MOMENTS OF AUTOMORPHIC L-FUNCTIONS AND RELATED PROBLEMS

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I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

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Abstract

We present in this dissertation several theorems on the subject of moments of automorphic L-functions.

In chapter 1 we give an overview of this area of research and summarize our results.

In chapter 2 we give asymptotic main term estimates for several different moments of central values of L-functions of a fixed GL_2 holomorphic cusp form f twisted by quadratic characters. When the sign of the functional equation of the twist $L(s, f \otimes \chi_d)$ is -1, the central value vanishes and one instead studies the derivative $L'(1/2, f \otimes \chi_d)$. We prove two theorems in the root number -1 case which are completely out of reach when the root number is $+1$.

In chapter 3 we turn to an average of GL_2 objects. We study the family of cusp forms of level q^2 which are given by $f \otimes \chi$, where f is a modular form of prime level q and χ is the quadratic character modulo q. We prove a precise asymptotic estimate uniform in shifts for the second moment with the purpose of understanding the off-diagonal main terms which arise in this family.

In chapter 4 we prove an precise asymptotic estimate for averages of shifted convolution sums of Fourier coefficients of full-level GL_2 cusp forms over shifts. We find that there is a transition region which occurs when the square of the average over shifts is proportional to the length of the shifted sum. The asymptotic in this range depends very delicately on the constant of proportionality: its second derivative seems to be a continuous but nowhere differentiable function. We relate this phenomenon

to periods of automorphic forms, multiple Dirichlet series, automorphic distributions, and moments of Rankin-Selberg L-functions.

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Chapter 1

Introduction

1.1 L-functions in families

This dissertation is about L-functions in families. L-functions play a central and unifying role in modern number theory. Among the deepest problems in number theory are the generalized Riemann hypothesis, the Birch and Swinnerton-Dyer conjecture, and the Langlands program all of which concern L-functions.

A recurring philosophy in modern mathematics is that "to understand a difficult object, one should deform it into a family of objects and study that family". Consideration of individual L-functions as members of a family of L-functions has had a long and productive history, going all the way back to their first application: Dirichlet's theorem on primes in arithmetic progressions. For a complex number s with $\mathrm{Re}(s) > 1$ and a Dirichlet character $\chi(n)$, Dirichlet defined the L-functions

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \frac{1}{1-\chi(p)p^{-s}}.$$

To show that each arithmetic progression $a \pmod q$ with $(a,q)=1$ contains infinitely many primes, Dirichlet expands the indicator function of the residue class $a \pmod q$

into a basis of Dirichlet characters and shows that the trivial character χ_0 modulo q has $\lim_{s\to 1^+} L(s,\chi_0) = \infty$, whereas every other character χ modulo q satisfies $L(1,\chi) \neq 0$.

Since Dirichlet's time there have been many other spectacular applications of studying families of L-functions in number theory. In the case of function fields, we need look no further than the solution of the Weil conjectures by Deligne [Del74]. In particular, Deligne proved the Riemann hypothesis for zeta functions of varieties over finite fields, which has as a consequence the Ramanujan-Petersson conjecture for holomorphic modular forms. In this case, the notion of a family of zeta functions derives from that of a family of varieties over a base scheme in algebraic geometry.

We give one more example which is much closer to the subject of this dissertation, in which the spectral theory of GL_2 automorphic forms is applied to the classical case of Dirichlet (i.e. GL_1) L-functions. It was established in the 1960s by Burgess [Bur62, Bur63] that on the critical line the Dirichlet L-functions of conductor q satisfy the bound $L(s,\chi) \ll_\varepsilon |s|^A q^{3/16+\varepsilon}$ for some fixed $A > 0$. No improvement on the bound was made for 40 years until Conrey and Iwaniec in 2000 [CI00] established, in the special case when χ is primitive quadratic, the bound

$$L(s,\chi) \ll_\varepsilon |s|^{A'} q^{1/6+\varepsilon} \tag{1.1}$$

for some constant A'. They accomplished this by considering the Dirichlet L-function as a GL_2 object

$$|L(1/2+it,\chi)|^2 = L(1/2, E\otimes\chi)$$

where E is a unitary Eisenstein series of trivial central character and spectral parameter $1/2+it$. Embedding $L(1/2, E\otimes\chi)$ into a family of GL_2 L-functions for the congruence subgroup $\Gamma_0(q^2)$, they succeeded in establishing a Lindelöf-on-average upper bound for the third power average of this family of L-functions, thereby proving the bound (1.1). We will revisit this family of L-functions in chapter 3 of this dissertation.

1.1.1 Examples of families of L-functions

We shall assume that the reader is familiar with the most common examples of L-functions, and defer to section 1.1.3 the definition. As for *families* of L-functions, there is currently no precise definition generally accepted, although the question has received some attention in the past few years. Several perspectives on this issue can be found in the papers of Conrey, Farmer, Keating, Rubinstein and Snaith [CFK+05], Diaconu, Goldfeld and Hoffstein [DGH03], Iwaniec and Sarnak [IS00b], and Sarnak's note [Sar08].

In [Sar08] Sarnak writes "In practice a 'family' is investigated as it arises", and we will keep with that tradition in this dissertation. Following the philosophy of [CFK+05], we often have in mind a set of L-functions whose underlying arithmetic data is tied together by some spectral completeness or trace formula. Several typical examples are:

1. Primitive Dirichlet L-functions of fixed conductor. The spectral completeness rule is orthogonality of characters.

2. Quadratic Dirichlet L-functions. The trace formula is Poisson summation. The family of quadratic Dirichlet characters does not have a strong spectral completeness, however these L-functions are tied together because they appear as the Fourier coefficients of a weight 1/2 Eisenstein series, whose modularity relation follows from Poisson summation.

3. The L-functions of GL_2 automorphic forms of specified weight and level with

root number +1. The spectral completeness rule is the Petersson/Kuznetsov trace formula.

4. The L-functions of GL_2 automorphic forms of specified weight and level with root number -1. The spectral completeness rule is the Petersson/Kuznetsov trace formula.

5. Riemann zeta function in t aspect, or rather, any L-function at all in t aspect. We may view $L(1/2+it, f)$ as a twist of $L(1/2, f)$ by the characters n^{-it}, where $t \in \mathbb{R}$. This gives the only known continuous family of L-functions. The spectral completeness rule is integration.

1.1.2 Moments of L-functions

A classical method used to study the analytic properties of a family of L-functions is to compute its kth power mean value, which in our subject is often called the kth moment of the family. Hardy and Littlewood were the first to compute moments of the Riemann zeta function. In 1916 [HL16] they proved an asymptotic formula for the 2nd moment of the Reimann zeta function:

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^2 \, dt \sim \log T,$$

as $T \to \infty$. In 1926 Ingham [Ing26] derived an asymptotic main term for the 4th moment

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^4 \, dt \sim \frac{1}{2\pi^2}(\log T)^4.$$

For an overview of these results, see Titchmarsh [THB86] chapter 7. To this day, no asymptotic formula for any higher moment of the Riemann zeta function has been proven, even assuming the Riemann hypothesis. On the other hand, as of the early 2000s, there exist widely believed conjectures for the asymptotic expansions of all moments of the zeta function, see subsection 1.2.5 below.

Some progress has been made in the error term in the 4th moment. In 1979 Heath-Brown [Hea79] proved

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^4\,dt = \sum_{i=0}^{4} a_i(\log T)^i + O(T^{-1/8+\varepsilon})$$

for some explicit constants a_i. Heath-Brown's result required a careful analysis of "off-diagonal" terms which contribute to the asymptotic main terms above. The best current result is due to Ivić and Motohashi [IM95] who obtain an error of $\ll_\varepsilon T^{-1/3+\varepsilon}$. One expects an error term of size $\ll_\varepsilon T^{-1/2+\varepsilon}$. We revisit such "off-diagonal" analysis later in the introduction, see section 1.3.

In addition to the Riemann zeta function, one also considers moments of the various families of L-functions e.g. those described in subsection 1.1.1 above. We give the currently best-known examples in several indicative cases below.

Young [You11] proved the asymptotic formula for the 4th moment of primitive Dirichlet L-functions of conductor q: let $q \to \infty$ through odd primes, then for some explicit constants c_i

$$\sum_{\substack{\chi \pmod q \\ \chi \text{ primitive}}} |L(1/2,\chi)|^4 = q\sum_{i=0}^{4} c_i(\log q)^i + O_\varepsilon(q^{1-5/512+\varepsilon}).$$

Let $\gcd(d,\square) = 1$ denote the condition that d is square-free. A second recent result also due to Young [You12] gives an estimate for the third moment

$$\sum_{\gcd(d,2\square)=1} L(1/2,\chi_{8d})^3 F(8d/X) = X\sum_{i=0}^{6} b_i(\log X)^i + O_\varepsilon(X^{3/4+\varepsilon})$$

for some explicit constants b_i and $F \in C_c^\infty(\mathbb{R}_{>0})$ a smooth function with support contained in the interval $[1/2, 3]$. It should be possible to establish an asymptotic main term for the 4th moment of this family by the technique used in chapter 2 of this thesis, originally developed by Soundararajan and Young in [SY10]. The asymptotic size of the 4th moment will be a degree 10 polynomial in $\log X$. Just as in the work of Heath-Brown mentioned above, an intricate analysis of the off-diagonal and off-off-diagonal main terms is required, and will be carried out in a future paper.

Perhaps the most interesting family of central values of L-functions to study are GL_2 automorphic forms, due to their arithmetic interpretation via the Birch and Swinnerton-Dyer conjecture and its generalizations. In this case, it was established by Kowalski, Michel and VanderKam [KMV00] for f in a basis $H_2(q)$ of primitive weight 2 Hecke eigenforms for $\Gamma_0(q)$ that

$$\sum_{f\in H_2(q)} L(1/2, f)^4 = q\sum_{i=0}^{6} d_i(\log q)^i + O_\varepsilon(q^{11/12+\varepsilon}),$$

for explicit constants d_i.

1.1.3 Definitions

For the purpose of giving an intelligible discussion of the various applications of moments of L-functions in the following section 1.2.1, we now give the definition of an L-function, the definition of the analytic conductor, and two standard examples. We follow here Iwaniec and Kowalski section 5.1 [IK04] and sections 1 and 2 of Iwaniec and Sarnak [IS00b].

Definition 1 (L-function) *Consider a meromorphic function $L(s, f)$ in the variable s with the following data:*

1. *Dirichlet series and Euler product of degree $d > 0$. When $\mathrm{Re}(s) > 1$ the function*

$L(s, f)$ admits the expressions

$$L(s,f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\alpha_1(p)}{p^s}\right)^{-1} \cdots \left(1 - \frac{\alpha_d(p)}{p^s}\right)^{-1},$$

with $\lambda_f(n), \alpha_1(p), \ldots, \alpha_d(p)$ complex numbers associated to f and $\lambda_f(1) = 1$. We also assume that the Dirichlet series and Euler product converge absolutely when $\mathrm{Re}(s) > 1$, and that the local parameters $\alpha_i(p)$ satisfy

$$|\alpha_i(p)| < p.$$

2. *Gamma factors. Associated with $L(s, f)$ there exists complex numbers κ_j, which are either real or come in conjugate pairs and have $\mathrm{Re}(\kappa_j) > -1$. These are the archimedean analogues of the $\alpha_i(p)$ from part 1. These κ_j are used to form the gamma factors*

$$\gamma(s,f) = \pi^{-ds/2} \prod_{j=1}^{d} \Gamma\left(\frac{s+\kappa_j}{2}\right).$$

3. *Conductor. There exists a positive integer $q(f)$ called the conductor of $L(s, f)$ for which $\alpha_i(p) \neq 0$ when $p \nmid q(f)$, i.e. $L(s, f)$ is unramified at the primes dividing the conductor.*

The meromorphic function $L(s, f)$ is called an L-function if it has the data from (1),(2) and (3), and the "completed L-function" defined

$$\Lambda(s,f) = q(f)^{s/2}\gamma(s,f)L(s,f)$$

can be analytically continued to a meromorphic function on $\mathbb{C}$ of order 1 with at most *poles at $s = 0$ and $s = 1$ satisfying a functional equation*

$$\Lambda(s, f) = w(f)\Lambda(1 - s, \overline{f}),$$

where the object $\overline{f}$ is the dual of f in the sense that we have that $\lambda_{\overline{f}}(n) = \overline{\lambda_f(n)}$, $\gamma(s, \overline{f}) = \gamma(s, f)$ and $q(\overline{f}) = q(f)$. The complex number $w(f)$ has $|w(f)| = 1$ and is called the "root number" or "sign of the functional equation".

The "Langlands philosophy" predicts that all L-functions come from automorphic forms. The L-functions of degree $d < 2$ have been completely classified by the combined work of Conrey and Ghosh [CG93], Richert [Ric57], Kaczorowski and Perelli [KP99], Soundararajan [Sou05] and Kaczorowski and Perelli [KP11]. Extending this classification to $d = 2$ is considered a difficult open problem.

We now define an important real-number invariant of $L(s, f)$: the analytic conductor. We can define the analytic conductor intrinsically: the density of zeros of $L(s, f)$ at height $|s|$ in the critical strip is proportional to $\log \mathfrak{q}(f, s)$, and this quantity $\mathfrak{q}(f, s)$ we call the analytic conductor. In practice however, the analytic conductor $\mathfrak{q}(f, s)$ is read off from the functional equation of $L(s, f)$.

Definition 2 (Analytic Conductor) *Let $L(s, f)$ be an L-function as above. The analytic conductor of $L(s, f)$ is the positive real number*

$$\mathfrak{q}(s, f) = q(f) \prod_{j=1}^{d} \left(|s + \kappa_j| + 3 \right),$$

where $q(f), d, \kappa_j$ are as in Definition 1.

We give two examples which were mentioned previously in this introduction and will be used later.

Example 1 (Dirichlet L-function) *Let $\chi : \mathbb{Z} \to \mathbb{C}$ be a primitive Dirichlet character of conductor q. For $\mathrm{Re}(s) > 1$ let*

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \frac{1}{1-\chi(p)p^{-s}}.$$

1. *We have for this L-function that the local parameters are given by $\lambda_\chi(p) = \alpha(p) = \chi(p)$, and that $|\alpha(p)| = |\chi(p)| = 1 < p$ for all p.*

2. *We have the local factor at ∞ is $\kappa = \mathfrak{a}$ with $\mathfrak{a}$ defined by*

$$\mathfrak{a} = \begin{cases} 0 & \text{if } \chi(-1) = 1 \\ 1 & \text{if } \chi(-1) = -1. \end{cases}$$

   The gamma factor is then given by

$$\gamma(s) = \pi^{-s/2}\Gamma\left(\frac{s+\mathfrak{a}}{2}\right).$$

3. *The conductor of $L(s,\chi)$ is given by $q(\chi) = q$.*

The root number is given by

$$w(\chi) = i^{-\mathfrak{a}}\frac{\tau(\chi)}{\sqrt{q}}.$$

Note that $|\tau(\chi)| = \sqrt{q}$, so that $|w(\chi)| = 1$. Thus we have

$$\Lambda(s,\chi) = \left(\frac{q}{\pi}\right)^{s/2}\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)L(s,\chi)$$

and

$$\Lambda(s,\chi) = i^{-\mathfrak{a}}\frac{\tau(\chi)}{\sqrt{q}}\Lambda(1-s,\overline{\chi}).$$

Finally, the analytic conductor of $L(s,\chi)$ *is*

$$\mathfrak{q}(s,\chi) = q(|s+\mathfrak{a}|+3).$$

If s *is high in the critical strip, we often use the approximation*

$$\mathfrak{q}(s,\chi) \approx q|s|.$$

Example 2 (Modular L-function) *Let* f *be a classical holomorphic primitive cuspidal eigenform of even weight* κ *for the congruence subgroup* $\Gamma_0(q)$ *with trivial central character. For* $\mathrm{Re}(s) > 1$ *let*

$$L(s,f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(\frac{1}{1-\alpha(p)p^{-s}}\right)\left(\frac{1}{1-\beta(p)p^{-s}}\right),$$

where

1. *The algebraic numbers* $n^{(\kappa-1)/2}\lambda_f(n)$ *are the Fourier coefficients of* $f(z)$ *as a function on the upper half plane, or alternatively,* $p^{(\kappa-1)/2}\lambda_f(p)$ *is the eigenvalue of the* p*-th Hecke operator* T_p *acting on* f*. We have by Deligne's proof of the Weil conjectures [Del74] that*

$$|\alpha(p)| = |\beta(p)| = 1.$$

2. *The local factors at infinity are* $\kappa_1 = (\kappa-1)/2$ *and* $\kappa_2 = (\kappa+1)/2$*. Hence the gamma factors are*

$$\gamma(s,f) = \pi^{-s}\Gamma\left(\frac{s+\frac{\kappa-1}{2}}{2}\right)\Gamma\left(\frac{s+\frac{\kappa+1}{2}}{2}\right).$$

3. *The conductor of* $L(s,f)$ *is the level* q*.*

The root number of f is given by

$$w(f) = i^\kappa \eta,$$

where $\eta = \pm 1$ is the eigenvalue of the Fricke involution. When q is square-free we have that

$$\eta = \mu(q)\lambda_f(q)\sqrt{q},$$

see Proposition 14.16 of [IK04]. Thus we have

$$\Lambda(s,f) = \left(\frac{\sqrt{q}}{\pi}\right)^s \Gamma\left(\frac{s+\frac{\kappa-1}{2}}{2}\right)\Gamma\left(\frac{s+\frac{\kappa+1}{2}}{2}\right)L(s,f)$$

and

$$\Lambda(s,f) = i^\kappa \eta \Lambda(1-s,f).$$

Note: in the case that f has trivial central character $\lambda_f(n) \in \mathbb{R}$ and f is self-dual. The analytic conductor is

$$\mathfrak{q}(s,f) = q\left(\left|s+\frac{\kappa-1}{2}\right|+3\right)\left(\left|s+\frac{\kappa+1}{2}\right|+3\right).$$

When s is high in the critical strip, we may approximate

$$\mathfrak{q}(s,f) \approx q(|s|^2+\kappa^2).$$

The analytic conductor $\mathfrak{q}(s,f)$ of an L-function is a measure of the complexity of $L(s,f)$. More precisely, the approximate functional equation expresses the value $L(s_0,f)$ as a sum of length $\mathfrak{q}(s_0,f)^{1/2}$. For the general form of the approximate functional equation, see Theorem 5.3 from section 5.2 of [IK04]. In practice, it is best to work out the approximate functional equation on a case-by-case basis as it is very flexible and may be constructed tailor-fit to each particular application, see Lemma 1 of chapter 2. The analytic conductor therefore gives an important benchmark for various calculations.

We see from the many results in the litterature, some of which are given in in subsection 1.1.2, that the range of moments for which we have rigorous estimates of the right order of magnitude is extremely limited. There is apparently a barrier obstructing progress to higher moments, which has persisted despite a good deal of effort. Precisely, let $\mathcal{F}$ be a family of L-functions of size $\mu(\mathcal{F})$. As we saw in section 1.1.2, current technology for estimating the kth moment

$$\int_{f\in\mathcal{F}} |L(s,f)|^k \, d\mu$$

typically breaks down around

$$k = 4\frac{\log \mu(\mathcal{F})}{\log \mathfrak{q}(s,f)}$$

for f a "typical" element of $\mathcal{F}$, as $\mathfrak{q}(s,f) \to \infty$. This gives a benchmark for whether a moment estimation problem should be approachable or not.

1.2 Motivation for Moments of L-functions

1.2.1 Applications of Moments

In this section we briefly describe several applications of estimating moments of L-functions: the subconvexity problem, zero-density results and non-vanishing theorems.

By estimating moments of L-functions, we seek to understand the average size of central values of a family of L-functions. But on-average estimates are also related to the deep question of the size of a single L-function. Essentially the best point-wise estimates are obtained from the generalized Riemann hypothesis. Assuming GRH we have for any L-function with $\mathrm{Re}(s) = 1/2$ and $\varepsilon > 0$ that

$$L(s, f) \ll_\varepsilon \mathfrak{q}(s, f)^\varepsilon,$$

where the implied constants only depend on ε. This estimate is known as the Lindelöf hypothesis, and has deep arithmetic consequences. For the deduction of the Lindelöf hypothesis from the Riemann hypothesis, see [IK04] section 5.7.

Unconditionally, one derives from the functional equation and the Phragmen-Lindelöf convexity principle when $\mathrm{Re}(s) = 1/2$ and $\varepsilon > 0$ that

$$L(s, f) \ll_\varepsilon \mathfrak{q}(s, f)^{1/4+\varepsilon}.$$

This estimate is known as the "convexity bound". Typically, it just barely fails to give non-trivial results in applications. One thinks of the convexity bound as the "no arithmetic in, no arithmetic out" situation.

The subconvexity problem is to establish for some fixed $\delta > 0$ a bound of the form

$$L(s, f) \ll \mathfrak{q}(s, f)^{1/4-\delta}$$

on the critical line, where the implied constants depend only on the degree of f. In contrast to the convexity bound, subconvex bounds have many applications to equidistribution-type problems. These include: representation of integers by quadratic forms over number fields [IS00b], quantum unique ergodicity [HS10], and thus equidistribution of zeros of eigenforms [Rud05], first Fourier coefficient at which two distinct modular forms must differ [Mic07], smallest quadratic nonresidue [Mic07, MV10], bounds on Fourier coefficients of eigenfunctions on negatively curved spaces when restricted to closed geodesics [MV10], and the equidistribution of Heegner points

[Mic07].

Now let us return to moments of L-functions and discuss how they may be applied to solve instances of the subconvexity problem. As an example, consider the work of Conrey and Iwaniec [CI00] which was introduced in section 1.1. Let $H_\kappa(q)$ be a basis of Hecke eigenforms of weight κ for the congruence subgroup $\Gamma_0(q)$, and χ the quadratic character modulo q. Conrey and Iwaniec prove for the cubic moment

$$\sum_{f \in H_\kappa(q)} L(1/2, f \otimes \chi)^3 \ll_\varepsilon q^{1+\varepsilon}, \tag{1.2}$$

which is the Lindelöf-on-average upper bound. Each of the modular forms $f \otimes \chi$ is of the congruence subgroup $\Gamma_0(q^2)$, so each of these L-functions has conductor q^2, and analytic conductor $\mathfrak{q}(1/2, f \otimes \chi) \sim q^2 k^2$. Throwing away all but one term of the sum and taking cube roots, one obtains

$$L(1/2, f \otimes \chi) \ll_{\kappa,\varepsilon} q^{1/3+\varepsilon},$$

which is an impressive Weyl-type subconvexity bound. The convexity bound here is $\ll_{\kappa,\varepsilon} q^{1/2+\varepsilon}$. This technique is crude but nonetheless yields the best known subconvex bound for this family. Note that to attain a subconvex estimate in this manner, only an upper bound for the moment (1.2) is necessary. In contrast, the non-vanishing results we discuss below require precise main term estimates.

A more sophisticated version of the above argument is the amplification technique. The amplification technique was originally introduced by Iwaniec [Iwa92], and developed by (amongst many others) Duke, Friedlander and Iwaniec in [DFI94, DFI02] and by Kowalski, Michel and VanderKam in [KMV00, KMV02]. It consists basically of introducing an extra factor $M(f)$ to the summand of a moment for which $M(f_0)$ is large for one particular term f_0. As the subconvexity problem is not the subject of this dissertation, we avoid giving further details and refer the reader to the literature.

Today, the subconvexity problem has been completely resolved for GL_1 and GL_2 L-functions by the impressive work of Michel and Venkatesh [MV10]. Michel and Venkatesh employ the amplification method applied to period integrals of automorphic forms instead of moments. For GL_n, $n \geq 3$ these period relations are not available, and the subconvexity problem is still mostly open. A few results we have for GL_3 do make use of the method of moments however, see [Li11, Blo12].

A second basic problem in the theory of L-functions is to determine how often L-functions may vanish at certain special points. These special points may be up the critical line or at the center point of symmetry, but in both cases, moments of L-functions and the mollifier technique are key tools.

Mollifiers in the t-aspect were originally introduced to study the zeros of the Riemann zeta function by Bohr and Landau [BL14] and further used by Selberg [Sel89] to show that a positive proportion of the zeros of the Riemann zeta function are on the critical line. The state of the art today is that more than 41.28% of the zeros of the classical $\zeta(s)$ are on the $\mathrm{Re}(s) = 1/2$ line, a result due to Feng [Fen12], which builds on the ideas of Conrey [Con89].

Often we are most interested in the center point of symmetry of the functional equation, and it is in this case that we now go into somewhat greater detail.

To give an example of the flavor of results which we are interested in, consider the family of quadratic Dirichlet L-functions $L(s, \chi_d)$ associated with the quadratic character of fundamental discriminant d. It was shown by Jutila [Jut81] that there are $\gg X/\log X$ of the fundamental discriminants $|d| \leq X$ for which $L(1/2, \chi_d) \neq 0$. To show this, Jutila established asymptotic formulae for the first and second moments of quadratic Dirichlet L-functions. Let $\mathcal{D}$ denote the set of fundamental discriminants, i.e. those discriminants coming from quadratic fields. Jutila showed

$$\sum_{\substack{|d|\leq X\\ d\in\mathcal{D}}} L(1/2,\chi_d) \sim c_1 X \log X \tag{1.3}$$

and

$$\sum_{\substack{|d|\leq X\\ d\in\mathcal{D}}} |L(1/2,\chi_d)|^2 \sim c_2 X (\log X)^3 \tag{1.4}$$

for specific constants c_1 and c_2. Applying the Cauchy-Schwarz inequality one obtains the lower bound stated above on the number of fundamental discriminants for which $L(1/2,\chi_d)$ does not vanish.

Soundararajan [Sou00] introduced mollifiers to this line of reasoning, and thereby showed that 7/8-ths of fundamental discriminants have $L(1/2,\chi_d)\neq 0$. The idea is to introduce a sum

$$M(d)=\sum_{\ell\leq M} a_\ell \chi_d(\ell)$$

to the moments (1.3) and (1.4) where the a_ℓ are arbitrary coefficients. Theses coefficients a_ℓ are then chosen so that

$$\sum_{\substack{|d|\leq X\\ d\in\mathcal{D}}} L(1/2,\chi_d)M(d) \asymp \sum_{\substack{|d|\leq X\\ d\in\mathcal{D}}} |L(1/2,\chi_d)M(d)|^2 \asymp X.$$

Here the symbol $f\asymp g$ means that $\lim_{x\to\infty} f(x)/g(x)=c$ for some constant c. The idea is to choose the a_ℓ to make $M(d)$ small whenever $L(1/2,\chi_d)$ is large. Applying Cauchy-Schwarz now gives a 7/8-ths proportion of non-vanishing values $L(1/2,\chi_d)$.

Mollifiers have been used in many other cases to prove non-vanishing theorems in the case of automorphic L-functions, for example in the work of Iwaniec and Sarnak [IS00a] and Kowalski, Michel and VanderKam [KMV00]. For example, Iwaniec and Sarnak prove that the percentage of primitive holomorphic eigenforms of even weight

κ, square-free level N and root number $w(f) = +1$ for which

$$L(1/2, f) \geq (\log N)^{-2}$$

is at least 50 as $N \to \infty$ with $\phi(N) \sim N$. Iwaniec and Sarnak also show that if this proportion could be increased to strictly more than half of the forms, it would imply that there are no Laudau-Siegel zeros for the quadratic Dirichlet L-functions $L(s, \chi)$, thus giving a rapid solution to the class number problem! Non-vanishing results and estimates for moments of L-functions are therefore closely tied to some of the deepest questions in number theory.

All of these applications would be reason enough to study moments of L-functions, but perhaps the best motivation is that estimating moments of L-functions gives us a glimpse into the structure of a family of L-functions and its interaction with a trace formula or spectral completeness relation stringing the family together. Despite the restricted range of rigorous results on moments (see the end of subsection 1.1.3), there now exist widely believed conjectures predicting the full main term estimates of moments of L-functions in all families, based on the conjectural underlying structure of families of L-functions and random matrix theory.

1.2.2 Symmetries of Families

Studying moments sheds light on the structure of a family of L-functions given by their conjectured "spectral interpretation", about which almost nothing is known. Hilbert and Pólya suggested in the 1910s as an approach to RH that the nontrivial zeros of $\zeta(s)$ should corresponded to the eigenvalues of some naturally occurring unbounded self-adjoint operator. In their time this was pure speculation, but today we have some evidence to expect the existence of such an operator. If such an operator for L-functions does exist, one would expect the zeros of the L-functions $L(s, f)$ to behave like random eigenvalues as f varies through some family. The inspiration for this section of the dissertation comes from the survey article of Iwaniec and Sarnak [IS00b] sections 3 and 8, and the Bulletin article by Katz and Sarnak [KS99b].

We first consider the case of zeta functions for varieties over finite fields in subsection 1.2.3 to make an analogy with the number field L-functions in section 1.2.4 which are the subject of this thesis. The sought-after spectral interpretation and Riemann hypothesis are known for zeta functions of varieties over finite fields [Del74]. The purpose of section 1.2.3 is to show for curves over finite fields how the nature of the spectral interpretation manifests itself in the zero distribution laws of these zeta functions, due to work of Katz and Sarnak [KS99a]. In the next section on number fields 1.2.4 we describe how the same study of zeros statistics has been carried out for many families (see table 1.1) without any knowledge of a spectral interpretation. The striking similarity of the results in the function field and number field cases suggests that a spectral interpretation exists in the number field case as well.

In subsection 1.2.5 we connect these ideas back to moments of L-functions. One knows cf. Jensen's formula, that the growth of holomorphic functions is intimately connected with the location of their zeros. Therefore asymptotic estimates for the moments of L-functions are also controlled by the conjectured spectral interpretation. This has allowed the formulation of precise conjectures and this is made precise via random matrix theory, see the work of Conrey, Farmer, Keating, Rubinstein and Snaith [CF00, KS00a, KS00b, CFK$^+$05].

1.2.3 Finite Fields

Let C be a nonsingular projective curve over the finite field with q elements $\mathbb{F}_q$, and let N_n be the number of $\mathbb{F}_{q^n}$-points of C for $n \geq 1$. The zeta function of $C/\mathbb{F}_q$ is defined by

$$\zeta(T, C/\mathbb{F}_q) = \exp\left(\sum_{n\geq 1} \frac{N_n T^n}{n}\right).$$

The Riemann hypothesis is the statement that all zero $\zeta(\rho, C/\mathbb{F}_q) = 0$ have $|\rho| = q^{1/2}$, and was proved by Weil [Wei41]. In this case an operator whose eigenvalues correspond with the zeros of $\zeta(T, C/\mathbb{F}_q)$ does exist, and it is the Frobenius endomorphism acting on cohomology with ℓ-adic coefficients.

The Riemann hypothesis in the more difficult case of varieties over finite fields $V/\mathbb{F}_q$ was proved by Deligne [Del74]. In the case of varieties families now enter the picture, but in a much more precise sense that anything we know how to define in the number field case. One fibers V over $\mathbb{P}^1$ as a Lefshetz pencil

$$V \longrightarrow \mathbb{P}^1$$

with very mild singularities in the fibers V_t. The zeta function $\zeta(T, V/\mathbb{F}_q)$ is then related to the collection of $\zeta(T, V_t/\mathbb{F}_q)$ via the monodromy action, exploiting the structure of the total space $V \to \mathbb{P}^1$. The proof then proceeds by induction on dimension. We refrain from going into any detail, but the point is that the family and its symmetry are crucial in the known proof of the Riemann hypothesis for varieties over finite fields.

The distribution of the zeros of these zeta functions was studied extensively by Katz and Sarnak in [KS99a, KS99b] using the known spectral interpretation. Let $\mathcal{M}_g(\mathbb{F}_q)$ denote the finite set of isomorphism classes curves of genus g over the finite field $\mathbb{F}_q$. After re-scaling, the $2g$ zeros of $\zeta(T, C/\mathbb{F}_q)$ all lie on the unit circle in $S^1 \subset \mathbb{C}$. Katz and Sarnak found that the distribution of zeros of curves $C \in \mathcal{M}_g(\mathbb{F}_q)$ as $g, q \to \infty$ coincides with the distribution of eigenvalues of random unitary matrices, and moreover they identify the source of this behavior in the monodromy of the family.

Next we give an example result of Katz and Sarnak, and therefore need some definitions. Let $A \in \mathrm{U}(N)$ be an $N \times N$ unitary matrix. Its eigenvalues $e^{i\theta_1}, \ldots, e^{i\theta_j}$ all lie on the unit circle, and we order their angles

$$0 \leq \theta_1(A), \ldots, \theta_N(A) < 2\pi. \tag{1.5}$$

Define for A fixed the (discrete) consecutive spacing measure of an interval $[a, b]$

$$\mu_1(A)[a,b] = \frac{\#\{1 \leq j \leq N | \frac{N}{2\pi}(\theta_{j+1} - \theta_j) \in [a,b]\}}{N}.$$

Then we may define the measure on $\mathbb{R}$

$$\mu_1(\mathrm{GUE}) = \lim_{N \to \infty} \int_{\mathrm{U}(N)} \mu_1(A)\, dA.$$

In fact, Katz and Sarnak show that $\mathrm{U}(N)$ here could be replaced by any of the groups $\mathrm{USp}(2N)$, $\mathrm{SO}(2N)$ or $\mathrm{SO}(2N+1)$ and in any of these cases one obtains the GUE measure $\mu_1(\mathrm{GUE})$ regardless of the choice of algebraic group. In this sense the measure $\mu_1(\mathrm{GUE})$ is universal. Finally we need the Kolmogorov-Smirnov distance between two measures ν_1 and ν_2:

$$D(\nu_1, \nu_2) = \sup_{I \subset \mathbb{R}} \{|\nu_1(I) - \nu_2(I)| : I \text{ an interval } \subset \mathbb{R}\}.$$

One of the main results of Katz and Sarnak [KS99a] is

$$\lim_{g \to \infty} \lim_{q \to \infty} \frac{1}{\#\mathcal{M}_g(\mathbb{F}_q)} \sum_{C \in \mathcal{M}_g(\mathbb{F}_q)} D(\mu_1(C/\mathbb{F}_q), \mu_1(\mathrm{GUE})) = 0, \tag{1.6}$$

That is to say the distributions of consecutive spacings of zeros of $\zeta(T, C/\mathbb{F}_q)$ and consecutive spacings of eigenvalues of large unitary matrices coincide. Thus zero spacing statistics are not sensitive to the symmetry type of the family.

On the other hand, the symmetry type of a subfamily of $\mathcal{M}_g(\mathbb{F}_q)$ *is* detectable by studying low-lying zero statistics, that is, the nearest zeros to $1 \in S^1$. There are only four possibilities for the low-lying zero distribution law, which correspond to the classical compact groups: unitary, symplectic, even orthogonal and odd orthogonal. Again, Katz and Sarnak identify the source of this behavior in terms of the monodromy of the family.

We give an example of a subfamily of $\mathcal{M}_g(\mathbb{F}_q)$ with symplectic symmetry, and make precise what we mean by the distribution of zeros near $1 \in S^1$. Let $H_n(\mathbb{F}_q)$ denote the set of monic square-free polynomials of degree n over $\mathbb{F}_q$. For $\Delta(X) \in H_n(\mathbb{F}_q)$ consider the family of hyperelliptic curves given by

$$C_\Delta : Y^2 = \Delta(X)$$

where Δ runs over $H_n(\mathbb{F}_q)$. The zeta functions $\zeta(T, C_\Delta)$ can be thought of as zeta functions of the quadratic extensions $\mathbb{F}_q(t)(\sqrt{\Delta})$ of the field $\mathbb{F}_q(t)$. The family

$$\{\zeta(T, C_\Delta) | \Delta(x) \in H_n(\mathbb{F}_q)\}$$

is the finite field analogue of the family of quadratic Dirichlet L-functions. The genus $g(n)$ of one of the C_Δ is given by

$$\begin{cases} n = 2g + 1 & \text{if } n \text{ odd} \\ n = 2g + 2 & \text{if } n \text{ even.} \end{cases}$$

Each $\zeta(T, C_\Delta)$ has $2g$ zeros, which we re-normalize to lie on the unit circle. Let the angles corresponding to these zeros be

$$0 \leq \theta_1(\Delta) \leq \cdots \leq \theta_{2g}(\Delta) < 2\pi.$$

We study the distribution of the lowest lying zero $\theta_1(\Delta)$ as we vary $\Delta \in H_n(\mathbb{F}_q)$, which is the finite field analogue of studying the lowest-lying zero statistics of quadratic Dirichlet L-functions. The monodromy of this family of hyperelliptic curves is $\mathrm{Sp}(2g)$, and thus Katz and Sarnak prove in [KS99a] for any test function $f \in C_c^\infty(\mathbb{R}_{\geq 0})$ that

$$\lim_{n\to\infty} \lim_{q\to\infty} \frac{1}{\#H_n(\mathbb{F}_q)} \sum_{\Delta \in H_n(\mathbb{F}_q)} f\left(\frac{\theta_1(\Delta)2g(n)}{2\pi}\right) = \int_0^\infty f(x) d\nu_1(\mathrm{Sp})(x). \tag{1.7}$$

Here $\nu_1(\mathrm{Sp})$ is the measure

$$\nu_1(\mathrm{Sp}) = \lim_{N\to\infty} \nu_1(\mathrm{Sp}(2N))$$

where

$$\nu_1(\mathrm{Sp}(2N))([a,b]) = \mathrm{Haar}\{A \in \mathrm{USp}(2N) | \frac{\theta_1(A)2N}{2\pi} \in [a,b]\}$$

and $\theta_1(A)$ is the smallest angle of an eigenvalue of the unitary symplectic matrix A (see (1.5)).

1.2.4 Number Fields

We would like to know if the structures such as symmetry and the monodromy representation, which are crucial to the proof of the Riemann hypothesis of varieties over finite fields, exist for number field L-functions! It is remarkable that the symmetry of the underlying family *is* detectable in the number field case despite the fact that we have no idea what the spectral interpretation might be.

We begin by describing the analogue of the nearest neighbor spacings (1.6) for the

Riemann zeta function, that is, Montgomery's pair correlation conjecture [Mon73]. We are presenting these results historically out-of-order as the pair correlation conjecture was the original inspiration for the present subject. For the remainder of this subsection we assume the Riemann hypothesis for any L-functions under consideration.

Let γ_j be the imaginary parts of the zeros of the Riemann zeta function

$$\cdots\gamma_{-1} \leq 0 \leq \gamma_1 \leq \gamma_2 \cdots$$

with $\gamma_j = -\gamma_{-j}$. For example, $\gamma_1 = 14.13\ldots$. By the argument principle we know the mean spacing of zeros:

$$\#\{j|0 \leq \gamma_j \leq T\} \sim \frac{T \log T}{2\pi},$$

as $T \to \infty$. Thus we re-normalize the imaginary parts of the zeros γ_j to have mean spacing 1. The re-normalized imaginary parts are

$$\widehat{\gamma}_j = \frac{\gamma_j \log \gamma_j}{2\pi},$$

for $j \geq 1$. Montgomery in 1973 [Mon73] showed for a very restricted class of test functions that the pair correlation of zeros is equal to the pair correlation of random unitary matrices. For any Fourier transform pair of Schwartz class functions $\{\phi, \widehat{\phi}\}$ with the support of $\widehat{\phi}$ contained in $(-1, 1)$ Montgomery shows that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{1\leq j\neq k\leq N} \phi(\widehat{\gamma}_j - \widehat{\gamma}_k) = \int_{-\infty}^{\infty} \phi(x) r_2(\mathrm{GUE})(x)\, dx \tag{1.8}$$

where

$$r_2(\mathrm{GUE}) = 1 - \left(\frac{\sin \pi x}{\pi x}\right)^2$$

is the pair correlation measure for eigenvalues of random unitary matrices. The pair correlation conjecture is that (1.8) is true without any restrictions on the support of $\widehat{\phi}$. The pair correlation conjecture should be see as a number field analogue of Katz and Sarnak's function field result (1.6). Odlyzko [Odl01] collected a great deal of numerical data on the zeta function from the 1980s through 2001, and this data strongly supports the truth of the pair correlation conjecture. Rudnick and Sarnak [RS96] studied in general n-correlation for zeros of arbitrary GL_m automorphic forms high in the critical strip and found the same phenomena. The n-correlation conjecture for GL_2 forms was numerically investigated by Rubinstein [Rub98].

On the other hand, the distribution of low-lying zeros is sensitive to the symmetry type of the family in question. As above in the finite field case, we consider the family of L-functions given by the quadratic field extensions of $\mathbb{Q}$. Recall that $\mathcal{D}$ denotes the set of fundamental discriminants, i.e. those discriminants that come from quadratic fields. Consider the family of Dirichlet L-functions

$$\mathcal{F} = \{L(s, \chi_d) | d \in \mathcal{D}\}.$$

The analytic conductor at the central point of these is

$$\mathfrak{q}(1/2, \chi_d) = d(\mathfrak{a} + 7/2) \asymp d.$$

Let us assume GRH for this family, order the imaginary parts of the zeros

$$\cdots \gamma_{d,-1} \leq 0 \leq \gamma_{d,1} \leq \gamma_{d,2} \leq \cdots,$$

and study their re-normalizations

$$\widehat{\gamma}_{d,j} = \frac{\gamma_{d,j} \log d}{2\pi}.$$

For $\phi \in \mathcal{S}(\mathbb{R})$ a Schwartz class test function whose Fourier transform $\widehat{\phi}$ is supported in $(-2, 2)$ it was proved by Özlük and Snyder [ÖzlükS93] that

$$\lim_{X\to\infty} \frac{1}{\#\{|d| \leq X | d \in \mathcal{D}\}} \sum_{\substack{|d|\leq X \\ d\in\mathcal{D}}} \sum_{j\neq 0} \phi\left(\widehat{\gamma}_{d,j}\right) = \int_{-\infty}^{\infty} \phi(x) w(\mathrm{Sp})(x)\, dx \tag{1.9}$$

where

$$w(\mathrm{Sp})(x) = 1 - \frac{\sin 2\pi x}{2\pi x}$$

is known as the 1-level scaling density for eigenvalues of unitary symplectic matrices near 1. The result (1.9) should be seen as the number field analogue of the result of Katz and Sarnak (1.7) although for a very restricted class of test functions and for slightly different measures $\nu_1(\mathrm{Sp})$ versus $w(\mathrm{Sp})$. It is conjectured that (1.9) holds without restriction on the support of the test function. Extensive numerical evidence for this and many other such conjectures was given by Rubinstein [Rub98].

We define the measure $w(\mathrm{Sp})(x)\, dx$ in terms of eigenvalues for the sake of completeness of exposition. For $A \in \mathrm{USp}(2N)$ and $[a, b] \subset \mathbb{R}$ let

$$V(A)[a, b] = \#\{\theta(A) | e^{i\theta(A)} \text{ is an eigenvalue of } A, \text{ and } \frac{\theta(A)2N}{2\pi} \in [a, b]\}.$$

Then we take

$$W(\mathrm{Sp}(2N))[a, b] = \int_{\mathrm{USp}(2N)} V(A)[a, b]\, dA,$$

and finally

$$\lim_{N\to\infty} W(\mathrm{Sp}(2N))[a, b] = \int_a^b w(\mathrm{Sp})(x)\, dx.$$

We may also make the same definition with $\mathrm{Sp}(2N)$ replaced by $\mathrm{U}(N)$, $\mathrm{SO}(2N)$ or $\mathrm{SO}(2N + 1)$, see [KS99b].

The example of quadratic Dirichlet L-functions above is one of many families where the zero distribution law for a restricted class of test functions has been investigated. Exactly as in the finite field case of subsection 1.2.3, the symmetry type detected falls into one of four types corresponding to the classical compact groups. We summarize the literature in table 1.1 below. All of the results in this table assume the Riemann hypothesis for the relevant L-functions, and are only valid for a very restricted class of test functions. By contrast, in the function field case the Riemann hypothesis is known, and the results of Katz and Sarnak have no restrictions on the test functions.

Table 1.1:

Family	Symmetry Type	Zeros studied by
t-aspect for $\zeta(s)$	$\mathrm{U}(n)$	[Mon73]
Quadratic Dirichlet L-functions	$\mathrm{Sp}(2n)$	[ÖzlükS93]
t-aspect for GL_m automorphic forms	$\mathrm{U}(n)$	[RS96]
Modular forms f with $w(f) = +1$	$\mathrm{SO}(2n)$	[ILS00]
Modular forms f with $w(f) = -1$	$\mathrm{SO}(2n+1)$	[ILS00]
Symmetric square lift of f	$\mathrm{Sp}(2n)$	[ILS00]
Quad twists of f with $w(f \otimes \chi_d) = +1$	$\mathrm{SO}(2n)$	[Hea04]
Quad twists of f with $w(f \otimes \chi_d) = -1$	$\mathrm{SO}(2n+1)$	[Hea04]
Primitive Dirichlet L-functions	$\mathrm{U}(n)$	[CLLR12]
A specific family of Hecke Grössencharacters	SO(2n)	[CS12]

Remarks:

1. The example of primitive Dirichlet L-functions demonstrates that there are discrete families of L-functions with unitary symmetry.

2. The families of quadratic twists of a modular form show that the harmonic detection device does not determine the symmetry type of the family. Note that the family of quadratic Dirichlet L-functions has symplectic symmetry whereas the families of twists of a modular form are orthogonal. See also the discussion in subsection 1.3.1

3. The last example has been included to show how robust the tetrachotomy is. Even when we pick an unusual and thin subsequence, we nonetheless get one of the four classical compact groups as a symmetry type.

1.2.5 Moments and Random Matrix Theory

Asymptotic estimates for moments of L-functions also reveal the symmetry type of the family. From the theory of complex analytic functions one knows that the size of a holomorphic function is intimately connected to the spacing of its zeros. Thus it is not surprising that in addition to zero statistics random matrix theory also predicts the asymptotic expansions for all moments of L-functions based on their symmetry types. This point of view was explored, and conjectures were worked out in detail for the asymptotic size of all moments of L-functions by Conrey and Farmer [CF00], Keating and Snaith [KS00a, KS00b], and further refined by Conrey, Farmer, Keating, Rubinstein and Snaith [CFK$^+$05] and Gonek, Hughes and Keating [GHK07].

A strange feature of our subject is that despite having a very limited rigorous knowledge of the asymptotic estimates for moments (see section 1.1.2, and the comments at the end of subsection 1.1.3) we essentially "know all the answers" due to the predictions of random matrix theory.

The main principle behind the random matrix conjectures is that a primitive L-function $L(s, f)$ of analytic conductor $\mathfrak{q}(s, f)$ of symmetry type G can be modeled by the characteristic polynomial $\Lambda_A(s)$ of a $N \times N$ matrix $A \in \mathrm{G}(N)$, where $N \sim \frac{1}{2\pi} \log \mathfrak{q}(s, f)$. Indeed, if we assume A is unitary, and define the characteristic polynomial

$$\Lambda_A(s) = \det(I - A^* s) = \prod_{n=1}^{N} (1 - s e^{-i\theta_n}),$$

where A^* is the conjugate transpose of A, so that $A^* A = I$, then $\Lambda_A(s)$ has the following familiar properties.

1. *Dirichlet Series.* We may expand $\Lambda_A(s)$ as a polynomial

$$\Lambda_A(s) = \sum_{n=0}^{N} a_n s^n.$$

2. *Analytic Continuation.* The characteristic polynomial $\Lambda_A(s)$ is a polynomial, hence entire.

3. *Functional Equation.* Because A is unitary we have

$$\Lambda_A(s) = (-1)^N \det A^* s^N \Lambda_{A^*}(1/s).$$

4. *Riemann Hypothesis.* Because A is unitary, all its eigenvalues lie on the unit circle. The unit circle is the analogue of the critical line.

5. *Zero Distribution.* As we have seen in subsections 1.2.3 and 1.2.4, the zeros of $\Lambda_A(s)$ have the same statistics as L-functions.

6. *Central Value.* The point $s = 1$ is the fixed point of the functional equation and therefore should be seen as the analogue of $s = 1/2$ in the case of L-functions.

Note one important feature of L-functions which is lacking in characteristic polynomials is an Euler product. This will correspond below to the "arithmetic factor" a_k in the moment conjectures which is not predicted by random matrix theory, but is given by an Euler product. More recent work by Gonek, Hughes and Keating [GHK07] on "hybrid Euler-Hadamard products" puts the arithmetic factor more naturally in the context of random matrix theory.

We next give 3 sample conjectures produced by random matrix theory. In the below we only state the leading order constants for sake of brevity. These conjectures can be found in the papers of Conrey and Farmer [CF00], and Keating and Snaith

[KS00b, KS00a]. We emphasize, however, that our perspective is substantially influenced by the five author paper [CFK+05], in which the full main term polynomials are predicted, and the connection with random matrix polynomial averages is made more deeply.

Conjecture 1 (Unitary Example) *The Riemann zeta function $\zeta(1/2+it)$ in t-aspect is a unitary family of L-functions. For k a positive integer*

$$\int_0^T |\zeta(1/2+it)|^{2k}\,dt = T\mathcal{P}_k(\log T) + O(T^{1/2+\varepsilon}), \tag{1.10}$$

for some polynomial $\mathcal{P}_k$ of degree k^2 with leading coefficient $g_k a_k/k^2!$, where

$$a_k = \prod_p \left(1-\frac{1}{p}\right)^{k^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+k)}{m!\Gamma(k)}\right)^2 p^{-m}$$

and

$$g_k = k^2! \prod_{j=0}^{k-1} \frac{j!}{(k+j)!}.$$

The degree of the polynomial k^2 and the constant g_k are universal for any family with unitary symmetry, whereas a_k depends delicately on the specific family. In the case of the Riemann zeta function, the arithmetic constant was studied in detail by Conrey and Ghosh [CG92], Conrey and Gonek [CG01] and Gonek, Hughes and Keating [GHK07]. It can be written down in a similar fashion for any other family on a case-by-case basis. Up to these arithmetic factors, the moment conjecture for any unitary family of L-functions is identical to the average of $|\Lambda_A(1)|^{2k}$ over the unitary group $\mathrm{U}(N)$.

Conjecture 2 (Symplectic Example) *Let $\mathcal{D}$ denote the set of fundamental discriminants. The quadratic Dirichlet L-functions $L(s,\chi_d)$, with $d \in \mathcal{D}$ are an example* *of a symplectic family of L-functions. For k a positive integer*

$$\sum_{\substack{|d|\leq X \\ d\in\mathcal{D}}} L(1/2,\chi_d)^k = \frac{6}{\pi^2} X\,\mathcal{Q}_k(\log X) + o(X), \tag{1.11}$$

where $\mathcal{Q}_k$ is a polynomial of degree $\frac{1}{2}k(k+1)$ with leading coefficient $g_k a_k/(\frac{1}{2}k(k+1))!$,

$$a_k = \prod_p \frac{(1-1/p)^{k(k+1)/2}}{1+1/p}\left(\frac{(1-1/\sqrt{p})^{-k}+(1+1/\sqrt{p})^{-k}}{2}+\frac{1}{p}\right)$$

and

$$g_k = (\frac{1}{2}k(k+1))!\prod_{j=1}^{k}\frac{j!}{(2j)!}.$$

There is some disagreement over the size of the error term for these moments, so we have left a completely inexplicit error term of $o(X)$ above. In particular, the work of Diaconu, Goldfeld and Hoffstein [DGH03] predicts for $k=3$ that there is a lower-order term of the form $bX^{3/4}$, whereas the paper [CFK$^+$05] predicts square-root cancellation in the error term. A numerical study has been carried out by Alderson and Rubinstein [AR12], which shows some support for the $bX^{3/4}$ conjecture, but their study is by no means conclusive.

As with moments of the zeta function above, the degree of the polynomial $\mathcal{Q}_k$ and the geometric constant g_k are universal for families with symplectic symmetry, whereas a_k depends on the specific family but is accessible on a case-by-case basis. Up to these arithmetic factors, the moment conjecture for any symplectic family of L-functions is identical to the average of $\Lambda_A(1)^k$ over the unitary symplectic group $\mathrm{USp}(2N)$.

Conjecture 3 (Even Orthogonal Example) *Let $H_2(q)$ be a basis of Hecke newforms of weight 2 and level q. The family of L-functions $L(s,f)$, for $f\in H_2(q)$ is an* *example of an orthogonal family of L-functions. For k a positive integer*

$$\sum_{f\in H_2(q)} L(1/2,f)^k = \frac{1}{3}q\mathcal{R}_k(\log q) + O(q^{1/2+\varepsilon}), \tag{1.12}$$

where $\mathcal{R}_k$ is a polynomial of degree $\frac{1}{2}k(k-1)$ with leading coefficient $g_k a_k/(\frac{1}{2}k(k-1))!$,

$$a_k = \prod_{p\nmid q}\left(1-\frac{1}{p}\right)^{k(k-1)/2} \frac{2}{\pi}\int_0^{\pi}\left(\frac{e^{i\theta}(1-e^{i\theta}/\sqrt{p})^{-1} - e^{-i\theta}(1-e^{-i\theta}/\sqrt{p})^{-1}}{e^{i\theta}-e^{-i\theta}}\right)^k \sin^2\theta\, d\theta$$

and

$$g_k = 2^k(\frac{1}{2}k(k-1))!\prod_{j=1}^{k-1}\frac{j!}{(2j)!}.$$

Again, the degree of the polynomial $\mathcal{R}_k$ and the geometric constant g_k are universal for families with even orthogonal symmetry, whereas a_k depends on the specific family but is accessible on a case-by-case basis. Up to these arithmetic factors, the moment conjecture for any orthogonal family of L-functions is identical to the average of $\Lambda_A(1)^k$ over the special orthogonal group $\mathrm{SO}(2N)$.

Thus by computing the first few moments of a given family of L-functions, one determines the symmetry type of the family. The problem of computing moments is in many ways similar to the problem of computing the zero distribution law, discussed extensively in subsection 1.2.4. In fact, our inability to compute the kth moment past the barrier

$$k = 4\frac{\log\mu(\mathcal{F})}{\log\mathfrak{q}(s,f)}$$

mentioned at the end of subsection 1.1.3 is exactly the same barrier we face in the restriction of support on the test functions in the results (1.8), (1.9) and more generally in each of the results on table 1.1. As an additional similarity between moments and 1-level density computations, one may also obtain many of the same applications listed in subsection 1.2.1 via either technique. We have specifically in mind non-vanishing results at the central point. The disadvantage of using 1-level density results is that usually one must assume the Riemann hypothesis to get the technique off the ground.

1.3 Outline of Dissertation

The results of this thesis are divided into three chapters, which we summarize in the next three subsections. Chapter 2 is to appear in International Mathematics Research Notices [Pet12]. Chapter 3 will eventually become part of a larger paper by the author on the family of L-functions studied by Conrey and Iwaniec [CI00]. Chapter 4 is published in the Journal of Number Theory [Pet13].

Initially, each of these chapters was a separate project embarked upon for its own reasons, and the three have only become a dissertation after the fact. Nonetheless, all three chapters are related because they concern themselves with moments of automorophic L-functions. In addition and perhaps less obviously, there is one additional highly speculative, but significant thread that runs through all 3 chapters: the properties of off-diagonal main terms. These off-diagonal main terms were mentioned earlier in subsection 1.1.2 and in the last paragraph of subsection 1.2.5. Consider the following quote from the paper of Conrey, Farmer, Keating, Rubinstein and Snaith [CFK$^+$05]:

"In the theorems in the literature it is often the case that the simple part of the harmonic detector is sufficiently good to determine the first or second moment of the family. The terms involved here are usually called the 'diagonal' terms. But invariably the more complicated version is needed to determine the asymptotics of the third and fourth moments; in these situations one has gone 'beyond the diagonal'....We believe that as one steps up the moments of a family then at every one or two steps a new type of off-diagonal contribution will emerge. The whole process is poorly understood; we have only glimpses of a mechanism but no clear idea how or why it works."

Our goal here is to gain some understanding of how this mysterious mechanism works.

1.3.1 Moments of $L'(1/2, f \otimes \chi_d)$ over Quadratic Twists

In chapter 2 we estimate moments of L-functions with odd orthogonal symmetry, assuming the generalized Riemann hypothesis (GRH). Consider the quadratic character twists $\chi_d = \left(\frac{\cdot}{d}\right)$ of a fixed holomorphic cusp form f with trivial central character for the congruence subgroup $\Gamma_0(q)$. In this case, the root numbers of the quadratic twists are restricted to be $w(f \otimes \chi_d) = \pm 1$.

Soundararajan and Young [SY10] study the case of root numbers $w(f \otimes \chi_d) = +1$. Recall that the notation $\gcd(d, \square) = 1$ means d is square-free. They prove for f of full level and assuming GRH that

$$\sum_{\substack{\gcd(d,2\square)=1 \\ w(f\otimes\chi_{8d})=1}} L(1/2, f \otimes \chi_{8d})^2 F(8d/X) = C(f) X \log X + O_{f,\varepsilon}\left(X (\log X)^{3/4+\varepsilon}\right) \quad (1.13)$$

for an explicit constant $C(f)$. Our work her improves the exponent in the error term from 3/4 to 1/2.

In chapter 2 we study the case of root number $w(f \otimes \chi_d) = -1$, which has features which are distinct from the $+1$ case. In the -1 case the central values $L(1/2, f \otimes \chi_{8d})$ vanish automatically, and instead the natural object to study is the derivative $L'(1/2, f \otimes \chi_{8d})$. Let $F \in C_c^\infty(\mathbb{R}_{>0})$ be any any fixed smooth cut-off function with mass 1 and compact support. Following Soundararajan and Young's technique, and again assuming GRH we establish for any even weight κ and odd level q the estimate

$$\sum_{\substack{\gcd(d,2q\square)=1\\ w(f\otimes\chi_{8d})=-1}} L'(1/2, f\otimes\chi_{8d})^2 F(8d/X) = C_1(f)X(\log X)^3 + C_2(f)X(\log X)^2 + O_{f,\varepsilon}\left(X(\log X)^{1+\varepsilon}\right) \tag{1.14}$$

where $C_1(f)$ and $C_2(f)$ are constants which are given explicitly in chapter 2. Note that we are able to uncover not one but two main terms in the odd root number case.

Moreover, we find that several additional theorems are possible in the case of root number -1. For two distinct forms f and g we establish the estimate

$$\sum_{\substack{\gcd(d,2q_1q_2\square)=1\\ w(f\otimes\chi_{8d})=-1\\ w(g\otimes\chi_{8d})=-1}} L'(1/2, f\otimes\chi_{8d})L'(1/2, g\otimes\chi_{8d})F(8d/X) = C(f,g)X(\log X)^2 + O_{f,g,\varepsilon}\left(X(\log X)^{1+\varepsilon}\right) \tag{1.15}$$

where $C(f,g)$ is an explicitly given constant depending only on f and g. Next, applying these techniques to the first moment we obtain for any real $A > 0$ and f be an eigenform of even weight κ and odd level q the estimate

$$\sum_{\substack{\gcd(d,2q\square)=1\\ w(f\otimes\chi_{8d})=-1}} L'(1/2, f\otimes\chi_{8d})F(8d/X) = C_3(f)X\log\frac{X\kappa\sqrt{q}}{2\pi} + O_{A,\varepsilon}\left(X(\log X\kappa q)^{1/4+\varepsilon} + \frac{X^{13/17}(\kappa q)^{4/17}}{(\log X\kappa q)^A}\right) \tag{1.16}$$

where $C_3(f)$ is an explicit and easily controlled constant depending only on f. The estimates analogous to (1.15) and (1.16) in the root number $w(f\otimes\chi_d) = +1$ case are completely out of reach of current techniques. The estimate (1.16) above has non-vanishing applications owing to the explicit nature of the error term.

Note that these results are at the very edge of what is possible for the family of quadratic twists according to the benchmark $k = 4\log\mu(\mathcal{F})/\log\mathfrak{q}(1/2, f\otimes\chi_d)$ (see 1.2.1). To estimate any of these moments, one applies Poisson summation to execute the sum over quadratic characters χ_d. One thus obtains a main term plus a "dual moment" which needs to be estimated. In the above three theorems this dual moment is the 2nd or 1st moment of $L(1/2+it, f\otimes\chi_d)$, without the derivatives, and with shifts in t-aspect introduced. The introduction of shifts in t-aspect and the lack of derivative is exploited to obtain the error terms in the above theorems.

The result (1.13) of Soundararajan and Young [SY10] represents the first success at the edge case moment for a family involving an average of quadratic twists. In their result (1.13) no off-diagonal analysis is required as the main term is just barely larger than the error term and comes only from the diagonal. In the same fashion, no off-diagonal analysis is necessary in our results (1.14), (1.15), and (1.16). However, if the error term in (1.13) could be improved to $o(X)$, one sees from the proof given by Sound and Young that an off-diagonal main term of size cX would arise from the first unnumbered displayed equation on page 1109 of their paper. We copy that term here and speculatively replace U by X for the reader's convenience:

$$\begin{aligned} -\frac{X\widetilde{F}(1)}{2\pi^2}\frac{1}{2\pi i}\int_{(1/10)} \frac{\Gamma(\kappa/2+u)\Gamma(\kappa/2-u)}{u^2\Gamma(\kappa/2)^2} &L(1+2u,\mathrm{sym}^2 f)L(1,\mathrm{sym}^2 f) \\ &\times L(1-2u,\mathrm{sym}^2 f)Z_2(u,-u)\,du. \end{aligned} \tag{1.17}$$

Here $\widetilde{F}$ is the Mellin transform of the cut-off function F, and $Z_2(u,v)$ is an absolutely convergent Euler product symmetric in u and v. This term is not predicted by the random matrix conjectures, and it is expected to cancel out against an identical term elsewhere in the calculation in the final analysis. Note the striking similarity of the off-diagonal main term (1.17) to the off diagonal main terms (1.19) and (1.22) arising in subsections 1.3.2 and 1.3.3 below. A "hidden functional equation" in the spirit of

[DGH03] or the discussion on page 5 of [You11] should be required to evaluate these off-diagonal main terms.

It would be very interesting to see how these ideas carry over to the problem of estimating the fourth moment of quadratic Dirichlet L-functions, a very similar problem to the one studied in chapter 2, but having symplectic symmetry. This represents future work of the author.

1.3.2 The Second Moment of Automorphic L-Functions

Let $H_\kappa(q)$ be a basis of Hecke eigenforms of weight κ and level q, and χ the quadratic Dirichlet character of conductor q. Consider the family of forms $f \otimes \chi$ of weight κ and level q^2 where f varies over the forms $f \in H_\kappa(q)$. In this section, we study the family of L-functions

$$\mathcal{F} = \{L(s, f \otimes \chi) | f \in H_\kappa(q)\}.$$

This family is of special interest because it was studied by Conrey and Iwaniec, who (as we remarked in subsection 1.2.1) established a Lindelöf-on-average for the cubic moment of central values of these L-functions. This represents the only known technique to go beyond the typical barrier $\kappa = 4 \log \mu(\mathcal{F}) / \log \mathfrak{q}(1/2, f \otimes \chi_d)$. The cubic moment in this family represents replacing 4 here by a 6.

In chapter 3 we investigate the easier problem of the second moment of this family, and obtain square root cancellation in the error term. Let $q \to \infty$ through primes and f have even weight $\kappa \geq 6$. Let $\Lambda(s, f \otimes \chi)$ be the completed L-function, and ω_f be the harmonic weights which make the Petersson trace formula work out nicely. In chapter 3 we prove uniformly for $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) < 1/2$ that

$$\sum_{f \in H_\kappa(q)} \omega_f \Lambda(1/2+\alpha, f\otimes\chi)\Lambda(1/2+\beta, f\otimes\chi) = \sum_{\pm\alpha,\pm\beta} \left(\frac{q}{2\pi}\right)^{1\pm\alpha\pm\beta} \zeta_q(1\pm\alpha\pm\beta)\Gamma(\pm\alpha+\kappa/2)\Gamma(\pm\beta+\kappa/2) + O_\kappa(q^{1/2}), \tag{1.18}$$

where the sum on the right is over the 4 choices of signs. Note that the archimedian factors of these L-functions depend on κ and q but not the individual f, so they may be easily removed from both sides of the formula. However, it seems more elegant to retain the functional equation symmetries.

The purpose of this chapter is to showcase the off-diagonal analysis which is executed in section 3.3 of chapter 3. One might expect the collection of off-diagonal terms to be a messy bookkeeping problem, but for this family we actually found the cancellation to be quite elegant. The off diagonal main terms are given by the integrals (3.4):

$$\frac{1}{2\pi i}\int_{(2)} \frac{\Gamma(\kappa/2-s)\Gamma(\kappa/2+s)}{(s-\alpha)(s-\beta)}\, ds. \tag{1.19}$$

The reader should compare this with the integral (1.17) in the above subsection 1.3.1 and the integral (1.22) in the below subsection 1.3.3. It would be quite interesting to extend the off-diagonal analysis of chapter 3 to the cubic moment. This represents future work of the author.

1.3.3 Transition Mean Values of Shifted Convolution Sums

The results of chapter 4 are related to moments via the work of Sarnak [Sar01]. Let g be a fixed Hecke eigenform of full level, and K and M are large with $K^{151/165} \leq$

$M \le K^{1-\varepsilon}$. Sarnak proves that

$$\sum_{\kappa=K-M}^{K+M} \sum_{f \in H_\kappa(1)} |L(1/2+it, f \otimes g)|^2 \ll_{\varepsilon,g,t} (KM)^{1+\varepsilon}. \tag{1.20}$$

This is a Lindelöf-on-average upper bound which is within ε of the "truth". To establish (1.20) Sarnak must bound certain off-diagonal contributions to the moment, which are basically of the form

$$\sum_{h \doteqdot Y} \sum_{n \doteqdot X} \lambda_g(n) \lambda_g(n+h), \tag{1.21}$$

where $\lambda_g(p)$ are the Hecke eigenvalues of the form g, and by the symbol $\doteqdot X$ we mean a sum of length X with a smoothing that is not specified here. See Lemma 4.2 and equation (90) in the paper [Sar01]. The interior sum of (1.21) is called a "shifted convolution sum".

While Sarnak only needs an upper bound on (1.21), we examine this sum in chapter 4 much more closely and find some surprising structure. When $Y \gg \sqrt{X} \log X$, say, we find that the average of shifted convolution sums (1.21) has an asymptotic main term estimate as X and $Y \to \infty$. Meanwhile, when $X \gg Y^2 \log Y$ we obtain an upper bound on (1.21) improving on the estimates in [Sar01] where the sum over h is evaluated trivially. The most interesting case is when X and $Y \to \infty$ and $Y^2/X = \alpha$, a fixed constant. In this case we obtain an asymptotic main term for (1.21) which depends very delicately on the constant α. Let us choose a decaying exponential cut-off function. Then the leading order constant in this asymptotic estimate for (1.21) is

$$\frac{\pi^{3/2}}{2} \alpha \frac{1}{2\pi i} \int_{(2)} \frac{\Lambda(s, g \times g)}{\xi(2s)} \frac{\Gamma(s-1/2)}{\Gamma(2-s)} \left(\frac{\pi}{4}\alpha\right)^{-s} ds - 12 \operatorname*{Res}_{s=1} \Lambda(s, g \times g), \tag{1.22}$$

where $\Lambda(s, g \times g)$ is the completed Rankin-Selberg L-function of g with itself, see [Iwa97] section 13.8. Note the similarity with the off-diagonal terms (1.17) and (1.19) from the previous two subsections.

A second interesting feature is that the leading constant at the transition range (1.22) has very weird differential properties. We expect that if the cut-off functions in (1.21) could be taken to be sharp, then the leading term constant at the transition range analogous to (1.22) would be twice continuously differentiable in α, but have a second derivative which is almost-nowhere differentiable!

The inspiration for this non-differentiability conjecture is the paper of Conrey, Farmer and Soundararajan [CFS00]. For n and m positive odd integers, let $\left(\frac{m}{n}\right)$ denote the quadratic residue symbol or Jacobi symbol. Conrey, Farmer and Soundararajan find a uniform asymptotic formula for the double sum

$$\sum_{\substack{m \le X \\ m \text{ odd}}} \sum_{\substack{n \le Y \\ n \text{ odd}}} \left(\frac{m}{n}\right). \tag{1.23}$$

Similar sums to (1.23) arise as off-diagonal main terms in moments of quadratic Dirichlet L-functions in the work of Soundararajan [Sou00]. When X and $Y \to \infty$ and $Y/X = \alpha$ a fixed constant, the asymptotic main term they find depends delicately on α. In fact, they show that the leading constant in the asymptotic is once continuously differentiable, but its first derivative is almost-nowhere differentiable.

In chapter 4 we discuss both of these results, and a heuristic for why this strange non-differentiable behavior occurs in terms of Eisenstein series and automorphic distributions. This suggests that such non-differentiable off-diagonal main terms are not simply an accident but part of a more general story not yet understood. It would be very interesting to investigate other examples, and to try to make these heuristics rigorous. This represents future work of the author.

1.3.4 Conclusion

The point we are trying to make is that off-diagonal main terms are not only "garbage" introduced by imperfect harmonic detection devices, but instead have some structure. This structure should come from period integrals of automorphic forms, represent some interplay between the symmetry type and the orthogonality relation or trace formula used to average the family. Consider the the similarities noted by Young (see page 5 of [You11]) between the off-diagonal main terms of the 4th moment of $\zeta(1/2 + it)$ and off-diagonal main terms of the 4th moment of primitive Dirichlet L-functions, despite the very different orthogonality relations for these two families.

Other authors have also encountered off-diagonal main terms given by integrals similar to the above emphasized (1.17), (1.19) and (1.22). See again the comments on page 5 of Young's paper [You11], the discussion following (5.16) in Soundararajan [Sou00] and the comments at the end of the appendix in Kowalski's Ph.D. thesis [Kow98]. In each of these situations there are extra symmetries of the integrand which allow the explicit evaluation and cancellation of the off-diagonal contributions. These are the so-called "extra functional equations" in the work of Diaconu, Goldfeld and Hoffstein [DGH03].

Perhaps chapter 4 in this thesis is the best suggestion that there should be some structure coming from automorphic objects behind off-diagonal main terms. But all of these ideas are highly speculative inspirations for further work, and this is probably a good place to stop and let the mathematics speak for itself, below.

Chapter 2

Moments of $L'(1/2, f \otimes \chi_d)$ in the Family of Quadratic Twists

In this chapter we study the central values of derivatives of L-functions of holomorphic GL_2 modular forms varying over the family of quadratic twists. The mean value of this family has been studied successfully in the past by several authors, notably Bump, Friedberg and Hoffstein [BFH90], Murty and Murty [MM91], Iwaniec [Iwa90] and Munshi [Mun11a], [Mun11b].

When $f \otimes \chi_d$ has even functional equation an asymptotic formula for the second moment of $L(1/2, f \otimes \chi_d)$ was computed assuming the generalized Riemann hypothesis (GRH) by Soundararajan and Young [SY10]. Here, we apply their techniques to several moment problems of comparable difficulty when the sign of the functional equation is -1 and the derivative $L'(1/2, f \otimes \chi_d)$ is the correct object of study. The family of quadratic twists with root number $+1$ as considered by Soundararajan and Young has even orthogonal symmetry in the sense of random matrix theory, while the family we consider has root number -1 and odd orthogonal symmetry. Surprisingly, we find that stronger results are possible in the odd case: the analogues of theorems 2 and 3 of are out of reach when the root number of $f \otimes \chi_d$ is 1 and one studies the L-functions themselves. As in Soundararajan and Young, our work is conditional on GRH, but we only use this hypothesis to obtain a useful upper bound to the corresponding un-differentiated 1st and 2nd moment problems, see Conjectures 4 and 5. The deduction of the necessary upper bounds from GRH is due to Soundararajan [Sou09]. We restrict our attention to holomorphic forms in this chapter, but our results should carry over to Maass forms with only minor modifications to the proofs.

Before stating our results, let us fix some notation and recall some standard facts which can be found in chapter 14 of [IK04]. We consider the space of cuspidal holomorphic modular forms of even weight κ on the congruence subgroup $\Gamma_0(q)$ with trivial central character. Such forms have a Fourier expansion of the form

$$f(z) = \sum_{n\geq 1} \lambda_f(n) n^{(\kappa-1)/2} \exp(2\pi i z).$$

We fix a basis of newforms which are eigenfunctions of the Hecke operators and have $\lambda_f(1) = 1$. From now on, we assume all forms f which we work with are elements of this basis. The Hecke eigenvalues of f are all real (by the adjointness formula and multiplicity one principle), and hence f is self-dual. We study the family of twists of f by quadratic characters. Let d be a fundamental discriminant relatively prime to q, and let $\chi_d(\cdot) = \left(\frac{d}{\cdot}\right)$ denote the primitive quadratic character of conductor $|d|$. Then $f \otimes \chi_d$ is a newform on $\Gamma_0(q|d|^2)$ and the twisted L-function is defined for $\mathrm{Re}(s) > 1$ by

$$\begin{aligned} L(s, f \otimes \chi_d) &:= \sum_{n\geq 1} \frac{\lambda_f(n)}{n^s} \chi_d(n) \\ &= \prod_{p\nmid qd} \left(1 - \frac{\lambda_f(p)\chi_d(p)}{p^s} + \frac{1}{p^{2s}}\right)^{-1} \prod_{p|q} \left(1 - \frac{\lambda_f(p)\chi_d(p)}{p^s}\right)^{-1}. \end{aligned}$$

The completed L-function is defined by

$$\Lambda(s, f \otimes \chi_d) := \left(\frac{|d|\sqrt{q}}{2\pi}\right)^s \Gamma\left(s + \frac{\kappa - 1}{2}\right) L(s, f \otimes \chi_d).$$

It has the functional equation

$$\Lambda(s, f \otimes \chi_d) = i^\kappa \eta \chi_d(-q) \Lambda(1 - s, f \otimes \chi_d),$$

where η is given by the eigenvalue of the Fricke involution, which is independent of d and always ± 1. We denote the root number by $w(f \otimes \chi_d) := i^\kappa \eta \chi_d(-q)$. Note that if d is a fundamental discriminant, then $\chi_d(-1) = \pm 1$ depending as whether d is positive or negative. In this chapter we work with positive discriminants so that $\chi_d(-q) = \chi_d(q)$, but we could just as easily formulate our results with negative discriminants.

We are interested here in the derivative of the L-function, which also has a Dirichlet series convergent in a right half-plane:

$$L'(s, f \otimes \chi_d) = -\sum_{n=1}^{\infty} \frac{\lambda_f(n) \chi_d(n) \log n}{n^s}.$$

It also has a functional equation

$$\Lambda'(s, f \otimes \chi_d) = -i^\kappa \eta \chi_d(-q) \Lambda'(1 - s, f \otimes \chi_d)$$

with sign opposite to that of $L(s, f \otimes \chi_d)$. When $w(f \otimes \chi_d) = -1$, one has that $L(1/2, f \otimes \chi_d) = 0$ and $L'(1/2, f \otimes \chi_d)$ is the more appropriate object for study.

2.1 Statement of Main Results

In the results of this section we assume the generalized Riemann hypothesis (GRH) for the zeta function, the family of quadratic twists of f and g and the symmetric square of f and g. See also the comments immediately before and after Conjectures 4 and 5, below. Recall the notations $(d,\Box)=1$ and $\mathcal{D}$ denote the sets of square-free integers and fundamental discriminants, respectively. Let $F \in C_c^\infty(\mathbb{R}_{>0})$ be a fixed smooth function with compact support closely resembling the indicator function of the interval $[0,1]$, and let $\widetilde{F}(s) = \int_0^\infty F(x)x^{s-1}\,dx$ denote its Mellin transform. We formulate our results for the residue class of $\mathcal{D}$ of integers which are 4 times a $2 \mod 4$ square-free integer, but could have just as well picked out the other congruence classes which together constitute $\mathcal{D}$.

Theorem 1 *Assume GRH, and let $F \in C_c^\infty(\mathbb{R}_{>0})$ be a smooth approximation to the indicator function of $[0,1]$ with compact support. For any normalized cuspidal Hecke newform f with trivial central character, odd level q and even weight κ we have*

$$\sum_{\substack{(d,2q\Box)=1\\ w(f\otimes\chi_{8d})=-1}} L'(1/2, f\otimes\chi_{8d})^2 F(8d/X) = \frac{X}{\pi^2} L(1,\mathrm{sym}^2 f)^3 Z^*(0,0)\widetilde{F}(1)$$
$$\times\left(\frac{1}{3}\log^3 X + C_2(f)\log^2 X\right) + O_{\kappa,q,\varepsilon}\left(X(\log X)^{1+\varepsilon}\right).$$

In the above

$$C_2(f) = \frac{\Gamma'(\kappa/2)}{\Gamma(\kappa/2)} + \log\frac{\sqrt{q}}{2\pi} + \gamma + 3\frac{L'(1,\mathrm{sym}^2 f)}{L(1,\mathrm{sym}^2 f)} + \frac{\frac{d}{du}Z^*(u,0)|_{u=0}}{Z^*(0,0)} + \frac{\widetilde{F}'(1)}{\widetilde{F}(1)},$$

γ is Euler's constant, and $Z^*(u,v)$ is a holomorphic function defined by (2.2) and (2.3) for $\mathrm{Re}(u), \mathrm{Re}(v) > -1/4+\varepsilon$ given by a sum of two absolutely convergent Euler products and is uniformly bounded in u, v where it converges. Moreover, $Z^*(0,0)=0$ if and only if the root number $w(f)=1$ and q is square, in which case the moment

vanishes identically.

By the celebrated theorem of Gross and Zagier [GZ86], Theorem 1 also gives the variance of canonical heights of Heegner points on an elliptic curve associated with f. Note that the analogue of Theorem 1 without the derivative is the main result of Soundararajan and Young [SY10]. In this chapter, we compute the main terms in a slightly different manner than do Soundararajan and Young, and applying our technique to the second moment without derivatives improves the error term there to $\ll_{\kappa,\varepsilon} X(\log X)^{1/2+\varepsilon}$. Nonetheless, shifted moments are still crucial to the theorem of Soundararajan and Young, whereas they are not necessary here.

The next theorem is a moment for two distinct modular forms f and g. Theorem 2 is particularly interesting because the asymptotic formula for the analogous moment without derivatives is completely out of reach by current techniques.

Theorem 2 *Assume GRH, and let $F \in C_c^\infty(\mathbb{R}_{>0})$ be a smooth approximation to the indicator function of $[0,1]$ with compact support. For any two distinct normalized cuspidal Hecke newforms f and g with trivial central characters, odd levels q_1 and q_2, and even weights κ_1 and κ_2 we have*

$$\sum_{\substack{(d,2q_1q_2\square)=1 \\ w(f\otimes\chi_{8d})=-1 \\ w(g\otimes\chi_{8d})=-1}} L'(1/2, f\otimes\chi_{8d})L'(1/2, g\otimes\chi_{8d})F(8d/X)$$
$$= C(f,g)X\log^2 X + O_{f,g,\varepsilon}\left(X(\log X)^{1+\varepsilon}\right).$$

In the above

$$C(f,g) = \frac{1}{2\pi^2}L(1,\mathrm{sym}^2 f)L(1,\mathrm{sym}^2 g)L(1, f\otimes g)Z^*(0,0)\widetilde{F}(1),$$

where $Z^*(u,v)$ is a holomorphic function defined by (2.8) and (2.9) in $\mathrm{Re}(u),\mathrm{Re}(v) \geq -1/4+\varepsilon$, depending on f and g, given by a sum of four absolutely convergent Euler *products and uniformly bounded in u, v where it converges. Moreover, $Z^*(0,0) = 0$ if and only if either the root number $w(f) = 1$ and q_1 is square or the root number $w(g) = 1$ and q_2 is square. In either of these two cases the moment vanishes identically.*

Lastly, Theorem 3 below is a first moment in the twist aspect with controlled dependence on both the weight κ and level q. Again, the analogue of Theorem 3 without the derivative is completely out of reach, but would have interesting corollaries, see [LY11].

Theorem 3 *Assume GRH, and let $F \in C_c^\infty(\mathbb{R}_{>0})$ be a smooth approximation to the indicator function of $[0,1]$ with compact support. For any $A > 0$ and any normalized cuspidal Hecke newform f with trivial central character, odd level q and even weight κ we have*

$$\sum_{\substack{(d,2q\Box)=1 \\ w(f\otimes\chi_{8d})=-1}} L'(1/2, f\otimes\chi_{8d})F(8d/X) = C_3(f)X\left(\log\frac{X\kappa\sqrt{q}}{2\pi} + 2\frac{L'(1,\mathrm{sym}^2 f)}{L(1,\mathrm{sym}^2 f)}\right.$$

$$\left. + \frac{Z^{*\prime}(0)}{Z^*(0)}\right) + O_{A,\varepsilon}\left(X(\log X\kappa q)^{1/4+\varepsilon} + \frac{X^{13/17}(\kappa q)^{4/17}}{(\log X\kappa q)^A}\right),$$

In the above

$$C_3(f) = \frac{\widetilde{F}(1)}{2\pi^2}L(1,\mathrm{sym}^2 f)Z^*(0)$$

and $Z^*(u)$ is a holomorphic function defined by (2.13) and (2.17) as a sum of two absolutely convergent Euler products for $\mathrm{Re}(u) > -1/4+\varepsilon$. Moreover, $Z^*(0) = 0$ if and only if the root number $w(f) = 1$ and q is a square. If so, then the moment vanishes identically, and if not

$$Z^*(0) \gg \frac{\log\log q}{(\log q)^{1/2}},$$

uniformly in κ.

Thus our methods break convexity in the dependence on κ and q in the error term by an arbitrary power of log. Using GRH once again, we obtain non-vanishing results. By applying the technique from [IK04] Theorem 5.17 we have that

$$\frac{L'(1,\mathrm{sym}^2 f)}{L(1,\mathrm{sym}^2 f)} + \frac{Z^{*'}(0)}{Z^*(0)} \ll \log\log \kappa q.$$

These terms therefore may be subsumed into the error term in Theorem 3. In the same vein, by Theorem 5.19 of [IK04] one has the bound

$$L(1,\mathrm{sym}^2 f) \gg (\log\log \kappa q)^{-1}.$$

From these estimates and Theorem 3 the following Corollary is obtained.

Corollary 1 *Assume GRH. If the root number of f is 1 then assume also that the level of f is not an integer square. For any $A > 0$ there exists an odd square-free d relatively prime to q with $d \ll_A \kappa q/(\log \kappa q)^A$ for which*

$$w(f \otimes \chi_{8d}) = -1 \qquad \textit{and} \qquad L'(1/2, f \otimes \chi_{8d}) > 0.$$

If $E/\mathbb{Q}$ is an elliptic curve given by the Weierstauss equation $y^2 = f(x)$, we may define the twisted elliptic curve $E^d/\mathbb{Q}$ by the equation $dy^2 = f(x)$. By the work of Gross and Zagier [GZ86] and the modularity Theorem [BCDT01] we have the following Corollary.

Corollary 2 *Assume GRH. Let $E/\mathbb{Q}$ be an elliptic curve of odd conductor q. If the root number of E is 1, then assume also that the conductor q is not an integer square. For every $A > 0$ there exist odd square-free d relatively prime to q with $d \ll_A q/(\log q)^A$ for which the curve $E^{8d}/\mathbb{Q}$ has root number -1 and Mordell-Weil rank exactly 1.*

The convexity bound here to be a non-vanishing twist of size $d \ll_\varepsilon (\kappa q)^{1+\varepsilon}$, see e.g. Hoffstein and Kontorovich [HK10]. Our non-vanishing corollaries on GRH are, in fact, quite weak. As previously remarked by many authors, the method of moments is an inefficient way to produce non-vanishing theorems. If one is willing to assume GRH, the methods of Iwaniec, Luo and Sarnak [ILS00], Özlük and Snyder [ÖzlükS93, ÖzlükS99, ÖzlükS06] or Heath-Brown [Hea04] adapted to small nonvanishing twists should yield better results. We postpone carrying out this line of research to a future paper, and moreover, we believe that the theorems 1, 2 and 3 have interest independent of the corollaries.

We do not use the full strength of GRH in theorems 1, 2 or 3. In fact, in the case of the first two all we need is the following Conjecture.

Conjecture 4 *Let $\varepsilon > 0$, and t be a real number with $|t| \leq X$ and $1/2 \leq \sigma \leq 1/2 + 1/\log X$. Then*

$$\sum_{\substack{d\in\mathcal{D} \\ (d,q)=1 \\ |d|\leq X}} |L(\sigma + it, f \otimes \chi_d)|^2 \ll_{\kappa,q,\varepsilon} X(\log X)^{1+\varepsilon}.$$

Theorem 3 on the other hand is true if we assume than q is odd square-free and Conjecture 5 in place of GRH.

Conjecture 5 *Let $\varepsilon > 0$, and t be a real number with $|t| \leq X$ and $1/2 \leq \sigma \leq 1/2 + 1/\log X$. Then*

$$\sum_{\substack{d\in\mathcal{D} \\ (d,q)=1 \\ |d|\leq X}} |L(\sigma + it, f \otimes \chi_d)| \ll_{\varepsilon} X(\log X\kappa q)^{1/4+\varepsilon}.$$

The work of Soundararajan [Sou09] shows that Conjecture 4 follows from the GRH for the Riemann zeta function, the family of quadratic twists of f and the symmetric square of f. By keeping track of the dependence on κ and q in Soundararajan's proof, one finds that the GRH for quadratic twists of f, the Riemann zeta function, and the symmetric square of f implies Conjecture 5. Unconditionally, all that is known towards Conjectures 4 and 5 is a bound of the form $\ll_{f,\varepsilon} (X(1+|t|))^{1+\varepsilon}$ due to Heath-Brown's quadratic large sieve [Hea95]. It seems that obtaining the results of this chapter unconditionally should not be completely out of reach, but nonetheless, doing so requires additional ideas.

Let us briefly describe the main difficulties in proving the above theorems, some previous attacks on these difficulties, and the new input in our work which allows us to overcome them.

Take for example Theorem 1. After applying the approximate functional equation and pulling the sum over d inside one encounters a sum of the form

$$\sum_{d} \chi_d(n_1 n_2) F\left(\frac{d}{U}\right)$$

for some cut-off function F, where χ_d is the quadratic character modulo d. One wants to apply Poisson summation to this sum, but the length of the sum $U \approx X$ is comparable to the square root of the conductor $\sqrt{n_1 n_2}$, so the dual sum that one obtains is of the same shape as the original. This is the familiar "deadlock" situation described, for example, in the paper of Munshi [Mun11a], or by multiple Dirichlet series, for example in [DGH03]. This deadlock has been broken in some ways before. Soundararajan and Young find that the second moment of $L(1/2, f \otimes \chi_d)$ is transformed by Poisson summation to the dual problem of finding an estimate of the integral over shifts it_1 and it_2 of the same moment. They exploit this transformation using GRH to obtain upper bounds on shifted moments to prove their theorem. Munshi observes in the paper [Mun11a] that taking derivatives amplifies the main term of moments but does not affect the error term. He uses this fact to unconditionally obtain an asymptotic formula for the first moment of higher derivatives $\Lambda^{(\ell)}(1/2, f \otimes \chi_d)$ with $\ell \geq 8$ weighted by the number of representations of d as a sum of two squares (a situation with conductor of similar length to ours). Munshi also solves a similar problem in [Mun11b] obtaining an asymptotic for the first derivative in the special case that f corresponds to a CM elliptic curve.

In this chapter, we observe that taking a derivative concentrates the mass of $L'(1/2, f \otimes \chi_d)$ in the terms of the approximate functional equation with small n. When we truncate $U \leq X/(\log X)^{100}$ we gain something from Poisson summation, and treat the tail separately. The idea behind bounding the tail is that

$$L'(1/2, f \otimes \chi_d) \approx \sum_{n \leq |d|} \frac{\lambda_f(n)\chi_d(n) \log \frac{|d|}{n}}{n^{1/2}},$$

so that when $|d|/(\log |d|)^{100} \leq n \leq |d|$ we have that the $0 \leq \log |d|/n \ll \log\log |d|$ are quite small. These terms look essentially like the series for $L(1/2, f \otimes \chi_d)$, the moments of which are smaller than moments of the derivative. We are then able to use Soundararajan's upper bounds assuming GRH [Sou09] to bound the tail. The idea is that the dual sum of a moment of $L'(1/2, f \otimes \chi_d)$ looks like a moment of the un-differentiated $L(1/2, f \otimes \chi_d)$, which we exploit to obtain our results.

2.2 Approximate Functional Equation

We begin with a lemma which will be used in all three theorems.

Lemma 1 (Approximate functional equation) *Let f be a $\lambda_f(1) = 1$ normalized cuspidal newform on $\Gamma_0(q)$ with trivial central character and root number $w(f) = i^\kappa \eta$.*

Let $Z > 0$ be an arbitrary real number parameter. Define the cut-off function

$$W_Z(x) := \frac{1}{2\pi i}\int_{(3)} \frac{\Gamma(u+\kappa/2)}{\Gamma(\kappa/2)}\left(\frac{2\pi x}{Z\sqrt{q}}\right)^{-u}\frac{1-u\log Z}{u^2}\,du.$$

Then

$$\sum_{n\geq 1}\frac{\lambda_f(n)\chi_d(n)}{n^{1/2}}W_Z\left(\frac{n}{|d|}\right) - i^\kappa\eta\chi_d(-q)\sum_{n\geq 1}\frac{\lambda_f(n)\chi_d(n)}{n^{1/2}}W_{Z^{-1}}\left(\frac{n}{|d|}\right)$$
$$= \begin{cases} L'(1/2, f\otimes\chi_d) & \textit{if } w(f\otimes\chi_d) = -1 \\ 0 & \textit{if } w(f\otimes\chi_d) = 1. \end{cases}$$

Proof. We follow Iwaniec and Kowalski [IK04] Section 5.2. Take

$$\begin{aligned} I(Z,f,s) &:= \frac{1}{2\pi i}\int_{(3)}\Lambda(s+u, f\otimes\chi_d)Z^u\frac{1-u\log Z}{u^2}\,du \\ &= \Lambda'(s, f\otimes\chi_d) + \frac{1}{2\pi i}\int_{(-3)}\Lambda(s+u, f\otimes\chi_d)Z^u\frac{1-u\log Z}{u^2}\,du, \end{aligned}$$

so that by a change of variables and an application of the functional equation we have

$$I(Z,f,s) = \Lambda'(s, f\otimes\chi_d) + i^\kappa\eta\chi_d(-q)I(Z^{-1}, f, 1-s).$$

If the root number $w(f\otimes\chi_d) = -1$, we take $s = 1/2$ to find

$$L'(1/2, f\otimes\chi_d) = \sum_{n\geq 1}\frac{\lambda_f(n)\chi_d(n)}{n^{1/2}}W_Z\left(\frac{n}{|d|}\right) + \chi_d(-q)\sum_{n\geq 1}\frac{\lambda_f(n)\chi_d(n)}{n^{1/2}}W_{Z^{-1}}\left(\frac{n}{|d|}\right),$$

as in the statement of the Lemma. On the other hand, if the root number of $f\otimes\chi_d$ is 1, then $\Lambda'(1/2, f\otimes\chi_d) = 0$, hence

$$I(Z,f,1/2) - i^\kappa\eta\chi_d(-q)I(Z^{-1}, f, 1/2) = 0.$$

Thus the Lemma holds for both cases of root number of $f \otimes \chi_d$. $\square$

In the proof of Theorems 1 and 2 we will use $Z = 1$ so that the approximate functional equation takes a particularly simple form. Let q be the level of f. In the proof of Theorem 3 we take $Z = q^{1/2}$ to compensate for the asymmetry in estimates in level aspect introduced from averaging over root numbers. Note that the only difference in the approximate functional equation for $L'(1/2, f \otimes \chi_d)$ as opposed to that of $L(1/2, f \otimes \chi_d)$ is the sign of the root number, and the denominator of the integrand of $W(x)$, which becomes u^2 instead of u. Therefore, many of the calculations necessary for our results are identical to those in the paper of Soundararajan and Young [SY10].

2.3 Proof of Theorem 1

We prove Theorem 1 by splitting the sums in the approximate functional equation (Lemma 1), and using Proposition 1 below to compute the main terms.

Proof of Theorem 1. Let F be a smooth, nonnegative, compactly supported function on $\mathbb{R}_{>0}$, and recall the definition of $W(x) = W_1(x)$ from the approximate functional equation (Lemma 1). For a parameter $U \leq X/(\log X)^{100}$ define the truncated sum

$$\mathcal{A}_U(1/2, f \otimes \chi_{8d}) := (1 - i^\kappa \eta \chi_d(-q)) \sum_{n=1}^{\infty} \frac{\lambda_f(n)\chi_{8d}(n)}{\sqrt{n}} W\left(\frac{n}{U}\right),$$

and define the tail $\mathcal{B}_U(1/2, f \otimes \chi_{8d})$ by setting $L'(1/2, f \otimes \chi_{8d}) = \mathcal{A}_U(1/2, f \otimes \chi_{8d}) +$

$\mathcal{B}_U(1/2, f \otimes \chi_{8d})$. Define the sums

$$\begin{aligned}
\mathrm{I}_U(f) &:= \sum_{(d,2q\square)=1} L'(1/2, f \otimes \chi_{8d})\mathcal{A}_U(1/2, f \otimes \chi_{8d})F(8d/X) \\
\mathrm{II}_U(f) &:= \sum_{(d,2q\square)=1} \mathcal{A}_U(1/2, f \otimes \chi_{8d})^2 F(8d/X) \\
\mathrm{III}_U(f) &:= \sum_{(d,2q\square)=1} \mathcal{B}_U(1/2, f \otimes \chi_{8d})^2 F(8d/X).
\end{aligned}$$

so that we have the decomposition

$$\sum_{(d,2q\square)=1} L'(1/2, f \otimes \chi_{8d})^2 F(8d/X) = 2\mathrm{I}_U(f) - \mathrm{II}_U(f) + \mathrm{III}_U(f).$$

Using the below Proposition 1 we will be able to give asymptotic formulae for $\mathrm{I}_U(f)$ and $\mathrm{II}_U(f)$, and using Conjecture 4 we will obtain an upper bound on $\mathrm{III}_U(f)$ smaller than the main terms. Applying this decomposition in Soundararajan and Young's work improves the error term there to $O\left(X(\log X)^{1/2+\varepsilon}\right)$.

For $q' = 1$ or q, and $h(x, y, z)$ some smooth cut-off function let

$$S(q', h) := \sum_{(d,2q\square)=1} \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \frac{\lambda_f(n_1)\lambda_f(n_2)}{\sqrt{n_1 n_2}} \chi_{8d}(q' n_1 n_2) h(d, n_1, n_2).$$

Proposition 1 *Assume GRH or Conjecture 4. Let X, U_1, U_2 large, $U_1 U_2 \le X^2$, and q odd. Let $h(x, y, z)$ be a smooth function on $\mathbb{R}^3_{>0}$, with compact support in x, having all partial derivatives extending continuously to the boundary, satisfying*

$$x^i y^j z^k h^{(i,j,k)}(x, y, z) \ll_{i,j,k} \left(1 + \frac{x}{X}\right)^{-100} \left(\log \frac{U_1}{y}\right) \left(1 + \frac{y}{U_1}\right)^{-100} \times \left(\log \frac{U_2}{z}\right) \left(1 + \frac{z}{U_2}\right)^{-100}.$$

Set $h_1(y,z) = \int_0^\infty h(xX,y,z)\,dx$. *Then*

$$S(q',h) = \frac{4X}{\pi^2} \sum_{\substack{(n_1n_2,2)=1\\ q'n_1n_2=\square}} \frac{\lambda_f(n_1)\lambda_f(n_2)}{\sqrt{n_1n_2}} \prod_{p|qn_1n_2} \frac{p}{p+1} h_1(n_1,n_2)$$
$$+O_{\kappa,q}\left((U_1U_2)^{1/4}X^{1/2}(\log X)^{11}\right).$$

This Proposition and its proof are nearly identical to the main Proposition from the paper of Soundararajan and Young [SY10] (see Proposition 3.1 and the remarks in §5 of that paper) except for minor details of generalizing from full level to arbitrary level q, so we omit the proof. The main idea is to use Poisson summation (see Lemma 3) to evaluate the sum over discriminants d, and Conjecture 4 to bound the dual sum thereby obtained.

We now proceed to the computation of $\mathrm{I}_U(f)$ and $\mathrm{II}_U(f)$. Let

$$h(x,y,z) = F(8x/X)W(y/U)W(z/8x).$$

In the notation of Proposition 1 we have by the approximate functional equation that

$$\mathrm{I}_U(f) = 2S(1,h) - 2i^\kappa\eta S(q,h).$$

For notational ease, set $G(u) := \Gamma(\kappa/2+u)\Gamma(\kappa/2)^{-1}(\sqrt{q}/2\pi)^u$ which, recall, appears in the function $W(x)$. Let $\widetilde{F}(v) = \int_0^\infty F(x)x^{v-1}\,dx$ denote the Mellin transform and set

$$Z_{q'}(u,v) = \sum_{\substack{(n_1n_2,2)=1\\ q'n_1n_2=\square}} \frac{\lambda_f(n_1)\lambda_f(n_2)}{n_1^{1/2+u}n_2^{1/2+v}} \prod_{p|qn_1n_2} \frac{p}{p+1}.$$

Applying Proposition 1 and Mellin inversion, we find that

$$\mathrm{I}_U(f) = \frac{X}{\pi^2}\frac{1}{(2\pi i)^2}\int_{(1)}\int_{(1)}\frac{G(u)G(v)}{u^2v^2}U^uX^v\widetilde{F}(1+v)\left(Z_q(u,v)-i^\kappa\eta Z_1(u,v)\right)du\,dv + O_{\kappa,q}(X). \tag{2.1}$$

We compute for either $q'=1$ or q that $Z_{q'}(u,v)$ has the Euler product

$$\begin{aligned} Z_{q'}(u,v) = \prod_{p\nmid 2q}\Bigg(1+\frac{p}{p+1}\Bigg[\frac{1}{2}\left(1-\frac{\lambda_f(p)}{p^{1/2+u}}+\frac{1}{p^{1+2u}}\right)^{-1}\left(1-\frac{\lambda_f(p)}{p^{1/2+v}}+\frac{1}{p^{1+2v}}\right)^{-1} \\ +\frac{1}{2}\left(1+\frac{\lambda_f(p)}{p^{1/2+u}}+\frac{1}{p^{1+2u}}\right)^{-1}\left(1+\frac{\lambda_f(p)}{p^{1/2+v}}+\frac{1}{p^{1+2v}}\right)^{-1}-1\Bigg]\Bigg) \\ \times\prod_{p\mid q}\frac{p}{p+1}\Bigg[\frac{1}{2}\left(1-\frac{\lambda_f(p)}{p^{1/2+u}}\right)^{-1}\left(1-\frac{\lambda_f(p)}{p^{1/2+v}}\right)^{-1} \\ +(-1)^{\mathrm{ord}_p(q')}\frac{1}{2}\left(1+\frac{\lambda_f(p)}{p^{1/2+u}}\right)^{-1}\left(1+\frac{\lambda_f(p)}{p^{1/2+v}}\right)^{-1}\Bigg]. \end{aligned} \tag{2.2}$$

Hence we have that

$$Z_{q'}(u,v) = \zeta(1+u+v)L(1+2u,\mathrm{sym}^2 f)L(1+u+v,\mathrm{sym}^2 f)L(1+2v,\mathrm{sym}^2 f)Z_{q'}^*(u,v),$$

where $Z_q^*(u,v)$ and $Z_1^*(u,v)$ are given by some absolutely convergent Euler products and are uniformly bounded in the region $\mathrm{Re}(u),\mathrm{Re}(v)\geq -1/4+\varepsilon$ in u,v,κ and q. Set $Z(u,v):=Z_q(u,v)-i^\kappa\eta Z_1(u,v)$ and

$$Z^*(u,v) := Z_q^*(u,v)-i^\kappa\eta Z_1^*(u,v). \tag{2.3}$$

A careful inspection of (2.3) and (2.2), using positivity of $(1\pm\lambda_f(p)p^{-1/2})^{-1}$ shows that $Z^*(0,0) = 0$ if and only if $\varepsilon(f) = 1$ and q is a square.

We now compute by shifting contours of (2.1). Start the lines of integration at $\mathrm{Re}(u) = \mathrm{Re}(v) = 1/10$, and begin the computation with shifting the v integration to the $\mathrm{Re}(v) = -1/5$ line. We encounter poles at $v = 0$ and $v = -u$. The remaining double integral on the lines $\mathrm{Re}(v) = -1/5$ and $\mathrm{Re}(u) = 1/10$ is $\ll_{\kappa,q,\varepsilon} X^{-1/10+\varepsilon}$, and the contribution from the simple pole at $v = -u$ is $\ll_{\kappa,q} 1$. The main term comes from double pole at $v = 0$, giving

$$I_U = \frac{X}{\pi^2}\widetilde{F}(1)\frac{1}{2\pi i}\int_{(1/10)}\frac{G(u)}{u^2}U^u Z(u,0)\left(\log X + G'(0) + \frac{\widetilde{F}'(1)}{\widetilde{F}(1)} + \frac{\frac{d}{dv}Z(u,v)|_{v=0}}{Z(u,0)}\right)du$$
$$+O_{\kappa,q}(X).$$

Now $Z(u,0)$ has a single pole and $\frac{d}{dv}Z(u,v)|_{v=0}$ has a double pole. Combine these with u^2 in the denominator, and we encounter a triple and quadruple pole. The residue of the triple pole of

$$\frac{G(u)}{u^2}U^u Z(u,0)$$

at $u = 0$ is given by

$$L(1,\mathrm{sym}^2 f)^3 Z^*(0,0)\left(\frac{1}{2}\log^2 U + \left[\frac{\Gamma'(\kappa/2)}{\Gamma(\kappa/2)} + \log\frac{\sqrt{q}}{2\pi} + \gamma + 3\frac{L'(1,\mathrm{sym}^2 f)}{L(1,\mathrm{sym}^2 f)}\right.\right.$$
$$\left.\left. + \frac{\frac{d}{du}Z^*(u,0)|_{u=0}}{Z^*(0,0)}\right]\log U + O_{\kappa,q}(1)\right).$$

The residue of the quadruple pole of

$$\frac{G(u)}{u^2}U^u\frac{d}{dv}Z(u,v)|_{v=0}$$

at $u = 0$ is given by

$$-L(1, \mathrm{sym}^2 f) Z^*(0,0) \left(\frac{1}{6} \log^3 U + \frac{1}{2} \left[\frac{\Gamma'(\kappa/2)}{\Gamma(\kappa/2)} + \log \frac{\sqrt{q}}{2\pi} \right] \log^2 U + O_{\kappa,q}(\log U) \right).$$

By shifting the line of integration to $\mathrm{Re}(u) = -1/5$, we find that the the remaining integral is $\ll_{\kappa,q,\varepsilon} X^{-1/5+\varepsilon}$, hence collecting the above terms coming from residues, we find that

$$\begin{aligned} \mathrm{I}_U(f) = \frac{X}{\pi^2} L(1, \mathrm{sym}^2 f)^3 Z^*(0,0) \widetilde{F}(1) \Bigg(\frac{1}{2} \log X (\log U)^2 - \frac{1}{6} \log^3 U + \Bigg[\frac{\Gamma'(\kappa/2)}{\Gamma(\kappa/2)} \\ + \log \frac{\sqrt{q}}{2\pi} + \gamma + 3 \frac{L'(1, \mathrm{sym}^2 f)}{L(1, \mathrm{sym}^2 f)} + \frac{\frac{d}{du} Z^*(u,0)|_{u=0}}{Z^*(0,0)} \Bigg] \log X \log U \\ + \frac{1}{2} \frac{\widetilde{F}'(1)}{\widetilde{F}(1)} (\log U)^2 + O_{\kappa,q}(\log X) \Bigg). \end{aligned}$$

The sum $\mathrm{II}_U(f)$ is computed similarly, but with a different choice of $h(x, y, z)$. As above, the main term comes from the intersection of the two polar divisors $u = 0$ and $v = 0$. One finds

$$\begin{aligned} \mathrm{II}_U(f) = \frac{X}{\pi^2} L(1, \mathrm{sym}^2 f)^3 Z^*(0,0) \widetilde{F}(1) \Bigg(\frac{1}{3} \log^3 U + \Bigg[\frac{\Gamma'(\kappa/2)}{\Gamma(\kappa/2)} + \log \frac{\sqrt{q}}{2\pi} + \gamma \\ + 3 \frac{L'(1, \mathrm{sym}^2 f)}{L(1, \mathrm{sym}^2 f)} + \frac{\frac{d}{du} Z^*(u,0)|_{u=0}}{Z^*(0,0)} \Bigg] \log^2 U + O_{\kappa,q}(\log U) \Bigg). \end{aligned}$$

We now give an upper bound for the sum $\mathrm{III}_U(f)$ which, recall, involves $\mathcal{B}_U$. We have

$$\mathcal{B}_U(1/2, 8d) = (1 - i^\kappa \eta \chi_{8d}(q)) \frac{1}{2\pi i} \int_{(2)} \frac{G(s)}{s} L(1/2 + s, f \otimes \chi_{8d}) \left(\frac{(8d)^s - U^s}{s} \right) ds.$$

Recall that $L(1/2+s, f \otimes \chi_{8d})$ has root number -1 and vanishes at $s = 0$, therefore the integrand is actually entire and we move the line of integration to the $\mathrm{Re}(s) = 1/\log X$

line. On this line

$$\left|\frac{(8d)^s - U^s}{s}\right| \ll \log\left(8d/U\right),$$

uniformly in s, thus

$$\mathcal{B}_U(1/2, 8d) \ll |\log 8d/U| \int_{-\infty}^{\infty} \frac{\left|G\left(\frac{1}{\log X}+it\right)\right|}{\left|\frac{1}{\log X}+it\right|} \left|L\left(\frac{1}{2}+\frac{1}{\log X}+it, f\otimes\chi_{8d}\right)\right| dt.$$

Inserting this in $\mathrm{III}_U(f)$ we have that

$$\begin{aligned}\mathrm{III}_U(f) \ll (\log X/U)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\left|G\left(\frac{1}{\log X}+it_1\right)G\left(\frac{1}{\log X}+it_2\right)\right|}{\left|\left(\frac{1}{\log X}+it_1\right)\left(\frac{1}{\log X}+it_2\right)\right|} \\ \times \sum_{\substack{(d,2q\square)=1\\0<8d\leq X}} \left|L\left(\frac{1}{2}+\frac{1}{\log X}+it_1, f\otimes\chi_{8d}\right)L\left(\frac{1}{2}+\frac{1}{\log X}+it_2, f\otimes\chi_{8d}\right)\right| dt_1\, dt_2.\end{aligned} \tag{2.4}$$

Use Cauchy-Schwarz to split the sum over d above in two, so that it suffices to bound

$$\int_{-\infty}^{\infty} \frac{\left|G\left(\frac{1}{\log X}+it\right)\right|}{\left|\frac{1}{\log X}+it\right|} \left(\sum_{\substack{(d,2q\square)=1\\0<8d\leq X}} \left|L\left(\frac{1}{2}+\frac{1}{\log X}+it, f\otimes\chi_{8d}\right)\right|^2\right)^{1/2} dt.$$

We have that

$$\int_{-\infty}^{\infty} \frac{\left|G\left(\frac{1}{\log X}+it\right)\right|^{\frac{1}{2}}}{\left|\frac{1}{\log X}+it\right|} dt \ll \log\log X,$$

and

$$\left|G\left(\frac{1}{\log X}+it\right)\right| \sum_{\substack{(d,2q\square)=1\\0<8d\leq X}} \left|L\left(\frac{1}{2}+\frac{1}{\log X}+it, f\otimes\chi_{8d}\right)\right|^2 \ll_{f,\varepsilon} X\left(\log X\right)^{1+\varepsilon}$$

uniformly in t by Conjecture 4 and the sharp cut-off in $|G(1/\log X + it)|$ for large t. Bringing these estimates together we find that

$$\mathrm{III}_U(f) \ll_{\kappa,q,\varepsilon} X(\log X)^{1+\varepsilon}(\log X/U)^2.$$

Note that in contrast to the work of Soundararajan and Young, shifted moments are not necessary to prove our theorem.

Finally, set $U = X/(\log X)^{100}$. Note that

$$\left(\log X - \frac{2}{3}\log U\right)\log^2 U = \frac{1}{3}\log^3 X + O_\varepsilon((\log X)^{1+\varepsilon}),$$

so that pulling together our evaluations of $\mathrm{I}_U(f), \mathrm{II}_U(f)$ and $\mathrm{III}_U(f)$ we find

$$\sum_{(d,2q\square)=1} L'(1/2, f\otimes\chi_{8d})^2 F(8d/X) = \frac{X}{\pi^2} L(1,\mathrm{sym}^2 f)^3 Z^*(0,0)\widetilde{F}(1)\left(\frac{1}{3}\log^3 X\right.$$
$$+\left[\frac{\Gamma'(\kappa/2)}{\Gamma(\kappa/2)} + \log\frac{\sqrt{q}}{2\pi} + \gamma + 3\frac{L'(1,\mathrm{sym}^2 f)}{L(1,\mathrm{sym}^2 f)} + \frac{\frac{d}{du}Z^*(u,0)|_{u=0}}{Z^*(0,0)} + \frac{\widetilde{F}'(1)}{\widetilde{F}(1)}\right]\log^2 X$$
$$\left. + O_{\kappa,q,\varepsilon}(X(\log X)^{1+\varepsilon})\right).$$

□

2.4 Proof of Theorem 2

We turn to the moment for two different forms f and g of levels q_1 and q_2 respectively. Set $q = q_1q_2$. The proof of Theorem 2 is a slight variation on the proof of Theorem 1.

Proof of Theorem 2. Assume GRH or Conjecture 4, and that $U \le X/(\log X)^{100}$. We split the sum $L'(1/2, f\otimes\chi_{8d}) = \mathcal{A}_U(1/2, f\otimes\chi_{8d}) + \mathcal{B}_U(1/2, f\otimes\chi_{8d})$, where $\mathcal{A}_U(1/2, f\otimes\chi_{8d})$ and $\mathcal{B}_U(1/2, f\otimes\chi_{8d})$ are defined at the outset of Section 2.3. Take the decomposition

$$\begin{aligned}L'(1/2, f\otimes\chi_{8d})L'(1/2, g\otimes\chi_{8d}) &= L'(1/2, f\otimes\chi_{8d})\mathcal{A}_U(1/2, g\otimes\chi_{8d}) \\ &+\mathcal{A}_U(1/2, f\otimes\chi_{8d})L'(1/2, g\otimes\chi_{8d}) \\ &-\mathcal{A}_U(1/2, f\otimes\chi_{8d})\mathcal{A}_U(1/2, g\otimes\chi_{8d}) \\ &+\mathcal{B}_U(1/2, f\otimes\chi_{8d})\mathcal{B}_U(1/2, g\otimes\chi_{8d}).\end{aligned} \tag{2.5}$$

Summing over $(d, 2q\Box) = 1$, we have the 4 sums which we denote by

$$\mathrm{I}_U(f, g) := \sum_{(d,2q\Box)=1} L'(1/2, f\otimes\chi_{8d})\mathcal{A}_U(1/2, g\otimes\chi_{8d})F(8d/X),$$

$$\mathrm{I}_U(g, f) := \sum_{(d,2q\Box)=1} \mathcal{A}_U(1/2, f\otimes\chi_{8d})L'(1/2, g\otimes\chi_{8d})F(8d/X),$$

$$\mathrm{II}_U(f, g) := \sum_{(d,2q\Box)=1} \mathcal{A}_U(1/2, f\otimes\chi_{8d})\mathcal{A}_U(1/2, g\otimes\chi_{8d})F(8d/X),$$

and

$$\mathrm{III}_U(f, g) := \sum_{(d,2q\Box)=1} \mathcal{B}_U(1/2, f\otimes\chi_{8d})\mathcal{B}_U(1/2, g\otimes\chi_{8d})F(8d/X),$$

so that

$$\begin{aligned}\sum_{(d,2q\Box)=1} L'(1/2, f\otimes\chi_{8d})L'(1/2, g\otimes\chi_{8d})F(8d/X) &= \mathrm{I}_U(f, g) + \mathrm{I}_U(g, f) \\ &-\mathrm{II}_U(f, g) + \mathrm{III}_U(f, g).\end{aligned}$$

We can compute precise asymptotic estimates for $\mathrm{I}_U(f, g)$, $\mathrm{I}_U(g, f)$ and $\mathrm{II}_U(f, g)$, meanwhile $\mathrm{III}_U(f, g)$ can be reduced by Cauchy-Schwarz to the sum $\mathrm{III}_U(f)$ from the proof of Theorem 1. Hence

$$\mathrm{III}_U(f, g) \ll_{\kappa,q,\varepsilon} X(\log X)^{1+\varepsilon}.$$

We next state the Proposition which allows us to compute the sums $\mathrm{I}_U(f,g)$, $\mathrm{I}_U(g,f)$ and $\mathrm{II}_U(f,g)$. Let q' be one of the four choices $q'=1,q_1,q_2$, or q. Define

$$S_{f,g}(q',h) := \sum_{(d,2q\square)=1} \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \frac{\lambda_f(n_1)\lambda_g(n_2)}{\sqrt{n_1 n_2}} \chi_{8d}(q' n_1 n_2) h(d,n_1,n_2).$$

Proposition 2 *Assume GRH or Conjecture 4. Let X,U_1,U_2 large, $U_1U_2 \le X^2$, and $q=q_1q_2$ odd. Let $h(x,y,z)$ be a smooth function on $\mathbb{R}^3_{>0}$, with compact support in x, having all partial derivatives extending continuously to the boundary, satisfying*

$$x^i y^j z^k h^{(i,j,k)}(x,y,z) \ll_{i,j,k} \left(1+\frac{x}{X}\right)^{-100} \left(\log \frac{U_1}{y}\right) \left(1+\frac{y}{U_1}\right)^{-100} \\ \times \left(\log \frac{U_2}{z}\right) \left(1+\frac{z}{U_2}\right)^{-100}.$$

Set $h_1(y,z) = \int_0^\infty h(xX,y,z)\,dx$. Then

$$S_{f,g}(q',h) = \frac{4X}{\pi^2} \sum_{\substack{(n_1n_2,2)=1 \\ q'n_1n_2=\square}} \frac{\lambda_f(n_1)\lambda_g(n_2)}{\sqrt{n_1n_2}} \prod_{p\mid qn_1n_2} \frac{p}{p+1} h_1(n_1,n_2) \\ + O_{f,g}\left((U_1U_2)^{1/4} X^{1/2} (\log X)^{11}\right).$$

Proposition 2 is a slight variation on Proposition 1, so we omit the proof. The reader should take note of the remarks following Proposition 1, as they apply just as well to Proposition 2.

Now we proceed to use this Proposition to evaluate $\mathrm{I}_U(f,g)$, $\mathrm{I}_U(g,f)$ and $\mathrm{II}_U(f,g)$. Take for example the case $\mathrm{I}_U(f,g)$, for which we set

$$h(d,n_1,n_2) = F(8d/X)W(n_1/U)W(n_2/8d).$$

By the approximate functional equation (Lemma 1) with $Z = 1$ we have that

$$\begin{aligned}\sum_{(d,2q\square)=1} & L'(1/2, f\otimes\chi_{8d})\mathcal{A}_U(1/2, g\otimes\chi_d)F(8d/X) \\ &= S_{f,g}(1,h) - i^{\kappa_1}\eta_f S_{f,g}(q_1,h) - i^{\kappa_2}\eta_g S_{f,g}(q_2,h) + i^{\kappa_1+\kappa_2}\eta_f\eta_g S_{f,g}(q,h).\end{aligned} \tag{2.6}$$

Likewise, $\mathrm{I}_U(g,f)$ and $\mathrm{II}_U(f,g)$ are evaluated the same way with

$$h(d,n_1,n_2) = F(8d/X)W(n_1/8d)W(n_2/U)$$

and

$$h(d,n_1,n_2) = F(8d/X)W(n_1/U)W(n_2/U),$$

respectively.

Next, we evaluate the main terms of the various $S_{f,g}$ in (2.6) by contour integration. We set $G_f(u) := \Gamma(\kappa_1/2+u)\Gamma(\kappa_1/2)^{-1}(\sqrt{q_1}/2\pi)^u$ to be the Mellin transform of $W_1(x)$, and similarly for G_g. For $q' = 1, q_1, q_2$ or q define the Dirichlet series $Z_{q'}(u,v)$ by

$$Z_{q'}(u,v) := \sum_{\substack{(n_1n_2,2)=1\\ q'n_1n_2=\square}} \frac{\lambda_f(n_1)\lambda_f(n_2)}{n_1^{1/2+u}n_2^{1/2+v}} \prod_{p\mid qn_1n_2}\frac{p}{p+1}.$$

One has therefore that

$$S_{f,g}(q',h) = \frac{X}{2\pi^2}\frac{1}{(2\pi i)^2}\int_{(1)}\int_{(1)}\frac{G_g(u)G_f(v)}{u^2v^2}U^uX^v\widetilde{F}(1+v)Z_{q'}(u,v)\,du\,dv + O_{\kappa,q}(X). \tag{2.7}$$

Let χ_{0,q_i} be the trivial Dirichlet character mod q_i for $i = 1, 2$, that is to say,

$$\chi_{0,q_i}(p) = \begin{cases} 1 & \text{if } p\nmid q_i \\ 0 & \text{if } p\mid q_i. \end{cases}$$

Then the Euler product for $Z_{q'}(u,v)$ is given by

$$Z_{q'}(u,v) = \prod_{p\nmid 2q}\left(1+\frac{p}{p+1}\left[\frac{1}{2}\left(1-\frac{\lambda_f(p)}{p^{1/2+u}}+\frac{1}{p^{1+2u}}\right)^{-1}\left(1-\frac{\lambda_g(p)}{p^{1/2+v}}+\frac{1}{p^{1+2v}}\right)^{-1}\right.\right.$$
$$\left.\left.+\frac{1}{2}\left(1+\frac{\lambda_f(p)}{p^{1/2+u}}+\frac{1}{p^{1+2u}}\right)^{-1}\left(1+\frac{\lambda_g(p)}{p^{1/2+v}}+\frac{1}{p^{1+2v}}\right)^{-1}-1\right]\right)$$

$$\times\prod_{p\mid q}\frac{p}{p+1}\left[\frac{1}{2}\left(1-\frac{\lambda_f(p)}{p^{1/2+u}}+\frac{\chi_{0,q_1}(p)}{p^{1+2u}}\right)^{-1}\left(1-\frac{\lambda_g(p)}{p^{1/2+v}}+\frac{\chi_{0,q_2}(p)}{p^{1+2v}}\right)^{-1}\right.$$
$$\left.+(-1)^{\mathrm{ord}_p(q')}\frac{1}{2}\left(1+\frac{\lambda_f(p)}{p^{1/2+u}}+\frac{\chi_{0,q_1}(p)}{p^{1+2u}}\right)^{-1}\left(1+\frac{\lambda_g(p)}{p^{1/2+v}}+\frac{\chi_{0,q_2}(p)}{p^{1+2v}}\right)^{-1}\right]. \tag{2.8}$$

If $\alpha_f(p)$ and $\beta_f(p)$ are the local roots of f with $\alpha_f(p)+\beta_f(p)=\lambda_f(p)$, then we define for $\mathrm{Re}(s)>1$

$$L(s,f\otimes g)=\prod_p\left(1-\frac{\alpha_f(p)\alpha_g(p)}{p^s}\right)^{-1}\left(1-\frac{\alpha_f(p)\beta_g(p)}{p^s}\right)^{-1}$$
$$\times\left(1-\frac{\beta_f(p)\alpha_g(p)}{p^s}\right)^{-1}\left(1-\frac{\beta_f(p)\beta_g(p)}{p^s}\right)^{-1},$$

and for $\mathrm{Re}(s)\le 1$ by analytic continuation. Then in any of the four cases $q'=1,q_1,q_2$, or q, we have that

$$Z_{q'}(u,v)=L(1+u+v,f\otimes g)L(1+2u,\mathrm{sym}^2 f)L(1+2v,\mathrm{sym}^2 g)Z^*_{q'}(u,v),$$

where $Z^*_{q'}(u,v)$ is given by some absolutely convergent Euler product which is uniformly bounded in the region $\mathrm{Re}(u),\mathrm{Re}(v)\ge -1/4+\varepsilon$. Set $Z(u,v)=Z_1(u,v)-i^{\kappa_1}\eta_f Z_{q_1}(u,v)-i^{\kappa_2}\eta_g Z_{q_2}(u,v)+Z_q(u,v)$, and

$$Z^*(u,v)=Z^*_1(u,v)-i^{\kappa_1}\eta_f Z^*_{q_1}(u,v)-i^{\kappa_2}\eta_g Z^*_{q_2}(u,v)+i^{\kappa_1+\kappa_2}\eta_f\eta_g Z^*_q(u,v). \tag{2.9}$$

A careful inspection of (2.9) and (2.8) using positivity of $(1\pm\lambda_f(p)p^{-1/2})^{-1}$ shows that $Z^*(0,0)=0$ if and only if either root number $w(f)$ or $w(g)=1$, and the corresponding q_1 or q_2 is a square.

With this information about $Z_{q'}(u,v)$, one shifts contours of (2.7) as in the proof of Theorem 1 to compute the various $S_{f,g}$. We find that

$$\begin{aligned}\mathrm{I}_U(f,g) &= \sum_{(d,2q\square)=1} L'(1/2, f\otimes\chi_{8d})\mathcal{A}_U(1/2, g\otimes\chi_{8d})F(8d/X)\\ &= \frac{X}{2\pi^2}L(1,\mathrm{sym}^2 f)L(1,\mathrm{sym}^2 g)L(1,f\otimes g)Z^*(0,0)\widetilde{F}(1)\log X\log U\\ &\qquad +O_{f,g}(X\log X),\end{aligned}$$

and similarly for $\mathrm{I}_U(g,f)$. We also compute

$$\begin{aligned}\mathrm{II}_U(f,g) &= \sum_{(d,2q\square)=1} \mathcal{A}_U(1/2, f\otimes\chi_{8d})\mathcal{A}_U(1/2, g\otimes\chi_{8d})\\ &= \frac{X}{2\pi^2}L(1,\mathrm{sym}^2 f)L(1,\mathrm{sym}^2 g)L(1,f\otimes g)Z^*(0,0)\widetilde{F}(1)\log^2 U\\ &\qquad +O_{f,g}(X\log U).\end{aligned}$$

Finally, setting $U=X/(\log X)^{100}$ we obtain

$$\begin{aligned}&\sum_{(d,2q\square)=1} L'(1/2, f\otimes\chi_{8d})L'(1/2, g\otimes\chi_{8d})F(8d/X)\\ &= \frac{X}{2\pi^2}L(1,\mathrm{sym}^2 f)L(1,\mathrm{sym}^2 g)L(1,f\otimes g)Z^*(0,0)\widetilde{F}(1)\log^2 X\\ &\qquad +O_{f,g,\varepsilon}\left(X(\log X)^{1+\varepsilon}\right).\end{aligned}$$

□

2.5 Proof of Theorem 3

In this section, we apply the techniques of the previous two sections to the first moment of $L'(1/2, f \otimes \chi_{8d})$ over twists, keeping careful track of the dependence on both the weight κ and the level q.

Proof of Theorem 3. We prove the Theorem by splitting the sum into a main part and tail, and use the asymmetric approximate functional equation (Lemma 1) with $Z = q^{1/2}$. Assume GRH or Conjecture 5, and that both $\kappa q \le X$ and $U \le X/(\log X\kappa q)^{\frac{17}{4}(A+6)}$ for $A > 0$ fixed. Define the main part

$$\mathcal{A}_U(1/2, f\otimes\chi_{8d}) = \sum_{n\ge1} \frac{\lambda_f(n)\chi_{8d}(n)}{n^{1/2}} W_Z\left(\frac{n}{U}\right) - i^\kappa \eta \chi_{8d}(q) \sum_{n\ge1} \frac{\lambda_f(n)\chi_{8d}(n)}{n^{1/2}} W_{Z^{-1}}\left(\frac{n}{U}\right),$$

and the tail $\mathcal{B}_U(1/2, f\otimes\chi_{8d}) = L'(1/2, f\otimes\chi_{8d}) - \mathcal{A}_U(1/2, f\otimes\chi_{8d})$ as in Section 2.3. Following Soundararajan and Young again, we give the analogue of Propositions 1 and 2 for the first moment. Let $q' = 1$ or q, and for $h(x, y)$ a smooth function on $\mathbb{R}^2_{>0}$ set

$$T(q', h) := \sum_{(d,2q\Box)=1} \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{\sqrt{n}} \chi_{8d}(q'n) h(d, n).$$

We will use the following Proposition with z equal to either $Z = q^{1/2}$ when $q' = 1$, or $Z^{-1} = q^{-1/2}$ when $q' = q$.

Proposition 3 *Assume GRH or Conjecture 2. Let $z > 0$ be a parameter (cf. the asymmetric approximate functional equation), and let X and U be large. Suppose that q is odd, and that $U\kappa\sqrt{q}z \le X^2$. Let $h(x, y)$ be a smooth function on $\mathbb{R}^2_{>0}$ which is compactly supported in x, having all partial derivatives extending continuously to the boundary, and satisfying the partial derivative bounds*

$$x^i y^j h^{(i,j)}(x, y) \ll_{i,j} \left(1 + \frac{x}{X}\right)^{-100} \left(\log \frac{U\kappa q}{y}\right) \left(1 + \frac{y}{U\kappa\sqrt{q}z}\right)^{-100}.$$

Then, setting $h_1(y) := \int_0^\infty h(xX, y)\,dx$, we have

$$T(1,h) = \frac{4X}{\pi^2} \sum_{\substack{(n,2)=1\\ n=\square}} \frac{\lambda_f(n)}{\sqrt{n}} \prod_{p|qn} \frac{p}{p+1} h_1(n) + O\left(X^{9/17}(U\kappa\sqrt{q}z)^{4/17}(\log X\kappa q)^6\right),$$

$$T(q,h) = \frac{4X}{\pi^2} \sum_{\substack{(n,2)=1\\ qn=\square}} \frac{\lambda_f(n)}{\sqrt{n}} \prod_{p|qn} \frac{p}{p+1} h_1(n) + O\left(X^{1/2}(U\kappa q^{3/2}z)^{1/4}(\log X\kappa q)^6\right).$$

Proposition 3 is sufficiently different from Proposition 3.1 of Soundararajan and Young that we give a detailed proof in Section 2.6.

Let $h_L(x,y) := F(8x/X)W_{q^{1/2}}(y/U)$ and $h_S(x,y) := F(8x/X)W_{q^{-1/2}}(y/U)$ for "long" and "short", respectively. Recall for fundamental discriminants $d > 0$ that $\chi_d(-q) = \chi_d(q)$, so that we have in the notation of Proposition 3 that the main part of the moment is

$$\sum_{(d,2q\square)=1} \mathcal{A}_U(1/2, f\otimes\chi_{8d})F(8d/X) = T(1,h_L) - i^\kappa\eta T(q,h_S).$$

Recalling that $U\kappa\sqrt{q}z \le X^2$ and taking $z = q^{1/2}$ or $q^{-1/2}$ in Proposition 3 we have that

$$\begin{aligned}\sum_{(d,2q\square)=1} \mathcal{A}_U(1/2, f\otimes\chi_{8d})F(8d/X) &= \frac{X}{2\pi^2}\widetilde{F}(1) \sum_{\substack{(n,2)=1\\ n=\square}} \frac{\lambda_f(n)}{n^{1/2}} \prod_{p|qn} \frac{p}{p+1} W_{q^{1/2}}\left(\frac{n}{U}\right)\\ &+ \frac{X}{2\pi^2}\widetilde{F}(1) \sum_{\substack{(n,2)=1\\ qn=\square}} \frac{\lambda_f(n)}{n^{1/2}} \prod_{p|qn} \frac{p}{p+1} W_{q^{-1/2}}\left(\frac{n}{U}\right) + O_A\left(\frac{X^{13/17}(\kappa q)^{4/17}}{(\log X\kappa q)^A}\right).\end{aligned} \tag{2.10}$$

For $q' = 1$ or q define the Dirichlet series

$$Z_{q'}(u) := \sum_{\substack{(n,2)=1 \\ q'n=\square}} \frac{\lambda_f(n)}{n^{1/2+u}} \prod_{p|qn} \frac{p}{p+1}.$$

We compute from the definition of $W_Z(x)$ that

$$\frac{X}{2\pi}\widetilde{F}(1) \sum_{\substack{(n,2)=1 \\ n=\square}} \frac{\lambda_f(n)}{\sqrt{n}} \prod_{p|qn} \frac{p}{p+1} W_{q^{1/2}}\left(\frac{n}{U}\right) = \frac{X}{2\pi^2}\widetilde{F}(1)\frac{1}{2\pi i}\int_{(3)} \frac{\Gamma(u+\kappa/2)}{\Gamma(\kappa/2)}\left(\frac{2\pi}{Uq}\right)^{-u} \times Z_1(u)\frac{1-\frac{1}{2}u\log q}{u^2}\,du, \tag{2.11}$$

and in the same way that

$$\frac{X}{2\pi}\widetilde{F}(1) \sum_{\substack{(n,2)=1 \\ qn=\square}} \frac{\lambda_f(n)}{\sqrt{n}} \prod_{p|qn} \frac{p}{p+1} W_{q^{-1/2}}\left(\frac{n}{U}\right) = \frac{X}{2\pi^2}\widetilde{F}(1)\frac{1}{2\pi i}\int_{(3)} \frac{\Gamma(u+\kappa/2)}{\Gamma(\kappa/2)}\left(\frac{2\pi}{U}\right)^{-u} \times Z_q(u)\frac{1+\frac{1}{2}u\log q}{u^2}\,du. \tag{2.12}$$

The Dirichlet series $Z_{q'}(u)$ also has an Euler product

$$Z_{q'}(u) = \prod_{p\nmid 2q} 1 + \frac{p}{p+1}\left[\frac{1}{2}\left(1-\frac{\lambda_f(p)}{p^{1/2+u}}+\frac{1}{p^{1+2u}}\right)^{-1} + \frac{1}{2}\left(1+\frac{\lambda_f(p)}{p^{1/2+u}}+\frac{1}{p^{1+2u}}\right)^{-1} - 1\right] \times \prod_{p|q} \frac{p}{p+1}\left[\frac{1}{2}\left(1-\frac{\lambda_f(p)}{p^{1/2+u}}\right)^{-1} + (-1)^{\mathrm{ord}_p(q')}\frac{1}{2}\left(1+\frac{\lambda_f(p)}{p^{1/2+u}}\right)^{-1}\right]. \tag{2.13}$$

We have then that

$$Z_{q'}(u) = L(1+2u, \mathrm{sym}^2 f) Z^*_{q'}(u),$$

where $Z^*_{q'}(u)$ is given by some absolutely convergent Euler product in the region $\mathrm{Re}(u) > -1/4$. Moreover, inspecting the above Euler product, we see that

$$\frac{1}{\log\log q} \ll Z_1^*(0) \ll \log\log q$$

and

$$q^{-(1/2+\varepsilon)} \ll_\varepsilon Z_q^*(0) \ll \log\log q,$$

uniformly in κ.

With this information about $Z_{q'}(u)$, we shift the contours in (2.11) and (2.12) to $\mathrm{Re}(u) = -4/17$, and pick up the residue from the double pole at $u = 0$. The double pole in (2.11) or (2.12) contributes

$$\frac{X}{2\pi^2}\widetilde{F}(1)L(1, \mathrm{sym}^2 f)Z^*_{q'}(0)\left(\log\frac{U\kappa\sqrt{q}}{2\pi} + \frac{Z'_{q'}(0)}{Z_{q'}(0)} + O(\kappa^{-1})\right).$$

We must also bound the integrals

$$\frac{X}{2\pi^2}\widetilde{F}(1)\frac{1}{2\pi i}\int_{(-4/17)}\frac{\Gamma(u+\kappa/2)}{\Gamma(\kappa/2)}\left(\frac{2\pi}{Uq}\right)^{-u} L(1+2u, \mathrm{sym}^2 f)Z_1^*(u)\frac{1-\frac{1}{2}u\log q}{u^2}\,du \tag{2.14}$$

and

$$\frac{X}{2\pi^2}\widetilde{F}(1)\frac{1}{2\pi i}\int_{(-4/17)}\frac{\Gamma(u+\kappa/2)}{\Gamma(\kappa/2)}\left(\frac{2\pi}{U}\right)^{-u} L(1+2u, \mathrm{sym}^2 f)Z_q^*(u)\frac{1+\frac{1}{2}u\log q}{u^2}\,du. \tag{2.15}$$

These two are treated a little differently. Let us begin with the simpler case of (2.14). We have the convexity bound

$$L(9/17+it, \mathrm{sym}^2 f) \ll (\kappa^2 q^2(1+|t|)^4)^{4/17}(\log\kappa q)^2. \tag{2.16}$$

by estimating with the approximate functional equation of the symmetric square L-function, and the Deligne bound [Del74] for its coefficients (see for example, equation (5.22) of [IK04]). Hence, the integral (2.14) is

$$\ll_A X^{13/17}(\kappa q)^{4/17}/(\log X\kappa q)^A.$$

The integral (2.15) is a little more delicate, and we need to use the decay of $Z_q^*(u)$ with respect to q. When $\mathrm{Re}(u) > -1/4$, we have that

$$Z_q^*(u) = \prod_{p\nmid q}\left(1+O(p^{-(2+4u)})\right)\prod_{\substack{p\mid q\\ \mathrm{ord}_p(q)\text{ odd}}}\frac{\lambda_f(p)}{p^{1/2+u}}\left(1+O(p^{-(1+2u)})\right)$$
$$\times\prod_{\substack{p\mid q\\ \mathrm{ord}_p(q)\text{ even}}}\left(1+O(p^{-(1+2u)})\right),$$

so that

$$Z_q^*(u) \ll \left(\prod_{\mathrm{ord}_p(q)\text{ odd}} p\right)^{-1/2-\mathrm{Re}(u)}(\log q)^2.$$

Assuming e.g. that q is square-free, this shows that for fixed $\mathrm{Re}(u)$ $Z_q(u)$ decays as a function of q. If one is willing to assume Lindelöf, it is unnecessary to use the decay of $Z_q^*(u)$ with respect to q, and hence the restriction to square-free q may be omitted. Using this along with the convexity bound (2.16) for $L(1+2u,\mathrm{sym}^2 f)$, we find that (2.15) is

$$\ll_A X^{13/17}\kappa^{4/17}q^{7/34}/(\log X\kappa q)^A \ll_A X^{13/17}(\kappa q)^{4/17}/(\log X\kappa q)^A,$$

so that these integrals are subsumed into the error term in the Theorem.

Now set

$$Z^*(u) = Z_1^*(u) - i^\kappa\eta Z_q^*(u) \tag{2.17}$$

so that we have from (2.10) that

$$\sum_{(d,2q\square)=1} \mathcal{A}_U(1/2, f\otimes\chi_{8d})F(8d/X) = \frac{X}{2\pi^2}\widetilde{F}(1)L(1,\mathrm{sym}^2 f)Z^*(0)\left(\log\frac{U\kappa\sqrt{q}}{2\pi}\right.$$
$$\left.+2\frac{L'(1,\mathrm{sym}^2 f)}{L(1,\mathrm{sym}^2 f)}+\frac{Z^{*\prime}(0)}{Z^*(0)}\right)+O_A\left(\frac{X^{13/17}(\kappa q)^{4/17}}{(\log X\kappa q)^A}\right).$$

By carefully inspecting (2.13) and using that

$$\prod_{p|q}\left(1+\frac{2}{\sqrt{p}}\right)\ll\frac{(\log q)^{1/2}}{\log\log q}$$

we find that $Z^*(0)=0$ if and only if $i^\kappa\eta := w(f)=1$ and q is a square and that if $Z^*(0)\neq 0$, it is $\gg \log\log q/(\log q)^{1/2}$, uniformly in κ.

Now consider the tail

$$\sum_{(d,2q\square)=1}\mathcal{B}_U(1/2, f\otimes\chi_{8d})F(8d/X).$$

Recall the notation $G(u)=\Gamma(\kappa/2+u)\Gamma(\kappa/2)^{-1}(\sqrt{q}/2\pi)^u$ from the definition of $W_Z(x)$, and that we have set $Z=q^{1/2}$. We have in similar fashion to the two preceding theorems that

$$\mathcal{B}_U(1/2,8d)=\frac{1}{2\pi i}\int_{(2)}\frac{G(s)}{s}L(1/2+s,f\otimes\chi_{8d})\frac{(8d)^s-U^s}{s}$$
$$\times\left(Z^s(1-s\log Z)-i^\kappa\eta\chi_{8d}(q)Z^{-s}(1+s\log Z)\right)\,ds.$$

The integrand is entire, and we may shift the contour to the line $\mathrm{Re}(s)=1/\log X\kappa q$. On this line we have

$$\frac{(8d)^s-U^s}{s}\left(Z^s(1-s\log Z)-i^\kappa\eta\chi_{8d}(q)Z^{-s}(1+s\log Z)\right)\ll\log X/U$$

so that

$$\sum_{(d,2q\square)=1} \mathcal{B}_U(1/2, 8d) F(8d/X) \ll |\log X/U| \int_{-\infty}^{\infty} \frac{\left|G\left(\frac{1}{\log X\kappa q} + it\right)\right|}{\left|\frac{1}{\log X\kappa q} + it\right|} \times \sum_{\substack{(d,2q\square)=1 \\ 0<8d\le X}} \left|L\left(\frac{1}{2} + \frac{1}{\log X\kappa q} + it, f \otimes \chi_{8d}\right)\right| dt.$$

Set $U = X/(\log X\kappa q)^{\frac{17}{4}(A+6)}$. Using Conjecture 5 when t is small and the cut-off in $|G(1/\log X\kappa q) + it|^{1/2}$ when t is large we have

$$|G(1/\log X\kappa q) + it|^{1/2} \sum_{\substack{(d,2q\square)=1 \\ 0<8d\le X}} \left|L\left(\frac{1}{2} + \frac{1}{\log X\kappa q} + it, f \otimes \chi_{8d}\right)\right| dt \ll_\varepsilon X\left(\log X\kappa q\right)^{1/4+\varepsilon}.$$

We also have the estimate

$$\int_{-\infty}^{\infty} \frac{\left|G\left(\frac{1}{\log X\kappa q} + it\right)\right|^{\frac{1}{2}}}{\left|\frac{1}{\log X\kappa q} + it\right|} dt \ll \log\log X\kappa q,$$

so that pulling these estimates together we obtain

$$\sum_{(d,2q\square)=1} \mathcal{B}_U(1/2, f \otimes \chi_{8d}) F(8d/X) \ll_{\varepsilon,A} X\left(\log X\kappa q\right)^{1/4+\varepsilon}$$

hence the Theorem. $\square$

2.6 Proof of Proposition 3

We treat the two cases $q' = 1$ and $q' = q$ somewhat differently. In the case $q' = 1$, the dependence on q in $T(1, h)$ appears only in the relatively prime condition, which we may we treat solely by Möbuis inversion. In the case of $T(q,h)$, the dependence on q is carried through the average over quadratic characters, but there is one less inversion to preform, making the calculation a bit simpler.

Proof. The condition $(d, 2q\square) = 1$ has been introduced to the sum over twists to restrict $8d$ to lie in a large subsequence of fundamental discriminants. However, this condition is awkward to work with and our first task will be to remove it.

2.6.1 Preliminary simplifications, $q' = 1$ case

We start with the $q' = 1$ case. We use Möbius inversion to remove both the square-free and relatively prime to q conditions from the sum over d,

$$T(1,h) = \sum_{(a_1,2q)=1} \mu(a_1) \sum_{a_2|q} \mu(a_2) \sum_{(d,2)=1} \sum_{(n,a)=1} \frac{\lambda_f(n)}{n^{1/2}} \chi_{8da_2}(n) h(da_1^2a_2, n).$$

Split the sums over a_1 and a_2 at Y_1 and Y_2 in to tail and main term. This splitting results in 4 truncated sums:

$$T(1,h) = T_1(1,h) + T_{21}(1,h) + T_{22}(1,h) + T_{23}(1,h)$$

where we have defined

$$T_1(1,h) := \sum_{\substack{(a_1,2q)=1\\ a_1\le Y_1}} \mu(a_1) \sum_{\substack{a_2|q\\ a_2\le Y_2}} \mu(a_2) \sum_{(d,2)=1} \sum_{(n,a_1)=1} \frac{\lambda_f(n)}{n^{1/2}} \chi_{8da_2}(n) h(da_1^2a_2, n),$$

$$T_{21}(1,h) := \sum_{\substack{(a_1,2q)=1\\ a_1\le Y_1}} \mu(a_1) \sum_{\substack{a_2|q\\ a_2> Y_2}} \mu(a_2) \sum_{(d,2)=1} \sum_{(n,a_1)=1} \frac{\lambda_f(n)}{n^{1/2}} \chi_{8da_2}(n) h(da_1^2a_2, n),$$

$$T_{22}(1,h) := \sum_{\substack{(a_1,2q)=1\\ a_1> Y_1}} \mu(a_1) \sum_{\substack{a_2|q\\ a_2\le Y_2}} \mu(a_2) \sum_{(d,2)=1} \sum_{(n,a_1)=1} \frac{\lambda_f(n)}{n^{1/2}} \chi_{8da_2}(n) h(da_1^2a_2, n),$$

$$T_{23}(1,h) := \sum_{\substack{(a_1,2q)=1\\ a_1>Y_1}} \mu(a_1) \sum_{\substack{a_2|q\\ a_2>Y_2}} \mu(a_2) \sum_{(d,2)=1} \sum_{(n,a_1)=1} \frac{\lambda_f(n)}{n^{1/2}} \chi_{8da_2}(n) h(da_1^2a_2, n).$$

The main term will come from the most difficult sum $T_1(1,h)$. First, however, we estimate the other cases $T_{21}(1,h)$, $T_{22}(1,h)$ and $T_{23}(1,h)$.

Lemma 2 *Assume GRH or Conjecture 5. We have the bounds*

$$T_{21}(1,h) \ll \frac{X}{Y_2}(\log X\kappa q)^5,$$

$$T_{22}(1,h) \ll \frac{X}{Y_1}(\log X\kappa q)^5,$$

$$T_{23}(1,h) \ll \frac{X}{Y_1Y_2}(\log X\kappa q)^5.$$

Proof. Consider the case $T_{21}(1,h)$ and write $d = b_1^2b_2\ell$ with $(\ell, 2q\Box) = 1$, $(b_1,q) = 1$ and $b_2 \mid q$. Group the variables as $c_1 = a_1b_1$ and $c_2 = a_2b_2$ to obtain

$$T_{21}(1,h) = \sum_{(c_1,2q)=1} \sum_{c_2|q} \sum_{\substack{a_1|c_1\\ a_1\le Y_1}} \mu(a_1) \sum_{\substack{a_2|c_2\\ a_2>Y_2}} \mu(a_2) \sum_{(\ell,2q\Box)=1} \sum_{(n,c_1)=1} \frac{\lambda_f(n)}{n^{1/2}} \chi_{8\ell c_2}(n) h(\ell c_1^2 c_2, n).$$

Let f_{c_2} denote the newform given by the quadratic twist $f \otimes \chi_{c_2}$, which is of some level dividing q^2. Set $\check{h}(x,u) := \int_0^\infty h(x,y)y^{u-1}\,dy$, which by repeated partial integration can be estimated by

$$\check{h}(x,u) \ll \left(1+\frac{x}{X}\right)^{-100} \frac{(U\kappa\sqrt{q}z)^{\mathrm{Re}(u)}}{|u|^2(1+|u|)^{10}}.$$

We then have by Mellin inversion that $T_{21}(1,h)$ is

$$= \sum_{(c_1,2q)=1} \sum_{c_2|q} \sum_{\substack{a_1|c_1\\ a_1\le Y_1}} \mu(a_1) \sum_{\substack{a_2|c_2\\ a_2>Y_2}} \mu(a_2) \frac{1}{2\pi i} \int_{(1/2+\varepsilon)} \sum_{(\ell,2q\Box)=1} \check{h}(\ell c_1^2 c_2, u) L_{c_1}(1/2+u, f_{c_2}\otimes\chi_{8\ell})\,du,$$

where $L_{c_1}(1/2+u, f_{c_2}\otimes\chi_{8\ell})$ is the function formed from the same Euler product as $L(1/2+u, f_{c_2}\otimes\chi_{8\ell})$, but with those factors at primes dividing c_1 omitted. We have that

$$|L_{c_1}(1/2+u, f_{c_2}\otimes\chi_{8\ell})| \le d(c_1)|L(1/2+u, f_{c_2}\otimes\chi_{8\ell})|,$$

so that shifting the contour to the line $\mathrm{Re}(u) = 1/\log X\kappa q$, we have that $T_{21}(1,h)$ is

$$\ll (\log X\kappa q)^2 \sum_{(c_1,2q)=1} d(c_1) \sum_{c_2|q} \sum_{\substack{a_1|c_1\\ a_1\le Y_1}} \sum_{\substack{a_2|c_2\\ a_2>Y_2}} \int_{-\infty}^{\infty} \sum_{(\ell,2q\square)=1} \left(1+\frac{\ell c_1^2 c_2}{X}\right)^{-100} \times \frac{|L(1/2+1/\log X\kappa q+it, f_{c_2}\otimes\chi_{8\ell})|}{(1+|t|)^{10}}\,dt.$$

Using Conjecture 5 (i.e. GRH) we find that

$$\begin{aligned} T_{21}(1,h) &\ll X(\log X\kappa q)^3 \sum_{(c_1,2q)=1} \frac{d(c_1)}{c_1^2} \sum_{c_2|q} \frac{1}{c_2} \sum_{\substack{a_1|c_1\\ a_1\le Y_1}} \sum_{\substack{a_2|c_2\\ a_2>Y_2}} \\ &\ll X(\log X\kappa q)^5. \end{aligned}$$

The cases $T_{22}(1,h)$ and $T_{23}(1,h)$ are treated similarly. $\square$

2.6.2 Averaging quadratic characters

We now turn to $T_1(1,h)$. We quote two very useful Lemmas from [Sou00]. The first is Lemma 2.6 of [Sou00], which is the trace formula for quadratic characters.

Lemma 3 (Poisson Summation) *Let F be a smooth function with compact support on the positive real numbers, and suppose that n is an odd integer. Then*

$$\sum_{(d,2)=1} \left(\frac{d}{n}\right) F\left(\frac{d}{Z}\right) = \frac{Z}{2n}\left(\frac{2}{n}\right) \sum_{k\in\mathbb{Z}} (-1)^k G_k(n) \widehat{F}\left(\frac{kZ}{2n}\right),$$

where

$$G_k(n) = \left(\frac{1-i}{2} + \left(\frac{-1}{n}\right)\frac{1+i}{2}\right) \sum_{a \pmod n} \left(\frac{a}{n}\right) e\left(\frac{ak}{n}\right),$$

and

$$\widehat{F}(y) = \int_{-\infty}^{\infty} (\cos(2\pi xy) + \sin(2\pi xy))\, F(x)\, dx$$

is a Fourier-type transform of F*.*

The Gauss-type sum $G_k(n)$ has the following explicit evaluation from Lemma 2.3 of [Sou00]:

Lemma 4 *If* m *and* n *are relatively prime odd integers, then* $G_k(mn) = G_k(m)G_k(n)$*, and if* p^α *is the largest power of* p *dividing* k *(setting* $\alpha = \infty$ *if* $k = 0$*), then*

$$G_k(p^\beta) = \begin{cases} 0 & \text{if } \beta \le \alpha \text{ is odd} \\ \phi(p^\beta) & \text{if } \beta \le \alpha \text{ is even} \\ -p^\alpha & \text{if } \beta = \alpha + 1 \text{ is even} \\ \left(\frac{kp^{-\alpha}}{p}\right) p^\alpha \sqrt{p} & \text{if } \beta = \alpha + 1 \text{ is odd} \\ 0 & \text{if } \beta \ge \alpha + 2. \end{cases}$$

Applying these Lemmas to $T_1(1,h)$ we find that

$$\begin{aligned} T_1(1,h) = \frac{X}{2} \sum_{\substack{(a_1,2q)=1 \\ a_1 \le Y_1}} \frac{\mu(a_1)}{a_1^2} \sum_{\substack{a_2 | q \\ a_2 \le Y_2}} \frac{\mu(a_2)}{a_2} \sum_{k \in \mathbb{Z}} (-1)^k \sum_{(n,2a_1)=1} \frac{\lambda_f(n)}{n^{1/2}} \chi_{a_2}(n) \frac{G_k(n)}{n} \\ \times \int_0^\infty (\sin + \cos)\left(\frac{2\pi k x X}{2na_1^2 a_2}\right) h(xX, n)\, dx. \end{aligned} \tag{2.18}$$

2.6.3 The main term

The main term of $T_1(1,h)$ is from the $k = 0$ term of (2.18), which we extract and analyze. Call the $k = 0$ term $T_{10}(1,h)$, and observe from Lemma 4 that $G_0(n) \neq 0$ if and only if n is a square, in which case $G_0(n) = \phi(n)$. Setting $h_1(n) = \int_0^\infty h(xX, n)\,dx$, we find

$$\begin{aligned} T_{10}(1,h) &= \frac{X}{2} \sum_{\substack{(a_1,2q)=1 \\ a_1\le Y_1}} \frac{\mu(a_1)}{a_1^2} \sum_{\substack{a_2|q \\ a_2\le Y_2}} \frac{\mu(a_2)}{a_2} \sum_{\substack{(n,2a_1a_2)=1 \\ n=\Box}} \frac{\lambda_f(n)}{n^{1/2}} \prod_{p|n}\left(1-\frac{1}{p}\right) h_1(n) \\ &= \frac{2X}{3\zeta(2)} \sum_{\substack{(n,2)=1 \\ n=\Box}} \frac{\lambda_f(n)}{n^{1/2}} \prod_{p|nq} \frac{p}{p+1} h_1(n) + O\left(X\left(\frac{1}{Y_1}+\frac{1}{Y_2}\right) \sum_{\substack{(n,2)=1 \\ n=\Box}} \frac{d(n)}{n^{1/2}} |h_1(n)| \right), \end{aligned}$$

so that using the bounds on h in the statement of the Proposition we have

$$T_{10}(1,h) = \frac{4X}{\pi^2} \sum_{\substack{(n,2)=1 \\ n=\Box}} \frac{\lambda_f(n)}{n^{1/2}} \prod_{p|nq} \frac{p}{p+1} h_1(n) + O\left(X\left(\frac{1}{Y_1}+\frac{1}{Y_2}\right) (\log X\kappa q)^4 \right). \quad (2.19)$$

2.6.4 Bounding the dual sum

We now proceed to the $k \neq 0$ terms of $T_1(1,h)$, which we call $T_3(1,h)$. Our first task is to express the integral in (2.18) in terms of Mellin inverses.

Lemma 5 *Let $k \neq 0$, $X > 1$ and let $h(x,y)$ be as in the statement of the Theorem. Define the transform*

$$\widetilde{h}(s,u) := \int_0^\infty \int_0^\infty h(x,y) x^s y^u \frac{ds}{s}\,\frac{du}{u}.$$

Then we have

$$\begin{aligned} \int_0^\infty (\sin+\cos)\left(\frac{2\pi k x X}{2na_1^2a_2}\right) h(xX,n)\,dx = \frac{1}{X}\frac{1}{(2\pi i)^2} \int_{(\varepsilon)}\int_{(\varepsilon)} \widetilde{h}(1-s,u) \frac{1}{n^u} \left(\frac{na_1^2a_2}{\pi|k|}\right)^s \\ \times\Gamma(s)\,(\cos+\mathrm{sgn}(k)\sin)\left(\frac{\pi s}{2}\right)\,ds\,du. \end{aligned} \quad (2.20)$$

Moreover one has the bounds

$$\widetilde{h}(s,u) \ll \frac{(U\kappa\sqrt{q})^{\mathrm{Re}(u)} X^{\mathrm{Re}(s)}}{|u|^2(1+|u|)^{98}(1+|s|)^{98}}.$$

Proof. Use the formulae for the Mellin transforms of sin and cos and Mellin inversion. See Soundararajan and Young [SY10], Section 3.3. □

Inspecting Lemma 4, we find that for odd n, $G_k(n) = G_{4k}(n)$, so that inserting the formula of Lemma 5 in (2.18), one finds that

$$T_3(1,h) = \frac{1}{2} \sum_{\substack{(a_1,2q)=1 \\ a_1 \le Y_1}} \frac{\mu(a_1)}{a_1^2} \sum_{\substack{a_2 | q \\ a_2 \le Y_2}} \frac{\mu(a_2)}{a_2} \sum_{k\in\mathbb{Z}} (-1)^k \sum_{(n,2a_1)=1} \frac{\lambda_f(n)}{n^{1/2}} \chi_{a_2}(n) \frac{G_{4k}(n)}{n}$$
$$\times \frac{1}{(2\pi i)^2} \int_{(\varepsilon)} \int_{(\varepsilon)} \widetilde{h}(1-s,u) \frac{1}{n^u} \left(\frac{n a_1^2 a_2}{\pi|k|}\right)^s \Gamma(s) \left(\cos + \mathrm{sgn}(k) \sin\right) \left(\frac{\pi s}{2}\right) ds\, du.$$

Recall that we denote the set of fundamental discriminants by $\mathcal{D}$. Now set $4k = k_1 k_2^2 k_3$, where $k_1 k_3 \in \mathcal{D}$, and $(k_1, q) = 1$ but $k_3 \mid q$. Define the function

$$Z_1(\alpha,\gamma,b_1,b_2,k_1k_3) := \sum_{k_2=1}^{\infty} \sum_{(n,2b_1)=1} \frac{\lambda_f(n)\chi_{b_2}(n)}{n^{\alpha}|k_2|^{2\gamma}} \frac{G_{k_1k_2^2k_3}(n)}{n}, \tag{2.21}$$

and set

$$H(s) := \Gamma(s) \left(\cos + \mathrm{sgn}(k) \sin\right) \left(\frac{\pi s}{2}\right) \left(1 - 2^{1-2s}\right)^{-1} \ll |s|^{\mathrm{Re}(s)-1/2}. \tag{2.22}$$

Splitting up $4k$ in this manner and after a change of variables one finds that

$$\begin{aligned} T_3(1,h) = \frac{1}{2} \sum_{\substack{(a_1,2q)=1 \\ a_1 \le Y_1}} \frac{\mu(a_1)}{a_1^2} \sum_{\substack{a_2 | q \\ a_2 \le Y_2}} \frac{\mu(a_2)}{a_2} \sum_{\substack{k_3\in\mathcal{D} \\ k_3 | q}} \sum_{\substack{k_1\in\mathcal{D} \\ (k_1,q)=1}} (-1)^{k_1k_3} \frac{1}{(2\pi i)^2} \int_{(1/2+\varepsilon)} \int_{(\varepsilon)} \\ \widetilde{h}(1-s,u+s) \left(\frac{a_1^2 a_2}{\pi|k_1k_3|}\right)^s H(s) Z_1(1/2+u,s,a_1,a_2,k_1k_3)\, du\, ds. \end{aligned} \tag{2.23}$$

To estimate $T_3(1,h)$ by contour shifting, we must analyze the Dirichlet series Z_1.

Lemma 6 *Let k_1k_3 be a fundamental discriminant, where $k_3 \mid q$ but $(k_1,q)=1$, and b_1, b_2 positive integers where $b_2 \mid q$ and $(b_1, 2q) = 1$. Denote by $f_{k_3b_2}$ the newform defined by the quadratic twist $f \otimes \chi_{k_3b_2}$, which is of some level dividing q^3. For the Dirichlet series defined by (2.21) one has*

$$Z_1(\alpha,\gamma,b_1,b_2,k_1k_3) = \frac{L_{b_1b_2}(1/2+\alpha, f_{k_3b_2} \otimes \chi_{k_1})}{L_{b_1b_2}(1+2\alpha, \mathrm{sym}^2 f)} Z_1^*(\alpha,\gamma,b_1,b_2,k_1k_3),$$

where subscripts denote the omission of Euler factors, and Z_1^* is given by some Euler product absolutely convergent in $\mathrm{Re}(\alpha) \geq 0$ and $\mathrm{Re}(\gamma) \geq 1/2+\varepsilon$ and uniformly bounded in $b_1, b_2, k_1, k_3, \kappa$ and q.

Proof. By Lemma 4, the terms of the Dirichlet series defining Z are joint multiplicative in n and k_2, so that we may decompose Z as an Euler product. The generic Euler factor is given by

$$\sum_{k_2,n\geq 0} \frac{\lambda_f(p^n)\chi_{b_2}(p)^n}{p^{n\alpha+2\gamma k_2}} \frac{G_{k_1k_3p^{2k_2}}(p^n)}{p^n},$$

and we must check the several cases where p divides the various parameters q, b_1, b_2, k_1, k_3, or not. First, we consider the generic case where $p \nmid 2qb_1b_2k_1k_3$. By Lemma 4, we find that the terms $k_2 \geq 1$ contribute $\ll p^{-(1+2\varepsilon)}$, and the $k_2 = 0$ terms are exactly

$$1 + \frac{\lambda_f(p)\chi_{k_3b_2}(p)\chi_{k_1}(p)}{p^{1/2+\alpha}},$$

so that these Euler factors match those in the statement of the Lemma. Next, consider the cases $p \mid k_1$, $p \nmid 2qb_1b_2k_3$, or $p \mid k_3$, $p \nmid 2b_1b_2k_1$. In either of these two cases we check that such an Euler factor is

$$1 - \frac{\lambda_f(p^2)}{p^{1+2\alpha}} + O(p^{-(1+2\varepsilon)}),$$

which again matches the Euler factor in the Lemma. If $p \mid q$, but $p \nmid 2b_1b_2k_1k_3$, then this Euler factor is

$$1 + \frac{\lambda_f(p)\chi_{k_3b_2}(p)\chi_{k_1}(p)}{p^{1/2+\alpha}} + O(p^{-(1+2\varepsilon)}).$$

Observing that $\lambda_f(p^2) = \lambda_f(p)^2$ for primes dividing the level, the also matches the Euler factor from the statement of the Lemma. Finally, if $p \mid b_1b_2$, then all terms $n \geq 1$ vanish, and the contribution of such an Euler factor is $1 + O(p^{-(1+2\varepsilon)})$. $\square$

We now return to (2.23), and split the sum over k_1 at $U\kappa\sqrt{q}zY_1^2Y_2^{1/2}/X$. For the small k_1 terms, we shift the lines of integration to $\mathrm{Re}(u) = -1/2 + 1/\log X\kappa q$ and $\mathrm{Re}(s) = 3/4$, and for the large k_1 terms to $\mathrm{Re}(u) = -1/2 + 1/\log X\kappa q$ and $\mathrm{Re}(s) = 5/4$. Recall that $H(s) \ll |s|^{\mathrm{Re}(s)-1/2}$, and observe

$$|L_{a_1a_2}(1/2+\alpha, f_{k_3b_2} \otimes \chi_{k_1})| \leq d(a_1)d(a_2)|L(1/2+\alpha, f_{k_3b_2} \otimes \chi_{k_1})|.$$

Applying the result of Goldfeld, Hoffstein and Lieman [GHL94], the small k_1 terms are

$$\ll (\log X\kappa q)^3 (U\kappa\sqrt{q}z)^{1/4} X^{1/4} \sum_{\substack{(a_1,2q)=1 \\ a_1 \leq Y_1}} \frac{d(a_1)}{\sqrt{a_1}} \sum_{\substack{a_2 \mid q \\ a_2 \leq Y_2}} \frac{d(a_2)}{a_2^{1/4}} \sum_{\substack{k_3 \in \mathcal{D} \\ k_3 \mid q}} \frac{1}{|k_3|^{3/4}} \int_{(3/4)} \int_{(-1/2+1/\log X\kappa q)}$$
$$\times \sum_{\substack{|k_1| \leq U\kappa\sqrt{q}zY_1^2Y_2^{1/2}/X \\ k_1 \in \mathcal{D} \\ (k_1,q)=1}} \frac{|L(1+u, f_{k_3a_2} \otimes \chi_{k_1})|}{|k_1|^{3/4}} \frac{ds\, du}{(1+|s|)^{98}(1+|u|)^{98}}.$$

Using Conjecture 5, i.e. GRH, we find that this is $\ll (U\kappa\sqrt{q}z)^{1/2}Y_1Y_2^{1/8}(\log X\kappa q)^6$. Now consider the large k_1 terms. Similarly, their contribution is

$$\ll (\log X\kappa q)^3 \frac{(U\kappa\sqrt{q}z)^{3/4}}{X^{1/4}} \sum_{\substack{(a_1,2q)=1 \\ a_1\le Y_1}} d(a_1)\sqrt{a_1} \sum_{\substack{a_2|q \\ a_2\le Y_2}} d(a_2)a_2^{1/4} \sum_{\substack{k_3\in\mathcal{D} \\ k_3|q}} \frac{1}{|k_3|^{5/4}}$$
$$\times \int_{(5/4)} \int_{(-1/2+1/\log X\kappa q)} \sum_{\substack{|k_1|>U\kappa\sqrt{q}zY_1^2Y_2^{1/2}/X \\ k_1\in\mathcal{D} \\ (k_1,q)=1}} \frac{|L(1+u, f_{k_3a_2}\otimes\chi_{k_1})|}{|k_1|^{5/4}} \frac{ds\,du}{(1+|s|)^{98}(1+|u|)^{98}}.$$

Again, by Conjecture 5, this is $\ll (U\kappa\sqrt{q}z)^{1/2}Y_1Y_2^{1/8}(\log X\kappa q)^6$. Taking

$$Y_1 = Y_2 = \frac{X^{8/17}}{(U\kappa\sqrt{q}z)^{4/17}},$$

we find that

$$T_3(1,h) \ll X^{9/17}(U\kappa\sqrt{q}z)^{4/17}\left(\log X\kappa q\right)^6,$$

and drawing all error terms together we obtain the Proposition for $q'=1$.

2.6.5 The $q'=q$ case

The proof in the $T(q,h)$ case follows the same outline as in the $T(1,h)$ case, above. We need only Möbius invert the square-free condition and not the relatively prime to q condition, but we must keep careful track of the dependence on q in the analogue of Lemma 6. We sketch the argument, omitting those details which are similar to those of $T(1,h)$.

We begin by using Möbius inversion to remove the square-free condition and split the resulting sum at Y.

$$\begin{aligned} T(q,h) &= \left(\sum_{\substack{a\le Y\\(a,2q)=1}} + \sum_{\substack{a>Y\\(a,2q)=1}} \right) \mu(a) \sum_{(d,2q)=1} \sum_{(n,a)=1} \frac{\lambda_f(n)}{\sqrt{n}} \chi_{8d}(qn) h(da^2,n) \\ &=: T_1(q,h) + T_2(q,h). \end{aligned}$$

By a slight modification of Lemma 2, we find that

$$T_2(q,h) \ll \frac{X}{Y} (\log X\kappa q)^5,$$

and so we concentrate on $T_1(q,h)$. Applying Poisson summation (Lemma 3), we have that

$$T_1(q,h) = \frac{X}{2} \sum_{\substack{a\le Y\\(a,2q)=1}} \frac{\mu(a)}{a^2} \sum_{k\in\mathbb{Z}} (-1)^k \sum_{(n,2a)=1} \frac{\lambda_f(n)}{\sqrt{n}} \frac{G_k(qn)}{qn} \int_0^\infty (\cos+\sin)\left(\frac{2\pi k x X}{2qna^2}\right) \times h(xX,n)\,dx.$$

Now we pick out from $T_1(q',h)$ the main term, which is when $k=0$, and call it $T_{10}(q,h)$. By pulling the sum over a inside and computing as in subsection 2.6.3, we find that

$$T_{10}(q,h) = \frac{4X}{\pi^2} \sum_{\substack{(n,2)=1\\qn=\square}} \frac{\lambda_f(n)}{\sqrt{n}} \prod_{p|qn} \frac{p}{p+1} h_1(n) + O\left(\frac{X}{Y}(\log X\kappa q)^3\right).$$

Now we turn to the $k\neq 0$ terms of $T_1(q,h)$ and call them $T_3(q,h)$. Define

$$Z_q(\alpha,\gamma,b,k_1k_3) := \sum_{k_2=1}^{\infty} \sum_{(n,2b)=1} \frac{\lambda_f(n)}{n^\alpha} \left(\frac{q}{|k_2|^2}\right)^\gamma \frac{G_{k_1k_2^2k_3}(qn)}{qn}. \tag{2.24}$$

Recall the definition of $H(s)$ from (2.22) and apply the inversion formula (2.20) for the weight function, to find the analogue of formula (2.23):

$$T_3(q,h) = \frac{1}{2} \sum_{\substack{a\le Y \\ (a,2q)=1}} \frac{\mu(a)}{a^2} \sum_{\substack{k_3\in\mathcal{D} \\ k_3|q}} \sum_{\substack{k_1\in\mathcal{D} \\ (k_1,q)=1}} (-1)^{k_1k_3} \frac{1}{(2\pi i)^2} \int_{(\varepsilon)} \int_{(1/2+\varepsilon)} \widetilde{h}(1-s,u+s) \left(\frac{a^2}{\pi|k_1k_3|}\right)^s \times H(s) Z_q(1/2+u,s,a,k_1k_3)\, ds\, du. \tag{2.25}$$

In order to use contour shifting, we analyze the Dirichlet series Z_q, taking special care with the dependence on q.

Lemma 7 *Let k_1k_3 be a fundamental discriminant, where $k_3 \mid q$ but $(k_1,q)=1$, and b positive integer relatively prime to $2q$. Denote by f_{k_3} the newform defined by the quadratic twist $f\otimes\chi_{k_3}$, which is of some level dividing q^2. For the Dirichlet series defined by* (2.24) *one has*

$$Z_q(\alpha,\gamma,b,k_1k_3) = \frac{L_{bq}(1/2+\alpha, f_{k_3}\otimes\chi_{k_1})}{L_{bq}(1+2\alpha, \mathrm{sym}^2 f)} Z_q^*(\alpha,\gamma,b,k_1k_3),$$

where $Z_q^* \ll d(q) q^{\mathrm{Re}(\gamma)-1/2}$, uniformly in $b,\kappa,k_1,k_3,\mathrm{Re}(\gamma)>1/2+\varepsilon$, $\mathrm{Re}(\alpha)\ge 0$.

Proof. From Lemma 4 we see that the summand is within a constant of being jointly multiplicative in n,k_2, so that we may write an Euler product. We use the notation $p^r||q$ to mean that r is the largest power of p dividing q. Then Z_q is given by

$$\prod_{p\nmid 2q} \sum_{k_2,n\ge 0} \frac{\lambda_f(p^n)}{p^{n\alpha+2\gamma k_2}} \frac{G_{k_1k_3p^{2k_2}}(p^n)}{p^n} \prod_{p^r||q} \sum_{k_2,n\ge 0} \frac{\lambda_f(p^n)}{p^{n\alpha+2\gamma k_2 - r\gamma}} \frac{G_{k_1k_3p^{2k_2}}(p^{r+n})}{p^{r+n}}. \tag{2.26}$$

We must check all possible cases when p does or does not divide the parameters q,b,k_1 and k_3. Let us begin with the generic $p\nmid q$. Suppose first that $p\nmid 2bk_1$. We have that all of the terms where $k_2\ge 1$ contribute $\ll p^{-(1+2\varepsilon)}$, uniformly in all parameters. The

$k_2 = 0$ terms are exactly

$$1 + \lambda_f(p)\chi_{k_1k_3}(p)p^{-(1/2+\alpha)},$$

which matches the proposed Euler factor in the statement of the Lemma up to a uniformly bounded factor. Now we consider the terms with $p \nmid 2b$ but $p \mid k_1$. In this case, the Euler factor is given by

$$1 - \frac{\lambda_f(p^2)}{p^{1+2\alpha}} + O\left(\frac{1}{p^{1+2\varepsilon}}\right),$$

which exactly matches the Euler factor in the statement of the Lemma up to a uniformly bounded factor in q, κ, k_1. If $p \mid 2b$, then the Euler factor is $1+O(p^{-(1+2\varepsilon)})$.

Now we turn to the terms where $p \mid q$. Inspecting Lemma 4 we find four cases depending on whether p divides q to even or odd order and whether $p \mid k_3$ or not. When r is odd the second product of (2.26) is

$$\prod_{\substack{p^r||q \\ r \text{ odd} \\ p\nmid k_3}} \left(\chi_{k_1k_3}(p)p^{\gamma-1/2} + \frac{\lambda_f(p)}{p^{\alpha+\gamma}} + O\left(\frac{1}{p^{1+\varepsilon}}\right)\right) \prod_{\substack{p^r||q \\ r \text{ odd} \\ p|k_3}} \left(-\frac{\lambda_f(p)}{p^{\alpha}}p^{\gamma-1} + \frac{\lambda_f(p)}{p^{\alpha+\gamma}} + O\left(\frac{1}{p^{1+2\varepsilon}}\right)\right),$$

and when r is even this product is

$$\prod_{\substack{p^r||q \\ r \text{ even} \\ p\nmid k_3}} \left(1 + \frac{\lambda_f(p)\chi_{k_1k_3}(p)}{p^{\alpha+1/2}} + O\left(\frac{1}{p^{1+2\varepsilon}}\right)\right) \prod_{\substack{p^r||q \\ r \text{ even} \\ p|k_3}} \left(-p^{2\gamma-1} + 1 - \frac{1}{p} - \frac{\lambda_f(p^2)}{p^{2\alpha+1}} + O\left(\frac{1}{p^{1+2\varepsilon}}\right)\right).$$

If r is even then $r \geq 2$, so we have that $Z_q^* \ll d(q)q^{\mathrm{Re}(\gamma)-1/2}$. $\square$

Now we return to $T_3(q,h)$, and split the sum over k_1 at $U\kappa q^{3/2}zY^2/X$. When $|k_1| \leq U\kappa q^{3/2}zY^2/X$, shift the lines of integration to $\mathrm{Re}(s) = 3/4$ and $\mathrm{Re}(u) = -1/2+ 1/\log X\kappa q$, and for the tail k_1, shift to $\mathrm{Re}(s) = 5/4$ and $\mathrm{Re}(u) = -1/2 + 1/\log X\kappa q$.

We have that

$$\begin{aligned} Z_q(1/2+u,s,a,k_1) &\ll |L_{aq}(1+u, f_{k_3}\otimes\chi_{k_1})|(\log X\kappa q)^2 q^{\mathrm{Re}(\gamma)-1/2} \\ &\ll (\log X\kappa q)^3 \prod_{p|a}\left(1+\frac{10}{\sqrt{p}}\right)|L(1+u, f_{k_3}\otimes\chi_{k_1})|q^{\mathrm{Re}(\gamma)-1/2} \end{aligned}$$

unconditionally due to the work of Goldfeld, Hoffstein and Lieman [GHL94]. We also have the estimate $H(s) \ll |s|^{\mathrm{Re}(s)-1/2}$, so that the small k_1 of $T_3(q,h)$ are

$$\ll (XU\kappa\sqrt{q}z)^{1/4}q^{1/4}(\log X\kappa q)^5 \sum_{a\le Y}\frac{1}{\sqrt{a}}\prod_{p|a}\left(1+\frac{10}{\sqrt{p}}\right)\sum_{\substack{k_3\in\mathcal{D}\\ k_3|q}}\frac{1}{|k_3|^{3/4}}\int_{(3/4)}\int_{(-1/2+1/\log X\kappa q)}$$
$$\times \sum_{\substack{|k_1|\le U\kappa q^{3/2}zY^2/X\\ k_1\in\mathcal{D}\\ (k_1,q)=1}}\frac{1}{|k_1|^{3/4}}|L(1+u, f_{k_3}\otimes\chi_{k_1})|\frac{du\,ds}{(1+|s|)^{98}(1+|u|)^{98}}.$$

We have that by Conjecture 5 this is $\ll (U\kappa q^{3/2}z)^{1/2}Y(\log X\kappa q)^6$. Similarly the tail k_1 terms are

$$\ll (U\kappa\sqrt{q}z)^{3/4}X^{-1/4}q^{3/4}(\log X\kappa q)^5\sum_{a\le Y}\sqrt{a}\prod_{p|a}\left(1+\frac{10}{\sqrt{p}}\right)\sum_{\substack{k_3\in\mathcal{D}\\ k_3|q}}\frac{1}{|k_3|^{5/4}}$$
$$\times\int_{(5/4)}\int_{(-1/2+1/\log X\kappa q)}\sum_{\substack{|k_1|> U\kappa q^{3/2}zY^2/X\\ k_1\in\mathcal{D}\\ (k_1,q)=1}}\frac{1}{|k_1|^{5/4}}|L(1+u, f_{k_3}\otimes\chi_{k_1})|\frac{du\,ds}{(1+|s|)^{98}(1+|u|)^{98}},$$

which is $\ll (U\kappa q^{3/2}z)^{1/2}Y(\log X\kappa q)^6$ as well by Conjecture 5. Taking

$$Y = X^{1/2}/(U\kappa q^{3/2})^{1/4},$$

we find $T_3(q,h)\ll X^{1/2}(U\kappa q^{3/2}z)^{1/4}(\log X\kappa q)^6$.

□

Chapter 3

The Second Moment of Automorphic L-Functions

3.1 Introduction

In this chapter we are concerned with finding an asymptotic formula for the second moment of automorphic L-functions with shifts. The techniques here are very similar to those of Conrey and Iwaniec [CI00] who establish an upper bound for the third moment of such L-functions. Our motivation for working out the second moment is (1) to see how powerful a result one can obtain in this easier case, and more importantly (2) to observe how off-diagonal main terms spectacularly cancel to give an asymptotic estimate which matches that given by the "recipe" of CFKRS [CFK$^+$05], and the random matrix conjectures. We ignore the dependence of our error terms on the weight κ. However, we are careful check that our Theorem is valid for even weights $\kappa \geq 6$. It is likely that the results extend to $\kappa = 4$, and possibly $\kappa = 2$, although no additional effort has been invested in obtaining these cases.

Let $\chi = \left(\frac{\cdot}{q}\right)$ be the primitive quadratic character $\pmod q$, and let f be a cusp form of even weight κ, level q and trivial central character with Fourier coefficients

$a_f(n) = a_f(1)\lambda_f(n)$, which we will choose so that $||f||_{L^2(\Gamma_0(q)\backslash\mathcal{H})} = 1$. Now consider the form $f \otimes \chi$, which is a primitive form on $\Gamma_0(q^2)$, see [Li75]. Thus the length of the sum we consider is roughly the square root of the conductor. The twisted L-function is defined

$$L_f(s,\chi) := \sum_{n=1}^{\infty} \frac{\lambda_f(n)\chi(n)}{n^s} = \prod_{p \nmid q} \left(1 - \frac{\lambda_f(p)\chi(p)}{p^s} + \frac{1}{p^{2s}}\right)^{-1}.$$

The completed L-function is defined

$$\Lambda_f(s,\chi) := \left(\frac{q}{2\pi}\right)^s \Gamma\left(s + \frac{\kappa - 1}{2}\right) L_f(s,\chi).$$

In this chapter, we assume that $i^\kappa = \chi(-1)$ to force the sign of the functional equation to be positive, otherwise the central values vanish identically and the moment is identically zero. The equation $i^\kappa = \chi(-1)$ will be used crucially in the cancellation of main terms in section 3.3. The L-functions we are considering in this chapter have the simple functional equation

$$\Lambda_f(s,\chi) = \Lambda_f(1-s,\chi).$$

Let $\mathcal{F}_\kappa(q)$ denote an orthonormal eigenbasis of cusp forms of weight κ for $\Gamma_0(q)$, and let $\zeta_q(s)$ denote the Riemann zeta function with the Euler factors at primes dividing q omitted. Let

$$\omega_f := \frac{\Gamma(\kappa-1)}{(4\pi)^{\kappa-1}}|a_f(1)|^2$$

be the harmonic weights which make the Petersson formula ((2.9) of [CI00]) work out nicely, which can be removed easily if desired. Our result is the following:

Theorem 4 *Let $q \to \infty$ through primes and the weight be and even integer $\kappa \geq 6$.*

Uniformly for $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) < 1/2$ *we have*

$$\sum_{f\in\mathcal{F}_\kappa(q)} \omega_f \Lambda_f(1/2+\alpha,\chi)\Lambda_f(1/2+\beta,\chi)$$
$$= \sum_{\pm\alpha,\pm\beta} \left(\frac{q}{2\pi}\right)^{1\pm\alpha\pm\beta} \zeta_q(1\pm\alpha\pm\beta)\Gamma(\pm\alpha+\kappa/2)\Gamma(\pm\beta+\kappa/2) + O_\kappa(q^{1/2}),$$

where the sum on the right is over the 4 *choices of signs. Both sides remain analytic when* $\alpha = \pm\beta$, *in which case the right hand side is to be interpreted as the limit as* $\alpha \to \pm\beta$ *of the displayed expression. The subscript* ζ_q *means that the Euler factors at primes dividing* q *have been omitted.*

In section 3.3 we require q to be odd squarefree to obtain cancellation of the main terms. On the other hand, the estimation of the error terms in section 3.4 is driven by the Riemann hypothesis for curves over finite fields, suggesting a restriction to prime power level q. The intersection of these two sets is the primes, hence the restriction in the Theorem.

Of course, the moment of the L-functions themselves can be derived easily from that of the completed L-functions above, but the formula is somewhat less elegant.

3.2 Opening

We work with moments of the completed L-functions, which have a particularly symmetric approximate functional equation:

$$\Lambda_f(1/2+\alpha,\chi) = \sum_{\pm\alpha} \left(\frac{q}{2\pi}\right)^{\pm\alpha+1/2} \sum_{n=1}^{\infty} \frac{\lambda_f(n)\chi(n)}{n^{\pm\alpha+1/2}} V_{\pm\alpha+1/2}\left(\frac{n}{q}\right),$$

with

$$V_{\alpha+1/2}(x) := \frac{1}{2\pi i}\int_{(c)} \Gamma(s+\alpha+\kappa/2)(2\pi x)^{-s}\frac{ds}{s}.$$

The sum over $\pm\alpha$ has two terms, one for each choice of sign. The proof of this approximate functional equation is standard (see [IK04] §5.2) so we omit the proof.

Our goal is to compute

$$\sum_{f\in\mathcal{F}_\kappa(q)} \omega_f \Lambda_f(1/2+\alpha,\chi)\Lambda_f(1/2+\beta,\chi), \tag{3.1}$$

so that applying the approximate functional equation for each $\Lambda_f(1/2+\alpha,\chi)$ we have four symmetric terms.

Applying both the Petersson formula and Poisson summation one finds that

Proposition 4

$$\sum_{f\in\mathcal{F}_\kappa(q)} \omega_f \Lambda_f(1/2+\alpha,\chi)\Lambda_f(1/2+\beta,\chi) = \sum_{\pm\alpha,\pm\beta}\left(\mathcal{D} + \sum_{c\equiv 0(q)} \frac{\mathcal{S}(c)}{c^2}\right),$$

with

$$\mathcal{D} = \left(\frac{q}{2\pi}\right)^{1+\alpha+\beta} \sum_{n=1}^{\infty} \frac{\chi(n^2)}{n^{1+\alpha+\beta}} V_{\alpha+1/2}\left(\frac{n}{q}\right) V_{\beta+1/2}\left(\frac{n}{q}\right)$$

and

$$\mathcal{S}(c) := \sum_{m_1}\sum_{m_2} G(m_1,m_2,c)\check{W}_{\alpha,\beta}(m_1,m_2,c), \tag{3.2}$$

where m_1 and m_2 are the dual variables to the n_1 and n_2 coming from the approximate functional equation. Here $G(m_1,m_2,c)$ is a character sum which does not depend on the shifts (non-Archimedean part), and $\check{W}_{\alpha,\beta}(m_1,m_2,c)$ is defined by some Fourier integrals (Archimedean part). Precisely,

$$G(m_1,m_2,c) = \sum_{a_1 \pmod c} \sum_{a_2 \pmod c} \chi(a_1a_2)S(a_1,a_2,c)e_c(a_1m_1+a_2m_2),$$

and

$$\check{W}_{\alpha,\beta}(m_1,m_2,c) = \left(\frac{q}{2\pi}\right)^{1+\alpha+\beta} \int_0^\infty \int_0^\infty \frac{J(2\sqrt{x_1x_2})}{(cx_1)^\alpha (cx_2)^\beta} V_{\alpha+1/2}\left(\frac{cx_1}{q}\right) V_{\beta+1/2}\left(\frac{cx_2}{q}\right) \times e(-m_1x_1 - m_2x_2)\, dx_1\, dx_2$$

where $S(a_1,a_2,c)$ is the standard Kloosterman sum, and up to a simple factor the function $J(x)$ is the J-Bessel function, see (3.3)*.*

The off-diagonal main term will come from the case $m_1 = m_2 = 0$ and $c = q$, in which case $G(0,0,q) = \tau(\chi)^2\phi(q)$ (*Proof:* formula (3.7) below). Before computing the main terms, we give the proof of Proposition 4.

Proof of Proposition 4: Applying the approximate functional equation to (3.1), we find four terms cooresponding to $\pm\alpha, \pm\beta$. The $+\alpha, +\beta$ term is

$$\left(\frac{q}{2\pi}\right)^{1+\alpha+\beta} \sum_{n_1=1}^\infty \sum_{n_2=1}^\infty \frac{\chi(n_1n_2)}{n_1^{1/2+\alpha} n_2^{1/2+\beta}} V_{1/2+\alpha}\left(\frac{n_1}{q}\right) V_{1/2+\beta}\left(\frac{n_2}{q}\right) \sum_{f\in\mathcal{F}_\kappa(q)} \omega_f \lambda_f(n_1)\lambda_f(n_2).$$

The Petersson formula is

$$\sum_{f\in\mathcal{F}_\kappa(q)} \omega_f \lambda_f(n_1)\lambda_f(n_2) = \delta_{n_1,n_2} + \sqrt{n_1n_2} \sum_{c\equiv 0(q)} c^{-2} S(n_1,n_2,c) J(2\sqrt{n_1n_2}/c),$$

see e.g. [CI00] (2.9), where we define

$$J(x) = 4\pi i^\kappa x^{-1} J_{\kappa-1}(2\pi x), \tag{3.3}$$

with $J_\nu(z)$ being the standard J-Bessel function. At this point, one already sees that the diagonal term $\mathcal{D}$ is as in the statement of the Proposition. Pulling out the sum over c and the c^{-2} we have that

$$\mathcal{S}(c) = \left(\frac{q}{2\pi}\right)^{1+\alpha+\beta} \sum_{n_1=1}^\infty \sum_{n_2=1}^\infty \frac{\chi(n_1n_2)}{n_1^\alpha n_2^\beta} V_{1/2+\alpha}\left(\frac{n_1}{q}\right) V_{1/2+\beta}\left(\frac{n_2}{q}\right) S(n_1,n_2,c) J(2\sqrt{n_1n_2}/c).$$

Our goal is to separate the arithmetic terms from the analytic terms, which allows us to use Poisson summation in the variables n_1 and n_2. First, observe that the χ and the S are periodic $\pmod c$, so we replace n by $a + nc$:

$$\left(\frac{q}{2\pi}\right)^{1+\alpha+\beta} \sum_{a_1 \pmod c} \sum_{a_2 \pmod c} \chi(a_1 a_2) S(a_1, a_2, c) \sum_{n_1} \sum_{n_2} \frac{1}{(a_1 + n_1 c)^{\alpha} (a_2 + n_2 c)^{\beta}}$$
$$\times V_{1/2+\alpha}\left(\frac{a_1 + n_1 c}{q}\right) V_{1/2+\beta}\left(\frac{a_2 + n_2 c}{q}\right) J(2\sqrt{(a_1 + n_1 c)(a_2 + n_2 c)}/c).$$

The expression inside the n_i sums are now smooth, so we can apply Poisson summation. We compute Fourier transforms. After a change of variables, the transform of the smooth expression inside the n_i sums is

$$e_c(a_1 m_1 + a_2 m_2) c^{-(\alpha+\beta)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{J(2\sqrt{x_1 x_2})}{x_1^{\alpha} x_2^{\beta}} V_{1/2+\alpha}\left(\frac{c x_1}{q}\right) V_{1/2+\beta}\left(\frac{c x_2}{q}\right)$$
$$\times e(-m_1 x_1 - m_2 x_2)\, dx_1\, dx_2.$$

Hence we have proved the above formulae for $\check{W}_{\alpha,\beta}(m_1, m_2, c)$ and $G(m_1, m_2, c)$. $\square$

3.3 The Main Terms

The main term in our Theorem comes from the diagonal $\mathcal{D}$ and from the single term $m_1 = m_2 = 0$ and $c = q$ in (3.2). In this section we see in action how these terms combine and cancel to leave only the main term which one expects from the "recipie" given in chapter 4 of [CFK+05]. These cancellations are in general mysterious and not well understood, see the comments in the introduction of [CFK+05].

Let us start by computing the diagonal terms, for which we employ standard contour-shifting techniques. Assume throughout that $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) < 1/2$, and $\kappa \geq 6$.

We define the following quantity which will arise later:

$$H_\kappa(\alpha,\beta) := \frac{1}{2\pi i}\int_{(2)} \frac{\Gamma(\kappa/2-s)\Gamma(\kappa/2+s)}{(s-\alpha)(s-\beta)}\, ds. \tag{3.4}$$

It is interesting to note that the following Lemma is not logically necessary to our proof. We have included it because it may clear some confusion.

Lemma 8 *Assume $\alpha \neq \beta$ are complex numbers. We have*

$$H_\kappa(\alpha,\beta) - H_\kappa(-\alpha,-\beta) = \frac{\Gamma(\kappa/2-\alpha)\Gamma(\kappa/2+\alpha) - \Gamma(\kappa/2-\beta)\Gamma(\kappa/2+\beta)}{\alpha-\beta}.$$

The left side of the above expression is analytic even if $\alpha = \beta$, in which case we interpret the right side as the limit $\alpha \to \beta$.

Proof. Shift contours and change variables $s \leftrightarrow -s$. $\square$

Let $\zeta_q(s)$ denote the function defined by the Euler product for the Riemann zeta function with the factors at primes dividing q removed. In particular,

$$\operatorname*{Res}_{s=1} \zeta_q(s) = \prod_{p|q} \left(1 - p^{-1}\right) = \frac{\phi(q)}{q}.$$

By definition of the cut-off functions we have

$$\mathcal{D} = \left(\frac{1}{2\pi i}\right)^2 \int_{(2)}\int_{(2)} \left(\frac{q}{2\pi}\right)^{1+\alpha+\beta+u+v} \frac{\Gamma(u+\alpha+\kappa/2)}{u}\frac{\Gamma(v+\beta+\kappa/2)}{v}$$
$$\times \zeta_q(1+\alpha+\beta+u+v)\, du\, dv.$$

There are polar divisors of this meromorphic function on $\mathbb{C}^2$ at $u=0$, $v=0$, $u+v=\alpha+\beta$, and some divisors far to the left coming from the poles of the gamma functions which we may ignore. The main terms come from the divisors $u=0$ and $v=0$ while the terms coming from the pole of the zeta function at $u+v=\alpha+\beta$ cancel out later.

We begin by moving both lines of integration to the $\mathrm{Re}(u)=\mathrm{Re}(v)=1/4$ lines. Then move the line of integration in v all the way to $\mathrm{Re}(v)=-2$. In doing so we encounter poles at $v=-(\alpha+\beta+u)$ and $v=0$. The remaining double integral on the lines $\mathrm{Re}(u)=1/4$ and $\mathrm{Re}(v)=-2$ contributes $O(q^{1/4+\varepsilon})\leq O(q^{1/2})$. The contribution to $\mathcal{D}$ from the pole at $v=-(\alpha+\beta+u)$ is

$$-\frac{q}{2\pi}\frac{\phi(q)}{q}H_\kappa(\alpha,-\beta).$$

The contribution to $\mathcal{D}$ from the pole at $v=0$ is

$$\Gamma(\beta+\kappa/2)\frac{1}{2\pi i}\int_{(1/4)}\left(\frac{q}{2\pi}\right)^{1+\alpha+\beta+u}\frac{\Gamma(u+\alpha+\kappa/2)}{u}\zeta_q(1+\alpha+\beta+u)\,du,$$

which after shifting the line of integration to $\mathrm{Re}(u)=-2$ is

$$-\frac{q}{2\pi}\frac{\phi(q)}{q}\frac{\Gamma(\beta+\kappa/2)\Gamma(-\beta+\kappa/2)}{\alpha+\beta}+\left(\frac{q}{2\pi}\right)^{1+\alpha+\beta}\Gamma(\alpha+\kappa/2)\Gamma(\beta+\kappa/2)\zeta_q(1+\alpha+\beta)$$
$$+O(q^{1/2}),$$

where the first explicit term here comes from the pole of ζ_q at $u=-(\alpha+\beta)$ and the second explicit term comes from the pole at $u=0$. Note that, despite appearances, the main terms of $\mathcal{D}$ here are indeed symmetric in α and β by Lemma 8. Taking all 4 choices of signs together we record

$$\begin{aligned}\sum_{\pm\alpha,\pm\beta}\mathcal{D} &= -\frac{q}{2\pi}\frac{\phi(q)}{q}\left(H_\kappa(\alpha,\beta)+H_\kappa(\alpha,-\beta)+H_\kappa(-\alpha,\beta)+H_\kappa(-\alpha,-\beta)\right)\\ &\quad+\sum_{\pm\alpha,\pm\beta}\left(\frac{q}{2\pi}\right)^{1\pm\alpha\pm\beta}\zeta_q(1\pm\alpha\pm\beta)\Gamma(\pm\alpha+\kappa/2)\Gamma(\pm\beta+\kappa/2)+O(q^{1/2}).\end{aligned}$$

We now compute the off-diagonal main term. Observe that when $\kappa\geq 6$ we have by Mellin inversion the formula

$$J(2\sqrt{x_1x_2}) = i^\kappa \frac{1}{2\pi i}\int_{(2)} (2\pi)^{2s}\frac{\Gamma(\kappa/2-s)}{\Gamma(\kappa/2+s)}(x_1x_2)^{s-1}\,ds,$$

see [GR07] §17.43. We use this formula for J and the definition of V to compute $\check{W}_{\alpha,\beta}(0,0,q)$. One finds

$$\check{W}_{\alpha,\beta}(0,0,q) = i^\kappa\frac{q}{2\pi}\frac{1}{2\pi i}\int_{(2)}\frac{\Gamma(\kappa/2-s)\Gamma(\kappa/2+s)}{(s-\alpha)(s-\beta)}\,ds = i^\kappa\frac{q}{2\pi}H_\kappa(\alpha,\beta).$$

We will compute in section 3.4, formula (3.7), that $G(0,0,q)=\tau(\chi)^2\phi(q)$, where $\tau(\chi)$ is the Gauss sum of the character χ. We have that the sum over the 4 choices of signs for this off-diagonal main term is

$$\begin{aligned}\frac{1}{q^2}\sum_{\pm\alpha,\pm\beta} G(0,0,q)\check{W}_{\pm\alpha,\pm\beta}(0,0,q) &= i^\kappa\frac{\tau(\chi)^2\phi(q)}{q^2}\frac{q}{2\pi}\\ &\times\left(H_\kappa(\alpha,\beta)+H_\kappa(-\alpha,\beta)+H_\kappa(\alpha,-\beta)+H_\kappa(-\alpha,-\beta)\right).\end{aligned}$$

Finally, we require the explicit evaluation of the Gauss sum in the case of quadratic characters, so we must suppose now that q is odd and square-free. In this case we have $\tau(\chi)^2 = \chi(-1)q$. Recall that we chose $i^\kappa = \chi(-1)$ to fix the sign of the functional equation. Working these two equalities in above, one observes the cancellation of the terms involving $H_\kappa(\alpha,\beta)$, and we are left with a main term which matches the one found in the Theorem.

3.4 The Error Terms

The remainder of this chapter is devoted to rigorously establishing the error term claimed in Theorem 4. The estimate is essentially by the Riemann hypothesis for curves over finite fields, but we have chosen to restrict to curves over the prime field

$\mathbb{F}_p$ because the restriction to squarefree level was already made in section 3.3.

We bound the arithmetic sum $G(m_1, m_2, c)$ and the Fourier integral $\check{W}_{\alpha,\beta}(m_1, m_2, c)$. Let $c = qr$, and begin with the arithmetic sum.

Lemma 9 *In general, we have the trivial bound*

$$|G(m_1, m_2, qr)| \leq q^2 r^2. \tag{3.5}$$

Suppose that $q = p$ is a prime, and that m_1 and m_2 are not simultaneously $0 \mod p$. *Then we have that*

$$|G(m_1, m_2, qr)| \ll q^{3/2} r^2. \tag{3.6}$$

Proof. We open the Kloosterman sum.

$$G(m_1, m_2, qr) = \sum_{(d,c)=1} \left[\sum_{a_1} \chi(a_1) e\left(\frac{a_1(m_1 + d)}{c} \right) \right] \left[\sum_{a_2} \chi(a_2) e\left(\frac{a_2(m_2 + \overline{d})}{c} \right) \right].$$

Each set of brackets is a Gauss sum, so we compute

$$\sum_{a_1} \chi(a_1) e\left(\frac{a_1(m_1 + d)}{c} \right) = \begin{cases} r\tau(\chi)\chi\left(\frac{d+m_1}{r}\right) & \text{if } d + m_1 \equiv 0 \mod r \\ 0 & \text{if } d + m_1 \not\equiv 0 \mod r \end{cases}$$

and

$$\sum_{a_2} \chi(a_2) e\left(\frac{a_2(m_2 + \overline{d})}{c} \right) = \begin{cases} r\tau(\chi)\chi\left(\frac{\overline{d}+m_2}{r}\right) & \text{if } \overline{d} + m_2 \equiv 0 \mod r \\ 0 & \text{if } \overline{d} + m_2 \not\equiv 0 \mod r. \end{cases}$$

So, we have that

$$G(m_1, m_2, qr) = r^2\tau(\chi)^2 \sum_{\substack{(d,c)=1 \\ d\equiv -m_1\,(r) \\ 1\equiv -m_2 d\,(r)}} \chi(d)\chi\left(\frac{d+m_1}{r}\right)\chi\left(\frac{1+dm_2}{r}\right). \tag{3.7}$$

Let $d = -m_1 + ur$ and let u run over residue classes $\pmod q$. Then

$$G(m_1, m_2, qr) = r^2\tau(\chi)^2 \sum_{u \mod q} \chi(-m_1+ur)\chi(u)\chi\left(\frac{1-m_1m_2}{r} + um_2\right), \tag{3.8}$$

with the understanding that this sum vanishes if $Q := (1-m_1m_2)/r$, is not an integer. From 3.8 the first bound in the Lemma is established.

We now assume that q is prime so that we are summing over a finite field, and our problem reduces to counting $\mathbb{F}_p$ points on the (sometimes degenerate!) curve defined by the equation

$$v^2 = rm_2u^3 + (1-2m_1m_2)u^2 - m_1Qu.$$

We may thus appeal to algebraic geometry to bound the character sums. Rather than work out each degenerate case by hand, we appeal to Theorem 11.13 and Theorem 11.23 of [IK04], which are general enough to treat all of the degenerate cases of (3.8) except for those in the statement of Lemma 9. $\square$

Having removed the term $m_1 = m_2 = 0$, $r = 1$ we now return to the off-diagonal sum from Proposition 4:

$$\frac{1}{q^2}\sum_{r>1}\frac{1}{r^2}\sum_{\substack{m_1,m_2 \\ (m_1,m_2)\neq(0,0)}} G(m_1,m_2,qr)\check{W}_{\alpha,\beta}(m_1,m_2,qr).$$

Note we have recorded that the sum vanishes if $m_1 = m_2 = 0$ and $r > 1$ by the conditions on the sum in (3.7). Split the sums over m_1 and m_2 at q and use the RH bound (3.6) on the terms with either $|m_1|$ or $|m_2| < q$. On the terms for which both $|m_1|$ and $|m_2| \geq q$ we use the trivial bound (3.5).

Now we turn to the Fourier integral. Observe that $\check{W}_{\alpha,\beta}(m_1, m_2, qr)$ has very simple dependence on q:

$$\check{W}_{\alpha,\beta}(m_1, m_2, qr) = \frac{q}{2\pi}(2\pi r)^{-(\alpha+\beta)} \int_0^\infty \int_0^\infty \frac{J(2\sqrt{x_1 x_2})}{x_1^\alpha x_2^\beta} V_{1/2+\alpha}(rx_1) V_{1/2+\beta}(rx_2) \times e(-m_1 x_1 - m_2 x_2)\, dx_1\, dx_2.$$

Let us re-define our cut off functions slightly. Let

$$U_{1/2+\alpha}(rx) = (2\pi rx)^{-\alpha} V_{1/2+\alpha}(rx) = \frac{1}{2\pi i} \int_{(2)} \Gamma(s + \kappa/2)(2\pi rx)^{-s} \frac{ds}{s-\alpha},$$

so that we have

$$\check{W}_{\alpha,\beta}(m_1, m_2, qr) = \frac{q}{2\pi} \int_0^\infty \int_0^\infty J(2\sqrt{x_1 x_2}) U_{1/2+\alpha}(rx_1) U_{1/2+\beta}(rx_2) \times e(-m_1 x_1 - m_2 x_2)\, dx_1\, dx_2.$$

By shifting the contour we have for $\mathrm{Re}(\alpha) < 1/2$ that

$$x^i U_{1/2+\alpha}^{(i)}(x) \ll_i x^{-\mathrm{Re}(\alpha)} (1 + |x|)^{-100}.$$

We show

Lemma 10 *Uniformly for* $\mathrm{Re}(\alpha)$ *and* $\mathrm{Re}(\beta) < 1/2$ *we have*

$$\check{W}_{\alpha,\beta}(m_1, m_2, qr) \ll \frac{q}{r^2(1 + |m_1|)^2(1 + |m_2|)^2}.$$

Proof. At this point, we assume that $\kappa \geq 6$ so that $J(x)$ vanishes at the origin to order at least 4. We have by integration by parts that

$$\check{W}_{\alpha,\beta}(m_1, m_2, qr) = \frac{1}{m_1^2 m_2^2} \frac{q}{2\pi} \int_0^\infty \int_0^\infty \frac{\partial^4}{\partial x_1^2 \partial x_2^2} \left(J(2\sqrt{x_1 x_2}) U_{1/2+\alpha}(rx_1) U_{1/2+\beta}(rx_2) \right) \times e(-m_1 x_1 - m_2 x_2)\, dx_1\, dx_2.$$

Expanding the derivatives, and applying the bounds for $U(x)$, we see that

$$\begin{aligned} \check{W}_{\alpha,\beta}(m_1, m_2, qr) \ll\ & \frac{q}{m_1^2 m_2^2} \int_0^\infty \int_0^\infty \Bigg(\frac{|J(2\sqrt{x_1 x_2})|}{x^2 y^2} + \frac{|J'(2\sqrt{x_1 x_2})|}{x^{3/2} y^{3/2}} \\ & + \frac{|J''(2\sqrt{x_1 x_2})|}{xy} + \frac{|J^{(3)}(2\sqrt{x_1 x_2})|}{x^{1/2} y^{1/2}} + |J^{(4)}(2\sqrt{x_1 x_2})| \Bigg) \\ & \times (rx_1)^{-\operatorname{Re}(\alpha)} (rx_2)^{-\operatorname{Re}(\beta)} \left(1 + |rx_1|\right)^{-100} \left(1 + |rx_2|\right)^{-100} dx_1\, dx_2 \\ \ll\ & \frac{q}{m_1^2 m_2^2} \int_0^{1/r} (rx_1)^{-\alpha}\, dx_1 \int_0^{1/r} (rx_2)^{-\beta}\, dx_2 \\ \ll\ & \frac{q}{r^2 m_1^2 m_2^2}. \end{aligned}$$

Likewise, by only differentiating in x_1 or x_2 and not the other variable, or not at all we also obtain

$$\check{W}_{\alpha,\beta}(m_1, m_2, qr) \ll \frac{q}{r^2 m_1^2}$$

or

$$\check{W}_{\alpha,\beta}(m_1, m_2, qr) \ll \frac{q}{r^2 m_2^2}$$

or

$$\check{W}_{\alpha,\beta}(m_1, m_2, qr) \ll \frac{q}{r^2},$$

thus we obtain the stated bound. $\square$

Split the sums over m_1 and m_2 at q, and apply Lemmae 9 and 10. We find that the terms where one of $|m_1|$ or $|m_2|$ is small is $O(q^{1/2})$, whereas the terms where both $|m_1|$ and $|m_2|$ are large are $O(1)$. Hence we obtain the Theorem.

Chapter 4

Transition Mean Values of Shifted Convolution Sums

Let f be a classical GL_2 holomorphic Hecke cusp form of full level and even weight κ, and let $e(z) := \exp(2\pi i z)$. The cusp form f admits a Fourier expansion of the form

$$f(z) = \sum_{n\geq 1} n^{\frac{\kappa-1}{2}} \lambda_f(n) e(nz),$$

where the $\lambda_f(n)$ are normalized so that $|\lambda_f(n)| \leq d(n)$ and $\lambda_f(1) = 1$. In this chapter, we study "shifted convolution sums", i.e. sums of the form

$$\sum_n \lambda_f(n)\lambda_f(n+h).$$

These sums have many applications in analytic number theory. They often arise in subconvexity results, and in off-diagonal terms in moment calculations for GL_1 and GL_2 L-functions, see Sarnak [Sar01], or Michel in [Mic07], lecture 4. For a thorough study of the shifted convolution problem, see the work of Blomer and Harcos [BH08]. In this chapter we obtain asymptotic estimates for shifted convolution sums after averaging over many shifts h, i.e. we study sums of the form

$$\sum_{h \doteqdot Y} \sum_{n \doteqdot X} \lambda_f(n)\lambda_f(n+h), \tag{4.1}$$

where the notation $n \doteqdot X$ indicates a sum over n of length X with a for-now-unspecified smoothing. A variant of the sum (4.1) twisted by Dirichlet characters $\chi(h)$ was studied by Michel in his chapter [Mic04] on the subconvexity problem for Rankin-Selberg L-functions.

We show that when X and Y grow large in such a manner that $Y^2/X \to \infty$, the double sum (4.1) has an asymptotic formula with well controlled error terms, and we obtain nontrivial asymptotic upper bounds when $Y^2/X \to 0$.

The transition range when $Y^2/X = c$ is a fixed constant is the most interesting case. In this situation, we derive an asymptotic main term for (4.1) which depends delicately on the constant c. One may interpret the sum (4.1) as varying on the open first quadrant of the (X, Y^2)-plane, and view asymptotic estimates as X and Y^2 go to infinity as a description of the singularity at infinity in this quarter-plane. The asymptotic behavior of (4.1) varies continuously on a blowup of the point at infinity in the quarter-(X, Y^2)-plane.

Our results for shifted convolution sums are very similar to the interesting results of Conrey, Farmer and Soundararajan in [CFS00] on transition mean values of the Jacobi symbol. They study the sum

$$S(X,Y) := \sum_{\substack{m \le X \\ m \text{ odd}}} \sum_{\substack{n \le Y \\ n \text{ odd}}} \left(\frac{m}{n}\right), \tag{4.2}$$

and similarly find asymptotic formulae when one of either X or Y grow much faster than the other, and a transition region when X/Y is a fixed constant. The leading term in the asymptotic they derive depends continuously on this constant. We will discuss in the next section how (4.1) and (4.2) are in fact quite similar to each other.

First, however, we state the main result of this chapter precisely. Define the function c_f on the positive real numbers

$$c_f(\alpha) := \frac{\pi^{\frac{3}{2}}}{2}\alpha \sum_{n\geq 1} \lambda_f(n)^2 W_\kappa\left(\pi^2 n\alpha\right), \tag{4.3}$$

with

$$W_\kappa(x) := \frac{1}{2\pi i}\int_{(2)} \frac{\Gamma(s+\kappa-1)\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(2-s)} x^{-s}\, ds.$$

By shifting contours and using standard facts about the L-function $L(s, f\otimes f)$ (see [Iwa97] section 13.8), one finds that $c_f(\alpha)$ is a smooth function on the positive real numbers which

- as $\alpha\to\infty$ decays faster than any polynomial,

- as $\alpha \to 0$
$$c_f(\alpha) = \frac{\Gamma(\kappa)L(1,\mathrm{sym}^2 f)}{2\zeta(2)} + E_\kappa(\alpha),$$
where $E_\kappa(\alpha) = o_\kappa\left(\alpha^{\frac{1}{2}}\right)$ unconditionally, and $E_\kappa(\alpha) = O_{\kappa,\varepsilon}\left(\alpha^{\frac{3}{4}-\varepsilon}\right)$ for any $\varepsilon > 0$ assuming the Riemann hypothesis for the classical Riemann zeta function.

In the below Theorem ψ is a test function on $\mathbb{R}_{>0}$ which one chooses so that the expression

$$(n(n+h))^{\frac{\kappa-1}{2}} \int_0^\infty \psi(y) e^{-4\pi(n+h)y} y^{\kappa-2}\, dy$$

becomes a cut-off function for the variable n.

Theorem 5 *Let ψ be any measurable function on $\mathbb{R}_{>0}$ such that the incomplete Poincaré series*

$$P_h(z|\psi) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} e(h\gamma z)\psi(\mathrm{Im}(\gamma z))$$

is a smooth and bounded L^2 function on $\Gamma\backslash\mathcal{H}$. Denote the average of shifted convolution sums

$$S_f(\psi,Y) := \sum_{h=1}^{Y}\sum_{n\geq 1}\lambda_f(n)\lambda_f(n+h)(n(n+h))^{\frac{\kappa-1}{2}}\int_0^\infty \psi(y)e^{-4\pi(n+h)y}y^{\kappa-2}\,dy.$$

Then we have that

$$S_f(\psi,Y) = \frac{1}{(4\pi)^\kappa}\int_0^\infty\left(c_f\left(4\pi yY^2\right)-\frac{\Gamma(\kappa)L(1,\mathrm{sym}^2 f)}{2\zeta(2)}\right)\frac{\psi(y)}{y^2}\,dy$$
$$+O_\kappa\left(Y^{\frac{1}{3}(1+\theta)}\int_0^\infty\frac{|\psi(y)|}{y^{\frac{3}{2}}}\,dy\right),$$

where $c_f(\alpha)$ is the function defined by (4.3) *(see also Lemma 11 and Lemma 12), and $\theta = 0$ or $7/64$ depending on whether we assume the truth of the generalized Ramanujan Conjecture or not.*

It should be noted that the size of the exponent of Y in the error term here depends on the sharp cut-off in h, and could be made smaller upon smoothing that sum.

As an example, one obtains the following Corollary by choosing ψ to be a sequence of approximations to a point mass at $y = (4\pi X)^{-1}$.

Corollary 3 *Uniformly for $Y \ll X$ we have*

$$\sum_{h=1}^{Y}\sum_{n\geq 1}\lambda_f(n)\lambda_f(n+h)\left(\frac{n(n+h)}{X^2}\right)^{\frac{\kappa-1}{2}}e^{-\frac{n+h}{X}} = \left(c_f\left(\frac{Y^2}{X}\right)-\frac{\Gamma(\kappa)L(1,\mathrm{sym}^2 f)}{2\zeta(2)}\right)X$$
$$+O_\kappa\left(X^{\frac{1}{2}}Y^{\frac{1}{3}(1+\theta)}\right)$$

where $c_f(\alpha)$ is defied above and $\theta = 0$ or $= 7/64$ depending on whether one assumes the generalized Ramanujan Conjecture for Maass forms of $SL_2(\mathbb{Z})$ or not.

As a benchmark, the best point-wise estimates for shifted convolution sums give

$$\sum_{n \asymp X} \lambda_f(n)\lambda_f(n+h) \ll_{f,\varepsilon} (X+h)^{\frac{1}{2}+\theta+\varepsilon}, \tag{4.4}$$

where θ is as in Theorem 5 above. The bound (4.4) follows from Theorem A.1 of Sarnak's paper [Sar01] and Mellin inversion. See also Michel [Mic07] section 4.4.2.

In Corollary 3, observe that if Y^2 is large compared to X we have that

$$\sum_{h=1}^{Y}\sum_{n\geq 1} \lambda_f(n)\lambda_f(n+h)\left(\frac{n(n+h)}{X^2}\right)^{\frac{\kappa-1}{2}} e^{-\frac{n+h}{X}} \sim -\frac{\Gamma(\kappa)L(1,\mathrm{sym}^2 f)}{2\zeta(2)}X$$

and if Y^2 is small compared to X then

$$\sum_{h=1}^{Y}\sum_{n\geq 1} \lambda_f(n)\lambda_f(n+h)\left(\frac{n(n+h)}{X^2}\right)^{\frac{\kappa-1}{2}} e^{-\frac{n+h}{X}} = o_\kappa\left(X^{\frac{1}{2}}Y\right) + O_\kappa\left(X^{\frac{1}{2}}Y^{\frac{1}{3}(1+\theta)}\right),$$

or better if we assume the Riemann hypothesis. In the transition region when Y^2 is a constant multiple of X, the asymptotic growth is controlled by the function $c_f(\alpha)$.

Now we describe the work of Conrey, Farmer and Soundararajan. Their result is

Theorem 6 (Conrey, Farmer and Soundararajan) *Uniformly for all large X and Y, we have*

$$S(X,Y) = \frac{2}{\pi^2}C\left(\frac{Y}{X}\right)X^{\frac{3}{2}} + O\left((XY^{\frac{7}{16}} + YX^{\frac{7}{16}})\log XY\right),$$

where for $\alpha \geq 0$ we define

$$C(\alpha) = \sqrt{\alpha} + \frac{1}{2\pi}\sum_{k=1}^{\infty}\frac{1}{k^2}\int_0^{\alpha}\sqrt{y}\left(1-\cos\left(\frac{2\pi k^2}{y}\right)+\sin\left(\frac{2\pi k^2}{y}\right)\right)dy.$$

An alternate expression for $C(\alpha)$ is

$$C(\alpha) = \alpha + \alpha^{\frac{3}{2}} \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{k^2} \int_0^{\frac{1}{\alpha}} \sqrt{y} \sin\left(\frac{\pi k^2}{2y}\right) \, dy.$$

From the first expression, one finds that after integrating by parts that

$$C(\alpha) = \sqrt{\alpha} + \frac{\pi}{18}\alpha^{\frac{3}{2}} + O\left(\alpha^{\frac{5}{2}}\right)$$

as $\alpha \to 0$. The second expression gives the limiting behavior

$$C(\alpha) = \alpha + O\left(\alpha^{-1}\right)$$

as $\alpha \to \infty$.

The Theorem proved by Conrey, Farmer and Soundararajan has one surprising feature which is not yet apparent the case of shifted convolution sums. The function $C(\alpha)$ appearing in their result is once continuously differentiable everywhere, but it is twice differentiable at $\alpha\mathbb{Q}$ if and only if $\alpha = 2p/q$ with p and q both odd, see section 6 of their paper. It is necessary that the sum $S(X, Y)$ has a sharp cut-off for $C(\alpha)$ to have such strange differentiability properties. In our work, we do not prove an asymptotic result for a sharp cut-off, but we do have some flexibility in our choice of cut-off functions. However, we expect that it should be possible to refine Theorem 5 to a sharp cut-off and in that case expect that the transition function should be twice continuously differentiable, but that the second derivative will be almost nowhere differentiable. A heuristic for this is given in the next section.

4.1 Heuristic Connection of the Two Theorems

There are many parallels between our work and that of Conrey, Farmer and Soundararajan, and moreover both results can be interpreted as averages of Fourier coefficients of an appropriate Eisenstein series.

We first discuss the double sum of the Jacobi symbol. We have that

$$S(X,Y) = \frac{1}{(2\pi i)^2}\int_{(c)}\int_{(c)} Z(s,w)\frac{X^s}{s}\frac{Y^w}{w}\,ds\,dw$$

where $c > 1$ and

$$Z(s,w) := \sum_{\substack{m,n\geq 1\\ m,n \text{ odd}}} \frac{\left(\frac{m}{n}\right)}{n^s m^w}. \tag{4.5}$$

This is a simple example of a multiple Dirichlet series, and information about the analytic properties of $Z(s,w)$ would determine the asymptotic behavior of $S(X,Y)$. Goldfeld and Hoffstein in [GH85] derive the analytic properties of $Z(s,w)$ (actually, they work with a slightly modified series), which crucially follow from studying weight 1/2 Eisenstein series for the congruence subgroup $\Gamma_0(4)$. If $E_{\frac{1}{2}}(z,s)$ is the weight 1/2 Eisenstein series at the cusp 0 (for details, see [GH85]), it has the Fourier expansion

$$E_{\frac{1}{2}}(z,s) = \sum_{m\geq 1} a_m(s,y)e(mx)$$

where

$$a_m(s,y) = \frac{L_2(2s,\chi_m)}{\zeta_2(4s)}\frac{y^s}{4^s}K_m(s,y),$$

where the subscript 2 indicates the omission of the Euler factor at the prime 2. This Fourier coefficient is essentially $L(2s,\chi_m)$ times a well-understood K-Bessel function.

The residues of the pole at $s = 1/2$ of this half-integer weight Eisenstein series can be given in terms of the classical theta function. Next, we explain how the classical theta function gives rise to the almost-nowhere differentiable function which was found by Conrey, Farmer and Soundararajan.

In Conrey, Farmer and Soundararajan's work the transition function $C(\alpha)$ is expressed in terms of Riemann's classical non-differentiable function

$$f(x) := \sum_{n \neq 0} \frac{1}{\pi n^2} \sin(\pi n^2 x)$$

by the relation

$$\frac{d}{d\alpha}\left(\alpha^{-3/2} C(\alpha)\right) = -\frac{1}{2}\alpha^{-3/2} - \alpha^{-5/2} f(\alpha/2).$$

Thus the transition function $C(\alpha)$ in Conrey, Farmer and Soundararajan's result inherits the non-differentiable properties of Riemann's function. Moreover, Riemann's non-differentiable function $f(x)$ arises as the limit behavior of the classical theta function

$$\theta(z) = \sum_{n \in \mathbb{Z}} e(n^2 z/2)$$

pushed towards the real axis. Indeed, the natural complexification of Riemann's non-differentiable function $f(x)$ is given by

$$\phi(x) = \sum_{n=1}^{\infty} \frac{1}{i\pi n^2} e(n^2 x/2),$$

where we note that $-2\,\mathrm{Re}(\phi(x)) = f(x)$. We then have the "equation"

$$\phi'(x) = \frac{1}{2}(\theta(x) - 1) \tag{4.6}$$

which holds formally between the derivative of an almost-nowhere differentiable function, and a series which fails to converge when x is real. A very nice explanation of this phenomenon, with pictures, is given by Duistermaat in his expository article

[Dui91].

Nonetheless, the equation (4.6) does make sense in terms of the theory of automorphic distributions developed by Miller and Schmid, see [MS08]. Miller and Schmid in their paper [MS04] find that almost-nowhere differentiable functions arise as antiderivatives of automorphic distributions generally. The series (4.5) inherits the pole of the weight 1/2 Eisenstein series, and thus the main terms of $S(X, Y)$ inherit the non-differentiable behavior of the theta function.

In the proof of our Theorem below, the reader will find that our transition function $c_f(\alpha)$ arises from the pole of an Eisenstein series as well, specifically, a pole of the real analytic Eisenstein series of weight 0 and full level. With X the length of the shifted convolution sum as in the previous section, the reader will find in the proof below that we work with Fourier coefficients of this Eisenstein series at a height of $1/X$ above the real axis, so that as $X \to \infty$, we are indeed working with the corresponding automorphic distribution.

The situation in this chapter and that of Conrey, Farmer and Soundararajan are not exactly the same. As an example, the Eisenstein series appears in the present chapter via the Rankin-Selberg theory of $f \otimes f$, but there is no Rankin-Selberg in the work of Conrey, Farmer and Soundararajan. Nonetheless, we expect that if refined to sharp cut-offs the transition function for averages of shifted convolution sums should also have similarly strange differentiability properties, and that our heuristics can be made rigorous.

4.2 Preliminaries

Our main approach to the Main Theorem is to take the Petersson inner product of $y^\kappa |f|^2$ against an incomplete Poincaré series $P_h(\cdot|\psi)$. This approach was first introduced by Selberg [Sel65], revisited by Sarnak [Sar01], and has been successfully used by many other authors to study shifted convolution sums in the past (for an overview, see [Mic07]). Throughout this chapter we set $\Gamma = SL_2(\mathbb{Z})$, and let $\mathcal{H}$ denote the upper half plane with its hyperbolic metric. We work in the Hilbert space $L^2(\Gamma\backslash\mathcal{H})$ of square integrable measurable functions with the Petersson inner product

$$\langle u, v\rangle = \int_{\Gamma\backslash\mathcal{H}} u(z)\overline{v(z)}\, d_\mu z.$$

The symmetric operator

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

acts on the subspace of smooth functions and moreover has a unique self-adjoint extension to all $L^2(\Gamma\backslash\mathcal{H})$, see Iwaniec [Iwa02] chapter 4. Given a classical holomorphic normalized cuspidal eigenform f of weight κ, set $F(z) := y^{\frac{\kappa}{2}} f(z)$. We have that $|F| \in L^2(\Gamma\backslash\mathcal{H})$, but on the other hand it is no longer holomorphic. Let $\psi(y)$ be an infinitely differentiable compactly supported function on $\mathbb{R}_{>0}$, and let Γ_∞ denote the stabilizer in Γ of the cusp at infinity. Then we define the incomplete Poincaré series

$$P_h(z|\psi) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} e(h\gamma z)\psi\left(\mathrm{Im}(\gamma z)\right),$$

which is a smooth and bounded function on $\Gamma\backslash\mathcal{H}$. By unfolding the inner product

$$\langle FP_h(\cdot|\psi), F\rangle = \int_{\Gamma\backslash\mathcal{H}} y^\kappa |f(z)|^2 P_h(z|\psi)\, d_\mu z$$

on the Poincaré series, we find that

$$\langle FP_h(\cdot|\psi), F\rangle = \sum_{n=1}^{\infty} \lambda_f(n)\lambda_f(n+h)(n(n+h))^{\frac{\kappa-1}{2}} \int_0^\infty \psi(y) e^{-4\pi(n+h)y} y^{\kappa-2}\, dy,$$

i.e. this inner product is a smoothed shifted convolution sum with cut-off function given in terms of an integral transform (similar to the Laplace transform) of ψ. The behavior of $\psi(y)$ as y tends to 0 is crucial to control the length of the shifted convolution sum. In connection with the previous section, it should be noted that if ψ were a delta function, then taking the inner product against $P_h(z|\psi)$ is equivalent to taking the h-th Fourier coefficient.

Let u_j be a complete orthonormal system of cusp forms which are eigenfunctions of the Laplace operator and all Hecke operators. Because we are only working in level 1, we need not worry about old forms or the Hecke operators whose index divides the level. Define the real analytic Eisenstein series by

$$E(z,s) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma z)^s$$

for $\mathrm{Re}(s) > 1$, and in general by analytic continuation. For each $s \neq 0, 1$, the Eisenstein series are also eigenfunctions for the Laplace operator and all the Hecke operators. We have then that

$$\sum_{j=1}^{\infty} \langle P_h(\cdot|\psi), u_j \rangle u_j(z) + \frac{1}{4\pi} \int_{-\infty}^{\infty} \langle P_h(\cdot|\psi), E(\cdot, 1/2+it) \rangle E(z, 1/2+it)\, dt$$

converges to $P_h(z|\psi)$ in the norm topology on $L^2(\Gamma\backslash\mathcal{H})$, see Theorems 4.7 and 7.3 in [Iwa02]. Then we have that

$$\begin{aligned} &\langle F P_h(\cdot|\psi), F \rangle \\ &= \sum_{j=1}^{\infty} \langle P_h(\cdot|\psi), u_j \rangle \langle F u_j, F \rangle + \frac{1}{4\pi} \int_{-\infty}^{\infty} \langle P_h(\cdot|\psi), E(\cdot, 1/2+it) \rangle \langle F E(\cdot, 1/2+it), F \rangle\, dt, \end{aligned} \tag{4.7}$$

and we will see later that the convergence is absolute and uniform in h.

If $u \in L^2(\Gamma\backslash\mathcal{H})$ and $\Delta u = \lambda u$ with $\lambda = s(1-s)$, then $u(z)$ has a Fourier expansion given in terms of the K-Bessel function $K_\nu(z)$, which is an exponentially decaying solution to the differential equation

$$z^2 f'' + zf' - \left(z^2 + \nu^2\right) f = 0.$$

We will primarily be interested in the K-Bessel function for purely imaginary ν, and note some useful properties of these functions: first, that $K_\nu(z)$ is real for real z, second, that as $\mathrm{Im}(\nu) \to \infty$ in a fixed vertical strip, $K_\nu(z)$ is decaying exponentially, and last, that for $\nu = it$, with $t \in \mathbb{R}$, $K_{it}(z)$ has a branch cut, which we take to be along the negative real axis in the z-plane. As $z \to 0$, we have that

$$|K_{it}(z)| \sim \pi \left| \frac{\sin(t \log z/2)}{\Gamma(1+it)\sinh(\pi t)} \right|$$

so long as one avoids the branch cut.

4.3 Proof of Theorem

We now proceed to the proof of the Main Theorem of the chapter by summing the right side of (4.7) over h. Pointwise, the largest term comes from the discrete spectrum (see [Sar01]), however, on average, the continuous spectrum dominates. We start with the continuous spectrum contribution to (4.7).

4.3.1 Eisenstein Series

The Eisenstein series $E(z,s)$ has a Fourier expansion given by

$$E(z,s) = \varphi(0,s) + \sum_{n \neq 0} \varphi(n,s) W_s(nz),$$

where

$$W_s(z) = 2y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi y) e(x).$$

If $n = 0$ and $s \neq 1/2$ then

$$\varphi(0, s) = y^s + \frac{\xi(2s-1)}{\xi(2s)} y^{1-s},$$

and if $n \neq 0$, the n-th Fourier coefficient of $E(z, s)$ is given by

$$\varphi(n, s) = \xi(2s)^{-1} |n|^{-1/2} \sum_{ab=|n|} \left(\frac{a}{b}\right)^{s-1/2},$$

where $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ denotes the completed Riemann zeta function which has the functional equation $\xi(s) = \xi(1-s)$. If $s = 1/2 + it$, we find by unfolding the Poincaré series that the inner product

$$\langle P_h(\cdot|\psi), E(\cdot, 1/2+it)\rangle = \frac{2}{\xi(2s-1)} \sum_{ab=h} \left(\frac{a}{b}\right)^{s-\frac{1}{2}} \int_0^\infty \frac{\psi(y)}{y^{3/2}} e^{-2\pi h y} K_{s-\frac{1}{2}}(2\pi h y)\, dy$$

where we have used that $\overline{\xi(2s)} = \xi(2\bar{s}) = \xi(2-2s) = \xi(2s-1)$. We can also unfold the second inner product on the Eisenstein series. Following Iwaniec [Iwa97] Chapter 13, set

$$L(s, f \times f) := \zeta(s) L(s, \mathrm{sym}^2 f) = \zeta(2s) L(s, f \otimes f) = \zeta(2s) \sum_{n \geq 1} \frac{\lambda_f(n)^2}{n^s},$$

where the last equality is valid only for $\mathrm{Re}(s) > 1$. This L-function admits the functional equation

$$\Lambda(s, f \times f) = \Lambda(1-s, f \times f),$$

where

$$\Lambda(s, f \times f) = L_\infty(s, f \times f) L(s, f \times f)$$

with

$$L_\infty(s, f \times f) := (2\pi)^{-2s}\Gamma(s)\Gamma(s+\kappa-1).$$

By unfolding when $\mathrm{Re}(s) > 1$, and in general after analytic continuation,

$$\langle FE(\cdot, s), F\rangle = \frac{\Lambda(s, f \times f)}{(4\pi)^{\kappa-1}\xi(2s)}.$$

Going back to the spectral expansion (4.7) and pulling these two inner products together, we have that the Eisenstein series contribution to $\langle FP_h(\cdot|\psi), F\rangle$ is

$$E_{f,h}(\psi) := \frac{1}{(4\pi)^{\kappa-1}}\frac{1}{2\pi i}\int_{(\frac{1}{2})}\frac{\Lambda(s, f\times f)}{\xi(2s)\xi(2s-1)}\sum_{ab=h}\left(\frac{a}{b}\right)^{s-\frac{1}{2}}\int_0^\infty \frac{\psi(y)}{y^{3/2}}e^{-2\pi hy} \times K_{s-\frac{1}{2}}(2\pi hy)\,dy\,ds.$$

The completed L and zeta functions have the same number of gamma factors in the numerator and denominator, and Bessel function decays rapidly as $|\,\mathrm{Im}(s)| \to \infty$, so the contour integral converges rapidly. It is interesting to note that the s integral only makes sense because $\xi(s)^{-1}$ has no poles and at most polynomial growth on the $\mathrm{Re}(s) = 1$ line, i.e. due to the prime number theorem. Indeed, one sees that the contour is constrained between the poles in the critical strips of the two $\xi(s)^{-1}$ functions appearing here. Due to the mysterious nature of the residues at these poles, shifting contours seems to be a futile approach to understanding the asymptotic size of $E_{f,h}(\psi)$.

Introducing a sum over h clears this obstruction. We have to compute

$$\sum_{h=1}^{Y} E_{f,h}(\psi) = \frac{1}{(4\pi)^{\kappa-1}}\int_0^\infty \frac{\psi(y)}{y^{\frac{3}{2}}}\frac{1}{2\pi i}\int_{(\frac{1}{2})}\frac{\Lambda(s, f\times f)}{\xi(2s)\xi(2s-1)}\sum_{h=1}^{Y}\sum_{ab=h}\left(\frac{a}{b}\right)^{s-\frac{1}{2}} e^{-2\pi yh} \times K_{s-\frac{1}{2}}(2\pi yh)\,ds\,dy, \tag{4.8}$$

and can evaluate the sum over h by standard techniques. We have by adapting Theorem 12.4 from [THB86] (due to Van Der Corput) that

$$\sum_{h\le x}\sum_{ab=h}\left(\frac{a}{b}\right)^{s-\frac{1}{2}}=\zeta(2s)\frac{x^{s+\frac{1}{2}}}{s+\frac{1}{2}}+\zeta(2-2s)\frac{x^{-s+\frac{3}{2}}}{-s+\frac{3}{2}}+O_{s,\varepsilon}\left(x^{\frac{27}{82}+|\operatorname{Re}(s)-\frac{1}{2}|+\varepsilon}\right),$$

where the implied constants depend at most polynomially on $|s|$. It should be noted that the error term here is not the best currently known, however, it is sufficiently small as to not contribute to the final result of this chapter. By partial summation,

$$\begin{aligned}
&\sum_{h=1}^{Y}\sum_{ab=h}\left(\frac{a}{b}\right)^{s-\frac{1}{2}}e^{-2\pi yh}K_{s-\frac{1}{2}}(2\pi yh)\\
=\ &\zeta(2s)\int_0^Y u^{s-\frac{1}{2}}e^{-2\pi yu}K_{s-\frac{1}{2}}(2\pi yu)\,du+\zeta(2-2s)\int_0^Y u^{\frac{1}{2}-s}e^{-2\pi yu}K_{s-\frac{1}{2}}(2\pi yu)\,du\\
+\ &O_{s,\varepsilon}\left(e^{-2\pi yY}K_{s-\frac{1}{2}}(2\pi yY)Y^{\frac{27}{82}+|\operatorname{Re}(s)-\frac{1}{2}|+\varepsilon}\right).
\end{aligned}$$

Note that the two integrals appearing in the displayed equation are interchanged under the transformation $s\longleftrightarrow 1-s$, by symmetry of the Bessel function. The contour integral over s in $\sum E_{f,h}(\psi)$ is also symmetric under $s\longleftrightarrow 1-s$ so that these two integrals are identical in the overall sum, and we need only work one of them out. The first integral can be evaluated explicitly, and the answer will be in terms of hypergeometric functions. The confluent hypergeometric function that will appear below is defined by the power series

$${}_1F_1(a,b,z)=\sum_{n=0}^{\infty}\frac{a^{(n)}z^n}{b^{(n)}n!}$$

where

$$a^{(n)}=a(a+1)(a+2)\cdots(a+n-1)=\frac{\Gamma(a+n)}{\Gamma(a)}$$

is called either the 'rising factorial' or 'Pochhammer symbol'. The theory of hypergeometric functions is developed in detail in [Erd53]. It is entire on $\mathbb{C}$ separately in each variable except for simple poles at $b = 0, -1, -2, \ldots$ by the absolute and uniform convergence of the defining series, hence it is meromorphic on $\mathbb{C}^3$. We have that the residues at these poles are given by

$$\operatorname*{Res}_{b=-n} {}_1F_1(a,b,z) = \frac{\Gamma(a+n+1)(-1)^n}{\Gamma(a)\Gamma(n+2)\Gamma(n+1)} z^{n+1} {}_1F_1(a+n+1, n+2, z),$$

see Gradshteyn and Ryzhik [GR00], 9.214.

We have the formulae

$$K_\nu(z) = \frac{\pi}{2} \frac{i^\nu J_{-\nu}(iz) - i^{-\nu} J_\nu(iz)}{\sin \pi\nu},$$

$$e^{-z} \frac{\Gamma\left(s+\frac{1}{2}\right)}{\left(\frac{iz}{2}\right)^{s-\frac{1}{2}}} J_{s-\frac{1}{2}}(iz) = {}_1F_1(s, 2s, -2z),$$

$$\frac{d}{dz} {}_1F_1(a,b,z) = \frac{a}{b} {}_1F_1(a+1, b+1, z),$$

and

$$\frac{d}{dz}\left(z^{b-1} {}_1F_1(a,b,z)\right) = (b-1) z^{b-2} {}_1F_1(a, b-1, z),$$

where $J_\nu(x)$ is the J-Bessel function, the second formula can be found in [GR00], 9.215 #3 and the last two formulae can be found in [Erd53] section 2.1.2. From these it follows that

$$\int_0^Y u^{s-\frac{1}{2}} e^{-2\pi y u} K_{s-\frac{1}{2}}(2\pi y u)\, du = \frac{1}{4(\pi y)^{\frac{1}{2}} s} \left(A_f(y,s) + B_f(y,Y,s) + C_f(y,Y,s)\right),$$

where

$$A_f = A_f(y,s) = (\pi y)^{-s} \Gamma(1/2 + s)$$

$$B_f = B_f(y, Y, s) = (\pi y)^{-s}\Gamma(1/2+s)_1F_1(-s, 1-2s, -4\pi yY)$$

and

$$C_f = C_f(y, Y, s) = (\pi y Y^2)^{s}\Gamma(1/2-s)_1F_1(s, 1+2s, -4\pi yY).$$

The $\zeta(2s)$ from the evaluation of the sum over h cancels against the $\zeta(2s)$ in the denominator of (4.8), eliminating its poles. Given that $A_f + B_f + C_f$ is holomorphic in s past the $\mathrm{Re}(s) = 0$ line, we are now free to pass the contour to the left. Picking up a residue at $s = 0$ we get

$$\begin{aligned}\sum_{h=1}^{Y} E_{f,h}(\psi) = &-\frac{\Gamma(\kappa)}{(4\pi)^{\kappa}}\frac{L(1,\mathrm{sym}^2 f)}{2\zeta(2)}\int_0^\infty \frac{\psi(y)}{y^2}\,dy + \frac{1}{(4\pi)^{\kappa-\frac{1}{2}}}\int_0^\infty \frac{\psi(y)}{y^2} \\ &\times \frac{1}{2\pi i}\int_{(-a)} \frac{\Lambda(s, f\times f)}{\xi(2s-1)}\frac{(A_f(y,s)+B_f(y,Y,s)+C_f(y,Y,s))}{\pi^{-s}\Gamma(s+1)}\,ds\,dy \\ &+O_{\kappa,\varepsilon}\left(Y^{\frac{27}{82}+\varepsilon}\int_0^\infty \frac{|\psi(y)|}{y^{\frac{3}{2}}}\,dy\right),\end{aligned}$$

where $0 < a < 1/2$, and we have again made use of the prime number theorem in estimating the error term.

The integrals over y converge at ∞, so the asymptotic size of $\sum E_{f,h}(\psi)$ depends only on the behavior of $\psi(y)$ for small y. We now make some estimates assuming that y is small. The remaining contour integral

$$\frac{1}{2\pi i}\int_{(-a)} \frac{\Lambda(s, f\times f)}{\xi(2s-1)}\frac{(A_f(y,s)+B_f(y,Y,s)+C_f(y,Y,s))}{\pi^{-s}\Gamma(s+1)}\,ds$$

is a sum of three terms coming from A_f, B_f and C_f. The term coming from A_f is $\ll_\kappa y^{\frac{1}{2}}$ as $y \to 0$. The hypergeometric function appearing in B_f is bounded above and below by universal constants in the half-plane $\mathrm{Re}(s) < 0$ and when $0 \le yY \le 4\pi$, thus term coming from B_f is also $\ll y^{\frac{1}{2}}$, uniformly in Y. Thus it remains to inspect the term coming from C_f. Explicitly, let

$$\begin{aligned} C_{f,1}(y,Y) &:= \frac{1}{2\pi i}\int_{(-a)} \frac{\Lambda(s, f\times f)}{\xi(2s-1)} \frac{C_f(y,Y,s)}{\pi^{-s}\Gamma(s+1)}\, ds \\ &= \frac{1}{2\pi i}\int_{(-a)} \frac{\Lambda(s, f\times f)}{\xi(2s-1)} \frac{\Gamma\left(\frac{1}{2}-s\right)}{\Gamma(s+1)} {}_1F_1(s, 1+2s, -4\pi yY)\left(\pi^2 yY^2\right)^s\, ds. \end{aligned}$$

Thus

$$\begin{aligned} \sum_{h=1}^{Y} E_{f,h}(\psi) &= -\frac{\Gamma(\kappa)}{(4\pi)^\kappa}\frac{L(1,\mathrm{sym}^2 f)}{2\zeta(2)}\int_0^\infty \frac{\psi(y)}{y^2}\, dy + \frac{1}{(4\pi)^{\kappa-\frac{1}{2}}}\int_0^\infty \frac{\psi(y)}{y^2} C_{f,1}(y,Y)\, dy \\ &+ O_{\kappa,\varepsilon}\left(Y^{\frac{27}{82}+\varepsilon}\int_0^\infty \frac{|\psi(y)|}{y^{\frac{3}{2}}}\, dy\right). \end{aligned}$$

In the transition region, $C_{f,1}(y,Y)$ is the crucial term.

Lemma 11 *Suppose that $y, Y \in \mathbb{R}_{>0}$ with Y becoming large and y becoming small. Then*

$$C_{f,1}(y,Y) = \frac{1}{2\sqrt{\pi}} c_f\left(4\pi yY^2\right) + O_\kappa\left(y^{\frac{1}{2}}\right),$$

where

$$c_f(\alpha) = \frac{\pi^{\frac{3}{2}}}{2}\alpha\sum_{n\geq 1}\lambda_f(n)^2 W_\kappa(\pi^2 n\alpha),$$

and

$$W_\kappa(x) = \frac{1}{2\pi i}\int_{(1+a)} \frac{\Gamma(s+\kappa-1)\Gamma(s-\frac{1}{2})}{\Gamma(2-s)} x^{-s}\, ds$$

for any fixed $a > 0$.

Proof. We apply the functional equation for the L-function to find that

$$\begin{aligned} C_{f,1}(y,Y) = \pi^2 yY^2 \frac{1}{2\pi i}\int_{(1+a)} &\frac{L(s, f\times f)}{\zeta(2s)} \frac{\Gamma(s+\kappa-1)\Gamma(s-\frac{1}{2})}{\Gamma(2-s)} \\ &\times {}_1F_1(1-s, 3-2s, -4\pi yY)\left(4\pi^3 yY^2\right)^{-s}\, ds. \end{aligned}$$

From the definition one finds that ${}_1F_1(1-s, 3-2s, u) = 1 + O_s(u)$, which we use to eliminate the hypergeometric function from the above expression. We proceed in two slightly different ways depending on whether yY^2 becomes large or becomes small. If yY^2 remains bounded, either approach is acceptable. First, assume that yY^2 is becoming small. In this case, choose $a = 1/4$, and observe that the s-dependence in the hypergeometric function is uniformly bounded along the line $\mathrm{Re}(s) = 5/4$. Together with the rapid decay of the integrand of $C_{f,1}(y, Y)$, this gives us that

$$\left| C_{f,1}(y, Y) - \frac{1}{2\sqrt{\pi}} c_f \left(4\pi y Y^2\right) \right| \ll_\kappa y^{\frac{1}{2}} (yY^2)^{\frac{1}{4}}.$$

If yY^2 becomes large shift the line of integration to the right, past the pole of the hypergeometric function at $s = 3/2$ to $a = 3/4$. The contribution to $C_{f,1}(y, Y)$ coming from this residue is $\ll_\kappa y^{\frac{1}{2}}$, uniformly in Y, and the s-dependence in the hypergeometric function is uniformly bounded along the line $\mathrm{Re}(s) = 7/4$. We find in this case that

$$\left| C_{f,1}(y, Y) - \frac{1}{2\sqrt{\pi}} c_f \left(4\pi y Y^2\right) \right| \ll_\kappa y^{\frac{1}{2}} \left(1 + (yY^2)^{-\frac{1}{4}}\right).$$

In either case, we obtain the error term stated in the Lemma. □

Lemma 12 *The function $c_f(\alpha)$ defined above is $C^\infty(\mathbb{R}_{>0})$. As $\alpha \to \infty$ it decays faster than any polynomial, and as $\alpha \to 0$*

$$c_f(\alpha) = \frac{\Gamma(\kappa) L(1, \mathrm{sym}^2 f)}{2\zeta(2)} + E_\kappa(\alpha)$$

where $E_\kappa(\alpha) = o_\kappa\left(\alpha^{\frac{1}{2}}\right)$ unconditionally, and $E_\kappa(\alpha) = O_{\kappa,\varepsilon}\left(\alpha^{\frac{3}{4}-\varepsilon}\right)$ for any $\varepsilon > 0$ assuming the Riemann hypothesis for the classical Riemann zeta function.

Proof. The $W_\kappa(x)$ defined above is $C^\infty(\mathbb{R})$ and its integrand has no poles to the right, thus $W_\kappa(x)$ is rapidly decaying as $x \to +\infty$. After differentiating the series for $c_f(\alpha)$ in the second line of Lemma 1 arbitrarily many times, the resulting series for $c_f^{(n)}(\alpha)$ converges absolutely for any $\alpha > 0$, so $c_f \in C^\infty(\mathbb{R}_{>0})$. The rapid decay of $c_f(\alpha)$ follows from that of $W_\kappa(x)$. By shifting the line of integration in the definition of $c_f(\alpha)$ to the left, we investigate the behavior of $c_f(\alpha)$ near $\alpha = 0$. The main term comes from the residue of the pole of $L(s, f \times f)$ at $s = 1$, and the error term is estimated by pushing the contour just past the $\mathrm{Re}(s) = 1/2$ line, or to the $\mathrm{Re}(s) = 1/4 + \varepsilon$ line if one assumes the Riemann hypothesis. $\square$

Thus we have proved the following Proposition.

Proposition 5 *For $E_{f,h}(\psi)$ and $c_f(\alpha)$ as defined above, we have that*

$$\sum_{h=1}^{Y} E_{f,h}(\psi) = \frac{1}{(4\pi)^\kappa} \int_0^\infty \left(c_f\left(4\pi y Y^2\right) - \frac{\Gamma(\kappa) L(1, \mathrm{sym}^2 f)}{2\zeta(2)} \right) \frac{\psi(y)}{y^2}\, dy$$
$$+ O_{\kappa,\varepsilon}\left(Y^{\frac{27}{82}+\varepsilon} \int_0^\infty \frac{\psi(y)}{y^{\frac{3}{2}}}\, dy \right).$$

The term involving $27/82$ is smaller than the remainder terms coming from Maass forms, as we will see in the next section.

4.3.2 Maass Forms

The discrete spectrum of Δ is spanned by Maass cusp forms. The Hecke algebra acting on $L^2(\Gamma\backslash\mathcal{H})$ is defined to be the algebra generated by the commuting self-adjoint bounded operators T_n, where for $u \in L^2(\Gamma\backslash\mathcal{H})$, define T_n by

$$(T_n u)(z) := \frac{1}{\sqrt{n}} \sum_{ad=n} \sum_{b \,(\mathrm{mod}\ d)} u\left(\frac{az+b}{d}\right).$$

These operators commute with Δ as well, so in fact our basis of Maass forms can be taken to be eigenfunctions of the Hecke algebra as well, and we denote the Hecke eigenvalues of the Maass form u_j by $\lambda_{u_j}(n)$. A Maass form of Laplace eigenvalue $\lambda_j = s_j(1-s_j) = 1/4 + t_j^2$ is cuspidal, so it has a Fourier expansion of the form

$$u_j(z) = \sum_{n \neq 0} a_{u_j}(n) W_{s_j}(nz),$$

where

$$W_{s_j}(z) = 2y^{\frac{1}{2}} K_{it_j}(2\pi y) e(x).$$

For $\Gamma = SL_2(\mathbb{Z})$, it was known to Selberg in the early 50s that the smallest Laplace eigenvalue λ_1 is $> 1/4$, hence $t_j \in \mathbb{R}$. For a proof of this fact, see [Hej83], chapter 11. Computationally, it has been verified that $t_1 = 9.53369526\ldots$, see for example [Hej91]. To apply the spectral theorem we must assume the normalization $||u_j||_{L^2}^2 = 1$, in which case the Fourier coefficient and Hecke eigenvalue are related by

$$a_{u_j}(n) = \left(\frac{\cosh \pi t_j}{2|n| L(1, \mathrm{sym}^2 u_j)} \right)^{\frac{1}{2}} \lambda_{u_j}(n),$$

where the symmetric square L-function appearing here is defined

$$L(s, \mathrm{sym}^2 u_j) = \prod_p \left(1 - \frac{\alpha_{u_j}(p)^2}{p^s}\right)^{-1} \left(1 - \frac{\alpha_{u_j}(p)\beta_{u_j}(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_{u_j}(p)^2}{p^s}\right)^{-1}.$$

The Hecke eigenvalues conform to the bound $|\lambda_{u_j}(n)| \leq d(n) n^{\theta}$, where the generalized Ramanujan Conjecture implies that $\theta = 0$ is admissible, and the best known unconditional bound is due to Kim and Sarnak [Kim03], which gives $\theta = 7/64$.

We call the Maass form contribution to (4.7)

$$M_{f,h}(\psi) := \sum_{j=1}^{\infty} \langle P_h(\cdot|\psi), u_j\rangle \langle F u_j, F\rangle.$$

By unfolding we have

$$\langle P_h(\cdot|\psi), u_j\rangle = \left(\frac{\cosh \pi t_j}{L(1, \mathrm{sym}^2 u_j)}\right)^{\frac{1}{2}} \lambda_{u_j}(h) \int_0^\infty \frac{\psi(y)}{y^{\frac{3}{2}}} e^{-2\pi h y} K_{it_j}(2\pi h y)\, dy$$

so that

$$\sum_{h=1}^{Y} M_{f,h}(\psi) = \int_0^\infty \frac{\psi(y)}{y^{\frac{3}{2}}} \sum_{j=1}^{\infty} \left(\frac{\cosh \pi t_j}{L(1, \mathrm{sym}^2 u_j)}\right)^{\frac{1}{2}} \langle F u_j, F\rangle \sum_{h=1}^{Y} \lambda_{u_j}(h) e^{-2\pi h y} \times K_{it_j}(2\pi h y)\, dy.$$

Lemma 13 *The spectral sum*

$$\sum_{j=1}^{\infty} \left(\frac{\cosh \pi t_j}{L(1, \mathrm{sym}^2 u_j)}\right)^{\frac{1}{2}} \langle F u_j, F\rangle \sum_{h=1}^{Y} \lambda_{u_j}(h) e^{-2\pi h y} K_{it_j}(2\pi h y)$$

appearing in $\sum M_{f,h}(\psi)$ converges absolutely.

Proof. There are three factors in the summand: that which involves $\cosh \pi t_j$, the inner product, and the sum over h. We show that the first of these two balance each other, and then show that the sum over h decays rapidly in $|t_j|$, uniformly in the other variables. To study $|\langle F u_j, F\rangle|$ we will use a beautiful formula of Watson [Wat01], but follow a classical work-out of it from Soundararajan's paper [Sou10]. In that paper both f and u_j are normalized to have mass 1, but in this chapter we take f to be Hecke normalized so that Watson's formula is

$$|\langle F u_j, F\rangle|^2 = \frac{1}{8}\left(\frac{\Gamma(\kappa)}{(4\pi)^\kappa} \frac{\mathrm{Vol}\,(\Gamma\backslash\mathcal{H})}{\zeta(2)}\right)^2 \frac{\Lambda(\frac{1}{2}, f\times f\times u_j)}{L_\infty(1, \mathrm{sym}^2 f)^2 \Lambda(1, \mathrm{sym}^2 u_j)},$$

where

$$\Lambda(s, f\times f\times u_j) = L_\infty(s, f\times f\times u_j)L(s, f\times f\times u_j),$$

$$L_\infty(s,\mathrm{sym}^2 f) = \pi^{-\frac{3}{2}s}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s+\kappa-1}{2}\right)\Gamma\left(\frac{s+\kappa}{2}\right),$$

$$\Lambda(s,\mathrm{sym}^2 u_j) = L_\infty(s,\mathrm{sym}^2 u_j)L(s,\mathrm{sym}^2 u_j),$$

$$\begin{aligned} &L_\infty(s, f\times f\times u_j)\\ &\quad= \pi^{-4s}\prod_\pm \Gamma\left(\frac{s+\kappa-1\pm it_j}{2}\right)\Gamma\left(\frac{s+\kappa\pm it_j}{2}\right)\Gamma\left(\frac{s+1\pm it_j}{2}\right)\Gamma\left(\frac{s\pm it_j}{2}\right),\end{aligned}$$

$$L_\infty(s,\mathrm{sym}^2 u_j) = \pi^{-3s/2}\Gamma\left(\frac{s-2it}{2}\right)\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+2it}{2}\right)$$

and

$$L(s, f\times f\times u_j) = \prod_p \left(1-\frac{\alpha_f(p)^2\alpha_{u_j}(p)}{p^s}\right)^{-1}\left(1-\frac{\alpha_{u_j}(p)}{p^s}\right)^{-2}\left(1-\frac{\beta_f(p)^2\alpha_{u_j}(p)}{p^s}\right)^{-1}$$

$$\times\left(1-\frac{\alpha_f(p)^2\beta_{u_j}(p)}{p^s}\right)^{-1}\left(1-\frac{\beta_{u_j}(p)}{p^s}\right)^{-2}\left(1-\frac{\beta_f(p)^2\beta_{u_j}(p)}{p^s}\right)^{-1}.$$

The archimedian parts evaluate to

$$\frac{L_\infty(\frac{1}{2}, f\times f\times u_j)}{L_\infty(1,\mathrm{sym}^2 f)^2 L_\infty(1,\mathrm{sym}^2 u_j)} = 4\pi^2\frac{|\Gamma(\kappa-\frac{1}{2}+it_j)|^2}{\Gamma(\kappa)^2}$$

after repeated application of the duplication formula. Using this, Watson's formula simplifies to

$$|\langle Fu_j, F\rangle| = \sqrt{2}\frac{|\Gamma(\kappa-\frac{1}{2}+it_j)|}{(4\pi)^\kappa}\left(\frac{L(\frac{1}{2}, f\times f\times u_j)}{L(1,\mathrm{sym}^2 u_j)}\right)^{\frac{1}{2}} \tag{4.9}$$

for f Hecke normalized, and u_j mass 1 normalized. By applying Stirling's formula, we find that $|\Gamma(\kappa-1/2+it_j)|(\cosh\pi t_j)^{\frac{1}{2}}$ is polynomially bounded as $|t_j|\to\infty$. Together with standard convexity bounds in the $|t_j|$ aspect for $L(\frac{1}{2}, f \times f \times u_j)$, we find that

$$\left| \frac{\cosh \pi t_j}{L(1, \mathrm{sym}^2 u_j)} \right|^{\frac{1}{2}} |\langle F u_j, F \rangle|$$

is polynomially bounded as $|t_j|$ gets large.

Now we turn to the sum over h. We have by a "folklore" result written down by Hafner and Ivić [HI89] that

$$\sum_{h=1}^{Y} \lambda_{u_j}(h) \ll_{u_j} Y^{\frac{1}{3}(1+\theta)}$$

where $\theta = 0$ or $= 7/64$ as above, and where the implied constants depend at most polynomially on $|t_j|$. As on the Eisenstein series side, the conjectural truth is $O_{u,\varepsilon}(Y^{\frac{1}{4}+\varepsilon})$, but this seems very difficult. By partial summation

$$\sum_{h=1}^{Y} \lambda_u(h) e^{-2\pi h y} K_{it_j}(2\pi h y) \ll_{u_j} e^{-2\pi y Y} K_{it_j}(2\pi y Y) Y^{\frac{1}{3}(1+\theta)} + \int_{\frac{1}{2}}^{Y} u^{\frac{1}{3}(1+\theta)} \left| \frac{\partial}{\partial u} e^{-2\pi y u} K_{it_j}(2\pi y u) \right| du,$$

where the implied constants again depend at most polynomially on $|t_j|$. We have that $|K_{it}(u)| \sim \pi \frac{|\sin(t \log u/2)|}{|\Gamma(1+it)\sinh(\pi t)|}$ for small u, thus after taking derivatives, changing variables, and using the power series expansion for the lower incomplete gamma function, we have that

$$\sum_{h=1}^{Y} \lambda_u(h) e^{-2\pi h y} K_{it_j}(2\pi h y) \leq P(t_j) e^{-\frac{\pi}{2}|t_j|} Y^{\frac{1}{3}(1+\theta)}, \tag{4.10}$$

uniformly in y, where $P(t)$ is a real-valued function on $\mathbb{R}$ that grows at most polynomially as $|t|$ becomes large. From these estimates together with Weyl's law (see, e.g.

Iwaniec [Iwa02])

$$\sum_{|t_j|\le T} 1 = \frac{\mathrm{Vol}\,(\Gamma\backslash\mathcal{H})}{4\pi} T^2 + O_\varepsilon\left((1+T)^{1+\varepsilon}\right)$$

the Lemma follows. $\square$

We apply trivial estimates along with (4.10) to obtain the following proposition.

Proposition 6 *For $M_{f,h}(\psi)$ defined above we have*

$$\sum_{h=1}^{Y} M_{f,h}(\psi) \ll_\kappa Y^{\frac{1}{3}(1+\theta)} \int_0^\infty \frac{|\psi(y)|}{y^{\frac{3}{2}}}\, dy.$$

The application of trivial bounds is justified by the absolute convergence given by Lemma 3.

Drawing together the Propositions from the two preceding sections, we obtain Theorem 5.

4.4 Proof of Corollaries

Now we make some choices for ψ, and record the results as corollaries. First, we give the result stated in the introduction.

Proof of Corollary 3. Let ψ be a smooth approximation to a point mass. Specifically, let ψ be smooth, non-negative, supported on a set of radius X^{-4} about the point $y = (4\pi X)^{-1}$ and have mass 1. Then for any continuously differentiable function ϕ on $\mathbb{R}_{>0}$, we have

$$\left|\phi\left(\frac{1}{4\pi X}\right) - \int_0^\infty \psi(y)\phi(y)\, dy\right| \ll \left|\phi'\left(\frac{1}{4\pi X}\right)\right| X^{-4},$$

where the implied constants are absolute. First, let $\phi(u) = u^{\kappa-2}e^{-4\pi(n+h)u}$, so that

$\left|\phi'\left(\frac{1}{4\pi X}\right)\right| \ll_\kappa (n+h)X^{3-\kappa}e^{-\frac{n+h}{X}}$, and thus we find that

$$\left|S_f(\psi, Y) - \sum_{h=1}^{Y}\sum_{n\geq 1}\lambda_f(n)\lambda_f(n+h)\frac{(n(n+h))^{\frac{\kappa-1}{2}}}{X^{\kappa-2}}e^{-\frac{n+h}{X}}\right| \ll_{\kappa,\varepsilon} X^{1+\varepsilon},$$

hence the difference between the left hand sides of the Main Theorem and Corollary 3 is $\ll_\varepsilon X^\varepsilon$. Secondly, let

$$\phi(u) = \left(c_f(4\pi u Y^2) - \frac{\Gamma(\kappa)L(1, \mathrm{sym}^2 f)}{2\zeta(2)}\right)\frac{1}{u^2}.$$

From the definition of $c_f(\alpha)$ as a sum one sees that $|c_f'(\alpha)| \ll_f \alpha^{-1}$ as $\alpha \to 0$, so that $\left|\phi'\left(\frac{1}{4\pi X}\right)\right| \ll_f X^3$. The error term can be treated similarly. Hence, the difference between the right hand side of Theorem 5 and the right hand side of Corollary 3 is $\ll_f X^{-2}$ with this choice of ψ. □

Let

$$\Gamma(s, x) = \int_x^\infty e^{-t}t^s\,\frac{dt}{t}$$

denote the incomplete gamma function.

Corollary 4 *Let*

$$\Sigma_f(X, Y) := \sum_{h=1}^{Y}\sum_{n\geq 1}\lambda_f(n)\lambda_f(n+h)\left(1 - \frac{h}{n+h}\right)^{\frac{\kappa-1}{2}}\frac{\Gamma(\kappa-1, (\kappa-1)\frac{n+h}{X})}{\Gamma(\kappa-1)}.$$

Then

$$\Sigma_f(X, Y) = -\frac{L(1, \mathrm{sym}^2 f)}{2\zeta(2)}X + \frac{Y^2}{\Gamma(\kappa-1)}\int_{(\kappa-1)\frac{Y^2}{X}}^\infty \frac{c_f(u)}{u^2}\,du + O_\kappa\left(X^{\frac{1}{2}}Y^{\frac{1}{3}(1+\theta)}\right).$$

The integral appearing here either cancels the main term if Y^2/X approaches 0, or decays rapidly as a function of Y^2/X if this parameter grows without bound.

Proof. Let $\mathbb{1}_{>x}$ denote the indicator function of the open set $(x, \infty) \subset \mathbb{R}$. In similar fashion to the previous proof, let $\psi(y)$ be a smooth approximation to

$$\frac{1}{4\pi\Gamma(\kappa-1)}\mathbb{1}_{>\frac{\kappa-1}{4\pi X}}(y),$$

and compute the answer. $\square$

The spectral theorem as used in this chapter holds for L^2 functions which are also C^∞ and bounded, however, it should also hold for a much wider class of functions, and thus Theorem 5 should in fact hold for a much wider class of functions. In practice however, one may always obtain a corollary for a specific choice of ψ by elementary arguments similar to the proof of Corollary 3, so we do not pursue the problem of expanding the class of functions for which Theorem 5 holds.

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