SIAM J. MATH. ANAL.
Vol. 53, No. 6, pp. 6724–6770

EIGENVALUES OF THE TRUNCATED HELMHOLTZ SOLUTION OPERATOR UNDER STRONG TRAPPING*

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Abstract. For the Helmholtz equation posed in the exterior of a Dirichlet obstacle, we prove that if there exists a family of quasimodes (as is the case when the exterior of the obstacle has stable trapped rays), then there exist near-zero eigenvalues of the standard variational formulation of the exterior Dirichlet problem (recall that this formulation involves truncating the exterior domain and applying the exterior Dirichlet-to-Neumann map on the truncation boundary). Our motivation for proving this result is that (a) the finite-element method for computing approximations to solutions of the Helmholtz equation is based on the standard variational formulation, and (b) the location of eigenvalues, and especially near-zero ones, plays a key role in understanding how iterative solvers such as the generalized minimum residual method (GMRES) behave when used to solve linear systems, in particular those arising from the finite-element method. The result proved in this paper is thus the first step towards rigorously understanding how GMRES behaves when applied to discretizations of high-frequency Helmholtz problems under strong trapping (the subject of the companion paper [P. Marchand et al., *Adv. Comput. Math.*, to appear]).

Key words. Helmholtz equation, trapping, quasimodes, eigenvalues, resonances, semiclassical analysis

AMS subject classifications. 35J05, 35P15, 35B34, 35P25

DOI. 10.1137/21M1399658

1. Introduction.

1.1. Preliminary definitions. Let $\Omega_- \subset \mathbb{R}^d, d \geq 2$, be a bounded open set such that its open complement $\Omega_+ := \mathbb{R}^d \setminus \overline{\Omega_-}$ is connected. Let $\Gamma_D := \partial\Omega_-$, where the subscript D stands for "Dirichlet." Let Ω_1 be another bounded open set with a connected open complement and such that $\mathrm{conv}(\Omega_-) \Subset \Omega_1$, where conv denotes the convex hull and $\Subset$ denotes compact containment. Let $\Omega_{\rm tr} := \Omega_1 \setminus \Omega_-$, and let $\Gamma_{\rm tr} := \partial\Omega_1$, where the subscript tr stands for "truncated." We assume throughout that Γ_D and $\Gamma_{\rm tr}$ are both C^∞. Let γ_0^D and $\gamma_0^{\rm tr}$ denote the Dirichlet traces on Γ_D and $\Gamma_{\rm tr}$, respectively, and let γ_1^D and $\gamma_1^{\rm tr}$ denote the respective Neumann traces, where the normal vector points out of $\Omega_{\rm tr}$ on both Γ_D and $\Gamma_{\rm tr}$. Let

$$H^1_{0,D}(\Omega_{\rm tr}) := \big\{ v \in H^1(\Omega_{\rm tr}) : \gamma_0^D v = 0 \big\}.$$

Let $\mathcal{D}(k) : H^{1/2}(\Gamma_{\rm tr}) \to H^{-1/2}(\Gamma_{\rm tr})$ be the Dirichlet-to-Neumann map for the equation $\Delta u + k^2 u = 0$ posed in the exterior of Ω_1 with the Sommerfeld radiation condition

$$\frac{\partial u}{\partial r}(x) - \mathrm{i}ku(x) = o\left(\frac{1}{r^{(d-1)/2}}\right) \tag{1.1}$$

*Received by the editors February 18, 2021; accepted for publication (in revised form) July 26, 2021; published electronically November 30, 2021.
https://doi.org/10.1137/21M1399658
Funding: The work of the second and third authors was supported by EPSRC grant EP/R005591/1.
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as $r := |x| \to \infty$, uniformly in $\widehat{x} := x/r$. We say that a function satisfying (1.1) is *k-outgoing*. When $\Gamma_{\rm tr} = \partial B_R$, for some $R > 0$, the definition of $\mathcal{D}(k)$ in terms of Hankel functions and polar coordinates (when $d = 2$)/spherical polar coordinates (when $d = 3$) is given in, e.g., [39, equations 3.7 and 3.10].

Definition 1.1 (eigenvalues of the truncated exterior Dirichlet problem). *We say that μ_ℓ is an* eigenvalue of the truncated exterior Dirichlet problem at frequency $k_\ell > 0$, *with corresponding eigenfunction u_ℓ, if $u_\ell \in H^1_{0,D}(\Omega_{\rm tr}) \setminus \{0\}$ and $\mu_\ell \in \mathbb{C}$ satisfies*

$$(\Delta + k_\ell^2)u_\ell = \mu_\ell u_\ell \quad \textit{in } \Omega_{\rm tr} \quad \textit{and} \quad \gamma_1^{\rm tr} u_\ell = \mathcal{D}(k_\ell)(\gamma_0^{\rm tr} u_\ell).$$

Definition 1.2 (quasimodes). *A* family of quasimodes of quality $\epsilon(k)$ *is a sequence $\{(u_\ell, k_\ell)\}_{\ell=1}^\infty \subset H^2(\Omega_{\rm tr}) \cap H^1_{0,D}(\Omega_{\rm tr}) \times \mathbb{R}$ such that the frequencies $k_\ell \to \infty$ as $\ell \to \infty$ and there is a compact subset $\mathcal{K} \Subset \Omega_1$ such that, for all ℓ,* supp $u_\ell \subset \mathcal{K}$,

$$\left\|(\Delta + k_\ell^2)u_\ell\right\|_{L^2(\Omega_{\rm tr})} \le \epsilon(k_\ell) \quad \textit{and} \quad \|u_\ell\|_{L^2(\Omega_{\rm tr})} = 1.$$

Remark 1.3. By [5, Theorem 2], we can assume that there exist $S_1, S_2 > 0$ such that $\epsilon(k) \ge S_1 \exp(-S_2 k)$.

Definition 1.4 (quasimodes with multiplicity). *Let $\{(u_\ell, k_\ell)\}_{\ell=1}^\infty$ be a quasimode with quality $\epsilon(k)$, and let $\{(m_j, k_j^-, k_j^+)\}_{j=1}^\infty \subset \mathbb{N} \times \mathbb{R}^2$ be such that $k_j^- \to \infty$ and $k_j^- \le k_j^+$. Define*

$$\mathcal{W}_j := \left\{\ell \,:\, k_\ell \in [k_j^-, k_j^+]\right\}.$$

We say that u_ℓ has multiplicity m_j in the window $[k_j^-, k_j^+]$ if

$$|\mathcal{W}_j| = m_j, \qquad |\langle u_{\ell_1}, u_{\ell_2}\rangle_{L^2(\Omega_{\rm tr})}| \le \epsilon(k_j^-) \quad \textit{for } \ell_1 \ne \ell_2,\ \ \ell_1, \ell_2 \in \mathcal{W}_j.$$

We assume throughout that the quality, $\epsilon(k)$, of a quasimode is a decreasing function of k; this can always be arranged by replacing $\epsilon(k)$ by $\tilde{\epsilon}(k) := \sup_{\tilde{k} \ge k} \epsilon(\tilde{k})$.

We use the notation that $A = \mathcal{O}(k^{-\infty})$ as $k \to \infty$ if, given $N > 0$, there exist C_N and k_0 such that $|A| \le C_N k^{-N}$ for all $k \ge k_0$, i.e., A decreases superalgebraically in k.

1.2. The main results.

Theorem 1.5 (from quasimodes to eigenvalues). *Let $\alpha > 3(d+1)/2$. Suppose there exists a family of quasimodes of quality $\epsilon(k)$ with*

$$\epsilon(k) \ll k^{1-\alpha}.$$

Then there exists $k_0 > 0$ (depending on α) such that if ℓ is such that $k_\ell \ge k_0$, then there exists an eigenvalue of the truncated exterior Dirichlet problem at frequency k_ℓ satisfying

$$|\mu_\ell| \le k_\ell^\alpha \epsilon(k_\ell).$$

We now give three specific cases when the assumptions of Theorem 1.5 hold. The first two cases are via the quasimode constructions of [4, Theorem 2.8, equations 2.20 and 2.21] and [7, Theorem 1] for obstacles whose exteriors support elliptic-trapped rays. The third case is via the "resonances to quasimodes" result of [44, Theorem 1]; recall that the resonances of the exterior Dirichlet problem are the poles of the meromorphic continuation of the solution operator from $\operatorname{Im} k \ge 0$ to $\operatorname{Im} k < 0$; see, e.g., [15, Theorem 4.4. and Definition 4.6].

LEMMA 1.6 (specific cases when the assumptions of Theorem 1.5 hold).

(i) *Let* $d = 2$. *Given* $a_1 > a_2 > 0$, *let*

$$E := \left\{ (x_1, x_2) \, : \, \left(\frac{x_1}{a_1}\right)^2 + \left(\frac{x_2}{a_2}\right)^2 < 1 \right\}. \tag{1.2}$$

If Γ_D *coincides with the boundary of* E *in the neighborhoods of the points* $(0, \pm a_2)$, *and if* Ω_+ *contains the convex hull of these neighborhoods, then the assumptions of Theorem* 1.5 *hold with*

$$\epsilon(k) = \exp(-C_1 k)$$

for some $C_1 > 0$ *(independent of* k*).*[1]

(ii) *Suppose* $d \geq 2$, $\Gamma_D \in C^\infty$, *and* Ω_+ *contains an elliptic-trapped ray such that* (a) Γ_D *is analytic in a neighborhood of the ray, and* (b) *the ray satisfies the stability condition* [7, (H1)]. *If* $q > 11/2$ *when* $d = 2$ *and* $q > 2d + 1$ *when* $d \geq 3$, *then the assumptions of Theorem* 1.5 *hold with*

$$\epsilon(k) = \exp(-C_2 k^{1/q})$$

for some $C_2 > 0$ *(independent of* k*).*

(iii) *Suppose there exists a sequence of resonances* $\{\lambda_\ell\}_{\ell=1}^\infty$ *of the exterior Dirichlet problem with*

$$0 \leq -\operatorname{Im} \lambda_\ell = \mathcal{O}\big(|\lambda_\ell|^{-\infty}\big) \quad \textit{and} \quad \operatorname{Re} \lambda_\ell \to \infty \quad \textit{as} \quad \ell \to \infty. \tag{1.3}$$

Then there exists a family of quasimodes of quality $\epsilon(k) = \mathcal{O}(k^{-\infty})$, *and thus the assumptions of Theorem* 1.5 *hold.*

Remark 1.7 (resonances $\iff$ quasimodes $\iff$ eigenvalues). Part (iii) of Lemma 1.6 is the "resonances to quasimodes" result of [44, Theorem 1]. The converse implication, i.e., that a family of quasimodes of quality $\epsilon(k) = \mathcal{O}(k^{-\infty})$ implies a sequence of resonances satisfying (1.3), was proved in [47, 43] (following [45, 46]); see also [15, Theorem 7.6]. Therefore, the "quasimodes to eigenvalues" result of Theorem 1.5 is equivalent to a "resonances to eigenvalues" result. In fact, in Appendix A we show that the existence of $\mathcal{O}(k^{-\infty})$ eigenvalues implies the existence of quasimodes of quality $\mathcal{O}(k^{-\infty})$. We therefore have that resonances $\iff$ quasimodes $\iff$ eigenvalues.

With $\{\mu_j(k)\}_j$ the set of eigenvalues, counting multiplicities, of the truncated exterior Dirichlet problem at frequency k (with $\mu_j(k)$ depending continuously on k for each j), let

$$\mathcal{E}(\varepsilon_1, \varepsilon_0, k_-, k_+) := \Big\{ j \, : \, \mu_j(k) \in (-2\varepsilon_1, 2\varepsilon_1) - \mathrm{i}(0, 2\varepsilon_0) \text{ for some } k \in [k_-, k_+] \Big\}; \tag{1.4}$$

$|\mathcal{E}|$ is therefore the counting function of the eigenvalues, $\mu_j(k)$, that pass through a rectangle next to zero in μ as k varies in the interval $[k_-, k_+]$; see Figure 1.[2]

[1] In [4, Theorem 2.8], Ω_+ is assumed to contain the whole ellipse E. However, inspecting the proof, we see that the result remains unchanged if E is replaced with the convex hull of the neighborhoods of $(0, \pm a_2)$. Indeed, the idea of the proof is to consider a family of eigenfunctions of the ellipse localizing around the periodic orbit $\{(0, x_2) : |x_2| \leq a_2\}$.

[2] In Figure 1, we have drawn the paths of the eigenvalues as arbitrary curves. We see later in Figure 7 an example where the paths appear to be horizontal lines; this is consistent with the intuition that eigenvalues should be shifted resonances.

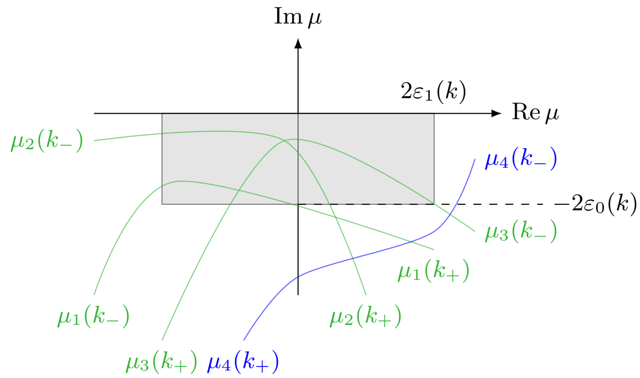


FIG. 1. *Paths of the eigenvalues, μ_j, of the truncated problem are shown as functions of $k \in [k_-, k_+]$. Those eigenvalues shown in green correspond to members of the box $\mathcal{E}$ defined by* (1.4) *(shaded), while the eigenvalue in blue is not in $\mathcal{E}$. (Color is available online only.)*

THEOREM 1.8 (from quasimodes to eigenvalues, with multiplicities). *Let $k_j^-, k_j^+ \to \infty$ such that there exists $C > 0$ satisfying $k_j^- \le k_j^+ \le Ck_j^-$. Suppose there exists a family of quasimodes of quality $\epsilon(k) \ll k^{-(5d+3)/2}$ and multiplicity m_j in the window $[k_j^-, k_j^+]$ (in the sense of Definition* 1.4*). If $\epsilon_0(k)$ is such that, for some $S > 0$,*

$$\epsilon_0(k) \le Sk^{-(d+1)/2} \text{ for all } k \quad \text{and} \quad \epsilon_0(k) \gg k^{2d+1}\epsilon(k) \text{ as } k \to \infty,$$

then there exists $k_0 > 0$ such that if $k_j^- \ge k_0$, then

$$\left|\mathcal{E}\Big((k_j^-)^{(d+1)/2}\epsilon_0(k_j^-)\,,\, \epsilon_0(k_j^-)\,,\, k_j^-\,,\, k_j^+\Big)\right| \ge m_j.$$

Observe that if $k_j^+ = k_j^-$, then (up to algebraic powers of k) Theorem 1.8 reduces to Theorem 1.5, except that now multiplicities are counted; therefore, the "quasimodes to eigenvalues" result holds with multiplicities (just as the "quasimodes to resonances" result of [43] includes multiplicities).

The ideas used in the proofs of Theorems 1.5 and 1.8 are discussed in section 1.5 below.

Remark 1.9. The reason why both the constant α in Theorem 1.5 and the exponent in the bound on the quality in Theorem 1.8 depend on d is because the right-hand side of the bound (1.15) below on the solution operator of the truncated problem depends on d, which in turn comes from the fact that the trace-class norm of compactly supported pseudodifferential operators depends on d.

1.3. Numerical experiments illustrating the main results.

Description of the obstacles Ω_-. In this section, Ω_- is one of the two "horseshoe-shaped" 2-d domains shown in Figure 2. We define the *small cavity* as the region between the two elliptic arcs

$$(\cos(t), 0.5\sin(t)), \quad t \in [-\phi_0, \phi_0], \quad \text{and} \quad (1.3\cos(t), 0.6\sin(t)), \quad t \in [-\phi_1, \phi_1],$$
$$\text{with } \phi_0 = 7\pi/10 \quad \text{and} \quad \phi_1 = \arccos\left(\frac{1}{1.3}\cos(\phi_0)\right);$$

this corresponds to the interior of the solid lines in Figure 2. We define the *large cavity* as the region between the two arcs now with $\phi_0 = 9\pi/10$. (Note that our small cavity

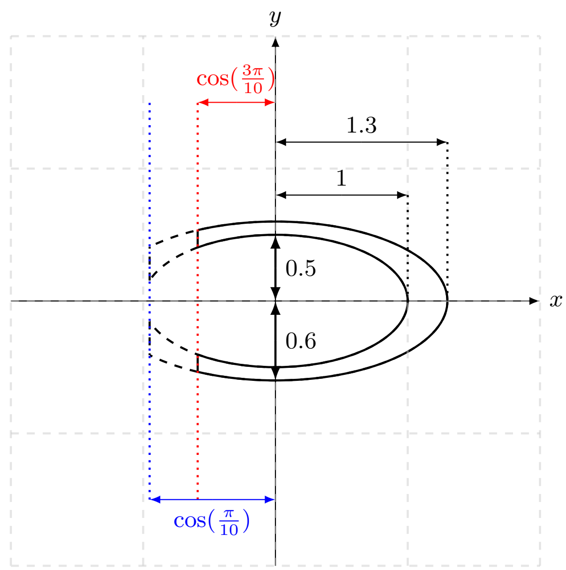


FIG. 2. *The two obstacles Ω_- considered in the numerical experiments.*

is the same as the cavity considered in the numerical experiments in [4, section IV].) Recall that Theorems 1.5 and 1.8 require Γ_D to be smooth, and thus these results do not strictly apply to the small and large cavities; however, they do apply to smoothed versions of these.

For both the small and the large cavities, Γ_D coincides with the boundary of the ellipse E (1.2) with $a_1 = 1$ and $a_2 = 0.5$ in the neighborhood of its minor axis. Part (i) of Lemma 1.6 (i.e., the results of [4]) then implies that there exist quasimodes with exponentially small quality.

We choose these particular Ω_- because we can compute the frequencies k_ℓ in the quasimode. Indeed, the functions u_ℓ in the quasimode construction in [4] are based on the family of eigenfunctions of the ellipse localizing around the periodic orbit $\{(0, x_2) : |x_2| \le a_2\}$; when the eigenfunctions are sufficiently localized, the eigenfunctions multiplied by a suitable cut-off function form a quasimode, with frequencies k_ℓ equal to the square roots of eigenvalues of the ellipse. By separation of variables, k_ℓ can be expressed as the solution of a multiparametric spectral problem involving Mathieu functions; see see [4, Appendix A] and [38, Appendix E].

When giving specific values of k_ℓ below, we use the notation from [4, Appendix A] and [38, Appendix E] that $k^e_{m,n}$ and $k^o_{m,n}$ are the frequencies associated with the eigenfunctions of the ellipse that are even/odd, respectively, in the angular variable, with m zeros in the radial direction (other than at the center or the boundary) and n zeros in the angular variable in the interval $[0, \pi)$.

Plots of the eigenvalues and eigenfunctions. Figures 3 and 4 plot the near-zero eigenvalues of the truncated exterior Dirichlet problem for the small and large cavities, respectively, at frequencies corresponding to eigenvalues of the ellipse. Figures 5 and 6 plot the corresponding eigenfunctions. In all these figures, $\Gamma_{\rm tr} = \partial B(0, 2)$.

Figure 4 shows that the large cavity has an eigenvalue very close to zero at each of the four frequencies considered, qualitatively illustrating Theorem 1.5. In contrast, Figure 3 shows that the small cavity only has an eigenvalue very close to zero at the

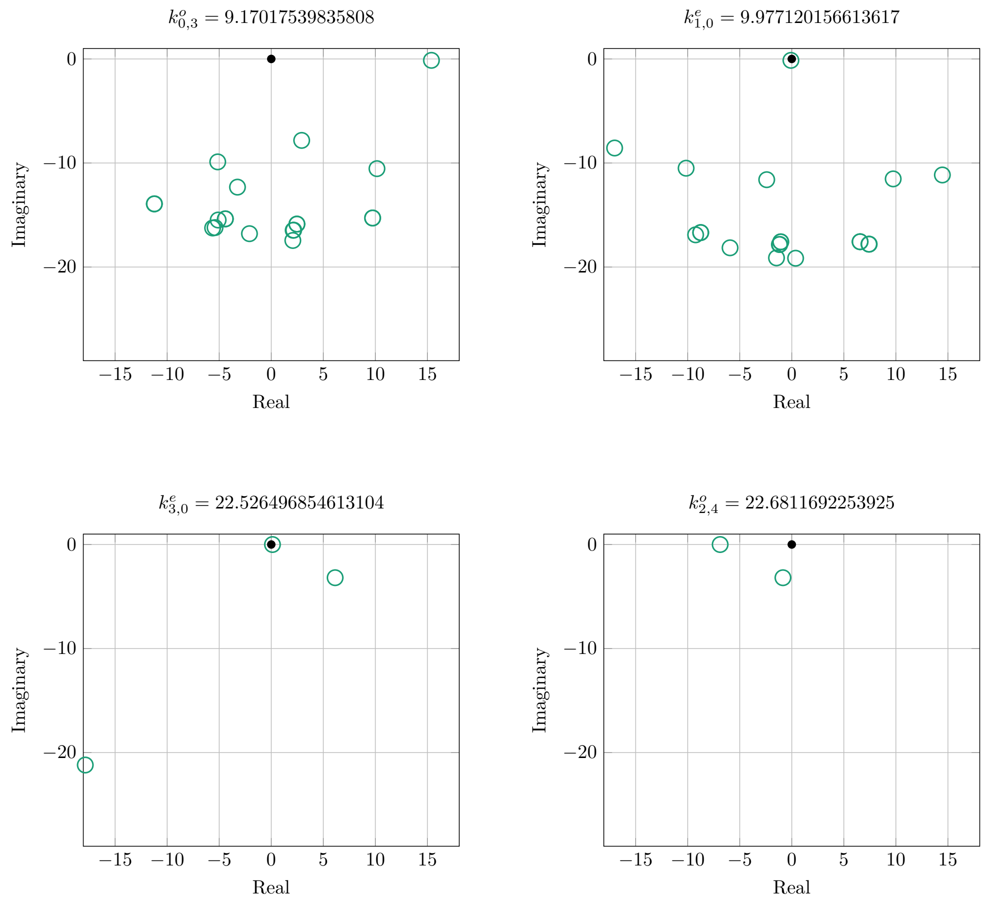


FIG. 3. *The eigenvalues of the truncated exterior Dirichlet problem (Definition* 1.1*) near the origin when* Γ_D *is equal to the small cavity. The eigenvalues are plotted at several frequencies,* k*, corresponding to eigenvalues of the ellipse. In each plot, the origin is marked with a black dot, and the eigenvalues are shown as green circles. (Color is available online only.)*

frequencies $k^e_{1,0}$ and $k^e_{3,0}$ (top right and bottom left in the figures) and not at $k^e_{0,3}$ and $k^o_{2,4}$ (top left and bottom right). The reason for this is clear from the plots of the eigenfunctions of the truncated exterior Dirichlet problem: looking at Figure 5, we see that at $k^e_{0,3}$ and $k^o_{2,4}$ the eigenfunctions are not well localized around the minor axis of the ellipse to be inside the small cavity—in the top left and bottom right of Figure 5, we see them "leaking out" of the small cavity. However, looking at Figure 6, we see that the corresponding eigenfunctions are localized sufficiently to be inside the large cavity and thus generate an eigenvalue very close to zero. In these plots, the eigenfunctions are normalized so that their $L^2(\Omega_{\rm tr})$ norm equals one.

Figure 7 plots the trajectories of the near-zero eigenvalues as functions of k for both the small cavity (left plot) and the large cavity (right plot) for $k \in (2.5, 12.5)$, with the spectra computed every 0.025. For Figure 7, $\Gamma_{\rm tr} = \partial B(0, 1.5)$; this change (compared to $\Gamma_{\rm tr} = \partial B(0, 2)$ for the earlier figures) is to reduce the cost of each eigenvalue solve because each of the two plots in Figure 7 requires 400 such solves. Since we use the exact (up to discretization error) Dirichlet-to-Neumann map on $\Gamma_{\rm tr}$, we expect there to be no difference between choosing $\Gamma_{\rm tr} = \partial B(0, 1.5)$ and $\Gamma_{\rm tr} =$

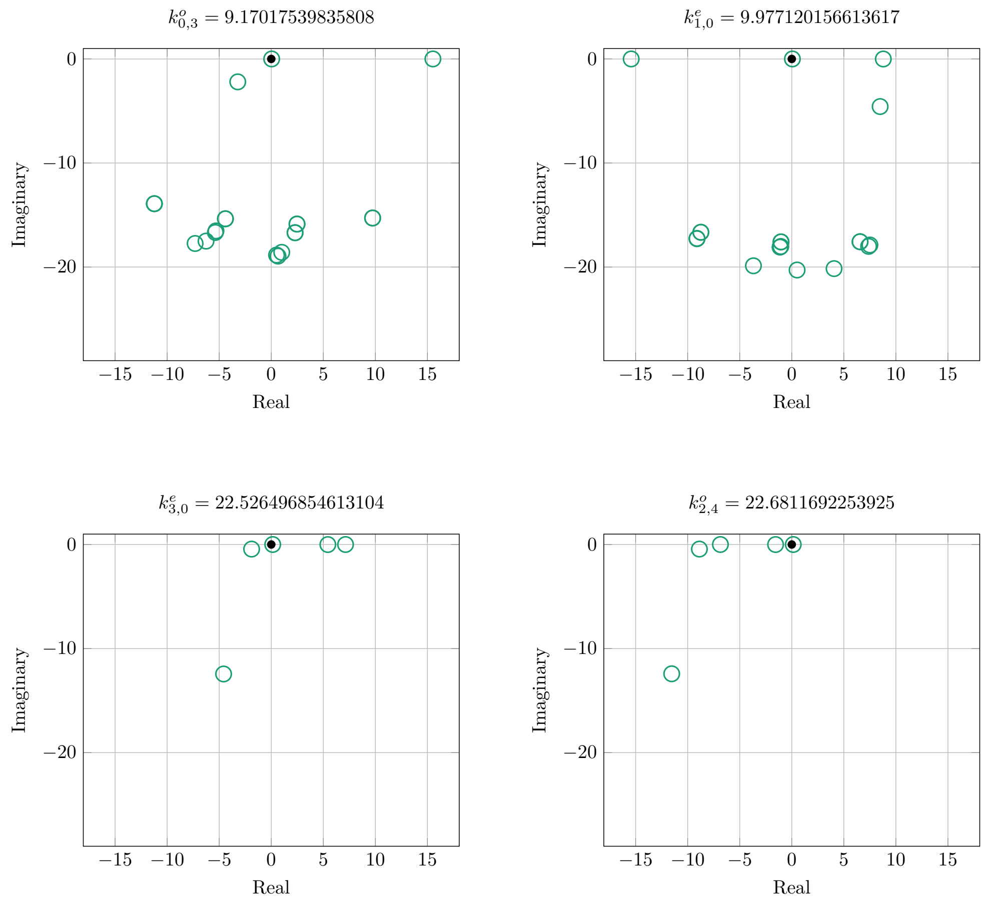


FIG. 4. *The eigenvalues of the truncated exterior Dirichlet problem (Definition* 1.1*) near the origin when* Γ_D *is equal to the large cavity. The eigenvalues are plotted at several frequencies,* k*, corresponding to eigenvalues of the ellipse. In each plot, the origin is marked with a black dot, and the eigenvalues are shown as green circles. (Color is available online only.)*

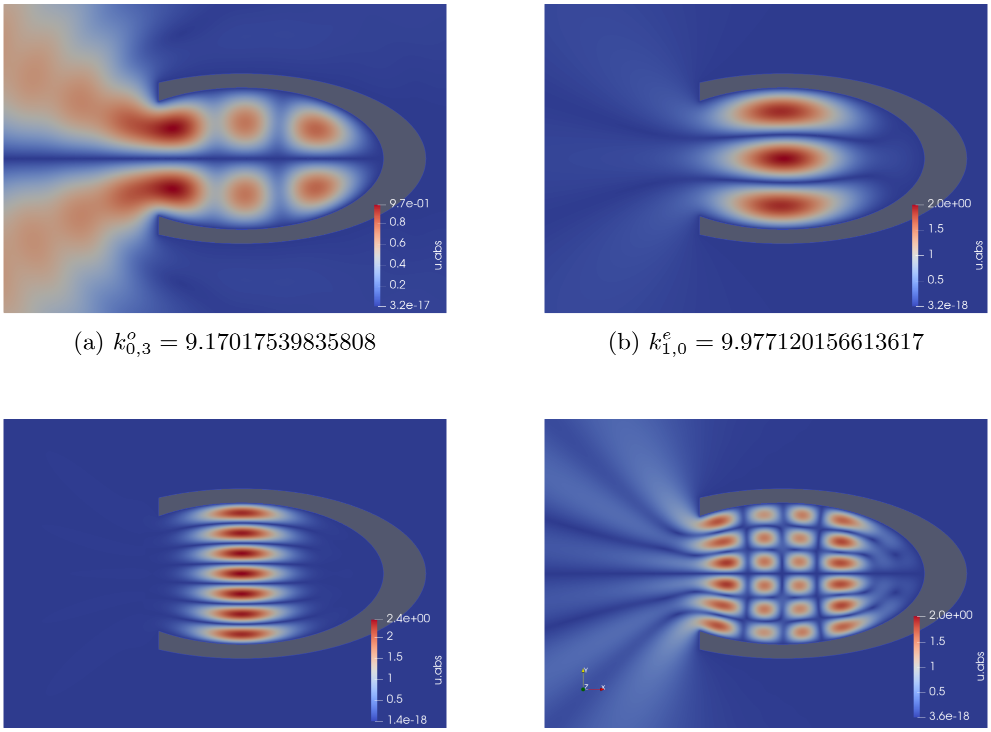
$\partial B(0,2)$ (in particular, Figures 3 and 4 are unchanged when $\Gamma_{\rm tr}$ is changed from $\partial B(0,2)$ to $\partial B(0,1.5)$).

The eigenvalues that enter the red rectangle in Figure 7 are colored green; these are members of $\mathcal{E}(0.2, 0.05, 2.5, 12.5)$, where $\mathcal{E}$ is defined by (1.4). Similar to the eigenvalues plots in Figures 3 and 4, Figure 7 shows that the large cavity has more near-zero eigenvalues for the range of k considered than the small cavity. This is expected since a larger number of the eigenfunctions of the ellipse are localized in the large cavity than in the small cavity.

How the eigenvalues and eigenfunctions were computed. Definition 1.1 (of the eigenvalues of the truncated Dirichlet problem) implies that if μ_ℓ is an eigenvalue at frequency k_ℓ, and with corresponding eigenfunction u_ℓ, then

$$a(u_\ell, v) = \mu_\ell (u_\ell, v)_{L^2(\Omega_{\rm tr})} \quad \text{for all } v \in H^1_{0,D}(\Omega_{\rm tr}), \tag{1.5}$$

where the sesquilinear form $a(\cdot,\cdot)$ is that appearing in the standard variational (i.e., weak) formulation of the Helmholtz exterior Dirichlet problem.



(a) $k^o_{0,3} = 9.17017539835808$ (b) $k^e_{1,0} = 9.977120156613617$

(c) $k^e_{3,0} = 22.526496854613104$ (d) $k^o_{2,4} = 22.6811692253925$

FIG. 5. *Absolute value of the eigenfunction of the truncated exterior Dirichlet problem associated with the smallest eigenvalue for the small cavity.*

DEFINITION 1.10 (variational formulation of Helmholtz exterior Dirichlet problem). *Given $k > 0$, Ω_- as above, and $F \in (H^1_{0,D}(\Omega_{\rm tr}))^*$, let $u \in H^1_{0,D}(\Omega_{\rm tr})$ be the solution of the following variational problem:*

$$\text{find } u \in H^1_{0,D}(\Omega_{\rm tr}) \quad \text{such that} \quad a(u,v) = F(v) \quad \text{for all } v \in H^1_{0,D}(\Omega_{\rm tr}), \tag{1.6}$$

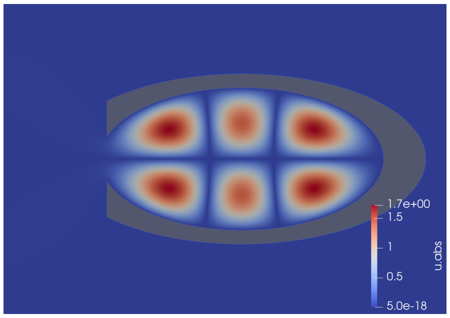
where

$$a(u,v) := \int_{\Omega_{\rm tr}} \left(\nabla u \cdot \overline{\nabla v} - k^2 u\overline{v}\right) - \big\langle \mathcal{D}(k)(\gamma_0^{\rm tr} u), \gamma_0^{\rm tr} v\big\rangle_{\Gamma_{\rm tr}}, \tag{1.7}$$

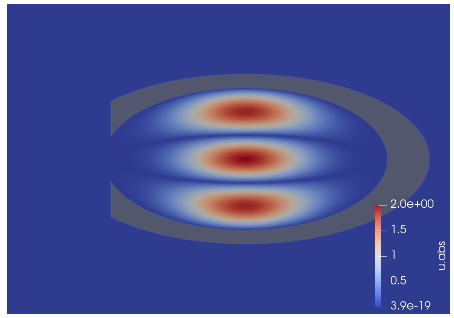
where $\langle\cdot,\cdot\rangle_{\Gamma_{\rm tr}}$ denotes the duality pairing on $\Gamma_{\rm tr}$ that is linear in the first argument and antilinear in the second.

The figures above were created by solving the eigenvalue problem (1.5) using the finite-element method with continuous piecewise-linear elements (i.e., the polynomial degree, p, equals one) and meshwidth h equal to $(2\pi/30)k^{-3/2}$. The Dirichlet-to-Neumann map, $\mathcal{D}(k)$, in $a(\cdot,\cdot)$ was computed using boundary integral equations—see Appendix B for details. The accuracy, uniform in frequency, of the finite-element method applied the variational problem (1.6) with $p = 1$ and $hk^{3/2}$ sufficiently small has been known empirically for a long time and was recently proved in [34] for the case when the Dirichlet-to-Neumann map is realized exactly.

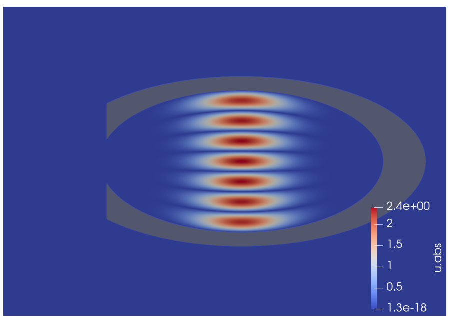
Since computing the Dirichlet-to-Neumann map is relatively expensive, in practice one often approximates it using a perfectly matched layer (PML) or an absorb-



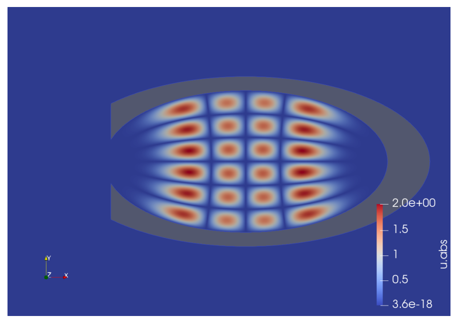
(a) $k^o_{0,3} = 9.17017539835808$



(b) $k^e_{1,0} = 9.977120156613617$



(c) $k^e_{3,0} = 22.526496854613104$



(d) $k^o_{2,4} = 22.6811692253925$

FIG. 6. *Absolute value of the eigenfunction of the truncated exterior Dirichlet problem associated with the smallest eigenvalue for the large cavity.*

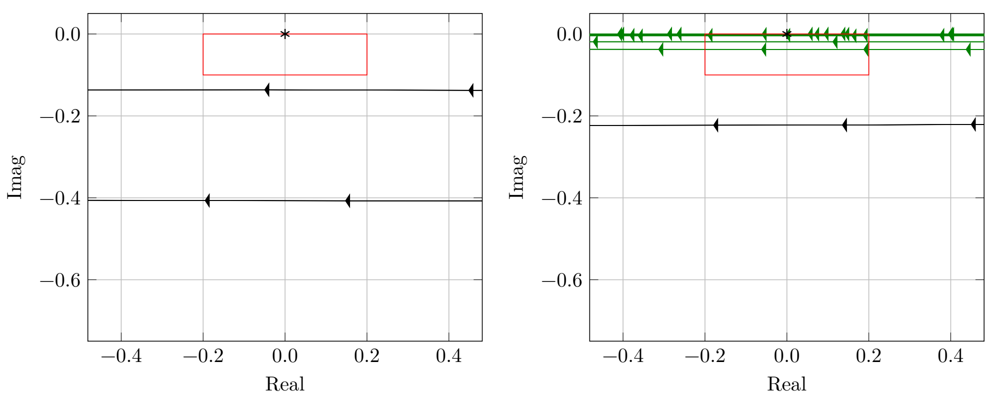


FIG. 7. *Paths of the eigenvalues for* $k \in (2.5, 12.5)$ *for the small cavity (left) and the large cavity (right). The eigenvalues that enter the red rectangle are colored green. (Color is available online only.)*

ing boundary condition (such as the impedance boundary condition). The plots of the eigenfunctions and near-zero eigenvalues of the corresponding truncated exterior Dirichlet problems are very similar to those above; this too is expected since the quasimode is supported in a neighborhood of the obstacle.

1.4. Implications of the main results for numerical analysis of the Helmholtz exterior Dirichlet problem. Theorems 1.5 and 1.8 are the first step towards rigorously understanding how iterative solvers such as the generalized minimum residual method (GMRES) behave when applied to discretizations of high-frequency Helmholtz problems under strong trapping (the subject of the companion paper [38]). We now explain this in more detail.

As we saw in (1.5), the eigenvalues of truncated exterior Dirichlet problem (in the sense of Definition 1.1) correspond to eigenvalues of the sesquilinear form of the standard variational formulation (Definition 1.10). The standard variational formulation is the basis of the finite-element method for computing approximations to the solution of the variational problem (1.6). Indeed, the finite-element method consists of choosing a piecewise-polynomial subspace of $H^1_{0,D}(\Omega_{\rm tr})$ and solving the variational problem (1.6) in this subspace.

A very popular way of solving the linear systems resulting from the finite-element method applied to the Helmholtz scattering problems is via iterative solvers such as GMRES [42]; this choice is made because the linear systems are (i) large and (ii) non-self-adjoint. Regarding (i), the systems are large since the number of degrees of freedom must be $\gtrsim k^d$ to resolve the oscillations in the solution; see, e.g., the literature review in [34, section 1.1]. Regarding (ii), the non-self-adjointness of the linear systems arises directly from the non-self-adjointness of the underlying Helmholtz scattering problem; GMRES is applicable to such systems, unlike the conjugate gradient method.

There is currently large research interest in understanding how iterative methods behave when applied to Helmholtz linear systems and in designing good preconditioners for these linear systems; see the literature reviews [19, 21, 25], [27, section 1.3].

The location of eigenvalues, especially near-zero ones, is crucial in understanding the behavior of iterative methods. In the Helmholtz context, eigenvalue analyses of iterative methods applied to nontrapping problems include, for finite-element discretizations, [17, 16, 20, 49, 21, 51, 11, 37] and, for boundary-element discretizations, [10, 12, 8].

The paper [38] analyzes GMRES applied to discretizations of Helmholtz problems with strong trapping, using the "cluster plus outliers" GMRES convergence theory from [6] (with this idea arising in the context of the conjugate gradient method [32] and used subsequently in, e.g., [18]). The paper [38] obtains bounds on how the number of GMRES iterations depends on the frequency under various assumptions about the eigenvalues. In particular, Theorem 1.5 rigorously justifies [38, Observation O2(b)] for the standard variational formulation of the truncated exterior Dirichlet problem. We highlight that, although the results in [38] are about unpreconditioned systems, they give insight into the design of preconditioners. Indeed, a successful preconditioner for Helmholtz problems with strong trapping will need to specifically deal with the near-zero eigenvalues created by trapping. Theorems 1.5 and 1.8 give information about the location and multiplicities of these eigenvalues, and [38] shows how these locations and multiplicities affect GMRES.

1.5. The ideas behind the proof of Theorem 1.5.

Semiclassical notation. Instead of working with the parameter k and being interested in the large-k limit, the semiclassical literature usually works with a parameter $h := k^{-1}$ and is interested in the small-h limit. So that we can easily recall results from this literature, we also work with the small parameter k^{-1}, but to avoid a notational clash with the meshwidth of the finite-element method, we let $\hbar := k^{-1}$ (the notation $\hbar$ comes from the fact that the semiclassical parameter is sometimes related to Planck's constant, which is written as $2\pi\hbar$; see, e.g., [52, section 1.2]). Theorem 1.5 is then restated in semiclassical notation as Theorem 2.2 below.

The solution operator of the truncated problem. Let $R_{\Omega_{\rm tr}}(\lambda, z) : L^2(\Omega_{\rm tr}) \to L^2(\Omega_{\rm tr})$ be the solution operator for the truncated problem

$$\begin{cases} (-\hbar^2\Delta - \lambda^2 - z)u = f & \text{in } \Omega_{\rm tr}, \\ \gamma_0^D u = 0, \\ \gamma_1^{\rm tr} u = \mathcal{D}(\lambda/\hbar)\gamma_0^{\rm tr} u; \end{cases} \tag{1.8}$$

that is, $R_{\Omega_{\rm tr}}(\lambda, z)$ satisfies

$$\begin{cases} (-\hbar^2\Delta - \lambda^2 - z)R_{\Omega_{\rm tr}}(\lambda, z)f = f & \text{in } \Omega_{\rm tr}, \\ \gamma_0^D R_{\Omega_{\rm tr}}(\lambda, z)f = 0, \\ \gamma_1^{\rm tr} R_{\Omega_{\rm tr}}(\lambda, z)f = \mathcal{D}(\lambda/\hbar)\gamma_0^D R_{\Omega_{\rm tr}}(\lambda, z)f. \end{cases}$$

Note that, at this point, it is not clear that the problem (1.8) is well posed and that the family of operators $R_{\Omega_{\rm tr}}(\lambda, z)$ is well defined. We address this in Lemma 1.11 below.

We study $R_{\Omega_{\rm tr}}(\lambda, z)$ by relating it to the solution operator of a more standard scattering problem. Namely, let $V \in L^\infty(\Omega_+)$ with $\operatorname{supp} V \Subset \mathbb{R}^d$, and consider the problem

$$\begin{cases} (-\hbar^2\Delta - \lambda^2 + V)u = f & \text{on } \Omega_+, \\ \gamma_0^D u = 0, \\ u \text{ is } \lambda/\hbar \text{ outgoing}. \end{cases} \tag{1.9}$$

By, e.g., [15, Chapter 4], the inverse of (1.9) is a meromorphic family of operators (for $\lambda \in \mathbb{C}$ when d is odd or λ in the logarithmic cover of $\mathbb{C} \setminus \{0\}$ when d is even) $R_V(\lambda) : L^2_{\rm comp}(\Omega_+) \to L^2_{\rm loc}(\Omega_+)$ with finite-rank poles satisfying

$$\begin{cases} (-\hbar^2\Delta - \lambda^2 + V)R_V(\lambda)f = f & \text{in } \Omega, \\ \gamma_0^D R_V(\lambda)f = 0, \\ R_V(\lambda)f \text{ is } \lambda/\hbar \text{ outgoing}. \end{cases} \tag{1.10}$$

Observe that, although both $R_{\Omega_{\rm tr}}(\lambda, z)$ and $R_V(\lambda)$ depend on $\hbar$, we omit this dependence in the notation to keep expressions compact.

The following two lemmas (proved in section 2.2) relate $R_{\Omega_{\rm tr}}(\lambda, z)$ and $R_V(\lambda)$ and then characterize the eigenvalues of the truncated exterior Dirichlet problem as poles of $R_{\Omega_{\rm tr}}(\lambda, z)$ as a function of z.

We use three indicator functions: $1_{\Omega_{\rm tr}}$ denotes the function in $L^\infty(\Omega_+)$ that is one on $\Omega_{\rm tr}$ and zero otherwise, $1^{\rm res}_{\Omega_{\rm tr}}$ denotes the restriction operator $L^2(\Omega_+) \to L^2(\Omega_{\rm tr})$, and $1^{\rm ext}_{\Omega_{\rm tr}}$ denotes the extension-by-zero operator $L^2(\Omega_{\rm tr}) \to L^2(\Omega_+)$.

LEMMA 1.11. *Define*

$$R(\lambda, z) := R_V(\lambda) \quad \textit{with } V(z) = -z 1_{\Omega_{\rm tr}}. \tag{1.11}$$

Then

$$R_{\Omega_{\rm tr}}(\lambda, z) = 1^{\rm res}_{\Omega_{\rm tr}} R(\lambda, z) 1^{\rm ext}_{\Omega_{\rm tr}}, \tag{1.12}$$

and thus $R_{\Omega_{\rm tr}}(\lambda, z)$ is a meromorphic family of operators in λ for $\lambda \in \mathbb{C}$ when d is odd and λ in the logarithmic cover of $\mathbb{C} \setminus \{0\}$ when d is even.

LEMMA 1.12. *For $\lambda \in \mathbb{R}\setminus\{0\}$, $z \mapsto R(\lambda, z)$ is a meromorphic family of operators $L^2_{\rm comp}(\Omega_+) \to L^2_{\rm loc}(\Omega_+)$ with finite rank poles.*

COROLLARY 1.13. *If z_j is a pole of $z \mapsto R_{\Omega_{\rm tr}}(1, z)$, then $\mu_\ell := -\hbar_j^{-2} z_j$ is an eigenvalue of the truncated exterior Dirichlet problem (in the sense of Definition* 1.1*).*

The key point is that we are interested in $R_{\Omega_{\rm tr}}(\lambda, z)$ as a meromorphic family in the variable z, in contrast to the more familiar study of $R_V(\lambda)$ as a meromorphic family in the variable λ.

Recap of "from quasimodes to resonances." Recall that resonances of $-\hbar^2\Delta + V$ are defined as poles of the meromorphic continuation of $R_V(w)$ into $\operatorname{Im} w < 0$; see [15, sections 4.2 and 7.2]. The "quasimodes to resonances" argument of [47] (following [45, 46]; see also [15, Theorem 7.6]) shows that existence of quasimodes (as in Definition 1.2) implies existence of resonances close to the real axis; the additional arguments in [43] then prove the corresponding result with multiplicities.

These arguments use the *semiclassical maximum principle* (a consequence of the maximum principle of complex analysis; see Theorem 2.7 below) combined with the bounds

(1.13)
$$\|\chi R_V(\lambda)\chi\|_{L^2\to L^2} \le C\exp\Big(C\hbar^{-d}\log\delta^{-1}\Big), \qquad \lambda^2 \in \Omega \Big\backslash \bigcup_{w\in \operatorname{Res}(-\hbar^2\Delta+V)} B(w,\delta),$$

for $\Omega \Subset \{\operatorname{Re} w > 0\}$, and

$$\|R_V(\lambda)\|_{L^2\to L^2} \le \frac{1}{\operatorname{Im}(\lambda^2)} \quad \text{for } \operatorname{Im}(\lambda^2) > 0; \tag{1.14}$$

see [47, Lemma 1], [48, Proposition 4.3], and [15, Theorem 7.5].

From quasimodes to eigenvalues. Theorems 1.5 and 1.8 are proved using the same ideas as in the "quasimodes to resonances" arguments, except that now we work in the complex z-plane (with real λ) instead of the complex λ-plane. The analogues of the bounds (1.13) and (1.14) are given in the following lemma.

LEMMA 1.14 (bounds on $R_{\Omega_{\rm tr}}(\lambda, z)$). *Let $0 < a < b$, and let $z_j(\hbar, \lambda)$ be the poles of $R_{\Omega_{\rm tr}}(\lambda, z)$ (as a meromorphic function of z). Then there exist $C_1, \varepsilon_1 > 0$ such that for all $0 < \hbar < 1$, $\lambda^2 \in [a, b]$, and $\delta > 0$,*

$$\begin{aligned}\|R_{\Omega_{\rm tr}}(\lambda, z)\|_{L^2(\Omega_{\rm tr})\to L^2(\Omega_{\rm tr})} &\le \exp\Big(C_1\hbar^{-d}\log\delta^{-1}\Big)\\ &\quad \text{for } z \in B(0, \varepsilon_1\hbar)\Big\backslash \bigcup_j B(z_j(\hbar,\lambda),\delta).\end{aligned} \tag{1.15}$$

Furthermore, there exists $C_2 > 0$ such that

$$\|R_{\Omega_{\rm tr}}(\lambda, z)\|_{L^2(\Omega_{\rm tr})\to L^2(\Omega_{\rm tr})} \le C_2\frac{\langle z\rangle}{\operatorname{Im} z} \quad \text{for } \operatorname{Im} z > 0, \tag{1.16}$$

where $\langle z\rangle := (1 + |z|^2)^{1/2}$.

The bound (1.15) is proved by finding a parametrix for $-\hbar^2\Delta - \lambda^2 - z1_{\Omega_{\rm tr}}$ (i.e., an approximation to $R_{\Omega_{\rm tr}}(\lambda, z)$) via a boundary complex absorbing potential. While parametrices based on complex absorption are often used in scattering theory (see, e.g., [14, 13], [15, Theorem 7.4]), parametrices based on boundary complex absorption appear to be new in the literature. One of the main features of the argument below is that it relies on a comparison of the (in principle, trapping) billiard flow with the nontrapping free flow to obtain estimates on the parametrix. A similar argument should work for boundaries in *any* nontrapping background.

We also highlight that, while we consider the scattering by Dirichlet obstacles in this paper and therefore must use boundary complex absorption, smooth compactly supported perturbations of $-\Delta$, e.g., metric perturbations or semiclassical Schrödinger operators, can be handled similarly. Indeed, for these problems, the parametrix based on boundary absorption could be replaced by one based on simpler complex absorbing potentials.

1.6. Outline of the rest of the paper. In section 2, we prove Lemmas 1.11 and 1.12 and then collect preliminary results about the generalized bicharacteristic flow (section 2.4), the geometry of trapping (section 2.5), complex scaling (section 2.6), and defect measures (section 2.8). In section 3, we find a parametrix for $R_{\Omega_{\rm tr}}(\lambda, z)$ via a boundary complex absorbing potential. In section 4, we prove Lemma 1.14. In section 5, we prove Theorems 1.5 and 1.8 using Lemma 1.14 and the semiclassical maximum principle.

2. Preliminary results.

2.1. Restatements of Theorems 1.5 and 1.8 in semiclassical notation.

DEFINITION 2.1 (quasimodes in $\hbar$ notation). *A family of quasimodes of quality $\varepsilon(\hbar)$ is a sequence $\{(u_\ell, \hbar_\ell)\}_{\ell=1}^\infty \subset H^2(\Omega_{\rm tr}) \cap H^1_{0,D}(\Omega_{\rm tr}) \times \mathbb{R}$ such that $\hbar_\ell \to 0$ as $\ell \to \infty$ and there exists a compact subset $\mathcal{K} \Subset \Omega_1$ such that, for all ℓ,* $\operatorname{supp} u_\ell \subset \mathcal{K}$,

$$\big\|(-\hbar^2\Delta - 1)u_\ell\big\|_{L^2(\Omega_{\rm tr})} \le \varepsilon(\hbar_\ell) \quad \text{and} \quad \|u_\ell\|_{L^2(\Omega_{\rm tr})} = 1.$$

Let

$$\varepsilon(\hbar) := \hbar^2 \epsilon(\hbar^{-1}). \tag{2.1}$$

Remark 1.3 implies that we can assume that there exist $\widetilde{S}_1, \widetilde{S}_2 > 0$ such that

$$\varepsilon(\hbar) \ge \widetilde{S}_1 \exp(-\widetilde{S}_2/\hbar). \tag{2.2}$$

Theorem 1.5 is then equivalent to the following result in the sense that the following result holds if and only if Theorem 1.5 holds with $\mu_\ell := \hbar_\ell^{-2} z_\ell$.

THEOREM 2.2 (analogue of Theorem 1.5 in $\hbar$ notation). *Let $\alpha > 3(d+1)/2$. Suppose there exists a family of quasimodes in the sense of Definition 2.1 such that the quality $\varepsilon(\hbar)$ satisfies*

$$\varepsilon(\hbar) \ll \hbar^{1+\alpha}. \tag{2.3}$$

Then there exists $\hbar_0 > 0$ (depending on α) such that if ℓ is such that $\hbar_\ell \le \hbar_0$, then there exists $z_\ell \in \mathbb{C}$ and $0 \ne u_\ell \in H^1_{0,D}(\Omega_{\rm tr})$ with

$$(-\hbar_\ell^2\Delta - 1 + z_\ell)u_\ell = 0 \text{ in } \Omega_{\rm tr}, \quad \gamma_1^{\rm tr} u_\ell = \mathcal{D}(\hbar_\ell^{-1})(\gamma_0^{\rm tr} u_\ell), \quad \text{and} \quad |z_\ell| \le \hbar_\ell^{-\alpha}\varepsilon(\hbar_\ell). \tag{2.4}$$

DEFINITION 2.3 (quasimodes with multiplicity in $\hbar$ notation). *Let $0 \le a(\hbar) \le b(\hbar) < \infty$ be two functions of $\hbar$. A* family of quasimodes of quality $\varepsilon(\hbar)$ and multiplicity $m(\hbar)$ *in the window $[a(\hbar), b(\hbar)]$ is a sequence $\{\hbar_j\}_{j=1}^\infty$ such that $\hbar_j \to 0$ as $j \to \infty$*

and for every j there exist $\{(u_{j,\ell}, E_{j,\ell})\}_{\ell=1}^{m(\hbar_j)} \subset H^2(\Omega_{\rm tr}) \cap H^1_{0,D}(\Omega_{\rm tr}) \times [a(\hbar_j), b(\hbar_j)]$ *with*

$$\left\|(-\hbar_j^2\Delta - E_{j,\ell})u_{j,\ell}\right\|_{L^2(\Omega_{\rm tr})} = \varepsilon(\hbar_j), \qquad \|u_{j,\ell}\|_{L^2(\Omega_{\rm tr})} = 1,$$
$$\left|\langle u_{j,\ell_1}, u_{j,\ell_2}\rangle_{L^2(\Omega_{\rm tr})}\right| \leq \hbar_j^{-2}\varepsilon(\hbar_j) \text{ for } \ell_1 \neq \ell_2,$$

and supp $u_{j,\ell} \subset \mathcal{K}$ *for all j and ℓ, where* $\mathcal{K} \Subset \Omega_1$.

With $\{z_p(\hbar,\lambda)\}_p$ the set of poles of $z \mapsto R_{\Omega_{\rm tr}}(\lambda, z)$ counting multiplicities (with $z_p(\hbar,\lambda)$ depending continuously on λ for each p), let

$$\mathcal{Z}(\varepsilon_1, \varepsilon_0, a, b; \hbar) := \Big\{p \,:\, z_p(\hbar,\lambda) \in (-2\varepsilon_1, 2\varepsilon_1) - {\rm i}(0, 2\varepsilon_0) \text{ for some } \lambda^2 \in [a,b]\Big\}; \tag{2.5}$$

$|\mathcal{Z}|$ is therefore the counting function of the poles of $z \mapsto R_{\Omega_{\rm tr}}(\lambda, z)$ that enter a rectangle next to zero in z as λ^2 varies from a to b.

THEOREM 2.4 (analogue of Theorem 1.8 in $\hbar$ notation). *Let* $0 < a_0 \leq a(\hbar) \leq b(\hbar) < b_0 < \infty$*, and suppose there exists a family of quasimodes with quality*

$$\varepsilon(\hbar) \ll \hbar^{(5d+3)/2} \tag{2.6}$$

and multiplicity $m(\hbar)$ in the window $[a(\hbar), b(\hbar)]$ (in the sense of Definition 2.3). If $\varepsilon_0(\hbar)$ is such that, for some $\widetilde{S} > 0$,

$$\varepsilon_0(\hbar) \leq \widetilde{S}\,\hbar^{(d+1)/2} \quad \text{for all } \hbar, \quad \text{and} \quad \varepsilon_0(\hbar) \gg \hbar^{-2d-1}\varepsilon(\hbar) \quad \text{as } \hbar \to 0, \tag{2.7}$$

then there exists $\hbar_0 > 0$ such that if $\hbar_j \leq \hbar_0$, then

$$\left|\mathcal{Z}\Big(\hbar_j^{-(d+1)/2}\varepsilon_0(\hbar_j)\,,\,\varepsilon_0(\hbar_j)\,,\,a(\hbar_j)\,,\,b(\hbar_j)\,;\,\hbar_j\Big)\right| \geq m(\hbar_j).$$

Proof of Theorem 1.8 *from Theorem* 2.4. We first show that if there exists a family of quasimodes u_j with multiplicity m_ℓ in the window $[k_\ell^-, k_\ell^+]$ in k notation (i.e., in the sense of Definition 1.4), then there exists a family of quasimodes in $\hbar$ notation (in the sense of Definition 2.3).

Without loss of generality, each $k_\ell \in [k_j^-, k_j^+]$ for some j (if necessary by adding a window with $k_j^- = k_j^+ = k_\ell$), i.e., given ℓ in the index set of the quasimode, there exists j such that $\ell \in W_j$. We now index the quasimode with the index j describing the windows $[k_j^-, k_j^+]$. Let

$$\hbar_j := (k_j^-)^{-1}, \quad m(\hbar_j) := m_j, \quad a(\hbar_j) := 1, \quad b(\hbar_j) := \frac{(k_j^+)^2}{(k_j^-)^2},$$
$$\varepsilon(\hbar_j) := \hbar_j^2\epsilon(\hbar_j^{-1}), \quad \text{and} \quad E_{j,\ell} := \frac{(k_\ell)^2}{(k_j^-)^2} \text{ and } u_{j,\ell} := u_\ell \text{ for } \ell \in \mathcal{W}_j.$$

Then

$$\begin{aligned}\left\|(\hbar_j^2\Delta + E_{j,\ell})u_{j,\ell}\right\|_{L^2(\Omega_{\rm tr})} &= (k_j^-)^{-2}\left\|(\Delta + k_\ell^2)u_\ell\right\|_{L^2(\Omega_{\rm tr})} \\ &= (k_j^-)^{-2}\epsilon(k_\ell) \leq (k_j^-)^{-2}\epsilon(k_j^-) = \varepsilon(h_j),\end{aligned}$$

where we have used that $\epsilon(k)$ is a decreasing function of k. Therefore, we have shown that there exists a family of quasimodes with multiplicity $m(\hbar)$ in the window $[a(\hbar), b(\hbar)]$ in $\hbar$ notation (i.e., in the sense of Definition 2.3).

The result of Theorem 1.8 then follows from the result of Theorem 2.4 since (a) if $\lambda^2 \in [a(\hbar), b(\hbar)]$ and $\lambda/\hbar = k$, then $k \in [k_j^-, k_j^+]$, and (b) if

$$z \in \mathcal{Z}\Big(\hbar_j^{-(d+1)/2}\varepsilon_0(\hbar_j)\,,\, \varepsilon_0(\hbar_j)\,,\, a(\hbar_j)\,,\, b(\hbar_j)\,;\, \hbar_j\Big),$$

then

$$\mu := \hbar_j^{-2} z \in \mathcal{E}\Big((k_j^-)^{(d+1)/2}\epsilon_0(k_j^-)\,,\, \epsilon_0(k_j^-)\,,\, k_j^-\,,\, k_j^+\Big).$$

∎

2.2. Results about meromorphic continuation.

Proof of Lemma 1.11. Once we show (1.12), the meromorphicity of $R_{\Omega_{\rm tr}}(\lambda, z)$ in λ follows from the corresponding result for $R_V(\lambda)$ [15, Theorem 4.4].

We first show that the appropriate extension of a solution of (1.8) is a solution of (1.9) with $V(z) = -z1_{\Omega_{\rm tr}}$. We then show that the appropriate restriction of the solution of (1.9) with $V(z) = -z1_{\Omega_{\rm tr}}$ is a solution of (1.8).

Given $f \in L^2(\Omega_{\rm tr})$, suppose that u solves (1.8). Then, by the definition of the operator $\mathcal{D}$, there exists a $\lambda/\hbar$-outgoing function $v \in H^2_{\rm loc}(\mathbb{R}^d \setminus \Omega_1)$ such that

$$(-\hbar^2\Delta - \lambda^2)v = 0 \quad \text{on } \mathbb{R}^d \setminus \overline{\Omega_1}, \qquad \text{and} \qquad \gamma_0^{\rm tr} v = \gamma_0^{\rm tr} u, \quad \gamma_1^{\rm tr} v = \gamma_1^{\rm tr} u.$$

Therefore,

$$\widetilde{v} := 1^{\rm ext}_{\Omega_{\rm tr}} u + 1^{\rm ext}_{\mathbb{R}^d\setminus\Omega_1} v$$

is in $H^2_{\rm loc}(\Omega_+)$ (since both its Dirichlet and Neumann traces match across $\partial\Omega_1$) and

$$(-\hbar^2\Delta - \lambda^2)\widetilde{v} = z1_{\Omega_{\rm tr}}\widetilde{v} + 1^{\rm ext}_{\Omega_{\rm tr}} f \quad \text{on } \Omega_+.$$

By the definition of $R(\lambda, z)$ as the solution of (1.10) with $V(z) = -z1_{\Omega_{\rm tr}}$,

$$\widetilde{v} = R(\lambda, z)1^{\rm ext}_{\Omega_{\rm tr}} f, \qquad \text{which implies that} \qquad u = 1^{\rm res}_{\Omega_{\rm tr}} R(\lambda, z) 1^{\rm ext}_{\Omega_{\rm tr}} f.$$

Now suppose $\widetilde{f} \in L^2(\Omega_+)$. Then, by (1.11) and (1.9),

$$\begin{cases} (-\hbar^2\Delta - \lambda^2 - z1_{\Omega_{\rm tr}})R(\lambda, z)\widetilde{f} = \widetilde{f} & \text{in } \Omega, \\ R(\lambda, z)\widetilde{f} = 0 & \text{on } \Gamma_D, \\ R(\lambda, z)\widetilde{f} \text{ is } \lambda/\hbar\text{-outgoing}. \end{cases} \tag{2.8}$$

Therefore, if $\widetilde{f} = 1^{\rm ext}_{\Omega_{\rm tr}} f$ and $v := R(\lambda, z)\widetilde{f}$, then $(-\hbar^2\Delta - \lambda^2)R(\lambda, z)\widetilde{f} = 0$ in $\mathbb{R}^d \setminus \overline{\Omega_1}$ and v is $\lambda/\hbar$-outgoing. This last fact implies that

$$\gamma_1^{\rm tr}(1^{\rm res}_{\mathbb{R}^d\setminus\overline{\Omega_{\rm tr}}} v) = \mathcal{D}(\lambda/\hbar)\gamma_0^{\rm tr}(1^{\rm res}_{\mathbb{R}^d\setminus\overline{\Omega_{\rm tr}}} v). \tag{2.9}$$

Since $v = R(\lambda, z)\widetilde{f} \in H^2_{\rm loc}(\Omega_+)$, the Dirichlet and Neumann traces of v across $\Gamma_{\rm tr}$ do not have jumps, so that (2.9) implies that

$$\gamma_1^{\rm tr}(1^{\rm res}_{\Omega_{\rm tr}} v) = \mathcal{D}(\lambda/\hbar)\gamma_0^{\rm tr}(1^{\rm res}_{\Omega_{\rm tr}} v). \tag{2.10}$$

Then, by (2.8) and (2.10), $u := 1^{\rm res}_{\Omega_{\rm tr}} v$ solves (1.8) and the proof is complete. ∎

Proof of Lemma 1.12. Since

$$(-\hbar^2\Delta - \lambda^2 - z1_{\Omega_{\rm tr}})R(\lambda, 0) = I - z1_{\Omega_{\rm tr}} R(\lambda, 0),$$

the definition of $R(\lambda, z)$ (1.11) implies that

$$R(\lambda, z) = R(\lambda, 0)\big(I - z1_{\Omega_{\rm tr}} R(\lambda, 0)\big)^{-1}. \tag{2.11}$$

We now claim that, for any $\rho \in C^\infty(\overline{\Omega_+})$ with $\operatorname{supp} \rho \Subset \mathbb{R}^d$ and $\rho \equiv 1$ on $\Omega_{\rm tr}$,

$$\big(I - z1_{\Omega_{\rm tr}} R(\lambda, 0)\big)^{-1} = \big(I - z1_{\Omega_{\rm tr}} R(\lambda, 0)\rho\big)^{-1}\big(I + z1_{\Omega_{\rm tr}} R(\lambda, 0)(1-\rho)\big). \tag{2.12}$$

Indeed,

$$I - z1_{\Omega_{\rm tr}} R(\lambda, 0) = \Big(I - z1_{\Omega_{\rm tr}} R(\lambda, 0)(1-\rho)\big(I - z1_{\Omega_{\rm tr}} R(\lambda, 0)\rho\big)^{-1}\Big)\big(I - z1_{\Omega_{\rm tr}} R(\lambda, 0)\rho\big),$$

and thus

$$\begin{aligned}&\big(I - z1_{\Omega_{\rm tr}} R(\lambda, 0)\big)^{-1} \\ &\quad= \big(I - z1_{\Omega_{\rm tr}} R(\lambda, 0)\rho\big)^{-1}\Big(I - z1_{\Omega_{\rm tr}} R(\lambda, 0)(1-\rho)\big(I - z1_{\Omega_{\rm tr}} R(\lambda, 0)\rho\big)^{-1}\Big)^{-1}.\end{aligned} \tag{2.13}$$

Observe that since $\rho R(\lambda, 0)\rho : L^2(\Omega_+) \to L^2(\Omega_+)$ is compact, $1_{\Omega_{\rm tr}} R(\lambda, 0)\rho : L^2(\Omega_+) \to L^2(\Omega_+)$ is compact, and the analytic Fredholm theorem [15, Theorem C.8] implies that

$$z \mapsto (I - z1_{\Omega_{\rm tr}} R(\lambda, 0)\rho)^{-1} \text{ is a meromorphic family of operators for } z \in \mathbb{C} \tag{2.14}$$

with finite rank poles.

Now, since $(1-\rho)1_{\Omega_{\rm tr}} = 0$, for $|z|$ small enough,

$$(1-\rho)\big(I - z1_{\Omega_{\rm tr}} R(\lambda, 0)\rho\big)^{-1} = (1-\rho)\sum_{j=0}^{\infty}(z1_{\Omega_{\rm tr}} R(\lambda, 0)\rho)^k = (1-\rho). \tag{2.15}$$

However, by (2.14) both the left- and the right-hand sides of (2.15) are meromorphic for $z \in \mathbb{C}$. Therefore, (2.15) holds for all $z \in \mathbb{C}$, and hence

$$\big(I - z1_{\Omega_{\rm tr}} R(\lambda, 0)(1-\rho)\big)^{-1} = I + z1_{\Omega_{\rm tr}} R(\lambda, 0)(1-\rho). \tag{2.16}$$

Using (2.15) and (2.16) in (2.13), we obtain (2.12). Therefore, for $\chi \equiv 1$ on $\Omega_{\rm tr}$ and $\rho \equiv 1$ on $\operatorname{supp} \chi$, (2.11), (2.12), and (2.15) imply that

$$\chi R(\lambda, z)\chi = \chi R(\lambda, 0)\rho\big(I - z1_{\Omega_{\rm tr}} R(\lambda, 0)\rho\big)^{-1}\chi.$$

Using (2.14) again completes the proof. ∎

With $z_0(\hbar, \lambda)$ a pole of $R_{\Omega_{\rm tr}}(\lambda, z)$, let

$$\Pi_{z_0(\hbar,\lambda)} := -\frac{1}{2\pi \mathrm{i}}\oint_{z_0(\hbar,\lambda)} R_{\Omega_{\rm tr}}(\lambda, z)\, dz \quad \text{and} \quad m_R\big(z_0(\hbar, \lambda)\big) := \operatorname{rank} \Pi_{z_0(\hbar,\lambda)}, \tag{2.17}$$

where $\oint_{z_0(\hbar,\lambda)}$ denotes integration over a circle containing z_0 and no other pole of $R_{\Omega_{\rm tr}}(\lambda, z)$.

The following result then holds by, e.g., [15, Theorem C.9].

LEMMA 2.5. *For $\lambda \in \mathbb{R}\backslash\{0\}$, $\Pi_{z_0(\hbar,\lambda)} : L^2(\Omega_{\rm tr}) \to L^2(\Omega_{\rm tr})$ is a bounded projection with finite rank.*

The next result concerns the singular behavior of $R_{\Omega_{\rm tr}}(\lambda, z)$ near its poles in z and is analogous to (parts of) [15, Theorem 4.7] concerning the singular behavior of $R_V(\lambda)$ near its poles in λ.

Lemma 2.6. *For $\lambda \in \mathbb{R} \setminus \{0\}$, if $z_0 = z_0(\hbar, \lambda)$ and $m_R(z_0) > 0$, then there exists $M_{z_0} > 0$ such that*

$$R_{\Omega_{\rm tr}}(\lambda, z) = -\sum_{\ell=1}^{M_{z_0}} \Pi_{z_0} \frac{(-\hbar^2\Delta - \lambda^2 - z)^{\ell-1}}{(z - z_0)^\ell} + A(z, z_0, \lambda),$$

where $z \mapsto A(z, z_0, \lambda)$ is holomorphic near z_0.

Proof. By Lemma 1.12, for $\lambda \in \mathbb{R} \setminus \{0\}$, $z \mapsto R_{\Omega_{\rm tr}}(\lambda, z)$ is a meromorphic family of operators (in the sense of [15, Definition C.7]) from $L^2(\Omega_{\rm tr}) \to L^2(\Omega_{\rm tr})$, and thus there exist $M_{z_0} > 0$, finite-rank operators $A_\ell(\lambda) : L^2(\Omega_{\rm tr}) \to L^2(\Omega_{\rm tr})$, $\ell = 1, \ldots, M_{z_0}$, and a family of operators $z \mapsto A(z, z_0, \lambda)$ from $L^2(\Omega_{\rm tr}) \to L^2(\Omega_{\rm tr})$, holomorphic near z_0, such that

$$R_{\Omega_{\rm tr}}(\lambda, z) = \sum_{\ell=1}^{M_{z_0}} \frac{A_\ell(\lambda)}{(z - z_0)^\ell} + A(z, z_0, \lambda).$$

By integrating around z_0 and using the residue theorem, we have $A_1 = -\Pi_{z_0}$. Then, with $\equiv$ denoting equality up to holomorphic operators,

$$\begin{aligned} R_{\Omega_{\rm tr}}(\lambda, z)(-\hbar^2\Delta - \lambda^2 - z) &\equiv \sum_{\ell=1}^{M_{z_0}} \left(\frac{A_\ell(-\hbar^2\Delta - \lambda^2 - z_0)}{(z - z_0)^\ell} - \frac{A_\ell}{(z - z_0)^{\ell-1}} \right), \\ &= \sum_{\ell=1}^{M_{z_0}} \frac{A_\ell(-\hbar^2\Delta - \lambda^2 - z_0) - A_{\ell+1}}{(z - z_0)^\ell}, \end{aligned}$$

where we define $A_{M_{z_0}+1} := 0$. Since $R_{\Omega_{\rm tr}}(\lambda, z)(-\hbar^2\Delta - \lambda^2 - z) = I$ on $H^2(\Omega_{\rm tr}) \cap H_0^1(\Omega_+)$, $A_{\ell+1} = A_\ell(-\hbar^2\Delta - \lambda^2 - z)$, $\ell = 1, \ldots, M_{z_0}$, and the result follows from the density of $H^2(\Omega_{\rm tr}) \cap H_0^1(\Omega_+)$ in $L^2(\Omega_{\rm tr})$. ∎

2.3. The semiclassical maximum principle.

The following result is the semiclassical maximum principle of [47, Lemma 2], [48, Lemma 4.2] (see also [15, Lemma 7.7]).

Theorem 2.7 (semiclassical maximum principle). *Let $\mathcal{H}$ be a Hilbert space and $z \mapsto Q(z, \hbar) \in \mathcal{L}(\mathcal{H})$ an holomorphic family of operators in a neighborhood of*

$$\Omega(\hbar) := \big(w - 2\beta(\hbar), w + 2\beta(\hbar)\big) + {\rm i}\big(-\delta(\hbar)\hbar^{-L}, \delta(\hbar)\big), \tag{2.18}$$

where

$$0 < \delta(\hbar) < 1 \qquad \text{and} \qquad \beta(\hbar)^2 \geq C\hbar^{-3L}\delta(\hbar)^2 \tag{2.19}$$

for some $L > 0$ and $C > 0$. Suppose that

$$\|Q(z, \hbar)\|_{\mathcal{H}\to\mathcal{H}} \leq \exp(C\hbar^{-L}), \qquad z \in \Omega, \tag{2.20}$$

$$\|Q(z, \hbar)\|_{\mathcal{H}\to\mathcal{H}} \leq \frac{C}{\operatorname{Im} z}, \qquad \operatorname{Im} z > 0, \quad z \in \Omega. \tag{2.21}$$

Then

$$\|Q(z, \hbar)\|_{\mathcal{H}\to\mathcal{H}} \leq \frac{C}{\delta(\hbar)} \exp(C + 1) \quad \text{for all } z \in \big[w - \beta(\hbar), w + \beta(\hbar)\big]. \tag{2.22}$$

References for proof. Let $f, g \in \mathcal{H}$ with $\|f\|_{\mathcal{H}} = \|g\|_{\mathcal{H}} = 1$, and let

$$F(z, \hbar) := \langle Q(z + w, h)g, f \rangle_{\mathcal{H}}.$$

The result (2.22) follows from the "three-line theorem in a rectangle" (a consequence of the maximum principle) stated as [15, Lemma D.1] applied to the holomorphic family $(F(\cdot, h))_{0<h\ll 1}$ with

$$R = 2\beta(\hbar), \qquad \delta_+ = \delta(\hbar), \qquad \delta_- = \delta(\hbar)h^{-L},$$
$$M = M_- = \exp(C\hbar^{-L}), \qquad M_+ = C\,\delta(\hbar)^{-1}.$$

∎

2.4. The generalized bicharacteristic flow. Recall that

$$T^*_{\overline{\Omega_+}}\mathbb{R}^d := \left\{(x, \xi) \in T^*\mathbb{R}^d, x \in \overline{\Omega_+}\right\} = \left\{x \in \overline{\Omega_+}, \xi \in \mathbb{R}^d\right\}$$

and

$$S^*_{\overline{\Omega_+}}\mathbb{R}^d := \left\{(x, \xi) \in S^*\mathbb{R}^d, x \in \overline{\Omega_+}\right\} = \left\{x \in \overline{\Omega_+}, \xi \in \mathbb{R}^d \text{ with } |\xi| = 1\right\}.$$

We write $\varphi_t : S^*_{\Omega_+}\mathbb{R}^d \to S^*_{\Omega_+}\mathbb{R}^d$ for the generalized bicharacteristic flow associated with a symbol p (see, e.g., [31, section 24.3]). Since the flow over the interior is generated by the Hamilton vector field H_p, for any symbol $b \in C_c^\infty(T^*_{\Omega_+}\mathbb{R}^d)$,

$$\partial_t(b \circ \varphi_t) = H_p b = \{p, b\}, \tag{2.23}$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket; see [52, section 2.4].

We primarily consider the case when p is the semiclassical principal symbol of the Helmholtz equation, namely $p = |\xi|^2 - 1$. By Hamilton's equations, away from the boundary of Ω_+, the corresponding flow satisfies $\dot{x}_i = 2\xi_i$ and $\dot{\xi}_i = 0$, and thus, for $\rho = (x, \xi)$ with x away from Γ_D, $\varphi_t(\rho) = x + 2t\xi$ for t sufficiently small; i.e., the flow has speed two.

We let $\pi_{\mathbb{R}}$ denote the projection operator onto the spatial variables; i.e.,

$$\pi_{\mathbb{R}} : T^*_{\overline{\Omega_+}}\mathbb{R}^d \to \overline{\Omega_+}, \quad \pi_{\mathbb{R}}((x, \xi)) = x.$$

2.5. Geometry of trapping. Let $\chi \in C^\infty(\overline{\Omega_+}; [0, 1])$ with $\operatorname{supp}\chi \Subset \mathbb{R}^d$ and $\chi \equiv 1$ near Ω_-, and define $\mathbf{r} : T^*_{\Omega_+}\mathbb{R}^d \to \mathbb{R}$ by

$$\mathbf{r}(x, \xi) := (1 - \chi(x))|x|$$

so that there exists $c > 0$ such that for $r_0 > c$,

$$\{x \,:\, \mathbf{r} > r_0\} = \mathbb{R}^d \setminus B(0, r_0).$$

Moreover, note that $\{\mathbf{r} \le c\}$ is compact for every c. Next, define the *directly escaping sets*,

$$\mathcal{E}_\pm := \Big\{(x, \xi) \in S^*\mathbb{R}^d \mid \mathbf{r}(x, \xi) \ge r_0, \quad \pm\langle x, \xi\rangle_{\mathbb{R}^d} \ge 0\Big\},$$
$$\mathcal{E}^o_\pm := \Big\{(x, \xi) \in S^*\mathbb{R}^d \mid \mathbf{r}(x, \xi) \ge r_0, \quad \pm\langle x, \xi\rangle_{\mathbb{R}^d} > 0\Big\}.$$

Then

$$\rho \in \mathcal{E}_\pm \text{ implies that } \varphi_{\pm t}(\rho) \in \mathcal{E}_\pm \text{ and } \mathbf{r}(\varphi_{\pm t}(\rho)) \geq \sqrt{\mathbf{r}(\rho)^2 + 4t^2} \text{ for all } t \geq 0. \tag{2.24}$$

Therefore, $\mathbf{r}(\varphi_t(\rho)) \to \infty$ as $t \to \pm\infty$, and hence $\rho \in \mathcal{E}_\pm$ escapes forward/backward in time. This, in particular, implies that

$$\mathbf{r}(\rho) \geq r_0,\ \mathbf{r}(\varphi_{\mp t_0}(\rho)) \leq \mathbf{r}(\rho) \text{ for some } t_0 > 0 \quad \Rightarrow \quad \pm\langle x(\rho), \xi(\rho)\rangle > 0. \tag{2.25}$$

We now define the *outgoing tail* $\Gamma_+ \subset S^*_\Omega\mathbb{R}^d$, the *incoming tail* $\Gamma_- \subset S^*_\Omega\mathbb{R}^d$, and the *trapped set* K by

$$\Gamma_\pm := \{q \in S^*_\Omega\mathbb{R}^d \mid \mathbf{r}(\varphi_t(q)) \not\to \infty,\ t \to \mp\infty\}, \qquad K := \Gamma_+ \cap \Gamma_-; \tag{2.26}$$

i.e., the outgoing tail is the set of trajectories that do not escape as $t \to -\infty$, the incoming tail is the set of trajectories that do not escape as $t \to \infty$, and the trapped set is the set of trajectories that do not escape in either time direction.

We now recall some basic properties of $\Gamma_\pm$ and K, with these proved in a more general setting in [15, section 6.1].

Lemma 2.8.
(i) *The sets* $\Gamma_\pm, K$ *are closed in* $S^*_\Omega\mathbb{R}^d$, *and* $K \subset \{\mathbf{r} < r_0\}$.
(ii) *Suppose that* $\rho_n \in S^*_{\Omega_+}\mathbb{R}^d$ *with* $\rho_n \to \rho$ *and there exist* $t_n \to \infty$ *such that* $\varphi_{\pm t_n}(\rho_n) \to \rho_\infty$. *Then* $\rho \in \Gamma_\mp$.

Proof. (i) We show that Γ_- is closed in $S^*_\Omega\mathbb{R}^d$. Suppose that $\rho_0 \in S^*_\Omega\mathbb{R}^d \setminus \Gamma_-$. Then $\mathbf{r}(\varphi_t(\rho_0)) \to \infty$ as $t \to \infty$. In particular, there exist $0 < t_1 < t_2$ such that $\mathbf{r}(\varphi_{t_2}(\rho_0)) \geq r_0$ and $\mathbf{r}(\varphi_{t_1}(\rho_0)) \leq \mathbf{r}(\varphi_{t_2}(\rho_0))$. So, applying (2.25) with $\rho = \varphi_{t_2}(\rho_0)$, we have $\varphi_{t_2}(\rho_0) \in \mathcal{E}^o_+$. Since $\mathcal{E}^o_+$ is open and φ_{t_2} is continuous, we have $\varphi_{t_2}(\rho) \in \mathcal{E}^o_+$ for all ρ sufficiently close to ρ_0, and hence, by (2.24), $\rho \notin \Gamma_-$. Therefore, Γ_- is closed. By an identical argument, Γ_+ and hence $\Gamma_- \cap \Gamma_+$ are closed.

Now we show that $K \subset \{\mathbf{r} < r_0\}$. Note that $S^*_\Omega\mathbb{R}^d \cap \{\mathbf{r} \geq r_0\} \subset \mathcal{E}_+ \cup \mathcal{E}_-$. But $\mathcal{E}_+ \cap \Gamma_- = \emptyset$ and $\mathcal{E}_- \cap \Gamma_+ = \emptyset$, and hence $S^*_\Omega\mathbb{R}^d \cap \{\mathbf{r} \geq r_0\} \cap \Gamma_+ \cap \Gamma_- = \emptyset$ as claimed.

(ii) We prove the result for $t_n \to \infty$; the proof of the other case is similar. Seeking a contradiction, assume that $\rho \notin \Gamma_-$. Then there exists $T > 0$ such that $\mathbf{r}(\varphi_T(\rho)) \in \mathcal{E}^o_+$, and hence, since φ_T is continuous, and $\mathcal{E}^o_+$ is open, for n large enough, $\varphi_T(\rho_n) \in \mathcal{E}^o_+$. But then, by (2.24) and (2.25), for $t \geq T$, $\mathbf{r}(\varphi_t(\rho_n)) \geq \sqrt{r_0^2 + 4(t-T)^2}$. In particular, for n large enough,

$$\mathbf{r}(\varphi_{T_n}(\rho_n)) \geq \sqrt{r_0^2 + 4(T_n - T)^2} \to \infty,$$

which contradicts the fact that $\mathbf{r}(\varphi_{T_n}(\rho_n)) \to \rho_\infty$. ∎

2.6. Complex scaling. We now review the method of complex scaling following [15, section 4.5]. We first fix a small angle of scaling, $\theta > 0$, and the radius, $r_1 > r_0$, where the scaling starts; without loss of generality, we assume that $\Omega_1 \Subset \{x : \mathbf{r} \leq r_1\}$. Let $f_\theta \in C^\infty([0,\infty)$ satisfy

$$\begin{aligned} &f_\theta(r) \equiv 0, \quad r \leq r_1; \qquad &&f_\theta(r) = r\tan\theta, \quad r \geq 2r_1; \\ &f'_\theta(r) \geq 0, \quad r \geq 0; \qquad &&\{f'_\theta(r) = 0\} = \{f_\theta(r) = 0\}. \end{aligned}$$

Then consider the totally real submanifold (see [15, Definition 4.28])

$$\Gamma_\theta := \left\{x + \mathrm{i} f_\theta(|x|)\frac{x}{|x|}\ :\ x \in \mathbb{R}^d\right\} \subset \mathbb{C}^d$$

and note that we identify Ω_- with its image on Γ_θ. We define the complex scaled operator P_θ on Ω by the Dirichlet realization of

$$P_\theta := \left(\frac{1}{1+\mathrm{i}f_\theta'(r)}\hbar D_r\right)^2 - \frac{(d-1)\mathrm{i}}{(r+\mathrm{i}f_\theta(r))(1+\mathrm{i}f_\theta'(r))}\hbar^2 D_r - \frac{\hbar^2\Delta_\phi}{(r+\mathrm{i}f_\theta(r))^2}, \qquad \{\mathbf{r}\geq r_0\},$$

where Δ_ϕ denotes the Laplacian on the round sphere S^{d-1}. Note that P_θ is a semiclassical differential operator of second order such that on $r\leq r_1$, $P_\theta = -\hbar^2\Delta$ with principal symbol, p_θ, satisfying $p_\theta(x,\xi) = |\xi|^2$ on $\{\mathbf{r}\leq r_1\}$, and in polar coordinates $x = r\phi$,

$$p_\theta(r,\phi,\xi_r,\xi_\phi) = \frac{\xi_r^2}{(1+\mathrm{i}f_\theta'(r))^2} + \frac{|\xi_\phi|^2}{(r+\mathrm{i}f_\theta(r))^2}. \tag{2.27}$$

Now, by, e.g., [15, Theorems 4.36 and 4.38], for $\mathrm{Im}(\mathrm{e}^{\mathrm{i}\theta}\lambda) > 0$,

(2.28) $P_\theta - \lambda^2 : H^2(\Omega_+)\cap H_0^1(\Omega_+) \to L^2(\Omega_+)$ is a Fredholm operator of index zero.

In particular, for $V\in L^\infty(\mathbb{R}^d)$, $\operatorname{supp} V\subset\{r<r_1\}$, this implies that
(2.29)
$P_\theta - \lambda^2 + V : H^2(\Omega_+)\cap H_0^1(\Omega_+)\to L^2(\Omega_+)$ is a Fredholm operator of index zero.

Moreover, by [15, Theorem 4.37], $(P_\theta-\lambda^2+V)^{-1}$ has the same poles as $R_V(\lambda)$ and, for $\chi\in C_c^\infty(\{x:\mathbf{r}\leq r_1\})$ with $\operatorname{supp}\chi \Subset \mathbb{R}^d$,

$$\chi(P_\theta-\lambda^2+V)^{-1}\chi = \chi R_V(\lambda)\chi. \tag{2.30}$$

2.7. Semiclassical pseudodifferential operators. For simplicity of exposition, we begin by discussing semiclassical pseudodifferential operators on $\mathbb{R}^d$ and then outline below how to extend the results from $\mathbb{R}^d$ to a manifold Γ (with these results then applied with $\Gamma=\Gamma_D$ or $\Gamma=\Gamma_{\mathrm{tr}}$).

A symbol is a function on $T^*\mathbb{R}^d := \mathbb{R}^d\times(\mathbb{R}^d)^*$ that is also allowed to depend on $\hbar$ and thus can be considered as an $\hbar$-dependent family of functions. Such a family $a=(a_\hbar)_{0<\hbar\leq\hbar_0}$, with $a_\hbar\in C^\infty(T^*\mathbb{R}^d)$, is a *symbol of order* m, written as $a\in S^m(\mathbb{R}^d)$, if for any multi-indices α,β

$$|\partial_x^\alpha\partial_\xi^\beta a_\hbar(x,\xi)| \leq C_{\alpha,\beta}\langle\xi\rangle^{m-|\beta|} \quad \text{for all } (x,\xi)\in T^*\mathbb{R}^d \text{ and for all } 0<\hbar\leq\hbar_0, \tag{2.31}$$

where $\langle\xi\rangle := (1+|\xi|^2)^{1/2}$ and $C_{\alpha,\beta}$ does not depend on $\hbar$; see [52, page 207], [15, section E.1.2].

For $a\in S^m$, we define the *semiclassical quantisation* of a, denoted by $\mathrm{Op}_\hbar(a) : \mathcal{S}(\mathbb{R}^d)\to\mathcal{S}(\mathbb{R}^d)$, as

$$\mathrm{Op}_\hbar(a)v(x) := (2\pi\hbar)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\exp\big(\mathrm{i}(x-y)\cdot\xi/\hbar\big)\,a(x,\xi)v(y)\,dyd\xi; \tag{2.32}$$

see [52, section 4.1], [15, section E.1 (in particular page 543)]. The integral in (2.32) need not converge, and can be understood *either* as an oscillatory integral in the sense of [52, section 3.6], [30, section 7.8], *or* as an iterated integral, with the y integration performed first; see [15, page 543].

Conversely, if A can be written in the form above, i.e., $A=\mathrm{Op}_\hbar(a)$ with $a\in S^m$, we say that A is a *semiclassical pseudodifferential operator of order* m and we write

$A \in \Psi_\hbar^m$. We use the notation $a \in \hbar^l S^m$ if $\hbar^{-l} a \in S^m$; similarly, $A \in \hbar^l \Psi_\hbar^m$ if $\hbar^{-l} A \in \Psi_\hbar^m$. We write $\Psi_\hbar^{-\infty} = \cap_m \Psi_\hbar^{-m}$.

Let the quotient space $S^m/\hbar S^{m-1}$ be defined by identifying elements of S^m that differ only by an element of $\hbar S^{m-1}$. For any m, there exists a linear, surjective map

$$\sigma_\hbar^m : \Psi_\hbar^m \to S^m/\hbar S^{m-1},$$

called the *principal symbol map*, such that, for $a \in S^m$,

$$\sigma_\hbar^m\big(\operatorname{Op}_\hbar(a)\big) = a, \quad \mod \hbar S^{m-1}; \tag{2.33}$$

see [52, page 213], [15, Proposition E.14] (observe that (2.33) implies that $\ker(\sigma_\hbar^m) = \hbar\Psi_\hbar^{m-1}$). When applying the map $\sigma_\hbar^m$ to elements of $\Psi_\hbar^m$, we denote it by $\sigma_\hbar$ (i.e., we omit the m dependence) and we use $\sigma_\hbar(A)$ to denote one of the representatives in S^m (with the results we use then independent of the choice of representative). Key properties of the principal symbol that we use below are that

$$\sigma_\hbar(AB) = \sigma_\hbar(A)\sigma_\hbar(B) \quad \text{and} \quad \hbar^{-1}\sigma_\hbar\big(\big[\operatorname{Op}_\hbar(a), \operatorname{Op}_\hbar(b)\big]\big) = -\mathrm{i}\{a, b\}, \tag{2.34}$$

where (as in section 2.4) $\{\cdot,\cdot\}$ denotes the Poisson bracket; see [15, Proposition E.17] and [15, equation E.1.44], [52, page 68].

While the definitions above are written for operators on $\mathbb{R}^d$, semiclassical pseudodifferential operators and all of their properties above have analogues on compact manifolds (see, e.g., [52, section 14.2], [15, section E.1.7]). Roughly speaking, the class of semiclassical pseudodifferential operators of order m on a compact manifold Γ, $\Psi_\hbar^m(\Gamma)$ are operators that, in any local coordinate chart, have kernels of the form (2.32) where the function $a \in S^m$ modulo a remainder operator R that has the property

$$\|R\|_{H_\hbar^{-N} \to H_\hbar^N} \le C_N \hbar^N. \tag{2.35}$$

We say that an operator R satisfying (2.35) is $O(\hbar^\infty)_{\Psi_\hbar^{-\infty}}$.

Semiclassical pseudodifferential operators on manifolds continue to have a natural principal symbol map

$$\sigma_\hbar : \Psi_\hbar^m \to S^m(T^*\Gamma)/\hbar S^{m-1}(T^*\Gamma),$$

where now $S^m(T^*\Gamma)$ is a class of functions on $T^*\Gamma$, the cotangent bundle of Γ, which satisfies the estimates (2.31). Furthermore, (2.34) holds as before.

Finally, there is a noncanonical quantization map $\operatorname{Op}_\hbar : S^m(T^*\Gamma) \to \Psi_\hbar^m(\Gamma)$ that satisfies

$$\sigma_\hbar(\operatorname{Op}_\hbar(a)) = a,$$

and for all $A \in \Psi_\hbar^m(\Gamma)$, there exists $a \in S^m(T^*\Gamma)$ such that

$$A = \operatorname{Op}_\hbar(a) + O(\hbar^\infty)_{\Psi_\hbar^{-\infty}}.$$

2.8. Defect measures. We say that a sequence $\{u_{\hbar_n}\}_{n=1}^\infty$ with $\|u_{\hbar_n}\|_{L^2(\mathbb{R}^d)} \le C$ for all n (with C independent of n) has *defect measure* μ if for all $a \in C_c^\infty(T^*\mathbb{R}^d)$,

$$\big\langle \operatorname{Op}_{\hbar_n}(a) u_{\hbar_n}, u_{\hbar_n}\big\rangle_{L^2(\mathbb{R}^d)} \to \int a\, d\mu,$$

where $\mathrm{Op}_\hbar(a)$ is defined by (2.32). By, e.g., [52, Theorem 5.2], μ is a positive Radon measure on $T^*\mathbb{R}^d$. We say that $u_{\hbar_n}$ and $f_{\hbar_n}$ have joint defect measure μ^j if

$$\big\langle \mathrm{Op}_{\hbar_n}(a)u_{\hbar_n}, f_{\hbar_n}\big\rangle_{L^2(\mathbb{R}^d)} \to \int a\, d\mu^j. \tag{2.36}$$

We usually suppress the n in the notation and instead write that $u_\hbar$ has defect measure μ and that $u_\hbar$ and $f_\hbar$ have joint defect measure μ^j.

Lemma 2.9 (see [52, Theorem 5.3]). *Let $\mathsf{P} \in \Psi^m_\hbar(\mathbb{R}^d)$, and suppose that $u_\hbar$ has defect measure μ and satisfies*

$$\|\mathsf{P}u_\hbar\|_{L^2(\mathbb{R}^d)} = o(1).$$

Then $\operatorname{supp}\mu \subset \{\sigma_\hbar(\mathsf{P}) = 0\}$, *where $\sigma_\hbar(\mathsf{P})$ is the semiclassical principal symbol of P.*

The following lemma is the defect-measure analogue of the propagation of singularities result [15, Theorem E.47].

Lemma 2.10. *Let $\mathsf{P} \in \Psi^m_\hbar(\mathbb{R}^d)$ with $\operatorname{Im}\sigma_\hbar(\mathsf{P}) \le 0$. There exists $C > 0$ such that the following holds: suppose that $u_\hbar$ has defect measure μ and satisfies*

$$\mathsf{P}u_\hbar = \hbar f_\hbar,$$

where $\|f_\hbar\|_{L^2(\mathbb{R}^d)} \le C_1$ and $u_\hbar$ and $f_\hbar$ have joint defect measure μ^j. Then, for all real valued $a \in C^\infty_c(T^*\mathbb{R}^d)$,

$$\mu(H_{\operatorname{Re}\sigma_\hbar(\mathsf{P})}a^2) \ge -2\operatorname{Im}\mu^j(a^2) - C\mu\big(\langle\xi\rangle^{m-1}a^2\big).$$

Proof. Let $A = \mathrm{Op}_\hbar(a)$. Since $\sigma_\hbar(A^*) = a$ (by [15, equation E.1.45]) and thus $\sigma_\hbar(A^*A) = a^2$ (by [15, equation E.1.43]), by the definition of the joint measure (2.36),

$$2\hbar^{-1}\operatorname{Im}\big\langle A^*Au_\hbar, \mathsf{P}u_\hbar\big\rangle = 2\operatorname{Im}\mu^j(a^2) + o(1), \tag{2.37}$$

and, by (2.34) and (2.23),

$$\hbar^{-1}\operatorname{Im}\big\langle [A^*A, \operatorname{Re}\mathsf{P}]u_\hbar, u_\hbar\big\rangle = \mu(H_{\operatorname{Re}\sigma_\hbar(\mathsf{P})}a^2).$$

Since $2\operatorname{Im} z = \operatorname{Im}(z - \overline{z})$ and $\mathsf{P} = \operatorname{Re}\mathsf{P} + \mathrm{i}\operatorname{Im}\mathsf{P}$ with $\operatorname{Re}\mathsf{P}$ and $\operatorname{Im}\mathsf{P}$ both self-adjoint,

$$\begin{aligned}
&-2\hbar^{-1}\operatorname{Im}\big\langle A^*Au_\hbar, \mathsf{P}u_\hbar\big\rangle,\\
&= \hbar^{-1}\operatorname{Im}\Big(\big\langle \mathsf{P}u_\hbar, A^*Au_\hbar\big\rangle - \big\langle A^*Au_\hbar, \mathsf{P}u_\hbar\big\rangle\Big),\\
&= \hbar^{-1}\operatorname{Im}\Big(\big\langle (A^*A\operatorname{Re}\mathsf{P} - \operatorname{Re}\mathsf{P}A^*A)u_\hbar, u_\hbar\big\rangle + \mathrm{i}\big\langle (A^*A\operatorname{Im}\mathsf{P} + \operatorname{Im}\mathsf{P}A^*A)u_\hbar, u_\hbar\big\rangle\Big),\\
&= \hbar^{-1}\operatorname{Im}\big\langle (A^*A\operatorname{Re}\mathsf{P} - \operatorname{Re}\mathsf{P}A^*A)u_\hbar, u_\hbar\big\rangle + 2\hbar^{-1}\operatorname{Re}\big\langle A^*A\operatorname{Im}\mathsf{P}u_\hbar, u_\hbar\big\rangle),\\
&= \mu(H_{\operatorname{Re}\sigma_\hbar(\mathsf{P})}a^2) + o(1) + 2\hbar^{-1}\operatorname{Re}\big\langle \operatorname{Im}\mathsf{P}Au_\hbar, Au_\hbar\big\rangle + 2\hbar^{-1}\operatorname{Re}\big\langle A^*[A, \operatorname{Im}\mathsf{P}]u_\hbar, u_\hbar\big\rangle,\\
&\le \mu(H_{\operatorname{Re}\sigma_\hbar(\mathsf{P})}a^2) + o(1) + 2\hbar^{-1}\operatorname{Re}\big\langle A^*[A, \operatorname{Im}\mathsf{P}]u_\hbar, u_\hbar\big\rangle + C\|Au\|^2_{H^{(m-1)/2}},
\end{aligned} \tag{2.38}$$

where the last line follows from the sharp Garding inequality (see, e.g., [15, Proposition E.32]) and the fact that $\operatorname{Im}\sigma_\hbar(\mathsf{P}) \le 0$. By (2.34),

$$\operatorname{Re}\hbar^{-1}\sigma_\hbar\big(A^*[A, \operatorname{Im}\mathsf{P}]\big) = \operatorname{Re}\big(-\mathrm{i}a\{a, \operatorname{Im}\sigma_\hbar(\mathsf{P})\}\big) = 0,$$

and therefore, since the kernel of $\sigma_\hbar : \Psi^{-\infty}_\hbar \to S^{-\infty}/\hbar S^{-\infty}$ is $\hbar\Psi^{-\infty}_\hbar$, $\hbar^{-1}\operatorname{Re}A^*[A, \operatorname{Im}\mathsf{P}] \in \hbar\Psi^{-\infty}_\hbar$ and, in particular,

$$\operatorname{Re}\langle A^*[A, \operatorname{Im}\mathsf{P}]u_\hbar, u_\hbar\rangle = \mathcal{O}(\hbar^2). \tag{2.39}$$

The result follows by combining (2.38) with (2.39) and (2.37) and sending $\hbar \to 0$. ∎

COROLLARY 2.11. *Let $\Xi \geq 0$, and suppose the assumptions of Lemma* 2.10 *hold and, in addition, $\mu^j = 0$. Then, with φ_t the bicharacteristic flow corresponding to the symbol* $\operatorname{Re}\sigma_\hbar(\mathsf{P})$, *for any $B \subset T^*\mathbb{R}^d \cap \{|\xi| \leq \Xi\}$,*

$$\mu\big(\varphi_t(B)\big) \leq e^{Ct\langle\Xi\rangle^{m-1}}\mu(B) \quad \text{for } t \geq 0. \tag{2.40}$$

Corollary 2.11 shows that, under the assumptions of Lemma 2.10, we have information about the defect measures of sets moving forward under the flow.

Proof of Corollary 2.11. Let $a \in C_c^\infty(T^*\mathbb{R}^d \cap \{|\xi| \leq \Xi\})$. By (2.23),

$$\begin{aligned}\partial_t\left(e^{Ct\langle\Xi\rangle^{m-1}}\int(a^2\circ\varphi_t)\,d\mu\right) &= \int \partial_t(a^2\circ\varphi_t) + (C\langle\Xi\rangle^{m-1}a^2)\circ\varphi_t\,d\mu \\ &\geq \int \partial_t(a^2\circ\varphi_t) + (C\langle\xi\rangle^{m-1}a^2)\circ\varphi_t\,d\mu \\ &= \mu\big(H_{\operatorname{Re}\sigma_\hbar(\mathsf{P})}a^2 + C\langle\xi\rangle^{m-1}a^2\big) \geq 0,\end{aligned}$$

and thus

$$e^{Ct\langle\Xi\rangle^{m-1}}\int a^2\,d\mu \geq \int (a^2\circ\varphi_{-t})\,d\mu. \tag{2.41}$$

Let 1_B be the indicator function of $B \subset T^*\mathbb{R}^d \cap \{|\xi| \leq \Xi\}$. By approximating 1_B by squares of smooth, compactly supported symbols, (2.41) holds with $a^2 = 1_B$. Since $1_B \circ \varphi_{-t} = 1_{\varphi_t(B)}$, the result (2.40) follows. More precisely, we first let B be open and $K_n \Subset B$ compact with $K_n \uparrow B$ and choose $a_n \in C_c^\infty(T^*\mathbb{R}^d \cap \{|\xi| \leq \Xi\})$ with $a_n \equiv 1$ on K_n and $\operatorname{supp} a_n \subset B$. The result for B open follows by monotonicity of measure from below; the result for general B follows by outer regularity of μ. ∎

We now review some recent results from [22] about defect measures when $u_\hbar$ satisfies the Helmholtz equation. Let $f_\hbar \in L^2_{\text{comp}}(\mathbb{R}^d)$ be such that $\|f_\hbar\|_{L^2(\mathbb{R}^d)} \leq C$.

We use Riemannian/Fermi normal coordinates (x_1, x') in which Γ_D is given by $\{x_1 = 0\}$ and Ω_+ is $\{x_1 > 0\}$. The conormal and cotangent variables are given by (ξ_1, ξ'). Recall the definition of the hyperbolic set

$$\mathcal{H}_{\Gamma_D} := \big\{(x', \xi') \in T^*\Gamma_D \,:\, |\xi'|_g < 1\big\} \subset T^*\Gamma_D$$

(where the metric g is that induced by Γ_D) and the definition of the gliding set

$$\mathcal{G} := \Big\{x_1 = H_p x_1 = 0, \quad H_p^2 x_1 < 0, \quad |\xi| = 1\Big\} \subset S^*_{\Gamma_D}\mathbb{R}^d.$$

Let $\mathcal{N} \in \Psi_\hbar^m(\Gamma_D)$ and $\mathcal{D} \in \Psi_\hbar^{m+1}(\Gamma_D)$ have real-valued principal symbols satisfying

$$\begin{gathered}|\sigma_\hbar(\mathcal{N})|^2\langle\xi'\rangle^{-2m} + |\sigma_\hbar(\mathcal{D})|^2\langle\xi'\rangle^{-2m-2} > c > 0 \text{ on } T^*\Gamma_D, \\ \sigma_\hbar(\mathcal{N})\sigma_\hbar(\mathcal{D}) > 0 \text{ on } \overline{B^*\Gamma_D},\end{gathered} \tag{2.42}$$

where $B^*\Gamma_D := \{(x', \xi') : |\xi'|_g < 1\}$ and $|\xi'|_g$ denotes the norm of ξ' in the metric, g, induced on Γ_D from $\mathbb{R}^d$. Let $u \in L^2_{\text{loc}}(\Omega_+)$ be a solution to

$$(-\hbar^2\Delta - 1)u_\hbar = \hbar f_\hbar \quad \text{in } \Omega_+, \qquad (\mathcal{N}hD_\nu - \mathcal{D})u_\hbar|_{\Gamma_D} = o(1).$$

Later we restrict our attention to specific $\mathcal{N}$ and $\mathcal{D}$, but we consider more general operators here because we believe some of our intermediate results (specifically Lemma 3.3) are of independent interest; see [23].

Suppose that $1^{\rm ext}_{\Omega_+}u_\hbar$ has defect measure μ and that $1^{\rm ext}_{\Omega_+}u_\hbar$ and $f_\hbar$ have joint defect measure μ^j. On Γ_D, let ν_j be the joint measure associated with the Dirichlet and Neumann traces and ν_n be the measure associated with the Neumann trace; see [22, Theorem 2.3]. In what follows, we only use the fact that $\dot{n}^j\nu_n = \nu_j$, where $\dot{n}^j = \sigma_\hbar(\mathcal{N})/\sigma_\hbar(\mathcal{D})$ is bounded (see [22, Lemma 2.14 and section 4]).

With u as above, let $\mu^{\rm in/out}$ be the positive measures on $T^*\Gamma_D$, supported in the hyperbolic set $\mathcal{H}_{\Gamma_D}$ and defined in [22, Lemma 2.9], [40, Proposition 1.7, part (ii)].

In the following lemma, ${}^bT^*\Omega_+$ denotes the b-cotangent bundle to Ω_+ and $\pi : T^*\Omega_+ \to {}^bT^*\Omega_+$ is defined in local coordinates by $\pi(x_1,x',\xi_1,\xi') := (x_1,x',x_1\xi_1,\xi')$ (for more details about ${}^bT^*\Omega_+$, see, e.g., [31, section 18.3], [24, section 4B]).

LEMMA 2.12. *With $u_\hbar$, μ, μ^j, $\mu^{\rm in}$, $\mu^{\rm out}$, and $\dot{n}^j$ as above, the following hold:*

(i) $\operatorname{supp}\mu \subset S^*\Omega_+$.

(ii) *For all $\chi \in C^\infty_c(\mathbb{R}^d\setminus\Omega_-)$, $\lim_{\hbar\to0}\|\chi u_\hbar\|^2_{L^2} = \mu(|\chi|^2)$.*

(iii) *For all $a \in C^\infty_c({}^bT^*\Omega_+)$,*

$$\begin{aligned}&\pi_*\mu(a\circ\varphi_t) - \pi_*\mu(a)\\&\quad = \int_0^t \bigg(-2\,\mathrm{Im}\,\pi_*\mu^j + \delta(x_1)\otimes(\mu^{\rm in}-\mu^{\rm out}) + \frac{1}{2}\frac{\sigma_\hbar(\mathcal{N})}{\sigma_\hbar(\mathcal{D})}H_p^2 x_1\mu 1_{\mathcal{G}}\bigg)(a\circ\varphi_s)\,ds\end{aligned}$$

(where the integral is understood as the integral of distributions acting on smooth functions).

(iv) *On $\mathcal{H}_{\Gamma_D}$, $\mu^{\rm out} = \alpha\,\mu^{\rm in}$, where*

$$\alpha := \left|\frac{\sqrt{1-|\xi'|^2_g}\,\sigma_\hbar(\mathcal{N})(x',\xi') - \sigma_\hbar(\mathcal{D})(x',\xi')}{\sqrt{1-|\xi'|^2_g}\,\sigma_\hbar(\mathcal{N})(x',\xi') + \sigma_\hbar(\mathcal{D})(x',\xi')}\right|^2. \tag{2.43}$$

References for the proof. Parts (i) and (ii) are proved in [24, Lemma 4.2]. Part (iii) is proved in [22, Theorem 2.15] (following [24, Lemma 4.8]), and part (iv) is proved in [22, Lemmas 2.12 and 2.18] (following [40, Proposition 1.10, part (iii)]). ◻

3. Parametrix for $(P_\theta - \lambda^2)$ via boundary complex absorption. We now find a parametrix for $(P_\theta-\lambda^2)$ using a complex absorbing potential on the boundary Γ_D. We then obtain by perturbation a parametrix for $(P_\theta-\lambda^2-z1_{\Omega_{\rm tr}})$ for z sufficiently small.

First, let

$$\mathcal{P}_\theta(\lambda) := \begin{pmatrix} P_\theta-\lambda^2\\ \gamma_0^D\end{pmatrix} : H^2(\Omega_+)\to L^2(\Omega_+)\oplus H^{3/2}(\Gamma_D).$$

Then let $E : H^{3/2}(\Gamma_D)\to H^2(\Omega_+)$ be an extension operator satisfying

$$\gamma_0^D Eg = g, \qquad g\in H^{\frac32}(\partial\Omega).$$

Simple calculation then implies that

$$(\mathcal{P}_\theta(\lambda))^{-1} = \Big(\mathcal{R}_\theta(\lambda),\, E - \mathcal{R}_\theta(\lambda)(P_\theta-\lambda^2)E\Big), \tag{3.1}$$

where $\mathcal{R}_\theta(\lambda) := (P_\theta-\lambda^2)^{-1}$ is the inverse of (2.28).

LEMMA 3.1. *The operator $\mathcal{P}_\theta(\lambda)$ is Fredholm with index zero.*

Proof. Recall that the map (2.28) is Fredholm with index zero. First, note that if $\mathcal{P}_\theta(\lambda)u = 0$, then $u \in H_0^1(\Omega_+) \cap H^2(\Omega_+)$ and in particular, $u \in \ker(P_\theta - \lambda^2)$. Therefore, since $P_\theta - \lambda^2 : H_0^1(\Omega_+) \cap H^2(\Omega_+) \to L^2(\Omega_+)$ is Fredholm, $\ker \mathcal{P}_\theta(\lambda)$ is finite dimensional. To see that the cokernel $L^2(\Omega_+) \oplus H^{3/2}(\Gamma_D)\big/\mathcal{P}_\theta(\lambda)H^2(\Omega_+)$ is finite dimensional, define the map

$$\pi : L^2(\Omega_+) \oplus H^{3/2}(\Gamma_D)\Big/\mathcal{P}_\theta(\lambda)H^2(\Omega_+) \to L^2(\Omega_+)\Big/(P_\theta - \lambda^2)\big(H_0^1(\Omega_+) \cap H^2(\Omega_+)\big),$$
$$(f,g) + \mathcal{P}_\theta(\lambda)H^2(\Omega_+) \mapsto f - (P_\theta - \lambda^2)Eg + (P_\theta - \lambda^2)\big(H_0^1(\Omega_+) \cap H^2(\Omega_+)\big).$$

First, observe that this map is well defined since if $(f_1, g_1)+\mathcal{P}_\theta(\lambda)H^2(\Omega_+) = (f_2, g_2)+\mathcal{P}_\theta(\lambda)H^2(\Omega_+)$, then there exists $u \in H^2(\Omega_+)$ such that

$$(f_1 - f_2, g_1 - g_2) = \big((P_\theta - \lambda^2)u, \gamma_D u\big).$$

In particular,

$$(f_1-f_2)-(P_\theta-\lambda^2)E(g_1-g_2) = (P_\theta-\lambda^2)(u-E(g_1-g_2)) \in (P_\theta-\lambda^2)\big(H_0^1(\Omega_+)\cap H^2(\Omega_+)\big),$$

so $\pi(f_1, g_1) = \pi(f_2, g_2)$.

Now suppose that $\pi(f, g) = 0$. Then there exists $u \in H_0^1(\Omega_+) \cap H^2(\Omega_+)$ such that

$$f - (P_\theta - \lambda^2)Eg = (P_\theta - \lambda^2)u.$$

Therefore,

$$(f, g) - \mathcal{P}_\theta(\lambda)Eg = (f - (P_\theta - \lambda^2)Eg, 0) = ((P_\theta - \lambda^2)u, 0) \in \mathcal{P}_\theta(\lambda)H^2(\Omega_+),$$

and π is injective. For an injective operator,

$$\dim(\text{domain}) \le \dim(\text{range}) \le \dim(\text{codomain});$$

therefore,

$$\dim\Big(L^2(\Omega_+) \oplus H^{3/2}(\Gamma_D)\Big/\mathcal{P}_\theta(\lambda)H^2(\Omega_+)\Big)$$
$$\le \dim\Big(L^2(\Omega_+)\Big/(P_\theta - \lambda^2)\big(H_0^1(\Omega_+) \cap H^2(\Omega_+)\big)\Big) < \infty.$$

Since $P_\theta - \lambda^2 : H_0^1(\Omega_+) \cap H^2(\Omega_+) \to L^2(\Omega_+)$ is Fredholm, $\mathcal{P}_\theta(\lambda)$ is Fredholm. To see that $\mathcal{P}_\theta(\lambda)$ has index zero, recall that the index is constant in λ by, e.g., [15, Theorem C.5], and observe that the formula (3.1) implies that the inverse exists for some λ. ◻

We now define our complex absorbing operator. Let $\psi \in C_c^\infty(\mathbb{R}; [0, 1])$ with $\psi \equiv 1$ on $[-b, b]$ and $\operatorname{supp}\psi \subset [-2b, 2b]$. It will be convenient to have a specific notation for the Neumann trace with the standard derivative operator replaced by $D := -i\hbar\partial$. We therefore let $\gamma_{1,\hbar}^D := -i\hbar\gamma_1^D$. Let

$$\mathcal{P}_{\theta,Q}(\lambda) := \begin{pmatrix} P_\theta - \lambda^2 \\ Q_b\gamma_{1,\hbar}^D + \gamma_0^D \end{pmatrix} : H^2(\Omega_+) \to L^2(\Omega_+) \oplus H^{3/2}(\Gamma_D),$$

where $Q_b \in \Psi_\hbar^{\mathrm{comp}}(\Gamma_D)$ with symbol

$$\sigma_\hbar(Q_b) = -\psi(|\xi'|_g). \tag{3.2}$$

Note that

$$\mathcal{P}_{\theta,Q}(\lambda) = \mathcal{P}_{\theta}(\lambda) + \begin{pmatrix} 0 \\ Q_b\gamma^D_{1,\hbar} \end{pmatrix}$$

and hence $\mathcal{P}_{\theta,Q}(\lambda)$ is a compact perturbation of $\mathcal{P}_{\theta}(\lambda)$. Therefore, by Lemma 3.1, $\mathcal{P}_{Q}(\lambda)$ is Fredholm with index zero.

LEMMA 3.2. *Let Q_b be as above, and let $0 < a < b$ and $C_1 > 0$. Then there exists $C > 0$ such that for all $\lambda \in [a, b] + \mathrm{i}[-C_1\hbar, C_1\hbar]$,*

(3.3)
$$\|\gamma^D_{1,\hbar}u\|_{L^2(\Gamma_D)}+\|u\|_{H^2_\hbar(\Omega_+)} \leq C\hbar^{-1}\left\|(P_\theta-\lambda^2)u\right\|_{L^2(\Omega_+)}+C\left\|(Q_b\gamma^D_{1,\hbar}+\gamma^D_0)u\right\|_{H^{3/2}_\hbar(\Gamma_D)}.$$

In particular, since $\mathcal{P}_{\theta,Q}(\lambda)$ is Fredholm with index zero,

$$\mathcal{R}_{\theta,Q}(\lambda) := (\mathcal{P}_{\theta,Q}(\lambda))^{-1}$$

exists and satisfies

(3.4)
$$\|\gamma^D_{1,\hbar}\mathcal{R}_{\theta,Q}(\lambda)(f,g)\|_{L^2(\Gamma_D)}+\|\mathcal{R}_{\theta,Q}(\lambda)(f,g)\|_{H^2_\hbar(\Omega_+)} \leq C\Big(h^{-1}\|f\|_{L^2(\Omega_+)}+\|g\|_{H^{3/2}_\hbar(\Gamma_D)}\Big).$$

Observe that the bound (3.4) has the same $\hbar$-dependence as the standard nontrapping resolvent estimate.

Before proving Lemma 3.2, we show how a parametrix for the operator $(P_\theta - \lambda^2 - z1_{\Omega_{\rm tr}})$ can be expressed in terms of $\mathcal{R}_{\theta,Q}(\lambda)$. Let

$$\mathcal{P}_{\theta,Q}(\lambda, z) := \begin{pmatrix} P_\theta - \lambda^2 - z1_{\Omega_{\rm tr}} \\ Q_b\gamma^D_{1,\hbar} + \gamma^D_0 \end{pmatrix} : H^2(\Omega_+) \to L^2(\Omega_+) \oplus H^{3/2}(\Gamma_D).$$

By Lemma 3.2, the bound (3.4), and inversion by Neumann series, for $|z| \leq \hbar/(2C)$ (where C is the constant from Lemma 3.2),

$$\mathcal{R}_{\theta,Q}(\lambda, z) := (\mathcal{P}_{\theta,Q}(\lambda, z))^{-1}$$

exists and satisfies

$$\begin{aligned} \|\gamma^D_{1,\hbar}\mathcal{R}_{\theta,Q}(\lambda,z)(f,g)\|_{L^2(\Gamma_D)} + \|\mathcal{R}_{\theta,Q}(\lambda,z)(f,g)\|_{H^2_\hbar(\Omega_+)} \\ \leq 2C\Big(h^{-1}\|f\|_{L^2(\Omega_+)} + \|g\|_{H^{3/2}_\hbar(\Gamma_D)}\Big). \end{aligned} \tag{3.5}$$

Next, let

$$\mathcal{P}_\theta(\lambda, z) := \begin{pmatrix} P_\theta - \lambda^2 - z1_{\Omega_{\rm tr}} \\ \gamma^D_0 \end{pmatrix} : H^2(\Omega_+) \to L^2(\Omega_+) \oplus H^{3/2}_\hbar(\Gamma_D). \tag{3.6}$$

If $\mathcal{R}_{\theta,Q}(\lambda, z)$ exists, then

$$\mathcal{P}_\theta(\lambda, z) = \big(I + K(\lambda, z)\big)\mathcal{P}_{\theta,Q}(\lambda, z),$$

where

$$K(\lambda, z) := Q\mathcal{R}_{\theta,Q}(\lambda, z) \quad \text{and} \quad Q := \begin{pmatrix} 0 \\ -Q_b\gamma^D_{1,\hbar} \end{pmatrix}. \tag{3.7}$$

Since $K(\lambda,z): L^2(\Omega_+)\oplus H^{3/2}(\Gamma_D)\to L^2(\Omega_+)\oplus H^{3/2}(\Gamma_D)$ is compact, $(I+K(\lambda,z))^{-1}$ is a meromorphic family of operators by [15, Theorem C.8]. Therefore, for $|z|\le \hbar/(2C)$,

$$\mathcal{P}_\theta(\lambda,z)^{-1}=\mathcal{R}_{\theta,Q}(\lambda,z)\big(I+K(\lambda,z)\big)^{-1}. \tag{3.8}$$

Let $R_\theta(\lambda,z)$ be the inverse of the map (2.29) with $V=-z1_{\Omega_{\rm tr}}$, i.e.,

$$R_\theta(\lambda,z):=(P_\theta-\lambda^2-z1_{\Omega_{\rm tr}})^{-1}. \tag{3.9}$$

Then, for $|z|\le \hbar/(2C)$,

$$R_\theta(\lambda,z)=\mathcal{P}_\theta(\lambda,z)^{-1}\begin{pmatrix}I\\0\end{pmatrix}=\mathcal{R}_{\theta,Q}(\lambda,z)\big(I+K(\lambda,z)\big)^{-1}\begin{pmatrix}I\\0\end{pmatrix}, \tag{3.10}$$

which is the required parametrix.

Proof of Lemma 3.2. Suppose that the estimate (3.3) fails with the left-hand side replaced by $\|u\|_{L^2(\Omega_+)}$; then there exist $\hbar_n\to 0$, $\lambda_n\in[a,b]+\mathrm{i}[-C\hbar,C\hbar]$, and $(\tilde u_n,\tilde f_n,\tilde g_n)\in H^2(\Omega_+)\oplus L^2(\Omega_+)\oplus H^{3/2}_\hbar(\Gamma_D)$ with

$$\|\tilde f_n\|_{L^2(\Omega_+)}+\|\tilde g_n\|_{H^{3/2}_\hbar(\Gamma_D)}=1,\qquad \|\tilde u_n\|_{L^2}=n,$$

and with

$$\mathcal{P}_{\theta,Q}(\tilde u_n)=(\hbar_n\tilde f_n,\tilde g_n).$$

In particular, renormalizing $u_n:=\tilde u_n/n$, $f_n:=\tilde f_n/n$, and $g_n:=\tilde g_n/n$,

$$\|f_n\|_{L^2(\Omega_+)}=\hbar^{-1}\left\|(P_\theta-\lambda_n^2)u_n\right\|_{L^2(\Omega_+)}\le\frac{1}{n}$$

and

$$\|g_n\|_{L^2(\Gamma_D)}=\left\|(Q_b\gamma^D_{1,\hbar}+\gamma_D)u_n\right\|_{L^2(\Gamma_D)}\le\frac{1}{n}.$$

Now, since $0<a\le\operatorname{Re}\lambda_n\le b$, we may rescale $\hbar_n$ to $\tilde\hbar:=\hbar_n/\operatorname{Re}\lambda_n$ and hence replace $\operatorname{Re}\lambda_n$ by 1. Note that this rescaling does not cause any issues since $b^{-1}\hbar_n\le\tilde\hbar_n\le a^{-1}\hbar_n$. Extracting a subsequence, we can assume that $1^{\rm ext}_{\Omega_+}u_n$ has defect measure μ (see, e.g., [52, Theorem 5.2]), and $\hbar_n^{-1}\operatorname{Im}\lambda_n\to\operatorname{Im}\beta_\infty$, and $\operatorname{Re}\lambda_n=1$. Since $\|f_{\hbar_n}\|_{L^2}\to 0$, $\mu^j=0$.

Let $\chi,\chi_0\in C_c^\infty(\mathbb{R}^d;[0,1])$ with $\operatorname{supp}\chi\Subset\mathbb{R}^d$ and $\chi,\chi_0\equiv 1$ in a neighborhood of $\{\mathbf{r}\le 2r_1\}$ and $\operatorname{supp}\chi_0\subset\{\chi\equiv 1\}$. We first show that

$$\|(1-\chi)u_n\|_{L^2(\Omega_+)}=\mathcal{O}(\hbar_n). \tag{3.11}$$

To do this, observe that, by (2.27),

$$|\sigma_\hbar(P_\theta-\lambda_n^2)(x,\xi)|=\left|\frac{|\xi|^2}{(1+\mathrm{i}\tan\theta)^2}-1\right|\ge c(|\xi|^2+1),\qquad \mathbf{r}(x,\xi)\ge 2r_1. \tag{3.12}$$

Therefore, by ellipticity, for W a neighborhood of $\operatorname{supp}\partial\chi$,

$$\|u_n\|_{H^2_\hbar(W)}\le C\big(\left\|(P_\theta-\lambda^2)u_n\right\|_{L^2(\Omega_+)}+\|u_n\|_{L^2(\Omega_+)}\big). \tag{3.13}$$

Now, by (3.12) and the definitions of χ and χ_0,

$$\left|\sigma\Big(\operatorname{Op}_\hbar\left((1+|\xi|^2)^{-1}\right)(1-\chi_0)(P_\theta-\lambda_n^2)(1-\chi_0)-\mathrm{i}\chi_0\Big)\right|\geq c.$$

Therefore, by [52, Theorem 4.29],

$$\begin{aligned}\|(1-\chi)u_n\|_{L^2(\Omega_+)}&\\ \leq C\left\|\left[\operatorname{Op}_\hbar((1+|\xi|^2)^{-1})(1-\chi_0)(P_\theta-\lambda_n^2)(1-\chi_0)-\mathrm{i}\chi_0\right](1-\chi)u_n\right\|_{L^2(\mathbb{R}^d)}&\\ = C\left\|\operatorname{Op}_\hbar((1+|\xi|^2)^{-1})(1-\chi_0)(P_\theta-\lambda_n^2)(1-\chi)u_n\right\|_{L^2(\mathbb{R}^d)}.&\end{aligned}\tag{3.14}$$

But

$$\begin{aligned}&\left\|\operatorname{Op}_\hbar((1+|\xi|^2)^{-1})(1-\chi_0)(P_\theta-\lambda_n^2)(1-\chi)u_n\right\|_{L^2(\mathbb{R}^d)}\\ &\leq C\left\|(1-\chi)\hbar_n f_n\right\|_{L^2(\Omega_+)}+\|[P_\theta,\chi]u_n\|_{H_\hbar^{-2}(\Omega_+)}\\ &\leq C\left\|(1-\chi)\hbar_n f_n\right\|_{L^2(\Omega_+)}+C\hbar_n\|u_n\|_{L^2(\Omega_+)}=\mathcal{O}(\hbar_n),\end{aligned}\tag{3.15}$$

where we have used that, by direct computation, $\|[P_\theta,\chi]\|_{H_\hbar^s(\Omega_+)\to H_\hbar^{s-1}(\Omega_+)}\leq C\hbar$ in the second inequality; (3.11) then follows from combining (3.14) and (3.15).

We now show that $\mu(T^*\mathbb{R}^d)=1$. First, observe that

$$(P_\theta-\lambda_n^2)\chi u_n=[P_\theta,\chi]u_n+o(\hbar_n)_{L^2}.\tag{3.16}$$

Consequently, using (3.13) in (3.16) we find that

$$(P_\theta-\lambda_n^2)\chi u_n=\mathcal{O}(\hbar_n)_{L^2}.$$

Since $(P_\theta-\lambda^2)=(-\hbar^2\Delta-\lambda^2)$ on $\operatorname{supp}\chi$, we can now apply Lemma 2.12 (with u in that lemma replaced by χu_n here) to find that

$$\mu(\chi^2)=\lim_{\hbar\to 0}\|\chi u_n\|^2_{L^2(\Omega_+)}=\lim_{\hbar\to 0}\|u_n\|^2_{L^2(\Omega_+)}=1,$$

where we have used (3.11) in the second equality. Moreover,

$$\mu(T^*\mathbb{R}^d)\leq\lim_{\hbar\to 0}\|u_n\|^2_{L^2(\Omega_+)}=1,$$

so that in fact $\mu(T^*\mathbb{R}^d)=1$.

We now show that $\mu=0$, which is a contradiction. To do this, we start by observing that (3.11) implies that $\mu(\{\mathbf{r}\geq 2r_1\})=0$. In fact, by Lemma 2.9, $\mu(\{\sigma_\hbar(P_\theta)\neq 0\})=0$, and therefore $\operatorname{supp}\mu\subset S^*_{\overline{\Omega_+}}\mathbb{R}^d\cap\{\mathbf{r}\leq 2r_1\}$.

Now, Lemma 2.12, along with Lemma 2.10 together with the fact that $\operatorname{Im}\sigma_\hbar(P_\theta)\leq 0$, allows us to propagate forward along the generalized bicharacteristic flow (in the sense of Corollary 2.11) but not backward. In particular, since $\mu(\{\mathbf{r}\geq 2r_1\}=0)$, this implies that $\operatorname{supp}\mu\subset\Gamma_+$. Indeed, suppose that $A\subset S^*_{\overline{\Omega_+}}\mathbb{R}^d$ is compact and $A\cap\Gamma_+=\emptyset$. Then, by the definition of Γ_+ (2.26), for each $\rho\in A$ there exists $t_\rho>0$ such that $\mathbf{r}(\varphi_{-t_\rho}(\rho))>\max(2r_1,\mathbf{r}(\rho))$. Hence, by (2.24), for $t\geq t_\rho$, $\mathbf{r}(\varphi_{-t}(\rho))>2r_1$, and by the continuity of φ_{-t_ρ}, there is a neighborhood U_ρ of ρ such that $\varphi_{-t}(U_\rho)\subset\{\mathbf{r}>2r_1\}$ for $t\geq t_\rho$. In particular, by the compactness of A, there exists $T>0$ such that $\varphi_{-T}(A)\subset\{\mathbf{r}>2r_1\}$. By the compactness of A in the ξ variable and (2.40), there exists $C>0$ such that $\mu(A)\leq\exp(CT)\mu(\varphi_{-T}(A))=0$.

Now, by Lemma 2.8, Γ_+ is closed, and hence we may write $(\Gamma_+)^c = \cup_n A_n$ with A_n compact. In particular, $\mu((\Gamma_+)^c) = 0$ by monotonicity from below.

Next, note that since $\operatorname{Im}\sigma_\hbar(P_\theta - \lambda^2) < 0$ on $\{f_\theta \neq 0\}$,

$$\operatorname{supp}\mu \subset \{f_\theta = 0\}$$

by Lemma 2.9. In particular, by the definition of f_θ,

$$\operatorname{supp}\mu \subset \{\mathbf{r} < 2r_1\}.$$

To complete the proof, we need to show that in fact $\mu(\Gamma_+) = 0$. This is where the boundary term Q_b is used.

We claim that there are $T, c > 0$ such that

$$\mu(\varphi_{-T}(A)) \geq \mathrm{e}^c \mu(A) \tag{3.17}$$

for all A. Once this is done, we have that $\mu \equiv 0$. To see this, observe that if $\mu(A) > 0$, then by induction $\mu(\varphi_{-nT}(A)) \geq \mathrm{e}^{nc}\mu(A)$. Taking $N > -(\log \mu(A))/c$, we have $\mu(\varphi_{-NT}(A)) > 1$, which is a contradiction to $\mu(T^*\mathbb{R}^d) = 1$.

We now prove (3.17). First, note that the statement is empty if $\mu(A) = 0$. Therefore, we can assume that $\mu(A) > 0$. Since $\operatorname{supp}\mu \subset \Gamma_+$, we assume that $A \subset \Gamma_+$; since Γ_+ is closed, we can assume that A is compact. Now, by (2.24), (2.25), and (2.26),

$$\Gamma_+ \cap \{r_0 \leq \mathbf{r} \leq 2r_1\} \subset \bigcup_{t=0}^{\sqrt{(2r_1)^2 - r_0^2}} \varphi_t(\Gamma_+ \cap \{\mathbf{r} \leq r_0\}).$$

Therefore, increasing T by $\sqrt{(2r_1)^2 - r_0^2}$, we may assume that $A \subset \{\mathbf{r} < r_0\} \cap \Gamma_+$.

Letting $\mathcal{N} = Q_b$ and $\mathcal{D} = -1$ and recalling (3.2), we see that $\mathcal{N}$ and $\mathcal{D}$ satisfy (2.42). Therefore, the proof of (3.17) is completed by the next lemma.

Lemma 3.3. *Suppose that $\mathcal{N}$ and $\mathcal{D}$ are as in* (2.42)*, μ satisfies the conclusions of parts* (iii) *and* (iv) *of Lemma* 2.12 *with $\mu^j = 0$, and $A \subset \{\mathbf{r} < r_0\} \cap \Gamma_+$. Then there exist $T, c > 0$ such that* (3.17) *holds.*

Proof. We claim that there exist $\varepsilon_1, T > 0$ such that for all $\rho \in \Gamma_+$ with $\mathbf{r}(\rho) < r_0$,

$$\int_0^T \Big(-\frac{1}{2}\frac{\sigma_\hbar(\mathcal{N})}{\sigma_\hbar(\mathcal{D})} H_p^2 x_1 1_{\mathcal{G}}(\varphi_{-t}(\rho)) + |H_p x_1(\varphi_{-t}(\rho))|^{-1}\delta\big(x_1(\varphi_{-t}(\rho))\big) \log \alpha\big(\pi_{\Gamma_D}(\varphi_{-t}(\rho))\big)\Big) dt \leq -\varepsilon_1, \tag{3.18}$$

where $\pi_{\Gamma_D} : S^*_{\Gamma_D}\mathbb{R}^d \to T^*\Gamma_D$ is the orthogonal projection and α is given by (2.43).

Once (3.18) is proved, we claim that Lemma 2.12 implies (3.17) with $(c, T) = (\varepsilon_1, T)$. Indeed, suppose that (3.18) holds and that $\mu(A) > 0$, $A \subset \Gamma_+ \cap \{\mathbf{r} < r_0\}$, and A is closed. Then let $0 \leq a \in C_c^\infty({}^bT^*\mathbb{R}^d \setminus \Omega_-)$ with $a \equiv 1$ on A and

$$\bigcup_{t \in [0, -T]} \varphi_t(\operatorname{supp} a) \subset \left\{\mathbf{r} < \frac{r_0 + r_1}{2}\right\}. \tag{3.19}$$

Now let $\chi \equiv 1$ on $\{\mathbf{r} \leq \frac{r_0 + r_1}{2}\}$ with $\operatorname{supp}\chi \subset \{\mathbf{r} < r_1\}$. Then

$$(-\hbar^2\Delta - 1)\chi u = [-\hbar^2\Delta, \chi]u + \hbar f, \qquad \chi u|_{\Gamma_D} = 0$$

with $f = o(1)_{L^2}$, and hence by Lemma 2.12

$$\pi_*\mu(\chi^2(a\circ\varphi_t)) - \pi_*\mu(\chi^2 a)$$
$$= \int_0^t \Big(-4\langle \xi, \partial\chi\rangle\mu + \delta(x_1)\otimes(\mu^{\rm in}-\mu^{\rm out}) + \frac{\sigma_\hbar(\mathcal{N})}{2\sigma_\hbar(\mathcal{D})}H_p^2 x_1 \mu 1_{\mathcal{G}}\Big)(\chi^2(a\circ\varphi_s))\,ds.$$

But, by (3.19), $\chi^2 \equiv 1$ on $\operatorname{supp} a\circ\varphi_t$ for $t\in[0,T]$. In particular, for $t\in[0,T]$,

$$\pi_*\mu(a\circ\varphi_t) - \pi_*\mu(a) = \int_0^t \Big(\delta(x_1)\otimes(\mu^{\rm in}-\mu^{\rm out}) + \frac{\sigma_\hbar(\mathcal{N})}{2\sigma_\hbar(\mathcal{D})}H_p^2 x_1\mu 1_{\mathcal{G}}\Big)(a\circ\varphi_s)\,ds.$$

Finally, since A is closed, we may approximate 1_A by smooth, compactly supported functions to obtain

$$\pi_*\mu(\varphi_{-t}(A)) - \pi_*\mu(A) = \int_0^t \Big(\delta(x_1)\otimes(\mu^{\rm in}-\mu^{\rm out}) + \frac{\sigma_\hbar(\mathcal{N})}{2\sigma_\hbar(\mathcal{D})}H_p^2 x_1\mu 1_{\mathcal{G}}\Big)(1_A\circ\varphi_s)\,ds. \tag{3.20}$$

Now, to study (3.20), we first assume that A and t are such that for all $\rho\in A$ and $s\in[0,2t]$, $\varphi_{-s}(\rho)$ does not lie in the glancing region $(H_p x_1 = 0)$ and each trajectory intersects Γ_D exactly once and does so for $s\in(0,t)$. Shrinking the support of a further if necessary, we can find $\Sigma \subset^b T^*\mathbb{R}^d\setminus\Omega_-$ transverse to the vector field H_p such that

$$F : [-t,t]\times\Sigma \ni (s,\rho)\mapsto \varphi_{-s}(\rho)\in^b T^*_{\Gamma_D}\mathbb{R}^d$$

are smooth coordinates and $\varphi_{-s}(A)$ is in the image of F for all $s\in[0,t]$. Then (3.20) reads as

$$\pi_*\mu(\varphi_{-t}(A)) - \pi_*\mu(A)$$
$$= \int_0^t \Big(\delta(x_1)\otimes(\mu^{\rm in}-\mu^{\rm out})\Big)(1_A\circ\varphi_{t'})\,dt'$$
$$= \int_0^t\int_{-t}^t\int_\Sigma \Big(|H_p x_1|(s,\rho)\delta(s)\otimes(1_A(s-t',\rho))\Big)d(\mu^{\rm in}-\mu^{\rm out})(\rho)\,ds\,dt'$$
$$= \int_\Sigma\int_0^t \Big(|H_p x_1|(0,\rho)(\alpha^{-1}(\rho)-1)1_A(-t',\rho)\Big)d\mu^{\rm out}(\rho)dt'.$$

Now, arguing as in [22, Lemma 2.16], we obtain that $\pi_*\mu = |H_p x_1|\mu^{\rm out}1_{s<0}ds + |H_p x_1|\mu^{\rm in}1_{s>0}ds$, and hence

$$\pi_*\mu(A) = \int_\Sigma\int_0^t |H_p x_1|(0,\rho)1_A(-t',\rho)d\mu^{\rm out}(\rho)dt'.$$

Therefore,

$$\pi_*\mu(\varphi_{-t}(A)) \ge \inf_{F^{-1}(A)}(\alpha(\rho))^{-1}\pi_*\mu(A)$$
$$= \inf_A e^{-\int_0^t (|H_p x_1|(\varphi_{-t'}(\rho))^{-1}\delta(x_1(\varphi_{-t'}(\rho)))\log\alpha(\pi_{\Gamma_D}(\varphi_{-t'}(\rho)))dt'}\pi_*\mu(A),$$

where this last equality comes from evaluating the integral using the fact that F is well defined (since each trajectory intersects Γ_D exactly once):

$$A\subset \Big\{\varphi_s\big(\{x_1 = H_p x_1 = 0\}\big)\setminus\big\{H_p x_1\neq 0,\, x_1 = 0\big\} : s\in[0,t]\Big\},$$

so that, in particular, trajectories from A do not intersect the hyperbolic set. In this case, (3.20) implies that

$$\partial_s \pi_*\mu(\varphi_{-s}(A)) = \Big(\frac{\sigma_\hbar(\mathcal{N})}{2\sigma_\hbar(\mathcal{D})} H_p^2 x_1 1_{\mathcal{G}} \pi_*\mu\Big)\big(\varphi_{-s}(A)\big). \tag{3.21}$$

In particular, shrinking A if necessary, we may choose $\Sigma \subset \{x_1 = H_p x_1 = 0\}$ transverse to H_p and work in coordinates

$$[0,t] \times \Sigma \ni (s,\rho) \mapsto \varphi_{-s}(\rho) \in \Big\{\varphi_{-s}\big(\{x_1 = H_p x_1 = 0\}\big) \,:\, s \in [0,t]\Big\}.$$

In these coordinates, (3.21) implies that $\pi_*\mu$ is absolutely continuous with respect to t in the sense that there is a family of measures $t \mapsto \nu_t$ on Σ such that $\nu_t(\Sigma) \in L^1$ and $\mu = \nu_t dt$. Moreover,

$$\int_B d\nu_s(\rho) = \int_B \exp\Big(\int \frac{\sigma_\hbar(\mathcal{N})}{2\sigma_\hbar(\mathcal{D})} H_p^2 x_1 1_{\mathcal{G}}(\varphi_{-s}(\rho)) ds\Big)\, d\nu(\rho).$$

In particular,

$$\pi_*\mu(\varphi_{-t}(A)) \geq \inf_A \exp\Big(\int \frac{\sigma_\hbar(\mathcal{N})}{2\sigma_\hbar(\mathcal{D})} x_1 1_{\mathcal{G}}(\varphi_{-s}(\rho)) ds\Big)\, \pi_*\mu(A).$$

Putting everything together, we have for all A and $0 \leq t \leq T$,

$$\begin{aligned} &\pi_*\mu(\varphi_{-t}(A)) \\ &\geq \inf_A \exp\Big(- \int_0^t (|H_p x_1|(\varphi_{-t}(\rho))^{-1} \delta(x_1(\varphi_{-t}(\rho))) \log \alpha(\pi_{\Gamma_D}(\varphi_t(\rho))) \\ &\qquad\qquad - \frac{\sigma_\hbar(\mathcal{N})}{2\sigma_\hbar(\mathcal{D})} x_1 1_{\mathcal{G}}(\varphi_{-s}(\rho))) ds\Big) \pi_*\mu(A) \\ &\geq \mathrm{e}^{\varepsilon_1} \pi_*\mu(A) \end{aligned}$$

as claimed.

Therefore, it is enough to prove (3.18). Seeking a contradiction, we assume that for every $\varepsilon_1 > 0$ and $T > 0$ there exists $\rho \in \Gamma_+$ with $\mathbf{r}(\rho) < r_0$ such that

$$\begin{aligned} &\int_0^T \Big(- \frac{\sigma_\hbar(\mathcal{N})}{2\sigma_\hbar(\mathcal{D})} H_p^2 x_1 1_{\mathcal{G}}(\varphi_{-t}(\rho)) \\ &\qquad + |H_p x_1(\varphi_{-t}(\rho))|^{-1} \delta(x_1(\varphi_{-t}(\rho))) \log \alpha(\pi_{\Gamma_D}(\varphi_{-t}(\rho)))\Big) dt \geq -\varepsilon_1. \end{aligned} \tag{3.22}$$

Note that since both terms are nonpositive (since $\alpha \leq 1$ and $\sigma_\hbar(\mathcal{N})\sigma_\hbar(\mathcal{D}) > 0$), this implies that each term is $\geq -\varepsilon_1$.

Now, if $\varphi_{-t}(\rho) \in \mathcal{G}$ for $t \in [t_1, t_2]$, then, since the flow in $\mathcal{G}$ is given by the flow of the vector field

$$H_p^G := H_p + \frac{H_p^2 x_1}{H_{x_1}^2 p} H_{x_1}, \qquad p = |\xi|^2 - 1$$

(see [31, Definition 24.3.6]), we obtain, using that $\sigma_\hbar(\mathcal{N})/\sigma_\hbar(\mathcal{D}) > c > 0$ on $\mathcal{G}$ (since $\mathcal{G} \subset \overline{B^*\Gamma_D}$),

$$\begin{aligned}\varphi_{-t_2}(\rho) &= \exp(-(t_2-t_1)H_{|\xi|^2}(\rho)) + \mathcal{O}\left(\int_{t_1}^{t_2} H_p^2 x_1(\varphi_{-t}(\rho))dt\right)\\ &= \exp(-(t_2-t_1)H_{|\xi|^2}(\rho)) + \mathcal{O}\left(\int_{t_1}^{t_2} \frac{\sigma_\hbar(\mathcal{N})}{2\sigma_\hbar(\mathcal{D})} H_p^2 x_1(\varphi_{-t}(\rho))dt\right),\end{aligned}$$

where both here and in the rest of this proof we write $a = b + \mathcal{O}(c)$ if $|a-b| \leq Cc$ for some $C > 0$ depending only on $\sigma_\hbar(\mathcal{N})$ and $\sigma_\hbar(\mathcal{D})$. On the other hand, if $\varphi_{-t}(\rho) \notin \mathcal{G}$ for $t \in [t_1, t_2]$, and has exactly one intersection with Γ_D, then

$$\varphi_{-t_2}(\rho) = \exp(-(t_2-t_1)H_{|\xi|^2}(\varphi_{-t_1}(\rho)) + \mathcal{O}\left(|t_2-t_1| 2\sqrt{1-|\xi_r'|_g^2}\right),$$

where $|\xi_r'|_g$ is measured at the point of reflection. All together, since $\sigma_\hbar(\mathcal{N})\sigma_\hbar(\mathcal{D}) > c > 0$ on $|\xi'|_g \leq 1$, and thus there exists $c > 0$ such that

$$\log\alpha = -4\sqrt{1-|\xi'|_g^2}\frac{\sigma_\hbar(\mathcal{N})}{\sigma_\hbar(\mathcal{D})} + \mathcal{O}\big(1-|\xi'|_g^2\big) \leq -c\sqrt{1-|\xi'|_g^2}\frac{\sigma_\hbar(\mathcal{N})}{\sigma_\hbar(\mathcal{D})} + \mathcal{O}\big(1-|\xi'|_g^2\big),$$

we obtain from (3.22) that

$$\varphi_{-T}(\rho) = \exp(-TH_{|\xi|^2}(\rho)) + \mathcal{O}(\varepsilon_1).$$

Therefore, choosing $T \gg r_0$, and ε_1 small enough, we obtain

$$\operatorname{dist}(\pi_{\mathbb{R}}(\varphi_{-T}(\rho)), \pi_{\mathbb{R}}(\rho)) > 3r_0,$$

which is a contradiction to $\rho \in \Gamma_+ \cap \{\mathbf{r} \leq r_0\}$. ◻

We have therefore proved that

$$\|u\|_{L^2(\Omega_+)} \leq C\hbar^{-1}\left\|(P_\theta-\lambda^2)u\right\|_{L^2(\Omega_+)} + C\left\|(Q_b\gamma_{1,\hbar}^D + \gamma_0^D)u\right\|_{H_\hbar^{3/2}(\Gamma_D)}, \tag{3.23}$$

where here, and in the rest of the proof, C denotes a constant, independent of $\hbar$, λ, and z, whose value may change from line to line. To complete the proof of Lemma 3.2, we now need to obtain a bound on the $H_\hbar^2$ norm of u, as opposed to just the L^2 norm in (3.23). By a standard elliptic parametrix construction, for $\chi_1 \in C^\infty(\overline{\Omega_+})$ supported away from Γ_D, we have

$$\begin{aligned}\|\chi_1 u\|_{H_\hbar^2(\Omega_+)} &\leq C\left\|(P_\theta-\lambda^2)u\right\|_{L^2(\Omega_+)} + C\|u\|_{L^2(\Omega_+)}\\ &\leq C\hbar^{-1}\left\|(P_\theta-\lambda^2)u\right\|_{L^2(\Omega_+)} + C\left\|(Q_b\gamma_{1,\hbar}^D+\gamma_0^D)u\right\|_{H_\hbar^{3/2}(\Gamma_D)}\end{aligned}$$

by (3.23). Finally, using the trace estimate from [22, Corollary 4.2] we have for $\chi_2 \in C^\infty(\{x : \mathbf{r} \leq r_0\})$ with $\operatorname{supp}\chi_2 \Subset \mathbb{R}^d$,

$$\left\|\gamma_{1,\hbar}^D u\right\|_{L^2(\Gamma_D)} \leq C\|\chi_2 u\|_{L^2(\Omega_+)} + \left\|(-\hbar^2\Delta-1)\chi_2 u\right\|_{L^2(\Omega_+)}.$$

Elliptic regularity for the Laplacian then implies that

$$\begin{aligned}\|\chi_2 u\|_{H_\hbar^2(\Omega_+)} &\leq C\left\|(-\hbar^2\Delta-\lambda^2)\chi_2 u\right\|_{L^2} + C\|\chi u\|_{L^2} + C\left\|\gamma_0^D u\right\|_{H_\hbar^{3/2}(\Gamma_D)}\\ &\leq C\hbar^{-1}\left\|(P_\theta-\lambda^2)u\right\|_{L^2} + C\left\|(Q_b\gamma_{1,\hbar}^D+\gamma_0^D)u\right\|_{H_\hbar^{3/2}(\Gamma_D)},\end{aligned}$$

where we have used (3.23). Combining the bounds on $\|\chi_1 u\|_{H_\hbar^2(\Omega_+)}$, $\|\chi_2 u\|_{H_\hbar^2(\Omega_+)}$, and $\|\gamma_{1,\hbar}^D u\|_{L^2(\Gamma_D)}$, we obtain (3.4). ◻

4. Proof of Lemma 1.14. With $R(\lambda, z)$ defined by (1.11), $R_\theta(\lambda, z)$ defined by (3.9), and $\chi \in C^\infty$ with $\operatorname{supp}\chi \subset \{x : \mathbf{r} \le r_1\}$ and $\operatorname{supp}\chi \Subset \mathbb{R}^d$, (2.30) implies that

$$\chi R_\theta(\lambda, z)\chi = \chi R(\lambda, z)\chi. \tag{4.1}$$

Recalling (1.12), we see that to prove the bounds (1.15), (1.16) it is sufficient to bound

$$\|R_\theta(\lambda, z)\|_{L^2(\Omega_{\rm tr})\to L^2(\Omega_{\rm tr})}.$$

We first focus on proving the bound for $\operatorname{Im} z > 0$ (1.16). By the definitions of $\mathcal{P}_\theta(\lambda, z)$ (3.6) and $R_\theta(\lambda, z)$ (3.9), the bound (1.16) follows if we can prove the following.

LEMMA 4.1. *There exists $C > 0$ such that if $\operatorname{Re}\lambda > 0$, $\operatorname{Im}\lambda = 0$, then*

$$\|\mathcal{P}_\theta(\lambda, z)^{-1}\|_{L^2(\Omega_{\rm tr})\otimes H_\hbar^{3/2}(\Gamma_D)\to L^2(\Omega_{\rm tr})} \le C\langle z\rangle(\operatorname{Im} z)^{-1} \quad \textit{for } \operatorname{Im} z > 0. \tag{4.2}$$

Moreover, there exists $\varepsilon > 0$ small enough such that if $\operatorname{Re}\lambda > 0$, then $\operatorname{Im}\lambda = 0$ and

$$\|\mathcal{P}_\theta(\lambda, z)^{-1}\|_{L^2(\Omega_+)\otimes H_\hbar^{3/2}(\Gamma_D)\to H_\hbar^2(\Omega_+)} \le C(\operatorname{Im} z)^{-1} \quad \textit{for } \operatorname{Im} z > 0 \textit{ and } |z| \le \varepsilon\hbar. \tag{4.3}$$

To prove Lemma 4.1, we need the following result about the sign of the Dirichlet-to-Neumann map.

LEMMA 4.2. *For $\operatorname{Re}\lambda > 0$ and $\operatorname{Im}\lambda \ge 0$, we have $\operatorname{Im}\mathcal{D}(\lambda/\hbar) \ge 0$.*

Proof. Let $G(\lambda)$ be the meromorphic continuation from $\operatorname{Im}\lambda > 0$ of the solution operator satisfying

$$(-\hbar^2\Delta - \lambda^2)G(\lambda)g = 0 \text{ in } \mathbb{R}^d \setminus \overline{\Omega_1}, \qquad G(\lambda)g|_{\Gamma_{\rm tr}} = g,$$

and G is $\lambda/\hbar$-outgoing; then $\mathcal{D}(\lambda/\hbar) = \gamma_1^{\rm tr} G(\lambda)$. Note that for $\operatorname{Im}\lambda > 0$, $G(\lambda) : H^{1/2}(\Gamma_{\rm tr}) \to H^1(\mathbb{R}^d \setminus \Omega_1)$. Therefore, for $\operatorname{Re}\lambda > 0$ and $\operatorname{Im}\lambda > 0$, by integration by parts,

$$\begin{aligned} 0 &= \big\langle(-\hbar^2\Delta - \lambda^2)G(\lambda)g, G(\lambda)g\big\rangle_{\mathbb{R}^d\setminus\Omega_1} \\ &= \|\hbar\nabla G(\lambda)g\|^2_{L^2(\mathbb{R}^d\setminus\Omega_1)} - \lambda^2\|G(\lambda)g\|^2_{L^2(\mathbb{R}^d\setminus\Omega_1)} + \hbar^2\big\langle\mathcal{D}(\lambda/\hbar)g, g\big\rangle_{\Gamma_{\rm tr}}. \end{aligned}$$

Therefore, taking imaginary parts,

$$2\operatorname{Re}\lambda\operatorname{Im}\lambda\|G(\lambda)g\|_{L^2(\mathbb{R}^d\setminus\Omega_1)} = \hbar^2\operatorname{Im}\langle\mathcal{D}(\lambda/\hbar)g, g\rangle_{\Gamma_{\rm tr}},$$

and in particular, for $\operatorname{Re}\lambda > 0$ and $\operatorname{Im}\lambda > 0$,

$$0 \le \operatorname{Im}\langle\mathcal{D}(\lambda/\hbar)g, g\rangle_{\Gamma_{\rm tr}}.$$

Now, since the right-hand side continues analytically from $\operatorname{Im}\lambda > 0$ to $\operatorname{Im}\lambda = 0$, we have

$$\operatorname{Im}\langle\mathcal{D}(\lambda/\hbar)g, g\rangle_{\Gamma_{\rm tr}} \ge 0$$

for $\operatorname{Re}\lambda > 0$ and $\operatorname{Im}\lambda = 0$. ∎

Proof of Lemma 4.1. Let $u \in H^2_{\rm loc}(\Omega_+)$. Then let $v = u - E\gamma_D u \in H^2_{\rm loc}(\Omega_+) \cap H^1_{0,\rm loc}(\Omega_+)$. By integration by parts,

$$\begin{aligned} -\operatorname{Im}\big\langle(P_\theta - \lambda^2 - z1_{\Omega_{\rm tr}})v, v\big\rangle_{\Omega_{\rm tr}} &= -\operatorname{Im}\big\langle(-\hbar^2\Delta - \lambda^2 - z1_{\Omega_{\rm tr}})v, v\big\rangle_{\Omega_{\rm tr}} \\ &= (\operatorname{Im} z)\|v\|^2_{L^2(\Omega_{\rm tr})} + \hbar^2\operatorname{Im}\big\langle\mathcal{D}(\lambda/\hbar)v, v\big\rangle_{\Gamma_{\rm tr}} \\ &\ge (\operatorname{Im} z)\|v\|^2_{L^2(\Omega_{\rm tr})}. \end{aligned}$$

Therefore, there exist $C, C_1, C_2 > 0$ such that for $\operatorname{Im} z > 0$,

$$\begin{aligned}\|u\|_{L^2(\Omega_{\rm tr})} &\leq \|v\|_{L^2(\Omega_{\rm tr})} + \|E\gamma_0^D u\|_{L^2(\Omega_{\rm tr})} \\ &\leq (\operatorname{Im} z)^{-1}\|(-\hbar^2\Delta - \lambda^2 - z1_{\Omega_{\rm tr}})v\|_{L^2(\Omega_{\rm tr})} + C_1\|\gamma_0^D u\|_{H_\hbar^{3/2}(\Gamma_D)} \\ &\leq (\operatorname{Im} z)^{-1}\|(P_\theta - \lambda^2 - z1_{\Omega_{\rm tr}})u\|_{L^2(\Omega_{\rm tr})} + C_2\langle z\rangle(\operatorname{Im} z)^{-1}\|E\gamma_0^D u\|_{H_\hbar^2(\Omega_{\rm tr})} \\ &\qquad + C_1\|\gamma_0^D u\|_{H_\hbar^{3/2}(\Gamma_D)} \\ &\leq C\langle z\rangle(\operatorname{Im} z)^{-1}\,\|\mathcal{P}_\theta(\lambda, z)\|_{L^2(\Omega_{\rm tr})\oplus H_\hbar^{3/2}(\Gamma_D)}\end{aligned}$$

by the definition of $\mathcal{P}_\theta(\lambda, z)$ (3.6). Having obtained the bound (4.2) on $\|u\|_{L^2(\Omega_{\rm tr})}$, we now prove the bound (4.3) on $\|u\|_{H_\hbar^2(\Omega_{\rm tr})}$. Using, e.g., the trace estimate from [22, Corollary 4.2] (in a way similar to the end of the proof of Lemma 3.2), we have

$$\|\gamma_{1,\hbar}^D u\|_{L^2(\Gamma_D)} \leq C\hbar^{-1}\|(-\hbar^2\Delta - \lambda^2 - z1_{\Omega_{\rm tr}})u\|_{L^2(\Omega_{\rm tr})} + C\langle z\rangle\|u\|_{L^2(\Omega_{\rm tr})}. \tag{4.4}$$

Furthermore, by (3.5) there exists $\varepsilon > 0$ small enough such that for $\operatorname{Im} z > 0$ and $|z| \leq \varepsilon\hbar$, $(\mathcal{P}_{\theta,Q}(\lambda, z))^{-1}$ exists, and then, by Lemma 3.2 and reducing ε further if necessary,

$$\|u\|_{H_\hbar^2(\Omega_+)} \leq C\hbar^{-1}\left\|(P_\theta - \lambda^2 - z1_{\Omega_{\rm tr}})u\right\|_{L^2(\Omega_+)} + C\left\|(Q_b\gamma_{1,\hbar}^D + \gamma_0^D)u\right\|_{H_\hbar^{3/2}(\Gamma_D)}.$$

By (3.2) and the Calderon–Vaillancourt theorem (see, e.g., [15, Proposition E.24], [52, Theorem 13.13]), $\|Q_b\|_{L^2(\Gamma_D)\to H_\hbar^{3/2}(\Gamma_D)} \leq C$. Using this along with (4.4), the fact that $P_\theta = -\hbar^2\Delta$ on $\Omega_{\rm tr}$, and (4.2), we obtain

$$\|u\|_{H_\hbar^2(\Omega_+)} \leq C\big(\hbar^{-1} + \langle z\rangle^2(\operatorname{Im} z)^{-1}\big)\left\|(P_\theta - \lambda^2 - z1_{\Omega_{\rm tr}})u\right\|_{L^2(\Omega_+)} + C\left\|\gamma_0^D u\right\|_{H_\hbar^{3/2}(\Gamma_D)},$$

which implies (4.3); the proof is complete. ∎

Having proved the bound (1.16), we now prove the bound (1.15). From (3.10),

$$R_\theta(\lambda, z) = \mathcal{R}_{\theta,Q}(\lambda, z)(I + K(\lambda, z))^{-1}\begin{pmatrix} I \\ 0\end{pmatrix}, \tag{4.5}$$

where $K(\lambda, z)$ is defined by (3.7). Since we have the bound (3.5) on $\mathcal{R}_{\theta,Q}(\lambda, z)$, to bound $R_\theta(\lambda, z)$ we only need to bound $(I + K(\lambda, z))^{-1}$.

Let $\mathcal{H} := L^2(\Omega_+) \oplus H_\hbar^{3/2}(\Gamma_D)$. Recalling the definition of trace class operators (see [15, Definition B.17]) and [15, equation B.4.7], since $\mathcal{R}_{\theta,Q}(\lambda, z)$ exists for $|z| \leq \varepsilon\hbar$, $K(\lambda, z)$ defined by (3.7) is trace class for $|z| \leq \varepsilon\hbar$ with

$$\begin{aligned}\|K(\lambda, z)\|_{\mathcal{L}_1(\mathcal{H};\mathcal{H})} &\leq \|Q_b\|_{\mathcal{L}_1(L^2(\Gamma_D);H^{3/2}(\Gamma_D))}\|\gamma_{1,\hbar}^D R_{\theta,Q}(\lambda, z)\|_{\mathcal{H}\to L^2(\Gamma_D)} \\ &\leq C\big\|\langle hD\rangle^{3/2}Q_b\big\|_{\mathcal{L}_1(L^2(\Gamma_D))}\|\gamma_{1,\hbar}^D R_{\theta,Q}(\lambda, z)\|_{\mathcal{H}\to L^2(\Gamma_D)}.\end{aligned}$$

Then, using reasoning similar to that in [15, page 434] to bound the norm of $\langle hD\rangle^{3/2}Q_b$ together with the bound (3.5) on $\gamma_{1,\hbar}^D R_{\theta,Q}(\lambda, z)$, we have

$$\|K(\lambda, z)\|_{\mathcal{L}_1(\mathcal{H};\mathcal{H})} \leq C\hbar^{1-d}\hbar^{-1} \leq C\hbar^{-d}. \tag{4.6}$$

Furthermore, by [15, equation B.5.21] and [15, equation B.5.19],

$$\begin{aligned}\big\|(I + K(\lambda, z))^{-1}\big\|_{\mathcal{H}\to\mathcal{H}} &\leq \det\big(I + K(\lambda, z)\big)^{-1}\det\big(I + [K(\lambda, z)^*K(\lambda, z)]^{1/2}\big) \\ &\leq \det\big(I + K(\lambda, z)\big)^{-1}\exp\big(\|[K(\lambda, z)^*K(\lambda, z)]^{1/2}\|_{\mathcal{L}_1(\mathcal{H})}\big) \\ &\leq \det\big(I + K(\lambda, z)\big)^{-1}\exp\big(\|[K(\lambda, z)\|_{\mathcal{L}_1(\mathcal{H})}\big),\end{aligned} \tag{4.7}$$

where we have used the definition of the trace class norm $\|\cdot\|_{\mathcal{L}_1}$ in terms of singular values (see [15, equation B.4.2]) to write

$$\left\|[K(\lambda,z)^*K(\lambda,z)]^{1/2}\right\|_{\mathcal{L}_1(\mathcal{H})} = \|K(\lambda,z)\|_{\mathcal{L}_1(\mathcal{H})}.$$

Using (4.6) in (4.7), we find that

$$\left\|(I+K(\lambda,z))^{-1}\right\|_{L^2\to L^2} \le \det\big(I+K(\lambda,z)\big)^{-1}\exp(C\hbar^{-d}) \quad \text{for } |z|\le\varepsilon\hbar. \tag{4.8}$$

To estimate $\det(I+K(\lambda,z))^{-1}$, we use the same idea used to prove the bound (1.13), namely the following complex-analysis result.

Lemma 4.3 (see [15, equation D.1.13]). *Let $\Omega_0 \Subset \Omega_1 \Subset \mathbb{C}$, let f be holomorphic in a neighborhood of Ω_1 with zeros $z_j, j=1,2,\ldots,$ and let $z_0\in\Omega_1$. There exists $C=C(\Omega_0,\Omega_1,z_0)$ such that for any $\delta>0$ sufficiently small,*

$$\log|f(z)| \ge -C\log\left(\delta^{-1}\right)\left(\max_{z\in\Omega_1}\log|f(z)| - \log|f(z_0)|\right) \quad \textit{for } z\in\Omega_0\setminus\bigcup_j B(z_j,\delta).$$

Applying this result with $f(z)=\det(I+K(\lambda,z))$, we see that to get an upper bound on $\log\det(I+K(\lambda,z))^{-1}$ we only need a lower bound on $\det(I+K(\lambda,z_0))$ for some $|z_0|\le\varepsilon\hbar$ and an upper bound on $\det(I+K(\lambda,z))$ for all $|z|\le\varepsilon\hbar$.

To obtain the upper bound for all $|z|\le\varepsilon\hbar$, we again use [15, equation B.5.19] and (4.6) to obtain

$$|\det(I+K(\lambda,z))| \le \exp(\|K(\lambda,z)\|_{\mathcal{L}_1}) \le \exp(C\hbar^{-d}) \quad \text{for } |z|\le\varepsilon\hbar. \tag{4.9}$$

To obtain the lower bound for some $|z_0|\le\varepsilon\hbar$, we first observe that, from (3.8),

$$\big(I+K(\lambda,z)\big)^{-1} = \mathcal{P}_{\theta,Q}(\lambda,z)\mathcal{P}_\theta(\lambda,z)^{-1} = I - Q\mathcal{P}_\theta(\lambda,z)^{-1},$$

so that

$$\left|\det\big(I+K(\lambda,z)\big)\right|^{-1} = \left|\det\big(I-Q\mathcal{P}_\theta(\lambda,z)^{-1}\big)\right|.$$

Since $Q\mathcal{P}_\theta(\lambda,z)$ is trace class, we use [15, equation B.5.19], [15, equation B.4.7], (4.6), and (4.3) to obtain

$$\log\left|\det(I+K(\lambda,z_0))\right|^{-1} \le \|Q\|_{\mathcal{L}_1(H^2_\hbar(\Omega_+);\mathcal{H})}\left\|\mathcal{P}_\theta(\lambda,z_0)^{-1}\right\|_{\mathcal{H}\to H^2_\hbar(\Omega_+)} \le C\hbar^{-d} \tag{4.10}$$

for $z_0 = \mathrm{i}\varepsilon\hbar$. Therefore, combining Lemma 4.3, (4.9), and (4.10), we have

$$\log\left|\det\big(I+K(\lambda,z)\big)^{-1}\right| \le C\hbar^{-d}\log\delta^{-1}, \qquad z\in B(0,\varepsilon_1\hbar)\Big\backslash\bigcup_{z_j}B(z_j,\delta),$$

where z_j are the poles of $(I+K(\lambda,z))^{-1}$. Therefore, combining this last bound with (4.5), (4.8), and (3.5), we have

$$\|R_\theta(\lambda,z)\|_{L^2(\Omega_+)\to L^2(\Omega_+)} \le \exp\left(C\hbar^{-d}\log\delta^{-1}\right) \quad \text{for } z\in B(0,\varepsilon_1\hbar)\Big\backslash\bigcup_{z_j}B(z_j,\delta),$$

where z_j are the poles of $\mathcal{R}_\theta(\lambda,z)$. The bound (1.15) and the fact that z_j are the poles of $R_{\Omega_{\rm tr}}(\lambda,z)$ then follow from the relation (4.1) and Lemma 1.11.

5. Proofs of Theorems 2.2 and 2.4.

5.1. Proof of Theorem 2.2. With Lemma 1.14 in hand, this proof is very similar to [15, Proof of Theorem 7.6], except that now we work in the complex z plane as opposed to the complex λ plane. In addition, in this proof, the roles of ε_0 and ε are swapped compared to [15, Proof of Theorem 7.6].

Let

$$\varepsilon_0(\hbar) := \hbar^{-\alpha}\varepsilon(\hbar), \tag{5.1}$$

with $\alpha > 3(d+1)/2$ (we see later where this requirement comes from). The lower bound (2.2) then implies that, given $\hbar_0$, there exists C' (depending on $\hbar_0$ and α) such that

$$\log\left(\frac{2}{\varepsilon_0(\hbar)}\right) \leq \frac{C'}{\hbar} \quad \text{for all } 0 < \hbar \leq \hbar_0. \tag{5.2}$$

Seeking a contradiction, we assume that when $\hbar = \hbar_j$ there are no eigenvalues in $B(0, \varepsilon_0(\hbar_j))$ (the exponential lower bound on $\varepsilon_0(\hbar)$ leading to (5.2) therefore limits how small this ball can be). Our goal is to show that this assumption implies that

$$\|R(1,0)\|_{L^2(\Omega_{\rm tr})\to L^2(\Omega_{\rm tr})} < \frac{1}{2}\big(\varepsilon(\hbar_j)\big)^{-1}. \tag{5.3}$$

Indeed, since $\operatorname{supp} u_\ell \Subset \Omega_1$,

$$R(1,0)(-\hbar_j^2\Delta - 1)u_\ell = u_\ell. \tag{5.4}$$

Then, by taking the norm of (5.4) and using (5.3), we obtain that $\|u_\ell\|_{L^2(\Omega_{\rm tr})} < 1/2$, which contradicts $\|u_\ell\|_{L^2(\Omega_{\rm tr})} = 1$. We prove (5.3) by using Theorem 2.7, where $\Omega(\hbar)$ is a box (to be specified below) in $B(0, \varepsilon_0(\hbar)/2)$ with Lemma 1.14 providing the bounds (2.20) and (2.21).

We first use the bound (1.15) from Lemma 1.14. This bound is valid for $z \in B(0, \varepsilon_1\hbar)$ and away from the poles. The definition of $\varepsilon_0(\hbar)$ (5.1) and the upper bound in (2.3) implies that $B(0, \varepsilon_0(\hbar)/2) \subset B(0, \varepsilon_1\hbar)$ for $\hbar$ sufficiently small. We then choose δ in (1.15) equal to $\varepsilon_0(\hbar)/2$ and use (5.2) so that, for all $\hbar_j$ sufficiently small,

$$\|R(1,z)\|_{L^2(\Omega_{\rm tr})\to L^2(\Omega_{\rm tr})} \leq \exp\left(C_1 C' \hbar_j^{-(d+1)}\right) \quad \text{for all } z \in B(0, \varepsilon_0(\hbar_j)/2) \tag{5.5}$$

and thus for all $z \in \Omega(\hbar_j)$ (since $\Omega(\hbar_j) \subset B(0, \varepsilon_0(\hbar_j)/2)$). We now let

$$Q(z,\hbar) := R_{\Omega_{\rm tr}}(1,z), \quad L := d+1, \quad \text{and } C := \max\big\{C_1C',\, C_2\, c\big\},$$

where $c = c(\hbar_0)$ is chosen large enough such that $\langle z\rangle \leq c$ for all $z \in B(0, \varepsilon_0(\hbar)/2)$ and $\hbar \leq h_0$; these choices ensure that the right-hand sides of the bounds (5.5) and (1.16) are bounded by the right-hand sides of (2.20) and (2.21), respectively. We then let

$$w = 0, \quad 2\beta(\hbar) = \frac{1}{4}\varepsilon_0(\hbar), \quad \text{and} \quad \delta(\hbar) = M\varepsilon(\hbar)$$

with M chosen (sufficiently large) later in the proof. For the assumptions of Theorem 2.7 to hold at $\hbar = \hbar_j$, we need that (i) the box $\Omega(\hbar_j)$ defined by (2.18) is inside

$B(0,\varepsilon_0(\hbar_j)/2)$ (so that the bound (2.20) follows from (5.5)), and (ii) the second inequality in (2.19) is satisfied. The first requirement is ensured if

$$\delta(\hbar_j)\hbar_j^{-(d+1)} \ll \frac{1}{2}\varepsilon_0(\hbar_j), \quad \text{that is,} \quad M\varepsilon(\hbar_j)\hbar_j^{-(d+1)} \ll \frac{1}{2}\hbar_j^{-\alpha}\varepsilon(\hbar_j),$$

which is satisfied if $\hbar_j$ is sufficiently small since $\alpha > d+1$. The second requirement is

$$\frac{1}{8}h^{-2\alpha}\varepsilon(\hbar)^2 \geq C\hbar^{-3(d+1)}M\varepsilon(\hbar)^2;$$

given M, this inequality is satisfied when $\hbar$ is sufficiently small since $\alpha > 3(d+1)/2$.

Therefore, the assumptions of Theorem 2.7 are all satisfied at $\hbar = \hbar_j$ (for $\hbar_j$ sufficiently small), and the result is that the bound (2.22) holds for all $z \in [-\beta(\hbar_j), \beta(\hbar_j)]$ and thus, in particular, at $z = 0$. Therefore, for all $\hbar_j$ sufficiently small,

$$\|R(1,0)\|_{L^2(\Omega_{\rm tr})\to L^2(\Omega_{\rm tr})} \leq \frac{C}{M\varepsilon(\hbar_j)}\exp(1+C).$$

We now choose

$$M := 2C\exp(1+C)$$

and obtain (5.3), i.e., the desired contradiction to there being no eigenvalues in $B(0,\varepsilon_0(\hbar_j))$.

5.2. Proof of Theorem 2.4. We first recall the following lemma proved in [43, Lemma 4]; see also [35, Lemma AII.20].

LEMMA 5.1. *Let $f_1,\dots,f_N$ be N vectors in a Hilbert space $\mathcal{H}$ with*

$$\big|\langle f_i, f_j\rangle_{\mathcal{H}} - \delta_{ij}\big| \leq \varepsilon \quad \text{for all } \ i,j = 1,\dots,N.$$

If $\varepsilon < N^{-1}$, then $f_1,\dots,f_N$ are linearly independent.

We use Lemma 5.1 both in the proof of Theorem 2.4 below and in the proof of the following preparatory result.

LEMMA 5.2. *Let $m(\hbar_j)$ and $\varepsilon(\hbar)$ be as in Theorem* 2.4 *(so that, in particular, $\varepsilon(\hbar) \ll \hbar^{(5d+3)/2}$ as $\hbar \to 0$). Then there exists $C > 0$ (independent of $\hbar_j$) such that*

$$m(\hbar_j) \leq C\hbar_j^{-d}. \tag{5.6}$$

Proof. First observe that it is sufficient to prove the result for sufficiently small $\hbar_j$ (equivalently, sufficiently large j). Let $P(\hbar_j) = -\hbar_j^2\Delta$ with zero Dirichlet boundary conditions on Γ_D and $\Gamma_{\rm tr}$. $P(\hbar_j)$ is therefore self-adjoint with a discrete spectrum and, since $\operatorname{supp} u_{j,\ell} \subset \mathcal{K} \Subset \Omega_1$,

$$\big\|\big(P(\hbar_j) - E_{j,\ell}\big)u_{j,\ell}\big\|_{L^2(\Omega_{\rm tr})} = \varepsilon(\hbar_j) \quad \text{for all } j,\ell.$$

Let $\mu > c > 0$, let $\Pi(\hbar_j)$ be the orthogonal projection onto the eigenspaces corresponding to all eigenvalues of $P(\hbar_j)$ in $[a_0-\mu, b_0+\mu]$, and let $M(\hbar_j)$ be the number of these eigenvalues (counting multiplicities). By the Weyl law (with no remainder term) on manifolds with boundary (see, e.g., [31, Theorem 17.5.3]),

$$M(\hbar_j) \leq C\hbar_j^{-d}.$$

Furthermore, $\operatorname{rank}\Pi(\hbar_j) \le M(\hbar_j)$, and thus to prove the result (5.6) it is sufficient to prove that $m(\hbar_j) \le \operatorname{rank}\Pi(\hbar_j)$. To keep expressions compact, we now write P and Π instead of $P(\hbar_j)$ and $\Pi(\hbar_j)$.

Since Π commutes with $(P - E_{j,\ell})^{-1}$, and $(P - E_{j,\ell})$ is invertible on $(I-\Pi)L^2$,

$$(I-\Pi)u_{j,\ell} = \big(P - E_{j,\ell}\big)^{-1}\big(I-\Pi\big)\big(P-E_{j,\ell}\big)u_{j,\ell}. \tag{5.7}$$

Since P is self-adjoint, the spectral theorem (see, e.g., [15, Theorem B.8]) implies that

$$\big\|\big(P - E_{j,\ell}\big)^{-1}\big(I-\Pi\big)\big\|_{L^2(\Omega_{\rm tr})\to L^2(\Omega_{\rm tr})} \le \frac{1}{\mu}. \tag{5.8}$$

Therefore, combining (5.7) and (5.8), we have

$$\big\|\big(I-\Pi\big)u_{j,\ell}\big\|_{L^2(\Omega_{\rm tr})\to L^2(\Omega_{\rm tr})} \le \frac{\varepsilon(\hbar_j)}{\mu}$$

(compare to [35, equation 32.2] and the first displayed equation in [43, section 3]). Then, for $\ell_1, \ell_2 \in \{1,\dots,m(\hbar_j)\}$,

$$\begin{aligned}\big|\langle \Pi u_{j,\ell_1}, \Pi u_{j,\ell_2}\rangle_{L^2(\Omega_{\rm tr})} - \delta_{\ell_1\ell_2}\big| &\le \big|\langle u_{j,\ell_1}, u_{j,\ell_2}\rangle_{L^2(\Omega_{\rm tr})} - \delta_{\ell_1\ell_2}\big| \\ &\quad + \big|\langle u_{j,\ell_1}, (I-\Pi)u_{j,\ell_2}\rangle_{L^2(\Omega_{\rm tr})}\big| \\ &\quad + \big|\langle (I-\Pi)u_{j,\ell_1}, \Pi u_{j,\ell_2}\rangle_{L^2(\Omega_{\rm tr})}\big| \\ &\le \hbar_j^{-2}\varepsilon(\hbar_j) + \frac{2}{\mu}\varepsilon(\hbar_j) \\ &\ll \hbar_j^{(5d-1)/2} \qquad \text{as } j\to\infty,\end{aligned} \tag{5.9}$$

where we have used that $\|\Pi\|_{L^2(\Omega_{\rm tr})\to L^2(\Omega_{\rm tr})} \le 1$ since Π is orthogonal. By Lemma 5.1, any subset of $\{\Pi u_{j,\ell}\}_{\ell=1}^{m(\hbar_j)}$ with cardinality $\ll \hbar_j^{-(5d-1)/2}$ is linearly independent. Seeking a contradiction, assume that (5.6) does not hold, i.e., for all $C>0$, there exists j such that $m(\hbar_j) > C\hbar_j^{-d}$. Choose a subset of $\{\Pi u_{j,\ell}\}_{\ell=1}^{m(\hbar_j)}$ with cardinality $\lfloor C\hbar_j^{-d} + 1\rfloor$. By the above argument, this subset is linearly independent, and thus $\lfloor C\hbar_j^{-d}+1\rfloor \le \operatorname{rank}\Pi(\hbar_j) = M(\hbar_j) \le C\hbar_j^{-d}$, which is the required contradiction. ∎

Proof of Theorem 2.4. The proof is similar to that of the corresponding "quasimodes to resonances" result [43, Theorem 1] (see also [15, section 7.7, Exercise 1]), except that we use the semiclassical maximum principle in the z plane (as in the proof of Theorem 2.2), and now we also work in an interval in λ (as opposed to at $\lambda = 1$ in the proof of Theorem 2.2). To keep the expressions compact, we write $\hbar$ instead of $\hbar_j$ and write functions of the index j as functions of $\hbar$; in particular, we drop the subscript j on $\hbar_j, E_{j,\ell}$, and $u_{j,\ell}$.

Let

$$\mathcal{Z} := \mathcal{Z}\big(\varepsilon_1(\hbar)\,,\, \varepsilon_0(\hbar)\,,\, a(\hbar),\, b(\hbar)\,;\, \hbar\big),$$

where $\mathcal{Z}(\varepsilon_1,\varepsilon_0,a,b;\hbar)$ is defined by (2.5), $\varepsilon_0(\hbar)$ is as in the statement of the theorem, and $\varepsilon_1(\hbar) \ll \hbar$ will be fixed later. We assume throughout that $|\mathcal{Z}| < \infty$ since otherwise the proof is trivial. Let $\Pi(\hbar)$ denote the orthogonal projection onto

$$\bigcup_{p\in\mathcal{Z}} \Pi_{z_p}(L^2(\Omega_{\rm tr})),$$

where Π_{z_p} is defined in (2.17). Let $\widetilde{\mathcal{Z}}(\lambda)$ be the set of distinct values of $z_p(\hbar,\lambda)$ such that $p\in\mathcal{Z}$. (While $\mathcal{Z}$ is independent of λ, $\widetilde{\mathcal{Z}}$ depends on λ since the poles of $z\mapsto R_{\Omega_{\rm tr}}(z,\lambda)$ depend on λ.) Note that for $z_p\neq z_q$, $\operatorname{rank}(\Pi_{z_p}+\Pi_{z_q})=\operatorname{rank}\Pi_{z_p}+\operatorname{rank}\Pi_{z_q}$; therefore,

$$\operatorname{rank}\Pi(\hbar)=\sum_{z_p\in\widetilde{\mathcal{Z}}(\lambda)}\operatorname{rank}\Pi_{z_p(\hbar,\lambda)}=\sum_{z_p\in\widetilde{\mathcal{Z}}(\lambda)}m_R\big(z_p(\hbar,\lambda)\big)=|\mathcal{Z}|,$$

where $m_R(z_0)$ is defined in (2.17). To prove the theorem, therefore, it is sufficient to show that $m(\hbar)\leq\operatorname{rank}\Pi(\hbar)$.

Seeking a contradiction, we assume that $\operatorname{rank}\Pi(\hbar)<m(\hbar)$. By Lemma 2.6, near z_p, the singular part of $R_{\Omega_{\rm tr}}(\lambda,z)$ is in the range of $\Pi_{z_p}(\hbar,\lambda)$, and therefore $z\mapsto(I-\Pi(\hbar))R_{\Omega_{\rm tr}}(\lambda,z)$ is holomorphic on

$$\Omega(\hbar):=\big(-2\varepsilon_1(\hbar),\,2\varepsilon_1(\hbar)\big)-\mathrm{i}\big(0,2\varepsilon_0(\hbar)\big)$$

for all $\lambda^2\in[a(\hbar),b(\hbar)]$. Let $\widetilde{\Omega}(\hbar)\subset\Omega(\hbar)$ be defined by

$$\widetilde{\Omega}(\hbar):=\big(-\varepsilon_1(\hbar),\,\varepsilon_1(\hbar)\big)-\mathrm{i}\big(0,\varepsilon_0(\hbar)\big).$$

Our goal is to apply the semiclassical maximum principle (Theorem 2.7) in subsets of $\widetilde{\Omega}(\hbar)$ with $Q(z,\hbar)=(I-\Pi(\hbar))R_{\Omega_{\rm tr}}(\lambda,z)$.

By Lemma 1.14, the fact that $\max(\varepsilon_0,\varepsilon_1)\ll\hbar$, and the fact that $\Pi(\hbar)$ is orthogonal (and so $\|I-\Pi(\hbar)\|_{L^2(\Omega_{\rm tr})\to L^2(\Omega_{\rm tr})}\leq 1$),

$$\big\|\big(I-\Pi(\hbar)\big)R_{\Omega_{\rm tr}}(\lambda,z)\big\|_{L^2(\Omega_{\rm tr})\to L^2(\Omega_{\rm tr})}\leq\exp\Big(C_1\hbar^{-d}\log\delta^{-1}\Big)\quad\text{for } z\in\widetilde{\Omega}(\hbar)\Big\backslash\bigcup_m B(z_m(\hbar,\lambda),\delta)\tag{5.10}$$

and for $\lambda^2\in[a(\hbar),b(\hbar)]$, where the $z_m(\hbar,\lambda)$ are the poles of $R_{\Omega_{\rm tr}}(\lambda,z)$ such that $B(z_m(\hbar,\lambda),\delta)\cap\widetilde{\Omega}(\hbar)\neq\emptyset$. If $\delta>\min\{\varepsilon_0(\hbar),\varepsilon_1(\hbar)\}$, then these $z_m(\hbar,\lambda)$ might include poles that are not equal to $z_p(\hbar,\lambda)$ for some $p\in\mathcal{Z}$, but we restrict δ so that this is not the case. Indeed, we now choose $\delta>0$ so that the bound in (5.10) holds for all $z\in\widetilde{\Omega}(\hbar)$ and for all $\lambda^2\in[a(\hbar),b(\hbar)]$.

If δ and z_m are such that $B(z_m,\delta)\Subset\Omega(\hbar)$, then the bound in (5.10) holds on $\partial B(z_m,\delta)$, and then, since $z\mapsto(I-\Pi(\hbar))R_{\Omega_{\rm tr}}(\lambda,z)$ is holomorphic in $\Omega(\hbar)$, the maximum principle implies that the bound in (5.10) holds in $B(z_m,\delta)$. We now restrict δ so that there cannot be a connected union of $B(z_m,\delta)$ that intersects both $\widetilde{\Omega}(\hbar)$ and $\partial\Omega(\hbar)$. Once this is ruled out, the maximum principle and the fact that $z\mapsto(I-\Pi(\hbar))R_{\Omega_{\rm tr}}(\lambda,z)$ is holomorphic in $\Omega(\hbar)$ imply that the bound in (5.10) holds in $\widetilde{\Omega}(\hbar)$. Since we have assumed that $\operatorname{rank}\Pi(\hbar)<m(\hbar)$, and $m(\hbar)\leq C\hbar^{-d}$ by (5.6), there exists a maximum of $C\hbar^{-d}$ of balls of radius δ. In particular, the maximum distance between any two points in such a connected union is bounded by $2C\delta\hbar^{-d}$, and hence a connected union intersecting both $\partial\Omega(\hbar)$ and $\tilde{\Omega}(\hbar)$ is ruled out if

$$2C\delta\hbar^{-d}<\min\big\{\varepsilon_0(\hbar),\varepsilon_1(\hbar)\big\}.\tag{5.11}$$

We now assume that $\varepsilon_0(\hbar)\leq\varepsilon_1(\hbar)$ and set

$$\delta:=\frac{\varepsilon_0(\hbar)\,\hbar^d}{4C},$$

so that (5.11) holds. The lower bound on $\varepsilon_0(\hbar)$ in (2.7) and the lower bound on $\varepsilon(\hbar)$ (2.2) imply that, given $\hbar_0$, there exists C' (depending on $\hbar_0$) such that

$$\log\delta^{-1} \le \frac{C'}{\hbar} \quad \text{for all } 0 < \hbar \le \hbar_0.$$

Therefore, the end result is that if $\hbar$ is sufficiently small,

$$\left\| \big(I - \Pi(\hbar)\big) R_{\Omega_{\rm tr}}(\lambda, z) \right\|_{L^2(\Omega_{\rm tr}) \to L^2(\Omega_{\rm tr})} \le \exp\left(C\hbar^{-d-1}\right)$$

for $z \in \widetilde{\Omega}(\hbar)$ and $\lambda^2 \in [a(\hbar), b(\hbar)]$, where $C := \max\{C_1 C', c\, C_2\}$, where, as in the proof of Theorem 2.2, $c = c(\hbar_0)$ is chosen large enough such that $\langle z\rangle \le c$ for all $z \in \widetilde{\Omega}(\hbar)$ and $\hbar \le h_0$.

We apply the semiclassical maximum principle (Theorem 2.7) with

$$w = 0, \quad \beta(\hbar) = \varepsilon_1(\hbar), \quad \delta(\hbar) = \hbar^{d+1}\varepsilon_0(\hbar), \quad \text{and} \quad L = d+1,$$

and we now fix $\varepsilon_1(\hbar)$ as

$$\varepsilon_1(\hbar) := \frac{\hbar^{(d+1)/2}\varepsilon_0(\hbar)}{C};$$

observe that this definition of $\varepsilon(\hbar)$ satisfies both the second requirement in (2.19) and our previous assumption that $\varepsilon_0(\hbar) \le \varepsilon_1(\hbar)$. The result of Theorem 2.7 is that

$$\begin{aligned} &\left\| \big(I - \Pi(\hbar)\big) R_{\Omega_{\rm tr}}(\lambda, z) \right\|_{L^2(\Omega_{\rm tr}) \to L^2(\Omega_{\rm tr})} \le C \exp(C+1) \frac{\hbar^{-(d+1)}}{\varepsilon_0(\hbar)} \\ &\qquad \text{for } z \in \big[-\varepsilon_1(\hbar), \varepsilon_1(\hbar)\big] \quad \text{and} \quad \lambda^2 \in [a(\hbar), b(\hbar)]. \end{aligned} \tag{5.12}$$

The definitions of E_ℓ and u_ℓ imply that

$$\big(I - \Pi(\hbar)\big) R_{\Omega_{\rm tr}}\big(\sqrt{E_\ell}, 0\big)\big(-\hbar^2\Delta - E_\ell\big) u_\ell = \big(I - \Pi(\hbar)\big) u_\ell$$

for $\ell = 1, \dots, m(\hbar)$. Since $E_\ell \in [a(\hbar), b(\hbar)]$ for all ℓ, the fact that the bound (5.12) holds for all $\lambda^2 \in [a(\hbar), b(\hbar)]$ implies that

$$\left\| (I - \Pi(\hbar)) u_\ell \right\|_{L^2(\Omega_{\rm tr}) \to L^2(\Omega_{\rm tr})} \le C \exp(C+1) \hbar^{-(d+1)} \frac{\varepsilon(\hbar)}{\varepsilon_0(\hbar)}$$

for $\ell = 1, \dots, m(\hbar)$. Therefore,

$$\left| \big\langle \Pi(\hbar) u_{\ell_1}, \Pi(\hbar) u_{\ell_2} \big\rangle_{L^2(\Omega_{\rm tr})} - \delta_{\ell_1 \ell_2} \right| \le \varepsilon(\hbar) + 2C \exp(C+1) \hbar^{-(d+1)} \frac{\varepsilon(\hbar)}{\varepsilon_0(\hbar)}$$

(compare to (5.9), but note that now the projection Π is different). Using the inequality (2.6) and the second inequality in (2.7), we have

$$\left| \big\langle \Pi(\hbar) u_{\ell_1}, \Pi(\hbar) u_{\ell_2} \big\rangle_{L^2(\Omega_{\rm tr})} - \delta_{\ell_1 \ell_2} \right| \ll \hbar^d,$$

and thus

$$\left| \big\langle \Pi(\hbar) u_{\ell_1}, \Pi(\hbar) u_{\ell_2} \big\rangle_{L^2(\Omega_{\rm tr})} - \delta_{\ell_1 \ell_2} \right| \le \frac{\hbar^d}{C},$$

where C is the constant in (5.6). By (5.6) and Lemma 5.1, $\{\Pi(h) u_\ell\}_{\ell=1}^{m(\hbar_j)}$ is linearly independent, and thus $\operatorname{rank} \Pi(\hbar) \ge m(\hbar)$, which is the desired contradiction to the assumption that $\operatorname{rank} \Pi(\hbar) < m(\hbar)$. ∎

Appendix A. From eigenvalues to quasimodes.

LEMMA A.1 (from eigenvalues to quasimodes in $\hbar$ notation). *Suppose that there exist $z = \mathcal{O}(\hbar^\infty)$ and u satisfying* (2.4) *with $\|u\|_{L^2(\Omega_{\rm tr})} = 1$. Let $\chi \in C_c^\infty(\Omega_1)$ with $\chi \equiv 1$ in a neighborhood of $\pi_{\mathbb{R}}(K)$. Then χu is a quasimode (in the sense of Definition* 2.1*) of quality $\varepsilon(\hbar) = \mathcal{O}(\hbar^\infty)$ satisfying*

$$\|u - \chi u\|_{H^2_\hbar(\Omega_{\rm tr})} = \mathcal{O}(\hbar^\infty).$$

Proof. The proof is similar to the proof of the "resonances to quasimodes" result of [44, Theorem 1], except that we avoid using results about $\mathcal{D}$ for strictly convex obstacles that are used in [44] and instead use a commutator argument.

First observe that

$$(-\hbar^2\Delta - 1 - z)u = 0 \qquad \text{in } \Omega_{\rm tr},$$

so that

$$u = 1^{\rm res}_{\Omega_{\rm tr}} R(1,0) 1^{\rm ext}_{\Omega_{\rm tr}}\, z\, u.$$

Therefore,

$$u = 1^{\rm res}_{\Omega_{\rm tr}} R_\theta(1,0) 1^{\rm ext}_{\Omega_{\rm tr}}\, z\, u$$

by (2.30) and the definition of $R_\theta(\lambda, z)$ (3.9). Let

$$v = R_\theta(1,0) 1^{\rm ext}_{\Omega_{\rm tr}} z\, u, \tag{A.1}$$

and observe that $v = u$ on $\Omega_{\rm tr}$.

We now claim that, since $z = \mathcal{O}(\hbar^\infty)$ and $\Omega_{\rm tr} \Subset \mathbb{R}^d$, $\mathrm{WF}_\hbar(v) \subset \Gamma_+$ (defined by (2.26)). By the definition of the wavefront set [15, Definition E.36], this is equivalent to $Av = \mathcal{O}(\hbar^\infty)$ for all A with $\mathrm{WF}_\hbar(A) \subset (\Gamma^+)^c$. This then follows by noting that $(P_\theta - 1)v = \mathcal{O}(\hbar^\infty)_{L^2_{\rm comp}}$ and applying [15, Theorem E.47], [31, section 24.4], and [50, Theorem 8.1][3] (with, in the notation of [15, Theorem E.47], $B_1 = I$, $B = \mathbf{P} = P_\theta - 1$), together with the fact that $\sigma_\hbar(\mathrm{Im}(P_\theta - 1)) \leq 0$ and that $P_\theta - 1$ is elliptic on $\{\mathbf{r} \geq 2r_1\}$ (so that if $(x_0, \xi_0) \in \mathrm{WF}_\hbar(A)$, then there exists $T \geq 0$ such that $\varphi_{-T}(x_0, \xi_0) \in \mathrm{ell}_\hbar(P_\theta - 1)$).

Now let $\chi \in C_c^\infty(\Omega_1)$ with $\chi \equiv 1$ in a neighborhood of $\pi_{\mathbb{R}}(K)$. We claim that $\chi v = \chi u$ is a quasimode with quality $\varepsilon(\hbar) = \mathcal{O}(\hbar^\infty)$. To prove this, since

$$\|u - \chi u\|_{H^2_\hbar(\Omega_{\rm tr})} = \|(1-\chi)v\|_{H^2_\hbar(\Omega_{\rm tr})} = \|(1-\chi)v\|_{H^2_\hbar(\Omega_{\rm tr}\setminus\{\chi\equiv 1\})}\,, \tag{A.2}$$

it is sufficient to prove that v is $\mathcal{O}(\hbar^\infty)_{H^2_{\hbar,\rm loc}}$ outside a compact set.

Our first step is to prove that, with $r_0 < a < b < r_1$, for $\hbar$ sufficiently small,

$$\|v\|_{L^2(\mathbf{r}>a)} \leq C\hbar^{-1}\|(P_\theta - 1)v\|_{L^2(\Omega_+)} + C\|v\|_{L^2(a<\mathbf{r}<b)}, \tag{A.3}$$

where here, and in the rest of the proof, C denotes a constant, independent of $\hbar$ and z, whose value may change from line to line. To prove (A.3), first observe that, since

[3] Strictly speaking, [15, Theorem E.47] is used away from the boundary and [50, Theorem 8.1] is written for the time dependent problem, but the semiclassical version can be easily recovered by applying the time dependent results to $e^{it/\hbar}v(x)$. It is then necessary to use the arguments in [31, section 24.4] to obtain the "diffractive improvement," i.e., that singularities hitting a diffractive point follow only the flow of H_p rather than sticking to the boundary. A careful examination of [31, Lemma 24.4.7] shows that the norm on the error term on $(P_\theta - 1)v$ is correct.

$P_\theta - 1$ is elliptic on $\mathbf{r} \geq 2r_1$, by [15, Theorem E.33] (more precisely its proof together with the calculus from [52, Chapter 4]),

$$\|v\|_{L^2(\mathbf{r}>3r_1)} \leq C\|(P_\theta - 1)v\|_{L^2(\Omega_+)} + C_N h^N \|v\|_{L^2(\mathbf{r}>2r_1)}$$

and hence

$$\|v\|_{L^2(\mathbf{r}>3r_1)} \leq C\|(P_\theta - 1)v\|_{L^2(\Omega_+)} + C_N h^N \|v\|_{L^2(2r_1<\mathbf{r}<4r_1)}. \tag{A.4}$$

Next, observe that there exists $T > 0$ such that for all $\rho \in \Gamma_+ \cap \{\frac{a+b}{2} < \mathbf{r} < 4r_1\}$, there exists $0 \leq t \leq T$ such that $a < \mathbf{r}(\varphi_{-t}(\rho)) < b$. In particular, using [15, Theorem E.47] again, we have

$$\|v\|_{L^2(\frac{a+b}{2}<\mathbf{r}<4r_1)} \leq C\hbar^{-1}\|(P_\theta - 1)v\|_{L^2(\Omega_+)} + \|v\|_{L^2(a<\mathbf{r}<b)} + C_N h^N \|v\|_{L^2(\mathbf{r}>a)}.$$

Using this and (A.4) in

$$\|v\|_{L^2(\mathbf{r}>a)} \leq \|v\|_{L^2(a<\mathbf{r}<b)} + \|v\|_{L^2(\frac{a+b}{2}<\mathbf{r}<4r_1)} + \|v\|_{L^2(\mathbf{r}>3r_1)},$$

we obtain (A.3) for $\hbar$ sufficiently small.

The next part of the proof involves using a commutator argument to control (up to h^∞ errors) $\|v\|_{L^2(a<\mathbf{r}<b)}$ by the norm on a slightly bigger region and with a gain of $\hbar$ (see (A.5) below). Let $\psi \in C_c^\infty(-r_1, r_1)$ with $\psi \equiv 1$ on $\{|x| \leq r_0\}$, $x\psi'(x) \leq 0$, and $x\psi'(x) < 0$ on $a \leq |x| \leq b$. Then

$$\begin{aligned}
&2\hbar^{-1} \operatorname{Im} \big\langle (-\hbar^2\Delta - 1)v, \psi(\mathbf{r})v \big\rangle_{L^2(\Omega_+)} \\
&\quad = -\mathrm{i}\hbar^{-1}\Big(\big\langle (-\hbar^2\Delta - 1)v, \psi(\mathbf{r})v \big\rangle_{L^2(\Omega_+)} - \big\langle \psi(\mathbf{r})v, (-\hbar^2\Delta - 1)v \big\rangle_{L^2(\Omega_+)} \Big) \\
&\quad = \mathrm{i}\hbar^{-1}\big\langle [-\hbar^2\Delta, \psi(\mathbf{r})]v, v \big\rangle_{L^2(\Omega_+)} \\
&\quad = \big\langle \big(2\psi'(\mathbf{r})\hbar D_r - \mathrm{i}\hbar[\Delta(\psi(\mathbf{r}))]\big)v, v \big\rangle_{L^2(\Omega_+)}.
\end{aligned}$$

By the definition of Γ_+ (2.26), $\sigma_\hbar(\psi'(\mathbf{r})hD_r) = \psi'(\mathbf{r})\langle \xi, \frac{x}{|x|}\rangle < -c < 0$ on $\Gamma_+ \cap \{a \leq \mathbf{r} \leq b\}$. Therefore, since $\mathrm{WF}_\hbar(v) \subset \Gamma_+$, for $\psi_1 \in C_c^\infty(r_0 < \mathbf{r} < r_1)$ with $\psi_1 \equiv 1$ in a neighborhood of $\operatorname{supp} \partial\psi(\mathbf{r})$,

$$\begin{aligned}
&2\hbar^{-1} \operatorname{Im} \big\langle (-\hbar^2\Delta - 1)v, \psi(\mathbf{r})v \big\rangle_{L^2(\Omega_+)} \\
&\qquad \leq -c\|v\|^2_{L^2(a<\mathbf{r}<b)} + C\hbar\|\psi_1 v\|^2_{L^2(\Omega_+)} + C_N \hbar^N \|v\|^2_{L^2(\Omega_+)}
\end{aligned}$$

by the microlocal Garding inequality [15, Proposition E.34] (with $A = -\psi'(\mathbf{r})\hbar D_r - c$ and B supported in $\langle \xi, x/|x|\rangle < \epsilon$, i.e., away from Γ^+, and in $\{r_0 \leq \mathbf{r} \leq r_1\}$). Therefore, by Young's inequality,

$$\|v\|^2_{L^2(a<\mathbf{r}<b)} \leq C\hbar^{-N-2}\|(-\hbar^2\Delta - 1)v\|^2_{L^2(\mathbf{r}<r_1)} + C\hbar\|\psi_1 v\|^2_{L^2(\Omega_+)} + C_N \hbar^N \|v\|^2_{L^2(\Omega_+)}. \tag{A.5}$$

We now use the propagation estimate again to control (up to $\hbar^\infty$ errors) $\|\psi_1 v\|^2_{L^2(\Omega_+)}$ by $\|v\|^2_{L^2(a<\mathbf{r}<b)}$. Suppose that $\rho \in \mathbf{r}^{-1}(\{\operatorname{supp} \psi_1\}) \cap \Gamma_+$. Then there exists $|t| \leq \sqrt{r_1^2 - r_0^2}$ such that $\varphi_t(\rho) \in \{a < \mathbf{r} < b\}$. Therefore, by standard propagation estimates [15, Theorem E.47], again using that $\mathrm{WF}_\hbar(v) \subset \Gamma_+$, we have

$$\|\psi_1 v\|^2_{L^2(\Omega_+)} \leq C\hbar^{-1}\|(-\hbar^2\Delta - 1)v\|^2_{L^2(\mathbf{r}\leq r_1)} + C\|v\|^2_{L^2(a<\mathbf{r}<b)} + C_N \hbar^N \|v\|^2_{L^2(\Omega_+)}. \tag{A.6}$$

We next use the propagation estimate again to control $\|v\|_{L^2(\{\mathbf{r}\le r_1\}\setminus\{\chi\equiv 1\})}$ by $\|v\|^2_{L^2(a<\mathbf{r}<b)}$. To do this, we need that there exists $T>0$ such that for all $\rho \in S^*_{\Omega_+\setminus\{\chi\equiv 1\}}\Omega_+$ with $\mathbf{r}(\rho)\le r_1$ there exists $|t|\le T$ with $a<\mathbf{r}(\varphi_t(\rho))<b$. Suppose not; then there exists $\rho_n \in S^*_{\Omega_+\setminus\{\chi\equiv 1\}}\Omega_+$ with $\mathbf{r}(\rho_n)\le r_1$ and $T_n\to\infty$ such that

$$\bigcup_{|t|\le T_n}\varphi_t(\rho_n)\cap\{a<\mathbf{r}<b\}=\emptyset.$$

By (2.24), we have $\mathbf{r}(\rho_n)\le r_0$ and also $\mathbf{r}(\varphi_{\pm T_n}(\rho_n))\le r_0$. In particular, we may assume that $\rho_n\to\rho\in\{\mathbf{r}\le r_0\}\setminus K$ (since $\pi_{\mathbb{R}}(K)\Subset\{\chi\equiv 1\}$) and $\varphi_{\pm T_n}(\rho_n)\to\rho_\pm$. Then, by Lemma 2.8, $\rho\in\Gamma_+\cap\Gamma_-=K$, which is a contradiction. Applying the propagation estimate (using the existence of the uniform time T), we have

(A.7)
$$\|v\|^2_{L^2(\{\mathbf{r}\le r_1\}\setminus\{\chi\equiv 1\})}\le C\hbar^{-1}\|(-\hbar^2\Delta-1)v\|^2_{L^2(\mathbf{r}\le r_1)}+C\|v\|^2_{L^2(a<\mathbf{r}<b)}+C_N\hbar^N\|v\|^2_{L^2(\Omega_+)}.$$

Finally, we control $\|v\|_{L^2(\Omega_+\setminus\Omega_{\rm tr})}$. For this, note that $v=u1_{\Omega_{\rm tr}}+v1_{(\Omega_{\rm tr})^c}$ and by (A.3) and (A.7) we have

$$\|v\|_{L^2(\Omega_+\setminus\Omega_{\rm tr})}\le C\hbar^{-1}\|(P_\theta-1)v\|_{L^2(\Omega_+)}+C\|v\|_{L^2(a<\mathbf{r}<b)}+C_N\hbar^N\|u\|_{L^2(\Omega_{\rm tr})}. \tag{A.8}$$

Now, using (A.6) in (A.5), and then using the definition of v (A.1) and that $v=u$ on $\Omega_{\rm tr}$, we have

$$\begin{aligned}\|v\|^2_{L^2(a<\mathbf{r}<b)}&\le C\hbar^{-N-2}\|(-\hbar^2\Delta-1)v\|^2_{L^2(\mathbf{r}\le r_1)}+C_N\hbar^N\|v\|^2_{L^2(\Omega_+)}\\&=C\hbar^{-N-2}\|(P_\theta-1)v\|^2_{L^2(\Omega_+)}+C_N\hbar^N\|u\|^2_{L^2(\Omega_{\rm tr})}+C_N\hbar^N\|v\|^2_{L^2(\Omega_+\setminus\Omega_{\rm tr})}.\end{aligned}$$

Then, using (A.8),

$$\|v\|^2_{L^2(a<\mathbf{r}<b)}\le C_N\hbar^N\|u\|^2_{L^2(\Omega_{\rm tr})}+C_N\hbar^N\|v\|_{L^2(a<\mathbf{r}<b)},$$

and, taking $\hbar$ small enough, we obtain

$$\|v\|_{L^2(a<\mathbf{r}<b)}\le C_N\hbar^N\|u\|_{L^2(\Omega_{\rm tr})}\le C_N\hbar^N$$

since $\|u\|_{L^2(\Omega_{\rm tr})}=1$. Therefore, using (A.7), (A.8), the definition of v (A.1), and the fact that $z=\mathcal{O}(\hbar^\infty)$, we have

$$\|\psi(\mathbf{r})v\|^2_{L^2(\Omega_+\setminus\{\chi\equiv 1\})}=\mathcal{O}(\hbar^\infty),$$

so that, since $\mathrm{WF}_\hbar(v)\subset S^*\mathbb{R}^d$ (which is fibre compact),

$$\|\psi(\mathbf{r})v\|^2_{H^2_\hbar(\Omega_+\setminus\{\chi\equiv 1\})}=\mathcal{O}(\hbar^\infty);$$

the result then follows from (A.2). ∎

Appendix B. Details of how the eigenvalues/eigenfunctions were computed in section 1.3. When discretizing the sesquilinear form $a(\cdot,\cdot)$ defined by (1.7), we need to calculate the Dirichlet-to-Neumann map $\mathcal{D}(k)$. Instead of approximating $\mathcal{D}(k)$ using either a PML or an absorbing boundary condition, we use boundary integral operators to find $\mathcal{D}(k)$ "exactly" (i.e., up to the discretization of these integral operators).

Recall that the single-layer potential on $\Gamma_{\rm tr}$ is defined for $\varphi \in L^1(\Gamma)$ by

$$\mathcal{S}_k\varphi(x) := \int_{\Gamma_{\rm tr}} \Phi_k(x,y)\varphi(y)\, ds(y) \qquad \text{for all } x \in \mathbb{R}^d \setminus \Gamma_{\rm tr},$$

where, in two dimensions, $\Phi_k(x,y) := {\rm i}H_0^{(1)}(k|x-y|)/4$, where $H_0^{(1)}$ is the order zero Hankel function of the first kind. The single-layer and adjoint-double-layer operators are then defined, respectively, by $S_k := \gamma_0^{\rm tr}\mathcal{S}_k$ and $D'_k := \gamma_1^{\rm tr}\mathcal{S}_k - I/2$, where the traces are taken from inside $\Omega_{\rm tr}$. With these definitions, for values of k for which $S_k : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is invertible,

$$\mathcal{D}(k) = \left(-\frac{1}{2}I + D'_k\right) S_k^{-1}; \tag{B.1}$$

see, e.g., [9, page 136].

To avoid the operator product in (B.1), we introduce the auxiliary variable $\varphi_\ell = S_k^{-1}(\gamma_0^{\rm tr}(u_\ell)) \in H^{-1/2}(\Gamma_{\rm tr})$. The eigenvalue problem (1.5) can therefore be rewritten as follows: find $u_\ell \in H^1_{0,D}(\Omega_{\rm tr})$ and $\varphi_\ell \in H^{-1/2}(\Gamma_{\rm tr})$ such that

$$\begin{aligned}\big(\nabla u_\ell, \nabla v\big)_{L^2(\Omega_{\rm tr})} - k^2\big(u_\ell, v\big)_{L^2(\Omega_{\rm tr})} - \left\langle \left(-\frac{1}{2}I + D'_k\right)\varphi_\ell, \gamma_0^{\rm tr} v\right\rangle_{\Gamma_{\rm tr}} &= \mu_\ell\big(u_\ell, v\big)_{L^2(\Omega_{\rm tr})} \\ \text{and} \qquad \big\langle \gamma_0^{\rm tr} u_\ell, \psi\big\rangle_{\Gamma_{\rm tr}} - \big\langle S_k\varphi_\ell, \psi\big\rangle_{\Gamma_{\rm tr}} &= 0\end{aligned} \tag{B.2}$$

for all $v \in H^1_{0,D}(\Omega_{\rm tr})$ and $\psi \in H^{-1/2}(\Gamma_{\rm tr})$. We note that this formulation is the transpose of the Johnson–Nédélec FEM-BEM coupling [33] applied to the eigenvalue problem (1.5); see, e.g., [26, equation 9].

We use continuous piecewise-linear basis functions to discretize (B.2) and obtain the following generalized eigenvalue problem:

$$\widetilde{\mathbf{A}}\mathbf{u}_\ell = \begin{pmatrix} \mathbf{A}_k & \frac{1}{2}(\mathbf{M}^{\rm tr})^T - \mathbf{D}'_k \\ \mathbf{M}^{\rm tr} & -\mathbf{S}_k \end{pmatrix}\mathbf{u}_\ell = \mu_\ell \begin{pmatrix} \mathbf{M} & 0 \\ 0 & 0\end{pmatrix}\mathbf{u}_\ell =: \mu_\ell \mathbf{B}\mathbf{u}_\ell, \tag{B.3}$$

where $\mathbf{M}$ is the mass matrix on $\Omega_{\rm tr}$, $\mathbf{S}_k$ is a discretization of the single-layer operator, and $\mathbf{A}_k$ is the Galerkin matrix corresponding to the discretization of $(\nabla u_\ell, \nabla v) - k^2(u_\ell, v)$. The matrices $\mathbf{M}^{\rm tr}$ and $\mathbf{D}'_k$ are defined, for $i = 1, \dots, M$ and $j = 1, \dots, N$, by

$$(\mathbf{M}^{\rm tr})_{i,j} = \langle \gamma_0^{\rm tr} v_j, \psi_i\rangle_{\Gamma_{\rm tr}} \quad \text{and} \quad (\mathbf{D}'_k)_{j,i} = \big\langle D'_k\psi_i, \gamma_0^{\rm tr} v_j\big\rangle_{\Gamma_{\rm tr}},$$

where v_j and ψ_i are, respectively, continuous piecewise-linear basis functions of the Galerkin discretizations $V_h(\Omega_{\rm tr}) \subset H^1_{0,D}(\Omega_{\rm tr})$ and $V_h(\Gamma_{\rm tr}) \subset H^{-1/2}(\Gamma_{\rm tr})$; the dimensions of these spaces are denoted by N and M, respectively.

To build the matrices in (B.3) and solve this problem, we use PETSc [3, 2, 1] and the eigensolver SLEPc [41, 29] via the software FreeFEM [28]. Since we are interested in the eigenvalues near the origin, we use the shift-and-invert technique; i.e, we compute the largest eigenvalues of the problem $(\widetilde{\mathbf{A}})^{-1}\mathbf{B}\mathbf{u}_\ell = \nu_\ell \mathbf{u}_\ell$ and then set $\mu_\ell = 1/\nu_\ell$. To obtain the action of $(\widetilde{\mathbf{A}})^{-1}$, we use SuperLU [36] to compute the LU factorization of $\widetilde{\mathbf{A}}$.

Acknowledgments. EAS gratefully acknowledges discussions with Alex Barnett (Flatiron Institute) that started his interest in eigenvalues of discretizations of the

Helmholtz equation under strong trapping. JG thanks Maciej Zworski (UC Berkeley) for bringing to his attention the paper [44]. PM thanks Pierre Jolivet (IRIT, CNRS) for his help with the software FreeFEM. The authors thank the referees for their careful reading of the paper and constructive comments. This research made use of the Balena High Performance Computing (HPC) Service at the University of Bath.

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