last inequality follows from the hypothesis (3.3.1). Hence f belongs to $S^*(\alpha)$. This completes the proof of the theorem. □

Remark If we put $a_2=0$ and $\alpha=1$ in the above theorem, then (3.3.1) only gives a rather poor estimate for δ. The best δ that we can derive from (3.3.1) with $a_2=0$, $\alpha=1$ is approximately 1.8.

Another observation is that we can estimate $arg\Big(\frac{f(z)}{z}\Big)$ the same way as we have done to $arg\Big(z\frac{f'(z)}{f(z)}\Big)$. Since

$$\left| arg \, \frac{f(z)}{z} \right| = \left| arg \, \frac{u(z)}{z\{cu(z)+v(z)\}} \right|$$

$$\leq \left| arg \, \frac{u(z)}{z} \right| + \left| arg \, \{cu(z)+v(z)\}^{-1} \right|$$

$$= \left| arg \, \frac{u(z)}{z} \right| + \left| arg \, \{cu(z)+v(z)\} \right|$$

$$\leq sin^{-1}\Big(\frac{\delta \, exp(\delta/2)}{2}\Big) + sin^{-1}\Big(\eta+(1+\eta)\,\frac{\delta \, exp(\delta/2)}{2}\Big),$$

this estimate is exactly the same as (3.3.1) of Theorem 3.3.2. Hence we obtain

<u>Corollary 3.3.3</u> *Let $f\in N$, suppose $0<\alpha\leq 1$ and $|a_2|=\eta< sin\ \alpha\pi/2$. Suppose*

$$\sup_{z\in\Delta} |S(f,z)| = 2\delta(\eta),$$

where $\delta(\eta)$ is some positive number which satisfies the inequality

$$sin^{-1}\Big[\frac{\delta(\eta) exp\delta(\eta)}{2}\Big] + sin^{-1}\Big[\eta+\frac{(1+\eta)exp(\delta(\eta)/2)\delta(\eta)}{2}\Big] \leq \frac{\alpha\pi}{2}.$$

Then $\left| \arg \frac{f(z)}{z} \right| \leq \frac{\alpha\pi}{2}$.

§ 3.4 Applications to Convexity

We shall now consider another class of analytic functions.

Definition 3.4.1 If f is analytic and defined in Δ, then it is called *convex univalent of order* μ ($0 \leq \mu \leq 1$) if and only if

$$\Re\left(1+z\frac{f''(z)}{f'(z)}\right)>\mu \quad \forall z\in\Delta.$$

The class of functions is denoted by $\mathfrak{K}(\mu)$. Clearly $\mathfrak{K}(0)=\mathfrak{K}$ is the class of *convex univalent* functions as in Chapter 2(see Duren [1]).

Note that the above definition is irrespective of the normalization of f.

We have, from the results of Nehari and Gabriel, the following relations:

$$2.73\approx 2c_0<2\delta(\boldsymbol{N},\ \textit{starlike})<2\delta(\boldsymbol{N},\ \textit{univalence})=\pi^2/2.$$

We would like to find a lower estimate for $2\delta(\boldsymbol{N},\ \textit{convex})$ under the additional assumption when a_2 is small. Needless to say we expect it to be less than $2c_0$.

Theorem 3.4.1 *Let* $f(z) = z+a_2z^2+ \cdots \in \boldsymbol{N}$ *and suppose that* $|a_2|=\eta<\frac{1}{3}$.
Let

$$\sup_{z\in\Delta} |S(f,z)| = 2\delta(\eta),$$

where $\delta(\eta)$ *satisfies*

$$6\eta+ 5(1+\eta)\ \delta(\eta)\ \exp\Big(\delta(\eta)/2\Big) < 2. \tag{3.4.1}$$

Then

$$f \in \mathfrak{K}\left(\frac{2-6\eta-5(1+\eta)\ \delta(\eta)\ \exp\{\delta(\eta)/2\}}{2-2\eta-(1+\eta)\ \delta(\eta)\ \exp\{\delta(\eta)/2\}}\right). \tag{3.4.2}$$

In particular if $a_2=0$ *then* $0.6712 \leq 2\delta(f\in\boldsymbol{N},\ a_2=0;\ f\ \textit{convex})$.

Remark 1 Note that (3.4.2) holds if (3.4.1) holds so that the quotient appearing in (3.4.2) is positive.

Remark 2 Unlike Theorem 3.3.2, Theorem 3.4.1 is valid only for $\eta<\frac{1}{3}$ and not for $\eta<1$.

Remark 3 J.G. Clunie has proved the special case of theorem when $a_2=0$ and he has improved the constant 0.6712 to $5/6=0.833\cdots$. We shall look at this again in Chapter 4.

We consider the example of Theorem 3.3.3 again in which we take

$$g(z)=\frac{z}{1-cz}=z+cz^2+c^2z^3+\cdots,\ |c|=1.$$

But g maps the unit disc onto a rotation of right hand half plane passing through $-\frac{1}{2}\overline{c}$, and so it is clearly a convex mapping with $|a_2|=1$. However the hypotheses in the above theorem require $|a_2|<\frac{1}{3}$. Hence it is not sharp.

Proof of the Theorem Let us assume that f satisfies the hypotheses of the theorem. As in the proof of Theorem 3.3.2 , we consider the differential equation (3.3.3) with $A=\frac{1}{2}S(f,z)$

$$y''+\frac{1}{2}S(f,z)\ y=0.$$

Using exactly the same argument as before we can write f as

$$f(z)=\frac{u(z)}{cu(z)+v(z)},$$

where $u(z)$, $v(z)$ are linearly independent solutions of the differential equation with the normalization $u(0)=0=u'(0)-1$, $v(0)-1=0=v'(0)$. It is easy to show that

$$1+z\frac{f''(z)}{f'(z)}=1-2z\frac{cu'(z)+v'(z)}{cu(z)+v(z)}. \tag{3.4.3}$$

We shall prove that $\Re\left(1+z\frac{f''(z)}{f'(z)}\right)>0$. In view of (3.4.3), it is sufficient to prove that

$$\left|\frac{cu'(z)+v'(z)}{cu(z)+v(z)}\right|<\frac{1}{2}.$$

We note that $\eta+\frac{1}{2}(1+\eta)\delta(\eta)exp\{\delta(\eta)/2\}<1$ since η and $\delta(\eta)$ satisfy (3.4.1). Let us recall that u, v have the following forms

$$u(z)=z+\int_0^z(\zeta-z)\ A(\zeta)\ u(\zeta)\ d\zeta, \qquad v(z)=1+\int_0^z(\zeta-z)\ A(\zeta)\ v(\zeta)\ d\zeta.$$

Hence

$$\frac{cu'(z)+v'(z)}{cu(z)+v(z)}=\frac{c-\int_0^z A(\zeta)\left(cu(\zeta)+v(\zeta)\right)d\zeta}{1+cz+\int_0^z(\zeta-z)\ A(\zeta)\left(cu(\zeta)+v(\zeta)\right)d\zeta}.$$

Since $|A|<\delta$ by hypotheses we deduce from (3.3.9) (after applying Lemma 3.3.1), that

$$\left| cz+\int_0^z (\zeta-z)\ A(\zeta)\ \{cu(\zeta)+v(\zeta)\}\ d\zeta \right|$$

$$< |c| + \int_0^r (r-t)\ |A(te^{i\theta})|\ (1+|c|)exp\left(\frac{\delta t^2}{2}\right)\ dt$$

$$\leq\ \eta\ +\ (1+\eta)\ \delta\ exp(\delta/2)\int_0^r (r-t)\ dt$$

$$<\ \eta\ +\ (1+\eta)\ \delta\ exp(\delta/2)/2 < 1. \tag{3.4.4}$$

The last inequality follows from hypothesis (3.4.1).

Thus $$\left|\frac{cu'(z)+v'(z)}{cu(z)+v(z)}\right| = \left| \frac{c-\int_0^z A(\zeta)\left(cu(\zeta)+v(\zeta)\right)\ d\zeta}{1+cz+\int_0^z (\zeta-z)\ A(\zeta)\left(cu(\zeta)+v(\zeta)\right)\ d\zeta} \right|$$

$$\leq\ \frac{|c-\int_0^z (\zeta-z)\ A(\zeta)\left(cu(\zeta)+v(\zeta)\right)\ d\zeta\ |}{1-|cz+\int_0^z (\zeta-z)\ A(\zeta)\left(cu(\zeta)+v(\zeta)\right)\ d\zeta\ |}$$

$$\leq \left\{\left| c-\int_0^z A(\zeta)\ [cu(\zeta)+v(\zeta)]\ d\zeta\right|\right\} \left\{\sum_{n=0}^{\infty}\left| cz+\int_0^z (\zeta-z)\ A(\zeta)\ [cu(\zeta)+v(\zeta)]\ d\zeta\right|^n\right\}$$

$$\leq \left\{\eta+(1+\eta)\ \delta\ exp(\delta/2)\right\}\left\{\sum_{n=0}^{\infty}\left(\eta+(1+\eta)\ \delta\ exp(\delta/2)/2\right)^n\right\}$$

$$= \frac{\eta+(1+\eta)\ \delta\ exp(\delta/2)}{1-\ \eta-\ \frac{(1+\eta)\ \delta\ exp(\delta/2)}{2}}\ =\ \frac{2\{\eta+(1+\eta)\ \delta\ exp(\delta/2)\}}{2-\ 2\eta-(1+\eta)\ \delta\ exp(\delta/2)}.$$

Because of (3.4.4), the above geometric progression converges. Moreover

$$\frac{2\left(\eta+(1+\eta)\ \delta\ exp(\delta/2)\right)}{2-\ 2\eta-(1+\eta)\ \delta\ exp(\delta/2)} < \frac{1}{2}$$

if and only if (3.4.1) holds, and so f is convex univalent. Now

$$\Re\left(1\ +\ z\frac{f''(z)}{f'(z)}\right) = \Re\left(1\ -\ 2z\frac{cu'(z)+v'(z)}{cu(z)+v(z)}\right)$$

$$\geq 1-\ 2\left(\frac{2\left[\eta+(1+\eta)\ \delta\ exp(\delta/2)\right]}{2-\ 2\eta-(1+\eta)\ \delta\ exp(\delta/2)}\right)$$

$$=\frac{2-6\eta-5(1+\eta)\ \delta\ exp(\delta/2)}{2-\ 2\eta-(1+\eta)\ \delta\ exp(\delta/2)}.$$

If we now put $-a_2=c=0$ in the above argument, it follows from (3.4.1) that

$$5\delta\ exp(\delta/2)\ <\ 2,$$

where δ can be calculated. Numerical calculations give $\delta\ <\ 0.3365$. Hence $|S(f,z)|<0.6712$ implies that f is convex univalent. This completes the proof of the Theorem. □

We summarize the above relations

$$0.6712<2\delta(N,\ convex)<2c_0<2\delta(N,\ starlike)<2\delta(N,\ univalence)=\pi^2/2.$$

§ 3.5 *An estimation on the Area and the Coefficients of f*

We have seen, in the last section, that when the Schwarzian and the second coefficient of f is small, then f is a α-strongly starlike function where α depends on a_2. Brannan and Kirwan [1] have shown that, if $\alpha<1$, an α-strongly starlike function is necessarily a bounded analytic function. They even showed that the boundary of $f(\Delta)$ is rectifiable and bounded by $2\pi M(\alpha)\ Sec(\alpha\pi/2)$ where $M(\alpha)$ ia a constant depending only on α. We shall give an upper bound for the area of $f(\Delta)$ by using the estimations in the last section. Then by using a theorem of Clunie and Keogh, we also give an upper bound of the coefficients of f.

<u>Theorem</u> <u>3.5.1</u> *Let $f \in N$ and $|a_2|=\eta<1$. Let*

$$\sup_{z\in\Delta}\ |S(f,z)|\ =\ 2\delta(\eta),$$

where $\delta(\eta)$ satisfies

$$2\eta+(1+\eta)\ \delta(\eta)\ exp(\delta(\eta)/2)\ <\ 2. \tag{3.5.1}$$

Then

$$\sum_{n=1}^{\infty} n|a_n|^2\ \leq \left[\ \frac{2\ exp(\delta/6)}{2-2\eta-(1+\eta)\ \delta(\eta)\ exp(\delta(\eta)/2)}\right]^3. \tag{3.5.2}$$

Remark It is well-known that if f is a bounded univalent function in Δ then the area of its image has the representation $\pi\sum_1^\infty n|a_n|^2\ (<\infty)$.

Proof Suppose that $f\in N$ and satisfies the hypotheses of the theorem. Also let u, v be normalized linearly independent solutions of the differential equation (3.3.3), as in the proof of theorem 3.3.2 . Then

$$f(z)=\frac{u(z)}{cu(z)+v(z)}.$$

By Green's formula (see Duren [1] p. 15), the area of $f(\Delta)$ can be written as

$$\iint_\Delta |f'(z)|^2\ dxdy = \lim_{r\to 1}\frac{1}{2i}\int_{\partial\Delta_r}\overline{f}(z)f'(z)\ dz$$

$$\le \lim_{r\to 1}\max_{|z|=r}\frac{2\pi r}{2}\,|\overline{f}(z)|\ |f'(z)|$$

$$=\lim_{r\to 1}\max_{|z|=r}\pi\left|\frac{u(z)}{cu(z)+v(z)}\right|\left|\frac{1}{\big(cu(z)+v(z)\big)}\right|^2$$

$$=\lim_{r\to 1}\max_{|z|=r}\pi\ \frac{|u(z)|}{|cu(z)+v(z)|^3}.$$

But $|A(z)|<\delta(\eta)$ and (3.5.1) is satisfied, so by the same argument as in the last proof , we deduce $|u(z)|\le exp(\delta/2)$ and $|cu(z)+v(z)|^{-1}\le\sum_0^\infty\Big(\eta+(1+\eta)\delta exp(\delta/2)/2\Big)^n$.

So

$$\iint_\Delta |f'(z)|^2\ dxdy \le \pi\ exp(\delta/2)\left(\sum_0^\infty\Big(\eta+(1+\eta)\delta exp(\delta/2)/2\Big)^n\right)^3$$

$$=\pi\left(\frac{2\ exp(\delta/6)}{2-2\eta-(1+\eta)\delta exp(\delta/2)}\right)^3.$$

However

$$\iint_\Delta |f'(z)|^2\ dxdy = \lim_{r\to 1}\int_0^r\int_0^{2\pi}|f'(re^{i\theta})|^2\ dxdy$$

$$=2\pi\int_0^r\left[1+\sum_2^\infty n^2|a_n|^2r^{2n-2}\right]rdr = \pi\sum_2^\infty n|a_n|^2. \qquad \square$$

We quote the following result:

<u>Theorem</u> <u>3.5.2</u> (*Clunie J. & Keogh F.G.* [1]) *Suppose $f(z)=z+\sum_2^\infty a_nz^n$ is starlike in Δ and maps Δ onto a domain of area Q. Then*

$$|a_n|\le\frac{2}{n-1}\sqrt{\frac{Q}{\pi}}\qquad n\ge 2.$$

We see that $|a_n| = O(\frac{1}{n})$ when $f(\Delta)$ has a finite area and this result is also best possible. On the other hand, if we fixed the area Q and consider all the starlike functions f maps Δ to Q then it is not best possible in the sense that one can produce an example (in Clunie and Keogh [1]) which has infinitely many $|a_n| > \frac{1}{n}$.

As a consequence we have

<u>Theorem</u> <u>3.5.3</u> *We assume the same notations and hypotheses as in the Theorem* 3.3.2. *Then*

$$|a_n| \leq \frac{2^{5/2}}{n-1}\left[\frac{exp(\delta/6)}{2-2\eta-(1+\eta)\delta exp(\delta/2)}\right]^{3/2} \qquad n\geq 2.$$

Remark We mention that the above estimate is not sharp when we put $n=2$, since $|a_2|$ is assumed to be less then $sin(\alpha\pi/2)$.

Proof f is a α-strongly starlike function for some $\alpha<1$, since it satisfies the hypotheses of the Theorem 3.3.2. Hence it must be a bounded univalent function. Condition (3.3.1) implies condition (3.5.1) with the same $\delta(\eta)$. Therefore the hypotheses of Theorem 3.4.1 are also satisfied. So from (3.5.2)

$$\text{Area of } f(\Delta) = Q \leq \pi \left[\frac{2\ exp(\delta/6)}{2-2\eta-(1+\eta)\delta exp(\delta/2)}\right]^{3}.$$

Now apply Theorem 3.5.2 to complete the proof. □

§ *3.6 On the Second Coefficient of f*

Theorem 3.4.1 was proved under the assumptions that $|a_2|$ is small and the Schwarzian derivative is also small depending on $|a_2|$. However, it is also clear from the above proofs that $|a_2|$ does not necessarily depend on $|S(f,z)|$ without further restriction on f. We show that this is indeed the case and there is a strong relation with quasiconformal extension of f (if f has one) as defined in Chapter 1. Let us recall from §2.3 the class $S_K(\infty)$ that are those $f\in S$ such that f has a K-quasiconformal extension to $\bar{\mathbb{C}}$, where its complex dilatation $\mu_f(z)$ satisfies $|\mu_f(z)| \leq k = \frac{K-1}{K+1}$ and its extension $\tilde{f}(\infty) = \infty$. We discuss the problem of Schwarzian radius of convexity of f in $S_K(\infty)$. That is we consider

$$2\delta(f\in S_K(\infty);\ convex) = sup\ \{\ 2\tilde{\delta} : |S(f,z)| \le 2\tilde{\delta},\ f \in S_K(\infty) \Rightarrow\ f(\Delta)\ is\ convex\ \}.$$

Unlike the previous cases where 2δ depends on a_2, this time K also depends on a_2.

Theorem 3.6.1 *Let $f \in S_K(\infty)$ where $K=\frac{1+\delta}{1-\delta} \le 1.24$ and suppose that*

$$\sup_{z\in\Delta} |S(f,z)| = 2\delta(\eta) < 0.217,$$

then f is convex in Δ.

Proof The first part of the proof is identical to that of Theorem 3.4.1. We also use the same notations as those in Theorem 3.4.1. But by our hypotheses

$$(1-|z|^2)^2\ |S(f,z)| \le |S(f,z)| \le 2\delta,\ \ \delta < 1\ \ \forall z\in\Delta.$$

It follows from the Theorem 1.4.1 of Ahlfors and Weill, that f admits a $\frac{1+\delta}{1-\delta}$-quasiconformal extension to $\bar{\mathbb{C}}$. But since $\tilde{f}(\infty)=\infty$, it follows from the Lemma 2.3.2 that $|a_2|=\eta \le 2\delta$. It follows that f is convex or $\Re\left(1+z\frac{f''}{f'}\right) > 0$, if only if (3.4.1) is valid. However we now have

$$6\eta + 5(1+\eta)\ \delta(\eta)\ \exp\left[\delta(\eta)/2\right] < 12\delta + 5(1+2\delta)\ \delta(\eta)\ \exp\left[\delta(\eta)/2\right].$$

Hence we only need to solve the last inequality for δ so that the last expression is less than 2. Numerical calculations show that this is true if $2\delta \le 0.217$. So f is convex univalent in Δ. This completes the proof of the theorem. □

§ 3.7 *On the applications of the Second Coefficient of f.*

It is well known that the image of Δ under any function in the class of S always contains a disc of radius ρ centred at the origin, here ρ is an absolute constant. This remarkable fact was first discovered by Koebe and later proved by Bieberbach. He found that the constant ρ is at least equal to $1/4$, the bound is attained only by the Koebe function $K(z)= z/(1-z)^2$ and its rotation and so it is sharp. The original proof made use of a well known result on the second coefficient of f, that is $|a_2| \le 2$ for all $f\in S$. We shall use the same method to obtain a special version of Koebe-$1/4$ theorem when the Schwarzian derivative is small.

Theorem 3.7.1 *(a) Suppose that* $f \in N$ *and* $|S(f,z)| \le 2\delta,\ \delta<1\ \forall z\in\Delta$. *Let* $\kappa = \frac{1}{\pi}\cos^{-1}\delta$ *so that* $\kappa \in (0,\frac{1}{2}]$. *Then the image* $f(\Delta)$ *of the unit disc contains a disc of radius*

$$\frac{1}{4(1-2\kappa(\delta)^2)},$$

centred at the origin.

(b) Let $f \in S_K(\infty)$ *and* $|S(f,z)| \le 2\delta,\ \delta<1\ \forall z\in\Delta$. *Then* $f(\Delta)$ *contains a disc of radius at least*

$$\frac{1}{2(1+\delta-2\kappa(\delta)^2)},$$

centred at the origin.

Proof (a) Suppose f satisfies hypotheses (a) and that there exists a complex number w so that

$$f(z) \ne w \quad \forall z\in\Delta.$$

Let

$$g(z) = \frac{wf(z)}{w-f(z)} = z + \left(a_2+\tfrac{1}{w}\right) z^2 + \cdots. \tag{3.7.1}$$

Notice that $g\in N$ again and $S(g,z) \equiv S(f,z)$. So

$$(1-|z|^2)^2|S(g,z)| = (1-|z|^2)^2|S(f,z)| \le |S(f,z)| \le 2\delta,\quad \delta < 1 \quad \forall z\in\Delta.$$

It follows from Theorem 1.4.1 again that both $g(z)$ and $f(z)$ have $\frac{1+\delta}{1-\delta}$-quasiconformal extensions to $\bar{\mathbb{C}}\setminus\Delta$. Since $g\in S_K$, we obtain from Lemma 2.4.2 that

$$\left| a_2+\frac{1}{w} \right| \le 2-4\kappa^2.$$

So

$$\left|\frac{1}{w}\right| \le 2-4\kappa^2 + |a_2| \le 2(2-4\kappa^2).$$

Hence

$$|w| \ge \frac{1}{4(1-2\kappa^2)}.$$

(b) If $f\in S_K(\infty)$ then $g\in S_K$ where g is defined in (3.7.1). By a similar argument and using the fact that $|a_2| \le 2\delta$ we deduce from Lemma 2.3.2 that

$$\left| a_2+\frac{1}{w} \right| \le 2-4\kappa^2.$$

$$\left|\frac{1}{w}\right| \le 2-4\kappa^2 + |a_2| \le 2-4\kappa^2+2\delta$$

$$|w| \ge \frac{1}{2\left[1+\delta-2\kappa^2\right]}. \qquad \square$$

Remark 1 Since we used only the fact that f admits a K-qc extension, we may weaken our assumption to $(1-|z|^2)^2|S(f,z)|\leq 2\delta$.

Remark 2 In the case (a), suppose $K\to 1$ as $\delta\to 0$, then $\kappa\to\frac{1}{2}$; this implies $f(\Delta)$ contains a disc of radius $1/2$. This coincides with the class

$$S_1 = \{\, f: f(z) = \frac{z}{1-cz} \quad |c| \leq 1\},$$

the normalized class which has 1-quasiconformal extension (analytic continuation) to $\bar{\mathbb{C}}$. On the other hand, let $K\to\infty$ as $\delta\to 1$, then $\kappa\to 0$ and f contains a disc of radius $1/4$. But S_K is dense in S in the topology of locally uniform convergence as $K\to\infty$ (see Schober [1] p.148), so this yields the classical Koebe constant. Therefore we see that (a) is sharp in the limiting cases. However in the case when $a_2=0$ we obtain from Theorem 3.4.1 that $|S(f,z)|<0.6$ implies f is convex and hence f contains a disc of radius equal to $1/2$ already.

Remark 3 In the case (b) $K\to 1$ as $\delta\to 0$. This implies that the radius tends to 1. The only function in $S_K(\infty)$ whose image contains the whole unit disc is $f(z)=z$. Also $\bigcup_{1\leq k<\infty} S_K(\infty)$ is dense in S in the topology of locally uniform convergence, since $f(kz)/k\in S_K(\infty)$. This time the radius approaches $1/4$ as $K\to\infty$ which is again what we expect.

Remark 4 We note that Schiffer and Schober also obtained another version of Koebe-1/4 theorem in (Schiffer and Schober [1]). They proved, by using the method of calculus of variations, that if $f\in S_K$, then

$$\bigcap_{f\in S_K} f(\Delta) = \left\{ w : |w| < \frac{1}{4}\, exp\left(\int_0^1 \left(x^{-\kappa/2} - x^{\kappa/2}\right)\frac{dx}{1-x^2}\right)\right\},$$

where κ is defined in Lemma 3.6.1. This also gives the same limits for the radii of discs as $K\to\infty$ and $K\to 1$ as in Remark 2. We have a stronger hypothesis than that of Schiffer and Schober but our proof is perhaps more straightforward.

§ *3.8 Estimations on the lower bounds of the δ's.*

We shall consider two examples which have same constant Schwarzians. So we can investigate how small the Schwarzian we require in order that the function be starlike or convex. We recall that two functions have a same Schwarzian derivative if and only if they differ by an arbitrary Möbius transformation T. If, however, we restrict ourselves to the class S, then one fixed the transformation T which takes $f \in S$ into S again. In this case, $T(z) = \frac{z}{1-kz}$ with $|k| \leq 4$. Since $(T \circ f)(z) \in S$, we have $1-kf \neq 0$ i.e. $f \neq 1/k$, therefore $|1/k| \geq 1/4$ by Koebe's-1/4 theorem. If we impose a further condition on T and f so that $f''(0) = 0$ and $\left(T \circ f\right)''(0) = 0$, then T must be the identity mapping. This can be easily verified.

Firstly, we observe that the normalized function

$$f(z) = \frac{exp(i2\sqrt{\delta}z) - 1}{i2\sqrt{\delta}},$$

has Schwarzian $S(f,z) \equiv 2\delta$. By Theorem 3.2.1, $f \in S$ if $S(f,z) \equiv 2\delta \leq \pi^2/2 \approx 4.9$. It is well-known that any function in S is starlike in the smaller disc of radius $|z| \leq tanh\ (\pi/4)$ and convex in $|z| \leq 2-\sqrt{3}$. Both constants are best possible (see Duren [1] p.95). In view of the above remark, it is easy to see that f must be starlike when

$$|S(f,z)| \leq \frac{\{tanh\ (\pi/4)\}^2}{2} \approx 0.215. \tag{3.8.1}$$

For we can choose δ sufficiently small so that $|i2\sqrt{\delta}z| \leq |2\sqrt{\delta}| \leq \left[tanh\ (\pi/4)\right]^2/2$, then according to the radius of starlikeness, the function $exp(w) - 1$ with $w = 2\sqrt{\delta}iz$ must be starlike. Since $\frac{exp(w)-1}{i2\sqrt{\delta}}$ remains starlike, hence it is sufficient to assume (3.8.1) for f to be starlike in Δ. This is of course is weaker than Corollary 3.2.3. Similarly we also have

$$|S(f,z)| \leq \frac{\left[2 - \sqrt{3}\right]^2}{2} \approx 0.036$$

implies that f is convex.

Our second example is $\qquad g(z) = \frac{1}{\sqrt{\delta}}\ tan\ (\sqrt{\delta}z).$

This is in fact obtained under a Möbius transformation from the above f and hence $S(g,z) \equiv 2\delta$. Notice that $g''(0) = 0$. Unlike f, we show that the bounds for Schwarzian derivatives when g becomes starlike or convex are much larger.

It is easy to obtain

$$z\frac{g'(z)}{g(z)} = \frac{\sqrt{\delta}z}{sin(\sqrt{\delta}z)\ cos(\sqrt{\delta}z)}$$

$$= \frac{2\sqrt{\delta}z}{sin(2\sqrt{\delta}z)}.$$

We require to show that $\Re\left(z\,\dfrac{g'(z)}{g(z)}\right) > 0\ \forall z\in\Delta$ when δ is small. This is equivalent to finding the smallest disc $|w| < r$ such that $\Re\left(\frac{\sin w}{w}\right)$ is non-negative, $2\sqrt{\delta}z = w = \xi+i\mu$.

Let $$h(\xi,\mu) = \Re\left(\frac{\sin w}{w}\right) = \Re\left(\frac{\sin\xi\cos i\mu - \cos\xi\sin i\mu}{\xi+i\mu}\right)$$

$$= \frac{\xi\sin\xi\cosh\mu - \mu\cos\xi\sinh\mu}{\xi^2+\mu^2}\ .$$

Let $$F(\xi,\mu) = \xi\sin\xi\cosh\mu - \mu\cos\xi\sinh\mu.$$

We apply the method of the *Lagrange multiplier* to $F(\xi,\mu)$ subject to $\xi^2 + \mu^2 = r^2$ for some r. At these point $F(\xi,\mu)$ must be extremal. Now let

$$\phi(\xi,\mu) = F(\xi,\mu) + \lambda(\xi^2 + \mu^2 - r^2),$$

We need to solve $\phi_\xi = 0$, $\phi_\mu = 0$, $\phi_\lambda = 0$.

$$\phi_\xi = \cosh\mu\,(\xi\cos\xi + \sin\xi) - \mu\sin\xi\sinh\mu + 2\lambda\xi = 0,$$

$$\phi_\mu = \xi\sin\xi\sinh\mu + \cos\xi\,(\mu\cosh\mu + \sinh\mu) + 2\lambda\mu = 0,$$

$$\phi_\lambda = \xi^2 + \mu^2 - r^2 = 0.$$

Multiply the first equation by μ and the second equation by ξ, and equate them, then

$$(\xi^2+\mu^2)\sin\xi\sinh\mu = -\xi\cos\xi\,(\mu\cosh\mu + \sinh\mu) + \mu\cosh\mu\,(\xi\cos\xi + \sin\xi)$$

$$= -\xi\cos\xi\sinh\mu) + \mu\cosh\mu\sin\xi$$

i.e. $$\xi^2+\mu^2 = \frac{\mu}{\tanh\mu} - \frac{\xi}{\tan\xi}\ . \qquad (3.8.2)$$

Hence

$$h(\xi,\mu) = \frac{1}{\xi^2+\mu^2}\left(\xi\sin\xi\cosh\mu + \mu\cos\xi\sinh\mu\right)$$

$$= \frac{1}{\xi^2+\mu^2}\left(\xi\sin\xi\,\frac{\sinh\mu}{\mu}\frac{(\xi^2+\mu^2)\sin\xi + \xi\cos\xi}{\sin\xi} + \mu\cos\xi\sinh\mu\right)$$

$$= \frac{1}{\xi^2+\mu^2}\left(\frac{\sinh\mu}{\mu}\Big((\xi^2+\mu^2)\,\xi\sin\xi + \xi^2\cos\xi + \mu^2\cos\xi\Big)\right)$$

$$= \frac{1}{\xi^2+\mu^2}\ \frac{\sinh\mu}{\mu}\,(\xi^2+\mu^2)\,(\xi\sin\xi + \cos\xi)$$

$$= \frac{\sinh\mu}{\mu}\,(\xi\sin\xi + \cos\xi).$$

As $\frac{sinh\ \mu}{\mu}$ is an even function and so always positive, we have that $h(\xi,\mu) < 0$ if and only if $\xi sin\xi + \xi < 0$. This is equivalent to finding the first positive root of the equation

$$\xi\ tan\ \xi = -1.$$

Substitute back into (3.8.2), we have

$$\xi^2 + \mu^2 = \frac{\mu}{tanh\ \mu} - \frac{\xi}{\left(-\frac{1}{\xi}\right)} = \frac{\mu}{tanh\ \mu} + \xi^2,$$

i.e. to solve μ tanh $\mu = 1$. Here the problem reduces to solve the following transcendental equations

$$\xi\ tan\ \xi = -1,$$

$$\mu\ tanh\ \mu = 1.$$

Numerical calculation gives

$$2.79 < \xi < 2.8,$$

$$1.199 < \mu < 1.2.$$

So

$$3.037 < (\xi^2+\mu^2)^{1/2} < 3.046. \tag{3.8.3}$$

So $\Re\left(\frac{sin\ w}{w}\right)$ will first become negative when w lies in the annula defined by (3.8.3). Hence if we require δ such that

$$|2\sqrt{\delta}\ z| = |w| \leq 3.037$$

i.e.

$$|2\sqrt{\delta}| \leq 3.037$$

i.e.

$$\delta \leq 2.3,$$

then g is a starlike function if $2\delta \leq 4.6$. It is also clear that g is not starlike if $2\delta \geq 4.64$.

We can similarly consider the convexity case. We show that $\Re\left(1+z\frac{g''}{g'}\right) > 0$ when the Schwarzian is small. Now we consider, as before that

$$1+z\frac{g''}{g'} = 1+ 2\sqrt{\delta}z\,\frac{sin\sqrt{\delta}z}{cos\sqrt{\delta}z}$$

$$= 1+ 2\sqrt{\delta}z\ tan\sqrt{\delta}z.$$

Again let $w=\sqrt{\delta}z$. It is sufficient to find the smallest r such that $\Re(w\ tan\ w) > -\frac{1}{2}$. It is easy to obtain

$$\Re(w\ tan\ w) = \frac{\xi\ tan\ \xi\ (1-tanh^2\mu) - \mu\ tanh\ \mu\ (1+tan^2\xi)}{1+\ tan^2\xi\ tanh^2\mu}.$$

Unlike the starlike case, it is much more difficult to work out precisely the first r such that

$\Re(w tan\ w) = -\frac{1}{2}$. Elementary calculations show that, if we set $\xi = 0$,

$$\Re(w\ tan\ w) = -\mu\ tanh\ \mu = -\frac{1}{2}.$$

i.e. we solve $\mu\ tanh\ \mu = \frac{1}{2}$. The approximate solution is $0.7715 < \mu < 0.773$. Hence if $\mu > 0.7715$, $\Re(w\ tan\ w)$ could be less than $-\frac{1}{2}$. i.e. if $\delta > 0.5952$, g need not be convex univalent.

Summarizing the above analyses, we deduce

Proposition 3.8.1 *Let* $f(z) = \dfrac{exp(2i\sqrt{\delta}z)-1}{2i\sqrt{\delta}}$ *and* $g(z) = \dfrac{1}{\sqrt{\delta}}\ tan\sqrt{\delta}z$.

(i) $\quad$ *If* $|S(f,z)| = 2\delta \le \frac{1}{2}\ tanh^2\frac{\pi}{4}$, $\forall z\in\Delta$ *then* $f \in S^*$,

$\quad$ *If* $|S(f,z)| = 2\delta \le \frac{1}{2}\left[2-\sqrt{3}\right]^2$, $\forall z\in\Delta$ *then* $f \in \mathfrak{K}$.

(ii) $\quad$ *If* $|S(g,z)| = 2\delta < 4.6$, $\forall z\in\Delta$ *then* $f \in S^*$,

where $\delta = (x^2+y^2)^{1/2}$ *and* x, y *are the first positive roots of the transcendental equations*

$$2\sqrt{\delta}x\ tan(2\sqrt{\delta}x) = -1,\ \ 2\sqrt{\delta}y\ tanh(2\sqrt{\delta}y) = 1.$$

And finally if for some $z_0\in\Delta$ *such that* $|S(g,z_0)| > 1.2$, *then* g *need not be convex univalent.*

Although the above calculations are not sharp, it shows that the Schwarzian radii of starlikeness when $a_2=0$ must be less than 4.6, and that of convexity must be less than 1.2. We shall also see in next chapter that if the coefficients of f, $a_2=a_3= \cdots =a_n=0$, then larger Schwarzian radii are obtained, increasing as a function of n.

Chapter Four

Further Schwarzian Derivatives and Related Results

§ *4.1 Introduction*

The first three sections of this chapter should be considered as a continuation of the last chapter. There were initiated by private communications with Professor J.G. Clunie and Professor T. Sheil-Small. Each of them has given his own proof for the lower bounds of Schwarzian radius of convex functions, under additional assumption on f. We then find that if the growth of the Schwarzian derivative is slower than that of $1+zf''/f'$, when the second coefficient is equal to zero, then f is also a convex function. In § 4, we apply the Clunie-Jack principle to give some alternative proofs of some results, concerning convexity, obtained by S.S. Miller and P.T. Mocanu [1]. In some cases, new criteria about convexity, involving the Schwarzian are also obtained. We shall look at another similar problem involving the logarithmic derivative and the logarithmic radii for univalence and starlikeness criteria. In § 6, 7 and 8 we consider a subclass of strongly gamma starlike functions that has qc extension. We mention that the main theme of this chapter is the applications of the Clunie-Jack principle.

In Chapter 3 we have already defined the class N, however we shall adopt a more general definition for it as follows: $N(n)$ consists of the normalized analytic functions defined in the unit disc Δ and have the expression

$$f(z) = z + \sum_{k=n}^{\infty} a_k z^k \qquad n \geq 2. \tag{4.1.1}$$

Each of the Clunie's and Sheil-Small's proof was originally given under the hypotheses that $n=3$ and if $S(f,z)$ is small then f is a convex function. However it is not difficult to extend their proofs to consider such f for any integer $n\geq 3$, and to conclude that f is convex univalent of order μ i.e. $f \in \mathfrak{K}(\mu)$. Their methods do not appear to extend to the case when a_2 in (4.4.1) is small as in Chapter 3. On the other hand the differential equation method of Theorem 3.4.1 fails to consider the case when $n\geq 4$. Also Clunie's proof gives a better result than the other two methods, when $n=3$.

§ *4.2 Clunie's Method*

Let us define the maximum modulus of f to be

$$M(r) = \max_{|z|=r} |f(z)| ,$$

where f in (4.1.1) is an analytic function defined in Δ. We require some lemmas. The first is an old result dating back to the beginning of this century.

<u>Lemma 4.2.1</u> (*Blumenthal see Hayman* [1]) *$M(r)$ is itself an analytic function of r, except at an isolated number of points $r_1 < r_2 < \cdots$, and $M(r)$ is represented by distinct analytic funtions in the intervals $[r_1,r_2]$, $[r_2,r_3]$, $[r_3,r_4]$, $\cdots$.*

<u>Lemma 4.2.2</u> *Let f be analytic and* $M'(r) = \max_{|z|=r} |f'(z)|$,

then

$$M(r) \leq \int_0^r M'(t)\ dt. \tag{4.2.1}$$

Proof Let f be defined in (4.1.1), hence $M(0)= 0$. By Lemma 4.2.1, $M(r)$ is continuous apart from countably many points in $(0,r)$. It is also clear that by the maximum principle $M(r)$ must be a monotonic increasing function in $[0,r]$, hence the integral in (4.2.1) must exist. Now

$$\begin{aligned} M(r) = M(r) - M(0) &= \max_{|z|=r} |\ f(z) - f(0)\ | = \max_{|z|=r}\left|\int_0^r f'(te^{i\theta})e^{i\theta}\ dt\right| \\ &\leq \max_{|z|=r}\int_0^r |\ f'(te^{i\theta})|\ dt \leq \int_0^r \max_{0\leq\theta\leq 2\pi} |\ f'(te^{i\theta})|\ dt \\ &= \int_0^r M'(t)\ dt. \end{aligned}$$

□

Although we are all familiar with Schwarz's lemma, we need the following extended version of it; since its proof is just an easy exercise, we shall omit its proof.

<u>Lemma</u> <u>4.2.3</u> (*Schwarz's lemma*)

Let
$$g(z) = b_n z^n + b_n z^{n+1} + \cdots$$

be an analytic function defined in Δ. *Then when* $0 < \rho \le r$,

$$M(\rho,g) \le \frac{\rho^n}{r^n} M(r,g).$$

<u>Theorem</u> <u>4.2.4</u> (*J.G. Clunie* [2]) *Let* f *be as defined in* (4.1.1) *with* $n \ge 3$. *If*

$$|S(f,z)| < \frac{(1-\mu)(4n-7+\mu)}{4n-6}, \tag{4.2.2}$$

where $0 \le \mu \le 1$, *then* $f \in \mathcal{K}(\mu)$.

Proof We require to proof that $\Re\left(1+z\frac{f''}{f'}\right) > \mu$. Let

$$\varphi(z) = \frac{f''(z)}{f'(z)} = \frac{n(n-1)\ a_n z^{n-2} + (n+1)n\ a_{n+1} z^{n-1} + \cdots}{1 + n\ a_n z^{n-1} + (n+1)\ a_{n+1} z^n + \cdots}$$

$$= n(n-1)\ a_n z^{n-2} + \cdots$$

This implies
$$|\varphi(z)| < \frac{r^{n-2}}{r_1^{\,n-2}} M(r_1,\varphi), \qquad r<r_1<1, \tag{4.2.3}$$

by Lemma 4.2.3. Then by the assumption we have

$$|\varphi'(z) - \tfrac{1}{2}\varphi^2(z)| = |S(f,z)| < \frac{(1-\mu)\ (4n-7+\mu)}{4n-6} \quad \forall z \in \Delta.$$

So, we have

$$|\varphi'(z)| - \tfrac{1}{2}|\varphi^2(z)| < \frac{(1-\mu)\ (4n-7+\mu)}{4n-6} \quad \forall z \in \Delta,$$

It follows that for $|z| = r$,

$$M(r,\varphi') - \tfrac{1}{2}M(r,\varphi)^2 < \frac{(1-\mu)\ (4n-7+\mu)}{4n-6}\ .$$

Integrate from 0 to r_1 on both sides,

$$\int_0^{r_1} M(t,\varphi')\ dt - \frac{1}{2}\int_0^{r_1} M(t,\varphi)^2\ dt \le \frac{(1-\mu)\ (4n-7+\mu)}{4n-6} r_1.$$

We apply (4.2.3) and Schwarz's lemma 4.2.3 to obtain

$$M(r_1,\varphi) - \frac{M(r_1,\varphi)^2}{2r_1^{\,2n-4}}\int_0^{r_1} t^{2(n-2)}\ dt < \frac{(1-\mu)\ (4n-7+\mu)}{4n-6} r_1.$$

That is $$M(r_1,\varphi) - \frac{M(r_1,\varphi)^2 r_1}{4n-6} < \frac{(1-\mu)\,(4n-7+\mu)}{4n-6} r_1. \tag{4.2.4}$$

Now consider the continuous function $F(t) = t - \dfrac{t^2 r_1}{2(2n-3)}$, so that $F'(t) = 1 - \dfrac{t r_1}{2n-3}$. We see that F is an increasing function of t as long as $tr_1 \leq 2n-3$, for $n \geq 3$. Suppose $M(r_1,\varphi) \geq 1-\mu$ then $M(r_1,\varphi) - \dfrac{M(r_1,\varphi)^2 r_1}{2(2n-3)} \geq (1-\mu) - \dfrac{(1-\mu)^2 r_1}{2(2n-3)} > \dfrac{(1-\mu)\,(4n-7+\mu)}{4n-6}$. This is a contradiction. Hence $M(r_1,\varphi) < 1-\mu$ and since r_1 is arbitrary so $\Re\left(1+z\dfrac{f''}{f'}\right) > \mu$. This completes the proof of the Theorem. □

Remark When n tends to infinity, the condition $|S(f,z)| < \epsilon < 1$ is sufficient to guarantee f to be convex univalent.

§ *4.3 Sheil-Small's Method*

Let us first state the result.

Theorem 4.3.1 (*T. Sheil-Small* [1]) *Let f be as defined in* (4.1.1) *with $n \geq 3$. Suppose that*

$$|S(f,z)| \leq \frac{(1-\mu)\,(n-2-\mu)}{2}\ ,\ 0 \leq \mu \leq 1, \tag{4.3.1}$$

then $f \in \mathcal{K}(\mu)$.

The original proof given by Sheil-Small was a special case of the above theorem when $n=3$ and $\mu=0$. The method of proof make use of a maximum principle known as *Clunie-Jack principle* (*see Jack I.S.* [1]). Similar methods involving Clunie-Jack principle have been used successfully by other mathematicians to solve wide variety of problems, among others, see J.G. Clunie [1], S.S Miller [1], S.S. Miller and P.T. Mocanu [1].

Lemma 4.3.2 (*Clunie-Jack principle*)

Let $$w(z) = b_m z^m + b_{m+1} z^{m+1} + \cdots, \qquad m \geq 1$$

be an analytic function defined in a unit disc Δ. Suppose w attains its maximum value at z_0, i.e. $|w(z_0)| = \max\limits_{|z| \leq r_0} |w(z)|$, $z_0 = r_0 e^{i\theta_0}$, *then $z_0\, w'(z_0)/w(z_0)$ is real and*

$$z_0\,\frac{w'(z_0)}{w(z_0)} = t \geq m \geq 1.$$

We note that part of the Clunie-Jack principle was also obtained by W.K. Hayman [1]. Instead of proving that (4.3.1) implies $f \in \mathfrak{K}(\mu)$, it is more convenient to consider the following equivalent statement: *the inequality*

$$|S(f,z)| < \frac{(1-\beta)\,(2n-5-\beta)}{8}, \tag{4.3.2}$$

implies that $f \in \mathfrak{K}\left[\frac{1}{2}(1+\beta)\right]$. *Here* $\mu = \frac{1}{2}\,(\beta+1)$, $-1 \leq \beta \leq 1$.

Proof of the Theorem: Let us consider the Möbius transformation

$$\zeta = \frac{1-\beta\,z}{1-z}$$

which maps the unit disc onto the half plane $\Re(\zeta) > \frac{1}{2}\,(\beta + 1)$. We shall only consider the case when $-1 \leq \beta < 1$.

Let $\qquad f(z) = z + a_n z^n + a_{n+1} z^{n+1} + \cdots \quad n \geq 3$

satisfy the hypotheses. We now defined $w(z)$ to be an analytic function defined in Δ, $w(0)=0$ such that

$$1 + z\,\frac{f''(z)}{f'(z)} = \frac{1 - \beta\,w(z)}{1 - w(z)}. \tag{4.3.3}$$

In view of the above remark, it is equivalent to assume f satisfies (4.3.2) and so we wish to show $1+\frac{z\,f''(z)}{f'(z)} \prec \frac{1-\beta z}{1 - z}$. Hence it is sufficient to prove $|w(z)| < 1$ for all $z \in \Delta$ in (4.3.3) by subordination. There are two possibilities: either $|w(z)| < 1$ for all $z \in \Delta$, or there exists a $z_0 = r_0 e^{i\theta_0} \in \Delta$ *such that* $|w(z)| < 1$ for all $|z| < r_0$, but $|w(z_0)|=1$. If it is the first case the theorem is proved by subordination (see Chapter 2). Hence we assume from now on the second case and get a contradiction. Now

$$z\,\frac{f''(z)}{f'(z)} = (1 - \beta)\,\frac{w(z)}{1 - w(z)} \tag{4.3.4}$$

and

$$z\,\frac{f''(z)}{f'(z)} = \frac{n\,(n - 1)\,a_n z^{n-1} + n\,(n+1)\,a_{n+1} z^n + \cdots}{1 + n\,a_n z^{n-1} + (n + 1)\,a_{n+1} z^n + \cdots}$$

$$= n\,(n - 1)\,a_n z^{n-1} + \cdots .$$

Since $w(0)=0$ we may suppose

$$w(z) = b_m z^m + b_{m+1} z^{m+1} + \cdots , \qquad m \geq 1.$$

Then

$$(1-\beta)\,\frac{w(z)}{1-w(z)} = (1-\beta)\Big(w(z) + w(z)^2 + w(z)^3 + \cdots\Big)$$

$$= (1-\beta)\Big(\,b_m z^m + b_{m+1} z^{m+1} + \cdots\Big).$$

Now compare the series expansions on both sides of (4.3.4). We deduce that $m = n-1$. Also

$$\Big(\frac{f''(z)}{f'(z)}\Big)' = (1-\beta)\bigg(\frac{zw'(z) - w(z) + w(z)^2}{2z^2(1-w(z))^2}\bigg)\,, \qquad \Big(\frac{f''(z)}{f'(z)}\Big)^2 = \frac{(1-\beta)^2\,w(z)^2}{z^2(1-w(z))^2}$$

and thus

$$S(f,z) = \Big(\frac{f''(z)}{f'(z)}\Big)' - \frac{1}{2}\Big(\frac{f''(z)}{f'(z)}\Big)^2$$

$$= (1-\beta)\bigg(\frac{2zw'(z) - 2w(z) + (1+\beta)\,w(z)^2}{2z^2(1-w(z))^2}\bigg).$$

Since $|w(z)| < 1$ for $|z| < r_0$, there exists a z_0, $|z_0| = r_0$ such that w attains its maximum at z_0 and $|w(z_0)| = 1$, $w(z_0) \neq 1$. For if $w(z_0) = 1$ then $1+z\dfrac{f''}{f'}$ is not an analytic function and hence f' will have a zero, this contradicts the fact that the $S(f,z)$ is uniformly bounded in Δ. It follows from the above analyses and the Clunie-Jack principle that $z_0w(z_0) = tw(z_0)$, where $t \geq n-1$. We consider $S(f,z)$ at z_0.

$$|S(f,z_0)| = (1-\beta)\left|\frac{2w(z_0)\,(t-1) + (1+\beta)\,w(z_0)^2}{2z_0^2\,(1-w(z_0)^2}\right|$$

$$\geq (1-\beta)\left(\frac{2(t-1) - \Big((1+\beta)\,|w(z_0)|\Big)\,|w(z_0)|}{2r_0^2\,(1+1)^2}\right)$$

$$> (1-\beta)\Big(\frac{2(t-1) - (1+\beta)}{8}\Big)$$

$$\geq (1-\beta)\,\frac{2n-4-1-\beta}{8} = (1-\beta)\,\frac{2n-5-\beta}{8}\ .$$

This contradicts (4.3.2) and shows the second case cannot happen. This completes the proof of the theorem. □

Let us compare Theorems 4.3.1 and 4.2.4 . We set $\mu = 0$ in both cases. When $n = 3$, a better bound for the Schwarzian radius $5/6$ is obtained by Clunie's method. However when $n \geq 4$, the criterion (4.2.2) is always bounded by 1 as we have already remarked. However, the criterion (4.3.1) not only gives a better estimate on the Schwarzian radii, when $n \geq 4$, it shows that the Schwarzian radii are *unbounded* as n approaches infinity. Hence when $n \geq 7$, if

$$(1-|z|^2)^2 \; |S(f,z)| \leq |S(f,z)| \leq \frac{7-2}{2} = 2.5$$

from (4.3.1) and Theorem 4.3.1 shows that f not only univalent, but also convex, a much stronger conclusion. However, the Nehari criterion (Theorem 3.1.1) fails to show that f is univalent. Hence our result complements that of Nehari.

Unlike the differential equation method used in the last chapter, the above methods do not seem to apply to the starlike case nor when $a_2 \neq 0$. But the advantage of the Clunie-Jack principle is that it could be used to consider different $n \geq 3$.

The result of Nehari shows that $|S(f,z)| \leq \frac{\pi^2}{2}$ is sufficient for f to be univalent and that $\frac{\pi^2}{2}$ is sharp. The above discussions suggest the following problem: What is the sharp bound $2\delta(n)$ for $|S(f,z)|$, depending on n, so that if f has the form (4.1.1) for each $n \geq 2$ then $|S(f,z)| \leq 2\delta(n) \Rightarrow f$ is univalent ?

i.e. $$2\delta(f \in N(n), n \geq 3;\ \textit{univalent})\ ?$$

In what follows, we shall use the Clunie-Jack principle to prove various results of this type, including some known ones. Among others, we first show that if the growth of $S(f,z)$ is slower than that of $1+z\frac{f''}{f'}$ where $n \geq 3$, then f is again convex univalent.

<u>Theorem</u> <u>4.3.3</u> *Let f be as defined in* (4.1.1) *and $f' \neq 0$ with $n \geq 3$. Suppose*

$$|\; z^2 S(f,z) \;| \leq (n-2)\left| 1 + z\frac{f''}{f'} \right| \quad \forall\ z \in \Delta, \tag{4.3.5}$$

then $f \in \mathcal{K}$.

Proof Let
$$1 + z\frac{f''(z)}{f'(z)} = \frac{1+w(z)}{1-w(z)} = h(z) \tag{4.3.6}$$

as before, where $w(z)$ is analytic in Δ. We require to prove $\Re(h) > 0$ for all $z \in \Delta$. Thus it is sufficient to show $|w(z)| < 1$, $\forall\ z \in \Delta$. Suppose this is not true as in the proof of the last theorem, then there exists a $r_0 < 1$ such that $|w(z)| < 1$ when $|z| < r_0 < 1$ and $w(z)$ attains its maximum at z_0, $|z_0| = r_0$ with $|w(z_0)| = 1$ and $w(z_0) \neq 1$.

Now $$z^2S(f,z) = zh'(z) - (h(z) - 1) - \frac{1}{2}(h(z) - 1)^2$$
$$= zh'(z) + \frac{1}{2}(1 - h(z)^2). \tag{4.3.7}$$

So $$\frac{z^2S(f,z)}{h(z)} = z\,\frac{h'(z)}{h(z)} + \frac{(1 - h(z)^2)}{2h(z)}$$
$$= \left(\frac{1 + w(z)}{1 - w(z)}\right)\left(\frac{2zw'(z)}{(1 - w(z))^2} + \frac{1}{2}\left(\frac{1-2w(z)+w(z)^2-1-2w(z)-w(z)^2}{(1 - w(z))^2}\right)\right)$$
$$= \frac{2zw'(z)}{1 - w(z)^2} - \frac{2w(z)}{1 - w(z)^2}.$$

Consider $z^2S(f,z)/h(z)$ at z_0. By the Clunie-Jack principle, we have

$$\left|\frac{z_0^2\ S(f,z_0)}{h(z_0)}\right| = \left|\frac{2\ tw(z_0) - 2w(z_0)}{1 - w(z_0)^2}\right| \qquad \text{where } t \geq n - 1$$
$$= \frac{2\ (t - 1)}{|1 - w(z_0)^2|}$$
$$> \frac{2\ (\mathrm{t} - 1)}{(1 + 1)} \geq n - 2.$$

The reason that $t \geq n - 1$ follows by comparing the series expansions from (4.3.6) as in the last proof. This contradiction completes the proof of the theorem. □

§ *4.4 On some theorems of S.S. Miller and P.T. Mocanu*

In 1978 S.S. Miller and P.T. Mocanu [1] developed some general differential inequalities which could be applied to different areas of function theory and differential equations. One of their basic tools is again the Clunie-Jack principle and subordination. Their methods served well to 'construct' some criteria for convexity (and starlikeness), but not for others like Theorem 4.3.3 proved in this chapter. We find that some of their results can be proved by applying the Clunie-Jack principle, directly, without going through their reasoning. In fact we also generalize some of their results and a result that does not appear to be obtained by their methods.

Let us first state their result.

Theorem 4.4.1 (*Miller S.S. & Mocanu P.T.* [1]) *Let* $u = u_1 + iv_2$, $v = v_1 + iv_2$, *and let* $\rho(u,v)$ *be a complex valued function satisfying:*

(*i*) $\rho(u,v)$ *is continuous in a domain* $\mathbb{D} \subset \mathbb{C}^3$, (*ii*) $(1,0) \in \mathbb{D}$ *and* $\Re\, \rho(1,0) > 0$,

(*iii*) $\Re\, \rho(iu_2, v_1) \le 0$ *when* $v_1 \le 0$.

Let f satisfy (4.1.1) *with* $f' \neq 0$ *and suppose that* $\left(1 + z\frac{f''}{f'},\ z^2 S(f,z)\right) \in \mathbb{D}$ *when* $z \in \Delta$.

If
$$\Re\left(\rho\left(1 + z\frac{f''}{f'},\ z^2 S(f,z)\right)\right) > 0 \ \ then\ \ \Re\left(1 + z\frac{f''}{f'}\right) > 0.$$

The following functions defined in $\mathbb{C}^3$ satisfy (i), (ii), and (iii) above as suggested in Miller S.S. & Mocanu P.T. [1],

$$\rho_1(u,v) = u^2 + v\,,$$

$$\rho_2(u,v) = u + \alpha v\,,\ \Re\,\alpha \ge 0,$$

$$\rho_3(u,v) = u\, e^{v}.$$

They imply the following criteria

$$\Re\left(\left(1 + z\frac{f''}{f'}\right)^2 + z^2 S(f,z)\right) > 0 \ \Rightarrow\ \Re\left(1 + z\frac{f''}{f'}\right) > 0$$

$$\Re\left(1 + z\frac{f''}{f'} + \alpha\, z^2 S(f,z)\right) > 0,\ \Re\alpha \ge 0 \ \Rightarrow\ \Re\left(1 + z\frac{f''}{f'}\right) > 0$$

and
$$\Re\left(\left(1 + z\frac{f''}{f'}\right) exp\left(z^2 S(f,z)\right)\right) > 0 \ \Rightarrow\ \Re\left(1 + z\frac{f''}{f'}\right) > 0\,.$$

We shall now give alternative proofs to these criteria and they also shed new light into the theorem. We shall discuss this point later in this section.

<u>Theorem</u> <u>4.4.2</u> *Let f be as defined in* (4.1.1) $f' \neq 0$, *then*

(*a*)
$$\Re\left(\left(1 + z\frac{f''}{f'}\right)^2 + z^2 S(f,z)\right) > 0 \ \Rightarrow\ \Re\left(1 + z\frac{f''}{f'}\right) > 0\,,\ \forall\ z \in \Delta$$

(*b*)
$$\Re\left(1 + z\frac{f''}{f'} + \alpha\, z^2 S(f,z)\right) > 0,\ \ \Re\alpha \ge 0 \ \Rightarrow\ \Re\left(1 + z\frac{f''}{f'}\right) > 0,\ \forall\ z \in \Delta$$

(*c*)
$$\Im\left(1 + z\frac{f''}{f'} + \alpha\, z^2 S(f,z)\right) > 0,\ \ \Im\alpha \ge 1 \ and\ n \ge 3 \ \Rightarrow\ \Re\left(1 + z\frac{f''}{f'}\right) > 0,\ \forall\ z \in \Delta$$

(*d*)
$$\Re\left(\left(1 + z\frac{f''}{f'}\right) exp\left(z^2 S(f,z)\right)\right) > 0 \ \Rightarrow\ \Re\left(1 + z\frac{f''}{f'}\right) > 0\ \forall\ z \in \Delta.$$

Remark The part (c) of above theorem does not appear to be proved directly by the methods of Miller and Mocanu.

Proof We shall prove (ii) and (iii) first as it is similar to Theorem 4.3.1. Define w to be an analytic function in Δ such that

$$1 + z\frac{f''(z)}{f'(z)} = \frac{1+w(z)}{1-w(z)} = h(z), \tag{4.4.1}$$

with $w(z) = b_m z^m + b_{m+1} z^{m+1} + \cdots$.

Under the hypotheses (ii) and (iii) we require to show $\Re\left(1+ zf''/f'\right) > 0$, i.e. that $|w|<1 \ \forall \ z \in \Delta$. We suppose this is not true. Then there exists a r_0, $0<r_0<1$ so that $|w| < 1$ for $|z|<r_0$ such that $|w(z_0)| = 1$ with $z_0 = r_0 e^{i\theta_0}$ where w is maximized. We can write $w(z_0)$ in the following form $w(z_0) = \cos\theta_0 + i\sin\theta_0$. It is easy to verify the following:

$$\frac{w(z_0)}{(1-w(z_0))^2} = \frac{-1}{2(1-\cos\theta_0)}, \quad \frac{1+w(z_0)}{1-w(z_0)} = i\frac{\sin\theta_0}{1-\cos\theta_0}, \tag{4.4.2 a}$$

$$\Re\left(\frac{w(z_0)}{1-w(z_0)}\right)^2 = \frac{\cos\theta_0}{2(\cos\theta_0 - 1)}, \ \Re\left(\frac{1}{(1-w(z_0))^2}\right) = \frac{\cos\theta_0}{2(\cos\theta_0 - 1)} \tag{4.4.2 b}$$

From (4.3.7) we have

$$z^2 S(f,z) = zh'(z) + \tfrac{1}{2}(1 - h(z)^2).$$

We also write $\alpha = \alpha_1 + i\alpha_2$, where $\alpha_1 \geq 0$ in (ii) and $\alpha_2 \geq 1$ when we consider case (iii). At z_0, we apply the Clunie-Jack principle and (4.4.2). We obtain

$$\begin{aligned}
\left(1 + z_0\frac{f''(z_0)}{f'(z_0)}\right) + \alpha z_0 S(f,z_0) &= h(z_0) + \alpha\left(z_0 h'(z_0) + \tfrac{1}{2}(1 - h(z_0)^2)\right). \\
&= \frac{1+w(z_0)}{1-w(z_0)} + \frac{2\alpha\,(z_0\, w'(z_0) - w(z_0))}{(1-w(z_0))^2} \\
&= \frac{1+w(z_0)}{1-w(z_0)} + \frac{2\alpha\,(t-1)w(z_0)}{(1-w(z_0))^2} \\
&= i\frac{\sin\theta_0}{1-\cos\theta_0} + 2\alpha(t-1)\frac{-1}{2(1-\cos\theta_0)} \\
&= \frac{-\alpha_1(t-1)}{1-\cos\theta_0} + i\frac{\sin\theta_0 - \alpha_2(t-1)}{1-\cos\theta_0}.
\end{aligned}$$

If we assume the hypothesis (ii) then

$$\Re\left(\left(1 + z_0\frac{f''(z_0)}{f'(z_0)}\right) + \alpha z_0{}^2 S(f,z_0)\right) = \frac{-\alpha_1(t-1)}{1-\cos\theta_0} \leq 0$$

since $\alpha_1 \geq 0$ and $t \geq m = 1$. If we now assume the hypothesis (iii) we have

$$\Im\left(\left(1 + z_0\frac{f''(z_0)}{f'(z_0)}\right) + \alpha z_0{}^2 S(f,z_0)\right) = \frac{\sin\theta_0 - \alpha_2(t-1)}{1-\cos\theta_0} \leq 0$$

since $\alpha_2 \geq 1$ and $t \geq m = 2$ or equivalently $n \geq 3$. This follows easily from comparing the power series expansions of (4.4.1). Hence in both cases we obtained contradictions. This proves (b) and (c).

Suppose f satisfies (a). We have

$$h(z)^2 + z^2 S(f,z) = zh'(z) + \tfrac{1}{2}\,(1 + h(z)^2)$$

where h is defined by (4.1.1). At z_0, we apply the Clunie-Jack principle as above to obtain

$$\begin{aligned} z_0 h'(z_0) + \tfrac{1}{2}\,(1 + h(z_0)^2) &= \frac{2\; z_0 w'(z_0)}{(1 - w(z_0))^2} + \frac{1}{2}\Big(\frac{2 + 2\; w(z_0)^2}{(1 - w(z_0))^2}\Big) \\ &= \frac{2\; tw(z_0)}{(1 - w(z_0))^2} + \frac{1 + w(z_0)^2}{(1 - w(z_0))^2}\,. \end{aligned}$$

From (4.2.2) we have

$$\begin{aligned} \Re\left(\Big(1 + z_0\frac{f''(z_0)}{f'(z_0)}\Big)^2 + z_0{}^2 S(f,z_0)\right) &= \frac{2\,t}{2(\cos\theta_0 - 1)} + \frac{2\cos\theta_0}{2(\cos\theta_0 - 1)} \\ &= \frac{-(t + \cos\theta_0)}{1 - \cos\theta_0} \leq 0 \quad \text{since } t \geq 1 \geq \cos\theta_0. \end{aligned}$$

We sketch the proof of (d). From (4.4.1), w attains its maximum at z_0 and we have from

$$\begin{aligned} \Big(1 + z_0\frac{f''(z_0)}{f'(z_0)}\Big)\, exp\Big(z^2 S(f,z_0)\Big) &= h(z_0)\; exp\,\Big(z_0 h'(z_0) + \tfrac{1}{2}\,(1 - h(z_0))^2\Big) \\ &= h(z_0)\; exp\,\Big(\frac{2\; z_0 w'(z_0)}{(1 - w(z_0))^2} - \frac{2\; w(z_0)}{(1 - w(z_0))^2}\Big) \\ &= \frac{1 + w(z_0)}{1 - w(z_0)}\; exp\,\Big(\frac{2(t - 1)\; w(z_0)}{(1 - w(z_0))^2}\Big), \end{aligned}$$

by the Clunie-Jack principle. But $\dfrac{w(z_0)}{(1 - w(z_0))^2} = \dfrac{-1}{2(1 - \cos\theta_0)}$ is real and $\dfrac{1 + w(z_0)}{1 - w(z_0)} = i\,\dfrac{\sin\theta_0}{1 - \cos\theta_0}$ is imaginary from (4.4.2).

Hence

$$\Re\left(\Big(1 + z_0\frac{f''(z_0)}{f'(z_0)}\Big)\, exp\Big(z_0{}^2 S(f,z_0)\Big)\right) = \Re\,\Big(i\,\frac{\sin\theta_0}{1 - \cos\theta_0}\; exp\,\Big(\frac{-(t - 1)}{1 - \cos\theta_0}\Big)\Big) = 0,$$

contradiction. This completes the proof of the theorem. □

Remark The last part of the the proof of the above theorem (i.e. (d)) seems to suggest that either $\Re\left(1+z\frac{f''}{f'}\right) exp\left(z^2 S(f,z)\right) > 0$ or < 0 is sufficient for f to become convex. However the second case when < 0 is meaningless when $z = 0$.

It is also true that many other functions which satisfy the hypotheses of Theorem 4.4.1 can be considered by means of the Clunie-Jack principle. We show that even Theorem 4.4.1 can be proved by the same method directly.

Let $\rho(u,v)$ satisfies (i) (ii) (iii) of Theorem 4.4.1 and let $1+z\frac{f''}{f'} = \frac{1+w}{1-w}$, where w is defined as before. We suppose w attains its maximum at $z_0 = r_0 e^{i\theta_0}$ and $|w| < 1$ for $|z|<r_0$. At z_0 and from (4.4.2) we have

$$1 + z_0\frac{f''(z_0)}{f'(z_0)} = i\,\frac{\sin\theta_0}{1-\cos\theta_0}\;,\quad {z_0}^2 S(f,z_0) = \frac{-(t-1)}{1-\cos\theta_0}.$$

Now by the hypothesis $\rho\left(i\,\frac{\sin\theta_0}{1-\cos\theta_0}, \frac{-(t-1)}{1-\cos\theta_0}\right) \leq 0$ with

$$u_2 = \frac{\sin\theta_0}{1-\cos\theta_0}\,,\quad v_1 = \frac{-(t-1)}{1-\cos\theta_0}\,.$$

Since $t \geq 1$, we have $-(t-1) \leq 0$; hence $v_1 \leq 0$. This contradicts the fact that $\rho(u,v) > 0$ $\forall$ $z \in \Delta$, and completes the proof.

We see that the condition $\rho(iu_2,v_1) \leq 0$ when $v_1 \leq 0$ is precisely where the contradiction occurs when we apply the Clunie-Jack principle. Theorem 4.4.1 actually characterises those functions in $\mathbf{C}^3$ which yield a contradiction to the hypotheses when the Clunie-Jack principle is applied. This also explains why the method works so well when proving some special cases. Before we go on to discuss the logarithmic derivative in the next section, we state two more results; their proofs have now become a matter of triviality.

<u>Theorem</u> <u>4.4.3</u> *Let f be as defined in* (4.1.1), $f' \neq 0$ *and* $n \geq 3$. *If*

$$\Re\left(z^2 S(f,z)\right) \geq 0 \quad \forall\ z \in \Delta,$$

then $\Re\left(1 + z\frac{f''}{f'}\right) > 0$.

Proof We simply consider (4.3.7) again. □

There are also some criteria for starlikeness.

Theorem 4.4.4 *Let f be as defined in* (4.1.1), $f' \neq 0$. *If f satisfies either of the following*

(a) $$\Re\left(1 + \frac{zf''}{f'} - \frac{zf'}{f}\right) > 0 \ \text{ or } \Re\left(1 + \frac{zf''}{f'}\right) > \Re\left(\frac{zf'}{f}\right) \quad \forall\ z \in \Delta,\ n \geq 2$$

or

(b) $$\Re\left[\frac{zf'}{f}\left(1 + \frac{zf''}{f'} - \frac{zf'}{f}\right)\right] \geq -(n-1) \qquad \forall\ z \in \Delta,\ n \geq 2$$

then $\Re\left(\frac{zf'}{f}\right) > 0$.

We shall omit the proofs.

§ 4.5 Applications to the Logarithmic Derivative

Let $f \in N$ with $f' \neq 0$ in Δ. Recall the *logarithmic derivative* is defined in § 1.11 as $T(f,z) = f''(z)/f'(z)$. It is invariant with respect to translations. We have seen in Chapter 1 that $T(f,z)$ and $S(f,z)$ have many similar properties. For example, both $(1-|z|^2)|T(f,z)| \leq 1$ and $(1-|z|^2)^2|S(f,z)| \leq 2$, proved by Becker and Nehari, lead to sharp univalence criteria. Theorem 3.2.1 showed that $|S(f,z)| \leq \frac{\pi^2}{2}$ is also a sharp criterion for univalence. It is therefore natural to ask for the sharp bound $\sigma > 0$, so that $|T(f,z)| \leq \sigma$ implies f to be univalent. We seek

$$\sigma(f \in N;\ univalent) = sup\{\tilde{\sigma} : |T(f,z)| \leq \tilde{\sigma} \Rightarrow f \text{ is univalent}\},$$

the *logarithmic radius of univalence* of *N*. Several papers have been devoted to this problem. The best result was proved by S.N. Kudryashov [1]; see also Avkhadiev and Aksent'ev [1] p.35 :

Theorem 4.5.1 *Let* $f \in N$ *be defined by* (4.1.1) *and suppose that*

$$|T(f,z)| \leq \sigma \quad \forall\ z \in \Delta,$$

where σ *is the root of the equation* $8\sqrt{x(x-2)^3} - 3(4-x)^2 = 12$, $\sigma \approx 3.05 \cdots$. *Then f is univalent in* $|z| < 1$.

This shows that $3.05 < \sigma(f \in N;\ univalent)$. Consider the function $f(z) = exp(\sigma z)$, then $T(f,z) = \sigma$. Hence f is univalent if and only if $\sigma < \pi$, for $\sigma \geq \pi$, f is not univalent. The sharp bound appears to be π but the problem of finding the best bound remains open. Now if the Schwarzian is small then f is convex as shown in Chapter 2. We can therefore ask a similar question, that is to find

$$\sigma(f\in N;\ starlike)=sup\{\tilde{\sigma} : |T(f,z)|\le\tilde{\sigma} \Rightarrow f \text{ is starlike}\}, \tag{4.5.1}$$

and similarly to define it as the *logarithmic radius of the class* S^*. Besides, we find that we can not solve this problem by the differential equation method, since we do not have the analogue of Lemma 2.5.3 for the logarithmic derivative. But the Clunie-Jack principle works again, and it enables us to find an lower bound of $\sigma(f\in N,\ n\ge 3;\ starlike)$ in (4.5.1).

<u>Theorem</u> <u>4.5.2</u> *Let f be as defined in* (4.1.1) *with* $n \ge 2$. *If*

$$|T(f,z)| \le n-1 \quad \forall\ z\in\Delta,$$

then f is starlike.

Remark When $n = 2$ the theorem becomes trivial, for $|T(f,z)| < 1$ implies $\Re\left(1+z\frac{f''}{f'}\right)>0$ and so f must be starlike.

Proof Let f satisfy the hypotheses. Define $w(z) = b_m z^m + \cdots$ to be analytic in Δ, such that

$$\frac{f(z)}{zf'(z)} = \frac{1+w(z)}{1-w(z)}. \tag{4.5.2}$$

We require to show $\Re\left(z\frac{f'}{f}\right) > 0$, and this is true if and only if $\Re\left(\frac{f}{zf'}\right) > 0$. Hence it is sufficient to prove $|w| < 1$ for $|z| < 1$ as before. Suppose this is not true. Then there exists a z_0 such that $|w| < 1$ when $|z| \le |z_0| = r_0$ and $|w(z_0)| = 1$.

Multiply z on both sides and then differentiate (4.5.2), we obtain :

$$1 - \frac{f''}{f'}\frac{f}{f'} = \frac{1+w}{1-w} + z\frac{2w'}{(1-w)^2}.$$

So

$$\frac{f''}{f'} = \frac{f'}{f}\left(1 - \frac{1+w}{1-w} - \frac{2zw'}{(1-w)^2}\right)$$

$$= \frac{1}{z}\left(\frac{1-w}{1+w}\right)\left(1 - \frac{1+w}{1-w} - \frac{2zw'}{(1-w)^2}\right)$$

$$= \frac{1}{z}\left(\frac{1-w}{1+w}\right)\left(\frac{(1-w)^2-(1-w)(1+w)-2zw'}{(1-w)^2}\right)$$

$$= \frac{1}{z}\frac{2\,(w^2-w-zw')}{1-w^2}$$

At z_0 ,where w attains its maximum, we can apply the Clunie-Jack principle that $z_0 w'(z_0)$ $= tw(z_0)$ and $t \geq n - 1$. This follows by comparing the coefficients on both sides of (4.5.2). Hence

$$\left| \frac{f''(z_0)}{f'(z_0)} \right| = \left| \frac{2}{z_0} \frac{w^2(z_0) - w(z_0) - t\ w(z_0)}{1 - w(z_0)^2} \right|$$

$$> \frac{2}{2} \mid t + 1 - w(z_0) \mid \geq t + 1 - 1 \geq n - 1.$$

This contradiction completes the proof. □

We shall consider functions defined in $|\zeta| > 1$, and apply the Clunie-Jack principle to solve an analogue of the above theorem. However the result is not so satisfactory in this case.

Let $$g(\zeta) = \zeta + \frac{b_n}{\zeta^n} + \frac{b_{n+1}}{\zeta^{n+1}} + \cdots \qquad n \geq 0,\ |\zeta| > 1. \tag{4.5.3}$$

The function is starlike if and only if $\Re\left(\zeta \frac{g'(\zeta)}{g(\zeta)}\right) > 0$, $\zeta \in \Delta^*$; see for example Pommerenke [1] p.47). We have the following result:

<u>Theorem</u> <u>4.5.3</u> *Let g be as defined in* (4.5.3) *with $n \geq 4$. If*

$$\left| \zeta \frac{g''(\zeta)}{g'(\zeta)} \right| < n - 3 \quad \forall\ z \in \Delta,$$

then g is starlike.

Proof The proof will be sketchy as it is similar to that of above theorem. Let g satisfy the hypotheses. Then

$$g'(\zeta) = 1 - \frac{nb_n}{\zeta^{n+1}} - \frac{(n+1)b_{n+1}}{\zeta^{n+2}} - \cdots \qquad n \geq 0,\ |\zeta| > 1. \tag{4.5.3}$$

Hence $$\zeta \frac{g'\zeta)}{g(\zeta)} = \frac{1 - \frac{nb_n}{\zeta^{n+1}} - \frac{(n+1)b_{n+1}}{\zeta^{n+2}} - \cdots}{1 + \frac{b_n}{\zeta^{n-1}} + \frac{b_{n+1}}{\zeta^{n-2}} + \cdots} = 1 - \frac{b_n}{\zeta^{n-1}} - \frac{b_{n+1}}{\zeta^n} - \cdots$$

$$\equiv 1 - b_n z^{n-1} - b_{n+1} z^n - \cdots$$

where we have made the substitution $\zeta = \frac{1}{z}$. Now consider

$$\frac{g(\zeta)}{\zeta g'(\zeta)} = \frac{1 + w(z)}{1 - w(z)} = \frac{1 + w(1/\zeta)}{1 - w(1/\zeta)}, \tag{4.5.4}$$

where $w(z) = q_m z^m + q_{m+1} z^{m+1} + \cdots, \quad m \geq 0$. By comparing the coefficients on both sides we deduce $m = n - 1$ again.

We use the same argument as before and by subordination it is sufficient to prove that $|w(z)| < 1$ for $\forall\ z \in \Delta$. Suppose this is not true then there exists a $\zeta_0 \in \Delta^*$ so that at the corresponding $1/\zeta_0 = z_0 = r_0 e^{i\theta_0} \in \Delta$. We have $|w(z)| < 1$ for $|z| < r_0$.

We now differentiate (4.5.4) with respect to ζ to obtain:

$$\frac{g''(\zeta)}{g'(\zeta)} = \frac{1}{\zeta}\left(\frac{1 - w(z)}{1 + w(z)}\right)\left(1 - \frac{1 + w(z)}{1 - w(z)} - \frac{2zw'(z)}{(1 - w(z))^2}\right).$$

Hence at z_0

$$\begin{aligned} \left|\zeta_0 \frac{g''(\zeta_0)}{g'(\zeta_0)}\right| &= \left|2\,\frac{w^2(z_0) - w(z_0) + t\,w(z_0)}{1 - w(z_0)^2}\right| \\ &\geq 2\frac{|t - 1 + w(z_0)|}{2} \\ &\geq \frac{2(t - 2)}{2} = t - 2 \geq n - 3. \end{aligned}$$

□

§ *4.6 On a subclass of Strongly Gamma Starlike Functions*

Let $f(z) = z + \sum_2^\infty a_n z^n$ and $f(z)f'(z) \neq 0$ be an analytic function in Δ. P.T. Mocanu [1 1969] defined a subclass of S such that the real part of the arithmetic mean of the quantities $z\frac{f'}{f}$ and $1 + z\frac{f''}{f'}$ is positive Δ:

$$\Re\left[(1 - \alpha)\, z\frac{f'}{f} + \alpha\left(1 + z\frac{f''}{f'}\right)\right] > 0 \quad \forall\ z \in \Delta, \tag{4.6.1}$$

where α is any fixed real number. Functions satisfying this condition are said to belong to the class of *alpha-starlike function* $\mathcal{M}_\alpha$, and they have been shown to be starlike for all α in Miller, Mocanu and Reade [1]. Also when $\alpha = 0$, $\mathcal{M}_0 = S^*$ and $\mathcal{M}_1 = \mathfrak{K}$. In 1979 Sakaguchi and Fukui [1] proved that if f satisfies (4.6.1) then $f(z)f'(z) \neq 0$ in Δ; hence this part of the hypotheses can be dropped.

Before we proceed further, it is necessary to recall some elementary facts; we define the *principal branch* of the argument of $z = re^{i\theta}$ be $-\pi < \theta \leq \pi$, and we denote it by $\theta = Arg\ z$. The *principal branch* of the *logarithm* is defined as $Log\ z = log\ r + i Arg\ z$ whereas the ordinary logarithm is denoted by $log\ z = log\ r + i arg\ z$. If no confusion arise we shall also sometime write $arg\ z = Arg\ z$ as in the rest of this thesis. We also define the complex exponential to be $z^\lambda = \lambda e^{Log\ z}$ where λ is a complex number. We have the following facts: $Arg\ z^\lambda = \lambda\ Arg\ z$ if λ is real where $0 < \lambda \leq 1$ and $Arg\ z_1 z_2 = Arg\ z_1 + Arg\ z_2$ if and only if $-\pi < Arg\ z_1 + Arg\ z_2 \leq \pi$.

Lewandowski, Miller and Złotkiewicz [1] defined another subclass of S such that the geometric mean of the quantities $z\frac{f'}{f}$ and $1+z\frac{f''}{f'}$ is positive i.e. f, f' and $1+z\frac{f''}{f'} \neq 0$ in $0<|z|<1$. Suppose γ is real and

$$\Re\left[\left(z\frac{f'}{f}\right)^{1-\gamma}\left(1 + z\frac{f''}{f'}\right)^{\gamma}\right] > 0 \quad \forall\ z \in \Delta.$$

Where the above quantities are raised to certain powers, the branch considered is meant to be the principal branch. Such functions are called *Gamma-Starlike functions* $\mathfrak{L}_\gamma$ and they too have been proved to be starlike for all real γ in Lewandowski, Miller and Złotkiewicz [1]. Clearly $\mathfrak{L}_0 \equiv S^*$ and $\mathfrak{L}_1 \equiv \mathfrak{K}$. Also in Lewandowski, Miller and Złotkiewicz [1], the following subclass of $\mathfrak{L}_\gamma$ was suggested:

Definition 4.6.1 *Let* $f(z) = z + \sum_{2}^{\infty} a_n z^n$ *be an analytic function in* Δ *and* f, f', *and* $1+z\frac{f''}{f'} \neq 0$ *in* $0<|z|<1$. *Suppose* γ , α *are real constants such that* $0 \leq \gamma, \alpha \leq 1$ *and that*

$$\left|(1-\gamma)\ Arg\left(z\frac{f'}{f}\right) + \gamma\ Arg\left(1+z\frac{f''}{f'}\right)\right| \leq \frac{\alpha\pi}{2} \qquad \forall\ z \in \Delta.$$

Then we say that f is a strongly gamma-starlike functions of order α, and we denoted the class of such functions by $\mathfrak{L}^*_\gamma(\alpha)$.

Note that $\mathfrak{L}^*_\gamma(\alpha) \subseteq \mathfrak{L}_\gamma$, and so strongly gamma-starlike function must be starlike. We shall show that for a subclass of $\mathfrak{L}^*_\gamma(\alpha)$, f is not only starlike but strongly-starlike of order β where β depends on γ. Strongly-starlike functions were defined in Chapter 1.

Let $$\mathfrak{G}^*_\gamma(\beta) = \left\{f \mid f \in \mathfrak{L}^*_\gamma(\alpha)\ where\ \alpha = \beta(1+\gamma)-\gamma\ ,\ 1 \geq \beta > \frac{\gamma}{1+\gamma}\right\},$$

i.e. if $f \in \mathfrak{G}^*_\gamma(\beta)$ then f satisfies

$$\left|(1-\gamma)\ Arg\left(z\frac{f'}{f}\right) + \gamma\ Arg\left(1+z\frac{f''}{f'}\right)\right| \leq (\beta(1+\gamma)-\gamma)\ \frac{\pi}{2} \qquad \forall\ z \in \Delta. \tag{4.6.2}$$

where $1 \geq \beta > \gamma/(1+\gamma)$.

We shall now state the theorem.

Theorem 4.6.1 $\mathcal{G}_\gamma^*(\beta) \subseteq S^*(\beta)$.

Proof The proof is similar to that of Lewandowski, Miller and Złotkiewicz [1], or Miller [1] and the theorems above. We let $f \in \mathcal{G}_\gamma^*(\beta)$ and

$$z\frac{f'(z)}{f(z)} = \Big(\frac{1+w(z)}{1-w(z)}\Big)^\beta = P(z) \quad 0 < \beta \le 1 \;\; \forall\; z \in \Delta. \tag{4.6.3}$$

Then $w(0)=0$ and $w \neq \pm 1$ is analytic in Δ as before. If $|w(z)|<1 \;\forall\; z\in\Delta$ then the theorem follows from subordination. Suppose not, then there exists a $z_0 = r_0 e^{i\theta_0} \in \Delta$ such that $|w(z)| < 1$ for $|z| < r_0$. Suppose also that w attains its maximum at z_0. Then by the Clunie-Jack principle we have at z_0,

$$z_0\frac{w'(z_0)}{w(z_0)} = T \ge 1 \quad \textit{and} \quad \frac{1+w(z_0)}{1-w(z_0)} = i\,\frac{\sin\theta_0}{1-\cos\theta_0} = iS\,, \tag{4.6.4}$$

where S is clearly a non-zero real number.

Note that we can write the left hand side of (4.6.2) in the equivalent form:

$$J\{\gamma, f(z)\} := \Big(z\frac{f'(z)}{f(z)}\Big)^{1-\gamma}\Big(1+z\frac{f''(z)}{f'(z)}\Big)^\gamma \quad 0 \le \gamma \le 1. \tag{4.6.5}$$

Differentiate (4.6.3) and substitute to obtain

$$\begin{aligned} J\{\gamma, f(z)\} &= P(z)^{1-\gamma}\Big(P(z) + z\frac{P'(z)}{P(z)}\Big)^\gamma \\ &= \Big(\frac{1+w(z)}{1-w(z)}\Big)^{(1-\gamma)\beta}\bigg(\Big(\frac{1+w(z)}{1-w(z)}\Big)^\beta + \beta z\Big(\frac{w'(z)}{1+w(z)} + \frac{w'(z)}{1-w(z)}\Big)\bigg)^\gamma. \end{aligned}$$

Applying (4.6.4) at z_0, we obtain

$$\begin{aligned} J\{\gamma, f(z_0)\} &= \Big(\frac{1+w(z_0)}{1-w(z_0)}\Big)^{(1-\gamma)\beta}\bigg(\Big(\frac{1+w(z_0)}{1-w(z_0)}\Big)^\beta + \beta z_0\Big(\frac{w'(z_0)}{1+w(z_0)} + \frac{w'(z_0)}{1-w(z_0)}\Big)\bigg)^\gamma \\ &= (iS)^{(1-\gamma)\beta}\bigg((iS)^\beta + T\beta\Big(\frac{w(z_0)}{1+w(z_0)} + \frac{w(z_0)}{1-w(z_0)}\Big)\bigg)^\gamma \\ &= (iS)^{(1-\gamma)\beta}\bigg((iS)^\beta + \frac{T\beta}{2}\Big(1-\frac{1}{iS} + iS-1\Big)\bigg)^\gamma \\ &= (iS)^{(1-\gamma)\beta}\bigg((iS)^\beta + i\,\frac{T\beta}{2}\Big(S + \frac{1}{S}\Big)\bigg)^\gamma. \end{aligned}$$

That is $\quad \Big(z_0\frac{f'(z_0)}{f(z_0)}\Big)^{1-\gamma} = (iS)^{(1-\gamma)\beta}, \; \Big(1+z_0\frac{f''(z_0)}{f'(z_0)}\Big)^\gamma = \bigg((iS)^\beta + i\,\frac{T\beta}{2}\Big(S + \frac{1}{S}\Big)\bigg)^\gamma.$

Since S could be either positive or negative, it is necessary to consider them in separate cases. We first consider S to be positive. Since both α and β are positive and less than or equal to 1, we clearly have $Arg\ (z_0f'(z_0)/f(z_0))^{1-\gamma}$ and $Arg\ (1+z_0f''(z_0)/f'(z_0))^{\gamma}$ less than $\pi/2$. Thus, taking the argument of (4.6.5),

$$Arg\ J\{\gamma, f(z_0)\} = (1-\gamma)\beta\ (Arg\ i + Arg\ S) + \gamma\ Arg\ i^{\beta}\left(S^{\beta} + i^{1-\beta}\frac{T\beta}{2}\left(S+\frac{1}{S}\right)\right),$$
$$=(1-\gamma)\beta\ Arg\ i + \gamma\ Arg\ i^{\beta}\left(S^{\beta} + i^{1-\beta}\frac{T\beta}{2}\left(S+\frac{1}{S}\right)\right).$$

Now $Arg\ i^{\beta} = \beta\frac{\pi}{2}$ and

$$Arg\left\{S^{\beta} + i^{1-\beta}\frac{T\beta}{2}\left(S+\frac{1}{S}\right)\right\} = tan^{-1}\left(\frac{T\beta/2\left(S+1/S\right)sin(1-\beta)\pi/2}{S^{\beta}+T\beta/2\left(S+1/S\right)cos(1-\beta)\pi/2}\right)$$

$$\leq tan^{-1}\left(\frac{T\beta/2\left(S+1/S\right)sin(1-\beta)\pi/2}{T\beta/2\left(S+1/S\right)cos(1-\beta)\pi/2}\right) = tan^{-1}\left(tan(1-\beta)\frac{\pi}{2}\right) = (1-\beta)\frac{\pi}{2}.$$

Hence the sum of the arguments of i^{β} and $\left(S^{\beta} + i^{1-\beta}\frac{T\beta}{2}\left(S+\frac{1}{S}\right)\right)$ is less then or equal to $\pi/2$ and each argument is positive. Thus we have

$$|Arg\ J\{\gamma, f(z_0)\}| = \left|(1-\gamma)\beta\frac{\pi}{2} + \gamma\beta\frac{\pi}{2} + \gamma Arg\left(S^{\beta} + i^{1-\beta}\frac{T\beta}{2}\left(S+\frac{1}{S}\right)\right)\right|$$
$$= \left|\beta\frac{\pi}{2} + \gamma\ Arg\left(S^{\beta} + \frac{T\beta}{2}\left(S+\frac{1}{S}\right)cos\left((1-\beta)\frac{\pi}{2}\right) + i\frac{T\beta}{2}\left(S+\frac{1}{S}\right)sin\left((1-\beta)\frac{\pi}{2}\right)\right)\right|$$
$$= \left|\beta\frac{\pi}{2} + \gamma\ tan^{-1}\left[\frac{\frac{T\beta}{2}\left(S+\frac{1}{S}\right)\ sin\left((1-\beta)\frac{\pi}{2}\right)}{S^{\beta} + \frac{T\beta}{2}\left(S+\frac{1}{S}\right)\ cos\left((1-\beta)\frac{\pi}{2}\right)}\right]\right|$$
$$\geq \beta\frac{\pi}{2} - \gamma\left|tan^{-1}\left[\frac{\frac{T\beta}{2}\left(S+\frac{1}{S}\right)\ sin\left((1-\beta)\frac{\pi}{2}\right)}{S^{\beta} + \frac{T\beta}{2}\left(S+\frac{1}{S}\right)\ cos\left((1-\beta)\frac{\pi}{2}\right)}\right]\right|$$
$$> \beta\frac{\pi}{2} - \gamma\left|tan^{-1}\left[\frac{\frac{T\beta}{2}\left(S+\frac{1}{S}\right)\ sin\left((1-\beta)\frac{\pi}{2}\right)}{\frac{T\beta}{2}\left(S+\frac{1}{S}\right)\ cos\left((1-\beta)\frac{\pi}{2}\right)}\right]\right|$$
$$= \beta\frac{\pi}{2} - \gamma\ tan^{-1}\left(tan\ \frac{\pi}{2}(1-\beta)\right)$$
$$= \beta\frac{\pi}{2} - \gamma\ (1-\beta)\frac{\pi}{2} = \frac{\pi}{2}\ \left(\beta(1+\gamma)-\gamma\right).$$

The above inequalities follow since tan^{-1} is an increasing function and S is a positive number. We must now consider S to be negative. Note that we can write $S = -|S| = e^{i\pi}|S|$ and hence

$iS = e^{-i\pi/2}|S|$. Similarly we have

$$|Arg\ J(\gamma, f(z_0))| = \left| Arg\left((e^{-i\pi/2}|S|)^{(1-\gamma)\beta}\left((e^{-i\pi/2}|S|)^{\beta} + e^{-i\pi/2}\frac{T\beta}{2}\left(|S| + \frac{1}{|S|}\right)\right)^{\gamma}\right)\right|$$

$$= \left| -\tfrac{\pi}{2}(1-\gamma)\beta - \tfrac{\pi}{2}\gamma\beta + \gamma\ Arg\left(|S|^{\beta} + \frac{T\beta}{2}\left(|S| + \frac{1}{|S|}\right) e^{i(-\frac{\pi}{2}+\beta\frac{\pi}{2})}\right)\right|$$

$$= \left| -\tfrac{\pi}{2}\beta + \gamma Arg\left[|S|^{\beta} + \frac{T\beta}{2}\left(|S| + \frac{1}{|S|}\right)\left(cos\ \tfrac{\pi}{2}(\beta-1) + i\ sin\ \tfrac{\pi}{2}(\beta-1)\right)\right]\right|$$

$$\geq \left| -\tfrac{\pi}{2}\beta \right| - \gamma \left| tan^{-1}\left[\frac{\frac{T\beta}{2}\left(|S| + \frac{1}{|S|}\right) sin\ \frac{\pi}{2}(\beta-1)}{|S|^{\beta} + \frac{T\beta}{2}\left(|S| + \frac{1}{|S|}\right) cos\frac{\pi}{2}(\beta-1)} \right]\right|$$

$$> \tfrac{\pi}{2}\beta - \gamma \left| tan^{-1}\left[\frac{\frac{T\beta}{2}\left(|S| + \frac{1}{|S|}\right) sin\ \frac{\pi}{2}(1-\beta)}{\frac{T\beta}{2}\left(|S| + \frac{1}{|S|}\right) cos\ \frac{\pi}{2}(1-\beta)} \right]\right|$$

$$= \tfrac{\pi}{2}\beta - \gamma tan^{-1}\left(tan\tfrac{\pi}{2}(1-\beta)\right)$$

$$= \tfrac{\pi}{2}\beta - \gamma(1-\beta)\tfrac{\pi}{2} = \tfrac{\pi}{2}\left(\beta(1+\gamma) - \gamma\right).$$

Hence, in both cases the above argument leads to contradictions at the same time. This completes the proof of the theorem. □

§ 4.7 *A Criterion for Quasiconformal Extension*

Since the class $S^*(\beta)$ admits a K-quasiconformal extension to $\bar{\mathbb{C}}$, with $k \leq sin(\beta\frac{\pi}{2})$, an immediate deduction from the theorem gives:

<u>Corollary 4.7.1</u> *The class $\mathfrak{G}^*_{\gamma}(\beta)$ admits a K-quasiconformal extension to $\bar{\mathbb{C}}$ with $k \leq sin(\beta\frac{\pi}{2})$.*

Note that $\mathfrak{G}^*_0(\beta) = S^*(\beta)$ and $\mathfrak{G}^*_1(\beta)$ is the class of f satisfying

$$\left| arg\left(1 + z\frac{f''(z)}{f'(z)}\right)\right| \leq (2\beta-1)\tfrac{\pi}{2}\ ,\ \tfrac{1}{2} < \beta \leq 1,$$

which is a subclass of *strongly convex function of order* $2\beta-1$. This condition implies that f

satisfies

$$\left| arg\left(z\frac{f'(z)}{f(z)}\right) \right| \leq \beta\frac{\pi}{2}.$$

As we can see the above implication only valid if $\frac{1}{2} < \beta \leq 1$. This leaves out the β in the range of $0 \leq \beta \leq \frac{1}{2}$. Hence it seems that the Theorem 4.7.1 is not best possible in the sense that a better estimate of β in the definition of $\mathfrak{G}^*_\gamma(\beta)$ could be obtained so that the theorem can include the missing range of β when $\gamma = 1$.

§ *4.8 A more General Inclusion*

The Theorem 4.7.1 showed that when $f \in \mathfrak{G}^*_\gamma(\beta) \Rightarrow f \in \mathfrak{G}^*_0(\beta) \equiv S^*(\beta)$. We now show that this is in fact a special case of the the following general inclusion.

<u>Theorem</u> <u>4.8.1</u> *If* $0 \leq \eta \leq \gamma$ *then* $\mathfrak{G}^*_\gamma(\beta) \subseteq \mathfrak{G}^*_\eta(\beta)$.

Proof The case when $\eta = 0$ has been proved in Theorem 4.7.1, so we only need to consider the case $0 < \eta \leq \gamma \leq 1$. By using the subordination principle, we find that we do not need to use the argument in Theorem 4.7.1 again.

Let $f \in \mathfrak{G}^*_\gamma(\beta)$, we shall prove that f satisfies

$$\left| (1-\eta)\ Arg\left(z\frac{f'}{f}\right) + \eta\ Arg\left(1+z\frac{f''}{f'}\right) \right| \leq \Big(\beta(1+\eta)-\eta\Big)\frac{\pi}{2} \qquad \forall\ z \in \Delta.$$

Since $f \in \mathfrak{G}^*_\gamma(\beta)$ if and only if there exists a function $P_1(z)$ such that

$$P_1(z) \in \mathfrak{P} \equiv \{\ P(z)\ |\ P(0)=1,\ P \textit{ is analytic in } \Delta \textit{ and } \Re(P(z))>0 \textit{ in } \Delta\}$$

the relation

$$\left(z\frac{f'(z)}{f(z)}\right)^{1-\gamma}\left(1+z\frac{f''(z)}{f'(z)}\right)^{\gamma} \equiv P_1(z)^{\beta(1+\gamma)-\gamma} \tag{4.8.1}$$

is satisfied. By Theorem 4.7.1, f also belongs to $S^*(\beta)$. Hence there also exists another $P_2(z)\in\mathfrak{P}$ so that

$$z\frac{f'(z)}{f(z)} \equiv P_2(z)^{\beta}. \tag{4.8.2}$$

Raise both sides of (4.8.1) to the power $\eta/\gamma \leq 1$ ($\eta \neq 0$) to obtain

$$\left(z\frac{f'(z)}{f(z)}\right)^{\frac{\eta}{\gamma}-\eta}\left(1+z\frac{f''(z)}{f'(z)}\right)^{\eta} \equiv P_1(z)^{\beta(\frac{\eta}{\gamma}+\eta)-\eta}. \tag{4.8.3}$$

Raise both sides of (4.8.2) to the power of $(1-\eta/\gamma) \leq 1$. We obtain

$$\left(z\frac{f'(z)}{f(z)}\right)^{1-\frac{\eta}{\gamma}} \equiv P_2(z)^{\beta(1-\frac{\eta}{\gamma})}. \tag{4.8.4}$$

We now multiply identities (4.8.3) to (4.8.4) and obtain

$$\left(z\frac{f'(z)}{f(z)}\right)^{1-\eta}\left(1+z\frac{f''(z)}{f'(z)}\right)^{\eta} \equiv P_1(z)^{\beta(\frac{\eta}{\gamma}+\eta)-\eta} P_2(z)^{\beta(1-\frac{\eta}{\gamma})}.$$

Let $P_3(z) \equiv P_1(z)^{\beta(\frac{\eta}{\gamma}+\eta)-\eta} P_2(z)^{\beta(1-\frac{\eta}{\gamma})}$, so $P_3(0)=1$. Note that both the powers are less than one. Now

$$\begin{aligned}
|Arg\ P_3(z)| &= \left| Arg\left(P_1(z)^{\beta(\frac{\eta}{\gamma}+\eta)-\eta} P_2(z)^{\beta(1-\frac{\eta}{\gamma})}\right)\right| \\
&\leq \left\{\beta\left(\frac{\eta}{\gamma}+\eta\right)-\eta\right\}|Arg\ P_1(z)| + \left(1-\frac{\eta}{\gamma}\right)\beta\ |Arg\ P_2(z)| \\
&< \left\{\beta\left(\frac{\eta}{\gamma}+\eta\right)-\eta\right\}\frac{\pi}{2} + \left(1-\frac{\eta}{\gamma}\right)\beta\ \frac{\pi}{2} \\
&= \frac{\pi}{2}\left(\beta(\eta+1)-\eta\right).
\end{aligned}$$

Since $f \in \mathcal{G}_\gamma^*(\beta)$, we have $1 \geq \beta \geq \frac{\gamma}{1+\gamma} \geq \frac{\eta}{1+\eta}$ as $\gamma \geq \eta$. This is because $q(x) = \frac{x}{1+x}$ is increasing for all positive x. Thus P_3 has positive real part and

$$|\ Arg\ P_3(z)\ | < \frac{\pi}{2}\left(\beta(\eta+1)-\eta\right) \leq \frac{\pi}{2}.$$

So $P_3(z) \in \mathcal{P}$ and $f \in \mathcal{G}_\eta^*(\beta)$. This completes the proof of the theorem. □

The whole class of $\mathcal{G}_\gamma^*(\beta)$ $0 \leq \gamma \leq 1$ is of special interest, since Corollary 4.7.1 shows that all functions in the class have a $K(\gamma)$-quasiconformal extension to $\bar{\mathbf{C}}$.

Note added to this chapter

After this thesis was written, I realized that Theorem 4.5.2 has been obtained independently by Miller and Mocanu in [2] without the assumption that $f''(0)=0$. Their method is more advanced and requires the use of Löwner chains. We note that our method allows us to consider $n \geq 3$.

Chapter Five

Second Order Linear Differential Equations with Transcendental Entire Coefficients

§ *5.1 Definitions and the Nevanlinna Theory*

We start by introducing some well known definitions and facts about the growth and value distribution theory of a meromorphic functions (or entire functions) in the complex plane **C**. We base our account of the theory on the standard text of meromorphic functions written by W.K. Hayman [2].

We consider a function f meromorphic and not identically equal to zero in the complex plane $\bar{\mathbf{C}}$. Let $n(t,\infty)$ be the number of poles of f in the disc $|z|\leq t$, counted with multiplicity. The *counting function for f* is defined as

$$N(r,f)=\int_0^r \frac{n(t,\infty)-n(0,\infty)}{t}\,dt + n(0,\infty)\log r.$$

Set $\log^+ u=\max\{0, \log u\}$, for $u>0$ and

$$m(r,f)=\frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{i\theta})|\,d\theta$$

which is called the *Nevanlinna proximity function.* Then it follows from the *Poisson-Jensen formula* , see Hayman [2] p.1 , that we have the relation

$$m(r,f)+N(r,f)=m(r,1/f)+N(r,1/f)+\log|c_\lambda|. \tag{5.1.1}$$

Here c_λ is the first non-zero coefficient in the *Laurent expansion* of $f(z)$.

Let us set $T(r,f)=m(r,f)+N(r,f)$. The function $T(r,f)$ is called the *Nevanlinna* *characteristic function of f*. So we can rewrite (5.1.1) as $T(r,f)=T(r,1/f)+O(1)$. We also defined $N(r,a)=N(r,1/(f-a))$ and $m(r,a)=m(r,1/(f-a))$ and $T(r,a)=T(r,1/(f-a))$ for any finite a. The first fundamental theorem states that (see Hayman [2] p.5) for any function meromorphic in the $|z|\leq R\leq\infty$, and for a finite or infinite, we have the relation

$$T(r,a)=T(r,f)+\epsilon(r,a) \qquad r<R$$

where
$$|\epsilon(r,a)|\leq log^{+}|a|+|log|c||+log\ 2,$$

and c is the first non-vanishing coefficient in the Laurant expression of $f-a$.

We have some basic properties of the characteristic functions:

$$T(r,f+g)\leq T(r,f)+T(r,g)+log\ 2,$$

$$T(r,fg)\leq T(r,f)+T(r,g),$$

and
$$T\left(r,\ \frac{af+b}{cf+d}\right)=T(r,f)+O(1) \quad ad-bc\neq 0.$$

Also $T(r,f)$ is continuous in r and increasing convex function of $log\ r$. The importance of the characteristic function is that it can determine the growth of function. For example, let f be any non-constant meromorphic function and satisfying $\varliminf_{r\to\infty}\frac{T(r,f)}{log\ r}\leq c$ for some constant c. Then f must be rational.

Let $s(r)$ be a non-negative increasing function. The *order* and *lower order* of $s(r)$ are defined as

$$\rho=\varlimsup_{r\to\infty}\frac{log\ s(r)}{log\ r} \quad \text{and} \quad \eta=\varliminf_{r\to\infty}\frac{log\ s(r)}{log\ r}$$

respectively. Then the *order* and *lower order of a meromorphic function f* are defined as

$$\rho(f)=\varlimsup_{r\to\infty}\frac{log\ T(r,f)}{log\ r} \quad \text{and} \quad \eta(f)=\varliminf_{r\to\infty}\frac{log\ T(r,f)}{log\ r}$$

respectively. Let $M(r,f)$ be the maximum modulus of f as defined in Chapter 2. We have the following relation between $T(r,f)$ and $M(r,f)$ for regular functions. If f is reguler for $|z|\leq R$, then $T(r,f)\leq log^{+}M(r,f)\leq\frac{R+r}{R-r}T(R,f)$ $(0\leq r<R)$; see also Hayman [2] p.18. This shows that $T(r,f)$ and $log^{+}M(r,f)$ have the same order.

We define the *Weierstrass primary factor* to be $E(z,q)=(1-z)e^{z+\frac{1}{2}z^{2}+\cdots+\frac{1}{q}z^{q}}$ with $E(z,0)=1-z$. Then we have the following important factorization theorem: *let f be meromorphic in* C *and* $\{a_v\}$, $\{b_\mu\}$ *are the zeros and the poles of f respectively such that* $\lim_{r\to\infty}\frac{T(r,f)}{r^{q+1}}=0$ *where q is a positive integer. Then*

$$f(z)=z^{k}e^{p(z)}\lim_{r\to\infty}\frac{\prod_{|a_v|\leq r}E\left(\frac{z}{a_v},q\right)}{\prod_{|b_v|\leq r}E\left(\frac{z}{b_\mu},q\right)},$$

where k is an integer and P(z) is a polynomial of degree at most q (see also Hayman p.20). The converse is also true. Given an increasing sequence of complex numbers $\{a_n\}$ whose moduli tends to infinity such that $\sum_1^\infty \frac{1}{|a_n|^q} = \infty$ and $\sum_1^\infty \frac{1}{|a_n|^{q+1}} < \infty$, where q is a positive integer, then the *canonical product* $\prod_1^\infty E\left(\frac{z}{a_n}, q\right)$ converges uniformly in any bounded region of $\mathbf{C}$. Let $\{a_n\}$ again be the sequence of zeros of f ordered with non-decreasing moduli and $|a_n| \to \infty$ as $n \to \infty$, we defined the *order of* $\{a_n\}$ to be the order of $n(r,0) = n(r,\frac{1}{f})$. We also define the *exponent of convergence of the zeros of f* to be $\lambda(f) = inf\left\{q : \sum_1^\infty \frac{1}{|a_n|^q} < \infty \text{ and } q \in \mathbf{R}^+\right\}$. The *genus* of the Weierstrass product is defined to be the integer q. Hence we have the relation $q \leq p \leq q+1$.

We now come to one of the most important theorems in the value distribution theory.

Theorem 5.1.1 (*Nevanlinna's Second fundamental theorem*)

Suppose that f is a non-constant meromorphic function in $|z| \leq r$. *Let* $a_1, a_2 \cdots, a_q$ *be distinct finite complex numbers,* $\delta > 0$ *and suppose that* $|a_\mu - a_\zeta| \geq \delta$ *for* $1 \leq \mu \leq \zeta \leq q$. *Then*

$$m(r,\infty) + \sum_1^q m(r,a_\mu) \leq 2T(r,f) - N_1(r) + S(r),$$

where $N_1(r)$ *is positive and is equal to* $N(r,1/f') + 2N(r,f) - N(r,f')$ *and*

$$S(r) = m(r,f'/f) + m\left\{r, \sum_1^q \frac{f'}{f-a_\mu}\right\} - q\log^+\frac{3q}{\delta} + \log 2 + \log\frac{1}{|f'(0)|}$$

with some modifications if $f(0)=0$ *or* ∞ *or* $f'(0)=0$.

Let us write $$\delta(a,f) = \lim_{r\to\infty} \frac{m(r,a)}{T(r,f)} = 1 - \overline{\lim_{r\to\infty}} \frac{N(r,a)}{T(r,f)}.$$

We call it the *deficiency of the value a of f.* The above theorem can be rewritten into a more useful form (see also Hayman [2] p.43 for details).

Theorem 5.1.1' (*Nevanlinna's Second fundamental theorem*)

If f is meromorphic for $z \neq \infty$, *then the deficiency* $\delta(a,f)$ *defined above vanishes for at most a countable set of values of a. The sum of all deficiencies is at most equal to* 2:

$$\sum_{a \in \overline{\mathbf{C}}} \delta(a,f) \leq 2.$$

This readily proves *Picard's theorem: If f is a meromorphic transcendental function in* $\mathbf{C}$, *then f takes every value infinitely often, except with at most two exceptions.* The values that f takes only finitely often are called *Picard exceptional values.* Notice that $0 \leq \delta(a,f) \leq 1$; if $\delta(a,f)$ is near to 1 then f takes a less often, when $\delta(a,f)$ is near to 0 then f takes a very often. For details about the relation between the deficiency of f at a and the growth, the asymptotic values of f, see W.H.J. Fuchs [3] 1982.

§ 5.2 Value Distribution of the solutions of Second Order Linear Differential Equations

We consider the second order differential equation:

$$y'' + Ay = 0$$

where A is either meromorphic or entire and of finite order. Much has been done when A is meromorphic or periodic, see Bank and Laine [2], [3]. We shall first consider A to be a polynomial.

Theorem 5.2.1 (*Bank and Laine* [1] 1982)

Let A be a polynomial of degree $n \geq 1$ and let f be a solution of $y'' + Ay = 0$. Then:

(a) The order of growth of f, $\rho(f) = \frac{n+2}{2}$;

(b) If n is odd, $\lambda(f) = \frac{n+2}{2}$;

(c) If n is even, and if f_1 , f_2 are linearly independent solutions of the equation, then $max\{\lambda(f_1), \lambda(f_2)\} = \frac{n+2}{2}$. If f is a solution such that $\lambda(f) < \frac{n+2}{2}$, then f has only finitely many zeros;

(d) If n is even, there are examples where some solution has no zeros and other examples where each solution $f \not\equiv 0$ has $\lambda(f) = \frac{n+2}{2}$.

We first note that if $n=0$, then the equation can possess two linearly independent solutions each of which has no zeros. For example consider $y'' - y = 0$ which has linearly independent solutions e^z, e^{-z}, non of which has any zeros. Part (a) of the above theorem was due to H. Wittich [1] using Wiman-Valiron theory, another proof can also be found in Gundersen [1]. Part (b) is a simple consequence of the Hadamard factorization theorem.

We now look at the result when A is transcendental entire.

Theorem 5.2.2 (*Bank and Laine* [1] 1982)

Let A be a transcendental entire functions of order $\rho(A)$.

(a) Let $\rho(A) < \infty$ and $\rho(A)$ is not an integer. Let f_1, f_2 be two linearly independent solutions of $y'' + Ay = 0$:

(i) if $\rho(A) \geq \frac{1}{2}$, then $max\{\lambda(f_1), \lambda(f_2)\} \geq \rho(A) \geq \frac{1}{2}$,

(ii) if $\rho(A) < \frac{1}{2}$, then $max\{\lambda(f_1), \lambda(f_2)\} = \infty$.

(b) Any solution $f \not\equiv 0$ of the equation has $\rho(f) = \infty$.

(c) Suppose $\lambda(A) < \rho(A)$ and f is any solution of the equation, then $\lambda(f) \geq \rho(A)$.

(d) Suppose that $\rho(A)$ is arbitrary and let $\overline{\lambda}(f)$ denotes $\lambda(f)$ but counted with the distinct zeros of

f, and that $\bar{\lambda}(A) \le \rho(A)$. Then $max\{\lambda(f_1), \lambda(f_2)\} \ge \rho(A)$.

(e) *If ρ is a positive integer or equal to ∞, then there exists a transcendental entire function A with $\rho(A)=\rho$ such that the equation $y''+Ay=0$ possesses two linearly independent solutions each having no zeros.*

Part (b) of the theorem is a simple consequence of the *logarithmic derivative lemma* which states that for any meromorphic function of finite order we have $m(r, \frac{f'}{f}) = O(\log r)$ for all r except possibly for a set r of finite measure. Following Hayman [2 p.36] let us say, if an inequality holds except for a set r of finite linear measure, that the inequality holds *nearly everywhere* or *n.e.* for short. Let f be any solution of $y''+Ay=0$, and suppose on the contrary that f has finite order $\rho(f)<\infty$. Then

$$T(r,A) = m(r,A) = m\left(r, \frac{f''}{f}\right) \le m\left(r, \frac{f''}{f'}\right) + m\left(r, \frac{f'}{f}\right) = O(\log r) \qquad n.e.\ r \to \infty.$$

Hence $T(r,A) = O(\log r)$ and A has only finitely many zeros and thus reduces to a polynomial. This is a contradiction.

Moreover, for any non-zero number b, from $y''+Ay=0$ we have

$$f''+A(f-b) = -Ab$$

or

$$\frac{f''}{f-b} + A = \frac{-Ab}{f-b}$$

or

$$\frac{1}{f-b} = \frac{-1}{Ab}\left(\frac{f''}{f-b}+A\right) = \frac{-1}{Ab}\left(\frac{(f-b)''}{f-b}+A\right) = \frac{-1}{Ab}\frac{(f-b)''}{f-b} + \frac{-1}{b}.$$

Thus

$$m\left(r, \frac{1}{f-b}\right) \le m\left(r, \frac{-1}{Ab}\left(\frac{f''}{f-b}+A\right)\right) + O(1) \le m\left(r, \frac{1}{A}\right) + m\left(r, \frac{(f-b)''}{f-b}\right) + O(1).$$

Since

$$T(r,A) = o(T(r,f)) \text{ and } m\left(r, \frac{(f-b)''}{f-b}\right) = o(T(r,f)) \qquad n.e.\quad r\to\infty$$

and so

$$m\left(r, \frac{1}{f-b}\right) = o(T(r,f)) \qquad n.e. \quad r\to\infty.$$

Hence

$$N\left(r, \frac{1}{f-b}\right) \sim T(r,f). \qquad n.e. \quad r\to\infty.$$

Thus any solution of $y''+Ay=0$ takes each non-zero complex number b infinitely often such that $\delta(b,f) = 1 - \overline{\lim}_{r\to\infty} \frac{N(r,b)}{T(r,f)} = 1$. So $z=0$ is the only possible exceptional value. We shall be interested in the exponent of the convergence of the zeros of the solutions of $y''+Ay=0$. As shown by Theorem 5.2.2 (a) case (*ii*) that when A is an entire transcendental function of order less than $1/2$, then $max\{\lambda(f_1), \lambda(f_2)\} = \infty$. This has been extended to $\rho(A) = \frac{1}{2}$ by J. Rossi [1]. In fact he proved a more general result.

Theorem 5.2.3 (*J. Rossi* [1] 1984) *Let A be a transcendental entire function of order $\rho(A)<1$. If f_1 and f_2 are the two linearly independent entire solutions of $y''+Ay=0$, then the exponent of convergence of $E=f_1f_2$ is either infinite or satisfies*

$$\frac{1}{\rho(A)}+\frac{1}{\lambda(E)}\leq 2.$$

In particular if $\rho(A)\leq\frac{1}{2} \Rightarrow max\{\lambda(f_1),\lambda(f_2)\}=\infty$.

It is still an open question whether the strong conclusion that both $\lambda(f_1)$, $\lambda(f_2)$ $=\infty$ as soon as $\rho(A)<1$. In fact Bank and Laine [1] 1982 conjectured that $max\{\lambda(f_1),\lambda(f_2)\}=\infty$ holds if $\rho(A)$ is any positive number not equal to an integer. However by Theorem 5.2.2 (e) there exists an A, $\rho(A)\in\mathbb{N}$ such that each of the linearly independent solution has no zeros. There are also results about the locations of zeros of the solutions which can be found in Hellerstein and Rossi [1].

§ 5.3 The Solutions of the Differential Equations when the Coefficient A is Transcendental Entire with some Growth Conditions and the Main Result

Bank, Laine and Langley investigated some growth conditions on A to ensure that $max\{\lambda(f_1),\lambda(f_2)\}=\infty$. They obtained the following.

Theorem 5.3.1 (*Bank, Laine and Langley* [1] 1986) *Let A be a transcendental entire function of finite order $\rho(A)<\infty$ with the following properties: there exists a set $H\in\mathbb{R}$ of measure zero such that for each real number $\theta\in\mathbb{R}\backslash H$, either*

(*i*) $\dfrac{|A(re^{i\theta})|}{r^N} \to \infty$ *as* $r \to \infty$ *for each* $N>0$,

or (*ii*) $\int_0^\infty r|A(re^{i\theta})|\, dr <+\infty$,

or (*iii*) *there exists a positive number K and b, and a non-negative real number n (all possibly depending on θ), such that $\frac{n+2}{2}<\rho(A)$ and*

$$|A(re^{i\theta})|\leq Kr^n \quad \textit{for all } r\geq b.$$

Then if f_1 and f_2 are linearly independent solutions of $y''+Ay=0$, we have $max\{\lambda(f_1),\lambda(f_2)\}=\infty$.

Roughly speaking Theorem 5.3.1 shows that if A behaves like e^P where P is a polynomial of degree n say. In the case of the equation $y''+e^{P}y=0$, it has been proved that every solution f satistfies $\lambda(f)=\infty$, for details please see Bank and Laine [1] 1982 p.10. Of course, Theorem 5.3.1 allows a much larger class of functions which will grow as rapidly as e^{z} or tend to zero as e^{-z} or behave like a polynomial of small degree in different regions of the plane C. The conclusion that $max\{\lambda(f_1),\lambda(f_2)\}=\infty$ is weaker than $\lambda(f)=\infty$ for every solution. We have the following useful corollary.

<u>Corollary</u> <u>5.3.2</u> (*Bank, Laine and Langley* [1] 1986) *Let P_1, P_2, $\cdots$, P_n $n\geq 1$ be non-constant polynomials whose degrees are deg $P_i=d_i$, $i=1,\cdots,n$ and suppose that $deg(P_i-P_j)=max\{d_i,d_j\}$ $i\neq j$. Set*

$$A(z)=\sum_{1}^{n} B_j(z)exp(P_j(z))$$

where for each j, B_j is a non-constant entire function with $\rho(B_j)\leq d_j$. Then if f_1 and f_2 are linearly independent solutions of $y''+Ay=0$, we have $max\{\lambda(f_1),\lambda(f_2)\}=\infty$.

If $y''+(A+P)y=0$ where P is a polynomial of degree m and $\frac{m+2}{2}<\rho(A)=\max_{j}\{d_j\}$, then the same conclusion holds. This is a best possible result.

Both the proofs of Theorems 5.2.3, 5.3.1 and part of 5.2.2 were based on the product $E=f_1f_2$. It is therefore reasonable to ask, if A being a transcendental entire function which behaves or has growth conditions similar to those of e^{P}or to the hypotheses of Theorem 5.3.1, the same conclusion will hold. A result of A. Edrei and W.H.J. Fuchs suggests that for an entire function A, if A omits 0 *too often*. For example if $\delta(0,A)=1=1-\overline{\lim}_{r\to\infty}\frac{N(r,0)}{T(r,f)}$, *or* $N(r,0)=o(T(r,f))$, then A grows like e^{P} where P is a polynomial in certain annuli. In fact they found

$$T(r,A)\sim|c(r)|r^{P}/2$$

for some r sufficiently large, where $c(r)$ may diverge.

We state our main result.

<u>Theorem</u> <u>5.3.3</u> *Let A be a transcendental entire function of finite order $\rho(A)$ satisfying $\delta(0,A)=1$. If f_1 and f_2 are linearly independent solutions of the equation $y''+Ay=0$. We have $max\{\lambda(f_1),\lambda(f_2)\}=\infty$.*

If f has only finitely many zeros, we have the following general result.

<u>Theorem</u> <u>5.3.4</u> (*Bank and Langley* [1] 1987) *If A is a transcendental entire function of finite*

order having finitely many zeros, all non-trivial solutions of $y^{(k)}+Ay=0$, $k\geq 2$ *satisfy* $\lambda(y)=\infty$.

Note that this settles the problem the equation $y''+e^{P}y=0$ where P is a polynomial. Not only do we have $max\{\lambda(f_1),\lambda(f_2)\}=\infty$ as Theorem 5.3.1 allows. Theorem 5.3.4 implies $\lambda(f)=\infty$ for every non-trivial solution.

§ 5.4 Results required for the Proof

Since $\delta(0,A)=1$, it is well-known that the order of the function is an integer (see Hayman [2] p.105). Let us quote the following lemma.

<u>Lemma</u> <u>5.4.1</u> (*A. Edrei and W.H.J. Fuchs* 1959 [1]) *Let A be a meromorphic function of finite lower order and $\delta(0,A)=1$ so that the order $\rho(A)=p$ is an integer say. Then A has the following properties:*

(i) A can be factorized as

$$A(z)=z^{k}e^{P(z)}\lim_{r\to\infty}\frac{\prod_{|a_v|\leq r}E\left(\frac{z}{a_v},p\right)}{\prod_{|b_\mu|\leq r}E\left(\frac{z}{b_\mu},p\right)},$$

where k is an integer and $P(z)=\alpha_0z^{p}+\cdots+\alpha_p$ is a polynomial of degree p.

(ii) Let
$$c(r)=\alpha_0+\tfrac{1}{p}\left\{\sum_{|a_v|\leq r}\frac{1}{a_v{}^{p}}-\sum_{|b_\mu|\leq r}\frac{1}{b_v{}^{p}}\right\},\tag{5.4.1}$$

then for any $0<\epsilon\leq 1$ we have

$$T(r,A)=(1+\eta(\epsilon))\frac{|c(r)|r^{p}}{\pi}\quad for\ r>r_0\ ,\ |\eta|<\epsilon\tag{5.4.2}$$

(iii) Let $\alpha=e^{\frac{1}{p+1}}$ and $c_j=c(\alpha^{j})$ where j is an integer. Then given the same $\epsilon>0$ as defined in (ii), there exists a $j_0(\epsilon)$ such that for all $j\geq j_0$ we have

$$\left|\log|A(z)|-\Re(c_jz^{j})\right|<4\epsilon|c_j|r^{p},\quad z\in\{\Gamma_j-E_j\}\tag{5.4.3}$$

for all j sufficiently large. Here

$$\Gamma_j=\{z|\ \alpha^{j}\leq r\leq\alpha^{j+3/2}\}\tag{5.4.4}$$

and E_j is a collection of finite number of discs whose sum of radii are less than $4e\delta\alpha^{j+3/2}$, where $0<\delta<1/e$ and δ can be chosen arbitrary small.

We shall introduce the concept of linear and logarithmic measure. Let $I=[1,\infty]$ and $F\subseteq I$, then

$$m(F(r))=\int_{t\in F(r)}dt\qquad\text{and}\qquad lm(F(r))=\int_{t\in F(r)}\tfrac{1}{t}\,dt,$$

where $F(r)=F\cap[1, r]$, *are the linear* and *logarithmic measure* respectively. We also define

$$ULD(F)=\overline{\lim_{r\to\infty}}\frac{lm(F(r))}{\log r} \quad\text{and}\quad LLD(F)=\underline{\lim}_{r\to\infty}\frac{lm(F(r))}{\log r}$$

to be the *upper* and *lower logarithmic density.*

Note that $$ULD(I)=LLD(I)=1$$

and $$ULD(F)=1-LLD(I-F).$$

We cite the following two well-known lemmas.

Lemma 5.4.2 (*W.H.J. Fuchs* [1]) *Suppose $h(z)$ is meromorphic in* $\mathbf{C}$ *and of finite order ρ. Then given $\zeta>0$ and δ with $0<\delta<1/2$, there exists a constant $K(\rho,\zeta)$ and a set of positive real numbers G of lower logarithmic density at least $1-\zeta$ i.e. $LLD(G)\geq 1-\zeta$ such that if $0\leq\theta_2-\theta_1\leq\delta$ and $r\in G$, then*

$$r\int_{\theta_1}^{\theta_2}\left|\frac{h'(re^{i\theta})}{h(re^{i\theta})}\right|d\theta < K(\rho,\zeta)\,\delta\,\log\tfrac{1}{\delta}\,T(r,h). \tag{5.4.5}$$

Lemma 5.4.3 (*Valiron* [1] *p.74-75*) *Let f be an entire function of finite order, then*

$$\left|\frac{f'(z)}{f(z)}\right| = O(r^k) \qquad r\in[0,\infty]-E \tag{5.4.6}$$

here k is some positive number and E is a set of real numbers of finite linear measure.

We also require the following lemma.

Lemma 5.4.4 *Let A be an entire function of finite order satisfying $\delta(0,A)=1$, then there exists a set H of r with positive lower logarithmic density, such that the inequalities* (5.4.3), (5.4.5) *and* (5.4.6) *hold at the same time for sufficiently large $r\in H$.*

Proof Let $$F_1=\{r|\ r\in I-G\} \tag{5.4.7}$$

that is the set of r such that the inequality (5.4.5) does not hold. We have

$$ULD(F_1)=1-LLD(I-F_1)=1-LLD(G)$$
$$\leq 1-(1-\zeta)=\zeta(>0).$$

Also let $$F_2=\{r|\ r\in E\} \tag{5.4.8}$$

that is the set r such that the inequality (5.4.6) does not hold. Since E has finite linear measure

we deduce $$\int_{t\in F_2(r)}\frac{dt}{t}\leq\int_{t\in F_2(r)}dt<\infty,$$

hence $ULD(F_2)=LLD(F_2)=0$.

Let us recall from Lemma 5.4.1 the definitions of $\Gamma_j=\{z|\ \alpha^j\le r\le\alpha^{j+3/2}\}$ and the set E_j which is a collection of finite number of discs whose sum of radii are less than $4e\delta_j\alpha^{j+3/2}$, where $0<\delta_j<1/e$ for each j. Here δ_j is defined for each E_j in Lemma 5.4.1.

Let
$$F_3=\bigcup_j\{r=|z|:\ z\in E_j\}. \tag{5.4.9}$$
Clearly r satisfies $\alpha^j\le r\le\alpha^{j+3/2}$ for each j. We let $q=\left[\frac{\log r}{\log\alpha}\right]=\left[\frac{\log r}{1/(p+1)}\right]$, where $[x]$ represents the integral part of $x\in\mathbb{R}$. Also since for each E_j, δ_j can be chosen arbitrary small, given any $\eta>0$ we may assume $\delta_j<\eta$ for all j. In order to prove the lemma it suffice to show $LLD(I-\bigcup_j F_j)>0$.

Consider
$$\begin{aligned}\int_{t\in F_3(r)}\frac{dt}{t} &= O(1)+\sum_{j=1}^{q}\int_{\alpha^j}^{\alpha^{j+3/2}}\frac{dt}{t}\\
&< O(1)+\sum_{j=1}^{q}\frac{1}{\alpha^j}\int_{\alpha^j}^{\alpha^{j+3/2}}dt\\
&\le O(1)+\sum_{j=1}^{q}\left(\frac{1}{\alpha^j}4e\delta_j\alpha^{j+3/2}\right)\\
&=O(1)+4e\alpha^{3/2}\sum_{j=1}^{q}\delta_j\\
&<O(1)+4e\alpha^{3/2}q\eta.\end{aligned}$$

Hence
$$\frac{1}{\log r}\int_{t\in F_3(r)}\frac{dt}{t} = O\left(\frac{1}{\log r}\right)+4e\alpha^{3/2}(p+1)\eta.$$

Since η is arbitrary we may let $r\to\infty$ to obtain $ULD(F_3)=0$. Hence
$$\begin{aligned}LLD(I-\bigcup_j F_j) &=1-ULD(\bigcup_j F_j)\ge 1-\sum_1^3 ULD(F_j)\\
&=1-(\zeta+0+0)=1-\zeta>0\end{aligned}$$
which completes the proof of the lemma. □

§ 5.5 *Proof of Theorem 5.3.3*

Our method of proof is to consider the product $E=f_1f_2$ where f_1 and f_2 are linearly independent solutions of $y''+Ay=0$ and we shall also use the Gronwall's lemma stated in Chapter 3. First we need to establish some basic facts about the function E.

We observe that by the *Abel's identity*, the *Wronskian* of f_1 and f_2 satisfies

$$W(f_1,f_2)=f_1f_2'-f_2f_1'=c$$

where c is a non-zero constant. Now

$$\frac{d}{dz}log\left(\frac{f_2}{f_1}\right)=\frac{f_2'}{f_2}-\frac{f_1'}{f_1}=\frac{f_1f_2'-f_2f_1'}{f_2f_1}=\frac{c}{E},$$

but
$$\frac{d}{dz}log\ E=\frac{d}{dz}\ log(f_1f_2)=\frac{f_2'}{f_2}+\frac{f_1'}{f_1}=\frac{f_1f_2'+f_2f_1'}{f_2f_1}=\frac{E'}{E}.$$

Adding the above two equalities, we obtain

$$2\frac{f_2'}{f_2}=\frac{c}{E}+\frac{E'}{E}.$$

Differentiate both sides and substitute for $A=-f_2''/f_2$, we have

$$2\left(\frac{f_2f_2''-(f_2')^2}{f_2^2}\right)=2\left(\frac{f_2''}{f_2}-\left(\frac{f_2'}{f_2}\right)^2\right)=\left(\frac{c}{E}\right)'+\left(\frac{E'}{E}\right)'.$$

Thus
$$-\frac{cE'}{E^2}-\left(\frac{E'}{E}\right)^2+\frac{E''}{E^2}=2\left(-A-\frac{1}{4}\left(\frac{c^2}{E^2}+2\ \frac{cE'}{E^2}+\left(\frac{E'}{E}\right)^2\right)\right).$$

i.e.
$$4A=\left(\frac{E'}{E}\right)^2-\frac{c^2}{E^2}-2\ \frac{E''}{E}. \tag{5.5.1}$$

This relation was first found by Bank and Laine in [1] in 1982. See also Chapter 2 section 5.

Now
$$\begin{aligned}2T(r,E) &= T(r,E^2)+O(1)\\
&=T\left(r,\frac{c^2}{E^2}\right)+O(1)\\
&=T\left(r,\ \left(\frac{E'}{E}\right)^2-2\ \frac{E''}{E}-4A\right)\\
&\le T\left(r,\left(\frac{E'}{E}\right)^2\right)+T\left(r,\ \frac{E''}{E}\right)+T(r,A)\Big)+O(1)\\
&=2N\left(r,\ \frac{E'}{E}\right)+N\left(r,\ \frac{E''}{E}\right)+T(r,A)+m\left(r,\ (\frac{E'}{E})^2\right)+m\left(r,\ \frac{E''}{E}\right)+O(1)\\
&=2N\left(r,\ \frac{E'}{E}\right)+N\left(r,\ \frac{E''}{E}\right)+T(r,A)+O(log\ r).\quad n.e.\ as\ r\to\infty.\end{aligned}$$

The last equality follows because of the applications of the logarithmic lemma. Since $W(f_1,f_2)\equiv c\neq 0$, this implies that all the zeros of f_1, f_2 are simple and different to each other. Hence all zeros of E are simple. So $N\left(r,\ \frac{E'}{E}\right)=\bar{N}\left(r,\ \frac{1}{E}\right)$ and $N\left(r,\ \frac{E''}{E}\right)\le\bar{N}\left(r,\ \frac{1}{E}\right)$ where $\bar{N}(r,f)$ denotes the distinct zeros of f. Therefore

$$2T(r,E)=3\bar{N}\left(r,\ \frac{1}{E}\right)+T(r,A)+O(\log r).\quad n.e.\ as\ r\to\infty.$$

or
$$T(r,E)=O\left(\bar{N}\left(r,\ \frac{1}{E}\right)+T(r,A)+O(\log r)\right).\quad n.e.\ as\ r\to\infty. \tag{5.5.2}$$

Also $E''=f_1''f_2+2f_1'f_2'+f_1f_2''=-Af_1f_2+2f_1'f_2'-Af_1f_2=-2AE+2f_1'f_2'$.

Similarly
$$E'''=-2A'E-4AE'.$$

Therefore E also satisfies the third order differential equation
$$E'''+4AE'+2A'E=0. \tag{5.5.3}$$

We also note that if a function y satisfies the equation $y^{(k)}+Ay=0$ where $k\geq 1$ and A is analytic in a domain $\mathcal{A}$ say, then integrating by parts many times, we can obtain the following

$$y(z)=c_0+c_1(z-z_0)+c_2(z-z_0)^2+\cdots+c_{k-1}(z-z_0)^{k-1}-\frac{1}{(k-1)!}\int_{z_0}^{z}(z-s)^{k-1}A(s)y(s)\ ds, \tag{5.5.4}$$

where the path of the integration is taken within the domain $\mathcal{A}$.

We now proceed to the proof of the theorem.

(a) Let f_1, f_2 be linearly independent solutions of the equation $y''+Ay=0$, where A is a transcendental entire function of finite order $\rho(A)$ and $\delta(0,A)=1$. We assume the contrary that $max\{\lambda(f_1),\lambda(f_2)\}<\infty$ and show that this leads to a contradiction. Consider $E=f_1f_2$. We have $\lambda(E)<\infty$. Since $\rho(A)=p$ is an integer and $\bar{N}(r,\frac{1}{E})=O(r^q)$ for some $q>0$, we can deduce from (5.5.2) that $\rho(E)<\infty$. However $\rho(E)\geq\rho(A)$, since from (5.5.1)

$$\begin{aligned}T(r,A)&=T(r,4A)+O(1)\\&\leq N\left(r,\ \left(\frac{E'}{E}\right)^2\right)+N\left(r,\ \frac{E''}{E}\right)+T\left(r,\ \frac{c^2}{E^2}\right)+O(\log r)\\&=3\bar{N}\left(r,\ \frac{1}{E}\right)+2T(r,E)+O(\log r)\\&\leq 5T(r,E)+O(\log r).\end{aligned}$$

So $\rho(E)\geq\rho(A)$. We shall use the same notations as in Lemma 5.4.1. Let us recall that

$$c(r)=\alpha_0+\frac{1}{p}\sum_{|a_v|\leq r}\frac{1}{a_v{}^p}\ , \tag{5.5.5}$$

where $\{a_v\}$ are the zeros of A. Also $c_j=c\left(e^{\frac{j}{p+1}}\right)=c(\alpha^j)$ and we have from (5.4.5) that given any $\epsilon>0$ there exists a j_0 such that for all $j\geq j_0$

$$\left|\log|A(z)|\ -\ \Re(c_jz^j)\right|<4\epsilon|c_j|r^p,\quad z\in\{\Gamma_j-E_j\}.$$

Let us write $z=re^{i\theta}$, $c_j=|c_j|e^{i\beta}$, so that $\Re(c_jz^p)=|c_j|r^p\cos(p\theta+\beta)$. We note that β may depend on j but we do not emphasize this. Suppose $\theta_1, \theta_2, \cdots, \theta_{2p}$ where $|\theta_j|\le\pi$ $j=1, 2, \cdots, 2p$ denote the $2p$ zeros of $\cos(p\theta+\beta)$. Given $1>\delta_2>0$, there exists a $\delta_1>0$ such that $|\theta-\theta_j|\ge\delta_1$ for all $j=1, 2, \cdots, 2p$ implies that $|\cos(p\theta+\beta)|\ge\delta_2>0$. Since $\epsilon>0$ is arbitrary, we may choose it to be $\epsilon\le\frac{\delta_2}{8}$. Hence

$$\begin{aligned}\left|\log|A(z)| - \Re(c_jz^p)\right| &< 4\epsilon|c_j|r^p, \quad z\in\{\Gamma_j-E_j\},\ j\ge j_0\\ &\le \frac{\delta_2}{2}|c_j|r^p\\ &< \frac{|c_j|r^p}{2}|\cos(p\theta+\beta)|\\ &=|\Re(c_jz^p)|/2.\end{aligned}$$

We divide the inequality into two cases.

(i) If $\cos(p\theta+\beta)>0$, we have for $z\in\{\Gamma_j-E_j\}$

$$0<\tfrac{1}{2}\Re(c_jz^p)<\log|A(z)|<\tfrac{3}{2}\Re(c_jz^p); \tag{5.5.6}$$

(ii) if $\cos(p\theta+\beta)<0$, we have for $z\in\{\Gamma_j-E_j\}$

$$\tfrac{3}{2}\Re(c_jz^p)<\log|A(z)|<\tfrac{1}{2}\Re(c_jz^p)<0. \tag{5.5.7}$$

The above inequalities show that one can divide up any particular annular region $\{\Gamma_j-E_j\}$ into different regions with j sufficiently large so that A is either very large or very small.

(b) To be more explicit, let us take an arbitrary Γ_j defined in (5.4.4) and

$$Q_i=\left\{z|\ \theta_i-\frac{\pi}{2p}\le\theta\le\theta_i+\frac{\pi}{2p}\right\},\ i=1, 2, \cdots, 2p.$$

We divide up Γ_j into $2p$ portions by $\Gamma_j\cap Q_i$ $i=1, 2,\cdots, 2p$. Since the behaviour of A in each of the $\Gamma_j\cap Q_i$ is essentially the same, we only consider $\Gamma_j\cap Q_1$ say. We aim to estimate the growth of A and E in each Q_i and to integrate $\log^+|E|$ over 0 to 2π i.e. to estimate $T(r,E)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|E|d\theta$. Let $H=I-(F_1\cup F_2\cup F_3)$ where F_j, $j=1,2,3$ were defined in lemma 5.4.4, so H has positive lower logarithmic density. We shall choose such $r\in H$ for the integration of $\log^+|E|$ and we shall always assume $r\in H$ in sequel. Note that $z\in\{\Gamma_j-E_j\}$ ($|z|=r\in H$) for some j sufficiently large. For convenience we define

$$H^*(\epsilon)=\{r\colon r\in H \text{ and } re^{i\theta}=z\in\Gamma_j-E_j,\ j\ge j_0(\epsilon)\}.$$

Without loss of generality we may assume $\cos\left(p(\theta_1+\frac{\pi}{2p})+\beta\right)=1$ and $\cos\left(p(\theta_1-\frac{\pi}{2p})+\beta\right)=-1$.

Now we estimate, with $r \in H^*(\epsilon)$,

$$\int_{re^{i\theta}\in\Gamma_j\cap Q_1} log^+|E(re^{i\theta})|\ d\theta = \int_{\theta_1-\frac{\pi}{2p}}^{\theta_1+\frac{\pi}{2p}} log^+|E(re^{i\theta})|\ d\theta$$

$$=\int_{\theta_1-\frac{\pi}{2p}}^{\theta_1-\delta_1} log^+|E(re^{i\theta})|\ d\theta+\int_{\theta_1-\delta_1}^{\theta_1+\delta_1} log^+|E(re^{i\theta})|\ d\theta+\int_{\theta_1+\delta_1}^{\theta_1+\frac{\pi}{2p}} log^+|E(re^{i\theta})|\ d\theta$$

$= I_1(r)+I_2(r)+I_3(r)$ say.

We first estimate $I_3(r)$. From (5.5.6)

$$\begin{aligned} log\ |A(z)| &> \tfrac{1}{2}\Re(c_j z^p)\ (>0) \\ &= \tfrac{1}{2}|c_j| r^p cos(p\theta+\beta) \\ &\geq \tfrac{1}{2}|c_j| r^p \delta_2. \end{aligned}$$

Also from (5.4.2) of lemma 5.4.1, for any $\epsilon>0$ and $r>r_0(\epsilon)$ we have $T(r,A)=(1+\eta(\epsilon))\frac{|c(r)|r^p}{\pi}$ for $|\eta|<\epsilon$. We note that A is of regular growth(see Edrei & Fuchs [1]) and $\rho(A)=p\in\mathbf{N}$. *i.e.* $\overline{\lim_{r\to\infty}}\frac{log\ T(r,A)}{log\ r}=\underline{\lim_{r\to\infty}}\frac{log\ T(r,A)}{log\ r}\geq 1$. We may further assume here that r and $r_0\in H^*(\epsilon)$.

Hence $\qquad log\ |A(z)| \geq \frac{1}{2}|c_j|r^p\delta_2=\frac{\pi T(r,A)\delta_2}{2(1+\eta)}$

$$>\frac{\pi\delta_2}{4}\ T(r,A)>\frac{\pi\delta_2}{4}\ r^{1-\delta_3}$$

for some $\delta_3(\epsilon)>0$ and it tends to zero as $r\in H^*(\epsilon)$, $r \to \infty$. Summarising the above observations we have

$$|A(z)|>exp\Big(\frac{\pi\delta_2}{4}\ r^{1-\delta_3}\Big) \tag{5.5.8}$$

as $r\to\infty$ n.e and $r\in H^*(\epsilon)$ and $\theta_1+\delta_1\leq\theta\leq\theta_1+\frac{\pi}{2p}$.

From (5.5.1) we can write

$$\left|4A-\left(\left(\frac{E'}{E}\right)^2-2\frac{E''}{E}\right)\right|=\left|\frac{c^2}{E^2}\right|$$

so that $\left|\frac{c^2}{E^2}\right| > |4A|-\left|\left(\frac{E'}{E}\right)^2-2\frac{E''}{E}\right|$. From Lemma 5.4.3 we deduce $\left|\left(\frac{E'}{E}\right)^2-2\frac{E''}{E}\right|=O(r^k)$ *n.e.* and for some $k>0$. Although this estimate only holds outside a set of finite measure, Lemma 5.4.4 asserts that it and (5.5.8) can hold at the same time for a set $r(\in H)$ of positive lower logarithmic density. Thus

$$\left|\frac{c^2}{E^2}\right|>exp\Big(\frac{\pi\delta_2}{4}\ r^{1-\delta_3}\Big)-const.(r^k) \tag{5.5.9}$$

as $r\to\infty$ *n.e.* and $r\in H^*(\epsilon)$ and $\theta_1+\delta_1\leq\theta\leq\theta_1+\frac{\pi}{2p}$. This implies $|E|\to 0$ outside those exceptional sets and for $\theta_1+\delta_1\leq\theta\leq\theta_1+\frac{\pi}{2p}$. In particular $|E|<1$ for r sufficiently large, so

$$I_3(r)=\int_{\theta_1+\delta_1}^{\theta_1+\frac{\pi}{2p}} \log^+|E(re^{i\theta})|\ d\theta=0, \qquad r\in H^*(\epsilon). \tag{5.5.10}$$

(c) Next we estimate $I_2(r)$. To do so we employ the Fuchs' small arc Lemma 5.4.2. Since $r\in H^*(\epsilon)$ we may choose δ_1 so small such that $2\delta_1<\delta<1/2$. For $\theta_1-\delta_1\le\theta\le\theta_1+\delta_1$, we have, for a fixed r,

$$\left|\log E(re^{i\theta}) - \log E(re^{i(\theta_1+\delta_1)})\right| = \left|-\int_{\theta}^{\theta_1+\delta_1} \frac{E'(re^{it})}{E(re^{it})}\ ire^{it}\ dt\right|$$

and

$$\log|E(re^{i\theta})| \le \log|E(re^{i(\theta_1+\delta_1)})| + \int_{\theta}^{\theta_1+\delta_1} r\left|\frac{E'(re^{it})}{E(re^{it})}\right|\ dt.$$

Hence

$$\begin{aligned}\log^+|E(re^{i\theta})| &\le \log^+|E(re^{i(\theta_1+\delta_1)})| + \int_{\theta}^{\theta_1+\delta_1} r\left|\frac{E'(re^{it})}{E(re^{it})}\right|\ dt.\\ &\le \log^+|E(re^{i(\theta_1+\delta_1)})| + \int_{\theta_1-\delta_1}^{\theta_1+\delta_1} r\left|\frac{E'(re^{it})}{E(re^{it})}\right|\ dt.\\ &\le \log^+|E(re^{i(\theta_1+\delta_1)})| + K(\rho(E),\zeta)\ \delta_1\ \log\frac{1}{\delta_1}\ T(r,E). \end{aligned} \tag{5.5.10a}$$

Here $K(\rho(E),\zeta)$ is a positive constant as defined in Lemma 5.4.2 and $\delta_1<1/4$.

Now $\log^+|E(re^{i(\theta_1+\delta_1)})|=0$, since from (5.5.9) we can choose $r\in H^*(\epsilon)$ sufficiently large so that $|E(re^{i(\theta_1-\delta_1)})|<1$. We deduce that

$$I_2(r)=\int_{\theta_1-\delta_1}^{\theta_1+\delta_1} \log^+|E(re^{i\theta})|\ d\theta \le 2K(\rho(E),\zeta)\ \delta_1{}^2\ \log\frac{1}{\delta_1}\ T(r,E). \tag{5.5.11}$$

(d) Finally we estimate $I_1(r)$. Recall that the function E itself satisfies a third order differential equation (5.5.3). Let us rewrite it into the following form

$$E'''+\phi(z)E=0,$$

where $\phi(z)=A\left(2\frac{A'}{A}+4\frac{E'}{E}\right)$. According to the formula (5.5.4) we may write

$$E(z)=d_0+d_1(z-z_0)+d_2(z-z_0)^2-\frac{1}{2!}\int_{z_0}^{z}(z-s)^2\phi(s)E(s)\ ds, \tag{5.5.12}$$

where $d_0=E(z_0)$, $d_1=E'(z_0)=E(z_0)\left(\frac{E'(z_0)}{E(z_0)}\right)$ and $d_2=\frac{1}{2}E''(z_0)=\frac{1}{2}E(z_0)\left(\frac{E''(z_0)}{E'(z_0)}\right)\left(\frac{E'(z_0)}{E(z_0)}\right)$. Since $r\in H$ we have from Lemma 5.4.3 that both $|d_1|=|E(z_0)|O(r^{k_1})$ and $|d_2|=|E(z_0)|O(r^{k_2})$ for some

k_1 and $k_2>0$ outside a set r of finite linear measure as $r \to \infty$.

We also have, from (5.4.2) and (5.5.7) with $r\in H^*(\epsilon)$ being chosen sufficiently large and $\theta_1-\delta_1\le\theta\le\theta_1-\frac{\pi}{2p}$, that

$$\begin{aligned} \log|A(z)| &< \tfrac{1}{2}\Re(c_j z^p)\ (<0) \\ &= \tfrac{1}{2}|c_j| r^p \cos(p\theta+\beta) \\ &\le -\tfrac{1}{2}|c_j| r^p \delta_2 = -\frac{\delta_2}{2}\frac{\pi T(r,A)}{1+\eta}, \quad |\eta|<\epsilon \\ &< -\pi\delta_2 T(r,A). \end{aligned}$$

i.e. $$|A(z)| \le e^{-\pi T(r,A)} < e^{-\pi\delta_2 r^{(1-\epsilon)}} \quad r\in H^*(\epsilon) \text{ and } \theta_1-\delta_1\le\theta\le\theta_1-\frac{\pi}{2p}.$$

We may choose z_0 with $arg(z_0)=\theta_1-\delta_1$ and the integration is taken along a circular arc with radius $r\in H^*(\epsilon)$. According to the above estimate we have from (5.5.12), that

$$|E(z)|\le|E(z_0)|\Big(1+O(r^{k_1})|z-z_0|+O(r^{k_1})|z-z_0|^2\Big)+ \\ +\frac{1}{2}\int_{\theta_1-\delta_1}^{\theta} |re^{i\theta}-re^{it}|^2|\phi(re^{it})||E(re^{it})|r\,dt.$$

Note that $|z-z_0|<\frac{\pi}{p}\alpha^{j+3/2}$, where $r<\alpha^{j+3/2}$. Hence

$$\begin{aligned} |E(z)| &\le |E(z_0)|\Big(1+O(r^{k_3})\Big)+\frac{1}{2}\int_{\theta_1-\delta_1}^{\theta} |re^{i\theta}-re^{it}|^2|\phi(re^{it})||E(re^{it})|r\,dt \\ &\le |E(z_0)|\Big(1+O(r^{k_3})\Big)+\frac{1}{2}\int_{\theta_1-\delta_1}^{\theta} r^3|e^{i\theta}-e^{it}|^2|\phi(re^{it})||E(re^{it})|\,dt \\ &\le |E(z_0)|\Big(1+O(r^{k_3})\Big)+2\int_{\theta_1-\delta_1}^{\theta} r^3|\phi(re^{it})||E(re^{it})|\,dt \\ &\le |E(z_0)|\Big(1+O(r^{k_3})\Big)\exp\left\{\int_{\theta_1-\delta_1}^{\theta} r^3|\phi(re^{it})|\,dt\right\}. \end{aligned}$$

Here we have used the Gronwall's lemma 3.3.1 in the last inequality for a fixed $r\in H^*(\epsilon)$. Recall, from Lemma 5.4.3, that

$$|\phi(z)|=|A(z)|\left|2\,\frac{A'}{A}(z)+4\,\frac{E'}{E}(z)\right| = O(e^{-\pi\delta_2 r^{(1-\epsilon)}})O(r^k)=O(r^k e^{-\pi\delta_2 r^{(1-\epsilon)}}).$$

Also since $arg(z_0)=\theta_1-\delta_1$, we can use the estimate (5.5.10a) in (c) above, in which case we have $\quad log|E(z_0)|=O\Big(\delta_1 \; log \, \frac{1}{\delta_1} \; T(r,E)\Big) \qquad r\in H^*(\epsilon)$ and $r<\alpha^{j+3/2}$.

Thus

$$|E(z)|\le O\Big[exp\Big(\delta_1 \; log \, \frac{1}{\delta_1} \; T(r,E)\Big)\Big]O\Big(1+O(r^{k_3})\Big) \; exp\Big(O(r^k e^{-\pi\delta_2 r^{(1-\epsilon)}})\Big).$$

So $I_1(r)=\displaystyle\int_{\theta_1-\frac{\pi}{2p}}^{\theta_1-\delta_1} log^+|E(re^{i\theta})| \; d\theta$

$$\le O(1)+O(log \; r)+O\Big(\delta_1 \; log \, \frac{1}{\delta_1} \; T(r,E)\Big)+O\Big(r^k e^{-\pi\delta_2 r^{(1-\epsilon)}}\Big)$$

$$\le O(log \; r)+O\Big(\delta_1 \; log \, \frac{1}{\delta_1} \; T(r,E)\Big)+O\Big(r^k e^{-\pi\delta_2 r^{(1-\epsilon)}}\Big). \tag{5.5.13}$$

(e) Combining the inequalities (5.5.10), (5.5.11) and (5.5.13) we finally obtain the estimate for $log^+|E|$ in $\Gamma_j\cap Q_1$. Since $r\in H^*(\epsilon)$, according to Lemma 5.4.4 it avoids all the exceptional sets arising from Lemmas 5.4.1, 5.4.2 and 5.4.3. Choosing $r\in H^*(\epsilon)$ we have

$$\begin{aligned}\int_{\theta_1-\frac{\pi}{2p}}^{\theta_1+\frac{\pi}{2p}} log^+|E(re^{i\theta})| \; d\theta &= I_1(r)+I_2(r)+I_3(r)\\ &=\Bigg(O(log \; r)+O\Big(\delta_1 \; log \, \frac{1}{\delta_1} \; T(r,E)\Big)+O\Big(r^k e^{-\pi\delta_2 r^{(1-\epsilon)}}\Big)\Bigg)+\\ &\qquad +O\Big(\delta_1^{\,2} \; log \, \frac{1}{\delta_1} \; T(r,E)\Big)+0.\end{aligned}$$

We can repeat the above analysis from (a)$-$(d) to the remaining $2p-1$ portions $\Gamma_j\cap Q_i$ $i=2,3,\cdots,2p$. So we have similar estimates with the *same* δ_1. Thus we get

$$\begin{aligned}T(r,E) &= \frac{1}{2\pi}\int_0^{2\pi} log^+|E(re^{i\theta})| \; d\theta\\ &=\frac{1}{2\pi}\sum_{i=0}^{2p}\int_{\theta_i-\frac{\pi}{2p}}^{\theta_i+\frac{\pi}{2p}} log^+|E(re^{i\theta})| \; d\theta\\ &= O(log \; r)+o\Big(r^{-N}\Big)+O\Big(\delta_1 \; log \, \frac{1}{\delta_1} \; T(r,E)\Big) \quad \textit{for each } N>0.\end{aligned}$$

Since $\epsilon>0$ is arbitrary, by choosing δ_1 sufficiently small the above calculations will remain valid and we can ensure that $O\Big(\delta_1 log\frac{1}{\delta_1} \; T(r,E)\Big)\le\frac{1}{3}T(r,E)$ say as $r\in H^*(\epsilon)$.

Thus

$$T(r,E) = O(log \; r)+o\Big(r^{-N}\Big) \quad \text{for each } N>0, \; r\in H^*(\epsilon) \text{ sufficiently large.}$$

$= O(\log r)$.

But since $T(r,E)$ is an increasing function, so $T(r,E)=O(\log r)$ for all r sufficiently large. This implies the order of $E(z)$ is 0. Hence E must be a polynomial and it follows from the expression (5.5.1) that A is a rational function, a contradiction. Hence $\max\{\lambda(f_1),\lambda(f_2)\}=\infty$. □

Remark One can also conclude that it contradicts the fact that $\rho(E)\geq\rho(A)\geq 1$.

§ 5.6 A Perturbation Result

In the second part of Corollary 6.3.2, equations of the form

$$y''(z)+\left(B(z)e^{P(z)}+P_1(z)\right)y=0 \tag{5.6.1}$$

were considered, where $B(z)$ is a non-constant entire function of order $\rho(B)$ and $P(z)$, $P_1(z)$ are polynomials of degrees $d\geq 2$ and $m\geq 0$ respectively, such that $\rho(B)<d$ and $\frac{m+2}{2}<d$. Then if f_1 and f_2 are linearly independent solutions of the equation (5.6.1) we have $\max\{\lambda(f_1),\lambda(f_2)\}=\infty$. That is if we perturb $B(z)e^{P(z)}$ carefully with a polynomial of small degree, we obtained the same conclusion for the zeros distribution of the solutions.

In view of the above observation and noting that an entire function $A(z)$ with $\delta(0,A)=1$ has behaviour similar to that of (5.6.1), it is reasonable to believe that if we consider $A+P$ instead of A, the same conclusion will hold. We obtain the following special case.

<u>Theorem</u> <u>5.6.1</u> *Let $A(z)$ be a transcendental entire function of finite order $\rho(A)$ satisfying $\delta(0,A)=1$ and all but finitely many zeros $\{a_n\}$ of A lying on certain half-line which has one end at the origin. Let $P(z)$ be a polynomial of degree $m\geq 0$ such that $\frac{m+2}{2}<\rho(A)$. If f_1 and f_2 are linearly independent solutions of the equation*

$$y''+(A+P)y=0.$$

We have $\max\{\lambda(f_1),\ \lambda(f_2)\}=\infty$.

Remark We note that $\rho(A)$ must be a positive integer ≥ 2 since $\delta(0,A)=1$.

To prove the theorem we require the following lemmas.

<u>Lemma</u> <u>5.6.2</u> (*Bank, Laine & Langley* [1]) *Suppose $B(z)$ is analytic in some sector S containing*

the ray $L=\{z: z=re^{i\theta},\ r>0\}$ *and that there exists a constant* $b>0$ *and an increasing function* $\eta(r)$ *for* $r\geq b$, *such that* $\eta(r)$ *has a continuous derivative,* $\eta(b)>0$ *and* $|B(re^{i\theta})|\leq\eta(r)$ *for* $r\geq b$. *Then any non-trivial solution* y *of* $y''+By=0$ *in* S *satisfies*

$$log^+|y(re^{i\theta})| \leq K+\int_b^r \Big(\eta(t)\Big)^{1/2} dt \tag{5.6.2}$$

for some $K>0$ *and* $r\geq b$.

Lemma 5.6.3 (*Edrei & Fuchs* [1]) *We assume the same hypotheses and notations as in Lemma* 5.4.1. *Given* $\epsilon>0$ *there exists a* $j_0(\epsilon)$ *such that*

$$log\ |A(z)| < \Re(c_jz^j) + 4\epsilon|c_j|r^p,\ \ z\in\Gamma_j \tag{5.6.3}$$

for all $j\geq j_0(\epsilon)$ *sufficiently large, where* p *is the order of* A.

Notice that in the above lemma the estimate (5.6.3) is true for all $z\in\Gamma_j$ without any exceptional sets E_j as in (5.4.3).

§ *5.7 Proof of Theorem 5.6.1*

As one may expect, the proof here will be similar to that of Theorem 5.3.3. In fact it is somewhat easier.

Let $A(z)$ and $P(z)$ satisfy the hypotheses of Theorem 5.6.1. Let f_1 and f_2 be the linearly independent solutions of the equation

$$y''+(A+P)y=0, \tag{5.7.1}$$

and $F=f_1f_2$. We assume the contrary that $\lambda(E)<\infty$. Hence $\rho(E)\geq\rho(A)\in\mathbb{N}$. We shall estimate $T(r,E)=\frac{1}{2\pi}\int_0^{2\pi} log^+|E|\ d\theta$ and arrive at a contradiction. The first half of the proof is the same as those in (a), (b) and (c) in § 5.5.

(a) This is the same as that in §5.5 (b) except that we choose $\delta_1>0$ such that $|cos(p\theta+\beta)|\geq\ \delta_2\geq5\epsilon>0$ when $|\theta-\theta_i|\geq\delta_1>0$. Recall that $\epsilon>0$ is arbitrary and there exists a $j_0(\epsilon)$ such that when $j\geq j_0$

$$\begin{aligned}\Big|log\ |A(z)|\ -\ \Re(c_jz^j)\Big| &< 4\epsilon|c_j|r^p,\ \ z\in\{\Gamma_j-E_j\}\\ &\leq \frac{4\delta_2}{5}\ |c_j|\ r^p\end{aligned}$$

$$\leq \frac{4}{5}\ |cos(p\theta+\beta)||c_j|\ r^p = \frac{4}{5}\Re(c_j r^p).$$

Hence when $\Re(c_j r^p)$ is positive with zeros $\theta_1,\cdots,\theta_{2p}$, we have, as in (5.5.5),

$$0<\frac{1}{5}\Re(c_j z^p)<log\ |A(z)|<\frac{9}{5}\Re(c_j z^p) \quad z\in\{\Gamma_j - E_j\}.$$

Hence from Lemma 5.4.1, we obtain

$$log\ |A(z)| > \frac{1}{5}|c_j|\ r^p cos(p\theta+\beta)$$

$$> \frac{1}{5}\frac{\delta_2\pi T(r,A)}{2(1+\eta)} > \frac{\delta_2\pi}{10}T(r,A) > \frac{\delta_2\pi}{10}\ r^{2-\delta_3} \qquad |\eta|<\epsilon,$$ where $\delta_3 \to 0$ as $r \to \infty$. Therefore $|A+P|>exp\left(\frac{\delta_2\pi}{10}\ r^{2-\delta_3}\right)$ as $r \geq r_0$ and $r\in H^*(\epsilon)$.

We also note that the quantity $c(r)$ in (5.5.5) has a fixed argument for all r sufficiently large because of the location of the zeros $\{a_n\}$ of A. Thus we see that the angle β does not depend on j. Hence the $2p$ zeros of $cos(p\theta+\beta)$ are fixed on all annuli regions Γ_j.

With the same notations and reasoning, we only consider the integral

$$\int_{\theta_1-\frac{\pi}{2p}}^{\theta_1+\frac{\pi}{2p}} log^+|F(re^{i\theta})|\ d\theta = \int_{\theta_1-\frac{\pi}{2p}}^{\theta_1-\delta_1} log^+|F(re^{i\theta})|\ d\theta + \int_{\theta_1-\delta_1}^{\theta_1+\delta_1} log^+|F(re^{i\theta})|\ d\theta +$$

$$+\int_{\theta_1+\delta_1}^{\theta_1+\frac{\pi}{2p}} log^+|F(re^{i\theta})|\ d\theta = I_1(r)+I_2(r)+I_3(r) \text{ say.}$$

Since the order of $A(z)$ is at least two and $|P(z)|=O(r^m)$ where m is the degree of P, it is easy to see, from the relation

$$4(A+P)=\left(\frac{F'}{F}\right)^2-\frac{c^2}{F^2}-2\ \frac{F''}{F^2}, \tag{5.7.2}$$

that $F \to 0$ as $r \to \infty$ outside the exceptional sets E_j for $\theta_1+\delta_1\leq\theta\leq\theta_1+\frac{\pi}{2p}$.

Hence $I_3(r)=0$ as $r \geq r_0$ and $r\in H^*(\epsilon)$. (5.7.3)

$$\text{Similarly } I_2(r)=\int_{\theta_1-\delta_1}^{\theta_1+\delta_1} log^+|F(re^{i\theta})|\ d\theta\leq 2K\Big(\rho(F),\zeta\Big)\delta_1{}^2 log\ \frac{1}{\delta}\ T(r,F), \tag{5.7.4}$$

where $\delta_1\leq 1/4$ for r belongs to a set of lower logarithmic density $\geq 1-\zeta$ as in (b) of the last proof by Lemma 5.4.2. Thus we can choose a $r\in H^*(\epsilon)$ (see Lemma 5.4.4) so that (5.7.3) and (5.7.4) can hold at the same time.

(b) If $cos(p\theta+\beta)<0$ then with the same $\epsilon>0$ and δ_1, δ_2 as in part (a), we deduce, from

Lemma 5.6.3, that

$$\begin{aligned} \log|A(z)| &< \Re(c_j z^j) + 4\epsilon|c_j|r^p, \quad z\in\Gamma_j \\ &= |c_j|r^p\Big(\cos(p\theta+\beta)+4\epsilon\Big) \\ &< |c_j|r^p(-5\epsilon+4\epsilon) = -\epsilon\,|c_j|r^p \leq 0 \end{aligned}$$

for all $r\in H^*(\epsilon)$ sufficiently large and $z\in\Gamma_j$.

So $$|(A+P)(re^{i\theta})| \leq Cr^m \text{ for } re^{i\theta}=z\in\Gamma_j \text{ and } \theta_1-\frac{\pi}{2p}\leq\theta\leq\theta_1-\delta_1.$$

Thus for any ray L from the origin and passing through Γ_j and $\theta_1-\pi/2p\leq\theta\leq\theta_1-\delta_1$ for all j sufficiently large, we have

$$|(A+P)(re^{i\theta})| = O(r^m).$$

We conclude from Lemma 5.6.2 with $\eta(r)=Kr^m$ that

$$\begin{aligned} \log^+|F(re^{i\theta})| &\leq \log^+|f_1(re^{i\theta})|+\log^+|f_2(re^{i\theta})| \\ &\leq K_1+\int_b^r (Kt^m)^{1/2}dt + K_2+\int_b^r (Kt^m)^{1/2}dt \\ &= O\Big(r^{\frac{m}{2}+1}\Big). \end{aligned}$$

We can subdivide the region into sections each with sufficiently small angle. Since F has finite order, an application of *Phragmen-Lindelöf principle* (see for example Titchmarsh E.C. [1] p.177) implies $\log^+|F(re^{i\theta})|=O\Big(r^{\frac{m}{2}+1}\Big)$ uniformly in $\theta_1-\pi/2p\leq\theta\leq\theta_1-\delta_1$ and $r\in H^*(\epsilon)$ sufficiently large.

Hence $$I_1(r)=\int_{\theta_1-\frac{\pi}{2p}}^{\theta_1-\delta_1} \log^+|F(re^{i\theta})|\,d\theta = O\Big(r^{\frac{m}{2}+1}\Big).$$

Combining the above calculation and repeating a similar argument for the remaining sections $\Gamma_j\cap Q_i$ $i=2, 3,\cdots,2p$, we have

$$T(r,F)=O\Big(I_1(r)+I_2(r)+I_3(r)\Big)\leq O\Big(r^{\frac{m}{2}+1}\Big)+O\Big(2K\Big(\rho(F),\zeta\Big)\delta_1{}^2\log\frac{1}{\delta_1}\,T(r,F)\Big)+0.$$

Hence by choosing δ_1 sufficiently small

$$T(r,F)=O\Big(r^{\frac{m}{2}+1}\Big) \text{ for } r\in H^*(\epsilon).$$

Thus $T(r,F)=O\Big(r^{\frac{m}{2}+1}\Big)$ for all r sufficiently large,

i.e. $$\rho(F)\leq\frac{m}{2}+1<\rho(A).$$

This contradicts the fact that $\rho(F)\geq\rho(A)$. Hence $max\{\lambda(f_1),\lambda(f_2)\}=\infty$. □

§ *5.8 Examples and Further Problems*

If we have the assumption that $A(z)$ is a transcendental entire function of order $\rho(A)=1$ and all but finitely many zeros of A lying on a half-line with one end at the origin, then clearly Theorems 5.3.3 and 5.6.1 are different. But if we assume $\rho(A)\geq 2$ and the degree of the polynomial $P(z)$ is 0 instead, then Theorem 5.6.1 actually includes 5.3.3. For example if the equation

$$y''+\left\{e^{z^2}+k\right\}y=0,$$

where k is any constant, has two linearly independent solutions f_1 and f_2 , then by Theorem 5.6.1, $max\{\lambda(f_1),\lambda(f_2)\}=\infty$. However the theorem does not cover the equation $y''+(e^z+k)y=0$. The difference is not just a gap due to the techniques used in the proof of Theorem 5.3.3 that do not seem to be able to be extended to that of Theorem 5.6.1. This gap in fact cannot be closed. This is because of

<u>Theorem</u> <u>5.8.1</u> (*Bank, Laine & Langley* [1]) *Let $c\in\mathbb{C}$ be a fixed constant, and suppose f is a non-trivial solution of* $\quad y''+(e^z-c)y=0.$
If $c=\frac{q^2}{16}$ for some positive odd integer q, then the equation has two linearly independent solutions f_1 and f_2 such that $max\{\lambda(f_1),\lambda(f_2)\}\leq 1$.

This shows that Theorem 5.3.3 is essentially the best possible with respect to the order of A.

Moreover, we note that Theorem 5.6.1 is also best possible in certain sense. We use an example due to Bank and Laine [1]; see also Bank, Laine and Langley [1] p.19 . Consider the equation

$$y''(z)+\left(e^{Q(z)}+P(z)\right)y(z)=0 \tag{5.8.1}$$

where $Q(z)$ and $P(z)$ are polynomials with degrees $degQ=n\geq 2$ and $degP=m\geq 0$. Suppose $m+2<2n$ then $max\{\lambda(f_1),\lambda(f_2)\}=\infty$ where f_1 and f_2 are linearly independent solutions of the equation.

Now let $F=e^{z^n}$ where $n\geq 2$ is a positive integer and we define $A_1(z)$ by (5.7.2). Then it is easy to check that

$$4A_1(z)=4(A(z)+P(z))=-e^{-2z^n}-n^2z^{2n-2}-2n(2n-1)z^{n-2}.$$

By Lemma 2.5.4 (b), F is a product of entire functions f_1 and f_2 which are linearly independent solutions of (5.8.1) with $A=-\frac{1}{4}e^{-2z^n}$ and $P=-\frac{1}{4}\left(n^2z^{2n-2}+2n(2n-1)z^{n-2}\right)$. Clearly $degQ=n$ and $degP=2n-2$. But $\dfrac{(2n-2)+2}{2}=n\not<n$ and F has no zeros at all.

On the other hand Bank, Laine and Langley investigated in [2] the equation

$$y''(z)+\left(e^{Q(z)}+\Pi(z)\right)y(z)=0$$

where $Q(z)$ is a polynomial of degree n and $\Pi(z)$ is an entire function of order $\rho(\Pi)<n$. They proved that if the equation admits a non-trivial solution f such that $\lambda(f)<n$, then f has no zeros and

$$\Pi(z)=-\frac{1}{16}(Q')^2+\frac{1}{4}Q''.$$

Thus an immediate question would be to ask: Let A be a transcendental entire function with $\delta(0,A)=1$ (hence $\rho(A)$ is an integer) and P a polynomial with $degP<n$. Suppose a non-trivial solution f such that $\lambda(f)<n$, satisfies the equation $y''(z)+\left(A+\Pi(z)\right)y(z)=0$. What conclusion about $\Pi(z)$ can we draw ? This question is under investigation.

A recent result of S. Hellerstein and J. Rossi [1] (Theorem 3) concerns about the location of zeros of the solutions of the equation $y''+Qe^P=0$ where Q and P are polynomials. They proved that in those sectors for which $A=Qe^P$ is small any solution will have only finitely many zeros. This resembles the real case when considering the equation $y''=0$.

Our function $A(z)$ with $\delta(0,A)=1$ has sectorial sets $\Gamma_j\cap Q_i$ $i=1,2,\cdots,2p$ in which A is either very large or very small. We would like to ask whether a similar conclusion is true that only finitely many zeros for the solutions of the equation $y''+Ay=0$ occur in sectors where A is small. For example, a particular simple case would be to consider when all the zeros of $A(z)$ are real, then the quantity $c(r)=\alpha_0+\frac{1}{p}\sum_{|a_v|\le r}\frac{1}{{a_v}^p}$ is a complex number and $\Re(c_jz^j)=|c_j|r^p cos(p\theta+\beta)$. This shows A will be large or small according to where the zeros of $cos(p\theta+\beta)$ are.

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