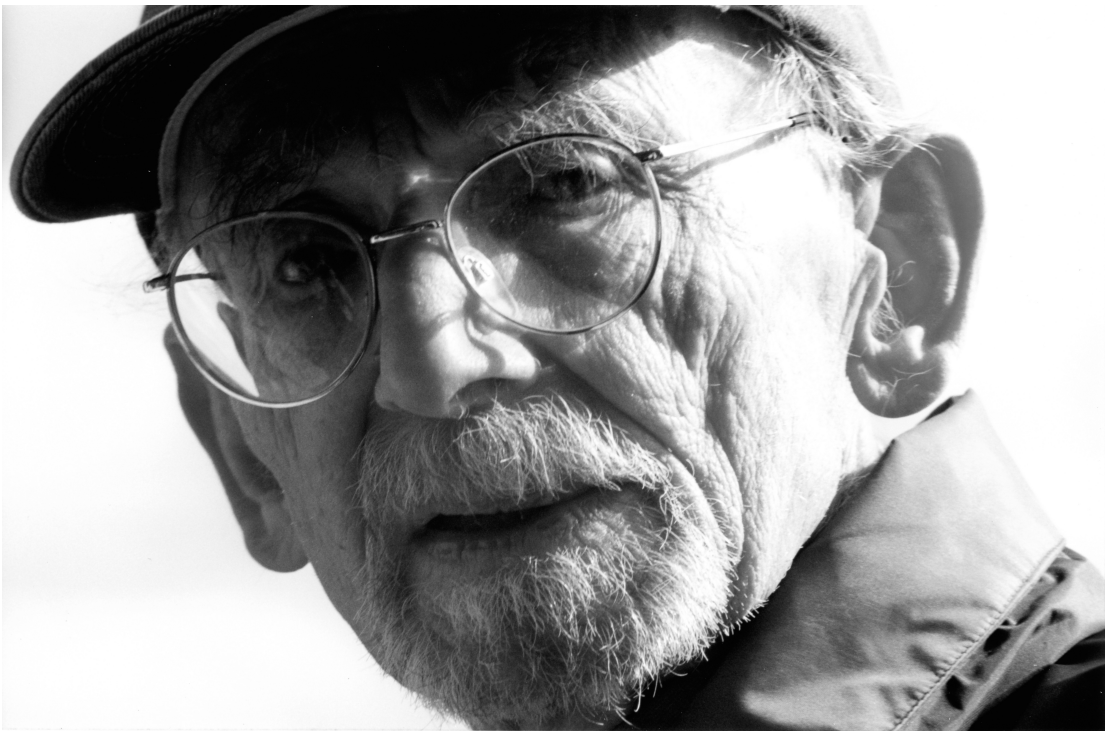


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Elected FRS 1980

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Gerard Friedlander was the son of Austrian communist intellectuals, who divorced when he was four. From the age of two he was raised by grandparents in Vienna, while his mother lived in Berlin as a communist organizer. Hitler came to power in 1933; Friedlander was sent to England, aged sixteen, in 1934; two years later, he won a scholarship to Trinity College, Cambridge. By 1940 he was a fully fledged applied mathematician who came to embrace both the European and British traditions of that subject. His work was marked by profound originality, by the importance of its applications and by the mathematical rigour of his treatment. The applications of his work changed over the years. The first papers (written in 1939-1941 but published only in 1946 for security reasons) were a contribution to Civil Defence: they presented entirely new and explicit results on the shielding effect of a wall from a distant bomb blast. The late papers were contributions to the general, more abstract theory of partial differential equations, but, characteristically, with concrete examples that illuminated obscure aspects of the general theory. Between these two, the middle years brought a flowering of results about the wave equation (including results for a curved space-time) of importance to both physicists and mathematicians.

INTRODUCTION

The first major conference in England on the modern theory of partial differential equations took place at the University of Durham in July 1976. The membership of the meeting consisted of twenty three celebrated experts from abroad, four master craftsmen from England and a mixture, numbering thirty four, of young Europeans and British mathematicians of widely varying ages and of widely varying competence in the theory. The four master craftsmen from England were Atiyah (later Sir Michael Atiyah OM, PRS 1990-1995), Eells, Friedlander and Friedlander's recent pupil Melrose, but Friedlander and Melrose were not widely known at the time.

Two Swedish giants of the subject, Gårding and Hörmander, were there; Friedlander and Melrose had corresponded with Hörmander, but they had not met previously. Gårding and Hörmander seemed surprised to find that Friedlander was fully conversant with Hörmander's recent work; perhaps even more surprised that this small, modest, bearded Englishman (as they thought) had just produced, in a paper of fifteen pages, an example that shed much light on a question that was troubling Hörmander. Those few, among Friedlander's colleagues, who took a close interest in his mathematics, were not surprised.

Gerard Friedlander came to England, aged sixteen and alone, in August 1934. He had spent his childhood in Vienna, living with grandparents while his mother, adopting the name Ruth Fischer, promoted communist causes in Berlin. He had spent much of the period 1929-1934 in Berlin and, after the rise of Hitler in 1933, in Paris with his mother and her lifelong lover Arkadij Maslow. By then Maslow was also a professional communist; previously he had been a youthful concert pianist and then a student of mathematics to doctoral level.

In the autumn of 1936 Gerard won a major scholarship to Trinity College, Cambridge. He went up in October 1937 and obtained a first-class degree in Mechanical Sciences in 1939. By the end of February 1940, G.I. Taylor (FRS 1919, later Sir Geoffrey Taylor), to whom Gerard was assistant (nominally for turbulence measurements in a wind tunnel), had submitted to the *Proceedings of the Cambridge Philosophical Society* and to the *Proceedings of the Royal Society* three papers by Gerard on the reflection and diffraction of sound pulses by various obstacles.

These papers contain mathematics of astonishing virtuosity; they led to Gerard's election to a Fellowship of Trinity in October 1940. At the time he was interned as an enemy alien in Canada; we defer to later pages the details of that affair and of the confidential war work to which he was co-opted upon his return to England in January 1941.

After a post-war year as a Fellow of Trinity and Assistant Lecturer in Cambridge, Gerard moved to a Lectureship at Manchester for eight years; it was a golden age of Applied Mathematics there. By the time that he returned to Cambridge in 1954, his work was exceptional in that it combined the European style of the subject (Courant-Hilbert, K.O. Friedrichs, Laurent Schwartz and, above all, Hadamard) with the more pragmatic British approach emphasizing explicit solution of particular problems. He was respected

and liked by his colleagues, but most of them paid little attention to his mathematics before the events at Durham in 1976. These led to his replacing Gårding at Lund for a semester in 1977, to his promotion to Reader in 1979 and to his election to the Royal Society in 1980.

Upon retirement in 1982, Gerard moved from Applied Mathematics and Theoretical Physics in Cambridge to Pure Mathematics and Mathematical Statistics and to an Honorary Research Fellowship at University College London.

FAMILY BACKGROUND AND EARLY LIFE

Gerard Friedlander was born in Vienna on 25 December 1917, the only child of Paul Friedländer and Elfriede (née Eisler). Much of what we know of his forebears comes from an autobiographical memoir kindly passed to us by Richard Melrose; a shorter version is deposited in the Churchill Archives Centre. His paternal grandfather, Jakob Friedländer, was born in 1859 near Karlovy Vary, in what was then the Sudetenland. He became a prosperous lawyer after studying in Vienna, and in 1890 married Bertha Weinstein, with whom he had four children: Paul and his twin Peter (born in 1891), Elizabeth and Alexander. Paul read law in Vienna but disliked it, switching to Philosophy, Sociology and Art History after one year. This was a particularly momentous change, for he then met Gerard's mother, Elfriede Eisler, who was also studying Sociology.

The Eislers came from Prague. Gerard's great-grandfather Ferdinand Eisler owned a textile factory in Bohemia but lived in Vienna, where his son Rudolf was born in 1873. Rudolf studied philosophy in Vienna and Prague, moving to Leipzig when twenty one to work under Wilhelm Wundt, often referred to as one of the founding fathers of modern psychology. There he fell in love with Maria Ida Fischer, the daughter of the owner of his lodgings. Ferdinand did not want his son to marry a gentile, and so when Elfriede arrived on 11 December 1895 her parents were not married; nevertheless she was registered as Elfriede Eisler. A softening of Ferdinand's attitude enabled the marriage to take place, after which two further children, Gerhart and Johannes (later to call himself Hanns), were born. The family moved to Vienna, but life was hard because despite academic distinction, Rudolf was unable to obtain a regular position at the University. The three children grew up in a liberal and agnostic household; soon after the war had begun in 1914, they founded a left-wing student group opposed to it. Paul was also a member of this group.

Paul and Elfriede were married on 10 July 1915. In September 1919 she moved to Berlin, leaving Gerard in the care of the Eisler grandparents, with whom he was to stay for the next ten years. Soon after arrival in Berlin she met Isaak Tschemerinsky. Russian-born and German-educated, he was a gifted pianist who studied mathematics to doctoral level at the University of Berlin. In 1919 his thesis was lost or stolen in Copenhagen; rather than rewrite it for formal submission he turned to politics and became an active communist party member. Elfriede and Isaak were instantly attracted to one another and within weeks they were living together. As the German Communist

Party had been forced to go underground, they had to live under assumed names, Elfriede becoming Ruth Fischer, while Tschemerinsky was transformed into Arkadij Maslow, known as Max by Gerard. These names stuck. Max's lifelong union with Ruth meant that, in effect, he became Gerard's stepfather. In 1921 Ruth decided to divorce Paul, and she and Max returned briefly to Vienna for the legal niceties. Matters did not go entirely smoothly, for initially Paul threatened to shoot himself if the proceedings went ahead, but it appears that eventually agreement was reached over a meal and divorce followed in 1922.

Ruth's life up to that point had been remarkable enough, but her subsequent activities proved to be even more singular. As a leader of the German Communist party and a member of the Reichsrat from 1924 to 1928, she became one of Europe's most prominent women. In 1933 when Hitler became Chancellor she fled to Paris, leaving for the United States in 1941. There she underwent a political epiphany and wrote books denouncing Stalinism. She formed the view that her two brothers were active in a Stalinist campaign against her and Max, who had been found dead in Havana on 21 November 1941. She was convinced (with some reason) that he had been murdered by Stalin's secret police. Eventually she testified to the House UnAmerican Activities Committee against her brother Hanns, an action that resulted in his blacklisting and deportation in 1948. This made quite a stir, for Hanns was by then a celebrated composer who, after studying with Schoenberg, had written much well-regarded classical and film music, and had collaborated extensively with Bertold Brecht. She also testified that her other brother, Gerhart, was a major communist agent. It appears that for about eight years from 1945 she worked for The Pond, an American secret service that was created during World War II by military intelligence as a counterweight to the Office of Strategic Services, the forerunner of the CIA. She left the USA for Paris in 1955 and died there on 12 March 1961.

In 1929 Gerard's prolonged stay with his grandparents came to an end and he joined his mother and Max in their flat in Berlin. There he went to school at the Kaiser-Friedrich-Realgymnasium, later to form part of what was called Karl-Marx-Schule at the behest of the Communist members of the local council. Although the prevailing idea at home was that he should become an engineer, Max pushed Gerard hard in mathematics, even taking him through parts of such weighty tomes as Jordan's *Cours d'Analyse*. In February 1933 a Nazi guard was posted at the gates of the school to bar entry to some staff; soon after, the Karl-Marx-Schule name was removed. Then came the Reichstag fire of 27 February 1933. When Gerard came home the following day he found the flat empty: in a phone message Ruth told him that she and Max would not be coming home and that he was to go and stay with a colleague of hers. He met Ruth and Max during the next few days and was taken by them to a flat in which they had been living as a temporary refuge. This served them well, but shortly after the Reichstag election (in which the Nazis gained only 44% of the vote) Gerard returned to find it ransacked; even worse, a Nazi living in the flat above had phoned the SA squad that had broken into the flat, alerting them to Gerard's presence. Gerard was arrested and taken to an SA prison, from which he was rescued by the regular police and transferred to a Juvenile

prison. Ruth and Max had, in the meantime, escaped and reached Prague, where they employed a lawyer to find out exactly where Gerard was being held. Grandfather Jakob then arranged for Gerard to be deported to Vienna as an ‘undesirable alien.’

He stayed in Vienna until the end of June 1934, when he was re-united with his father, Paul. On 29 June 1934 he left Vienna to join Ruth and Max in Paris. There he discovered that she had a plan to send him to London: with the aid of some friends she had persuaded the Jewish Refugees Committee in London to accept him as a bona fide refugee and to help him continue his education in England. He travelled to London on 17 August, where bizarrely the Committee initially suggested that he should be apprenticed to a tailor or articled to an architect. When he turned these ideas down the Committee arranged for him to be seen by Augustus Kahn, a retired mathematics master. Gerard told him that he wanted to study mathematics at university (the idea of engineering having been dropped), the interview went well and Kahn suggested that he should try for a Cambridge Entrance Scholarship, provided that a school could be found that would take him in the Sixth form to prepare for the examination. After rejection by the Central Foundation School, which did not rate his chances of success highly enough, he was accepted by Latymer Upper School in Hammersmith and went to the seventh (!) form of this direct grant school in September 1934. In January 1935 he had to leave the flat at 147 Abbey Road in which he had been staying with his uncle Hanns Eisler, and became a lodger with Mrs Hugo (the mother of the school secretary) at 2 Roman Road, Chiswick, remaining there until June, when he had to leave at rather short notice, apparently because Mrs Hugo needed the room. Once more the Committee came to his rescue: they arranged for him to spend a few weeks with a middle-class Jewish family in a large flat in Kensington. The results of his matriculation examinations came through in July 1935 and were entirely satisfactory except for a bare pass in arithmetic! Kahn was pleased with his performance and persuaded the Committee to fund a trip to Copenhagen, where he appears to have seen his mother and Max. On return he lodged at 36 Hazlitt Road and began his preparation for the Scholarship examination, constantly spurred on by Max, who emphasised the need to give elegant answers to the questions. At the request of the Committee he was entered for the group of colleges that included King’s and Trinity; the examinations went well and on 20 December 1936 he was able to telegraph the good news to Paris that he had been awarded a major scholarship at Trinity. The following October (1937) he went up to Trinity with the proposal that he should read Mechanical Sciences rather than Mathematics. There seem to be various reasons for this volte face: Gerard was unsure that he really wanted to study mathematics; Ruth and Max put the point to him that an engineering qualification would make him more globally employable; and Max also wrote along Marxist lines that engineering would involve him more with real workers. At any rate, support for this change came not only from Trinity but also from the Committee, which had promised some financial support, and the Director of Studies in Engineering, who arranged for Gerard to skip the first year because of his knowledge of mathematics and go immediately into the two-year honours course.

The first year Tripos examinations in 1938 went well: he won the prize for the

best script in the aeronautics paper. Because of this he was offered the chance to spend the remaining undergraduate year in the Aeronautics Lab instead of reading for another Tripos. G.I. Taylor suggested that he should help with his experimental work on turbulence: essentially he would be a research student but could not be formally registered as such since he was still an undergraduate. In April 1939 he was told that Trinity would support him for another year and that he would be registered with the Board of Graduate Studies. His B.A. degree was awarded in 1939. A change of direction came when Taylor, perhaps sensing that Gerard was not ideally suited to laboratory work, proposed that he should calculate the pressure in the shadow of a semi-infinite plane screen (or half-plane) resulting from the diffraction of a plane wave normally incident on the screen. This was an inspired suggestion, leading eventually to work on reflection and diffraction by various obstacles. His conversion to an applied mathematician was complete.

THE WAR YEARS

The war began in September 1939, and as Gerard had legally been a German citizen since the Austrian Anschluss, he was technically an ‘enemy alien’ and had to appear before a Home Office panel. His claim to be a genuine refugee from the Nazis was accepted and he was allowed to continue with his studies. In January 1940 Taylor suggested that he should submit his current work to Trinity as a Fellowship dissertation for the next election, in October. A draft version was almost finished by the end of April, but then the Nazi successes in Holland, Belgium and France led to the decision to intern all male Germans. Gerard was collected from his rooms and taken to the police headquarters in Bury St Edmunds, where such dangerous enemies of the state as Hermann Bondi (FRS 1959), Tommy Gold (FRS 1964) and Max Perutz (FRS 1954) were also assembled. They were then transferred to the Isle of Man, where an internment camp had been formed. Before long it was decided that all such people between twenty and thirty years old would be sent to Canada. Two days before sailing Gerard posted his fellowship dissertation to Trinity. In Canada, he was first lodged in Quebec in a disused army compound, and then in ‘Camp N’, an empty railway engine shed near Montreal. It was in Camp N that he received telegrams of congratulation on his fellowship election from G. I. Taylor and Duff (Senior Tutor at Trinity). The absurdity of deporting Gerard, a refugee from the Nazis who had just been elected to a Fellowship at Trinity, soon attracted comment. A letter to the Times from A. V. Hill pointed out that with no suspicion whatsoever of his loyalty and integrity he had been deported simply because he was of ‘enemy’ origin, and that the action of Trinity seemed more sensible than that of the Government. This was followed by critical pieces in the Spectator (17 and 24 October 1940), and in the House of Commons both the Foreign Secretary Ernest Bevin and the Home Secretary Herbert Morrison had to face hostile questioning (Hansard 365, 1120-2). Morrison stated on 24 October that a ‘communication’ had been sent to Canada ordering that Mr Friedlander should be released and, if he desired, sent back

to England. The Spectator commented on 28 November that on 21 November a tutor at Trinity had received a cable from Gerard to the effect that nothing had happened. The paper added that ‘something more than suave assurances that proposals have been adopted and action has been taken’ was needed. Action was then forthcoming, but it was not until January that Gerard was able to sail back from Halifax, in convoy, and eventually to exchange the pleasures of Camp N for those of the Trinity High Table.

Gerard’s war work proceeded in three stages, each of which followed recommendations by G.I. Taylor.

- (a) In January 1941, the papers (1) and parts I and II of (5) had already been accepted for publication. Now back in Cambridge, Gerard did the work reported in parts III and IV of (5), which were submitted to the Royal Society in November 1941. The four parts of (5) can certainly be described as war work; publication was delayed until 1946 for security reasons. At the same time Gerard prepared other parts of his fellowship dissertation for publication; these became the papers (2), (3) and (4).
- (b) Probably in 1942, Gerard was co-opted by R.E. Peierls (FRS 1945, later Sir Rudolf Peierls) to his team in Birmingham working on development of the atomic bomb. Gerard’s role was to advise on the accompanying problems in fluid dynamics; he worked mainly in his College rooms in Cambridge, with occasional visits to Birmingham.
- (c) When Peierls’s group was moved to Canada in 1943, Gerard preferred to stay in England. He became a Temporary Experimental Officer in two Admiralty research establishments: first, the Underwater Explosion Establishment in Rosyth dockyard, where he felt himself to be of little use, then the Admiralty Signal Establishment at Witley in Surrey. Naval radar was being developed at Witley; Gerard was in the Aerials section and had to learn the theory of electromagnetic waves and of geometrical optics. This led to the papers (6) and (7); perhaps more importantly, the similar subject of geometrical acoustics became a recurrent theme in his work.

SOUND PULSES, 1939-1941

In the summer of 1939, before the outbreak of war and approximately a year before Gerard’s internment, the scientific branch of the Home Office was charged with preparations for Civil Defence. The head of that branch, J.D. Bernal (FRS 1937), asked G.I. Taylor what was known about the shielding effect of a wall from a bomb blast. Since a sound pulse can approximate the weak shock wave and the disturbance behind it due to a distant explosion, Taylor suggested to Gerard that he compute the pressure behind a half-plane when a sound pulse impinges on the front side. The means to this end that Taylor suggested were

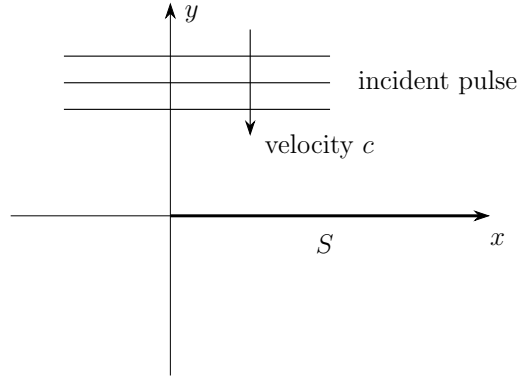


Figure 1

- (a) two papers by Horace Lamb (1906, 1910);
- (b) a Brunsviga calculator, which was a primitive, mechanical, hand-powered ancestor of today's machines.

The problem is illustrated in Figure 1. The half-plane (or screen) S is represented by the positive x -axis because its edge is the z -axis of Cartesian co-ordinates x, y, z . (In order to regard S as a wall, one must turn Figure 1 through 90° clockwise.) The flow is assumed to be independent of z . The incident pulse, propagating downwards from infinity in Figure 1, is constant along each moving line

$$y = \text{constant} - ct, \quad -\infty < t < \infty,$$

where t denotes time and c is the speed of sound. This incident pulse has velocity potential

$$\Phi_0(y, t) = F(y + ct),$$

where F is a given function; the pressure field of a typical incident pulse will be shown presently.

In Lamb's paper of 1910, the total velocity potential $\Phi(x, y, t)$ satisfies three conditions.

In the xy -plane (now also called $\mathbb{R}^2$) outside the closure $\bar{S} = S \cup \{(0, 0)\}$ of the screen S , the wave equation is satisfied:

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = 0 \quad \text{for } (x, y) \in \mathbb{R}^2 \setminus \bar{S} \text{ and } -\infty < t < \infty. \quad (\text{A})$$

On both sides of S (for $x > 0$ and $y \downarrow 0$ or $y \uparrow 0$) the velocity perpendicular to S is zero:

$$\frac{\partial \Phi}{\partial y} = 0 \quad \text{for } x > 0 \text{ and } y = 0 \pm. \quad (\text{B})$$

Far to the left in Figure 1, the incident pulse is not disturbed: with $x + iy = re^{i\theta}$ and $0 \leq \theta \leq 2\pi$,

$$\Phi(x, y, t) - F(y + ct) \rightarrow 0 \quad \text{as } r \rightarrow \infty \text{ and } \theta \rightarrow \pi. \quad (\text{C})$$

Lamb's solution is partly in terms of parabolic co-ordinates ξ, η defined by

$$x + iy = (\xi + i\eta)^2, \quad 0 \leq \arg(\xi + i\eta) \leq \pi;$$

it is

$$\begin{aligned} \Phi(x, y, t) = & \frac{1}{2} F(ct + y) + \int_0^{\xi+\eta} f(ct + y - v^2) dv \\ & + \frac{1}{2} F(ct - y) + \int_0^{\xi-\eta} f(ct - y - v^2) dv, \end{aligned} \quad (\text{D})$$

where f is determined from the given function F (which describes the incident pulse) by the integral equation

$$\int_0^\infty f(s - v^2) dv = \frac{1}{2} F(s), \quad -\infty < s < \infty. \quad (\text{E})$$

Lamb did not claim that (D) and (E) define the only solution of (A), (B) and (C). The only flaw in his beautifully lucid paper is the statement that, in the paper by Arnold Sommerfeld (1901) on the diffraction of X-rays, the boundary condition on S is not (B). In fact, Sommerfeld had considered two separate cases: $\Phi = 0$ on S and $\partial\Phi/\partial y = 0$ on S in the present notation; but it is not always easy to separate the wheat from the chaff of Sommerfeld's massive paper.

Gerard set to work with a will. The pressure rise above atmospheric pressure is given by

$$p = \rho \frac{\partial\Phi}{\partial t},$$

where ρ is the density of air, assumed to be constant. To compute numerical values of the pressure rise from (D) and (E) is straightforward in principle but difficult in practice. Gerard adopted a form of the solution f of (E) more useful than the form used by Lamb; thereby he could obtain explicit answers for functions F that are more realistic than Lamb's single explicit example. Gerard's favourite incident pulse has a non-dimensional pressure rise (above atmospheric pressure)

$$p_0(y + ct) = \begin{cases} 0 & \text{if } y < -ct, \\ \left(1 - \frac{y + ct}{\lambda}\right) \exp\left(-\frac{y + ct}{\lambda}\right) & \text{if } y > -ct, \end{cases} \quad (\text{F})$$

where $-\infty < t < \infty$ and λ is a "pulse thickness" such that p_0 passes through 0 when $y + ct = \lambda$. Figure 2 shows this pressure rise; evidently $y = -ct$ describes a wave front

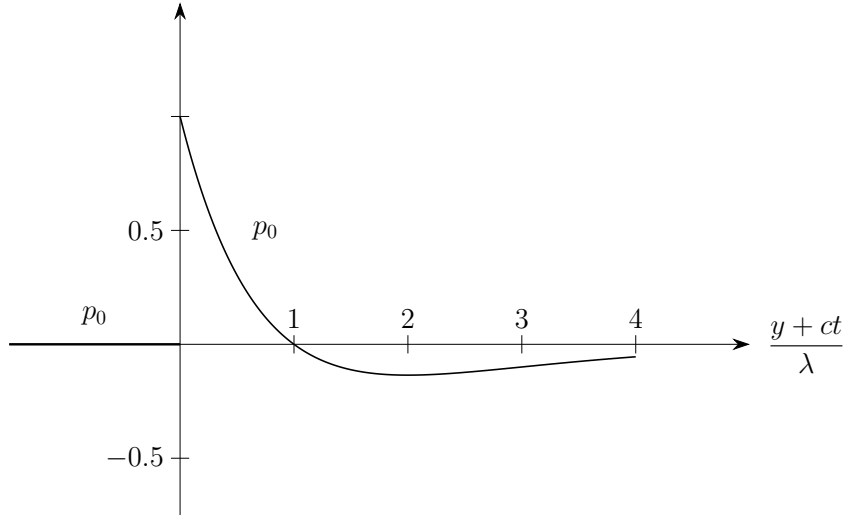


Figure 2

ahead of which the incident pulse vanishes. A form of Figure 2 now appears in Wikipedia with the legend: “A Friedlander waveform is the simplest form of a blast wave.”

In part I of (5), Gerard presented substantial tables of the total pressure field caused by this incident pulse, and by some others, according to (D) and (E). An anonymous, skilled draughtsman turned these tables into three-dimensional graphs of a kind that modern computers produce without effort, but that was not common in 1940. Gerard must have spun the Brunsviga with patience and skill.

After answering Bernal’s question fully in part I of (5) and dealing with a relevant integral in part III, Gerard went much further. The background of his paper (1) on reflection by parabolic obstacles is as follows.

Lamb’s paper of 1906 had concerned a half-plane with incident waves that are not pulses, but are sinusoidal wave trains with velocity potential

$$\Phi_0(y, t) = \operatorname{Re} e^{ik(y+ct)}, \quad -\infty < t < \infty,$$

for a given wave number $k > 0$. The symbol Re denotes “the real part of”, the exponential being more convenient than a cosine or sine. The Fourier integral theorem allows one to pass from sinusoidal wave trains to pulses, but only if one has accurate estimates of high-frequency Fourier integrals ($k \rightarrow \infty$); even then, the method is not a good one for numerical computation. It was this fact that caused Lamb to write his paper of 1910.

In the paper of 1906, but not that of 1910, Lamb had also considered parabolic cylinders and paraboloids of revolution. Following Gerard’s paper (1), we take our basic parabola to have the equation

$$y = \pm \sqrt{4a(x+a)} \quad \text{or} \quad r \equiv \sqrt{x^2 + y^2} = x + 2a \quad \text{for } -a \leq x < \infty, \quad (\text{G})$$

where $a > 0$ denotes the focal length. Henceforth the words *parabolic cylinder* will mean the convex solid bounded by the same curve (G) in each plane $z = \text{constant}$. The words

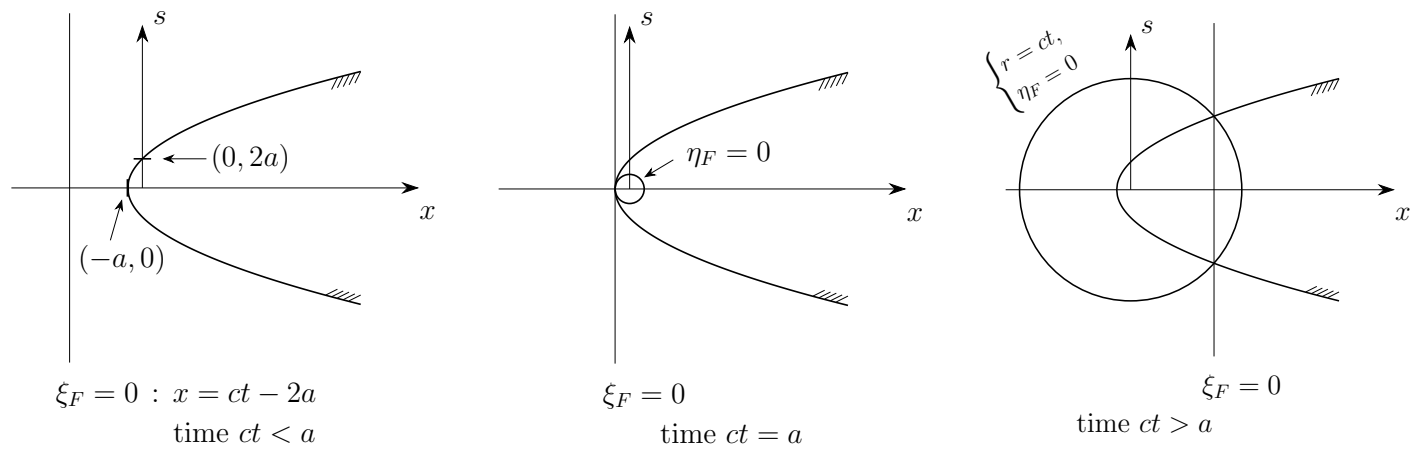


Figure 3

paraboloid of revolution will mean the convex solid bounded by the surface (Figure 3)

$$s \equiv \sqrt{y^2 + z^2} = \sqrt{4a(x+a)} \quad \text{or} \quad r \equiv \sqrt{x^2 + s^2} = x + 2a \quad \text{for } -a \leq x < \infty. \quad (\text{H})$$

For such obstacles, Lamb (1906) had considered incident wave trains with velocity potential

$$\Phi_0(x, t) = \text{Re } e^{-ik(x-ct)}, \quad -\infty < t < \infty.$$

In (1), Gerard solved the problem of the reflection, by a paraboloid of revolution or by a parabolic cylinder, of an incident pulse with velocity potential

$$\Phi_0(x, t) = F(x - ct) = F_0\left(\frac{ct - x - 2a}{a}\right), \quad -\infty < t < \infty;$$

the notation is that in (G) and (H). We sketch his construction for the (slightly harder) case of a paraboloid of revolution. Gerard introduced non-dimensional co-ordinates ξ_F , η_F (Figure 3) defined by

$$\xi_F = \frac{ct - x - 2a}{a}, \quad \eta_F = \frac{ct - r}{a}. \quad (\text{I})$$

We have added the suffix F to Gerard's ξ , η in order to avoid confusion with the parabolic co-ordinates ξ , η used in (D). There is no suggestion of this pulse problem, nor of these co-ordinates, in the papers of Sommerfeld (1895, 1901) and Lamb (1906, 1910). A priori, it seems far from obvious that

- (a) when the incident pulse has a sharp front, as was the case for the pressure rise (F), the reflected wave front emanates not from the leftmost point, $(x, s) = (-a, 0)$, of the paraboloid, but from the focus $(x, s) = (0, 0)$;
- (b) the condition of zero normal velocity $\partial\Phi/\partial n = 0$, on the boundary (H) of the paraboloid, transforms to

$$\frac{\partial\Phi}{\partial\xi_F} = \frac{\partial\Phi}{\partial\eta_F} \quad \text{for } \xi_F = \eta_F; \quad (\text{J})$$

- (c) the wave equation

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial s^2} + \frac{1}{s} \frac{\partial\Phi}{\partial s} - \frac{1}{c^2} \frac{\partial^2\Phi}{\partial t^2} = 0$$

transforms to an equation that can be integrated twice, by elementary means, to yield

$$\Phi(x, s, t) = F_0(\xi_F) + \int_{-\infty}^{\eta_F} \frac{\ell(v)}{\xi_F - v + 2} dv, \quad \eta_F \leq \xi_F, \quad (\text{K})$$

where ℓ is an unknown function that will be determined by (J);

- (d) there is a solution $\Phi(x, s, t) = \tilde{\Phi}(\xi_F, \eta_F)$ that depends only on the *two* variables ξ_F, η_F and satisfies not only the wave equation and the boundary condition (J), but also the condition that the incident pulse is undisturbed at infinity:

$$\tilde{\Phi}(\xi_F, \eta_F) - F_0(\xi_F) \rightarrow 0 \quad \text{as } \eta_F \rightarrow -\infty \text{ with } \infty < \xi_F < \infty. \quad (\text{L})$$

It is now safe to *omit the suffix F from ξ_F and η_F* .

Gerard established the surprising facts (a) to (d) and noted that (J) and (K) combine to the integral equation

$$\ell(\eta) + 2 \int_{-\infty}^{\eta} \frac{\ell(v)}{(\eta - v + 2)^2} dv = 2F'_0(\eta), \quad -\infty < \eta < \infty, \quad (\text{M})$$

and that (K) implies (L) when the integral in (K) exists. He showed that, when F_0 is chosen to represent a sinusoidal wave train, Lamb's solution of 1906 is recovered.

Next, restricting attention to incident pulses with sharp fronts, for which $F_0(\xi) = 0$ if $\xi < 0$ and for which (M) is restricted to $0 \leq \eta < \infty$ and $0 \leq v \leq \eta$, he extracted a “well known” solution of this form of (M) from tome 3 of Goursat's *Cours d'analyse*. From this solution he produced tables and graphs of pressure once more.

All this was done, it should be remembered, in addition to the answer to Bernal's question, within the seven months following Gerard's two -year course in Mechanical Sciences.

We turn now to parts II and IV of (5). In part II Gerard compared some of his results for the half-plane with corresponding results for an infinite wedge; necessarily, he restricted this work to the pressure at a point on the back face of the wedge when a pulse impinges on the front face. A solution had been found by Sommerfeld (1901), but there had been no numerical evaluation of the elaborate functions that form Sommerfeld's solution. Gerard presented graphs showing the variation with time of the pressure rise at a fixed point on the back face of the wedge. The *shape* of these curves is nearly independent of the wedge angle (which is zero for the half-plane), but the *magnitude* of the pressure rise increases with wedge angle.

In part IV Gerard explained a result that was considered at the time to be paradoxical. Suppose that in Figure 1 the incident pulse does not propagate vertically; rather, the direction of propagation makes an angle α with the screen S , so that the incident pulse has a velocity potential

$$\Phi_0(x, y, t) = F(y \sin \alpha - x \cos \alpha + ct), \quad 0 < \alpha \leq \frac{\pi}{2}.$$

For our previous case, $\alpha = \pi/2$. If the obstacle were a *complete* plane, represented by $-\infty < x < \infty, y = 0$, then the total velocity potential would be

$$\Phi(x, y, t) = F(y \sin \alpha - x \cos \alpha + ct) + F(-y \sin \alpha - x \cos \alpha + ct), \quad y \geq 0,$$

which satisfies the wave equation and the boundary condition on the plane. The direction of propagation of the pulse represented by the second term accords with the classical

rule for the reflection of light. The total pressure rise on the plane $y = 0$ would be twice that of the incident pulse, for every α in the interval $(0, \pi/2]$. Hence this would persist in the limit as $\alpha \downarrow 0$ and for the complete plane there is a paradox: the incident pulse with velocity potential $F(-x + ct)$ propagates in a direction parallel to the plane and therefore should be undisturbed by the plane, yet the total pressure rise on the plane is twice that of the incident pulse.

Gerard argued that the complete plane $(-\infty < x < \infty, y = 0)$ is too far from physical reality to be relevant. Using yet another solution of Sommerfeld (1901), he examined the case of a pulse incident at angle α to the half-plane S . Let $p(x, t; \alpha)$ denote the total pressure rise on the upper side ($y = 0+$) of S . In effect, Gerard's results were as follows (although he stated them differently). As $x \rightarrow \infty$ with $x \cos \alpha - ct$ fixed, $p(x, t; \alpha)$ approaches twice the pressure rise of the incident pulse, for every α in the interval $(0, \pi/4]$. However,

$$\lim_{x \rightarrow \infty} \lim_{\alpha \downarrow 0} p(x, t; \alpha) \neq \lim_{\alpha \downarrow 0} \lim_{x \rightarrow \infty} p(x, t; \alpha),$$

where it is to be understood that $x \rightarrow \infty$ with $x \cos \alpha - ct$ fixed. He derived similar results for some wedges.

One final remark must be made. It is not known whether Lamb's solution of (A), (B) and (C) is the only one, even though, for given F , equation (E) has only one solution f in a large class of functions. It *is* known that, if the problem is changed to (A), (B) and

$$\Phi(x, y, t) - F(y + ct) \rightarrow 0 \quad \text{as } r \rightarrow \infty \text{ and } r^{1/2} \cos \frac{\theta}{2} \rightarrow 0, \quad (\text{C}')$$

then there are infinitely many solutions of (A), (B) and (C'). Therefore it is of interest whether Sommerfeld and Lamb, following different paths and using partly different equations, reached the same conclusion from (A), (B) and (C). Sommerfeld (1901) gave an argument (pp. 20-22) that he seemed to regard as a uniqueness proof under an additional condition, but his argument is incomplete. In part I of (5), Gerard asserted that Sommerfeld's and Lamb's solutions are the same, which is true. However, in a rare lapse, Gerard failed to prove this assertion. Where four identities are required, Gerard gave only two, one of which is conspicuously wrong. In claiming that the false identity "agrees with the result given by Sommerfeld (1901, p. 40)", he referred to a page which does not contain Sommerfeld's solution, but discusses only one component of it.

On the other hand, in parts II and IV of (5) Gerard seemed familiar with every aspect of Sommerfeld's paper. It is just possible that the errors in part I were the result of hasty transcription. In any case, the important thing is that Lamb's solution, which Gerard used carefully to compute the pressure, is the same as Sommerfeld's.

MANCHESTER

According to our enquiries, no close colleague of Gerard's Manchester years (1946-1954) survives as we prepare this Memoir (2014-2016). However, one of us had some contact

with Applied Mathematicians at Manchester during the period 1950-1954 and remains in awe of the Department at that time. After wartime work on aerodynamics at the National Physical Laboratory, Sydney Goldstein (FRS 1937) was appointed to the Beyer Chair of Applied Mathematics in 1946. He recruited not only Gerard Friedlander but also D.S. Jones (FRS 1968), M.J. Lighthill (FRS 1953, later Sir James Lighthill), R.E. Meyer, F.J. Ursell (FRS 1972) and G.N. Ward. In 1950 Goldstein moved to Israel and was replaced by Lighthill, then aged 26. Lighthill had huge respect for Goldstein and for Goldstein's style; the Department continued much as before and flourished. We have the impression that Gerard was very much a part of this group and that he was a close observer of the mathematical techniques that each of his colleagues used. Possibly he stood a little apart because Hadamard's text *Lectures on Cauchy's problem in linear partial differential equations* (Yale 1923) was the greatest single influence on his work from 1939 onwards. However, it would not have occurred to him to lay claim to the rigour and width of vision that distinguished much of his work from the impressive (often dazzling) but more pragmatic work of his colleagues.

Towards the end of his time at Manchester, Gerard was offered the chance to spend six months doing research at New York University. The photograph shows Gerard, Yolande and Paul, then aged two, dining with another family aboard the Queen Mary on their way to New York. Soon after his return to England, Gerard was offered a lectureship at Cambridge.

CAMBRIDGE, 1954-1976

As a lecturer in Cambridge, Gerard cut a distinctive figure with his feathered hat and pack of French cigarettes. Though not the most organised teacher, he was certainly one of the most inspiring, with many insights and anecdotes to pass on to his audience. Only he could produce an analogy between the breakdown of the solution of the transport equation and the Jean-Luc Godard film *Weekend*!

To other lecturers, Gerard was a wonderfully knowledgeable colleague. Edward Fraenkel consulted him regularly about awkward mathematical points that standard texts treated dishonestly or incompletely. Invariably, Gerald gave a satisfactory answer. ("Yes, the local argument in the books of X and Y and of Z is not enough. But there is an old and difficult *global* theorem due to Kamke; I believe that there is a full reference to it in ...") Again, on the Lebesgue spine, which dates from 1912: "Yes, a uniqueness proof is needed; you will find it in a book of lectures that Hadamard gave in China in 1964.") Others found him as helpful in other ways.

In 1937 Gerard had met his future wife, Yolande Morris Moden, at the Footlights Club in Cambridge while he was an undergraduate. A talented artist, she never gained the recognition she deserved. They were married in St George's Church, Cambridge in 1944, and had three children: Paul, who describes himself as a scientific artist and has produced light installations all over the world; Peter, who teaches Hindi and Buddhism at the Australian National University; and Liz, who opted for a quiet life and still lives



**Photographed on board
R.M.S. QUEEN MARY.**

Yolande, Gerard and Paul (left side of table) on the Queen Mary, 1954.

in Cambridge. Yolande died tragically young in 1968 before all her children had grown up, so that Gerard had to bring up Liz and Peter largely on his own. His appearance at lectures complete with a shopping bag for grocery shopping after the lecture reinforced his image as a family man with a strong human side.

In the early 1970s the field of differential equations at Cambridge was re-energised by the introduction of two new courses: one on non-linear ordinary differential equations was given by Peter Swinnerton-Dyer (FRS 1967, later Sir Peter Swinnerton-Dyer) in Part II of the Tripos, while at Part III level Edward Fraenkel lectured on applications of functional analysis to continuum mechanics. Seminars were stimulating affairs, sometimes attended by Dame Mary Cartwright (FRS 1947); Stephen Hawking (FRS 1974) was a regular in the DAMTP coffee room and would occasionally ask Gerard for advice about a differential equation he had encountered in his work. It was around this time that Gerard took on Richard Melrose as a research student. As Richard soon made great strides in the abstract theory of hyperbolic differential equations, Gerard encouraged him to write to Lars Hörmander who, though initially less than enthusiastic, quickly realised that both Gerard and Richard were forces to be reckoned with. A lighter note was struck by Gerard's recollections of various pop concerts he had attended with Liz

and by his claim to prefer marking examination papers with Bob Dylan playing loudly in the background.

BICHARACTERISTIC STRIPS AND WAVE FRONT SETS

The purpose of this section is twofold: to give the reader with mathematical interests a sketch of some of the tools that Gerard used repeatedly and to prepare for a description of the remarkable papers (8) and (9). An account of this material seems in order because in Britain wave front sets are not widely known; here most Applied Mathematicians dismiss such things as Pure Mathematics, while most Pure Mathematicians dismiss them as Applied Mathematics.

Much of this section is taken from the first volume of Hörmander's treatise (Hörmander 1983), but with some change of notation and with a very different style of exposition.

Bicharacteristic strips are geometrical properties of an operator $\mathcal{L}$ defined by

$$\mathcal{L}u = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \text{lower-order terms},$$

$$x = (x_1, \dots, x_n) \in \Omega \subset \mathbb{R}^n, \quad a_{ij} \in C^\infty(\Omega),$$

where Ω is an open subset of the real n -dimensional space $\mathbb{R}^n$ (possibly the whole space) and functions in the set $C^\infty(\Omega)$ have continuous partial derivatives of all orders at each point x in Ω . The *principal symbol* of $\mathcal{L}$ is

$$L(x, \xi) = \xi \cdot \mathbf{a}(x) \cdot \xi = \sum_{i,j=1}^n \xi_i a_{ij}(x) \xi_j, \quad x \in \Omega, \xi \in \mathbb{R}^n.$$

For our purpose here, it will be sufficient to let $n = 3$ and to suppose that $\mathcal{L}$ is uniformly hyperbolic. This means that we may regard x_1, x_2 as space variables and x_3 as time, and that $\mathcal{L}u = 0$ is a wave equation. More precisely, it means that the symmetric matrix $\mathbf{a}(x)$ has two strictly positive eigenvalues and one strictly negative eigenvalue at each point x in Ω . Of course, specialists consider more general operators.

For a solution u of $\mathcal{L}u = 0$ or $\mathcal{L}u = f$, a *characteristic surface* is one that can act as a weak shock wave or as a carrier of other singularities; it is described by an equation $\psi(x) = \text{constant}$ if also

$$L(x, \nabla \psi(x)) = \sum_{i,j=1}^3 \frac{\partial \psi}{\partial x_i}(x) a_{ij}(x) \frac{\partial \psi}{\partial x_j}(x) = 0.$$

A family of bicharacteristic strips describes a characteristic surface in more detail: each strip is a pair $(x(s), \xi(s))$ such that $x = x(s)$ is the parametric form of a curve in the surface, while the $\xi(s)$ are (non-zero) vectors normal to the surface (perpendicular to the surface) along the bicharacteristic curve $x = x(s)$. More precisely, the pair $(x(s), \xi(s))$ is

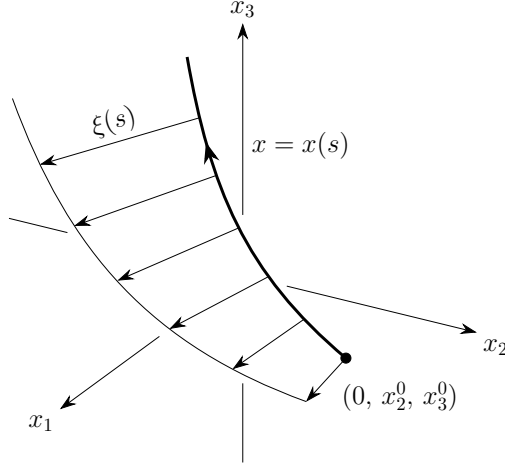


Figure 4

a *bicharacteristic strip* of the operator $\mathcal{L}$ if $x(s)$ is in the closure $\bar{\Omega}$ of Ω (now a subset of $\mathbb{R}^3$), if $\xi(s)$ is in $\mathbb{R}^3 \setminus \{0\}$ and if the pair is a solution of the Hamiltonian system

$$\dot{x}(s) = (\nabla_{\xi} L)(x(s), \xi(s)), \quad (1)$$

$$\dot{\xi}(s) = -(\nabla_x L)(x(s), \xi(s)), \quad (2)$$

$$L(x(s), \xi(s)) = 0, \quad (3)$$

where $(\dot{\cdot}) = d(\cdot)/ds$ and s need not be arc length. Since (1) and (2) imply that

$$\frac{d}{ds} L(x(s), \xi(s)) = 0,$$

condition (3) need be applied at only one point. Since ξ is merely a direction, it is often convenient to normalize it by $\xi_S = \xi/|\xi|$. (The suffix S denotes restriction to the unit sphere S^2 in $\mathbb{R}^3$.)

Here is an example. Reduced to two space variables x_1, x_2 and time x_3 , the partial differential equation in the paper (8) is

$$\mathcal{L}u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} - (1 + x_1) \frac{\partial^2 u}{\partial x_3^2} = 0,$$

in

$$\Omega = \{ x \in \mathbb{R}^3 \mid x_1 > 0, (x_2, x_3) \in \mathbb{R}^2 \},$$

so that

$$L(x, \xi) = \xi_1^2 + \xi_2^2 - (1 + x_1) \xi_3^2, \quad x_1 \geq 0.$$

Then the bicharacteristic strip with initial values

$$x(0) = (0, x_2^0, x_3^0), \quad \xi(0) = (1, 0, -1),$$

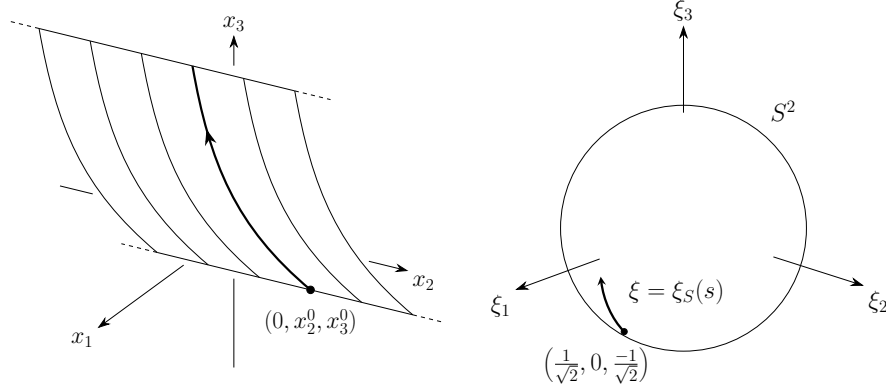


Figure 5

is

$$\left. \begin{aligned} x &= x(0) + \left(2s + s^2, 0, 2s + 2s^2 + \frac{2}{3}s^3 \right), \\ \xi &= (1 + s, 0, -1), \\ \xi_S &= \left(\frac{1 + s}{\sqrt{(2 + 2s + s^2)}}, 0, \frac{-1}{\sqrt{(2 + 2s + s^2)}} \right), \end{aligned} \right\} \quad 0 \leq s < \infty.$$

Figure 4 shows the curve $x = x(s)$ and the vectors $\xi(s)$ in a way that explains the word *strip*. Figure 5 shows the curves $x = x(s)$ in $\mathbb{R}^3$ and $\xi = \xi_S(s)$ in S^2 ; this representation will be more useful in what follows. Figure 5 also shows a piece of a characteristic surface, one of many in which our bicharacteristic curve might be embedded.

Other initial values $\xi(0)$ are possible, but it is necessary that $\xi_{S,3}(0) = \pm \frac{1}{\sqrt{2}}$.

The *wave front set of a distribution*, which is designed to indicate the singularities of a wave motion, is a more difficult object. Its mere definition involves five mathematical devices that are not elementary, but that we shall strive to explain, nevertheless. These five are usually called: infinitely differentiable function of compact support, distribution, Fourier transform of a distribution, frequency set of a distribution of compact support and, for more general distributions, frequency set at a point.

Again let Ω be an open subset of $\mathbb{R}^3$ (possibly the whole “ordinary” space $\mathbb{R}^3$). As many readers will know, a function is in the set $C_c^\infty(\Omega)$ and may be called an *infinitely differentiable function of compact support in Ω* if

- (a) it has continuous partial derivatives of all orders at each point of Ω ,
- (b) it equals zero at all points outside a bounded, closed subset of Ω .

The smallest such subset of Ω is the *support* of the function. When Ω has a boundary, there is room between the support of a function in $C_c^\infty(\Omega)$ and the boundary of Ω .

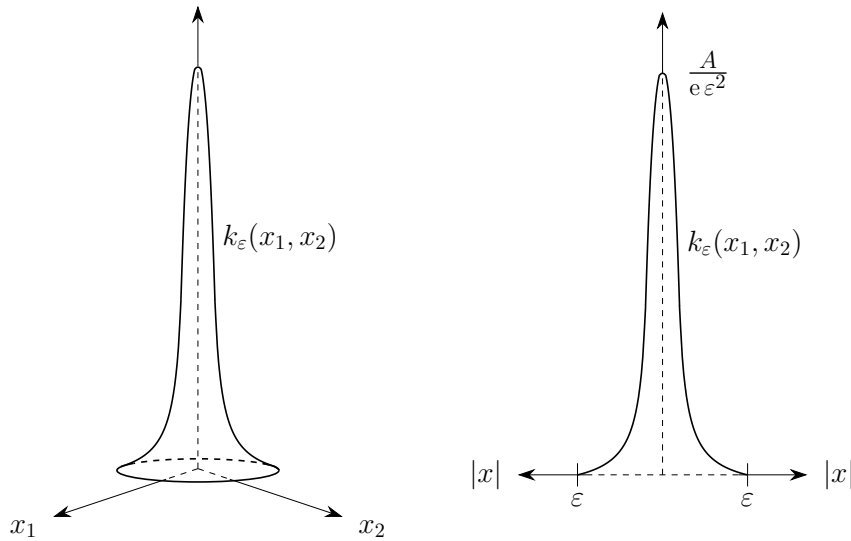


Figure 6

There is no shortage of functions in $C_c^\infty(\Omega)$. We define

$$k_1(x) = \begin{cases} A \exp\left(-\frac{1}{1-r^2}\right) & \text{if } r = |x| < 1, \\ 0 & \text{if } r \geq 1, \end{cases} \quad \text{and} \quad k_\varepsilon(x) = \frac{1}{\varepsilon^3} k_1\left(\frac{x}{\varepsilon}\right) \quad (\varepsilon > 0),$$

where the constant A is such that $\int_{r < 1} k_1(x) dx = 1$. If the origin of $\mathbb{R}^3$ is in Ω (if $0 \in \Omega$) and if ε is sufficiently small, then certainly $k_\varepsilon \in C_c^\infty(\Omega)$. But the importance of k_ε is that the recipe

$$f_\varepsilon(x) = \int_{|y-x| < \varepsilon} k_\varepsilon(x-y) f(y) dy$$

produces a function f_ε in $C_c^\infty(\Omega)$ that is arbitrarily close (when ε is sufficiently small) to any function f that we may find useful, or that may arise in a problem, provided that f equals zero outside a bounded, closed subset of Ω and is integrable on that subset. Figure 6 is a picture of k_ε when x is in $\mathbb{R}^2$ instead of $\mathbb{R}^3$ (when $x = (x_1, x_2)$). Figure 7 shows f and f_ε for a one-dimensional case: $x = x_1$ and $r = |x_1|$ in the definition of k_1 .

One often writes $\mathcal{D}(\Omega)$ in place of $C_c^\infty(\Omega)$.

In the context of Ω (still an open subset of $\mathbb{R}^3$) a *distribution* is a continuous*, linear functional, denoted by $\langle u, \cdot \rangle \in \mathcal{D}'(\Omega)$ and with values $\langle u, \phi \rangle$ for all ϕ in $\mathcal{D}(\Omega)$. If a function f is integrable on each bounded, closed subset of Ω , then the formula

$$\langle f, \phi \rangle = \int_{\text{supp}(\phi)} f(x) \phi(x) dx \quad \text{for all } \phi \in \mathcal{D}(\Omega)$$

*The precise definition of continuity in this context will be known to some readers; it can be found in the book (12) or in the text of Schwartz (1950-51). Such technicalities are beyond us here.

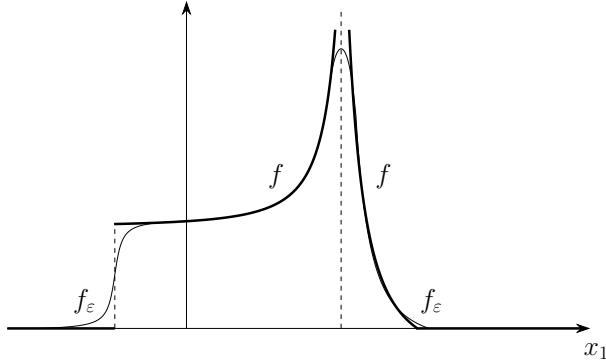


Figure 7

(in which $\text{supp}(\phi)$ denotes the support of ϕ) defines a distribution $\langle f, \cdot \rangle \in \mathcal{D}'(\Omega)$. Moreover, although the graph of f may be far from smooth, the distribution $\langle f, \cdot \rangle$ has partial derivatives of all orders; these are *defined* by the rule for integration by parts. For example, with the notation $(\partial_j \phi)(x) = \partial \phi(x) / \partial x_j$, the distribution $\langle \partial_1 \partial_3^2 f, \cdot \rangle$ is defined by

$$\langle \partial_1 \partial_3^2 f, \phi \rangle = - \int_{\text{supp}(\phi)} f(x) (\partial_1 \partial_3^2 \phi)(x) dx \quad \text{for all } \phi \in \mathcal{D}(\Omega).$$

Perhaps the most popular distribution in $\mathcal{D}'(\Omega)$ is the Dirac distribution $\langle \delta, \cdot \rangle$; if the origin of $\mathbb{R}^3$ is in Ω (if $0 \in \Omega$), then $\langle \delta, \cdot \rangle$ is defined by

$$\langle \delta, \phi \rangle = \phi(0) \quad \text{for all } \phi \in \mathcal{D}(\Omega). \quad (4)$$

Here δ is merely a *symbolic* function. We may secretly visualize a mystical object $\delta(x)$ that is infinite at the origin and zero elsewhere in such a way that $\int_{\Omega} \delta(x) dx = 1$, but the meaning of $\langle \delta, \cdot \rangle$ must be (4). The partial derivatives of $\langle \delta, \cdot \rangle$ are defined like those of $\langle f, \cdot \rangle$:

$$\langle \partial_1 \partial_3^2 \delta, \phi \rangle = -(\partial_1 \partial_3^2 \phi)(0) \quad \text{for all } \phi \in \mathcal{D}(\Omega).$$

The *support* $\text{supp}(u)$ of a distribution $\langle u, \cdot \rangle$ in $\mathcal{D}'(\Omega)$ is, loosely speaking, the smallest closed subset of Ω outside which the symbolic function u may be replaced by the genuine function 0. The *singular support* $\text{ss}(u)$ of $\langle u, \cdot \rangle$ is, loosely speaking, the smallest closed subset of Ω outside which the symbolic function u may be replaced by a genuine function that is infinitely differentiable in the open set $\Omega \setminus \text{ss}(u)$. For the Dirac distribution $\langle \delta, \cdot \rangle$, we have $\text{ss}(\delta) = \text{supp}(\delta) = \{0\}$.

For a more interesting example, we return to space variables x_1, x_2 and time x_3 and consider the solution $\langle u, \cdot \rangle \in \mathcal{D}'(\mathbb{R}^3)$ of

$$\langle \partial_1^2 u + \partial_2^2 u - \partial_3^2 u, \cdot \rangle = 0, \quad (5)$$

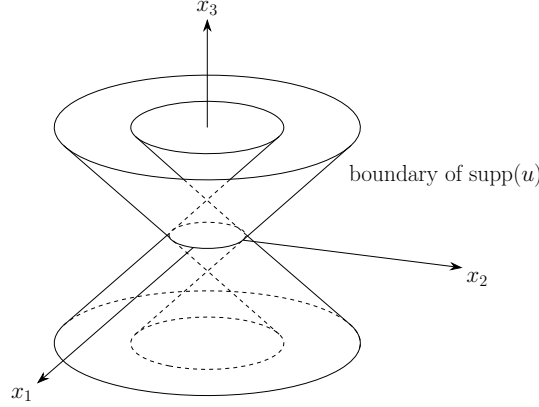


Figure 8

subject to

$$u(x_1, x_2, 0) = \begin{cases} 1 & \text{if } x_1^2 + x_2^2 < a^2, \\ 0 & \text{if } x_1^2 + x_2^2 > a^2, \end{cases} \quad (6)$$

$$(\partial_3 u)(x_1, x_2, 0) = 0 \quad \text{if } x_1^2 + x_2^2 \neq a^2. \quad (7)$$

Here (6) and (7) are legitimate because, as we shall see presently, the set $\{x_1^2 + x_2^2 \neq a^2, x_3 = 0\}$ is outside the singular support $\text{ss}(u)$. In fact, it is only the need to satisfy the wave equation in a weak sense on $\text{ss}(u)$ that has forced us to regard the solution as a distribution rather than a genuine function.

The support $\text{supp}(u)$ of this distribution is the set of points x in $\mathbb{R}^3$ that satisfy

$$x_1^2 + x_2^2 \leq (a + |x_3|)^2 \quad (-\infty < x_3 < \infty).$$

The singular support $\text{ss}(u)$ consists of the conical surfaces described by

$$\begin{aligned} x_1^2 + x_2^2 &= (x_3 + a)^2 & (-\infty < x_3 < \infty), \\ x_1^2 + x_2^2 &= (x_3 - a)^2 & (-\infty < x_3 < \infty). \end{aligned} \quad (8)$$

All this is illustrated in Figure 8. This description of $\text{supp}(u)$ and $\text{ss}(u)$ does not require knowledge of the solution $\langle u, \cdot \rangle$; it follows from the mere statement (5), (6) and (7).

When a distribution $\langle v, \cdot \rangle$ in $\mathcal{D}'(\mathbb{R}^3)$ has bounded support, usually called *compact support*, one writes $\langle v, \cdot \rangle \in \mathcal{E}'(\mathbb{R}^3)$ and admits values $\langle v, \phi \rangle$ for functions ϕ in $C^\infty(\mathbb{R}^3)$ (for infinitely differentiable functions that need not have compact support). Then $\langle v, \cdot \rangle$ has a genuine Fourier transform defined by

$$\begin{aligned} \hat{v}(\xi) &= \langle v, e_{-i\xi} \rangle, \\ e_{-i\xi}(x) &= \exp(-i\xi \cdot x), \end{aligned}$$

where $\xi \in \mathbb{R}^3$ and $\xi \cdot x = \xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3$. The *frequency set* $F(v)$ of a distribution $\langle v, \cdot \rangle$ in $\mathcal{E}'(\mathbb{R}^3)$ is the set of those (bad) directions $\xi_S = \xi/|\xi|$ in the unit sphere S^2 of

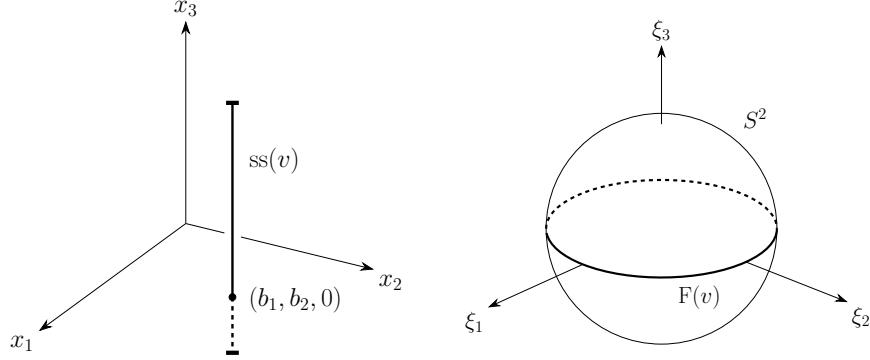


Figure 9

$\mathbb{R}^3$ for which $\hat{v}(\xi)$ fails to decrease rapidly as $|\xi| \rightarrow \infty$; here *rapidly* means faster than $|\xi|^{-m}$ for every integer $m \geq 0$. Accordingly, $\hat{\delta}(\xi) = 1$ and $F(\delta) = S^2$.

Here is a better example. Let $b = (b_1, b_2, 0)$ be a fixed point of $\mathbb{R}^3$, let $w \in C_c^\infty(\mathbb{R})$ (let w be a function of only one variable that is infinitely differentiable and has compact support) with $\int_{\text{supp}(w)} w(t) dt \neq 0$, and define

$$\langle v, \phi \rangle = \int_{\text{supp}(w)} \phi(b_1, b_2, x_3) w(x_3) dx_3 \quad \text{for all } \phi \in C^\infty(\mathbb{R}^3). \quad (9)$$

The support and singular support of $\langle v, \cdot \rangle$ are equal and are a vertical line segment (Figure 9):

$$\text{ss}(v) = \text{supp}(v) = \{ x \in \mathbb{R}^3 \mid x_1 = b_1, x_2 = b_2, x_3 \in \text{supp}(w) \}. \quad (10)$$

The Fourier transform of $\langle v, \cdot \rangle$ is

$$\begin{aligned} \hat{v}(\xi) &= \int_{\text{supp}(w)} \exp(-i\xi_1 b_1 - i\xi_2 b_2 - i\xi_3 x_3) w(x_3) dx_3 \\ &= \exp(-i\xi \cdot b) \int_{\text{supp}(w)} \exp(-i\xi_3 t) w(t) dt. \end{aligned}$$

If $\xi_{S,3} \neq 0$, then $\hat{v}(\xi)$ decreases rapidly as $|\xi| \rightarrow \infty$ because repeated integration by parts yields

$$\hat{v}(\xi) = \frac{\exp(-i\xi \cdot b)}{(i\xi_3)^k} \int_{\text{supp}(w)} \exp(-i\xi_3 t) w^{(k)}(t) dt \quad (\xi_3 \neq 0),$$

for every integer $k \geq 0$. If $\xi_{S,3} = 0$, then

$$|\hat{v}(\xi)| = \left| \int_{\text{supp}(w)} w(t) dt \right| \neq 0.$$

Thus the frequency set $F(v)$ is the equator of S^2 (Figure 9):

$$F(v) = \{ \xi_S \in S^2 \mid \xi_{S,3} = 0 \}. \quad (11)$$

In this exceptionally simple case, the definition of wave front set reduces to a Cartesian product:

$$\begin{aligned} \text{WF}(v) &= \text{ss}(v) \times F(v) \\ &= \{ (x, \xi_S) \in \mathbb{R}^3 \times S^2 \mid x \in \text{ss}(v), \xi_{S,3} = 0 \}, \end{aligned} \quad (12)$$

where $\text{ss}(v)$ is defined in equation (10) and both sets are displayed in Figure 9.

We turn to the difficult definition of *wave front set* $\text{WF}(u)$ for a distribution $\langle u, \cdot \rangle$ that is merely in $\mathcal{D}'(\Omega)$; in general, it lacks compact support. To this end, we fix an arbitrary point $z \in \Omega$ and truncate $\langle u, \cdot \rangle$ smoothly to have arbitrarily small support near z . First, we observe that, if $\langle u, \cdot \rangle \in \mathcal{D}'(\Omega)$ and $\psi \in C_c^\infty(\Omega)$, then $\langle \psi u, \cdot \rangle \in \mathcal{E}'(\mathbb{R}^3)$ (then $\langle \psi u, \cdot \rangle$ has compact support in $\mathbb{R}^3$) when it is defined to be the zero distribution in $\mathbb{R}^3 \setminus \text{supp}(\psi)$. Accordingly, we define a set $T(z)$ of *truncation multipliers at z* by

$$T(z) = \{ \psi \in C_c^\infty(\Omega) \mid \psi(z) \neq 0 \}, \quad (13a)$$

and then define the *frequency set of $\langle u, \cdot \rangle \in \mathcal{D}'(\Omega)$ at a point $z \in \Omega$* by

$$F_z(u) = \bigcap_{\psi \in T(z)} F(\psi u). \quad (13b)$$

If $\Omega = \mathbb{R}^3$ and $\psi \in T(z)$ and we apply multiplication by ψ to the distribution $\langle v, \cdot \rangle$ defined by equation (9), then, when $z \notin \text{ss}(v)$ (when z is not in the singular support of $\langle v, \cdot \rangle$), both $F(v)$ and $F(\psi v)$ will be the empty set; when $z \in \text{ss}(v)$, the distributions $\langle v, \cdot \rangle$ and $\langle \psi v, \cdot \rangle$ will usually differ, but their frequency sets $F(v)$ and $F(\psi v)$ will be equal, because the bad directions ξ_S of their Fourier transforms are the same.

At last we are in a position to define the *wave front set* $\text{WF}(u)$ of a distribution $\langle u, \cdot \rangle$ in $\mathcal{D}'(\Omega)$ as the set of those pairs (x, ξ_S) , with x in $\text{ss}(u)$ and ξ_S in S^2 , that are related by $\xi_S \in F_x(u)$. In other words,

$$\text{WF}(u) = \{ (x, \xi_S) \in \text{ss}(u) \times S^2 \mid \xi_S \in F_x(u) \}. \quad (13c)$$

In general, the wave front set is far more elusive than the example that we encountered in equation (12) for the distribution $\langle v, \cdot \rangle$ defined by (9). For that case, the equator was the relevant subset of S^2 for every point x of $\text{ss}(v)$. In general, the relevant subset of S^2 depends on the point x of $\text{ss}(u)$. For the example (9), localization by ψ did not change the frequency set of $\langle v, \cdot \rangle$; in general, there will be a change. The only simple, general facts that we know are (Hörmander 1983, p. 254)

- (a) the projection of $\text{WF}(u)$ into Ω is $\text{ss}(u)$;
- (b) when $\langle v, \cdot \rangle \in \mathcal{E}'(\mathbb{R}^3)$, the projection of $\text{WF}(v)$ into S^2 is $F(v)$.

Nevertheless, much is known. Reduced to $\Omega \subset \mathbb{R}^3$, Hörmander's definition of the *characteristic set* $\text{Char } \mathcal{L}$, which describes all bicharacteristic strips of an operator $\mathcal{L}$, amounts to the following[†]. $\text{Char } \mathcal{L}$ is the set of pairs (x, ξ_S) in $\Omega \times S^2$ such that $x(s)$ and $\xi(s)$ are related by the Hamiltonian equations (1), (2) and (3). In other words,

$$\text{Char } \mathcal{L} = \{ (x, \xi_S) \in \Omega \times S^2 \mid (x(s), \xi(s)) \text{ is a solution of (1), (2) and (3)} \}.$$

A theorem of Hörmander (1983, p. 271), the first of several in this direction, states that, *if $\langle u, \cdot \rangle \in \mathcal{D}'(\Omega)$ is a solution of $\langle \mathcal{L}u, \cdot \rangle = 0$, then*

$$\text{WF}(u) \subset \text{Char } \mathcal{L}. \quad (14)$$

In order to illustrate this, we return to the solution $\langle u, \cdot \rangle \in \mathcal{D}'(\mathbb{R}^3)$ of equations (5), (6), (7) and exploit the cylindrical symmetry of that problem. Let

$$x = (r \cos \theta, r \sin \theta, t) \quad \text{and} \quad \xi = (\rho \cos \lambda, \rho \sin \lambda, \tau).$$

Then the relevant operator and principal symbol are

$$\mathcal{L}_2 = \left(\frac{\partial}{\partial r} \right)^2 - \left(\frac{\partial}{\partial t} \right)^2 + \frac{1}{r} \frac{\partial}{\partial r} \quad \text{and} \quad L_2(x, \xi) = \rho^2 - \tau^2;$$

the relevant subset of $\mathbb{R}^2$ is

$$\Omega_2 = \{ (r, t) \mid r > 0, -\infty < t < \infty \}.$$

The corresponding bicharacteristic equations (1), (2) and (3), now for $(r(s), t(s))$ and $(\rho(s), \tau(s))$, are easily solved; one finds that the characteristic set of $\mathcal{L}_2$ is

$$\text{Char } \mathcal{L}_2 = \mathbb{R}^3 \times \Gamma, \quad (15a)$$

where (Figure 10)

$$\Gamma = \left\{ \xi \in S^2 \mid \rho = \frac{1}{\sqrt{2}}, \tau = \pm \frac{1}{\sqrt{2}}, \lambda = \text{constant} \right\}; \quad (15b)$$

the condition $\lambda = \text{constant}$ is included because it holds for each solution $\xi(s)$ of equation (2). The factor $\mathbb{R}^3$ occurs in (15a) because all lines

$$0 \leq r < \infty, \quad t = t_0 \pm r, \quad \theta = \text{constant}, \quad (15c)$$

for arbitrary $t_0 \in \mathbb{R}$, are bicharacteristic curves of $\mathcal{L}_2$; the points in these lines fill $\mathbb{R}^3$.

The definition (13c) and the theorem (14) combine, for the present case, to

$$\text{WF}(u) \subset \{ (x, \xi_S) \mid x \in \text{ss}(u), \xi_S \in \Gamma \}, \quad (16)$$

[†]Of the equivalent notions of a cone C in $\mathbb{R}^n \setminus \{0\}$ and its intersection $C_S = C \cap S^{n-1}$ with the unit sphere, Hörmander usually preferred C , while we find use of C_S more helpful.

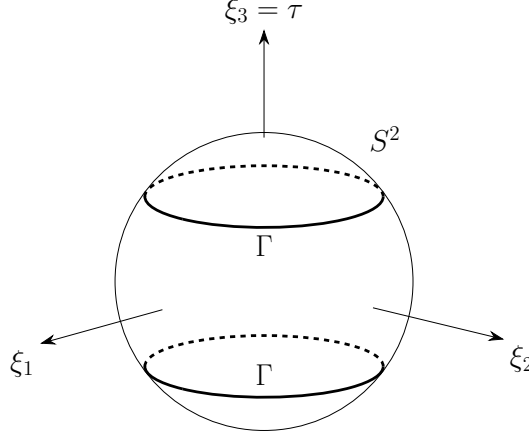


Figure 10

where $ss(u)$ is described by equation (8) and is shown in Figure 8. This result is remarkable, even though it does not describe $WF(u)$ exactly, because explicit formulae show that the singularities of $\langle u, \cdot \rangle$ along $ss(u)$ are complicated, making it unlikely that $WF(u)$ can be found exactly even in this relatively simple case.

Here is an indication of these singularities. Let

$$\vartheta = \tan^{-1} \frac{t-a}{r} \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right].$$

Then, for $0 < r < a$,

$$u(x) = \begin{cases} 1 & \text{if } \vartheta < -\frac{\pi}{4} \text{ and } t > r-a, \\ 1 - \frac{1}{2} \left(\frac{r}{a} \right)^{-1/2} \{1 + O(1 + \tan \vartheta)\} + O\left(\left(\frac{r}{a} \right)^{1/2} \right) & \text{as } \vartheta \downarrow -\pi/4, \\ 1 - \frac{1}{2\pi} \left(\frac{r}{a} \right)^{-1/2} \left(\log \frac{32}{1 - \tan \vartheta} \right) \{1 + O(1 - \tan \vartheta)\} + O\left(\left(\frac{r}{a} \right)^{1/2} \right) & \text{as } \vartheta \uparrow \pi/4, \\ 1 - \frac{1}{2\pi} \left(\frac{r}{a} \right)^{-1/2} \left(\log \frac{32}{\tan \vartheta - 1} \right) \{1 + O(\tan \vartheta - 1)\} + O\left(\left(\frac{r}{a} \right)^{1/2} \right) & \text{as } \vartheta \downarrow \pi/4. \end{cases}$$

The second and third estimates of this list follow from

$$u(x) = 1 - \frac{1}{\pi} \left(\frac{r}{a} \right)^{-1/2} K \left(\frac{1 + \tan \vartheta}{2} \right) + O\left(\left(\frac{r}{a} \right)^{1/2} \right) \quad \text{if } -\frac{\pi}{4} < \vartheta < \frac{\pi}{4},$$

where

$$K(m) = \int_0^1 \frac{1}{\sqrt{(1-y^2)(1-my^2)}} dy, \quad 0 \leq m < 1,$$

denotes the complete elliptic integral of the first kind.

We can shed a little light on the result (16) by noting that the Fourier transform of $\langle u, \cdot \rangle$ is, in merely symbolic form,

$$\hat{u}(\xi) = \pi g^\dagger(\rho) \{ \delta(\rho - \tau) + \delta(\rho + \tau) \}, \quad (17)$$

where

$$g^\dagger(\rho) = 2\pi \frac{a}{\rho} J_1(a\rho),$$

and where J_1 denotes the Bessel function of the first kind and of order 1. (This means that, for all rapidly decreasing ϕ in $C^\infty(\mathbb{R}^3)$,

$$\langle \hat{u}, \phi \rangle = \pi \iint_{\mathbb{R}^2} g^\dagger(\rho) \{ \phi(\xi_1, \xi_2, \rho) + \phi(\xi_1, \xi_2, -\rho) \} d\xi_1 d\xi_2 ;$$

it implies that $\langle \hat{u}, \phi \rangle = \langle u, \hat{\phi} \rangle$ for all such ϕ , and this last is the definition of $\langle \hat{u}, \cdot \rangle$ for the class of *tempered* distributions, to which our present example belongs.)

We observe from (15b) and (17) that *the set Γ , which contains ξ_S for all bicharacteristic strips of the operator $\mathcal{L}_2$, is precisely the set of directions for which $\hat{u}(\xi)$ differs from zero.*

GLANCING RAYS, 1976 AND 1977

In the previous section we have seen difficulties in description of the wave front set even for the case of equations (5), (6) and (7), which involve constant coefficients, cylindrical symmetry and the whole space $\mathbb{R}^3$. Naturally, such difficulties are compounded by variable coefficients, by lack of symmetry and by the presence of a boundary $\partial\Omega$ of the set Ω . However, by 1976 Hörmander (1971 a, b, c), with contributions from Duistermaat (1972) and Lax and Nirenberg (1973), had largely overcome these difficulties, with one notable exception. The open question was essentially this: for the solution $\langle u, \cdot \rangle$ in $\mathcal{D}'(\Omega)$ of an equation $\langle \mathcal{L}u, \cdot \rangle = 0$ or $\langle \mathcal{L}u, \cdot \rangle = \langle v, \cdot \rangle$, *what happens when a bicharacteristic curve of the wave front set $\text{WF}(u)$ meets the boundary $\partial\Omega$ tangentially?* The case in which a bicharacteristic curve of $\text{WF}(u)$ meets $\partial\Omega$ *transversely* was understood; there is reflection of the bicharacteristic curve. But it was feared that, if a bicharacteristic curve of $\text{WF}(u)$ meets $\partial\Omega$ tangentially, then this might cause various bicharacteristic curves of $\text{WF}(u)$ to emanate from $\partial\Omega$ beyond that point. In other words, it was feared that singularities might propagate along the boundary $\partial\Omega$.

In the paper (8), Gerard answered the question for a particular case, the simplest form of which (two space variables and time) proceeds as follows. Let

$$\Omega = \{ (x, y, t) \in \mathbb{R}^3 \mid x > 0, (y, t) \in \mathbb{R}^2 \}, \quad (18)$$

and let $\langle f, \cdot \rangle \in \mathcal{E}'(\mathbb{R}^2)$ be a distribution of compact support acting in the boundary $\partial\Omega$ (acting in the plane $\{x = 0\}$). Little would be lost if $\langle f, \cdot \rangle$ were replaced by a genuine,

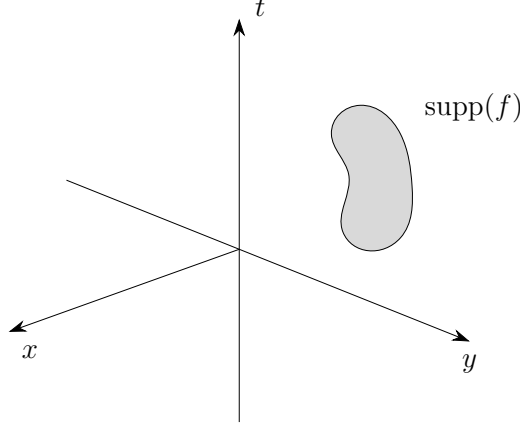


Figure 11

integrable function f of compact support in $\mathbb{R}^2$. The problem is to find a function u in $C^\infty([0, \infty) \rightarrow \mathcal{D}'(\mathbb{R}^2))^\ddagger$ such that, in this semi-distributional sense,

$$\mathcal{L}u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - (1+x) \frac{\partial^2 u}{\partial t^2} = 0 \quad \text{in } \Omega, \quad (19a)$$

subject to

$$\langle u(0, \cdot, \cdot), \cdot \rangle = \langle f, \cdot \rangle \quad \text{in } \mathcal{E}'(\mathbb{R}^2), \quad (19b)$$

$$u(x, y, t) = 0 \quad \text{if } t < \inf \{ z \mid (y, z) \in \text{supp}(f) \}. \quad (19c)$$

The situation is illustrated in Figure 11. We have already met equation (19a) as the source of the bicharacteristic strip in Figures 4 and 5.

If one can construct a fundamental solution k in $C^\infty([0, \infty) \rightarrow \mathcal{D}'(\mathbb{R}^2))$ that satisfies (again in a semi-distributional sense)

$$\mathcal{L}k = 0 \quad \text{in } \Omega, \quad (20a)$$

subject to

$$k(0, y, t) = \delta(y)\delta(t), \quad (20b)$$

$$k(x, y, t) = 0 \quad \text{if } t < 0, \quad (20c)$$

then the problem (19a, b, c) is solved by the distributional convolution (at fixed x)

$$\langle u(x, \cdot, \cdot), \cdot \rangle = k(x, \cdot, \cdot) * f; \quad (21a)$$

if f were a genuine, integrable function of compact support, then this would become

$$u(x, y, t) = \iint_{\text{supp}(f)} k(x, y - y', t - t') f(y', t') dy' dt' \quad (0 \leq x < \infty). \quad (21b)$$

[‡]This means that u has partly symbolic values $u(x, y, t)$ such that $\langle u(x, \cdot, \cdot), \cdot \rangle \in \mathcal{D}'(\mathbb{R}^2)$ for each fixed x in $[0, \infty)$ and that the map $x \mapsto \langle u(x, \cdot, \cdot), \cdot \rangle$ is infinitely differentiable with respect to x on $[0, \infty)$.

It seems that Gerard had no difficulty with solution of the problem that reduces to (20a, b, c) in the case of two space variables; his method was as follows. Under a Fourier-Laplace transform

$$\left. \begin{aligned} \hat{\varphi}(\beta, \gamma) &= \iint_{\mathbb{R} \times (0, \infty)} \exp(-i\beta y - i\gamma t) \varphi(y, t) dy dt, \\ \beta &\in \mathbb{R}, \gamma \in \mathbb{C}, \operatorname{Im} \gamma < 0, \end{aligned} \right\}$$

for functions φ such that $\hat{\varphi} \in \mathcal{C}_c^\infty(\mathbb{R}^2)$, and under the definition

$$\langle \hat{k}(x, \cdot, \cdot), \varphi \rangle = \langle k(x, \cdot, \cdot), \hat{\varphi} \rangle \quad \text{in } \mathcal{D}'(\mathbb{R}^2) \text{ for all such } \varphi,$$

the problem (20a, b, c) becomes

$$\left. \begin{aligned} \frac{d^2 \hat{k}}{dx^2} - \beta^2 \hat{k} + (1+x)\gamma^2 \hat{k} &= 0, \quad 0 < x < \infty, \\ \hat{k}(0, \beta, \gamma) &= 1. \end{aligned} \right\} \quad (22)$$

(We have changed Gerard's Fourier-Laplace transform a little in order to have an analogue of the definition $\langle \hat{v}, \varphi \rangle = \langle v, \hat{\varphi} \rangle$ for tempered distributions. As some readers will know, a Paley-Wiener theorem establishes the properties of φ when $\hat{\varphi} \in \mathcal{C}_c^\infty(\mathbb{R}^2)$.) The further transformation

$$\zeta = \gamma^{-4/3} \beta^2 - (1+x)\gamma^{2/3}, \quad \zeta_0 = \gamma^{-4/3} \beta^2 - \gamma^{2/3}, \quad (23a)$$

in which

$$\gamma = |\gamma| \exp \left\{ i \frac{3}{2} (\pi - \vartheta) \right\}, \quad |\vartheta| \leq \frac{\pi}{3}, \quad (23b)$$

yields Airy's equation

$$\frac{d^2 \hat{k}}{d\zeta^2} - \zeta \hat{k} = 0, \quad \text{with } \hat{k}|_{\zeta=\zeta_0} = 1, \quad (24)$$

and implies that, as x increases from 0 with $\operatorname{Im} \gamma \leq 0$, the point

$$\zeta(x) = \zeta_0 + |\gamma|^{2/3} x e^{-i\vartheta}$$

moves along a straight path as in Figure 12. The initial value ζ_0 is always in the closed sector to the right of AOB and $\zeta(x)$ is always in the closed sector to the right of $C\zeta_0 D$. Airy's equation has solutions $\operatorname{Ai}(\zeta)$ and $\operatorname{Bi}(\zeta)$ with asymptotic behaviour

$$\begin{aligned} \operatorname{Ai}(\zeta) &\sim \frac{1}{2} \pi^{-1/2} \zeta^{-1/4} \exp \left(-\frac{2}{3} \zeta^{3/2} \right) \quad \text{as } |\zeta| \rightarrow \infty \text{ with } |\arg \zeta| < \pi, \\ \operatorname{Bi}(\zeta) &\sim \pi^{-1/2} \zeta^{-1/4} \exp \left(\frac{2}{3} \zeta^{3/2} \right) \quad \text{as } |\zeta| \rightarrow \infty \text{ with } |\arg \zeta| < \frac{1}{3} \pi; \end{aligned}$$

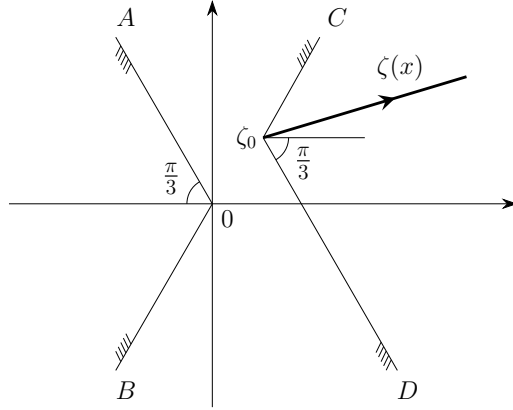


Figure 12

also, Ai is an entire function, the only zeros of which are on the (strictly) negative real axis. Accordingly,

$$\hat{k}(x, \beta, \gamma) = \frac{\text{Ai}(\zeta)}{\text{Ai}(\zeta_0)}. \quad (25)$$

Evidently the domain of $\hat{k}$ can be extended from $\text{Im } \gamma < 0$ to $\text{Im } \gamma \leq 0$; then $\hat{k}$ can be inverted as a Fourier transform rather than as a Fourier-Laplace transform.

Now, there is a bijection from the set $C^\infty([0, \infty) \rightarrow \mathcal{D}'(\mathbb{R}^2))$ of distribution-valued functions to a set $\mathcal{D}'_r((0, \infty) \times \mathbb{R}^2)$ of distributions. (This point is obscure in the paper (8) but clear in (9); distributions in $\mathcal{D}'_r((0, \infty) \times \mathbb{R}^2)$ are restrictions to $(0, \infty) \times \mathbb{R}^2$ of certain distributions in $\mathcal{D}'(\mathbb{R} \times \mathbb{R}^2)$ that are called *regular in x* in the paper (9).) A consequence of this is that a distribution $\langle K, \cdot \rangle$ in $\mathcal{D}'_r((0, \infty) \times \mathbb{R}^2)$ is defined by the inversion formula

$$\langle K, \hat{\psi} \rangle = \iiint_{(0, \infty) \times \mathbb{R}^2} \hat{k}(x, \beta, \gamma) \psi(x, \beta, \gamma) dx d\beta d\gamma \quad (26a)$$

for all

$$\hat{\psi} \in \mathcal{C}_c^\infty((0, \infty) \times \mathbb{R}^2) \quad (26b)$$

and

$$\psi(x, \beta, \gamma) = \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2} \exp(i\beta y + i\gamma t) \hat{\psi}(x, y, t) dy dt. \quad (26c)$$

The question is now: *what can be inferred from (25) and (26a, b, c) about the wave front set $\text{WF}(K)$ of K ?*

Gerard dealt with this question in six and a half formidable pages of the paper (8); these pages include a partition of $\hat{k}$ into four parts, skilful use of classical analysis for the integrals in (26a, c) and application of existing theorems of Hörmander. The result was this.

The wave front set $\text{WF}(K)$ is contained in the union of those bicharacteristic strips that project into $\bar{\Omega}$ as the bicharacteristic curves described by equations (28) and illustrated in Figure 13. (27)

In terms of $x \geq 0$ and a parameter $\lambda \in [0, 1]$, the bicharacteristic curves in question are

$$\left. \begin{aligned} y &= \pm 2\lambda^{1/2} \left(\sqrt{1-\lambda+x} - \sqrt{1-\lambda} \right), \\ t &= 2 \left\{ (1+x)\sqrt{1-\lambda+x} - \sqrt{1-\lambda} \right\} - \frac{4}{3} \left\{ (1-\lambda+x)^{3/2} - (1-\lambda)^{3/2} \right\}, \\ x &\geq 0, \quad 0 \leq \lambda \leq 1. \end{aligned} \right\} \quad (28)$$

We note that $t(x) \geq 0$ because $t(0) = 0$ and

$$t'(x) = (1+x)(1-\lambda+x)^{-1/2} > 0.$$

The most important of these curves is the widest:

$$\text{if } \lambda = 1, \quad \text{then} \quad y = \pm 2x^{1/2}, \quad t = 2x^{1/2} + \frac{2}{3}x^{3/2}. \quad (29)$$

This meets the boundary $\partial\Omega$ tangentially. If we begin with $y < 0$ and let y increase through 0, then this bicharacteristic curve causes nothing dreadful in the half-space $\{y > 0\}$.

Figure 14 shows a set of bicharacteristic curves that emanate from a curve $x = h(y)$, $t = 0$ closer to the boundary $\partial\Omega$ than the parabola $x = \frac{1}{4}y^2$, $t = 0$ appearing in (29) and Figure 13. The result (27) ensures that such bicharacteristic curves make no contribution to $\text{WF}(K)$. For the particular configuration in Figure 14,

$$h(y) = \begin{cases} 0 & \text{if } 0 < y \leq 1, \\ \exp\left(-\frac{1}{y-1}\right) & \text{if } 1 < y < \infty. \end{cases}$$

Gerard then applied this result for K to the more general problem (19a,b,c). There is a distribution $\langle U, \cdot \rangle$ in $\mathcal{D}'_r((0, \infty) \times \mathbb{R}^2)$ that corresponds to the distribution-valued function u , with values $k(x, \cdot, \cdot) * f$, in (21a). We recall that $\langle f, \cdot \rangle$ is the forcing distribution in the plane $\{x = 0\}$. The result for $\text{WF}(U)$ resembles closely the result for $\text{WF}(K)$; the family of bicharacteristic curves (28) emanating from the origin, for $\text{WF}(K)$, is replaced, for $\text{WF}(U)$, by a larger family of bicharacteristic curves emanating from the points of $\text{supp}(f)$.

The paper (9) made three further contributions; all concerned the solution $\langle u, \cdot \rangle$ in $\mathcal{D}'(\Omega)$ of $\langle \mathcal{L}u, \cdot \rangle = 0$ or of $\langle \mathcal{L}u, \cdot \rangle = \langle v, \cdot \rangle$, the operator $\mathcal{L}$ and the side conditions being those that reduce to (19a, b, c) for two space variables. The work of Melrose (1975, 1976) also played a part.

First, the paper made precise the relationship between the distribution $\langle u, \cdot \rangle$, which determines $\text{WF}(u)$, and the distribution-valued functions that form stepping-stones in

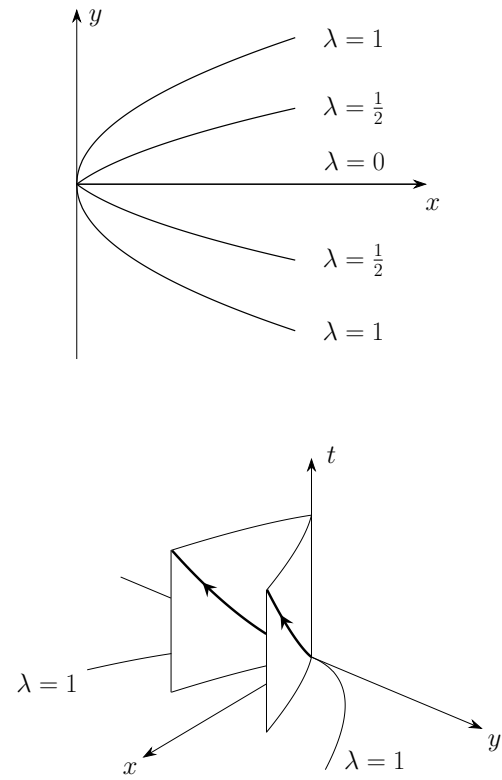


Figure 13

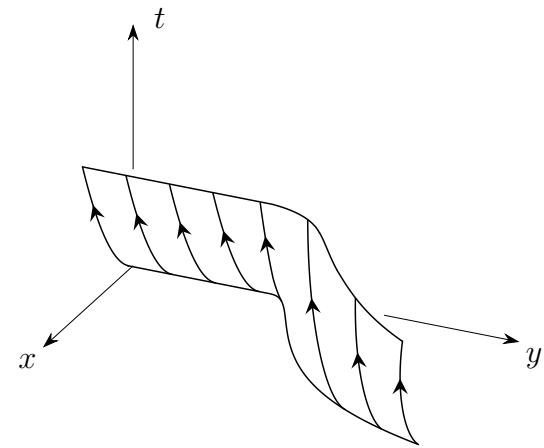


Figure 14

the estimation of $\text{WF}(u)$. As was noted above, this (rather delicate) relationship had been vague in the earlier paper.

Secondly, the paper (9) extended the earlier results for the homogeneous equation $\langle \mathcal{L}u, \cdot \rangle = 0$ to the inhomogeneous equation $\langle \mathcal{L}u, \cdot \rangle = \langle v, \cdot \rangle$. This was achieved by means of machinery (a pseudo-differential operator called the *boundary operator*) that could also shorten the earlier proofs, but that is far beyond us here.

The third contribution had its roots in the fact that functions in the set $C^\infty(\Omega)$ are by no means the best functions with domain Ω . That accolade is usually given to the set $A(\Omega)$ of *real-analytic* functions with domain Ω . (These are functions having a convergent Taylor series in a neighbourhood of every point of Ω .) For a distribution $\langle u, \cdot \rangle$ in $\mathcal{D}'(\Omega)$ one can define the singular support $\text{ss}_A(u)$ with respect to $A(\Omega)$ to be the smallest closed subset of Ω outside which the symbolic function u may be replaced by a genuine function that is real-analytic in the open set $\Omega \setminus \text{ss}_A(u)$. There are also *Gevrey classes* $G(s, \Omega)$, with $1 \leq s < \infty$ and $G(1, \Omega) = A(\Omega)$, that go some way towards filling the gap between $A(\Omega)$ and $C^\infty(\Omega)$. The definition of singular support $\text{ss}_{G(s)}(u)$ with respect to one of these is similar to the definitions of $\text{ss}(u)$ and $\text{ss}_A(u)$. Naturally,

$$\text{ss}(u) \subset \text{ss}_{G(s)}(u) \subset \text{ss}_A(u),$$

with strict inclusion on the right when $s > 1$.

By 1976 Hörmander had managed to give corresponding definitions of wave front sets $\text{WF}_A(u)$ and $\text{WF}_{G(s)}(u)$, and to prove that

$$\text{WF}(u) \subset \text{WF}_{G(s)}(u) \subset \text{WF}_A(u)$$

for every $\langle u, \cdot \rangle$ in $\mathcal{D}'(\Omega)$. This is surprising, because $A(\Omega)$ contains no non-trivial function of compact support in Ω and therefore contains no truncation multiplier at a point z of Ω .

Despite these huge difficulties, Friedlander and Melrose were able, in the third part of the paper (9), to refine the results of (8) by estimates of $\text{WF}_A(u)$ and $\text{WF}_{G(s)}(u)$ for their problem. There were parallel results by M.E. Taylor (1976) at essentially the same time. It turned out that $\text{WF}_{G(s)}(u)$ with $s \geq 3$ is not conspicuously different from $\text{WF}(u)$. But for $\text{WF}_A(u)$ certain *indirect curves* close to $\partial\Omega$, which make no contribution to $\text{WF}(u)$, now play a part. In the language of specialists in this field, analytic singularities can propagate along such indirect curves. That is to say, along such curves the symbolic function u may be replaced by a C^∞ function but not by a real-analytic function.

LUND

Gerard spent the period 1 January to 31 May 1977 at Lund, accompanied by Liz: details of his visit are based on information kindly given to us by Professor Nils Dencker. During this time Gerard was substitute professor for Lars Gårding and gave a course entitled “Topics in wave propagation” in which the various themes were

- Radiation fields of expanding waves
- Energy estimates (of Friedrichs-Lewy type)
- Lorenzian metrics
- Energy tensor
- Conformal spacetime
- Inverse problems of radiation fields
- Zonal harmonics
- The Funk-Hecke formula
- Radiation fields of compactly supported sources.

It seems that he had been invited partly because he was the supervisor of Richard Melrose, whose recent results on gliding rays had been a breakthrough; thus it was natural to want to learn about Friedlander's own results. However, the daunting task facing anyone coming to Lund and lecturing to Hörmander was underlined by Gårding when he jokingly asked Gerard "Do you want to go to hell?"

Dencker remembers Gerard as a very friendly and cultured professor of a characteristically British type, whose lunchtime conversations covered a wide range of topics, from the emerging politician Margaret Thatcher to the latest British punk bands (of which he knew more than Hörmander). Anecdotally, his tolerance of Swedish spiced aquavit, the so-called "snaps", was low.

BILLIARD BALL TRAJECTORIES IN POLYGONAL DOMAINS

This section is based on comments kindly given to us by Professor D. Vassiliev. In (10) and (11) Gerard studied the non-homogeneous wave equation in a planar domain with a boundary that is polygonal in the sense that it is the union of a finite number of pairwise disjoint Jordan polygons. Dirichlet boundary conditions were imposed together with an appropriate initial condition; the prescribed right-hand side of the wave equation was assumed to have compact support both in the spatial variables and in time. Such problems are of great interest for spectral theory as the propagation of singularities of the solution determines the asymptotic distribution of the eigenvalues of the Dirichlet Laplacian. At a formal level the relation between the wave equation and the spectral problem for the Laplacian can be seen by performing a Fourier transform in the time variable, which replaces differentiation in time by multiplication by the spectral parameter.

Singularities of solutions of the wave equation propagate along billiard ball trajectories (closed geodesics), reflecting from the boundary. A peculiar feature of polygonal boundaries is that diffraction from the vertices has to be taken into account: when a billiard trajectory hits a vertex whose angle is generic, all possible continuations of the trajectory from this vertex must be considered. In the papers mentioned above, Gerard paid special attention to periodic billiard trajectories as these are responsible for the clustering (that is, uneven distribution) of the eigenvalues of the Dirichlet Laplacian. He showed in (11) that if all the vertices are diffractive, then there are infinitely many

minimal closed diffractive geodesics, thereby settling (in the negative) a conjecture he made in (10). These two papers motivated a good deal of subsequent research by others on wave propagation on Riemannian manifolds with non-smooth boundaries.

RETIREMENT

After retirement in 1982, he and Liz lived in London to be closer to their family. Paul's first child, Naomi, had just been born and Gerard's other son, Peter, was also living in London. Once in London, Gerard missed having contact with a mathematics department but Ambrose Rogers (FRS 1959) was delighted to offer him an honorary position at UCL. There he met Bill Stephenson, mathematician and keen student of communism, who was stunned to learn that Gerard's mother was Ruth Fischer. Further links with the past came as a result of one of us (M.P.) attending a season of German silent films organised by the celebrated and knowledgeable film critic John Gillett. When John learned that Hanns Eisler was Gerard's uncle he was tremendously keen to talk to Gerard. They met before a screening of *Hangmen also die*, directed by Fritz Lang, based on a story by Bertold Brecht and with a score written by Eisler. A memorable conversation ensued with Gillett becoming more and more amazed at the detailed information poured out by Gerard in his usual low key, modest way about one of Gillett's film music heroes.

In his later years Gerard started to receive the credit he deserved for his path-breaking work on the Airy equation and its relevance to the modern theory of hyperbolic differential equations. In 2000, Richard Melrose organised a conference in his honour at MIT with speakers including Peter Lax and Lars Hörmander himself. At his funeral, Sir Hermann Bondi spoke eloquently about his friendship with him and recalled how Gerard, as the person in DAMTP who knew more pure mathematics than anyone else, was always willing to take time to provide the pure mathematics input for a colleague's paper, often to the detriment of his own career.

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