Dynamics of expanding gases

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Abstract

We report on a recent development in the dynamics of expanding compressible gases governed by the Euler and the gravitational Euler-Poisson system in the vacuum free boundary framework. We discuss expanding affine solutions that exist globally-in-time. And we explain the methodology of the global existence of expanding smooth non-affine solutions to the Euler system for $1 < \gamma \leq \frac{5}{3}$ and the Euler-Poisson system for $\gamma = \frac{4}{3}$.

1 Introduction

In this paper, we are interested in compactly supported *expanding* compressible inviscid gases with finite total mass and energy in the three-dimensional space. The goal is to present a series of recent mathematical advances on the dynamics of expanding gases in a unified manner. We shall in particular consider two physical models exhibiting the global-in-time expansion phenomenon. The first one is the well-known system of the compressible Euler equations:

$$\partial_t \rho + \text{div}\,(\rho \mathbf{u}) = 0 \tag{1.1a}$$

$$\rho\,(\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}) + \nabla p = 0 \tag{1.1b}$$

where ρ is the density, $\mathbf{u}$ is the velocity and p is the pressure of the gas. We shall restrict our attention to polytropic gases whose equation of state is given by

$$p = \rho^\gamma, \quad 1 < \gamma < 2. \tag{1.2}$$

The Euler equations of fluid mechanics are among the first few partial differential equations ever written. They describe the motion of fluids/gases and are often used to describe a range of important physical phenomena, such as turbulence, vortex filamentation, gas expansion, and many others. Their rich structure has been a source of inspiration for the development of many mathematical tools tying together ideas from nonlinear partial differential equations (PDE), dynamical systems, functional analysis, and geometry.

The second example is the gravitational Euler-Poisson system, a fundamental hydrodynamical model describing the dynamics of gaseous stars subject to Newtonian gravity:

$$\partial_t \rho + \text{div}\,(\rho \mathbf{u}) = 0 \tag{1.3a}$$

$$\rho\,(\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}) + \nabla p = -\rho \nabla \Phi \tag{1.3b}$$

$$\Delta \Phi = 4\pi\rho, \quad \lim_{|x|\to\infty} \Phi(t,x) = 0 \tag{1.3c}$$

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Again we shall assume that the equation of state is given by the γ-law (1.2). The gravitational Euler-Poisson system has been extensively used in the astrophysics literature in an attempt to understand the rich phenomenology associated with star dynamics, with the emphasis on finding special solutions subject to symmetry assumptions.

Two fundamental conserved quantities associated with the Euler and Euler-Poisson systems are the total mass

$$M(\rho) := \int_{\mathbb{R}^3} \rho \, dx, \tag{1.4}$$

and energy

$$E(\rho, \mathbf{u}) := \begin{cases} \int_{\mathbb{R}^3} \left[\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{1}{\gamma-1}\rho^\gamma\right] \, dx & \text{for (1.1)–(1.2)} \\ \int_{\mathbb{R}^3} \left[\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{1}{2}\rho\Phi + \frac{1}{\gamma-1}\rho^\gamma\right] \, dx & \text{for (1.3)–(1.2)} \end{cases} \tag{1.5}$$

We are interested in gases and fluids preserving the total mass and the energy along the evolution.

1.1 What are expanding solutions and why study them?

We use the term *expansion* to refer to the growth of the support of a gas or a fluid blob, while the amplitude of its density decreases due to the mass conservation. A possibility of such a dynamic behavior yields a rather exciting physical argument (and a hint) for finding global-in-time solutions to compressible fluid systems. Namely, the underlying expansion of the fluid blob forces the flow lines to spread away from each other, therefore counteracting the possible shock formation. It is therefore a natural first step to investigate properties of global solutions to (1.1) and (1.3) *assuming* that they exist. For the sake of brevity, we shall limit our discussion to the Euler system (1.1) and comment below on the known results for the Euler-Poisson system (1.3).

Proposition 1.1. *Let* $(\rho, \mathbf{u})$ *be a smooth global-in-time solution to the Euler system* (1.1)–(1.2) *with finite total mass and energy. Then the* L^1*-norm of the pressure decays to zero as* t *tends to infinity. More specifically, the following quantitative estimates hold for large times* $t > 1$,

$$\int_{\mathbb{R}^3} p dx \lesssim \begin{cases} \frac{1}{t^2} & \text{for } \gamma \geq \frac{5}{3} \\ \frac{1}{t^{3\gamma-3}} & \text{for } 1 < \gamma \leq \frac{5}{3} \end{cases}. \tag{1.6}$$

Proposition 1.1 is well-known [2, 30] and it can be proved by using virial type identities. Such arguments have been well-known in the physics literature and often used to prove robust decay statements for nonlinear dispersive equations. For the convenience of the reader, we provide a proof of Proposition 1.1 in Appendix. An immediate consequence of Proposition 1.1 is that the volume of the fluid domain grows in time.

Let $\Omega(t)$ denote the support of the density ρ:

$$\Omega(t) = \overline{\{x \in \mathbb{R}^3 : \rho(t,x) > 0\}}$$

Proposition 1.2. *Let* $(\rho, \mathbf{u})$ *be a smooth global-in-time solution to the Euler system* (1.1)–(1.2) *with finite total mass and energy. Then*

$$|\Omega(t)| \gtrsim \begin{cases} t^{\frac{2}{\gamma-1}} & \text{for } \gamma \geq \frac{5}{3} \\ t^3 & \text{for } 1 < \gamma \leq \frac{5}{3} \end{cases} \tag{1.7}$$

Proof. By Hölder inequality,

$$M = \int_{\mathbb{R}^3} \rho dx \le \left(\int_{\mathbb{R}^3} \rho^\gamma dx\right)^{\frac{1}{\gamma}} \left(\int_{\Omega(t)} 1 dx\right)^{\frac{\gamma-1}{\gamma}} = \left(\int_{\mathbb{R}^3} p dx\right)^{\frac{1}{\gamma}} |\Omega(t)|^{\frac{\gamma-1}{\gamma}},$$

which leads to

$$\frac{M^{\frac{\gamma}{\gamma-1}}}{\left(\int_{\mathbb{R}^3} p dx\right)^{\frac{1}{\gamma-1}}} \le |\Omega(t)|.$$

The result now follows from Proposition 1.1. □

We can also see that the diameter of the domain should grow in time. Let

$$D(t) := \sup_{x\in\Omega(t)} |x|. \tag{1.8}$$

Proposition 1.3. *Let* $(\rho, \mathbf{u})$ *be a smooth global-in-time solution to the Euler system* (1.1)–(1.2) *with finite total mass and energy. Then for each* $\gamma > 1$ *and for large times* t

$$D(t) \gtrsim t. \tag{1.9}$$

Proof. Let

$$F(t) := \int_{\mathbb{R}^3} \rho |x|^2 dx.$$

By a direct computation we obtain

$$F'(t) = 2\int_{\mathbb{R}^3} \rho \mathbf{u} \cdot x dx, \quad F''(t) = 4E + \frac{2(3\gamma-5)}{\gamma-1}\int_{\mathbb{R}^3} p dx.$$

We recall that E is a positive constant. If $\gamma \ge \frac{5}{3}$, $F''(t) \ge 4E > 0$ for all time t and if $1 < \gamma < \frac{5}{3}$, $F''(t) \ge 2E > 0$ for all $t \ge t_* > 0$. In both cases we have

$$F(t) \ge F(t_*) + F'(t_*)(t-t_*) + E(t-t_*)^2 \quad \text{for all } \ t \ge t_*.$$

Since $D(t)^2 M \ge F(t)$ and since $E, M > 0$, we obtain the desired result. □

An analogous argument applies to the gravitational Euler-Poisson system. One can show that [24, 5] when $\gamma \ge \frac{4}{3}$, the support of a global-in-time solution with finite mass and *positive* energy grows *at least* linearly-in-time in analogy to Proposition 1.3. Note that the physical energy for the Euler-Poisson flows can be negative due to the attractive nature of the gravitational energy (Φ is negative in (1.5)) in contrast to Euler flows where the energy is always positive for nonzero solutions. In particular, there exist interesting special solutions with a negative total energy - the most famous ones are the so-called Lane-Emden steady stars in the regime $\gamma > \frac{4}{3}$ [1].

The above results show that globally-in-time defined compactly supported solutions should expand if they exist. However, this statement is conditional upon the existence of global solutions and two natural questions present themselves:

Q1. *Are there any non-trivial examples of global-in-time solutions with a finite total mass and energy?*

Q2. *What is the right mathematical framework that captures the expansion phenomenon?*

As we shall see Q1 and Q2 are intricately related. In order for Q1 to make sense, one needs a suitable concept of solution, which is related to Q2. And within the right framework, one hopes to establish the existence of such solutions. A short answer to Q1 and Q2 is yes, one can find classes of special expanding solutions to both the Euler and the Euler-Poisson system, and a robust function analytic framework for their analysis has been recently developed in the context of so-called *vacuum free boundary problems*.

2 Vacuum free boundary problem

Motivated by the discussion above, we are led to consider a *vacuum free boundary* problem in which the unknowns include not only the density ρ and the velocity $\mathbf{u}$ but also the moving boundary $\partial\Omega(t)$ itself that evolves dynamically. In order for the problem to be well-posed, we need appropriate boundary conditions. We require the zero pressure on the vacuum boundary and the kinematic boundary condition for the moving boundary. The vacuum free boundary compressible Euler system takes the form:

$$\partial_t\rho + \text{div}\,(\rho\mathbf{u}) = 0 \quad \text{in } \Omega(t) \tag{2.10a}$$
$$\rho\,(\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}) + \nabla p = 0 \quad \text{in } \Omega(t) \tag{2.10b}$$
$$p = 0 \quad \text{on } \partial\Omega(t) \tag{2.10c}$$
$$\mathcal{V}_{\partial\Omega(t)} = \mathbf{u}\cdot\mathbf{n}(t) \quad \text{on } \partial\Omega(t) \tag{2.10d}$$
$$(\rho(0,\cdot),\mathbf{u}(0,\cdot)) = (\rho_0,\mathbf{u}_0)\,,\ \Omega(0) = \Omega_0 \tag{2.10e}$$

Similarly, the vacuum free boundary gravitational Euler-Poisson system takes the form:

$$\partial_t\rho + \text{div}\,(\rho\mathbf{u}) = 0 \quad \text{in } \Omega(t) \tag{2.11a}$$
$$\rho\,(\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}) + \nabla p = -\rho\nabla\Phi \quad \text{in } \Omega(t) \tag{2.11b}$$
$$\Delta\Phi = 4\pi\rho,\ \lim_{|x|\to\infty}\Phi(t,x) = 0 \quad \text{in } \mathbb{R}^3 \tag{2.11c}$$
$$p = 0 \quad \text{on } \partial\Omega(t) \tag{2.11d}$$
$$\mathcal{V}_{\partial\Omega(t)} = \mathbf{u}\cdot\mathbf{n}(t) \quad \text{on } \partial\Omega(t) \tag{2.11e}$$
$$(\rho(0,\cdot),\mathbf{u}(0,\cdot)) = (\rho_0,\mathbf{u}_0)\,,\ \Omega(0) = \Omega_0 \tag{2.11f}$$

Here $\partial\Omega(t)$ denotes the boundary of the moving domain, $\mathcal{V}_{\partial\Omega(t)}$ denotes the normal velocity of $\partial\Omega(t)$, and $\mathbf{n}(t)$ denotes the outward unit normal vector to $\partial\Omega(t)$. From now on, we refer to the system of equations (2.10) as the Euler system and (2.11) as the Euler-Poisson system.

The above two systems represent the Eulerian formulation of the vacuum free boundary problem in the context of compressible fluids without and with gravity, respectively.

2.1 Physical vacuum

We further examine the meaning of linear growth rate of the domain given in the previous section in the context of vacuum free boundary problem. From the boundary condition (2.10d), we observe that the boundary movement in the normal direction is determined by the normal velocity at the boundary. A linear growth rate of the diameter of the moving domain then implies that the linear growth in velocity, which in turn indicates the finite acceleration due to the enthalpy gradient $\frac{1}{\rho}\nabla p$ near the vacuum boundary. For our interest of polytropic gases, it means that the vanishing rate of the density at the vacuum interface is not arbitrary: the enthalpy $\rho^{\gamma-1}$ would have a bounded nonzero normal gradient, namely it behaves like a distance function towards the boundary. This particular behavior is a defining feature of the *physical vacuum*:

$$-\infty < \frac{\partial c^2}{\partial n}\Big|_{\partial\Omega} < 0 \tag{2.12}$$

where $c = \sqrt{\frac{d}{d\rho}p(\rho)}$ is the speed of the sound. The definition of physical vacuum can be tracked back to Liu and Yang [19]. We refer to [16] for more discussion on physical vacuum.

As it can be seen from (2.12), the enthalpy c^2 is not smooth across the vacuum boundary and moreover, the system becomes degenerate along the boundary. For these reasons, a rigorous understanding of physical vacuum states in compressible fluid dynamics has been a challenging problem. Only recently, a successful local-in-time well-posedness theory for the Euler system was developed by Coutand and Schkoller [3, 4] and Jang and Masmoudi [15, 17, 18] using the Lagrangian formulation of the Euler system in the vacuum free boundary framework. In the case of the Euler-Poisson system, see [13, 14, 22], where local-in-time spherically symmetric solutions were studied under the physical vacuum boundary condition (2.12). We also mention the works [25, 26] by Makino that rely on the Nash-Moser theory to prove the existence of local-in-time solutions.

A natural question, which is related to Q1 in the introduction, is whether global-in-time solutions exist or not in the vacuum boundary framework. This is not a trivial matter because the Euler and Euler-Poisson systems are well-known examples of hyperbolic conservation and balance laws where the shock singularity can be developed in a finite time [29, 24]. See also [20] for more recent development on the formation of shocks. Nevertheless, there do exist global-in-time, non-stationary solutions to both Euler and Euler-Poisson systems in the vacuum free boundary framework.

2.2 Expanding affine solutions

2.2.1 Sideris' affine solutions for the Euler system

Recently Sideris [30] constructed a class of *affine* fluid motions that solve (2.10)–(2.12) globally-in-time in the vacuum free boundary setting. To describe affine solutions, it is convenient to introduce the flow map ζ defined as a solution of ODEs

$$\partial_t\zeta = \mathbf{u}\circ\zeta.$$

By definition, the flow map and the velocity field of an affine solution take the following form

$$\zeta_A(t,x) = A(t)x, \ \ \mathbf{u}(t,x) = \dot{A}(t)A^{-1}(t)x, \ \ A(t)\in \mathrm{GL}^+(3). \tag{2.13}$$

With this ansatz, one can derive the family of solutions of the form

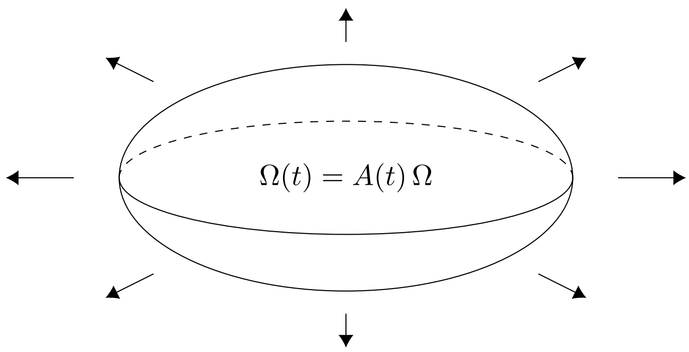
$$\rho_A(t,x) = \det A(t)^{-1}\left[\frac{(\gamma-1)}{2\gamma}(1-|A^{-1}(t)x|^2)\right]^{\frac{1}{\gamma-1}}, \quad \mathbf{u}_A(t,x) = \dot{A}(t)A^{-1}(t)x, \tag{2.14}$$

where $A(t)$ solves the Cauchy problem for a system of ODEs:

$$\ddot{A}(t) = \det A(t)^{1-\gamma}A(t)^{-\top}, \tag{2.15}$$

$$(A(0),\dot{A}(0)) = (A_0,A_1) \in \mathrm{GL}^+(3)\times\mathbb{M}^{3\times3}. \tag{2.16}$$

The density and the velocity field are both supported on a moving ellipse $\Omega(t)$ of the form $A(t)\Omega$, where $\Omega = B_1(\mathbf{0})$ is the unit ball in $\mathbb{R}^3$. Sideris proved that $A(t) \sim A_0 + A_1 t$ for large t. We denote the moduli space of expanding motions by $\mathscr{S}$ - it is clear from (2.15)–(2.16) that this set is parametrized by finitely many parameters.



Elements of $\mathscr{S}$

Although finite dimensional, this is a remarkable result. It shows the existence of a special class of expanding gases and it gives a concrete family of examples wherein the gas expansion overwhelms a possible onset of shock formation. Of course, this does not mean that such solutions are stable and generic. A big question that one can ask is whether one can construct *non-affine* solutions that exist for all time. In [10], we gave a positive answer to this question in the range $1 < \gamma \le \frac{5}{3}$: yes, there exist many non-affine solutions for $1 < \gamma \le \frac{5}{3}$. In the process we developed a robust approach to the question of stability of affine motions, see Section 3.

2.2.2 Homologous solutions for the Euler-Poisson system

Special solutions to the Euler-Poisson system have a long history due to their relevance in astrophysics [1, 31]. The key questions of interest are the dynamical stability of steady stars [5, 12, 13, 21, 28], qualitative description of stellar collapse and stellar expansion (as is the case with supernovae). A remarkable feature of the Euler-Poisson system is the existence of explicit expanding and collapsing solutions in the special case $\gamma = \frac{4}{3}$. As we shall see, the value $\gamma = \frac{4}{3}$ occupies a distinguished role in the study of the Euler-Poisson system as it corresponds to a mass-critical case with respect to a family of rescalings parametrized by

γ. In 1980 Goldreich and Weber [8] constructed a family of spherically symmetric, compactly supported solutions to the Euler-Poisson system expanding into vacuum (with $\gamma = \frac{4}{3}$), called *homologous* solutions. In the mathematics community, these special solutions were independently discovered and rigorously analyzed by Makino [23], Fu and Lin [7] and in the nonisentropic case by Deng, Xiang and Yang [6]. We distinguish two types of the special solutions according to their rate of expansion.

(a) Stars whose radius $\lambda(t)$ grows to infinity at a linear rate

$$\rho(t,x) = \lambda(t)^{-3} w_\delta^3(\lambda(t)^{-1}x), \;\; \mathbf{u}(t,x) = \dot\lambda(t)\lambda(t)^{-1}x, \;\; \lambda(t) \sim_{t\to\infty} t, \tag{2.17}$$

(b) Stars whose radius $\lambda(t)$ grows to infinity at a self-similar rate

$$\rho(t,x) = \lambda(t)^{-3} w_\delta^3(\lambda(t)^{-1}x), \;\; \mathbf{u}(t,x) = \dot\lambda(t)\lambda(t)^{-1}x, \;\; \lambda(t) \sim_{t\to\infty} t^{\frac{2}{3}}. \tag{2.18}$$

Here, the expanding radius of the special solutions satisfies the ODE:

$$\ddot\lambda\lambda^2 = \delta, \quad \lambda(0) = \lambda_0 > 0, \quad \dot\lambda(0) = \lambda_1 \tag{2.19}$$

and $w_\delta : [0,1] \to \mathbb{R}_+$ is a non-negative enthalpy function solving the generalized Lane-Emden equation:

$$w'' + \frac{2}{z}w' + \pi w^3 = -\frac{3}{4}\delta \;\text{ in }\; [0,1], \;\; w'(0) = 0, \;\; w(1) = 0. \tag{2.20}$$

Our use of the term "self-similar" will be justified in the next section. It was shown in [7] that there exists $-\infty < \delta_* < 0$ such that for any $\delta \geq \delta_*$, the solution to (2.20) exists. The parameter δ is related to the total mass of the star. Interestingly, the linear rate prevails over the self-similar rate in a sense that the self-similar expansion occurs only when the initial data satisfy the relationship $\lambda_1^2 + \frac{2\delta}{\lambda_0} = 0$ for $\delta \in (\delta_*, 0)$, which corresponds to the assumption that the initial star configuration has zero total energy. The difference between the self-similar rate and the linear rate can in fact be best understood at the level of physical energy, which for the homologous solutions takes the form

$$E(\lambda, \dot\lambda) = \left(\dot\lambda^2 + \frac{2\delta}{\lambda}\right) \int_0^1 2\pi w_\delta^3 z^4 dz. \tag{2.21}$$

In the case of the self-similar expansion (2.18) $E = 0$, while for the linearly expanding configurations (2.17) we have $E > 0$. Note that the solutions (2.17), (2.18) are in fact a special class of affine solutions in the sense of (2.13), wherein the flow matrix $A = \lambda I_{3\times 3}$ is diagonal.

Again these solutions are characterized by a finite number of parameters and an interesting question is whether such expanding stars are stable or not. For $E > 0$, the result of [24] suggests a possibility of globally defined expanding solutions, but note that the result does not apply to $E = 0$. In [9], we have shown the nonlinear radial stability results for the above expanding solutions, both linear rate and self-similar rate. For the self-similar expansion, the perturbations should honor the same energy level of the background homologous expansion. Namely, when $E = 0$ we obtain a co-dimension one stability and $\delta < 0$ has to be sufficiently close to 0, while for the linear rate, these restrictions are not necessary. See Section 3.

We note that by a simple time reversal, one can easily find a family of collapsing solutions to the Euler-Poisson system, where, in parallel to the above discussion, the collapse may occur at both a linear and a self-similar rate. A compact description of the special solutions discussed in this section is given in Figure 1.

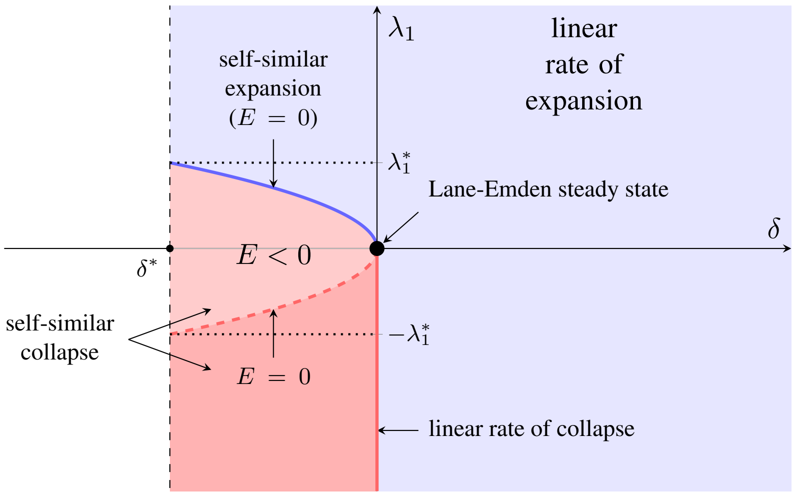


Figure 1: Bifurcation diagram, $\lambda_0 = 1$

2.3 Affine motions as a consequence of the symmetries of the system

It is perhaps not surprising that the existence of above described affine motions is intrinsically related to the symmetries of the Euler equations. In this section we proceed to explain this relationship for the Euler system, although an analogous result applies to the Euler-Poisson system.

For any invertible matrix $A \in \mathrm{GL}^+(3)$ and a given solution $(\rho, \mathbf{u})$ of (2.10) we consider a transformation

$$(\rho, \mathbf{u}) \mapsto (\tilde\rho, \tilde{\mathbf{u}}) \tag{2.22}$$

defined by

$$\rho(t,x) = \det A^{-1} \tilde\rho \left(\det A^{\frac{1-3\gamma}{6}} t,\ A^{-1}x\right) \tag{2.23}$$

$$\mathbf{u}(t,x) = \det A^{\frac{1-3\gamma}{6}} A \tilde{\mathbf{u}} \left(\det A^{\frac{1-3\gamma}{6}} t,\ A^{-1}x\right) \tag{2.24}$$

It is straightforward to check that if $(\rho, \mathbf{u})$ solve (2.10a)–(2.10b), then the pair $(\tilde\rho, \tilde{\mathbf{u}})$ solves the following generalized Euler system:

$$\partial_s \tilde\rho + \mathrm{div}\,(\tilde\rho \tilde{\mathbf{u}}) = 0 \ \text{ in } \tilde\Omega(s)\,; \tag{2.25a}$$

$$\tilde\rho\,(\partial_s \tilde{\mathbf{u}} + (\tilde{\mathbf{u}} \cdot \nabla)\tilde{\mathbf{u}}) + \Lambda \nabla(\tilde\rho^\gamma) = 0 \ \text{ in } \tilde\Omega(s)\,; \tag{2.25b}$$

where

$$\Lambda := \det A^{\frac{2}{3}} A^{-1} A^{-\top}, \ \ \tilde\Omega(s) = A^{-1}\Omega(s), \ \ s = \det A^{\frac{1-3\gamma}{6}} t. \tag{2.26}$$

Matrix Λ is clearly symmetric, positive definite and belongs to $\mathrm{SL}(3)$. Here $A^{-\top}$ is by definition the transpose of the inverse of A. An intriguing new feature of (2.25b) is the presence of the matrix Λ in (2.25b). In the special case when A is a *conformal* matrix (i.e. $A(\det A)^{-\frac{1}{3}} \in \mathrm{SO}(3)$) Λ is simply the identity matrix and the transformation (2.22) is an exact invariance. This of course is not surprising as the problem possesses both a scaling and a rotational symmetry. Conformal invariance corresponds to arbitrary compositions of these symmetries acting on the solution space of the Euler system.

Our claim is that the above rich scaling freedom of the compressible Euler flows is in fact responsible for the existence of the affine motions described in Section 2.2.1. Namely, since the transformation (2.23)–(2.24) leaves the total mass unchanged we may refer to it as *mass-critical*. It is therefore natural to look for paths $t \mapsto A(t) \in \mathrm{GL}^+(3)$ that give us special solutions of the Eeuler system of the form (2.23)–(2.24). Upon rescaling the time and the space according to the self-similar change of variables

$$\frac{ds}{dt} = \det A(t)^{\frac{1-3\gamma}{6}}, \;\; y = A(t)^{-1}x \tag{2.27}$$

we discover that the unknowns $(\tilde{\rho}(s,y), \tilde{\mathbf{u}}(s,y))$ solve a new and more complicated looking system of equations. The key insight is that one can introduce a *modified velocity*

$$\mathbf{U} = \tilde{\mathbf{u}} - A^{-1}A_s y, \tag{2.28}$$

whose form is intrinsically related to the Galilean invariance of the problem. By simple algebraic manipulations, one dsicovers that the unknowns $(\tilde{\rho}, \mathbf{U})$ in turn solve a considerably more pleasant system of equations, reminiscent of Euler system:

$$\partial_s \tilde{\rho} + \mathrm{div}\,(\tilde{\rho}\mathbf{U}) = 0 \tag{2.29}$$

$$\begin{aligned}&\partial_s \mathbf{U} + (\mathbf{U}\cdot\nabla)\mathbf{U} + \left(-\frac{3\gamma-1}{2}\frac{\mu_s}{\mu}\, I_{3\times 3} - 2B(s)\right)\mathbf{U} + \frac{\gamma}{\gamma-1}\Lambda\nabla(\tilde{\rho}^{\gamma-1}) \\ &\quad = \left[B_s + \left(-\frac{3\gamma-1}{2}\frac{\mu_s}{\mu}\, I_{3\times 3} - B\right)B\right]y,\end{aligned} \tag{2.30}$$

where $\Lambda = \Lambda(A)$ is defined in (2.26) and

$$\mu(s) = (\det A(s))^{\frac{1}{3}}.$$

For any $\delta > 0$ we can find a special solution of (2.29)–(2.30) by setting $\mathbf{U} = 0$, $\frac{\gamma}{\gamma-1}\nabla(\tilde{\rho}^{\gamma-1}) = -\delta y$ and

$$B_s + \left(-\frac{3\gamma-1}{2}\frac{\mu_s}{\mu}\, I_{3\times 3} - B\right)B = -\delta\Lambda(A). \tag{2.31}$$

A simple calculation shows that the equation (2.31) is just a restatement of the ODE (2.15) in the rescaled time variable s and this way we rediscover all the the affine motions from [30].

It is thus helpful to think of the elements of the moduli space $\mathscr{S}$ as steady state solutions of the suitably rescaled version of the Euler system. In this way, the question of dynamic stability of such solutions is naturally rephrased as a question of small data global existence for the system (2.29)–(2.30). It is instructive to draw a parallel, pointed out to us by P. Raphaël, to the study of self-similar Type I singular behavior in the nonlinear Schrödinger equation. In particular, a certain analogy to the work of Merle-Raphaël-Szeftel [27] can be traced, wherein the stable self-similar behavior of solutions to the slightly supercritical Schrödinger equation is investigated. There the analogue of the key transformation (2.28) leads to a reformulation of the nonlinear Schrödinger equation, analogous to (2.29)–(2.30). In the setting of [27] explicit formula for self-similar solutions is not available, but the new formulation is ideally suited for the study of their existence and stability. The modified velocity $\mathbf{U}$ defined by (2.28) differs from $\tilde{\mathbf{u}}$ by an affine transformation in analogy to the quantity P_b from [27] which arises by a suitable "conformal" change of variables honoring the symmetries of the nonlinear Schrödinger equation.

A related analysis can carried out for the Euler-Poisson system, where the fundamental scaling invariance is given by the following transformation. If $(\rho, \mathbf{u})$ is a solution of the Euler-Poisson system, so is the pair $(\tilde{\rho}, \tilde{\mathbf{u}})$ defined by

$$\rho(t,x) = \lambda^{-\frac{2}{2-\gamma}} \tilde{\rho}\Big(\frac{t}{\lambda^{\frac{1}{2-\gamma}}}, \frac{x}{\lambda}\Big), \quad \mathbf{u}(t,x) = \lambda^{-\frac{\gamma-1}{2-\gamma}} \tilde{\mathbf{u}}\Big(\frac{t}{\lambda^{\frac{1}{2-\gamma}}}, \frac{x}{\lambda}\Big) \tag{2.32}$$

for any $\lambda > 0$. The associated pressure $\tilde{p}$ and the gravitational potential $\tilde{\Phi}$ relate to p and Φ via:

$$p(t,x) = \lambda^{-\frac{2\gamma}{2-\gamma}} \tilde{p}\Big(\frac{t}{\lambda^{\frac{1}{2-\gamma}}}, \frac{x}{\lambda}\Big) \;\; \Phi(t,x) = \lambda^{-\frac{2\gamma-2}{2-\gamma}} \tilde{\Phi}\Big(\frac{t}{\lambda^{\frac{1}{2-\gamma}}}, \frac{x}{\lambda}\Big).$$

A simple computation reveals that when $\gamma = \frac{4}{3}$ we have $M(\rho) = M(\tilde{\rho})$, thus justifying the terminology "mass-critical" in reference to the $\gamma = \frac{4}{3}$-case. Ideas similar to the ones used for the Euler system, but now relying on the invariant transformation (2.32), allow us to realize the homologous solutions discovered by Goldreich and Weber as steady states of a suitably rescaled version of the Euler-Poisson system.

3 Stability results

In this section, we present the recent results obtained by the authors in [9, 10] on the stability of special expanding affine solutions discussed in the previous section. For the Euler system, we have

Theorem 3.1 (Global existence in the vicinity of affine motions [10])**.** *Assume that* $\gamma \in (1, \frac{5}{3}]$. *Then small perturbations of the expanding affine motions given by* (2.14)–(2.16) *give rise to unique globally-in-time defined solutions of the Euler system. Moreover, their support expands at a linear rate and they remain close to the underlying moduli space of affine motions.*

For the mass-critical Euler-Poisson system ($\gamma = \frac{4}{3}$), we have shown

Theorem 3.2 (Nonlinear stability of the linearly expanding homogeneous solutions [9])**.** *There exists an* $\tilde{\varepsilon} > 0$ *such that for any* $\delta > -\tilde{\varepsilon}$ *the linearly expanding homologous solutions* $(\tilde{\lambda}, \tilde{\rho}, \tilde{v})$ *are nonlinearly stable with respect to spherically symmetric small perturbations.*

Theorem 3.3 (Codimension-one stability of self-similar expanding solutions [9])**.** *The self-similar expanding solutions are globally-in-time nonlinearly stable with respect to spherically symmetric initial data* (ρ_0, v_0) *with vanishing energy* $E(\rho_0, v_0) = 0$. *Any such perturbation converges to a nearby self-similar expanding homologous solution.*

The above theorems provide the first global-in-time smooth solutions to the Euler and Euler-Poisson system in the vacuum free boundary framework. We have in particular shown that the expansion of the background affine solution stabilizes the whole system through a certain damping effect and counteracts the possibility of shock formation. In particular, in Theorem 3.1, we do not require any symmetry assumption on the data and they are truly three-dimensional non-affine global solutions. In fact, the method developed in [10] can be adapted to prove a *non-radial* stability result for the expanding stars, which is in its final stages [11].

An important feature of expanding solutions is that the domain (the support) is moving with the fluid. The problem is thus better understood in the material coordinates by following the fluid trajectories. The formulation of the problem in suitably rescaled Lagrangian coordinates is a key in our results. In the next section, we will give our setup designed for the study of stability of affine solutions for the Euler system and Theorem 3.1.

4 Lagrangian formulation

Lagrangian coordinates turn out to be useful in the study of free boundary problems because they allow us to pull the problem back onto a fixed domain. As a result, the problem turns into an initial-value problem for the flow map, posed on a fixed domain. This choice of coordinates was also fundamental for the development of the local well-posedness theory in the presence of the physical vacuum [3, 4, 15, 18]. Of course, one should not expect to have global solutions for arbitrary data, but in the vicinity of an affine motion. This is best accomplished if we develop a framework that transforms an affine solution to an equilibrium state of a new rescaled system, and a novelty of [9, 10] is the design of such a framework. As elaborated in Section 2.3, precisely this was accomplished using the (almost) invariance of the Euler system, which naturally led to the system (2.29)–(2.30). Here we will discuss a slight variation on the theme, a different strategy for the formulation of the stability for affine motions of the Euler system. A similar derivation will apply to the Euler-Poisson system.

Let ζ be the flow map driven by the original fluid velocity, i.e. $\partial_t \zeta = \mathbf{u} \circ \zeta$. Consider another flow map η as the pull-back of ζ by an affine motion A, i.e. $\eta := A^{-1}\zeta$. Note that when $\eta(t,x) = x$, $\zeta(t,x) = A(t)x$, which implies that the equilibrium state for η corresponds to the affine motion. See Appendix D in [10] for more detail. Whether we use this approach or introduce the Lagrangian variables for the modified velocity $\mathbf{U}$ introduced in Section 2.3, two approaches give rise to the same equation for η:

$$\eta_{tt} + 2A^{-1}A_t\eta_t + A^{-1}A_{tt}\eta + \tfrac{\gamma}{\gamma-1}(\det A)^{1-\gamma}A^{-1}A^{-\top}\mathscr{A}^{\top}\nabla(w\mathscr{J}^{1-\gamma}) = 0$$

where $\mathscr{A} = [D\eta]^{-1}$, $\mathscr{J} = \det\mathscr{A}$, w is the enthalpy function associated with the affine motion A. We then rescale the time variable t so that $1+t \sim e^{\mu_1\tau}$ by setting $\frac{d\tau}{dt} = (\det A)^{-\frac{1}{3}}$ and let $\theta = \eta - y$, $\mu = (\det A)^{\frac{1}{3}}$, $\Gamma^* = O^{-1}O_\tau$, $O = \mu^{-1}A$, $\Lambda = \det A^{\frac{2}{3}}A^{-1}A^{-T} \in \mathrm{SL}(3)$ to obtain the equation for the perturbation θ:

$$\begin{aligned} w^{\frac{1}{\gamma-1}}\mu^{3\gamma-3}\left(\partial_{\tau\tau}\theta_i + \frac{\mu_\tau}{\mu}\partial_\tau\theta_i + 2\Gamma^*_{ij}\partial_\tau\theta_j\right) + w^{\frac{1}{\gamma-1}}\Lambda_{i\ell}\theta_\ell \\ + \left(w^{\frac{\gamma}{\gamma-1}}\Lambda_{ij}\left(\mathscr{A}^k_j\mathscr{J}^{-\frac{1}{\alpha}} - \delta^k_j\right)\right)_{,k} = 0 \end{aligned} \tag{4.33}$$

equipped with the initial conditions

$$\theta(0,y) = \theta_0(y), \;\; \theta_\tau(0,y) = \mathbf{v}(0,y) = \mathbf{v}_0(y), \;\; (y \in \Omega = B_1(\mathbf{0}))\,.$$

Now the question is whether one can prove the global existence of θ in some function space. The equation (4.33) is a quasi-linear wave type equation, which requires the higher order energy for smooth solutions, and it is degenerate because w vanishes at the boundary, which requires some weighted space to handle boundary singularity. As mentioned earlier, even local existence result is a challenging problem. Fundamental difference compared to the unperturbed Euler system, however, is the existence of exponentially growing time weight μ (recall $\mu \sim e^{\mu_1 \tau}$ for large τ and some $\mu_1 > 0$) which follows from the background expansion of affine motion. This is in fact a saving grace for the global existence together with the local well-posedness framework developed by Jang and Masmoudi [18] based on a careful use of spatial derivatives. The two ingredients are crucial, and it is important to note that time derivatives do not commute well with the equation, due to the presence of time weights. This is a delicate technical point, but plays fundamentally into the set-up of our high-order energy method.

4.1 Stabilization of expanding background profile

To illustrate the gist of stabilizing effect, let us consider a simplified model

$$e^\tau \partial_{\tau\tau}\theta + ce^\tau \partial_\tau \theta + \mathcal{L}\theta = \mathcal{N}\theta$$

where $c > 0$ is a given constant, $\mathcal{L}\theta$ is non-negative self-adjoint linear operator with respect to an inner product $\langle , \rangle$ and $\mathcal{N}$ consists of nonlinear terms. A basic energy identity reveals

$$\frac{1}{2}\frac{d}{dt}\left(e^\tau \|\partial_\tau\theta\|^2\right) + (c - \frac{1}{2})e^\tau\|\partial_\tau\theta\|^2 + \langle \mathcal{L}\theta, \partial_\tau\theta\rangle = \langle \mathcal{N}\theta, \partial_\tau\theta\rangle$$

We make the following remarks. First, the constant c is important. In order to achieve the global existence, we need $c - \frac{1}{2} \geq 0$, otherwise, it gives an anti-damping effect, which may result in an undesirable growth of our norms. In the context of the Euler system, this restriction reduces to $\gamma \leq \frac{5}{3}$ and for the mass-critical Euler-Poisson system, the term is simply positive. Second, the background expansion is also beneficial in the control of the nonlinear right-hand side $\mathcal{N}$. Since $|\langle \mathcal{N}\theta, \partial_\tau\theta\rangle| \lesssim e^{-\frac{\tau}{2}}\|\mathcal{N}\theta\| \left(e^{\frac{\tau}{2}}\|\partial_\tau\theta\|\right)$, as long as $\|\mathcal{N}\theta\|$ and $e^{\frac{\tau}{2}}\|\partial_\tau\theta\|$ stay bounded, it becomes integrable in τ, which makes the continuity argument to $\tau = \infty$ accessible. The goal is then to establish the inequality

$$\mathcal{S}^N(\tau) \leq C_0 + C\int_0^\tau e^{-\mu_*\tau'}\mathcal{S}^N(\tau')\,d\tau' \quad \text{for some } \mu_* > 0 \tag{4.34}$$

for some high energy norm $\mathcal{S}^N(\tau)$.

Of course, a nontrivial work is necessary to confirm that indeed this is the case. We next discuss some of new difficulties associated with (4.33).

4.2 Difficulties and strategies

A central difficulty of getting the energy estimates lies in the fact that $\langle \mathcal{L}\theta, \partial_\tau\theta\rangle$ may not give a full control of the energy. In fact, in the case of the self-similar expansion of the Euler-Poisson system, one obtains a coercive energy functional only in the direction transversal to a one-dimensional subspace, which therefore needs to be modded out from our perturbations.

On the other hand, for the Euler system and the nonradial Euler-Poisson system, $\langle \mathcal{L}\theta, \partial_\tau \theta \rangle$ is not fully coercive, but it does provide the control of the norm of the divergence of the flow map η. To complete the argument, one needs an additional estimate for the curl of the flow map, which in turn satisfies a transport equation.

In the case of (4.33), the effect of the background affine motion is even more intriguing due to the presence of Λ. If $\Lambda = I_{3\times 3}$, the strategy in [18] would work fine without significant changes. However, in general Λ is not an identity matrix and new difficulties arise. First of all, the usual curl operator will not annihilate Λ term in (4.33). Instead, we use Λ-curl

$$[\text{curl}_{\Lambda\mathscr{A}} \mathbf{F}]^i := \epsilon_{ijk} (\Lambda\mathscr{A})_j^s \mathbf{F}^k,_s = \epsilon_{ijk} \Lambda_{jm} \mathscr{A}_m^s \mathbf{F}^k,_s \,, \tag{4.35}$$

where ϵ_{ijk} is the standard permutation symbol and this Λ-curl will satisfy a good transport equation

$$\text{Curl}_{\Lambda\mathscr{A}} \mathbf{V}_\tau + \frac{\mu_\tau}{\mu} \text{Curl}_{\Lambda\mathscr{A}} \mathbf{V} + 2\text{Curl}_{\Lambda\mathscr{A}} \left(\Gamma^* \mathbf{V} \right) = 0 \tag{4.36}$$

Note that the Λ-curl is nothing but the natural anti-symmetric 2 form that annihilates the Λ-gradient appearing in the rescaled Euler equation. One can further derive the equation for the Λ-curl of θ:

$$\begin{aligned} \partial_\tau \text{Curl}_{\Lambda\mathscr{A}} \theta(\tau) &= \frac{\mu(0) \text{Curl}_{\Lambda\mathscr{A}} \mathbf{V}(0)}{\mu} + [\partial_\tau, \text{Curl}_{\Lambda\mathscr{A}}] \theta \\ &\quad + \frac{1}{\mu} \int_0^\tau \mu [\partial_\tau, \text{Curl}_{\Lambda\mathscr{A}}] \mathbf{V} d\tau' - \frac{2}{\mu} \int_0^\tau \mu \text{Curl}_{\Lambda\mathscr{A}} \left(\Gamma^* \mathbf{V} \right) d\tau' \end{aligned} \tag{4.37}$$

where $[A, B] = AB - BA$ is the usual commutator. The fact that the Λ-curl is used for the energy has a consequence on the energy for $\partial_\tau \theta$. A correct way is to derive the estimates for $\sqrt{\Lambda^{-1}} \partial_\tau \theta$ rather than $\partial_\tau \theta$ and to make use of the diagonalization of Λ in order to extract the coercive energy for the full gradient of the flow map η. This is then combined with rather delicate curl estimates through geometric and algebraic structure of the Λ gradient of the pressure. See [10] for more details.

4.3 Function spaces and main result revisited

In order to deal with the degeneracy caused by physical vacuum condition for the Euler and Euler-Poisson system, it is inevitable to use weighted Sobolev spaces as the basic energy spaces, and Hardy-Sobolev embeddings as a basic tool in closing the estimates [4, 18]. Before we introduce the precise energy space, we first introduce some notations.

For any given integer $k \in \mathbb{N}$, a function $f : \Omega \to \mathbb{R}$ and any smooth, continuous, and non-negative function $\varphi : \Omega \to \mathbb{R}^+$ we introduce the notation

$$\|f\|_{k,\varphi}^2 := \int_\Omega \varphi w^k |f(y)|^2 \, dy. \tag{4.38}$$

The weight function φ will be taken as either ψ or $1 - \psi$ where $\psi : B_1(\mathbf{0}) \to [0, 1]$ is a smooth cut-off function satisfying

$$\psi = 0 \ \text{ on } \ B_{\frac{1}{4}}(\mathbf{0}) \ \text{ and } \ \psi = 1 \ \text{ on } \ B_1(\mathbf{0}) \setminus B_{\frac{3}{4}}(\mathbf{0}). \tag{4.39}$$

We define the *angular gradient* $\slashed{\nabla}$ as the projection of the usual gradient onto the plane tangent to the sphere of constant (t, r):

$$\slashed{\nabla} f(y) = \nabla f(y) - \frac{y}{r}(\frac{y}{r}\cdot\nabla)f(y) = \nabla f(y) - \frac{y}{r}\partial_r f(y), \tag{4.40}$$

where $\partial_r := \frac{y}{r}\cdot\nabla$ and $r = |y|$.

For any multi-index $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{Z}^3_{\geq 0}$ and any $a \in \mathbb{Z}_{\geq 0}$ we introduce the following differential operator

$$\partial_r^a \slashed{\nabla}^\beta := \partial_r^a \slashed{\nabla}_1^{\beta_1}\slashed{\nabla}_2^{\beta_2}\slashed{\nabla}_3^{\beta_3}. \tag{4.41}$$

Similarly for any multi-index $\nu = (\nu_1, \nu_2, \nu_3) \in \mathbb{Z}^3_{\geq 0}$we denote

$$\partial^\nu := \partial_{y_1}^{\nu_1}\partial_{y_2}^{\nu_2}\partial_{y_3}^{\nu_3}, \tag{4.42}$$

where ∂_{y_i}, $i = 1, 2, 3$ denote the usual Cartesian derivatives.

The high-order weighted Sobolev norm that measures the size of the deviation θ is given as follows. For any $N \in \mathbb{N}$, let

$$\begin{aligned}\mathcal{S}^N(\theta, \mathbf{V})(\tau) := \sum_{a+|\beta|\leq N}\sup_{0\leq\tau'\leq\tau}&\Big\{\mu^{3\gamma-3}\left\|\partial_r^a\slashed{\nabla}^\beta \mathbf{V}\right\|^2_{a+\frac{1}{\gamma-1},\psi} + \left\|\partial_r^a\slashed{\nabla}^\beta\theta\right\|^2_{a+\frac{1}{\gamma-1},\psi} \\ &+ \left\|\nabla_\eta\partial_r^a\slashed{\nabla}^\beta\theta\right\|^2_{a+\frac{1}{\gamma-1}+1,\psi} + \left\|\mathrm{div}_\eta\partial_r^a\slashed{\nabla}^\beta\theta\right\|^2_{a+\frac{1}{\gamma-1}+1,\psi}\Big\} \\ + \sum_{|\nu|\leq N}\sup_{0\leq\tau'\leq\tau}&\Big\{\mu^{3\gamma-3}\left\|\partial^\nu\mathbf{V}\right\|^2_{\frac{1}{\gamma-1},1-\psi} + \left\|\partial^\nu\theta\right\|^2_{\frac{1}{\gamma-1},1-\psi} \\ &+ \left\|\nabla_\eta\partial^\nu\theta\right\|^2_{\frac{1}{\gamma-1}+1,1-\psi} + \left\|\mathrm{div}_\eta\partial^\nu\theta\right\|^2_{\frac{1}{\gamma-1}+1,1-\psi}\Big\}\end{aligned}$$

The high-order quantity measuring the Λ-curl of $\mathbf{V}$ which is a priori not controlled by the norm $\mathcal{S}^N(\tau)$ is given as follows.

$$\begin{aligned}\mathcal{B}^N[\mathbf{V}](\tau) := &\sum_{a+|\beta|\leq N}\sup_{0\leq\tau'\leq\tau}\left\|\mathrm{Curl}_{\Lambda\mathscr{A}}\partial_r^a\slashed{\nabla}^\beta\mathbf{V}\right\|^2_{a+\frac{1}{\gamma-1}+1,\psi} \\ &+ \sum_{|\nu|\leq N}\sup_{0\leq\tau'\leq\tau}\left\|\mathrm{Curl}_{\Lambda\mathscr{A}}\partial^\nu\mathbf{V}\right\|^2_{\frac{1}{\gamma-1}+1,1-\psi}\end{aligned}$$

We are now ready to give a quantitative statement of Theorem 3.1:

Theorem 4.1 (Nonlinear stability of affine motions)**.** *Assume that* $\gamma \in (1, \frac{5}{3}]$ *and let* $N \geq 2\lceil\frac{1}{\gamma-1}\rceil + 12$ *be fixed. Let* A *of the affine motion be given, and let*

$$\mu_0 := \frac{3\gamma-3}{2}\mu_1, \ \ \mu_1 := \lim_{\tau\to\infty}\frac{\mu_\tau(\tau)}{\mu(\tau)}.$$

Then there exists an $\varepsilon_0 > 0$ *such that for any* $0 \leq \varepsilon \leq \varepsilon_0$ *and* $(\theta_0, \mathbf{V}_0)$ *satisfying* $\mathcal{S}^N(\theta_0, \mathbf{V}_0) + \mathcal{B}^N(\mathbf{V}_0) \leq \varepsilon$*, there exists a global-in-time solution to* (4.33) *and a constant* $C > 0$ *such that*

$$\mathcal{S}^N(\theta, \mathbf{V})(\tau) \leq C\varepsilon, \ \ 0 \leq \tau < \infty, \tag{4.43}$$

and

$$\mathcal{B}^N[\mathbf{V}](\tau) \le \begin{cases} C\varepsilon e^{-2\mu_0\tau} & \textit{if } 1<\gamma<\frac{5}{3} \\ C\varepsilon(1+\tau^2)e^{-2\mu_0\tau} & \textit{if } \gamma=\frac{5}{3} \end{cases}, \quad 0\le\tau<\infty. \tag{4.44}$$

Furthermore, there exists a τ-independent asymptotic attractor θ_∞ such that

$$\sum_{a+|\beta|\le N} \left\| \partial_r^a \slashed{\nabla}^\beta \theta(\tau,\cdot) - \partial_r^a \slashed{\nabla}^\beta \theta_\infty(\cdot) \right\|_{a+\frac{1}{\gamma-1},\psi} + \sum_{|\nu|\le N} \|\partial^\nu \theta(\tau,\cdot) - \partial^\nu\theta_\infty(\cdot)\|_{\frac{1}{\gamma-1},1-\psi} \longrightarrow 0 \;\; \textit{as} \;\; \tau\to 0.$$

5 Discussion

The existence of affine solutions discussed in Sections 2.2.1 and 2.2.2 has served as an important starting point for the development of a robust approach to the existence of non-affine global solutions described in Section 3 for both the Euler and the Euler-Poisson system. Several important open questions present themselves:

1. For the Euler system, the dynamics near the expanding affine solutions of Sideris for $\gamma > \frac{5}{3}$ is an open question. The above described damping effect is absent, yet this does not tell us what the solution actually *does*.

2. The stability of the collapsing solutions to the Euler-Poisson system in the mass-critical case $\gamma = \frac{4}{3}$ is not understood. This problem is of fundamental importance, as an understanding of generic rates of collapse would significantly deepen our qualitative understanding of the gravitational collapse.

3. When $\gamma \neq \frac{4}{3}$ there are no available examples of either collapsing or expanding stars. Various heuristic arguments can be made to suggest that the collapse should occur in the mass supercritical range $1<\gamma<\frac{4}{3}$, but there is no proof of such a statement.

Many other interesting problems arise in the presence of important physical effects such as the viscosity, or by including relativistic effects. We hope that the methods described in this paper will prove itself useful to other physically important models pertaining to the dynamics of free boundary fluids surrounded by vacuum.

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Appendix

Proof of Proposition 1.1. The proof relies on the varial identity argument. Let

$$H(t) := \int_{\mathbb{R}^3} |x - t\mathbf{u}(t,x)|^2 \rho(t,x)dx.$$

The first derivative of H is given by

$$\begin{aligned}
\dot{H}(t) &= \int_{\mathbb{R}^3} |x-t\mathbf{u}|^2 \partial_t \rho - 2(x-t\mathbf{u})\cdot(\mathbf{u}+t\partial_t\mathbf{u})\rho dx \\
&= \int_{\mathbb{R}^3} -|x-t\mathbf{u}|^2 \nabla\cdot(\rho\mathbf{u}) - 2(x-t\mathbf{u})\cdot(\rho\mathbf{u} - t(\rho(\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla p))dx \\
&= \int_{\mathbb{R}^3} 2\left((x-t\mathbf{u})\cdot\nabla(x-t\mathbf{u})\right)\cdot\rho\mathbf{u} - 2(x-t\mathbf{u})\cdot(\rho\mathbf{u} - t\rho\mathbf{u}\cdot\nabla\mathbf{u})dx \\
&\quad + 2t\int_{\mathbb{R}^3}(x-t\mathbf{u})\cdot\nabla p dx
\end{aligned}$$

The first integral at the last step is 0. By using $\nabla\cdot(xp) = 3p + x\cdot\nabla p$ and using the dynamics of $p=\rho^\gamma$: $\mathbf{u}\cdot\nabla p = \frac{\partial_t p}{\gamma-1} + \frac{\gamma}{\gamma-1}\nabla\cdot(p\mathbf{u})$, we obtain

$$\dot{H}(t) = -6t\int_{\mathbb{R}^3} p dx - 2t^2\frac{d}{dt}\int_{\mathbb{R}^3}\frac{p}{\gamma-1}dx.$$

This leads to

$$\frac{d}{dt}\left[\int_{\mathbb{R}^3}|x-t\mathbf{u}|^2\rho + \frac{2t^2p}{\gamma-1}dx\right] + \frac{(3\gamma-5)}{t}\int_{\mathbb{R}^3}\frac{2t^2p}{\gamma-1}dx = 0. \tag{5.45}$$

We first let $\gamma \geq \frac{5}{3}$. Then we deduce from (5.45) that

$$\int_{\mathbb{R}^3}\frac{2t^2p}{\gamma-1}dx \leq C \quad \Rightarrow \quad \int_{\mathbb{R}^3} p dx \lesssim \frac{1}{t^2} \text{ for large } t.$$

For $1<\gamma\leq\frac{5}{3}$, letting $H_1 = \int|x-t\mathbf{u}|^2\rho dx$ and $H_2 = \int\frac{2t^2p}{\gamma-1}dx$, we obtain

$$\frac{d}{dt}(t^{3\gamma-5}H_2) = -t^{3\gamma-5}\frac{d}{dt}H_1.$$

Integrating over $[1,t]$ $t\geq 1$, we see that

$$t^{3\gamma-5}H_2(t) = H_2(1) - t^{3\gamma-5}H_1(t) + H_1(1) + (3\gamma-5)\int_1^t s^{3\gamma-6}H_1(s)ds.$$

Now since $\gamma\leq\frac{5}{3}$, the right-hand side is less than $H_2(1)+H_1(1)$, and hence we deduce that

$$\int_{\mathbb{R}^3} p dx \lesssim \frac{1}{t^{3\gamma-3}}.$$

□

References

[1] S. CHANDRASEKHAR: *An Introduction to the Study of Stellar Structures.* University of Chicago Press, Chicago, 1938.

[2] CHEMIN, J-Y. Dynamique des gaz'a masse totale finie. *Asymptotic Anal.*, 3(3):215–220 (1990)

[3] COUTAND, D., SHKOLLER, S.: Well-posedness in smooth function spaces for the moving-boundary 1-D compressible Euler equations in physical vacuum, *Comm. Pure Appl. Math.* **64**, no. 3, 328–366 (2011)

[4] COUTAND, D., SHKOLLER, S.: Well-posedness in smooth function spaces for the moving boundary three-dimensional compressible Euler equations in physical vacuum. *Arch. Ration. Mech. Anal.* **206**, no. 2, 515–616 (2012)

[5] DENG, Y., LIU, T.P., YANG, T., YAO Z.: Solutions of Euler-Poisson equations for gaseous stars. *Arch. Ration. Mech. Anal.* **164**, no. 3, 261–285 (2002)

[6] DENG, Y., XIANG, J., YANG, T.: Blowup phenomena of solutions to Euler-Poisson equations. *J. Math. Anal. Appl.* **286** 295-306 (2003)

[7] FU, C.-C.; LIN, S.-S.: On the critical mass of the collapse of a gaseous star in spherically symmetric and isentropic motion. *Japan J. Indust. Appl. Math.* **15**, no. 3, 461-469 (1998)

[8] GOLDREICH, P., WEBER, S.: Homologously collapsing stellar cores, *Astrophys. J.* **238** 991 (1980)

[9] HADŽIĆ, M., JANG, J.: Nonlinear stability of expanding star solutions in the radially-symmetric mass-critical Euler-Poisson system, accepted for publication in *Communications on Pure and Applied Mathematics*

[10] HADŽIĆ, M., JANG, J.: Expanding large global solutions of the equations of compressible fluid mechanics, available online at https://arxiv.org/abs/1610.01666

[11] HADŽIĆ, M., JANG, J.: Nonradial stability of expanding stars in the mass-critical Euler-Poisson system, in preparation

[12] JANG, J.: Nonlinear Instability in Gravitational Euler-Poisson system for $\gamma = 6/5$, *Arch. Ration. Mech. Anal.* **188**, 265-307 (2008)

[13] JANG, J.: Nonlinear Instability Theory of Lane-Emden stars. *Comm. Pure Appl. Math.* **67**, no. 9, 1418–1465 (2014)

[14] JANG, J.: Time periodic approximations of the Euler-Poisson system near Lane-Emden stars. *Anal. PDE.* **9** (2016), no. 5, 1043–1078

[15] JANG, J., MASMOUDI, N. Well-posedness for compressible Euler equations with physical vacuum singularity, *Comm. Pure Appl. Math.* **62** (2009), 1327–1385

[16] J. JANG, N. MASMOUDI: Vacuum in Gas and Fluid dynamics, *Proceedings of the IMA summer school on Nonlinear Conservation Laws and Applications*, Springer (2011), 315-329

[17] J. JANG, N. MASMOUDI: Well and ill-posedness for compressible Euler equations with vacuum. *J. Math. Phys.* **53** (11) (2012) 115625

[18] JANG, J., MASMOUDI, N. Well-posedness of compressible Euler equations in a physical vacuum. *Communications on Pure and Applied Mathematics* **68**, no. 1, 61–111 (2015)

[19] LIU, T.-P., YANG, T.: Compressible flow with vacuum and physical singularity, *Methods Appl. Anal.* **7**, 495-509 (2000)

[20] LUK, J., SPECK, J.: Shock formation in solutions to the 2D compressible Euler equations in the presence of non-zero vorticity, Available online at https://arxiv.org/abs/1610.00737

[21] LIN, S.-S.: Stability of gaseous stars in spherically symmetric motions. *SIAM J. Math. Anal.* **28** (1997), no. 3, 539–569.

[22] T. LUO, Z. XIN, H. ZENG: Well-posedness for the motion of physical vacuum of the three-dimensional compressible Euler equations with or without self-gravitation. *Arch. Ration. Mech. Anal.* **213**, no. 3, 763-831 (2014)

[23] MAKINO, T.: Blowing up solutions of the Euler-Poisson equation for the evolution of gaseous stars. *Transport Theory Statist. Phys.*, **21**, 615-624 (1992)

[24] MAKINO, T., PERTHAME, B.: Sur les Solution á Symétrie Sphérique de l'Equation d'Euler-Poisson pour l'Evolution d'Etoiles Gazeuses. *Japan J. Appl. Math.* **7** 165–170 (1990)

[25] MAKINO, T.: On spherically symmetric motions of a gaseous star governed by the Euler-Poisson equations. *Osaka J. Math.* **52**, no. 2, 545–580 (2015)

[26] MAKINO, T.: An Application of the Nash-Moser Theorem to the Vacuum Boundary Problem of Gaseous Stars, *J. Differential Equations*, **262**, 803-843 (2017)

[27] MERLE, F., RAPHAËL, P., SZEFTEL, J. Stable self similar blow up dynamics for L^2-supercritical NLS equations. *Geom. Funct. Anal.* **20** no 4, 1028–1071 (2010)

[28] REIN, G.: Non-linear stability of gaseous stars. *Arch. Ration. Mech. Anal.* **168**, no. 2, 115–130 (2003)

[29] SIDERIS,T. C.: Formation of singularities in three-dimensional compressible fluids. *Comm. Math. Phys.* **101**(4), 475–485 (1985)

[30] SIDERIS,T., C.: Global existence and asymptotic behavior of affine motion of 3D ideal fluids surrounded by vacuum, *Available online at http://arxiv.org/abs/1512.03288*

[31] YA. B. ZEL'DOVICH, I. D. NOVIKOV: Relativistic Astrophysics.