Unique continuation problems and stabilised finite element methods

Mihai Nechita

A thesis submitted in conformity with the requirements for the degree of Doctor of Philosophy

University College London

2020

Disclaimer

I, Mihai Nechita, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

Signature ________________________

Date ________________________

Abstract

Numerical analysis for partial differential equations (PDEs) traditionally considers problems that are well-posed in the continuum, for example the boundary value problem for Poisson's equation. Computational methods such as the finite element method (FEM) then discretise the problem and provide numerical solutions. However, when a part of the boundary is inaccessible for measurements or no information is given on the boundary at all, the continuum problem might be ill-posed and solving it, in this case, requires regularisation.

In this thesis we consider the unique continuation problem with (possibly noisy) data given in an interior subset of the domain. This is an ill-posed problem also known as data assimilation and is related to the elliptic Cauchy problem. It arises often in inverse problems and control theory. We will focus on two PDEs for which the stability of this problem depends on the physical parameters: the Helmholtz and the convection–diffusion equations. We first prove conditional stability estimates that are explicit in the wave number and in the Péclet number, respectively, by using Carleman inequalities. Under a geometric convexity assumption, we obtain that for the Helmholtz equation the stability constants grow at most linearly in the wave number.

Then we present a discretise-then-regularise approach for the unique continuation problem. We cast the problem into PDE-constrained optimisation with discrete weakly consistent regularisation. The regularisation is driven by stabilised FEMs and we focus on the interior penalty stabilisation. For the Helmholtz and diffusion-dominated problems, we apply the continuum stability estimates to the approximation error and prove convergence rates by controlling the residual through stabilisation. For convection-dominated problems, we perform a different error analysis and obtain sharper weighted error estimates along the characteristics of the convective field through the data region, with quasi-optimal convergence rates. The results are illustrated by numerical examples.

Impact statement

The work presented in this thesis concerns the theoretical and numerical analysis of unique continuation problems. These are a class of ill-posed problems for partial differential equations that are of both mathematical and practical importance.

Unique continuation arises often in inverse problems and control theory, and potential applications of our results and of similar numerical methods could arise in acoustic problems with unknown scatterers or flow problems in which the full boundary is inaccessible for measurements but where local data (either in a subset of the domain or on a part of the boundary) can be obtained. Another source of practical applications could involve data assimilation problems in biomedical imaging.

The main results obtained for the Helmholtz and convection–diffusion equations have been published in [21] and [22], with the preprint [23] being under review. They have also been presented at several conferences: ICIAM (Valencia, 2019), Applied Inverse Problems (Grenoble, 2019) and Inverse Problems: Modeling and Simulation (Malta, 2018).

Acknowledgments

I would first like to thank my supervisors Lauri Oksanen and Erik Burman. I have been very lucky to work with them during the four years of my PhD and I have greatly benefited from their guidance and help. Their advice and encouraging support have been essential to obtaining the results presented in this thesis.

I would then like to thank the Department of Mathematics at UCL for funding my studies through a teaching assistantship and for providing a stimulating academic environment. The days spent in the KLB office were made extremely enjoyable by the people I've met there, and the lunch discussions and evening pints have shaped invaluable friendships.

I would also like to thank Emil Cătinaș, head of ICTP from the Romanian Academy, for his guidance during my master studies, and my school teachers Valer Pop and Petru Ivănescu for inspiring me.

Finally, the most personal thoughts are heading towards my family: thank you to my parents for their love and continuous support, to my brother Bogdan for being a role model and my best friend, and to Ivona for our love. Vă mulțumesc pentru tot și vă iubesc!

Cluj (due to the coronavirus pandemic)
August 2020

Familiei mele

Contents

Introduction

In this thesis we will consider the ill-posed unique continuation problem for two partial differential equations (PDEs) for which the stability depends on the physical parameters: the Helmholtz and the convection–diffusion equations. The numerical approximations that we propose are designed following a discretise-then-regularise approach in which the discrete regularisation is based on stabilised finite element methods (FEMs).

We begin in Chapter 1 by introducing well-posed problems in the setting of PDEs and recalling the necessary and sufficient conditions for well-posedness. We then discuss linear ill-posed problems and the classical theory of the Moore-Penrose generalised inverse based on least squares solutions. For linear continuous operators with non-closed range – such as the ones we will consider – there is no continuous dependence of the solution on data as the generalised inverse is discontinuous. However, instead of continuous dependence, one can have conditional stability. The problems we will consider are of this kind and an important example is the elliptic Cauchy problem, where Dirichlet and Neumann data are given on a subpart of the boundary. This is a severely ill-posed problem, but it is conditionally stable. In close connection to this, we introduce the main topic of this thesis: unique continuation problems with (perturbed) data given in an interior subset of the domain.

Conditional stability estimates for unique continuation are discussed in Chapter 2. These are obtained from Carleman estimates that are proven in an elementary way. For the convection–diffusion operator we derive three-ball inequalities in which the stability constant depends exponentially on the Péclet number. This is based on our work [22]. For the Helmholtz operator we assume a specific geometric setting for unique continuation inside the convex hull of the data set, and, following our paper [21], we prove stability estimates with constants that are uniform in the wave number or increase at most linearly in it, depending on the chosen norms. For both operators, to weaken the norms in these estimates we make use of semiclassical pseudodifferential operators that are briefly recalled in Appendix B.

In Chapter 3 we focus on the numerical approximation of the unique continuation problem. We present a general discretise-then-regularise approach in which the problem is written in the form of PDE-constrained optimisation. We first discretise the problem and introduce a Lagrangian functional to which we then add discrete regularisation terms based on stabilised FEMs. The discrete regularisation should enhance stability and also control certain residual quantities in order to provide error estimates. In particular, the stabilisation is obtained through the continuous interior penalty that acts on the discrete solution penalising the jumps of the normal gradient across interior faces. The optimality conditions then give a well-posed discrete FEM system.

Based on [21], we apply in Chapter 4 this methodology to the Helmholtz equation and we prove error estimates that are explicit in the wave number for the geometric setting considered in Chapter 2. To prove that the saddle points converge to the continuum solution (when it exists) we apply the continuum stability estimates to the approximation error. Bounding the residual by the stabilising terms, we obtain error estimates with the convergence order given by the Hölder stability exponent. Numerical examples demonstrate the results and highlight the difference between convex and non-convex directions for unique continuation in this case.

For the convection–diffusion equation we consider the diffusion-dominated regime in Chapter 5, which is based on [22]. We take a similar approach to the previous chapter and use continuous interior penalty as the key component of the discrete regularisation. The continuum stability is used to obtain error bounds by controlling the residual of the PDE through stabilisation. The constants depend explicitly on the Péclet number and the convergence order is again given by the Hölder stability exponent. When convection dominates, however, the stability constant grows exponentially and to overcome this we develop in Chapter 6, based on [23], an error analysis that captures the dominant transport phenomenon. For a simplified geometric setting, we obtain quasi-optimal weighted error estimates in a stability region along the characteristics of the convective field that go through the data set. Different convergence properties are proven and observed numerically for unique continuation upstream compared to downstream.

Chapter 1

Ill-posed inverse problems and unique continuation

We begin with an introductory chapter that briefly describes ill-posed inverse problems, setting the scene for the class of ill-posed problems that we will study in this thesis: *unique continuation problems*. We first recall in Section 1.1 the definition of well-posed problems related to PDEs and then consider in Section 1.2 the class of ill-posed problems that can be solved by using the generalised inverse. The problems we are interested in fall outside these two classical types of problems.

1.1 Well-posed problems

We first recall the definition of a well-posed problem given by Hadamard [35]: a problem is well-posed if it admits a unique solution and the solution depends continuously on data (stability property). To be more precise, let us consider an abstract linear problem

$$\mathcal{A}u = f, \tag{1.1}$$

where $\mathcal{A} : W \to F$ is a linear operator between two Banach spaces. We will denote the kernel of $\mathcal{A}$ by $\mathrm{Ker}(\mathcal{A})$ and the range by $\mathrm{Im}(\mathcal{A})$.

Definition 1.1 (Well-posed problem)**.** *Problem* (1.1) *is* well-posed *if the following hold true:*

1. *For any $f \in F$ there exists a solution $u \in W$.*

2. *The solution $u \in W$ is unique.*

3. *The solution $u \in W$ depends continuously on data, i.e. there exists $C_{st} > 0$ such that for any $f \in F$,*

$$\|u\|_W \leq C_{st}\|f\|_F.$$

In other words, problem (1.1) is *well-posed* if $\mathcal{A}$ is bijective and its inverse $\mathcal{A}^{-1}$ is continuous. If $\mathcal{A}$ is a *continuous* linear operator, then the condition for well-posedness is that $\mathcal{A}$ should be bijective – its inverse $\mathcal{A}^{-1}$ will necessarily be continuous by the Open Mapping Theorem.

For PDE-related problems, the operator $\mathcal{A}$ in the abstract problem (1.1) arises from the weak formulation, which typically reads as

$$\text{find } u \in W \text{ such that } a(u, v) = f(v), \text{ for any } v \in V, \tag{1.2}$$

where V is a reflexive Banach space, a is a continuous bilinear form on $W \times V$, and $f \in V'$ is a continuous linear form on V. Indeed, defining $\mathcal{A} : W \to V'$ by

$$\langle \mathcal{A}u, v\rangle_{V',V} := a(u, v), \tag{1.3}$$

problem (1.2) is equivalent to problem (1.1) with $F = V'$.

When the test space V and the solution space W are Hilbert spaces and coincide, a sufficient condition for well-posedness is given by the Lax-Milgram Lemma.

Lemma 1.2 (Lax-Milgram)**.** *Let V be a Hilbert space, $a : V \times V \to \mathbb{R}$ be a continuous bilinear form and $f \in V'$. If the bilinear form a is coercive, i.e. there exists $\alpha > 0$ such that for any $u \in V$,*

$$a(u, u) \geq \alpha\|u\|_V^2,$$

then problem (1.2) is well-posed with the a priori estimate

$$\|u\|_V \leq \frac{1}{\alpha}\|f\|_{V'}.$$

Proof. See [32, Lemma 2.2]. □

If, in addition to the assumptions in Lemma 1.2, the bilinear form a is symmetric, then the solution of problem (1.2) is the minimiser over V of the energy functional

$$J(v) = \tfrac{1}{2}a(v, v) - f(v).$$

Example 1.3 (Poisson's equation)**.** *Let $\Omega \subset \mathbb{R}^n$ be a domain, let $f \in H^{-1}(\Omega)$ and consider*

Poisson's equation with homogeneous Dirichlet boundary conditions

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Integrating by parts, the weak formulation of this problem reads as: find $u \in H^1_0(\Omega)$ *such that*

$$(\nabla u, \nabla v)_{L^2(\Omega)} =: a(u,v) = f(v), \text{ for any } v \in H^1_0(\Omega).$$

By the Lax-Milgram Lemma 1.2 this problem is well-posed*. Indeed, for the usual norm on* $H^1_0(\Omega)$ *given by* $\|v\|^2_{H^1_0(\Omega)} := \|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)}$*, the bilinear form* a *defined on* $H^1_0(\Omega)$ *is continuous by the Cauchy-Schwarz inequality and it is* coercive *since*

$$a(u,u) = \|\nabla u\|^2_{L^2(\Omega)} \geq \alpha \|u\|^2_{H^1_0(\Omega)}, \text{ with } \alpha = \tfrac{1}{1+C_p^2},$$

by the Poincaré inequality

$$\|u\|_{L^2(\Omega)} \leq C_p \|\nabla u\|_{L^2(\Omega)}, \text{ for any } u \in H^1_0(\Omega). \tag{1.4}$$

However, the coercivity condition in Lemma 1.2 is not an optimal condition for the well-posedness of problem (1.2); it is only a sufficient condition. Necessary and sufficient conditions are given by the following *inf-sup condition*, which is also known as the Babuška-Brezzi condition or the Ladyzhenskaya-Babuška-Brezzi condition. For consistency with [32], we adopt the Banach-Nečas-Babuška terminology proposed there.

Theorem 1.4 (Banach-Nečas-Babuška)**.** *Let* W *be a Banach space and* V *a reflexive Banach space. Let* $a : W \times V \to \mathbb{R}$ *be a continuous bilinear form and* $f \in V'$*. Then problem* (1.2) *is well-posed if and only if*

(BNB1) There exists $\alpha > 0$ *such that*

$$\inf_{w \in W} \sup_{v \in V} \frac{a(w,v)}{\|w\|_W \|v\|_V} \geq \alpha.$$

(BNB2) Let $v \in V$*. If* $a(w,v) = 0$ *for any* $w \in W$*, then* $v = 0$*.*

Moreover, the following a priori estimate holds

$$\|u\|_W \leq \frac{1}{\alpha} \|f\|_{V'}.$$

Proof. See [32, Theorem 2.6]. □

Note that the inf-sup condition in Theorem 1.4 is equivalent to

$$\sup_{v \in V} \frac{a(w,v)}{\|v\|_V} \geq \alpha \|w\|_W, \quad \forall w \in W,$$

which in terms of the operator $\mathcal{A} : W \to V'$ given in (1.3) can be written as

$$\|Aw\|_{V'} \geq \alpha \|w\|_W, \quad \forall w \in W.$$

By the Closed Range Theorem and the Open Mapping Theorem this is equivalent to $\mathcal{A}$ being injective and $\mathrm{Im}(\mathcal{A})$ being closed, see e.g. [32, Lemma A.39]. Considering the dual operator $\mathcal{A}^* : V \to W'$ defined by $\langle \mathcal{A}^* v, w \rangle_{W',W} = \langle \mathcal{A} w, v \rangle_{V',V}$, where we have identified V'' with V due to reflexivity, we thus have the following equivalences for the BNB conditions, see e.g [32, Appendix A]:

- (BNB1) $\iff$ $\mathrm{Ker}(\mathcal{A}) = \{0\}$ and $\mathrm{Im}(\mathcal{A})$ closed $\iff$ $\mathcal{A}^*$ surjective.
- (BNB2) $\iff$ $\mathcal{A}^*$ injective.

When the spaces V and W have the same finite dimension, problem (1.2) reduces to a linear system and the condition in the Lax Milgram Lemma 1.2 states that the matrix is positive definite, while the BNB conditions in Theorem 1.4 become equivalent and say that the matrix is invertible.

1.2 Linear inverse problems

We will now introduce linear ill-posed problems and briefly review the standard theory of least squares solutions and the Moore-Penrose generalised inverse, see e.g. [31, Chapter 2]. For the abstract problem (1.1), we herein suppose that $\mathcal{A} : W \to F$ is a continuous linear operator between two Hilbert spaces, and we denote by $\mathcal{A}^* : F \to W$ its adjoint operator defined by $\langle \mathcal{A} v, g \rangle_F = \langle v, \mathcal{A}^* g \rangle_W$, for any $v \in W$ and $g \in F$.

Definition 1.5 (Ill-posed problem)**.** *Problem* (1.1) *is* ill-posed *if at least one condition in Definition 1.1 is not satisfied. This means that at least one of the following holds:*

1. *Non-existence, i.e. there exists $f \in F$ such that $f \notin \mathrm{Im}(\mathcal{A})$.*
2. *Non-uniqueness, i.e.* $\mathrm{Ker}(\mathcal{A}) \neq \{0\}$.

3. *Instability, i.e. the solution does not depend continuously on data.*

The non-existence issue, in which the data is not in the range of the operator, can be overcome by considering *least squares solutions* that minimise the distance to the data. To deal with non-uniqueness, which arises when the operator is not injective, one of the least squares solutions – the one with the minimal norm – can be selected to obtain the *best approximation solution*. We recall these definitions and the conditions under which the best approximation solution exists.

We call $u \in W$ a *least squares solution* of problem (1.1) if

$$\|\mathcal{A}u - f\|_F \leq \|\mathcal{A}v - f\|_F, \text{ for any } v \in W.$$

We call $u^\dagger \in W$ a *best approximation solution* of problem (1.1) if $u^\dagger$ is a least squares solution and

$$\|u^\dagger\|_W \leq \|u\|_W, \text{ for any least squares solution } u.$$

Moreover, due to the convexity of the norm, the best approximation solution is unique in this case. When least squares solutions exist, they can be characterised by the *normal equation*

$$\mathcal{A}^*\mathcal{A}u = \mathcal{A}^*f.$$

However, least squares solutions might not always exist.

Lemma 1.6. *Least-squares solutions exist for problem* (1.1) *if and only if* $f \in \mathrm{Im}(\mathcal{A}) \oplus \mathrm{Im}(\mathcal{A})^\perp$.

Proof. See [31, Theorem 2.6] □

Notice that $\mathrm{Im}(\mathcal{A})^\perp = \overline{\mathrm{Im}(\mathcal{A})}^\perp$, hence $\mathrm{Im}(\mathcal{A}) \oplus \mathrm{Im}(\mathcal{A})^\perp$ is dense in F.

- If $\mathrm{Im}(\mathcal{A})$ is closed, then least squares solutions exist for any $f \in F$. Note that this is a necessary condition for well-posedness in Theorem 1.4 as well.
- When $\mathrm{Im}(\mathcal{A})$ is not closed, least squares solutions do not exist for $f \in \overline{\mathrm{Im}(\mathcal{A})} \setminus \mathrm{Im}(\mathcal{A})$.

We can thus define an operator mapping data $f \in \mathrm{Im}(\mathcal{A}) \oplus \mathrm{Im}(\mathcal{A})^\perp$ to the best approximation solution of $\mathcal{A}u = f$. This operator is called the *pseudoinverse* or the *Moore-Penrose generalised inverse*, and can be obtained by restricting the domain and codomain of $\mathcal{A}$ in order to obtain an invertible operator which is then extended to its maximal domain.

Definition 1.7 (Moore-Penrose generalised inverse)**.** *Let* $\mathcal{A} : W \to F$ *be a continuous linear operator and denote its restriction to* $\mathrm{Ker}(A)^{\perp}$ *by* $\tilde{A} : \mathrm{Ker}(A)^{\perp} \to \mathrm{Im}(\mathcal{A})$*. The Moore-Penrose generalised inverse*

$$\mathcal{A}^{\dagger} : D(\mathcal{A}^{\dagger}) \to W$$

is the unique linear extension of the inverse $\tilde{A}^{-1}$ *to* $D(\mathcal{A}^{\dagger}) := \mathrm{Im}(\mathcal{A}) \oplus \mathrm{Im}(\mathcal{A})^{\perp}$ *with* $\mathrm{Ker}(\mathcal{A}^{\dagger}) = \mathrm{Im}(\mathcal{A})^{\perp}$*. Moreover,* $\mathcal{A}^{\dagger}$ *is a linear operator with* $\mathrm{Im}(\mathcal{A}^{\dagger}) = \mathrm{Ker}(A)^{\perp}$*.*

Theorem 1.8. *Let* $f \in \mathrm{Im}(\mathcal{A}) \oplus \mathrm{Im}(\mathcal{A})^{\perp}$*. Then problem* (1.1) *has a unique best approximation solution which is given by*

$$u^{\dagger} = \mathcal{A}^{\dagger} f.$$

Proof. See [31, Theorem 2.5]. □

Even though the generalised inverse overcomes the issues of *non-existence* and *non-uniqueness* that can appear in the linear inverse problem (1.1), it cannot provide a remedy for the *instability* encountered when the range of the operator is not closed: the generalised inverse is discontinuous in this case.

Theorem 1.9. *The generalised inverse* $\mathcal{A}^{\dagger}$ *is continuous if and only if* $\mathrm{Im}(\mathcal{A})$ *is closed.*

Proof. See [31, Proposition 2.4]. □

An important class of problems that are ill-posed due to instability is given by the inversion of *compact* operators – their infinite-dimensional range cannot be closed, see e.g. [31, Section 2.2]. In this case, the generalised inverse can be expressed using the singular value decomposition of the operator and the ill-posedness can be quantified in terms of the decay to zero of the singular values:

- Problem (1.1) is *severely ill-posed* if the singular values decay to zero at an exponential rate.
- Problem (1.1) is *mildly ill-posed* if the singular values decay to zero at a polynomial rate.

Solving such unstable ill-posed problems requires *regularisation* and we will present a discrete regularisation framework in Chapter 3.

1.2.1 Conditionally stable problems

In this thesis we will focus on ill-posed problems that have *conditional stability*: the solution depends continuously on data assuming an *a priori* bound on the solution itself. We articulate this notion in the following definition.

Definition 1.10 (Conditionally stable problem)**.** *Problem* (1.1) *is* conditionally stable *with respect to a seminorm* $|\cdot|_W$ *on* W *if*

1. *For any* $f \in \mathrm{Im}(\mathcal{A})$*, the solution* u *is unique.*

2. *There exist a non-decreasing function* $C_{st} : [0, \infty) \to [0, \infty)$ *and a modulus of continuity* Υ*, such that for any* $f \in \mathrm{Im}(\mathcal{A})$*,*

$$|u|_W \leq C_{st}\left(\|u\|_W\right) \Upsilon\left(\|f\|_F\right).$$

We say that a function $\Upsilon : [0, \infty) \to [0, \infty)$ is a *modulus of continuity* if it is continuous with $\Upsilon(0) = 0$. The non-decreasing function C_{st} incorporates the a priori bound on the solution and acts similarly to a stability constant. Note that we only assume that the solution can be controlled in a seminorm $|\cdot|_W$. The dependence on data, i.e. the stability of the problem, is provided through the modulus of continuity Υ. The ill-posed problems that we will discuss henceforth have two types of conditional stability, namely:

- Hölder stability, if $\Upsilon(t) = t^\kappa$ for some $\kappa \in (0, 1)$.
- Logarithmic stability, if $\Upsilon(t) = |\log(t)|^{-\kappa}$ for some $\kappa \in (0, 1)$.

1.3 The Cauchy problem

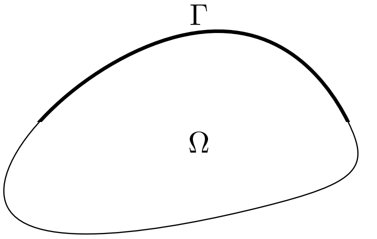


Figure 1.1: Sketch for a domain Ω with Cauchy data given on a part of the boundary $\Gamma \subset \partial\Omega$.

A well-known ill-posed problem which is conditionally stable is the *elliptic Cauchy problem*. For an open set $\Omega \subset \mathbb{R}^n$ with $\partial\Omega$ smooth, Dirichlet and Neumann data are given on a part of the boundary $\Gamma \subset \partial\Omega$, see Figure 1.1. The problem reads as follows: find $u \in H^1(\Omega)$ such that

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g_D & \text{on } \Gamma, \\ \nabla u \cdot n = g_N & \text{on } \Gamma, \end{cases} \tag{1.5}$$

where $f \in H^{-1}(\Omega)$, $g_D \in H^{\frac{1}{2}}(\Gamma)$, $g_N \in H^{-\frac{1}{2}}(\Gamma)$ and n is the unit outward normal.

The elliptic Cauchy problem (1.5) was considered by Hadamard in [35] where it was shown – by a now classical example – that it is ill-posed: there is no continuous dependence on data.

Example 1.11 (Hadamard's example)**.** *Let $\Omega = (0,\pi) \times (0,1)$ and consider a zero source term $f = 0$ and Cauchy data on $\Gamma = (0,\pi) \times \{0\}$ given by $g_D = 0$ and $g_N = \frac{1}{n}\sin(nx)$, together with the exact solution of* (1.5)

$$u(x,y) = \tfrac{1}{n^2}\sin(nx)\sinh(ny).$$

The solution doesn't depend continuously on data since

$$\|g_N\|_{H^{-\frac{1}{2}}(\Gamma)} \leq C\|g_N\|_{L^\infty(\Gamma)} \to 0 \textit{ as } n \to \infty,$$

while

$$\|u\|_{H^1(\Omega)} \to \infty \textit{ and } u(x,y) \to \infty \textit{ exponentially a.e. as } n \to \infty.$$

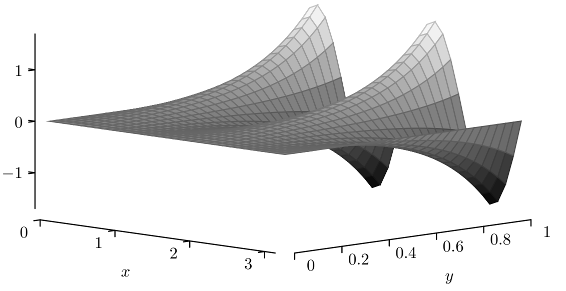


Figure 1.2: Plot of Hadamard's classical Example 1.11 on $[0,\pi] \times [0,1]$ for $n = 4$.

Moreover, the Cauchy problem is an example of a severely ill-posed problem. It was proven in [7] that for a smooth domain in $\mathbb{R}^2$ the problem is equivalent to inverting a compact operator with non-closed range and whose singular values decay to zero faster than any polynomial rate.

However, the Cauchy problem (1.5) is conditionally stable, i.e. assuming an a priori bound on the solution one can obtain continuous dependence on data, see e.g. [1] or [43, Chapter 3]. We recall a simplified version of [1, Theorem 1.7] which states that for a simply connected subset $B \subset \Omega$ for which $\overline{B} \cap (\partial\Omega \setminus \Gamma) = \emptyset$, there exist $C > 0$ and $\kappa \in (0,1)$ depending on B such that the following L^2-norm stability estimate holds

$$\|u\|_{L^2(B)} \leq C \left(\|u\|_{L^2(\Omega)}\right)^{1-\kappa} \left(\|f\|_{H^{-1}(\Omega)} + \|g_D\|_{H^{\frac{1}{2}}(\Gamma)} + \|g_N\|_{H^{-\frac{1}{2}}(\Gamma)}\right)^{\kappa}. \tag{1.6}$$

Since $\|u\|^{1-\kappa}_{L^2(\Omega)} \leq \|u\|^{1-\kappa}_{H^1(\Omega)}$ trivially, the problem is Hölder stable with respect to the local L^2-norm on B. For a global L^2-norm estimate on Ω, the stability deteriorates into logarithmic with an H^1-norm a priori bound on u.

1.4 Unique continuation

In close connection to the Cauchy problem, we introduce the *unique continuation problem* in which the restriction of the solution is given on an open subset and no data is given on the boundary, see Figure 1.3. This is also known as the *data assimilation problem*. We make no assumption on the regularity of the domain which, without loss of generality, can be replaced with a slightly smaller polyhedral domain that can be easily discretised. Considering again the Laplacian as the typical elliptic differential operator, for an open subset $\omega \subset \Omega \subset \mathbb{R}^n$ the problem reads as follows: find $u \in H^1(\Omega)$ such that

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \quad u = u_\omega & \text{in } \omega, \end{cases} \tag{1.7}$$

where $f \in H^{-1}(\Omega)$ and $u_\omega \in H^1(\omega)$. For a solution to exist, the data must be compatible: $-\Delta u_\omega = f$ in ω. Assuming there is a solution $u \in H^1(\Omega)$ for problem (1.7), its uniqueness follows by considering the difference w of two solutions, which satisfies

$$\begin{cases} \Delta w = 0 & \text{in } \Omega, \\ \quad w = 0 & \text{in } \omega. \end{cases}$$

We have that the difference w vanishes everywhere by analytic continuation as harmonic functions are real analytic. In general, we say that an elliptic differential operator $\mathfrak{L}$ has the *unique continuation property* if any solution to $\mathfrak{L}u = 0$ that vanishes on an open subset must vanish everywhere.

The unique continuation property can also be deduced from the uniqueness of the solution to a Cauchy problem. Indeed, if $w|_{\omega} = 0$ and $\Delta w = 0$ in Ω, then taking a ball $\omega' \subset \omega$, we obtain $w = 0$ in $\Omega \setminus \omega'$ as the unique solution of the equation $\Delta w = 0$ in $\Omega \setminus \omega'$ with zero Cauchy data on $\partial\omega'$. The reverse is also true: the unique continuation property implies uniqueness for Cauchy data, see e.g. [1].

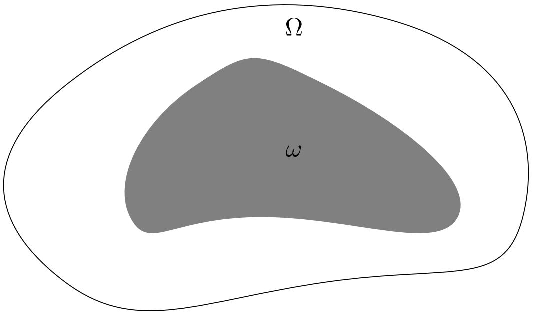


Figure 1.3: Sketch for unique continuation from the subset ω (grey).

The unique continuation problem is ill-posed. To prove this consider the continuous linear operator $\mathcal{A}_{uc} : H^1(\Omega) \to H^{-1}(\Omega) \times H^1(\omega)$ given by $\mathcal{A}_{uc}u = (-\Delta u, u|_{\omega})$. The range of this operator is not closed and its generalised inverse ${\mathcal{A}^{\dagger}}_{uc}$ is discontinuous. Indeed, if $\text{Im}(\mathcal{A}_{uc})$ was closed then by Theorem 1.9 there would exist a constant $C > 0$ such that for any $u \in H^1(\Omega)$

$$\|u\|_{H^1(\Omega)} \leq C\Big(\|\Delta u\|_{H^{-1}(\Omega)} + \|u\|_{H^1(\omega)}\Big).$$

Using Hadamard's Example 1.11 for a subset ω close to the origin, we have that the constant C above cannot be uniform and must actually depend on n, hence we obtain a contradiction.

However, if a solution $u \in H^1(\Omega)$ to problem (1.7) exists in the usual weak sense, i.e. not just in the sense of $\mathcal{A}_{uc}^{\dagger}$ acting on (f, u_{ω}), then conditional stability estimates hold and they are closely related to the ones for the Cauchy problem. We discuss conditional stability estimates for unique continuation in Chapter 2. The Cauchy problem can be used to derive stability estimates for unique continuation, as in [40, Corollary 1.2], while quantifying the reverse implication was at the heart of the method used in [1] in which stability estimates for unique continuation (Theorem 5.1 there) were used to derive stability estimates for the Cauchy problem (Theorem 1.7 there) by slightly extending the domain near the set with Cauchy data. As discussed, the Cauchy problem is severely ill-posed, and hence unique continuation is also severely ill-posed, due to the above equivalence.

Chapter 2

Conditional stability estimates for unique continuation

We have seen in Chapter 1 that unique continuation is ill-posed and closely related to the Cauchy problem. In this chapter we will study *quantitative unique continuation* via conditional stability estimates in a bounded, open, simply connected domain $\Omega \subset \mathbb{R}^n$ for a second order elliptic operator $\mathfrak{L}$ that will be either:

- the Laplacian $\mathfrak{L} = \Delta$.
- the convection–diffusion operator $\mathfrak{L} = \mu\Delta - \beta \cdot \nabla$, with $\mu > 0$ and $\beta \in [L^\infty(\Omega)]^n$.
- the Helmholtz operator $\mathfrak{L} = \Delta + k^2$, with the wave number $k > 0$.

Given an open subset $\omega \subset \Omega$ we want to find $u \in H^1(\Omega)$ that solves the unique continuation problem

$$\begin{cases} \mathfrak{L}u = f & \text{in } \Omega, \\ u = u_\omega & \text{in } \omega, \end{cases} \tag{2.1}$$

where $f \in H^{-1}(\Omega)$, $u_\omega \in H^1(\omega)$. For an open set $B \subset \Omega$ that contains ω such that $B \setminus \omega$ does not touch the boundary $\partial\Omega$ (see Figure 2.1) we will prove Hölder stability estimates that essentially state that

$$\|u\|_{L^2(B)} \leq C_{st} \left(\|u\|_{L^2(\omega)} + \|\mathfrak{L}u\|_{H^{-1}(\Omega)} \right)^\kappa \|u\|_{L^2(\Omega)}^{1-\kappa},$$

for a stability constant $C_{st} > 0$ and some $\kappa \in (0,1)$. When lower terms are included, i.e. for the convection-diffusion and the Helmholtz operators, we will be interested in making such estimates explicit in the physical parameters.

Let us mention that we will not focus on global stability estimates, which are of logarithmic type. We recall that it was proven in [44] that analytic continuation from a disc to a larger concentric disc has only logarithmic stability. From the numerical point of view, it was recently shown in [59] that an exponential loss in digits of accuracy can happen for analytic continuation when moving away from the subset where the function is known to a certain precision. This indicates that accurate computations in the logarithmic stability regime are extremely challenging. We will consider only Hölder stable problems.

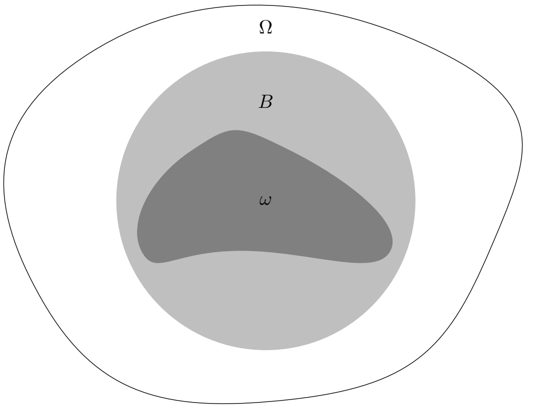


Figure 2.1: Sketch for unique continuation from the set ω (grey) to the set $B \subset \Omega$ (light grey).

We first prove, in an elementary way, a Carleman inequality that will then be used to prove quantitative unique continuation results. Sometimes semiclassical analysis is used to derive this kind of results, see e.g. [51]. Such techniques are very convenient when the estimates are shifted in the Sobolev scale, and we will use them in Sections 2.3.1 and 2.4.1 below. This chapter is based on [21] and [22].

2.1 A pointwise Carleman estimate for the Laplacian

In the seminal paper [26], Carleman introduced a new kind of inequality to prove unique continuation for a second order elliptic operator $\mathfrak{L}$ in $\mathbb{R}^2$. Roughly speaking, for a compactly supported function $w \in C_0^\infty(\mathbb{R}^n)$, the modern usage of the term Carleman estimate refers to an inequality of the form

$$\|e^{\tau\phi} w\|_{L^2(\mathbb{R}^n)} \leq C\|e^{\tau\phi}\mathfrak{L}w\|_{L^2(\mathbb{R}^n)},$$

where $\tau > 0$ is a large parameter and ϕ is a weight function. Quantitative unique continuation can be proven by applying Carleman estimates with suitable ϕ and cut-off χ.

In this section we present an elementary proof for a Carleman estimate for the Laplacian $\mathfrak{L} = \Delta$. The starting idea is to use an exponential weight function e^ℓ and study the expression

$$\Delta(e^\ell w) = e^\ell \Delta w + \text{lower order terms},$$

or the conjugated operator $e^{-\ell}\Delta e^\ell$ for a function ℓ that will be carefully chosen afterwards. For an overview of Carleman estimates we refer the reader to [51,58], the classical references are [38, Chapter 17] for second order elliptic equations, and [39, Chapter 28] for hyperbolic and more general equations. Let us also mention that a typical approach is to study commutator estimates for the real and imaginary part of the principal symbol of the conjugated operator. Our approach can be seen as an alternative to these estimates by considering a more elementary computation that leads first to a pointwise Carleman estimate similar those in [48, Chapter 2] or [50, Chapter 4].

Lemma 2.1 (Carleman-type identity)**.** *Let $\ell \in C^3(\Omega)$, $w \in C^2(\Omega)$ and $\sigma \in C^1(\Omega)$. We define $v = e^\ell w$ and*

$$a = \sigma - \Delta\ell, \quad q = a + |\nabla\ell|^2, \quad b = -\sigma v - 2(\nabla v, \nabla\ell), \quad B = (|\nabla v|^2 - qv^2)\nabla\ell.$$

Then

$$\begin{aligned}\tfrac{1}{2}e^{2\ell}(\Delta w)^2 = {}& \tfrac{1}{2}(\Delta v + qv)^2 + \tfrac{1}{2}b^2 \\ & + a|\nabla v|^2 + 2D^2\ell(\nabla v, \nabla v) + \left(-a|\nabla\ell|^2 + 2D^2\ell(\nabla\ell, \nabla\ell)\right) v^2 \\ & + \operatorname{div}(b\nabla v + B) + R,\end{aligned}$$

where $R = (\nabla\sigma, \nabla v)v + (\operatorname{div}(a\nabla\ell) - a\sigma)\, v^2$.

Proof. Step 1. Let us start by expanding $\Delta(e^\ell w)$ into

$$\begin{aligned}\Delta v = \Delta(e^\ell w) &= \operatorname{div}(\nabla(e^\ell w)) = \operatorname{div}(v\nabla\ell + e^\ell \nabla w) \\ &= (\nabla v, \nabla\ell) + v\Delta\ell + (\nabla e^\ell, \nabla w) + e^\ell \Delta w \\ &= 2(\nabla v, \nabla\ell) + (\Delta\ell - |\nabla\ell|^2)v + e^\ell \Delta w,\end{aligned}$$

where we have used the identity

$$\begin{aligned}(\nabla e^{\ell}, \nabla w) &= (e^{\ell}\nabla \ell, \nabla w) = (\nabla \ell, e^{\ell}\nabla w) = (\nabla \ell, \nabla(e^{\ell}w)) - (\nabla \ell, w\nabla e^{\ell}) \\ &= (\nabla \ell, \nabla v) - v|\nabla \ell|^2.\end{aligned}$$

Rearranging terms, then adding and subtracting σv, we have that

$$e^{\ell}\Delta w = \Delta v - 2(\nabla v, \nabla \ell) + (-\Delta \ell + |\nabla \ell|^2)v = \Delta v + b + qv.$$

Thus

$$\tfrac{1}{2}e^{2\ell}(\Delta w)^2 = \tfrac{1}{2}(\Delta v + qv + b)^2 = \tfrac{1}{2}(\Delta v + qv)^2 + \tfrac{1}{2}b^2 + b\Delta v + bqv. \tag{2.2}$$

We will now study the cross terms $b\Delta v$ and bqv.

Step 2. We start with the first cross term, $b\Delta v$, in (2.2). Let us begin by studying $\beta\Delta v$ where $\beta = -2(\nabla v, \nabla \ell)$. We write

$$\beta\Delta v = -(\nabla\beta, \nabla v) + \operatorname{div}(\beta\nabla v)$$

and

$$-(\nabla\beta, \nabla v) = 2(\nabla(\nabla v, \nabla \ell), \nabla v) = 2D^2v(\nabla v, \nabla \ell) + 2D^2\ell(\nabla v, \nabla v),$$

where $D^2\ell$ is the Hessian matrix of ℓ, and $D^2\ell(X, Y) = X^T(D^2\ell)Y$ for some vectors X and Y. We also have that

$$2D^2v(\nabla v, \nabla \ell) = (\nabla \ell, \nabla|\nabla v|^2) = \operatorname{div}(|\nabla v|^2\nabla \ell) - |\nabla v|^2\Delta \ell. \tag{2.3}$$

To summarize, for $\beta = -2(\nabla v, \nabla \ell)$ it holds that

$$\beta\Delta v = -\Delta \ell|\nabla v|^2 + 2D^2\ell(\nabla v, \nabla v) + \operatorname{div}(\beta\nabla v + |\nabla v|^2\nabla \ell). \tag{2.4}$$

Consider now $\beta\Delta v$ where $\beta = -\sigma v$. We have that

$$-(\nabla\beta, \nabla v) = (\nabla\sigma, \nabla v)v + \sigma|\nabla v|^2,$$

and

$$\beta\Delta v = \sigma|\nabla v|^2 + (\nabla\sigma, \nabla v)v + \operatorname{div}(\beta\nabla v). \tag{2.5}$$

Now (2.4) and (2.5) imply that

$$b\Delta v = a|\nabla v|^2 + 2D^2\ell(\nabla v, \nabla v) + \operatorname{div}(b\nabla v + c_0) + R_0, \tag{2.6}$$

where $c_0 = |\nabla v|^2\nabla\ell$ and $R_0 = (\nabla\sigma, \nabla v)v$.

Step 3. Let us now study the second cross term, bqv, in (2.2). We have

$$-2(\nabla v, \nabla\ell)qv = -(\nabla v^2, q\nabla\ell) = v^2\operatorname{div}(q\nabla\ell) - \operatorname{div}(v^2 q\nabla\ell),$$

hence, recalling that $q = a + |\nabla\ell|^2$ and $-a = -\sigma + \Delta\ell$,

$$\begin{aligned} bqv &= -\sigma qv^2 - 2(\nabla v, \nabla\ell)qv \\ &= (-\sigma q + \operatorname{div}(q\nabla\ell))v^2 + \operatorname{div} c_1 \\ &= (-|\nabla\ell|^2\sigma + \operatorname{div}(|\nabla\ell|^2\nabla\ell))v^2 + \operatorname{div} c_1 + R_1, \end{aligned} \tag{2.7}$$

where $c_1 = -qv^2\nabla\ell$ and $R_1 = (\operatorname{div}(a\nabla\ell) - a\sigma)v^2$. The identity (2.3) with $v = \ell$ implies that

$$\operatorname{div}(|\nabla\ell|^2\nabla\ell) = 2D^2\ell(\nabla\ell, \nabla\ell) + |\nabla\ell|^2\Delta\ell,$$

and since $\sigma = a + \Delta\ell$, we have that

$$-|\nabla\ell|^2\sigma + \operatorname{div}(|\nabla\ell|^2\nabla\ell) = -a|\nabla\ell|^2 + 2D^2\ell(\nabla\ell, \nabla\ell). \tag{2.8}$$

The claim follows by combining (2.8), (2.7), (2.6) and (2.2). □

Let us notice that from this identity we obtain an inequality that is very close to a pointwise Carleman estimate, namely

$$\begin{aligned} \tfrac{1}{2}e^{2\ell}(\Delta w)^2 \geq{} & a|\nabla v|^2 + 2D^2\ell(\nabla v, \nabla v) + \left(-a|\nabla\ell|^2 + 2D^2\ell(\nabla\ell, \nabla\ell)\right)v^2 \\ & + \operatorname{div}(b\nabla v + B) + R. \end{aligned}$$

For a weight function ℓ, we can now take the function a in such a way that the following bounds containing Hessian terms hold

$$
\begin{aligned}
a|\nabla v|^2+2D^2\ell(\nabla v,\nabla v) &\geq C\tau|\nabla v|^2,\\
(-a|\nabla \ell|^2+2D^2\ell(\nabla \ell,\nabla \ell))v^2 &\geq C\tau^3 v^2,
\end{aligned}
$$

with constants depending on ℓ, and from this we will obtain the following result.

Proposition 2.2 (Pointwise Carleman estimate)**.** *Let $\rho \in C^3(\Omega)$ and $K \subset \Omega$ be a compact set that does not contain critical points of ρ. Let $\alpha, \tau > 0$ and the weight function $\phi = e^{\alpha\rho}$. Let $w \in C^2(\Omega)$. Then there exists $C > 0$ such that*

$$
e^{2\tau\phi}((a_1\tau^3 - a_2\tau^2)w^2 + (b_1\tau - b_0)|\nabla w|^2) + \operatorname{div}(b\nabla v + B) \leq Ce^{2\tau\phi}(\Delta w)^2 \quad \textit{in } K.
$$

Proof. Step 1. Let $\ell = \tau\phi$, $v = e^{\tau\phi}w$, and let $\lambda > 0$ such that $|D^2\rho(X,X)|\leq \lambda|X|^2$. Using the product rule we have that

$$
D^2\phi(X,X) = \alpha\phi(\alpha(\nabla\rho, X)^2 + D^2\rho(X,X)),
$$

hence

$$
D^2\phi(X,X) \geq \alpha\phi D^2\rho(X,X) \geq -\alpha\lambda\phi|X|^2
$$

and

$$
D^2\phi(\nabla\phi,\nabla\phi) \geq \alpha^3\phi^3(\alpha|\nabla\rho|^4 - \lambda|\nabla\rho|^2).
$$

Choosing $\varepsilon > 0$ such that $\varepsilon \leq |\nabla\rho|^2 \leq \varepsilon^{-1}$ it holds

$$
D^2\phi(\nabla\phi,\nabla\phi) \geq \alpha^3\phi^3(\alpha\varepsilon^2 - \lambda\varepsilon^{-1}).
$$

Since

$$
2D^2\ell(\nabla v,\nabla v) \geq -2\alpha\lambda\phi\tau|\nabla v|^2,
$$

by choosing $a = 3\alpha\lambda\phi\tau$, i.e. $\sigma = \Delta\ell + 3\alpha\lambda\phi\tau$ in Lemma 2.1 we obtain the bounds

$$
\begin{aligned}
a|\nabla v|^2+2D^2\ell(\nabla v,\nabla v) &\geq \alpha\lambda\phi\tau|\nabla v|^2,\\
(-a|\nabla \ell|^2+2D^2\ell(\nabla \ell,\nabla \ell))v^2 &\geq (2\alpha\varepsilon^2 - 5\lambda\varepsilon^{-1})(\alpha\phi\tau)^3 v^2.
\end{aligned}
$$

Step 2. We now bound the last two terms

$$
(\operatorname{div}(a\nabla\ell) - a\sigma)v^2 = ((\nabla a,\nabla\ell) - a^2)v^2 = (3\alpha\lambda|\nabla\phi|^2 - 9\alpha^2\lambda^2\phi^2)\tau^2 v^2,
$$

and

$$(\nabla\sigma, \nabla v)v = (\nabla(\Delta\ell), \nabla v)v + 3\alpha\lambda(\nabla\ell, \nabla v)v \geq -(|\nabla(\Delta\phi)|+3\alpha\lambda|\nabla\phi|)\tau|\nabla v|\cdot|v|.$$

Combining this lower bound with

$$\tau|\nabla v|\cdot|v| \leq \tfrac{1}{2}(|\nabla v|^2+\tau^2|v|^2),$$

and taking α large enough, we obtain from Lemma 2.1 that

$$Ce^{2\tau\phi}(\Delta w)^2 \geq (a_1\tau^3 - a_2\tau^2)v^2 + (b_1\tau - b_0)|\nabla v|^2 + \operatorname{div}(b\nabla v + B), \tag{2.9}$$

with $a_j, b_j > 0$ depending only on α and ρ. In particular, $a_1 = 2\alpha^3\varepsilon^2\phi^3$ and $b_1 = \alpha\lambda\phi$.

Step 3. Using the elementary inequality

$$(x+y)^2 \geq \tfrac{1}{2}x^2 - y^2,$$

which gives that

$$|\nabla v|^2 = e^{2\tau\phi}|\tau w\nabla\phi + \nabla w|^2 \geq e^{2\tau\phi}\tfrac{1}{2}|\nabla w|^2 - e^{2\tau\phi}|\nabla\phi|^2\tau^2 w^2,$$

we conclude from (2.9) by taking α large enough such that a_1 can absorb $-Cb_1$ for some positive constant C independent of α. □

Corollary 2.3 (Carleman estimate)**.** *Let $\rho \in C^3(\Omega)$ and $K \subset \Omega$ be a compact set that does not contain critical points of ρ. Let $\alpha, \tau > 0$ and $\phi = e^{\alpha\rho}$. Let $w \in C_0^2(K)$. Then there exists $C > 0$ such that*

$$\int_K e^{2\tau\phi}(\tau^3 w^2 + \tau|\nabla w|^2)\,\mathrm{d}x \leq C\int_K e^{2\tau\phi}|\Delta w|^2\,\mathrm{d}x,$$

for α large enough and $\tau \geq \tau_0$, where $\tau_0 > 1$ depends only on α and ρ.

Proof. We integrate over K the pointwise inequality in Proposition 2.2 and then use that the integral of the divergence term vanishes by the divergence theorem since $w = 0$ on ∂K. □

2.2 Hölder stability estimates

Now we explore how the Carleman estimate in Corollary 2.3 can be used to obtain a conditional stability estimate for unique continuation for the Laplacian. The first estimate of this kind that we prove is a *three-ball inequality*. By a covering argument, see e.g. [52], it can be propagated to obtain Hölder stability in the interior of the domain for unique continuation from an open subset.

We first state an auxilliary log-convexity lemma that optimises the exponential parameters coming from the Carleman estimate and provides a Hölder-type inequality with an explicit stability constant.

Lemma 2.4. *Suppose that $a, b, c \geq 0$ and $p, q > 0$ satisfy $c \leq b$ and $c \leq e^{p\lambda} a + e^{-q\lambda} b$ for all $\lambda > \lambda_0 \geq 0$. Then there are $C > 0$ and $\kappa \in (0, 1)$ (depending only on p and q) such that*

$$c \leq C e^{q\lambda_0} a^\kappa b^{1-\kappa}.$$

Here $\kappa = q/(p + q)$.

Proof. We may assume that $a, b > 0$, since $c = 0$ if $a = 0$ or $b = 0$. The minimizer λ_* of the function $f(\lambda) = e^{p\lambda} a + e^{-q\lambda} b$ is given by

$$\lambda_* = \tfrac{1}{p+q} \log \frac{qb}{pa},$$

and writing $r = q/p$, the minimum value is

$$f(\lambda_*) = a \left(\frac{qb}{pa}\right)^{p/(p+q)} + b \left(\frac{qb}{pa}\right)^{-q/(p+q)} = \left(r^{p/(p+q)} + r^{-q/(p+q)}\right) a^{q/(p+q)} b^{p/(p+q)}.$$

This shows that if $\lambda_* > \lambda_0$ then

$$c \leq C_1 a^\kappa b^{1-\kappa},$$

where $\kappa = q/(p + q)$ and $C_1 = r^{p/(p+q)} + r^{-q/(p+q)}$. On the other hand, if $\lambda_* \leq \lambda_0$ then it holds that $e^{-q\lambda_0} \leq e^{-q\lambda_*} = a^{q/(p+q)} (rb)^{-q/(p+q)}$, or equivalently,

$$b^{q/(p+q)} \leq e^{q\lambda_0} a^{q/(p+q)} r^{-q/(p+q)}.$$

Therefore

$$c \leq b = b^{q/(p+q)} b^{p/(p+q)} \leq e^{q\lambda_0} r^{-q/(p+q)} a^{q/(p+q)} b^{p/(p+q)}.$$

That is, if $\lambda_* \leq \lambda_0$ then

$$c \leq C_2 e^{q\lambda_0} a^\kappa b^{1-\kappa},$$

where $C_2 = r^{-q/(p+q)}$. As $e^{q\lambda_0} \geq 1$ and $C_1 > C_2$, the claim follows by taking $C = C_1$. □

Corollary 2.5 (Three-ball inequality for the Laplacian)**.** *Let $x_0 \in \Omega$ and $0 < r_1 < r_2 < d(x_0, \partial\Omega)$. Define $B_j = B(x_0, r_j)$, $j = 1, 2$. Then there exist $C > 0$ and $\kappa \in (0, 1)$ such that for $u \in H^2(\Omega)$ it holds that*

$$\|u\|_{H^1(B_2)} \leq C \left(\|u\|_{H^1(B_1)} + \|\Delta u\|_{L^2(\Omega)} \right)^\kappa \|u\|_{H^1(\Omega)}^{1-\kappa}.$$

Proof. Due to the density of $C^2(\Omega)$ in $H^2(\Omega)$, it is enough to consider $u \in C^2(\Omega)$. Let now $0 < r_0 < r_1$ and $r_2 < r_3 < r_4 < d(x_0, \partial\Omega)$. We choose non-positive $\rho \in C^\infty(\Omega)$ such that $\rho(x) = -d(x, x_0)$ outside B_0. Since $|\nabla\rho| = 1$ outside B_0, ρ does not have critical points in $B_4 \setminus B_0$. Let $\chi \in C_0^\infty(B_4 \setminus B_0)$ satisfy $\chi = 1$ in $B_3 \setminus B_1$, and set $w = \chi u$. We apply Corollary 2.3 with $K = \bar{B}_4 \setminus B_0$ to get

$$\int_{B_4 \setminus B_0} (\tau^3 |w|^2 + \tau |\nabla w|^2) e^{2\tau\phi} \, \mathrm{d}x \leq C \int_{B_4 \setminus B_0} |\Delta w|^2 e^{2\tau\phi} \, \mathrm{d}x, \tag{2.10}$$

for $\phi = e^{\alpha\rho}$, with large enough $\alpha > 0$, and $\tau \geq \tau_0$ (where $\tau_0 > 1$ depends only on α and ρ). Since $\phi \leq 1$ everywhere, by defining $\Phi(r) = e^{-\alpha r}$ we now bound from below the left-hand side in (2.10) by

$$\int_{B_2 \setminus B_1} (\tau^3 |w|^2 + \tau |\nabla w|^2) e^{2\tau\phi} \, \mathrm{d}x \geq \tau e^{2\tau\Phi(r_2)} \|u\|_{H^1(B_2)}^2 - \tau e^{2\tau} \|u\|_{H^1(B_1)}^2.$$

An upper bound for the right-hand side in (2.10) is given by

$$\begin{aligned} &C \int_{B_4} |\Delta u|^2 e^{2\tau\phi} \, \mathrm{d}x + C \int_{(B_4 \setminus B_3) \cup B_1} |[\Delta, \chi] u|^2 e^{2\tau\phi} \, \mathrm{d}x \\ &\quad \leq C e^{2\tau} \|\Delta u\|_{L^2(B_4)}^2 + C e^{2\tau\Phi(r_3)} \|u\|_{H^1(B_4 \setminus B_3)}^2 + C e^{2\tau} \|u\|_{H^1(B_1)}^2. \end{aligned}$$

Combining the last two inequalities we thus obtain that, for $\tau \geq \tau_0$,

$$e^{2\tau\Phi(r_2)} \|u\|_{H^1(B_2)}^2 \leq C e^{2\tau} \left(\|u\|_{H^1(B_1)}^2 + \|\Delta u\|_{L^2(B_4)}^2 \right) + C e^{2\tau\Phi(r_3)} \|u\|_{H^1(B_4)}^2.$$

We conclude by Lemma 2.4 with $p = 1 - \Phi(r_2) > 0$ and $q = \Phi(r_2) - \Phi(r_3) > 0$. □

2.3 Convection–diffusion operator

Using the previous results, one can easily include lower order terms having low regularity in the differential operator $\mathfrak{L}$ and obtain a three-ball inequality similar to Corollary 2.5. However, *the stability constant will depend exponentially* on the size of these lower order terms. To be more precise, for the convection–diffusion operator

$$\mathfrak{L} = \mu\Delta - \beta \cdot \nabla,$$

with $\mu > 0$ and $\beta \in [L^\infty(\Omega)]^n$, we recall the Péclet number associated to a given length scale l given by

$$\mathrm{Pe}(l) := \frac{l\|\beta\|_{[L^\infty(\Omega)]^n}}{\mu}.$$

We have the following result.

Corollary 2.6. *Let $x_0 \in \Omega$ and $0 < r_1 < r_2 < d(x_0, \partial\Omega)$. Define $B_j = B(x_0, r_j)$, $j = 1, 2$. Then there are $C > 0$ and $\kappa \in (0, 1)$ such that for $\mu > 0$, $\beta \in [L^\infty(\Omega)]^n$ and $u \in H^2(\Omega)$ it holds that*

$$\|u\|_{H^1(B_2)} \le Ce^{C\tilde{\mathrm{Pe}}^2}\left(\|u\|_{H^1(B_1)} + \tfrac{1}{\mu}\|\mathfrak{L}u\|_{L^2(\Omega)}\right)^\kappa \|u\|^{1-\kappa}_{H^1(\Omega)},$$

where $\tilde{\mathrm{Pe}} = 1 + |\beta|/\mu$ *and* $|\beta| = \|\beta\|_{[L^\infty(\Omega)]^n}$.

Proof. Following the proof of Corollary 2.5 and using the same notation, we have that

$$\mu^2 \int_{B_4\setminus B_0} (\tau^3|w|^2 + \tau|\nabla w|^2)e^{2\tau\phi}\,\mathrm{d}x \le C\int_{B_4\setminus B_0} |\mu\Delta w|^2 e^{2\tau\phi}\,\mathrm{d}x, \tag{2.11}$$

for $\phi = e^{\alpha\rho}$, with large enough $\alpha > 0$, and $\tau \ge \tau_0$ (where $\tau_0 > 1$ depends only on α and ρ). We bound from above the right-hand side by a constant times

$$\int_{B_4\setminus B_0} |\mu\Delta w - \beta\cdot\nabla w|^2 e^{2\tau\phi}\,\mathrm{d}x + |\beta|^2\int_{B_4\setminus B_0} |\nabla w|^2 e^{2\tau\phi}\,\mathrm{d}x.$$

Taking $\tau \ge 2|\beta|^2/\mu^2$, the second term above is absorbed by the left-hand side of (2.11) to give

$$\mu^2 \int_{B_4\setminus B_0} (\tau^3|w|^2 + \frac{\tau}{2}|\nabla w|^2)e^{2\tau\phi}\,\mathrm{d}x \le C\int_{B_4\setminus B_0} |\mu\Delta w - \beta\cdot\nabla w|^2 e^{2\tau\phi}\,\mathrm{d}x. \tag{2.12}$$

Since $\phi \le 1$ everywhere, by defining $\Phi(r) = e^{-\alpha r}$ we now bound from below the left-hand side in (2.12) by

$$\mu^2 \int_{B_2 \setminus B_1} (\tau^3 |w|^2 + \tau |\nabla w|^2) e^{2\tau\phi} \, \mathrm{d}x \geq \mu^2 \tau e^{2\tau\Phi(r_2)} \|u\|^2_{H^1(B_2)} - \mu^2 \tau e^{2\tau} \|u\|^2_{H^1(B_1)}.$$

An upper bound for the right-hand side in (2.12) is given by

$$\begin{aligned} C \int_{B_4} &|\mu\Delta u - \beta \cdot \nabla u|^2 e^{2\tau\phi} \, \mathrm{d}x + C \int_{(B_4 \setminus B_3) \cup B_1} |(\mu[\Delta, \chi] - \beta \cdot \nabla \chi) u|^2 e^{2\tau\phi} \, \mathrm{d}x \\ &\leq C e^{2\tau} \|\mu\Delta u - \beta \cdot \nabla u\|^2_{L^2(B_4)} + C e^{2\tau\Phi(r_3)} (\mu^2 + |\beta|^2) \|u\|^2_{H^1(B_4 \setminus B_3)} \\ &\quad + C e^{2\tau} (\mu^2 + |\beta|^2) \|u\|^2_{H^1(B_1)}. \end{aligned}$$

Combining the last two inequalities we thus obtain that

$$\begin{aligned} \mu^2 e^{2\tau\Phi(r_2)} \|u\|^2_{H^1(B_2)} \leq & C e^{2\tau} \left((\mu^2 + |\beta|^2) \|u\|^2_{H^1(B_1)} + \|\mu\Delta u - \beta \cdot \nabla u\|^2_{L^2(B_4)} \right) \\ & + C e^{2\tau\Phi(r_3)} (\mu^2 + |\beta|^2) \|u\|^2_{H^1(B_4)}, \end{aligned}$$

for $\tau \geq \tau_0 + 2|\beta|^2/\mu^2$. We divide by μ^2 and conclude by Lemma 2.4 with $p = 1 - \Phi(r_2) > 0$ and $q = \Phi(r_2) - \Phi(r_3) > 0$, followed by absorbing the $\tilde{\mathrm{Pe}} = 1 + |\beta|/\mu$ factor into the exponential factor $e^{C\tilde{\mathrm{Pe}}^2}$. □

Note that a similar approach applied to the Helmholtz operator $\mathfrak{L} = \Delta + k^2$ would give a three-ball inequality for which the stability constant increases exponentially with the wave number k. However, estimates that are robust in the wave number – in the sense that the stability constant is uniform – can be obtained under certain convexity assumptions on the geometry. We discuss this in Section 2.4.

2.3.1 Shifting the norms

Our goal in Chapter 5 will be to design finite element methods for unique continuation that combine continuum Hölder stability estimates as in Corollary 2.6 with discrete regularisation, in order to obtain good convergence properties for the numerical solutions. For this aim, we will need to weaken the norms in which we measure data for unique continuation, from

$$\|u\|_{H^1(B_1)} \text{ and } \|\mathfrak{L}u\|_{L^2(\Omega)}$$

to

$$\|u\|_{L^2(B_1)} \text{ and } \|\mathfrak{L}u\|_{H^{-1}(\Omega)}.$$

This shift in the Sobolev indices will allow us to make the stability estimates valid for H^1 functions and apply them to the finite element approximation error and its residual. A similar argument to the one in this section will also be made in Section 2.4.1 for the Helmholtz operator and will be used in the error analysis in Chapter 4.

When shifting Carleman estimates, as we want to keep track of the large parameter τ, it is convenient to use the semiclassical version of pseudodifferential calculus where we write $\hbar > 0$ for the semiclassical parameter that satisfies $\hbar = 1/\tau$. The semiclassical pseudodifferential operators are pseudodifferential operators where, roughly speaking, each derivative is multiplied by $\hbar$. We recall the precise definition and the results that we will use in Appendix B. The scale of semiclassical Bessel potentials is defined by

$$J^s = (1 - \hbar^2 \Delta)^{s/2}, \quad s \in \mathbb{R},$$

and the semiclassical norms for Sobolev spaces are given by

$$\|u\|_{H^s_{\text{scl}}(\mathbb{R}^n)} = \|J^s u\|_{L^2(\mathbb{R}^n)}.$$

We will give a shifting argument that is similar to that in Section 4 of [29]. To this end, we introduce the following key estimates for semiclassical pseudodifferential operators, see e.g. (4.8) and (4.9) of [29]. Suppose that $\psi, \chi \in C_0^\infty(\mathbb{R}^n)$ and that $\chi = 1$ near $\text{supp}(\psi)$, and let $\mathcal{A}, \mathcal{B}$ be two semiclassical pseudodifferential operators of orders s, m, respectively. Then for all $p, q, N \in \mathbb{R}$, there exists $C > 0$ such that

$$\|(1-\chi)\mathcal{A}(\psi u)\|_{H^p_{\text{scl}}(\mathbb{R}^n)} \le C\hbar^N \|u\|_{H^q_{\text{scl}}(\mathbb{R}^n)}, \tag{2.13}$$

$$\|[\mathcal{A}, \mathcal{B}]u\|_{H^p_{\text{scl}}(\mathbb{R}^n)} \le C\hbar \|u\|_{H^{p+s+m-1}_{\text{scl}}(\mathbb{R}^n)}. \tag{2.14}$$

Both these estimates follow from the composition calculus, see Corollary B.3. The first inequality is sometimes called a pseudolocality estimate, and the second one is a commutator estimate.

Lemma 2.7. *Let $x_0 \in \Omega$ and $0 < r_1 < r_2 < d(x_0, \partial\Omega)$. Define $B_j = B(x_0, r_j)$, $j = 1, 2$. Then there are $C > 0$ and $\kappa \in (0,1)$ such that for $\mu > 0$, $\beta \in [L^\infty(\Omega)]^n$ and $u \in H^1(\Omega)$ it* *holds that*

$$\|u\|_{L^2(B_2)} \leq Ce^{C\tilde{\mathrm{Pe}}^2}\left(\|u\|_{L^2(B_1)} + \tfrac{1}{\mu}\|\mathfrak{L}u\|_{H^{-1}(\Omega)}\right)^{\kappa}\|u\|^{1-\kappa}_{L^2(\Omega)},$$

where $\tilde{\mathrm{Pe}} = 1 + |\beta|/\mu$ *and* $|\beta| = \|\beta\|_{[L^\infty(\Omega)]^n}$.

Proof. Let $0 < r_j < r_{j+1} < d(x_0, \partial\Omega)$, $j = 0, \dots, 4$ and $B_j = B(x_0, r_j)$, keeping B_1, B_2 unchanged. Let $\tilde{r}_j \in (r_{j-1}, r_j)$ and $\tilde{B}_j = B(x_0, \tilde{r}_j)$, $j = 0, \dots, 3$, where $r_{-1} = 0$. Choose $\rho \in C^\infty(\Omega)$ such that $\rho(x) = -d(x, x_0)$ outside $\tilde{B}_0$, and define $\phi = e^{\alpha\rho}$ for large enough α. Consider $v \in C_0^\infty(B_5 \setminus \tilde{B}_0)$. As in Lemma 2.1, by taking $\ell = \phi/\hbar$ and $\sigma = \Delta\ell + 3\alpha\lambda\phi/\hbar$, we obtain after integration that

$$C\int_{\mathbb{R}^n} |e^{\phi/\hbar}\Delta(e^{-\phi/\hbar}v)|^2 \,\mathrm{d}x \geq \int_{\mathbb{R}^n} (\hbar^{-1}|\nabla v|^2 + \hbar^{-3}v^2 - |\nabla v|^2 - \hbar^{-2}v^2)\,\mathrm{d}x.$$

Scaling this with $\mu^2\hbar^4$, we insert the convective term and obtain that

$$C\int_{\mathbb{R}^n} (\mu e^{\phi/\hbar}\hbar^2\Delta(e^{-\phi/\hbar}v) - e^{\phi/\hbar}\hbar^2\beta\cdot\nabla(e^{-\phi/\hbar}v))^2\,\mathrm{d}x$$

can be bounded from below by

$$\int_{\mathbb{R}^n} \hbar\mu^2(\hbar^2|\nabla v|^2 + v^2)\,\mathrm{d}x - \int_{\mathbb{R}^n} \hbar^2\mu^2(\hbar^2|\nabla v|^2 + v^2)\,\mathrm{d}x - \int_{\mathbb{R}^n} (e^{\phi/\hbar}\hbar^2\beta\cdot\nabla(e^{-\phi/\hbar}v))^2\,\mathrm{d}x.$$

Since

$$e^{\phi/\hbar}\hbar^2\beta\cdot\nabla(e^{-\phi/\hbar}v) = -\hbar(\beta\cdot\nabla\phi)v + \hbar^2\beta\cdot\nabla v,$$

introducing the conjugated operator $\mathcal{P}v = -\hbar^2 e^{\phi/\hbar}\mathcal{L}(e^{-\phi/\hbar}v)$, the previous bound implies

$$C\|\mathcal{P}v\|^2_{L^2(\mathbb{R}^n)} \geq \hbar\mu^2\|v\|^2_{H^1_{\mathrm{scl}}(\mathbb{R}^n)} - \hbar^2\mu^2\|v\|^2_{H^1_{\mathrm{scl}}(\mathbb{R}^n)} - \hbar^2|\beta|^2\|v\|^2_{H^1_{\mathrm{scl}}(\mathbb{R}^n)}.$$

The last two terms in the right-hand side can be absorbed by the first one when

$$\hbar \leq \frac{1}{2} \text{ and } \hbar \leq \frac{1}{2}\frac{\mu^2}{|\beta|^2}, \tag{2.15}$$

thus obtaining that

$$\sqrt{\hbar}\mu\|v\|_{H^1_{\mathrm{scl}}(\mathbb{R}^n)} \leq C\|\mathcal{P}v\|_{L^2(\mathbb{R}^n)}. \tag{2.16}$$

Let now $\eta, \vartheta \in C_0^\infty(B_5 \setminus \tilde{B}_0)$ and suppose that $\vartheta = 1$ near $B_4 \setminus B_0$ and $\eta = 1$ near $\mathrm{supp}(\vartheta)$. Let also $\chi \in C_0^\infty(B_4 \setminus B_0)$ satisfy $\chi = 1$ in $B_3 \setminus \tilde{B}_1$. Then there exists $\hbar_0 > 0$ such that for $v = \chi w$, $w \in C^\infty(\Omega)$, and $\hbar < \hbar_0$,

$$\|v\|_{L^2(\mathbb{R}^n)} \leq \|\eta J^{-1}v\|_{H^1_{\mathrm{scl}}(\mathbb{R}^n)} + \|(1-\eta)J^{-1}\vartheta v\|_{H^1_{\mathrm{scl}}(\mathbb{R}^n)} \leq C\|\eta J^{-1}v\|_{H^1_{\mathrm{scl}}(\mathbb{R}^n)}, \tag{2.17}$$

where we used (2.13) to absorb one term by the left-hand side. From (2.17) and (2.16) we have that

$$\sqrt{\hbar}\mu\|v\|_{L^2(\mathbb{R}^n)} \leq C\sqrt{\hbar}\mu\|\eta J^{-1}v\|_{H^1_{\mathrm{scl}}(\mathbb{R}^n)} \leq C\|P(\eta J^{-1}v)\|_{L^2(\mathbb{R}^n)}, \tag{2.18}$$

and the commutator estimate (2.14) gives that

$$\|[P, \eta J^{-1}]v\|_{L^2(\mathbb{R}^n)} \leq C\hbar\mu\|v\|_{L^2(\mathbb{R}^n)} + C\hbar^2|\beta|\|v\|_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^n)}.$$

Recalling the assumption (2.15), these terms can be absorbed by the left-hand side of (2.18), obtaining

$$\sqrt{\hbar}\mu\|v\|_{L^2(\mathbb{R}^n)} \leq C\|\eta J^{-1}(Pv)\|_{L^2(\mathbb{R}^n)} \leq C\|Pv\|_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^n)}. \tag{2.19}$$

We now combine this estimate with the technique used to prove Corollary 2.6. Consider $u \in C^\infty(\mathbb{R}^n)$ and set $w = e^{\phi/\hbar}u$. Take $\psi \in C_0^\infty(\Omega)$ supported in $B_1 \cup (B_5 \setminus \tilde{B}_3)$ with $\psi = 1$ in $(\tilde{B}_1 \setminus B_0) \cup (B_4 \setminus B_3)$. Recall that $\chi \in C_0^\infty(B_4 \setminus B_0)$ satisfies $\chi = 1$ in $B_3 \setminus \tilde{B}_1$. Using (2.14) to bound the commutator

$$\|[P,\chi]w\|_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^n)} \leq \|[P,\chi]\psi w\|_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^n)} \leq C\hbar(\mu + |\beta|)\|\psi w\|_{L^2(\mathbb{R}^n)},$$

we obtain from (2.19) that

$$\sqrt{\hbar}\mu\|\chi w\|_{L^2(\mathbb{R}^n)} \leq C\|\chi P w\|_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^n)} + C\hbar(\mu + |\beta|)\|\psi w\|_{L^2(\mathbb{R}^n)}.$$

This leads to

$$\sqrt{\hbar}\mu\left\|\chi e^{\phi/\hbar}u\right\|_{L^2(\mathbb{R}^n)} \leq C\left\|\chi e^{\phi/\hbar}(\mu\Delta u - \beta\cdot\nabla u)\right\|_{H^{-1}(\mathbb{R}^n)} + C\hbar(\mu + |\beta|)\left\|\psi e^{\phi/\hbar}u\right\|_{L^2(\mathbb{R}^n)},$$

where we used the norm inequality $\|\cdot\|_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^n)} \leq C\hbar^{-2}\|\cdot\|_{H^{-1}(\mathbb{R}^n)}$. Letting $\Phi(r) = e^{-\alpha r}$

and using a similar argument as in the proof of Corollary 2.6, we find that

$$\mu e^{\Phi(r_2)/\hbar}\|u\|_{L^2(B_2)} \leq Ce^{1/\hbar}\left((\mu+|\beta|)\|u\|_{L^2(B_1)} + \hbar^{-\frac{3}{2}}\|\mu\Delta u - \beta\cdot\nabla u\|_{H^{-1}(\Omega)}\right) + Ce^{\Phi(\tilde{r}_3)/\hbar}\hbar^{\frac{1}{2}}(\mu+|\beta|)\|u\|_{L^2(\Omega)},$$

when $\hbar$ satisfies (2.15) and is small enough. Absorbing the negative power of $\hbar$ in the exponential, we then use Lemma 2.4 and conclude by absorbing the $\tilde{\text{Pe}} = 1 + |\beta|/\mu$ factor into the exponential factor $e^{C\tilde{\text{Pe}}^2}$. □

Remark 2.8. *In the case of three-ball inequalities one can be more precise about the Hölder exponent κ in Lemma 2.7. We recall some known results for second-order elliptic equations: we refer to [1, Theorem 2.1] for the Laplace equation, and for the case including lower-order terms to [52, Theorem 3]. Let u be a homogeneous solution of $\mathfrak{L}u = 0$. For a constant C_{st} depending implicitly on the coefficients μ and β, the following three-ball inequality holds*

$$\|u\|_{L^2(B_2)} \leq C_{st}\|u\|^{\kappa}_{L^2(B_1)}\|u\|^{1-\kappa}_{L^2(B_3)}.$$

The constant C_{st} does not depend on the radii r_1, r_2, but it does depend on r_3. The exponent $\kappa \in (0,1)$ is given by

$$\kappa = \frac{\log\frac{r_3}{r_2}}{C_3\log\frac{r_2}{r_1} + \log\frac{r_3}{r_2}},$$

where $C_3 > 0$ is a constant depending on r_3. Notice also that when the radius $r_1 \to 0$, the exponent $\kappa \to 0$ as well and the three-ball inequality provides no useful information.

We can weaken the norms just in the right-hand side of Corollary 2.6 by making the additional coercivity assumption $\nabla\cdot\beta \leq 0$ and using the stability estimate for a well-posed convection-diffusion problem with homogeneous Dirichlet boundary conditions.

Corollary 2.9. *Let $x_0 \in \Omega$ and $0 < r_1 < r_2 < d(x_0, \partial\Omega)$. Define $B_j = B(x_0, r_j)$, $j = 1, 2$. Then there exist $C > 0$ and $\kappa \in (0,1)$ such that for $\mu > 0$, $\beta \in [W^{1,\infty}(\Omega)]^n$ having* $\operatorname{ess\,sup}_\Omega \nabla\cdot\beta \leq 0$, *and $u \in H^1(\Omega)$ it holds that*

$$\|u\|_{H^1(B_2)} \leq Ce^{C\tilde{\text{Pe}}^2}\left(\|u\|_{L^2(B_1)} + \tfrac{1}{\mu}\|\mathfrak{L}u\|_{H^{-1}(\Omega)}\right)^{\kappa}\left(\|u\|_{L^2(\Omega)} + \tfrac{1}{\mu}\|\mathfrak{L}u\|_{H^{-1}(\Omega)}\right)^{1-\kappa},$$

where $\tilde{\text{Pe}} = 1 + |\beta|/\mu$ and $|\beta| = \|\beta\|_{[L^\infty(\Omega)]^n}$.

Proof. Let the balls $B_0, B_3 \subset \Omega$ be such that $B_j \subset B_{j+1}$, for $j = 0, 2$. Consider the well-posed problem

$$\mathfrak{L}w = \mathfrak{L}u \text{ in } B_3, \quad w = 0 \text{ on } \partial B_3.$$

Since $\operatorname{ess\,sup}_\Omega \nabla \cdot \beta \leq 0$, as a consequence of the divergence theorem, we have

$$\|w\|_{H^1(B_3)} \leq C\frac{1}{\mu}\|\mathfrak{L}u\|_{H^{-1}(B_3)}.$$

Taking $v = u - w$, we have $\mathfrak{L}v = 0$ in B_3. The stability estimate in Corollary 2.6 used for B_0, B_2, B_3 reads as

$$\|v\|_{H^1(B_2)} \leq Ce^{C\tilde{\mathrm{Pe}}^2}\|v\|^{\kappa}_{H^1(B_0)}\|v\|^{1-\kappa}_{H^1(B_3)},$$

and the following estimates hold

$$\begin{aligned}\|u\|_{H^1(B_2)} &\leq \|v\|_{H^1(B_2)} + \|w\|_{H^1(B_2)}\\ &\leq Ce^{C\tilde{\mathrm{Pe}}^2}(\|u\|_{H^1(B_0)} + \tfrac{1}{\mu}\|\mathfrak{L}u\|_{H^{-1}(\Omega)})^{\kappa}(\|u\|_{H^1(B_3)} + \tfrac{1}{\mu}\|\mathfrak{L}u\|_{H^{-1}(\Omega)})^{1-\kappa}.\end{aligned}$$

Now we choose a cutoff function $\chi \in C_0^\infty(B_1)$ such that $\chi = 1$ in B_0. Then χu satisfies

$$\mathfrak{L}(\chi u) = \chi\mathfrak{L}u + [\mathfrak{L}, \chi]u, \quad \chi u = 0 \text{ on } \partial B_1,$$

and we obtain

$$\begin{aligned}\|u\|_{H^1(B_0)} \leq \|\chi u\|_{H^1(B_1)} &\leq C\frac{1}{\mu}\left(\|[\mathfrak{L},\chi]u\|_{H^{-1}(B_1)} + \|\chi\mathfrak{L}u\|_{H^{-1}(B_1)}\right)\\ &\leq C\frac{1}{\mu}\left((\mu + |\beta|)\|u\|_{L^2(B_1)} + \|\mathfrak{L}u\|_{H^{-1}(\Omega)}\right)\end{aligned}$$

The same argument for $B_3 \subset \Omega$ gives

$$\|u\|_{H^1(B_3)} \leq C\frac{1}{\mu}\left((\mu + |\beta|)\|u\|_{L^2(\Omega)} + \|\mathfrak{L}u\|_{H^{-1}(\Omega)}\right),$$

thus leading to the conclusion after absorbing the $\tilde{\mathrm{Pe}} = 1 + |\beta|/\mu$ factor into the exponential factor $e^{C\tilde{\mathrm{Pe}}^2}$. □

2.4 Helmholtz operator

It is well known, see e.g. [43], that if $\overline{B \setminus \omega} \subset \Omega$ then the unique continuation problem for the Helmholtz operator $\mathfrak{L} = \Delta + k^2$ is conditionally Hölder stable: for all $k \geq 0$ there exist $C_{st} > 0$ and $\kappa \in (0, 1)$ such that for all $u \in H^2(\Omega)$

$$\|u\|_{H^1(B)} \leq C_{st}\Big(\|u\|_{H^1(\omega)} + \|\Delta u + k^2 u\|_{L^2(\Omega)}\Big)^{\kappa}\|u\|^{1-\kappa}_{H^1(\Omega)}. \tag{2.20}$$

As discussed in the introduction of this chapter, if $B \setminus \omega$ touches the boundary $\partial\Omega$ then one can only expect logarithmic stability.

In general, the stability constant C_{st} and the Hölder exponent κ in (2.20) depend on the wave number k, and might actually blow up faster than any polynomial in k as we show in the following example.

Example 2.10. *Let $\varepsilon \in (0,1)$ be small and consider $\Omega = (-\varepsilon, 1+\varepsilon)^2$ and the domains $\omega = (0,1) \times (0,\varepsilon)$ and $B = (0,1)^2$. Take the ansatz $u(x,y) = e^{ikx}a(x,y)$, where for $n \in \mathbb{N}$,*

$$a(x,y) = a_0(x,y) + k^{-1}a_1(x,y) + \ldots + k^{-n}a_n(x,y).$$

We have that

$$\Delta u + k^2 u = e^{ikx}\left(2ik\partial_x a + \Delta a\right),$$

and we choose a_j, $j = 0, \ldots, n$ such that

$$\partial_x a_0 = 0, \quad 2i\partial_x a_j + \Delta a_{j-1} = 0, \quad j = 1, \ldots, n. \tag{2.21}$$

Then

$$\Delta u + k^2 u = e^{ikx}k^{-n}\Delta a_n$$

and $\|\Delta u + k^2 u\|_{L^2(\Omega)} = k^{-n}\|\Delta a_n\|_{L^2(\Omega)}$. Since a_j is chosen to be independent of k, for all $j = 0, \ldots, n$, we obtain for a generic constant $C > 0$ that

$$\|\Delta u + k^2 u\|_{L^2(\Omega)} \le Ck^{-n}.$$

We can solve (2.21) in such a way that $a_0(x,y) = a_0(y)$, $\operatorname{supp}(a_0) \subset (\varepsilon, 1)$ and

$$\operatorname{supp}(a_j) \subset [-\varepsilon, 1+\varepsilon] \times (\varepsilon, 1), \quad j = 1, \ldots, n..$$

Then $u|_\omega = 0$ and we have that

$$C^{-1}k \le \|u\|_{H^1(B)} \le \|u\|_{H^1(\Omega)} \le Ck.$$

for large k. The estimate (2.20) then becomes

$$Ck \le C_{st}(k^{-n})^\kappa k^{1-\kappa}, \quad \textit{i.e.}\ Ck^{\kappa(n+1)} \le C_{st}.$$

Choosing large n we see that C_{st} depends on k, and for any $N \in \mathbb{N}$, $C_{st} \le k^N$ cannot hold.

However, under suitable convexity assumptions on the geometry and direction of continuation, it is possible to prove that in (2.20) both the constants C_{st} and κ are independent of the wave number k – this is closely related to the so-called increased stability for unique continuation [40].

The proofs are based on a pointwise Carleman estimate that is a variation of [40, Lemma 2.2] but we give a self-contained proof in Corollary 2.12. In [40] the Carleman estimate was used to derive the increased stability estimate under suitable convexity assumptions on the geometry. We briefly recall these assumptions for completeness. Let $\Gamma \subset \partial\Omega$ be such that $\Gamma \subset \partial\omega$ and Γ is at some positive distance away from $\partial\omega \cap \Omega$. For a compact subset S of the open set Ω, let $P(\nu; d)$ denote the half space which has distance d from S and ν as the exterior normal vector. Let $\Omega(\nu; d) = P(\nu; d) \cap \Omega$ and denote by B the union of the sets $\Omega(\nu; d)$ over all ν for which $P(\nu; d) \cap \partial\Omega \subset \Gamma$. This geometric setting is illustrated in a general way in Figs. 1 and 2 of [40] where B is denoted by $\Omega(\Gamma; d)$. Under these assumptions it was proven that

$$\|u\|_{L^2(B)} \leq CF + Ck^{-1}F^{\kappa}\|u\|^{1-\kappa}_{H^1(\Omega)}, \tag{2.22}$$

where $F = \|u\|_{H^1(\omega)} + \|\Delta u + k^2 u\|_{L^2(\Omega)}$ and the constants C and κ are independent of k. As k grows, the first term on the right-hand side of (2.22) dominates the second one, and the stability is increasing in this sense.

As our focus in Chapter 4 will be to design a finite element method for this problem, we prefer to measure the size of the data in the weaker norm

$$E = \|u\|_{L^2(\omega)} + \|\Delta u + k^2 u\|_{H^{-1}(\Omega)}.$$

Taking u to be a plane wave solution to the Helmholtz equation suggests that

$$\|u\|_{L^2(B)} \leq CkE + CE^{\kappa}\|u\|^{1-\kappa}_{L^2(\Omega)},$$

could be the right analogue of (2.22) when both the data and the a priori bound are in weaker norms. Similarly to Section 2.3.1, using tools from semiclassical analysis we prove in Section 2.4.1 a stronger estimate with only the second term on the right-hand side, namely

$$\|u\|_{L^2(B)} \leq CE^{\kappa}\|u\|^{1-\kappa}_{L^2(\Omega)}.$$

Following the argument in Lemma 2.1, we first prove a Carleman-type identity for the Helmholtz operator.

Lemma 2.11. *Let $k \geq 0$. Let $\ell \in C^3(\Omega)$, $w \in C^2(\Omega)$ and $\sigma \in C^1(\Omega)$. We define $v = e^{\ell} w$, and*

$$a = \sigma - \Delta\ell, \quad q = k^2 + a + |\nabla\ell|^2, \quad b = -\sigma v - 2(\nabla v, \nabla\ell), \quad c = (|\nabla v|^2 - qv^2)\nabla\ell.$$

Then

$$\begin{aligned}\tfrac{1}{2}e^{2\ell}(\Delta w + k^2 w)^2 = \tfrac{1}{2}(\Delta v + qv)^2 + \tfrac{1}{2}b^2 \\ &+ a|\nabla v|^2 + 2D^2\ell(\nabla v, \nabla v) + \left(-a|\nabla\ell|^2 + 2D^2\ell(\nabla\ell, \nabla\ell)\right) v^2 - k^2 a v^2 \\ &+ \operatorname{div}(b\nabla v + c) + R,\end{aligned}$$

where $R = (\nabla\sigma, \nabla v)v + (\operatorname{div}(a\nabla\ell) - a\sigma)\, v^2$.

Proof. Following Step 1 in the proof of Lemma 2.1 we have

$$\tfrac{1}{2}e^{2\ell}(\Delta w + k^2 w)^2 = \tfrac{1}{2}(\Delta v + b + qv)^2 = \tfrac{1}{2}(\Delta v + qv)^2 + \tfrac{1}{2}b^2 + b\Delta v + bqv, \tag{2.23}$$

and it remains to study the cross terms $b\Delta v$ and bqv. For the first term we have from Step 2 in the proof of Lemma 2.1 that

$$b\Delta v = a|\nabla v|^2 + 2D^2\ell(\nabla v, \nabla v) + \operatorname{div}(b\nabla v + c_0) + R_0, \tag{2.24}$$

where $c_0 = |\nabla v|^2 \nabla\ell$ and $R_0 = (\nabla\sigma, \nabla v)v$. Let us now study the second cross term in (2.23). We have

$$-2(\nabla v, \nabla\ell)qv = -(\nabla v^2, q\nabla\ell) = v^2 \operatorname{div}(q\nabla\ell) - \operatorname{div}(v^2 q\nabla\ell),$$

hence, recalling that $q = k^2 + a + |\nabla\ell|^2$ and $-a = -\sigma + \Delta\ell$,

$$\begin{aligned}bqv &= (-\sigma q + \operatorname{div}(q\nabla\ell))v^2 + \operatorname{div} c_1 \\ &= (-|\nabla\ell|^2\sigma + \operatorname{div}(|\nabla\ell|^2\nabla\ell))v^2 - k^2 a v^2 + \operatorname{div} c_1 + R_1,\end{aligned} \tag{2.25}$$

where $c_1 = -qv^2\nabla\ell$ and $R_1 = (\operatorname{div}(a\nabla\ell) - a\sigma)v^2$. The identity (2.3) with $v = \ell$ implies that

$$\operatorname{div}(|\nabla\ell|^2\nabla\ell) = 2D^2\ell(\nabla\ell, \nabla\ell) + |\nabla\ell|^2\Delta\ell,$$

hence, recalling that $\sigma = a + \Delta\ell$,

$$-|\nabla\ell|^2\sigma + \operatorname{div}(|\nabla\ell|^2\nabla\ell) = -a|\nabla\ell|^2 + 2D^2\ell(\nabla\ell, \nabla\ell). \tag{2.26}$$

The claim follows by combining (2.26), (2.25), (2.24) and (2.23). □

In this section we use the identity in Lemma 2.11 with the choice $\sigma = \Delta\ell$, or equivalently $a = 0$, but we remark that a different choice was used in the proof of Proposition 2.2 for convection–diffusion.

Corollary 2.12 (Pointwise Carleman estimate)**.** *Let $\phi \in C^3(\Omega)$ be a strictly convex function without critical points, and choose $\rho > 0$ such that*

$$D^2\phi(X, X) \geq \rho|X|^2, \quad X \in \mathbb{R}^n.$$

Let $\tau > 0$ and $w \in C^2(\Omega)$. We define $\ell = \tau\phi$, $v = e^\ell w$, and

$$b = -(\Delta\ell)v - 2(\nabla v, \nabla\ell), \quad c = (|\nabla v|^2 - (k^2 + |\nabla\ell|^2)v^2)\nabla\ell.$$

Then

$$e^{2\tau\phi}\left((a_0\tau - b_0)\tau^2 w^2 + (a_1\tau - b_1)|\nabla w|^2\right) + \operatorname{div}(b\nabla v + c) \leq \tfrac{1}{2}e^{2\tau\phi}(\Delta w + k^2 w)^2,$$

where the constants $a_j, b_j > 0$, $j = 0, 1$, depend only on ρ,

$$\inf_{x\in\Omega}|\nabla\phi(x)|^2 \text{ and } \sup_{x\in\Omega}|\nabla(\Delta\phi(x))|^2.$$

Proof. We employ the equality in Lemma 2.11 with $\ell = \tau\phi$ and $\sigma = \Delta\ell$. With this choice of σ, it holds that $a = 0$. As the two first terms on the right-hand side of the equality are positive, it is enough to consider

$$\begin{aligned} &2D^2\ell(\nabla v, \nabla v) + 2D^2\ell(\nabla\ell, \nabla\ell)v^2 + R \\ &\quad \geq 2\rho\tau|\nabla v|^2 + 2\rho\tau^3|\nabla\phi|^2 v^2 - \tau|\nabla(\Delta\phi)|\cdot|\nabla v|\cdot|v|. \end{aligned}$$

The claim follows by combining this with

$$|\nabla v|^2 = e^{2\tau\phi}|\tau w\nabla\phi + \nabla w|^2 \geq e^{2\tau\phi}\tfrac{1}{3}|\nabla w|^2 - e^{2\tau\phi}\tfrac{1}{2}|\nabla\phi|^2\tau^2 w^2,$$

and

$$\tau|\nabla(\Delta\phi)|\cdot|\nabla v|\cdot|v| \leq C(|\nabla v|^2+\tau^2|v|^2).$$

□

When continuing the solution inside the convex hull of ω as in [40], we consider for simplicity a specific geometric setting defined in Corollary 2.13 below and illustrated in Figure 2.2. As shown in Example 2.10, without such a convexity assumption, the stability constant in estimates of the form (2.20) may depend on the wave number k and can even increase faster than any polynomial in k. A similar reasoning to the one in Corollary 2.6 would lead to a three-ball inequality with a stability constant that is exponential in k. However, in the geometric setting that we consider, we can derive a stability estimate that is robust in the wave number by starting from the pointwise Carleman estimate Corollary 2.12 and using a foliation by spheres in the convex direction.

The stability estimates we prove below in Corollaries 2.13 and 2.14, and Lemma 2.15 also hold in other geometric settings in which B is included in the convex hull of ω and $B\setminus\omega$ does not touch the boundary of Ω, such as the ones we consider for numerical experiments in Chapter 4 and for which we give a proof in Example 4.10.

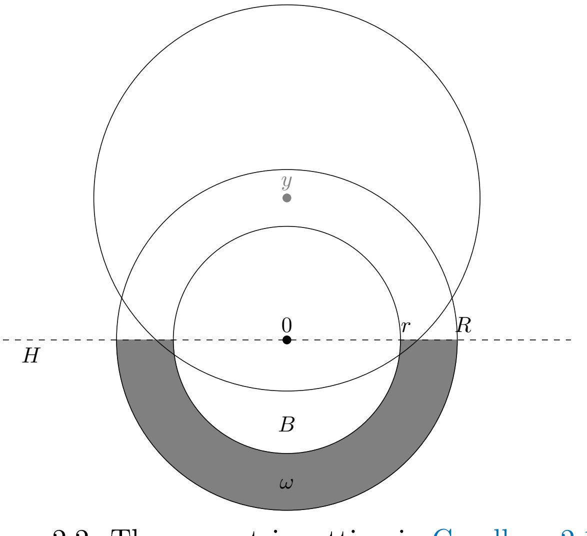


Figure 2.2: The geometric setting in Corollary 2.13.

We use the following notation for a half space

$$H := \{(x^1, \dots, x^n) \in \mathbb{R}^n;\ x^n < 0\}.$$

Corollary 2.13. *Let* $r > 0$, $\beta > 0$, $R > r$ *and* $\sqrt{r^2+\beta^2} < \rho < \sqrt{R^2+\beta^2}$. *Define* $y = (0, \dots, 0, \beta)$ *and*

$$\Omega = H \cap B(0,R), \quad \omega = \Omega \setminus \overline{B(0,r)}, \quad B = \omega \cup (\Omega \setminus \overline{B(y,\rho)}).$$

Then there exist $C > 0$ *and* $\kappa \in (0,1)$ *such that for all* $u \in C^2(\Omega)$ *and* $k \geq 0$

$$\|u\|_{H^1(B)} \leq C\Big(\|u\|_{H^1(\omega)} + \|\Delta u + k^2 u\|_{L^2(\Omega)}\Big)^{\kappa} \|u\|_{H^1(\Omega)}^{1-\kappa}.$$

Proof. Choose $\sqrt{r^2+\beta^2} < s < \rho$ and observe that $\partial\Omega \setminus B(y,s) \subset \overline{\omega}$. Define $\phi(x) = |x-y|^2$. Then ϕ is smooth and strictly convex in $\overline{\Omega}$, and it does not have critical points there.

Choose $\chi \in C_0^\infty(\Omega)$ such that $\chi = 1$ in $\Omega \setminus (B(y,s) \cup \omega)$ and set $w = \chi u$. Corollary 2.12 implies that for large $\tau > 0$

$$\int_\Omega (\tau^3 w^2 + \tau|\nabla w|^2) e^{2\tau\phi} dx \leq C \int_\Omega (\Delta w + k^2 w)^2 e^{2\tau\phi} dx, \tag{2.27}$$

a result also stated, without a detailed proof, in [43, Exercise 3.4.6]. The commutator $[\Delta, \chi]$ vanishes outside $B(y,s) \cup \omega$ and $\phi < s^2$ in $B(y,s)$. Hence the right-hand side of (2.27) is bounded by a constant times

$$\begin{aligned} &\int_\Omega |\Delta u + k^2 u|^2 e^{2\tau\phi} dx + \int_{B(y,s)\cup\omega} |[\Delta, \chi]u|^2 e^{2\tau\phi} dx \\ &\quad \leq Ce^{2\tau(\beta+R)^2}(\|\Delta u + k^2 u\|_{L^2(\Omega)}^2 + \|u\|_{H^1(\omega)}^2) + Ce^{2\tau s^2}\|u\|_{H^1(B(y,s))}^2. \end{aligned} \tag{2.28}$$

The left-hand side of (2.27) is bounded from below by

$$\int_{B\setminus\omega} \Big(\tau|\nabla u|^2 + \tau^3|u|^2\Big) e^{2\tau\phi}\, dx \geq e^{2\tau\rho^2}\|u\|_{H^1(B\setminus\omega)}^2. \tag{2.29}$$

Using the trivial bound $e^{2\tau\rho^2}\|u\|_{H^1(\omega)}^2 \leq e^{2\tau(\beta+R)^2}\|u\|_{H^1(\omega)}^2$, the inequalities (2.27)–(2.29) imply

$$\|u\|_{H^1(B)} \leq Ce^{p\tau}\Big(\|\Delta u + k^2 u\|_{L^2(\Omega)} + \|u\|_{H^1(\omega)}\Big) + Ce^{-q\tau}\|u\|_{H^1(\Omega)},$$

where $p = (\beta+R)^2 - \rho^2 > 0$ and $q = \rho^2 - s^2 > 0$. The claim follows from Lemma 2.4. □

Corollary 2.14. *Let* $\omega \subset B \subset \Omega$ *be defined as in Corollary 2.13. Then there exist* $C > 0$ *and* $\kappa \in (0,1)$ *such that for all* $u \in H^1(\Omega)$ *and* $k \geq 0$

$$\|u\|_{H^1(B)} \leq Ck(\|u\|_{L^2(\omega)} + \|\Delta u + k^2 u\|_{H^{-1}(\Omega)})^{\kappa}(\|u\|_{L^2(\Omega)} + \|\Delta u + k^2 u\|_{H^{-1}(\Omega)})^{1-\kappa}.$$

Proof. Let $\omega_1 \subset \omega \subset B \subset \Omega_1 \subset \Omega$, denote for brevity $\mathfrak{L} = \Delta + k^2$, and consider the following auxiliary problem

$$\begin{cases} \mathfrak{L}w = \mathfrak{L}u & \text{in } \Omega_1, \\ \partial_n w + \mathrm{i}kw = 0 & \text{on } \partial\Omega_1, \end{cases}$$

where ∂_n denotes the normal derivative, whose solution satisfies the estimate [5, Corollary 1.10]

$$\|\nabla w\|_{L^2(\Omega_1)} + k\|w\|_{L^2(\Omega_1)} \leq Ck\|\mathfrak{L}u\|_{H^{-1}(\Omega_1)},$$

which gives

$$\|w\|_{H^1(\Omega_1)} \leq Ck\|\mathcal{L}u\|_{H^{-1}(\Omega)}.$$

For $v = u - w$ we have $\mathfrak{L}v = 0$ in Ω_1. The stability estimate in Corollary 2.13 used for ω_1, B, Ω_1 reads as

$$\|v\|_{H^1(B)} \leq C\|v\|^{\kappa}_{H^1(\omega_1)}\|v\|^{1-\kappa}_{H^1(\Omega_1)},$$

and the following estimates hold

$$\begin{aligned} \|u\|_{H^1(B)} &\leq \|v\|_{H^1(B)} + \|w\|_{H^1(B)} \\ &\leq C(\|u\|_{H^1(\omega_1)} + \|w\|_{H^1(\omega_1)})^{\kappa}(\|u\|_{H^1(\Omega_1)} + \|w\|_{H^1(\Omega_1)})^{1-\kappa} + Ck\|\mathfrak{L}u\|_{H^{-1}(\Omega)} \\ &\leq C(\|u\|_{H^1(\omega_1)} + k\|\mathfrak{L}u\|_{H^{-1}(\Omega)})^{\kappa}(\|u\|_{H^1(\Omega_1)} + k\|\mathfrak{L}u\|_{H^{-1}(\Omega)})^{1-\kappa}. \end{aligned}$$

Now we choose a cutoff function $\chi \in C_0^{\infty}(\omega)$ such that $\chi = 1$ in ω_1 and χu satisfies

$$\mathfrak{L}(\chi u) = \chi\mathfrak{L}u + [\mathfrak{L}, \chi]u, \quad \partial_n(\chi u) + ik(\chi u) = 0 \text{ on } \partial\omega.$$

Since the commutator $[\mathfrak{L}, \chi]$ is of first order, using again [5, Corollary 1.10] we obtain

$$\begin{aligned} \|u\|_{H^1(\omega_1)} \leq \|\chi u\|_{H^1(\omega)} &\leq Ck\left(\|[\mathfrak{L}, \chi]u\|_{H^{-1}(\omega)} + \|\chi\mathfrak{L}u\|_{H^{-1}(\omega)}\right) \\ &\leq Ck\left(\|u\|_{L^2(\omega)} + \|\mathfrak{L}u\|_{H^{-1}(\omega)}\right). \end{aligned}$$

The same argument for $\Omega_1 \subset \Omega$ gives

$$\|u\|_{H^1(\Omega_1)} \leq Ck(\|u\|_{L^2(\Omega)} + \|\mathfrak{L}u\|_{H^{-1}(\Omega)}),$$

thus leading to the conclusion. □

2.4.1 Shifting the norms

In this section we prove an estimate similar to the one in Corollary 2.13, but with the Sobolev indices shifted down one degree. Compared to Corollary 2.14, we can win a factor of k by using a semiclassical argument that is in the same vein as the one in Section 2.3.1. Our starting point for this is again a Carleman inequality. Appendix B contains the definitions and the main results in semiclassical analysis that we will use. The stability estimate with weaker norms will be applied to the approximation error of the finite element method devised in Chapter 4.

Lemma 2.15. *Let $\omega \subset B \subset \Omega$ be defined as in Corollary 2.13. Then there exist $C > 0$ and $\kappa \in (0,1)$ such that for all $u \in H^1(\Omega)$ and $k \geq 0$*

$$\|u\|_{L^2(B)} \leq C(\|u\|_{L^2(\omega)} + \|\Delta u + k^2 u\|_{H^{-1}(\Omega)})^\kappa \|u\|_{L^2(\Omega)}^{1-\kappa}.$$

Proof. We take the semiclassical parameter to be $\hbar = 1/\tau$. Let ϕ be as in Corollary 2.12 and set $\ell = \phi/\hbar$ and $\sigma = \Delta \ell$ in Lemma 2.11. Then

$$\begin{aligned} \tfrac{1}{2}\Big(e^{\phi/\hbar}\Delta e^{-\phi/\hbar} v + k^2 v\Big)^2 &\geq 2\hbar^{-1} D^2\phi(\nabla v, \nabla v) + 2\hbar^{-3} D^2\phi(\nabla\phi, \nabla\phi) v^2 \\ &\quad + \operatorname{div}(b\nabla v + B) + \hbar^{-1}(\nabla\Delta\phi, \nabla v)v \end{aligned}$$

Consider the conjugated operator $\mathcal{P} = e^{\phi/\hbar}\hbar^2\Delta e^{-\phi/\hbar}$ and let $v \in C_0^\infty(\Omega')$ where $\Omega' \subset \mathbb{R}^n$ is open and bounded, and $\overline{\Omega} \subset \Omega'$. Then, rescaling by $\hbar^4$,

$$C\|\mathcal{P}v + \hbar^2 k^2 v\|_{L^2(\mathbb{R}^n)}^2 \geq \hbar\|\hbar\nabla v\|_{L^2(\mathbb{R}^n)}^2 + \hbar\|v\|_{L^2(\mathbb{R}^n)}^2 - C\hbar^2\|v\|_{H^1_{\mathrm{scl}}(\mathbb{R}^n)}^2,$$

and for small enough $\hbar > 0$ we obtain

$$\sqrt{\hbar}\|v\|_{H^1_{\mathrm{scl}}(\mathbb{R}^n)} \leq C\|\mathcal{P}v + \hbar^2 k^2 v\|_{L^2(\mathbb{R}^n)}.$$

Note that the conjugated operator $\mathcal{P}$ is a semiclassical differential operator,

$$\mathcal{P}u = e^{\phi/\hbar}\hbar^2 \operatorname{div}\nabla(e^{-\phi/\hbar}u) = \hbar^2\Delta u - 2(\nabla\phi, \hbar\nabla u) - \hbar(\Delta\phi)u + |\nabla\phi|^2 u.$$

Let $\chi, \psi \in C_0^\infty(\Omega')$ and suppose that $\psi = 1$ near Ω and $\chi = 1$ near $\operatorname{supp}(\psi)$. Then for $v \in C_0^\infty(\Omega)$,

$$\|v\|_{H^{1+s}_{\mathrm{scl}}(\mathbb{R}^n)} \leq \|\chi J^s v\|_{H^1_{\mathrm{scl}}(\mathbb{R}^n)} + \|(1-\chi)J^s\psi v\|_{H^1_{\mathrm{scl}}(\mathbb{R}^n)} \leq C\|\chi J^s v\|_{H^1_{\mathrm{scl}}(\mathbb{R}^n)}$$

where we used the pseudolocality (2.13) to absorb the second term on the right-hand side by the left-hand side. We have

$$\sqrt{\hbar}\|v\|_{H^{1+s}_{\mathrm{scl}}(\mathbb{R}^n)} \leq C\sqrt{\hbar}\|\chi J^s v\|_{H^1_{\mathrm{scl}}(\mathbb{R}^n)} \leq C\|(\mathcal{P}+\hbar^2 k^2)\chi J^s v\|_{L^2(\mathbb{R}^n)}, \tag{2.30}$$

and using the commutator estimate (2.14), we obtain

$$\|[\mathcal{P},\chi J^s]v\|_{L^2(\mathbb{R}^n)} \leq C\hbar\|v\|_{H^{1+s}_{\mathrm{scl}}(\mathbb{R}^n)}.$$

This can be absorbed by the left-hand side of (2.30). Thus

$$\sqrt{\hbar}\|v\|_{H^{1+s}_{\mathrm{scl}}(\mathbb{R}^n)} \leq C\|\chi J^s(\mathcal{P}+\hbar^2 k^2)v\|_{L^2(\mathbb{R}^n)} \leq C\|(\mathcal{P}+\hbar^2 k^2)v\|_{H^s_{\mathrm{scl}}(\mathbb{R}^n)}.$$

Take now $s=-1$ and let the cutoff χ and the weight ϕ be as in the proof of Corollary 2.13, with the following additional condition on χ: there exists $\psi \in C_0^\infty(B(y,s)\cup\omega)$ satisfying $\psi = 1$ whenever $[\mathcal{P},\chi] \neq 0$.

Let $u \in C^\infty(\mathbb{R}^n)$ and set $w = e^{\phi/\hbar}u$. Then the previous estimate becomes

$$\sqrt{\hbar}\|\chi w\|_{L^2(\mathbb{R}^n)} \leq C\|(\mathcal{P}+\hbar^2 k^2)\chi w\|_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^n)}.$$

We have

$$\|[\mathcal{P},\chi]w\|_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^n)} = \|[\mathcal{P},\chi]\psi w\|_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^n)} \leq C\hbar\|\psi w\|_{L^2(\mathbb{R}^n)}.$$

Using the norm inequality $\|\cdot\|_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^n)} \leq C\hbar^{-2}\|\cdot\|_{H^{-1}(\mathbb{R}^n)}$, we thus obtain

$$\begin{aligned}\sqrt{\hbar}\|\chi e^{\phi/\hbar}u\|_{L^2(\mathbb{R}^n)} &\leq C\left\|\chi(e^{\phi/\hbar}\Delta e^{-\phi/\hbar}+k^2)w\right\|_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^n)} + C\hbar\|\psi w\|_{L^2(\mathbb{R}^n)}\\ &\leq C\hbar^{-2}\left\|\chi e^{\phi/\hbar}(\Delta u+k^2 u)\right\|_{H^{-1}(\mathbb{R}^n)} + C\hbar\left\|\psi e^{\phi/\hbar}u\right\|_{L^2(\mathbb{R}^n)}\end{aligned}$$

Using the same notation as in the proof of Corollary 2.13, due to the choice of ψ we get

$$e^{\rho^2/\hbar}\|u\|_{L^2(B)} \leq Ce^{(\beta+R)^2/\hbar}\left(\hbar^{-\frac{7}{2}}\|\Delta u+k^2 u\|_{H^{-1}(\Omega)} + \hbar^{\frac{1}{2}}\|u\|_{L^2(\omega)}\right) + Ce^{s^2/\hbar}\hbar^{\frac{1}{2}}\|u\|_{L^2(\Omega)},$$

for small enough $\hbar > 0$. Absorbing the negative power of $\hbar$ in the exponential, and using Lemma 2.4, we conclude the proof.

□

Chapter 3

Discrete regularisation using stabilised finite element methods

In this chapter we change focus to numerical analysis and review a computational framework that can be used to solve ill-posed problems with an emphasis on unique continuation. We first describe two different approaches to regularising such ill-posed problems: regularise-then-discretise and discretise-then-regularise. Then we present in more details the latter strategy based on stabilised finite element methods, which will be employed in the subsequent chapters.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $\mathfrak{L}$ be a second order elliptic operator, that will either be the Hemlholtz operator in Chapter 4 or the convection–diffusion operator in Chapters 5 and 6. We consider the ill-posed unique continuation problem discussed previously, which reads as follows: given an open subset $\omega \subset \Omega$ and the data $u_\omega \in L^2(\omega)$ and $f \in H^{-1}(\Omega)$, find $u \in H^1(\Omega)$ such that

$$\begin{cases} \mathfrak{L}u = f & \text{in } \Omega, \\ u = u_\omega & \text{in } \omega. \end{cases} \tag{3.1}$$

To begin with, we cast this as a PDE-constrained optimisation problem:

$$\min_{u \in H^1(\Omega)} \tfrac{1}{2}\|u - u_\omega\|^2_{L^2(\omega)} \quad \text{subject to} \quad \mathfrak{L}u = f \text{ in } \Omega. \tag{3.2}$$

For compatible data u_ω and f, these two problems are equivalent. Consider the weak formulation of $\mathfrak{L}u = f$ given through the bilinear form a by

$$a(u, z) = f(z), \text{ for any } z \in V,$$

for a test space $V \subset H^1(\Omega)$ that could encode some boundary conditions, for example $V = H^1_0(\Omega)$. We can then introduce the test function z as a Lagrange multiplier and consider the naive Lagrangian functional

$$\tilde{L}(u,z) := \underbrace{\tfrac{1}{2}\|u-u_\omega\|^2_{L^2(\omega)}}_{\text{data term}} + \underbrace{a(u,z)-f(z)}_{\text{PDE constraint}}.$$

Solving the minimisation problem (3.2) is equivalent to finding the saddle-point of $\tilde{L}$, which we will denote by $(\tilde{u},\tilde{z}) \in H^1(\Omega)\times V$. The saddle point satisfies the optimality conditions

$$\begin{cases} 0 = \partial_u \tilde{L} v = (\tilde{u}-u_\omega, v)_{L^2(\omega)} + a(v,\tilde{z}), \\ 0 = \partial_z \tilde{L} w = a(\tilde{u},w) - f(w), \end{cases} \quad \forall (v,w) \in H^1(\Omega)\times V,$$

which can be written as: find $(\tilde{u},\tilde{z}) \in H^1(\Omega)\times V$ such that

$$\begin{cases} (\tilde{u}, v)_{L^2(\omega)} + a(v,\tilde{z}) = (u_\omega, v)_{L^2(\omega)}, \\ a(\tilde{u},w) = f(w), \end{cases} \quad \forall (v,w) \in H^1(\Omega)\times V,$$

This problem is still ill-posed, of course, and its discretisation will lead to linear systems that might be singular or very ill-conditioned (thus with very unstable numerical solutions).

To overcome the ill-posedness one uses *regularisation*. This can be performed on the continuum level or on the discrete level. On the continuum level, regularisation is achieved by adding some penalty terms to the objective function of the minimisation problem (3.2) to render it well-posed. The optimality conditions then give a well-posed problem that can be discretised with standard methods and solved numerically. This is a *regularise-then-discretise* approach, and one classical example of this kind is *Tikhonov regularisation*. For a general introduction, see e.g. [31, Chapter 5]. Introducing a regularisation parameter $\alpha_T > 0$ and penalising the norm of the gradient, this reads as follows:

$$\min_{u\in H^1(\Omega)} \tfrac{1}{2}\|u-u_\omega\|^2_{L^2(\omega)} + \tfrac{\alpha_T}{2}\|\nabla u\|^2_{L^2(\Omega)} \quad \text{subject to} \quad \mathfrak{L}u = f \text{ in } \Omega.$$

Notice that increasing the regularisation parameter α_T enhances stability in the detriment of data fitting. Writing the PDE constraint in weak form again and introducing a Lagrange multiplier z, we have the regularised Lagrangian functional

$$L_T(u,z) := \underbrace{\tfrac{1}{2}\|u-u_\omega\|^2_{L^2(\omega)}}_{\text{data term}} + \underbrace{\tfrac{\alpha_T}{2}\|\nabla u\|^2_{L^2(\Omega)}}_{\text{regularisation}} + \underbrace{a(u,z)-f(z)}_{\text{PDE constraint}}.$$

The optimality conditions for the saddle point $(u_{\alpha_T}, z_{\alpha_T}) \in H^1(\Omega) \times V$ are given by

$$\begin{cases} (u_{\alpha_T}, v)_{L^2(\omega)} + \alpha_T(\nabla u_{\alpha_T}, \nabla v)_{L^2(\Omega)} + a(v, z_{\alpha_T}) = (u_\omega, v)_{L^2(\omega)}, \\ \qquad\qquad\qquad\qquad\qquad\qquad a(u_{\alpha_T}, w) = f(w), \end{cases} \quad \forall (v,w) \in H^1(\Omega) \times V.$$

For any $\alpha_T > 0$ this system is well-posed by Theorem 1.4. If the unique continuation problem (3.1) has a solution u, then this will be recovered as the limit of the regularised solutions $\lim_{\alpha_T \to 0} u_{\alpha_T} = u$.

Another version of Tikhonov regularisation for problem (3.1) is to consider the minimisation problem

$$\min_{w \in D_{u_\omega}(\mathfrak{L})} \tfrac{1}{2}\|\mathfrak{L}w - f\|^2_{L^2(\Omega)} + \tfrac{\alpha_{qr}}{2}\|w\|^2_{H^1(\Omega)},$$

where $\alpha_{qr} > 0$ is a regularisation parameter and $D_g(\mathfrak{L}) = \{u \in D(\mathfrak{L}) : u = g \text{ in } \omega\}$, with $D(\mathfrak{L})$ being the domain of the operator $\mathfrak{L}$. The optimality condition for the solution $u_{qr} \in D_{u_\omega}(\mathfrak{L})$ is given by

$$(\mathfrak{L}u_{qr}, \mathfrak{L}v)_{L^2(\Omega)} + \alpha_{qr}(u_{qr}, v)_{H^1(\Omega)} = (f, \mathfrak{L}v)_{L^2(\Omega)}, \quad \forall v \in D_0(\mathfrak{L}). \tag{3.3}$$

The fourth order regularised problem (3.3) is well-posed for any $\alpha_{qr} > 0$. This regularisation method is known as *quasi-reversibility*. It was introduced in [49] for the Cauchy problem, and revisited in [47] and [10]. Mixed formulation of this method have also been considered, most recently in [11].

However, when solving regularised continuum problems, a regularisation parameter – such as $\alpha_T > 0$ and $\alpha_{qr} > 0$ above – must be chosen and making an optimal choice is a delicate issue that depends on the stability of the problem and the size of the perturbations in data. Moreover, when the problem is discretised one also has to balance the regularisation parameter with the mesh size. In the following we will see that such issues can be avoided by taking a different methodological approach.

3.1 Discretise-then-regularise

We now present a *discretise-then-regularise* approach in which the optimisation problem with PDE constraints is *first* discretised using finite elements and *then* regularising terms are added on the discrete level. These regularising terms draw upon stabilisation techniques for numerically unstable well-posed problems that we briefly review in

Sections 3.2 and 3.3. This framework was introduced in [14] and [15] to solve non-coercive boundary value problems as well as ill-posed problems such as the Cauchy problem. We will use it in Chapter 4 for a Helmholtz problem and in Chapters 5 and 6 for convection–diffusion problems. The methodology has also been proven feasible for unique continuation subject to other PDEs such as the heat equation [24], the wave equation [16], the Stokes equation [19] or the linearised Navier-Stokes equation [9]. For a comparison with Tikhonov regularisation see [20].

Consider first a finite element solution space V_h in which a discrete solution $u_h \in V_h$ for problem (3.1) is sought. Let W_h be the discrete test space and let a_h be the discrete bilinear form giving the weak formulation of $\mathfrak{L}u = f$ as

$$a_h(u_h, z_h) = f(z_h), \text{ for any } z_h \in W_h,$$

Introducing the test function z_h as a Lagrange multiplier, a naive discrete Lagrangian functional would be

$$L_h^0(u_h, z_h) = \underbrace{\tfrac{1}{2}\|u_h - u_\omega\|^2_{L^2(\omega)}}_{\text{data term}} + \underbrace{a_h(u_h, z_h) - f(z_h)}_{\text{PDE constraint}}.$$

To enhance stability we add some discrete regularising terms through the abstract bilinear forms $s : V_h \times V_h \to \mathbb{R}$ and $s^* : W_h \times W_h \to \mathbb{R}$ and obtain the discrete Lagrangian functional

$$L_h(u_h, z_h) := \underbrace{\tfrac{1}{2}\|u_h - u_\omega\|^2_{L^2(\omega)}}_{\text{data term}} + \underbrace{\tfrac{1}{2}s(u_h, u_h) - \tfrac{1}{2}s^*(z_h, z_h)}_{\text{discrete regularisation}} + \underbrace{a_h(u_h, z_h) - f(z_h)}_{\text{PDE constraint}}. \tag{3.4}$$

Here the stabilising forms s and s^* can be scaled with the parameters $\gamma > 0$ and $\gamma^* > 0$, respectively. These can be chosen experimentally and their values can be fixed throughout all the computations (when refining the mesh). They do not depend on the noise level in the measurements nor on the mesh size, and they do not play a role in the convergence of the method.

Any saddle point $(u_h, z_h) \in V_h \times W_h$ of the Lagrangian (3.4) satisfies the optimality conditions

$$\begin{cases} a_h(u_h, w_h) - s^*(z_h, w_h) = f(w_h), \\ (u_h, v_h)_{L^2(\omega)} + a_h(v_h, z_h) + s(u_h, v_h) = (u_\omega, v_h)_{L^2(\omega)}, \end{cases} \quad \forall (v_h, w_h) \in V_h \times W_h. \tag{3.5}$$

Notice that we have written first the equation corresponding to $\partial_{z_h} L_h w_h = 0$. Assuming that an exact solution to problem (3.1) exists, we see from this equation that the dual variable that we are approximating by z_h is trivially $\tilde{z} = 0$ for which the bilinear form s^* should vanish as well. As general required properties, the stabilisers s and s^* must:

- enhance stability and make the discrete problem well-posed.
- be *weakly consistent*, i.e converge to zero at an optimal rate for smooth solutions as the approximation spaces get refined.
- control, together with the data term, the residual of the weak formulation which, combined with conditional stability estimates on the continuum level, then provides convergence rates for the error.

Collecting the left-hand sides of the optimality conditions (3.5) in the bilinear form A, with

$$A[(u_h, z_h), (v_h, w_h)] := a_h(u_h, w_h) - s^*(z_h, w_h) + (u_h, v_h)_{L^2(\omega)} + s(u_h, v_h) + a_h(v_h, z_h),$$

the discrete system (3.5) can be written as

$$A[(u_h, z_h), (v_h, w_h)] = (u_\omega, v_h)_{L^2(\omega)} + f(w_h), \quad \forall (v_h, w_h) \in V_h \times W_h. \tag{3.6}$$

Note that

$$A[(u_h, z_h), (u_h, -z_h)] = \|u_h\|^2_{L^2(\omega)} + s(u_h, u_h) + s^*(z_h, z_h),$$

and by designing the stabilisers s and s^* such that

$$\|(u_h, z_h)\|_s^2 := \|u_h\|^2_{L^2(\omega)} + s(u_h, u_h) + s^*(z_h, z_h) \tag{3.7}$$

is a norm on $V_h \times W_h$, we obtain by Theorem 1.4 that problem (3.6) is *well-posed* – and problem (3.5) as well – since it satisfies the inf-sup condition

$$\sup_{(v_h, w_h) \in V_h \times W_h} \frac{A[(u_h, z_h), (v_h, w_h)]}{\|(v_h, w_h)\|_s} \geq \|(u_h, z_h)\|_s.$$

The linear system corresponding to problem (3.5) thus has a unique solution and its matrix version is

$$\left[\begin{array}{c|c} A & -S^* \\ \hline M_\omega + S & A^T \end{array}\right] \left[\begin{array}{c} U_h \\ \hline Z_h \end{array}\right] = \left[\begin{array}{c} F \\ \hline U_\omega \end{array}\right],$$

where A is the matrix representation of a_h, S corresponds to s, S^* to s^*, and M_ω to the scalar product on ω. By the capitals U_h, Z_h, F and U_ω we denote the vectors corresponding to u_h, z_h, f and u_ω.

3.2 Stabilised finite element methods

The required features of the stabilising operators s and s^* can be formulated in an abstract way and now we discuss how these operators can be designed to have these features, that is to provide the needed stability, to be weakly consistent, and to contribute to the control of the residual. A wide range of techniques from stabilised finite element methods come to help and possible choices depend on the differential operator $\mathfrak{L}$ as well as the approximation order of the finite element spaces. In this section we briefly discuss an example for a well-posed convection-dominated problem that requires discrete stabilisation. Then in Section 3.3 we discuss in more details how continuous interior penalty can provide discrete regularisation for ill-posed problems.

Example 3.1. *For the diffusivity $\mu > 0$ and the convection vector field $\beta \in [W^{1,\infty}(\Omega)]^n$ with $\operatorname{ess\,sup}_\Omega \nabla \cdot \beta \leq 0$, consider the convection–diffusion problem*

$$\begin{cases} -\mu\Delta u + \beta \cdot \nabla u = f & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega, \end{cases}$$

with the source term $f \in L^2(\Omega)$.

The weak formulation in this example reads as follows: find $u \in H_0^1(\Omega)$ such that

$$\mu(\nabla u, \nabla v)_{L^2(\Omega)} + (\beta \cdot \nabla u, v)_{L^2(\Omega)} =: a(u, v) = (f, v)_{L^2(\Omega)}, \quad \forall v \in H_0^1(\Omega).$$

Denoting by $|\beta| := \|\beta\|_{[L^\infty(\Omega)]^n}$, the bilinear form a is continuous by the Cauchy-Schwarz inequality since

$$a(u, v) \leq C(\mu + |\beta|)\|u\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)}.$$

Integrating by parts and using the divergence theorem, we have that

$$a(u, u) = \mu\|\nabla u\|^2_{L^2(\Omega)} - ((\tfrac{1}{2}\nabla \cdot \beta)u, u)_{L^2(\Omega)}.$$

Assuming that $\operatorname{ess\,sup}_\Omega \nabla \cdot \beta \leq 0$, we obtain by the Poincaré inequality (1.4) that the bilinear form a is coercive since

$$a(u, u) \geq \alpha\|u\|^2_{H_0^1(\Omega)}, \text{ with } \alpha = \tfrac{\mu}{1+C_p^2} > 0,$$

The problem is then well-posed by the Lax-Milgram Lemma 1.2 and we have the stability estimate

$$\|u\|_{H_0^1(\Omega)} \leq \frac{1}{\alpha}\|f\|_{L^2(\Omega)}.$$

When convection dominates and $\mu \ll |\beta|$, we see that there is a loss of coercivity ($\alpha \ll 1$) and the stability deteriorates ($\frac{1}{\alpha} \gg 1$).

Note that if $\nabla \cdot \beta = 0$, then by multiplying the equation with a constant function 1 and using the divergence theorem we have that $-\mu \int_{\partial\Omega} \nabla u \cdot n \, \mathrm{d}s = \int_\Omega f \, \mathrm{d}x$, showing that the gradient could blow up at the boundary as $\mu \to 0$, resulting in sharp boundary layers for the solution. We also have that $a(u,u) = \mu\|\nabla u\|^2_{L^2(\Omega)} = (f,u)_{L^2(\Omega)}$.

Even though on the continuum level the problem is well-posed in the convection-dominated regime, when considering numerical solutions the degenerate stability constant becomes an issue and instability arises. Spurious oscillations can appear as there is no robust control of the error. For example, considering a naive discretisation of the weak formulation using finite elements, the discrete solution u_h satisfies the stability estimate

$$\mu\|\nabla u_h\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)},$$

and $\mu\|\nabla u_h\|^2_{L^2(\Omega)} = (f,u_h)_{L^2(\Omega)}$ when $\nabla \cdot \beta = 0$. Since in general u_h does not satisfy the maximum principle, the term $(f,u_h)_{L^2(\Omega)}$ could be moderate and $\|\nabla u_h\|_{L^2(\Omega)}$ could be large due to spurious oscillations for small μ. One remedy comes from *stabilised methods*. Considering a solution space V_h with $u_h^s \in V_h$, a test space W_h and a discrete version a_h of the bilinear form a, the underlying idea of such methods is to introduce the stabilising forms a_h^s and l_h^s and consider the modified problem

$$a_h(u_h^s, v_h) + a_h^s(u_h^s, v_h) = l_h^s(v_h) + (f, v_h)_{L^2(\Omega)}, \quad \forall v_h \in W_h.$$

This should enhance the stability and provide estimates with an improved dependence on the physical parameters. Examples of such methods for singularly perturbed problems are reviewed in [55] and we mention the streamline diffusion method [46], also known as streamline upwind Petrov-Galerkin [12], Galerkin least squares, discontinuous Galerkin methods [28], local projection stabilisation, and *interior penalty*. In the following we will focus on the latter which was introduced in [30] and revisited in [18], where the following stability estimate for Example 3.1 was proven with $l_h^s = 0$ and $a_h^s = h\mathcal{J}_h$ as in (3.8) below,

$$\mu^{\frac{1}{2}}\|\nabla u_h^s\|_{L^2(\Omega)} + \|u_h^s\|_{L^2(\Omega)} + \|h^{\frac{1}{2}}\beta \cdot \nabla u_h^s\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)},$$

This improves the control of the gradient. It is also robust in the L^2-norm and the streamline derivative for fixed h, limiting the rate of blow-up of $\|\beta \cdot \nabla u_h^s\|_{L^2(\Omega)}$ when $h \to 0$.

3.3 Continuous interior penalty

We now present the discrete regularisation strategy based on *interior penalty*, which will be the cornerstone of the numerical methods discussed in Chapters 4 to 6.

To prepare the setting, let $\mathcal{T}_h = \{K\}$ be a triangulation of the polygonal domain Ω with elements K having maximal diameter h. Let $\mathbb{P}_1$ be the set of piecewise affine functions and consider the conforming finite element space

$$V_h := \Big\{ u \in C(\bar{\Omega}) : \, u|_K \in \mathbb{P}_1(K), K \in \mathcal{T}_h \Big\}.$$

We denote the set of interior element faces by $\mathcal{F}_i$. The jump $[\![\nabla u_h \cdot n]\!]_F$ of the normal derivative across an interior face $F \in \mathcal{F}_i$ is given by

$$[\![\nabla u_h \cdot n]\!]_F := \nabla u_h \cdot n_1|_{K_1} + \nabla u_h \cdot n_2|_{K_2},$$

with $K_1, K_2 \in \mathcal{T}_h$ being two elements such that $K_1 \cap K_2 = F$, and n_j the outward normal of K_j, $j = 1, 2$, see Figure 3.1 for a sketch.

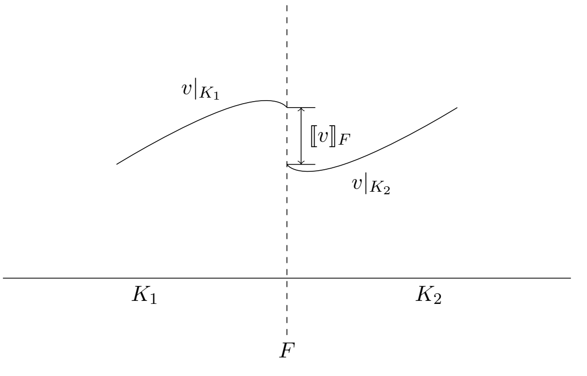


Figure 3.1: Sketch for the jump of a function across an interior face.

The key component in the discrete stabilisation that we will use is the *interior penalty* operator

$$\mathcal{J}_h(u_h, v_h) := \sum_{F \in \mathcal{F}_i} \int_F h [\![\nabla u_h \cdot n]\!]_F [\![\nabla v_h \cdot n]\!]_F \, \mathrm{d}s, \tag{3.8}$$

which acts on functions in the finite element space V_h and penalises the jumps of the normal gradient across interior faces.

The following Poincaré-type inequality says that the interior penalty $\mathcal{J}_h$ can be chosen as the primal stabiliser s since together with the L^2-seminorm on ω it provides a full norm on V_h. Taking the dual stabiliser s^* such that it gives a norm on the test space W_h, we obtain that the stabilising norm (3.7) defines indeed a norm on $V_h \times W_h$ and hence problem (3.5) is well-posed. To control the residual of the PDE, we will also need the norm given by s^* to be equivalent to the H^1-norm, preferably with constants independent of the mesh size h.

Lemma 3.2 (Poincaré-type inequality)**.** *There exists a constant $C > 0$ such that for all $h > 0$ and $v_h \in V_h$ there holds*

$$h\|v_h\|_{H^1(\Omega)} \leq C\Big(\|v_h\|_{L^2(\omega)} + \mathcal{J}_h(v_h, v_h)^{\frac{1}{2}}\Big).$$

Proof. See [20, Lemma 2]. □

The interior penalty $\mathcal{J}_h$ is also weakly consistent, as the following estimate shows.

Lemma 3.3 (Weak consistency)**.** *Let $i_h : H^1(\Omega) \to V_h$ be an interpolant that satisfies the standard approximation inequality*

$$\|\nabla(w - i_h w)\|_{L^2(\Omega)} \leq Ch|w|_{H^2(\Omega)}, \quad \forall w \in H^2(\Omega).$$

Then there exists a constant $C > 0$ such that for all $h > 0$ and $w \in H^2(\Omega)$ there holds

$$\mathcal{J}_h(i_h w, i_h w) \leq Ch^2|w|^2_{H^2(\Omega)}.$$

Proof. Notice first that since $w \in H^2(\Omega)$ its gradient has no jumps across faces $F \in \mathcal{F}_i$. Inserting w and using the trace inequality (A.2) and approximation, we have that

$$\begin{aligned}\sum_{F\in\mathcal{F}_i}\int_F h[\![\nabla i_h w\cdot n]\!]_F^2 \,\mathrm{d}s &= \sum_{F\in\mathcal{F}_i}\int_F h[\![\nabla (i_h w - w)\cdot n]\!]_F^2 \,\mathrm{d}s \\ &\leq C\Big(\|\nabla(i_h w - w)\|^2_{L^2(\Omega)} + h^2\|\nabla(i_h w - w)\|^2_{H^1(\Omega)}\Big) \\ &\leq Ch^2\|w\|^2_{H^2(\Omega)}.\end{aligned}$$

□

Chapter 4

Helmholtz equation

In this chapter we consider the unique continuation problem for the Helmholtz equation

$$\mathfrak{L}u := \Delta u + k^2 u = -f,$$

in an open, bounded and connected set $\Omega \subset \mathbb{R}^n$, and introduce a stabilised finite element method to solve the problem computationally. The method is explicit with respect to the wave number k and we prove convergence estimates with explicit dependence on k. This chapter is based on [21].

We recall first the unique continuation problem that we aim to approximate

$$\begin{cases} \Delta u + k^2 u = -f & \text{in } \Omega, \\ u = u_\omega & \text{in } \omega, \end{cases} \tag{4.1}$$

where $\omega \subset \Omega$ is open, and $f \in L^2(\Omega)$ and $u_\omega \in L^2(\omega)$ are given. For a solution to exist, a data compatibility condition must hold.

Following the discretise-then-regularise approach in Chapter 3, the crux of the computational method is to discretise this ill-posed problem as a PDE-constrained minimisation with additional discrete regularising terms. Continuous interior estimates with Hölder stability were proven in Section 2.4 using Carleman estimates in a geometric setting typical for continuation inside the convex hull of ω. Combined with the discrete properties of the stabilisers, these estimates lead, when applied to the approximation error, to error bounds with a (sub-linear) convergence order given by the Hölder exponent and which are explicit in the wave number. The chapter ends with some numerical examples illustrating the theoretical results and the importance of the geometric setting, i.e. continuing the solution inside vs outside the convex hull of the data set.

For the well-posed problem of the Helmholtz equation with Robin boundary conditions

$$\Delta u + k^2 u = -f \quad \text{in } \Omega \qquad \text{and} \qquad \partial_n u + \mathrm{i}ku = 0 \quad \text{on } \partial\Omega, \tag{4.2}$$

the following sharp bounds

$$\|\nabla u\|_{L^2(\Omega)} + k\|u\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)} \tag{4.3}$$

and

$$\|u\|_{H^2(\Omega)} \leq Ck\|f\|_{L^2(\Omega)} \tag{4.4}$$

hold for a star-shaped Lipschitz domain Ω and any wave number k bounded away from zero [5]. The error estimates that we derive in Section 4.2, e.g. $\|u-u_h\|_{H^1(B)} \leq C(hk)^\kappa \|u\|_*$ in Theorem 4.4, contain the term

$$\|u\|_* = \|u\|_{H^2(\Omega)} + k^2\|u\|_{L^2(\Omega)},$$

which corresponds to the term $k\|f\|_{L^2(\Omega)}$ in the well-posed case.

It is well known from the seminal works [4, 41, 42] that the finite element approximation of the Helmholtz problem is challenging also in the well-posed case due to the so-called pollution error. Indeed, to observe optimal convergence orders of H^1- and L^2-errors the mesh size h must satisfy a smallness condition related to the wave number k, typically for piecewise affine elements, the condition $k^2h \lesssim 1$. This is due to the dispersion error that is most important for low order approximation spaces. The situation improves if higher order polynomial approximation is used. Recently, the precise conditions for optimal convergence when using hp-refinement (p denotes the polynomial order of the approximation space) were shown in [53]. Under the assumption that the solution operator for Helmholtz problems is polynomially bounded in k, it is shown that quasi-optimality is obtained under the conditions that kh/p is sufficiently small and the polynomial degree p is at least $\mathcal{O}(\log k)$.

Another way to obtain stability without, or under mild, conditions on the mesh size of the approximate scheme is to use stabilisation. The continuous interior penalty stabilisation (CIP) was introduced for the Helmholtz problem in [60], where stability was shown in the $kh \lesssim 1$ regime, and was subsequently used to obtain error bounds for standard piecewise affine elements when $k^3h^2 \lesssim 1$. It was then shown in [25] that, in the one dimensional case, the CIP stabilisation can also be used to eliminate the pollution error, provided the penalty parameter is appropriately chosen. When deriving error estimates for the stabilised FEM that we herein introduce, we will use the mild condition $kh \lesssim 1$.

4.1 Discretisation

Consider a quasi-uniform family $\{\mathcal{T}_h\}$ of geometrically conformal triangulations of Ω, see e.g. [32, Definition 1.140]. Let

$$V_h := \left\{ u \in C(\bar{\Omega}) : u|_K \in \mathbb{P}_1(K), K \in \mathcal{T}_h \right\}$$

be the H^1-conformal approximation space based on the $\mathbb{P}_1$ finite element and let

$$W_h := V_h \cap H_0^1(\Omega).$$

Consider the orthogonal L^2-projection $\pi_h : L^2(\Omega) \to V_h$, which satisfies

$$(u - \pi_h u, v)_{L^2(\Omega)} = 0, \quad \forall u \in L^2(\Omega), \forall v \in V_h,$$
$$\|\pi_h u\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)}, \quad \forall u \in L^2(\Omega),$$

and the Scott-Zhang interpolant $\pi_{sz} : H^1(\Omega) \to V_h$, that preserves vanishing Dirichlet boundary conditions. Both operators have the following stability and approximation properties, see Appendix A,

$$\|p_h u\|_{H^1(\Omega)} \leq C\|u\|_{H^1(\Omega)}, \quad \forall u \in H^1(\Omega), \tag{4.5}$$
$$\|u - p_h u\|_{H^m(\Omega)} \leq Ch^{k-m}\|u\|_{H^k(\Omega)}, \quad \forall u \in H^k(\Omega), \tag{4.6}$$

where $p_h \in \{\pi_h, \pi_{sz}\}$, $k = 1, 2$ and $m = 0, k - 1$.

Following the approach presented in Chapter 3, the regularisation on the discrete level will be based on the L^2-control of the gradient jumps over element faces using the continuous interior penalty bilinear form on V_h

$$\mathcal{J}_h(u_h, v_h) := \sum_{F \in \mathcal{F}_i} \int_F h [\![\nabla u_h \cdot n]\!]_F [\![\nabla v_h \cdot n]\!]_F \, ds,$$

where $\mathcal{F}_i$ is the set of all interior faces. We recall the following trace inequality, see (A.2) in Appendix A,

$$\|v\|_{L^2(\partial K)} \leq C(h^{-\frac{1}{2}}\|v\|_{L^2(K)} + h^{\frac{1}{2}}\|\nabla v\|_{L^2(K)}), \quad \forall v \in H^1(K), \tag{4.7}$$

Lemma 4.1. *There exists $C > 0$ such that all $u_h \in V_h$, $v \in H_0^1(\Omega)$, $w \in H^2(\Omega)$ and $h > 0$ satisfy*

$$(\nabla u_h, \nabla v)_{L^2(\Omega)} \leq C\mathcal{J}_h(u_h, u_h)^{1/2}(h^{-1}\|v\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega)}). \tag{4.8}$$

Proof. This result was proven in [24, Lemma 2] and we include the proof for completeness. For the first inequality, let us begin by integrating by parts and using that $\Delta u_h = 0$ on each element K to write

$$(\nabla u_h, \nabla v)_{L^2(\Omega)} = \sum_{K \in \mathcal{T}_h} \int_K \nabla u_h \cdot \nabla v \, \mathrm{d}x = h^{-\frac{1}{2}} \sum_{F \in \mathcal{F}_i} \int_F h^{\frac{1}{2}} [\![\nabla u_h \cdot n]\!]_F \, v \, \mathrm{d}s.$$

By the Cauchy-Schwarz inequality and the trace inequality (4.7) we obtain that

$$(\nabla u_h, \nabla v)_{L^2(\Omega)} \leq C\mathcal{J}_h(u_h, v_h)^{\frac{1}{2}}(h^{-1}\|v\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega)}).$$

□

We denote, for brevity, the standard inner product $(\cdot,\cdot)_{L^2(\Xi)}$ by $(\cdot,\cdot)_\Xi$, and we introduce the bilinear form

$$a_h(u, z) := (\nabla u, \nabla z)_\Omega - k^2(u, z)_\Omega.$$

The weak formulation of the underlying Helmholtz equation $\Delta u + k^2 u = -f$ reads as follows: find $u \in H^1(\Omega)$ such that

$$a_h(u, z) = (f, z)_\Omega, \text{ for any } z \in H_0^1(\Omega).$$

As discussed in Chapter 3, our approach is to rewrite the unique continuation problem as a PDE-constrained minimisation with weakly consistent regularisation leading to a well-posed discrete system. To be more precise, we aim to find the saddle points of the Lagrangian functional

$$L(u_h, z_h) := \tfrac{1}{2}\|u_h - u_\omega\|_\omega^2 + \tfrac{1}{2}s(u_h, u_h) - \tfrac{1}{2}s^*(z_h, z_h) + a_h(u_h, z_h) - (f, z_h)_\Omega,$$

where s and s^* are discrete regularising terms for the primal and dual variables that should also be consistent and vanish at optimal rates. The optimality conditions for the saddle points $(u_h, z_h) \in V_h \times W_h$ of the Lagrangian L give rise to the following system

$$\left\{\begin{array}{rcl} a_h(u_h, w_h) - s^*(z_h, w_h) &=& (f, w_h)_\Omega, \\ a_h(v_h, z_h) + s(u_h, v_h) + (u_h, v_h)_\omega &=& (u_\omega, v_h)_\omega, \end{array}\right. \quad \forall (v_h, w_h) \in V_h \times W_h.$$

This discrete system can be written as

$$A[(u_h, z_h), (v_h, w_h)] = (u_\omega, v_h)_\omega + (f, w_h)_\Omega, \quad \forall (v_h, w_h) \in V_h \times W_h, \tag{4.9}$$

where A is the symmetric but indefinite bilinear form

$$A[(u_h, z_h), (v_h, w_h)] := (u_h, v_h)_\omega + s(u_h, v_h) + a_h(v_h, z_h) - s^*(z_h, w_h) + a_h(u_h, w_h).$$

The stabilisation must control certain residual quantities representing the data of the error equation. The primal stabiliser s will be based on the continuous interior penalty given by $\mathcal{J}_h$. It must also take into account the low order term of the Helmholtz operator. Notice that when the PDE-constraint is satisfied, it follows trivially that the stabiliser s^* is consistent since $z_h = 0$ is the solution for the dual variable of the saddle point. We make the following choice

$$s(u_h, v_h) := \gamma \mathcal{J}_h(u_h, v_h) + \gamma h^2 k^4 (u_h, v_h)_\Omega,$$

where the low order term is the only remaining part for piecewise affine functions of a Galerkin least squares stabilisation $\gamma h^2 (\mathfrak{L} u_h, \mathfrak{L} v_h)_\Omega$, and

$$s^*(z_h, w_h) := (\nabla z_h, \nabla w_h)_\Omega.$$

The parameter $\gamma > 0$ does not depend on the wave number and can be set empirically. It does not influence the convergence order of the method and will be absorbed in the generic constants. We define on V_h and W_h, respectively, the norms

$$\|u_h\|_{V_h} := s(u_h, u_h)^{1/2}, \quad u_h \in V_h, \quad \|z\|_{W_h} := s^*(z_h, z_h)^{1/2}, \quad z_h \in W_h,$$

together with the norm on $V_h \times W_h$ defined by

$$\|(u_h, z_h)\|_s^2 := \|u_h\|_{V_h}^2 + \|u_h\|_\omega^2 + \|z_h\|_{W_h}^2.$$

Since $A[(u_h, z_h), (u_h, -z_h)] = \|u_h\|_\omega^2 + \|u_h\|_{V_h}^2 + \|z_h\|_{W_h}^2$ we have the following inf-sup condition

$$\sup_{(v_h,w_h)\in V_h\times W_h} \frac{A[(u_h,z_h),(v_h,w_h)]}{\|(v_h,w_h)\|_s} \geq C\|(u_h,z_h)\|_s \tag{4.10}$$

that by Theorem 1.4 guarantees a unique solution in $V_h \times W_h$ for the discrete problem (4.9).

4.2 Error estimates

The strategy is to first prove an optimal bound for the stabilising quantity

$$\|(u_h - \pi_h u, z_h)\|_s$$

and then to obtain a bound for the residual $a_h(u-u_h,\cdot)$. Combining this with the conditional stability in Section 2.4 we thus obtain error estimates in the L^2- and H^1-norms.

We start by deriving some lower and upper bounds for the norm $\|\cdot\|_{V_h}$. For $u_h \in V_h$ and $z \in H_0^1(\Omega)$, we use (4.8) to bound

$$\begin{aligned} a(u_h,z) &= (\nabla u_h,\nabla z)_{L^2(\Omega)} - k^2(u_h,z)_{L^2(\Omega)} \\ &\leq C\mathcal{J}_h(u_h,u_h)^{1/2}(h^{-1}\|z\|_{L^2(\Omega)} + \|z\|_{H^1(\Omega)}) + k^2\|u_h\|_{L^2(\Omega)}\|z\|_{L^2(\Omega)}, \end{aligned}$$

and hence

$$a(u_h,z) \leq C\|u_h\|_{V_h}\Big(h^{-1}\|z\|_{L^2(\Omega)} + \|z\|_{H^1(\Omega)}\Big). \tag{4.11}$$

For $u \in H^2(\Omega)$, from (4.6) and the weak consistency Lemma 3.3 combined with the stability of the L^2-projection we have that

$$\|\pi_h u\|_{V_h}^2 = \mathcal{J}_h(\pi_h u,\pi_h u) + \|hk^2\pi_h u\|_{L^2(\Omega)}^2 \leq C\Big(h^2\|u\|_{H^2(\Omega)}^2 + \|hk^2 u\|_{L^2(\Omega)}^2\Big),$$

obtaining that

$$\|\pi_h u\|_{V_h} \leq Ch(\|u\|_{H^2(\Omega)} + k^2\|u\|_{L^2(\Omega)}) = Ch\|u\|_*, \tag{4.12}$$

where the star norm is defined by

$$\|u\|_* := \|u\|_{H^2(\Omega)} + k^2\|u\|_{L^2(\Omega)}.$$

Lemma 4.2. *Assume that $u \in H^2(\Omega)$ is a solution to (4.1) and let $(u_h,z_h) \in V_h \times W_h$ be the solution to (4.9). Then there exists $C > 0$ such that for all $h \in (0,1)$*

$$\|(u_h - \pi_h u, z_h)\|_s \leq Ch\|u\|_*.$$

Proof. Due to the inf-sup condition (4.10) it is enough to prove that for $(v_h, w_h) \in V_h \times W_h$,

$$A[(u_h - \pi_h u, z_h), (v_h, w_h)] \leq Ch\|u\|_* \|(v_h, w_h)\|_s.$$

The discrete weak form (4.9) gives that

$$A[(u_h - \pi_h u, z_h), (v_h, w_h)] = (u - \pi_h u, v_h)_\omega + a_h(u - \pi_h u, w_h) - s(\pi_h u, v_h).$$

Using (4.6) we bound the first term to get

$$(u - \pi_h u, v_h)_\omega \leq Ch^2 \|u\|_{H^2(\Omega)} \|v_h\|_\omega.$$

For the second term we use the L^2-orthogonality property of π_h, and (4.6) to obtain

$$a_h(u - \pi_h u, w_h) = (\nabla(u - \pi_h u), \nabla w_h)_{L^2(\Omega)} \leq Ch\|w_h\|_{W_h} \|u\|_{H^2(\Omega)},$$

while for the last term we employ (4.12) to estimate

$$s(\pi_h u, v_h) \leq \|\pi_h u\|_{V_h} \|v_h\|_{V_h} \leq Ch\|u\|_* \|v_h\|_{V_h}.$$

□

Theorem 4.3 (L^2-error estimate)**.** *Let $\omega \subset B \subset \Omega$ be defined as in Corollary 2.13. Assume that $u \in H^2(\Omega)$ is a solution to (4.1) and let $(u_h, z_h) \in V_h \times W_h$ be the solution to (4.9). Then there exist $C > 0$ and $\kappa \in (0, 1)$ such that for all k, $h > 0$ with $kh \lesssim 1$*

$$\|u - u_h\|_{L^2(B)} \leq C(hk)^\kappa k^{\kappa - 2} \|u\|_*.$$

Proof. Consider the residual $\langle r, w\rangle = a_h(u_h - u, w) = a_h(u_h, w) - (f, w)_\Omega$, for $w \in H^1_0(\Omega)$. Taking $v_h = 0$ in (4.9) we get $a_h(u_h, w_h) = (f, w_h)_\Omega + s^*(z_h, w_h)$, $w_h \in W_h$ which implies that

$$\begin{aligned}\langle r, w\rangle &= a_h(u_h, w) - (f, w)_\Omega - a_h(u_h, \pi_{sz} w) + a_h(u_h, \pi_{sz} w)\\ &= a_h(u_h, w - \pi_{sz} w) - (f, w - \pi_{sz} w)_\Omega + s^*(z_h, \pi_{sz} w), \quad w \in H^1_0(\Omega).\end{aligned}$$

Using (4.11) and (4.6) we estimate the first term

$$\begin{aligned} a_h(u_h, w - \pi_{sz} w) &\leq C\|u_h\|_{V_h}(h^{-1}\|w - \pi_{sz} w\|_{L^2(\Omega)} + \|w - \pi_{sz} w\|_{H^1(\Omega)}) \\ &\leq C\|u_h\|_{V_h}\|w\|_{H^1(\Omega)} \leq Ch\|u\|_*\|w\|_{H^1(\Omega)}, \end{aligned}$$

since, due to Lemma 4.2 and (4.12)

$$\|u_h\|_{V_h} \leq \|u_h - \pi_h u\|_{V_h} + \|\pi_h u\|_{V_h} \leq Ch\|u\|_*.$$

The second term is bounded by using (4.6)

$$(f, w - \pi_{sz} w)_\Omega \leq \|f\|_{L^2(\Omega)}\|w - \pi_{sz} w\|_{L^2(\Omega)} \leq Ch\|f\|_{L^2(\Omega)}\|w\|_{H^1(\Omega)}$$

and the last term by using Lemma 4.2, the Poincaré inequality and the H^1-stability (4.5)

$$s^*(z_h, \pi_{sz} w) \leq \|z_h\|_{W_h}\|\pi_{sz} w\|_{W_h} \leq Ch\|u\|_*\|w\|_{H^1(\Omega)}.$$

Hence the following residual norm estimate holds

$$\|r\|_{H^{-1}(\Omega)} \leq Ch(\|u\|_* + \|f\|_{L^2(\Omega)}) \leq Ch\|u\|_*.$$

Using the continuum estimate in Lemma 2.15 for $u - u_h$ we obtain the following error estimate

$$\|u - u_h\|_{L^2(B)} \leq C(\|u - u_h\|_{L^2(\omega)} + \|r\|_{H^{-1}(\Omega)})^\kappa \|u - u_h\|_{L^2(\Omega)}^{1-\kappa}.$$

By (4.6) and Lemma 4.2 we have the bounds

$$\begin{aligned} \|u - u_h\|_{L^2(\omega)} &\leq \|u - \pi_h u\|_{L^2(\omega)} + \|u_h - \pi_h u\|_{L^2(\omega)} \\ &\leq Ch\|u\|_{H^1(\Omega)} + Ch\|u\|_* \\ &\leq Ch\|u\|_* \end{aligned}$$

and

$$\begin{aligned}\|u-u_h\|_{L^2(\Omega)} &\leq \|u-\pi_h u\|_{L^2(\Omega)} + \|u_h-\pi_h u\|_{L^2(\Omega)} \\ &\leq Ch^2\|u\|_{H^2(\Omega)} + Ch^{-1}k^{-2}\|u_h-\pi_h u\|_{V_h} \\ &\leq C\Big((h^2+k^{-2})\|u\|_{H^2(\Omega)} + \|u\|_{L^2(\Omega)}\Big) \\ &\leq Ck^{-2}\|u\|_*\end{aligned}$$

thus leading to the conclusion. □

Theorem 4.4 (H^1-error estimate)**.** *Let $\omega \subset B \subset \Omega$ be defined as in Corollary 2.13. Assume that $u \in H^2(\Omega)$ is a solution to (4.1) and let $(u_h, z_h) \in V_h \times W_h$ be the solution to (4.9). Then there exist $C > 0$ and $\kappa \in (0,1)$ such that for all k, $h > 0$ with $kh \lesssim 1$*

$$\|u-u_h\|_{H^1(B)} \leq C(hk)^\kappa \|u\|_*.$$

Proof. We employ a similar argument as in the proof of Theorem 4.3 with the same estimates for the residual norm and the L^2-errors in ω and Ω, only now using the continuum estimate in Corollary 2.14 to obtain

$$\begin{aligned}\|u-u_h\|_{H^1(B)} &\leq Ck(\|u-u_h\|_{L^2(\omega)} + \|r\|_{H^{-1}(\Omega)})^\kappa (\|u-u_h\|_{L^2(\Omega)} + \|r\|_{H^{-1}(\Omega)})^{1-\kappa} \\ &\leq Ckh^\kappa (k^{-2}+h)^{1-\kappa}\|u\|_*,\end{aligned}$$

which ends the proof. □

Let us remark that under the assumption $k^2h \lesssim 1$, the estimate in Theorem 4.4 becomes

$$\|u-u_h\|_{H^1(B)} \leq C(hk^2)^\kappa k^{-1}\|u\|_*,$$

and combining Theorems 4.3 and 4.4 we obtain the following result.

Corollary 4.5. *Let $\omega \subset B \subset \Omega$ be defined as in Corollary 2.13. Assume that $u \in H^2(\Omega)$ is a solution to (4.1) and let $(u_h, z_h) \in V_h \times W_h$ be the solution to (4.9). Then there exist $C > 0$ and $\kappa \in (0,1)$ such that for all k, $h > 0$ with $k^2h \lesssim 1$*

$$k\|u-u_h\|_{L^2(B)} + \|u-u_h\|_{H^1(B)} \leq C(hk^2)^\kappa k^{-1}\|u\|_*.$$

Comparing this with the well-posed boundary value problem (4.2) and the sharp bounds (4.3) and (4.4), we note that the $k^{-1}\|u\|_*$ term in the above estimate is analogous to the well-posed case term $\|f\|_{L^2(\Omega)}$.

4.2.1 Data perturbations

The analysis above can also handle the perturbed data

$$\tilde{u}_\omega := u_\omega + \delta u, \quad \tilde{f} := f + \delta f,$$

with the unperturbed data u_ω, f in (4.1), and perturbations $\delta u \in L^2(\omega)$, $\delta f \in H^{-1}(\Omega)$ measured by

$$\delta(\tilde{u}_\omega, \tilde{f}) := \|\delta u\|_{L^2(\omega)} + \|\delta f\|_{H^{-1}(\Omega)}.$$

The saddle points $(u_h, z_h) \in V_h \times W_h$ of the correspondingly perturbed Lagrangian satisfy

$$A[(u_h, z_h), (v_h, w_h)] = (\tilde{u}_\omega, v_h)_\omega + \langle \tilde{f}, w_h \rangle_{H^{-1}, H_0^1}, \quad \forall (v_h, w_h) \in V_h \times W_h. \tag{4.13}$$

Lemma 4.6. *Assume that $u \in H^2(\Omega)$ is a solution to the unperturbed problem (4.1) and let $(u_h, z_h) \in V_h \times W_h$ be the solution to the perturbed problem (4.13). Then there exists $C > 0$ such that for all $h \in (0, 1)$*

$$\|(u_h - \pi_h u, z_h)\|_s \leq C(h\|u\|_* + \delta(\tilde{u}_\omega, \tilde{f})).$$

Proof. Proceeding as in the proof of Lemma 4.2, the weak form gives

$$\begin{aligned} A[(u_h - \pi_h u, z_h), (v_h, w_h)] &= (u - \pi_h u, v_h)_\omega + a_h(u - \pi_h u, w_h) - s(\pi_h u, v_h) \\ &\quad + (\delta u, v_h)_\omega + \langle \delta f, w_h \rangle. \end{aligned}$$

We bound the perturbation terms by

$$\begin{aligned} (\delta u, v_h)_\omega + \langle \delta f, w_h \rangle &\leq \|\delta u\|_\omega \|v_h\|_\omega + C\|\delta f\|_{H^{-1}(\Omega)} \|w_h\|_{W_h} \\ &\leq C\delta(\tilde{u}_\omega, \tilde{f}) \|(v_h, w_h)\|_s \end{aligned}$$

and we conclude by using the previously derived bounds for the other terms. □

Theorem 4.7 (L^2-error estimate)**.** *Let $\omega \subset B \subset \Omega$ be defined as in Corollary 2.13. Assume that $u \in H^2(\Omega)$ is a solution to the unperturbed problem (4.1) and let $(u_h, z_h) \in V_h \times W_h$ be the solution to the perturbed problem (4.13). Then there exist $C > 0$ and $\kappa \in (0, 1)$ such* *that for all* k, $h > 0$ *with* $kh \lesssim 1$

$$\|u - u_h\|_{L^2(B)} \leq C(hk)^\kappa k^{\kappa-2}(\|u\|_* + h^{-1}\delta(\tilde{u}_\omega, \tilde{f})).$$

Proof. Following the proof of Theorem 4.3, the residual satisfies

$$\langle r, w\rangle = a_h(u_h, w - \pi_{sz}w) - (f, w - \pi_{sz}w)_\Omega + s^*(z_h, \pi_{sz}w) + \langle \delta f, \pi_{sz}w\rangle, \quad w \in H_0^1(\Omega)$$

and

$$\|r\|_{H^{-1}(\Omega)} \leq C(\|u_h\|_{V_h} + h\|f\|_{L^2(\Omega)} + \|z_h\|_{W_h} + \|\delta f\|_{H^{-1}(\Omega)}).$$

Bounding the first term in the right-hand side by Lemma 4.6 and (4.12)

$$\|u_h\|_{V_h} \leq \|u_h - \pi_h u\|_{V_h} + \|\pi_h u\|_{V_h} \leq C(h\|u\|_* + \delta(\tilde{u}_\omega, \tilde{f}))$$

and the third one by Lemma 4.6 again, we obtain

$$\|r\|_{H^{-1}(\Omega)} \leq Ch(\|u\|_* + \|f\|_{L^2(\Omega)}) + C\delta(\tilde{u}_\omega, \tilde{f}) \leq C(h\|u\|_* + \delta(\tilde{u}_\omega, \tilde{f})).$$

The continuum estimate in Lemma 2.15 applied to $u - u_h$ gives

$$\|u - u_h\|_{L^2(B)} \leq C\left(h\|u\|_* + \delta(\tilde{u}_\omega, \tilde{f})\right)^\kappa \|u - u_h\|_{L^2(\Omega)}^{1-\kappa},$$

where $\|u - u_h\|_{L^2(\omega)}$ was bounded by using Lemma 4.6 and (4.6). Then the bound

$$\begin{aligned}
\|u - u_h\|_{L^2(\Omega)} &\leq \|u - \pi_h u\|_{L^2(\Omega)} + \|u_h - \pi_h u\|_{L^2(\Omega)} \\
&\leq C(h^2\|u\|_{H^2(\Omega)} + h^{-1}k^{-2}\|u_h - \pi_h u\|_{V_h}) \\
&\leq C(h^2\|u\|_{H^2(\Omega)} + k^{-2}\|u\|_* + h^{-1}k^{-2}\delta(\tilde{u}_\omega, \tilde{f})) \\
&\leq Ck^{-2}(\|u\|_* + h^{-1}\delta(\tilde{u}_\omega, \tilde{f}))
\end{aligned}$$

concludes the proof. □

Theorem 4.8 (H^1-error estimate)**.** *Let* $\omega \subset B \subset \Omega$ *be defined as in Corollary 2.13. Assume that* $u \in H^2(\Omega)$ *is a solution to the unperturbed problem* (4.1) *and let* $(u_h, z_h) \in V_h \times W_h$ *be the solution to the perturbed problem* (4.13)*. Then there exist* $C > 0$ *and* $\kappa \in (0, 1)$ *such that for all* k, $h > 0$ *with* $kh \lesssim 1$

$$\|u - u_h\|_{H^1(B)} \leq C(hk)^\kappa(\|u\|_* + h^{-1}\delta(\tilde{u}_\omega, \tilde{f})).$$

Proof. Following the proof of Theorem 4.7, we now use Corollary 2.14 to derive

$$\begin{aligned}\|u-u_h\|_{H^1(B)} &\le Ck(\|u-u_h\|_{L^2(\omega)} + \|r\|_{H^{-1}(\Omega)})^{\kappa}(\|u-u_h\|_{L^2(\Omega)} + \|r\|_{H^{-1}(\Omega)})^{1-\kappa} \\ &\le Ck\Big(h\|u\|_* + \delta(\tilde{u}_\omega, \tilde{f})\Big)^{\kappa}\Big((k^{-2}+h)(\|u\|_* + h^{-1}\delta(\tilde{u}_\omega, \tilde{f}))\Big)^{1-\kappa} \\ &\le Ckh^{\kappa}(k^{-2}+h)^{1-\kappa}(\|u\|_* + h^{-1}\delta(\tilde{u}_\omega, \tilde{f})),\end{aligned}$$

which ends the proof. □

Analogous to the unpolluted case, if $k^2 h \lesssim 1$ the above result becomes

$$\|u-u_h\|_{H^1(B)} \le C(hk^2)^{\kappa}k^{-1}(\|u\|_* + h^{-1}\delta(\tilde{u}_\omega, \tilde{f})),$$

and combining Theorems 4.7 and 4.8 gives the following.

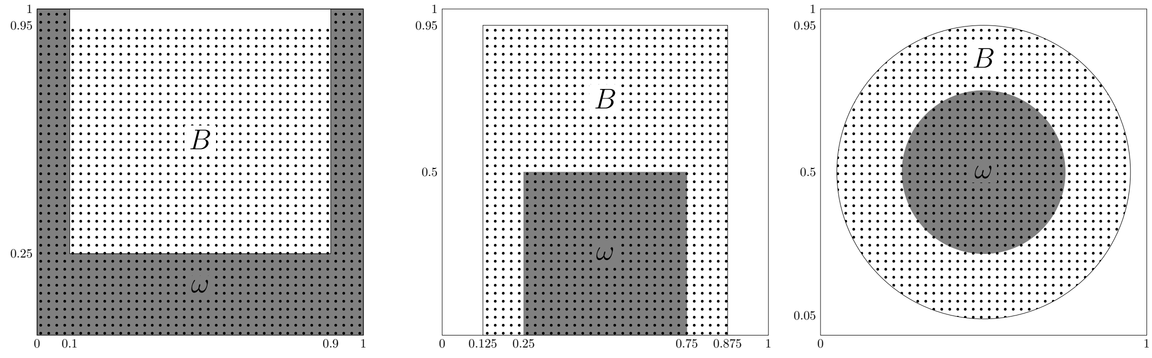
Corollary 4.9. *Let $\omega \subset B \subset \Omega$ be defined as in Corollary 2.13. Assume that $u \in H^2(\Omega)$ is a solution to the unperturbed problem (4.1) and $(u_h, z_h) \in V_h \times W_h$ be the solution to the perturbed problem (4.13). Then there exist $C > 0$ and $\kappa \in (0,1)$ such that for all k, $h > 0$ with $k^2 h \lesssim 1$*

$$k\|u-u_h\|_{L^2(B)} + \|u-u_h\|_{H^1(B)} \le C(hk^2)^{\kappa}k^{-1}(\|u\|_* + h^{-1}\delta(\tilde{u}_\omega, \tilde{f})).$$

4.3 Numerical examples

We illustrate the above theoretical results for the unique continuation problem (4.1) with some numerical examples. The implementation of our method and all the computations have been carried out in FEniCS [2]. The domain Ω is the unit square, and the triangulation is uniform with alternating left and right diagonals. Various numerical experiments indicate that for the stabiliser s the parameter $\gamma = 10^{-5}$ is a near-optimal value for different kinds of geometries and solutions. We show in Figure 4.2 an example of the effect of this parameter on the approximation error for a homogeneous solution.

In the light of the convexity assumptions in Section 2.4, we shall consider two different geometric settings: one in which the data is continued in the convex direction, inside the convex hull of ω, and one in which the solution is continued in the non-convex direction, outside the convex hull of ω.



(a) Convex direction (4.14). (b) Non-convex direction (4.15). (c) Non-convex direction (4.16).

Figure 4.1: Computational domains for Example 4.11. Data set ω (grey) and error measurement regions B (dotted).

In the convex setting, given in Figure 4.1a, we take

$$\omega = \Omega \setminus [0.1, 0.9] \times [0.25, 1], \quad B = \Omega \setminus [0.1, 0.9] \times [0.95, 1]. \tag{4.14}$$

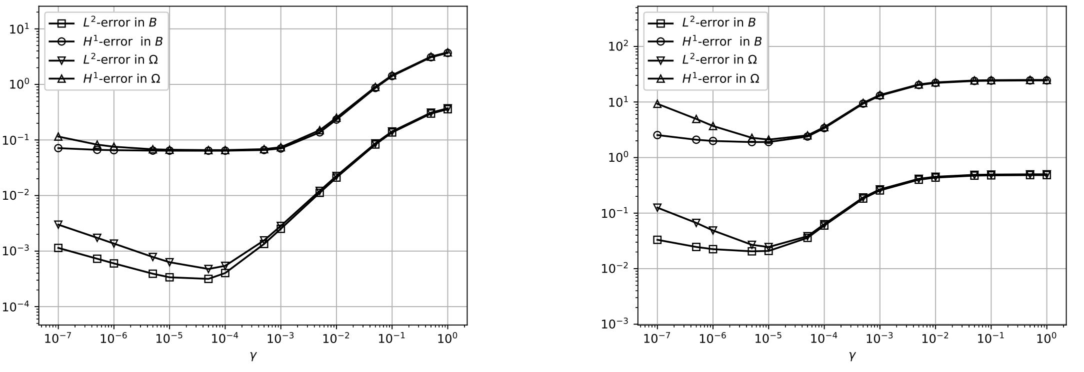
This example does not correspond exactly to the specific geometric setting in Corollary 2.13, but all the theoretical results are valid in this case as proven in the following.

Example 4.10. *Let $\omega \subset B \subset \Omega$ be defined by (4.14) (Figure 4.1a). Then the stability estimates in Corollaries 2.13 and 2.14, and Lemma 2.15 hold true.*

Proof. Consider an extended rectangle $\tilde{\Omega} \supset \Omega$ such that the unit square Ω is centred horizontally and touches the upper side of $\tilde{\Omega}$, and $\tilde{\omega} \supset \omega$ and $\tilde{B} \supset B$ are defined as in Corollary 2.13. Choose a smooth cutoff function χ such that $\chi = 1$ in $\Omega \setminus \omega$ and $\chi = 0$ in $\tilde{\Omega} \setminus \Omega$. Applying now Corollary 2.13 for $\tilde{\omega}$, $\tilde{B}$, $\tilde{\Omega}$ and χu we get

$$\begin{aligned} \|u\|_{H^1(B\setminus\omega)} &\le C\|\chi u\|_{H^1(\tilde{B}\setminus\tilde{\omega})} \le C(\|\chi u\|_{H^1(\tilde{\omega})} + \|\Delta(\chi u) + k^2 \chi u\|_{L^2(\tilde{\Omega})})^\kappa \|\chi u\|_{H^1(\tilde{\Omega})}^{1-\kappa} \\ &\le C(\|u\|_{H^1(\omega)} + \|\Delta u + k^2 u\|_{L^2(\Omega)})^\kappa \|u\|_{H^1(\Omega)}^{1-\kappa}, \end{aligned}$$

where we have used that the commutator $[\Delta, \chi]u$ is supported in ω. A similar proof is valid for the estimates in Corollary 2.14 and Lemma 2.15. □



(a) $k = 10$.

(b) $k = 50$.

Figure 4.2: Varying the stabilisation parameter γ for $u(x,y) = \sin(kx/\sqrt{2})\cos(ky/\sqrt{2})$ in the convex direction (4.14) and mesh size $h \approx 0.005$.

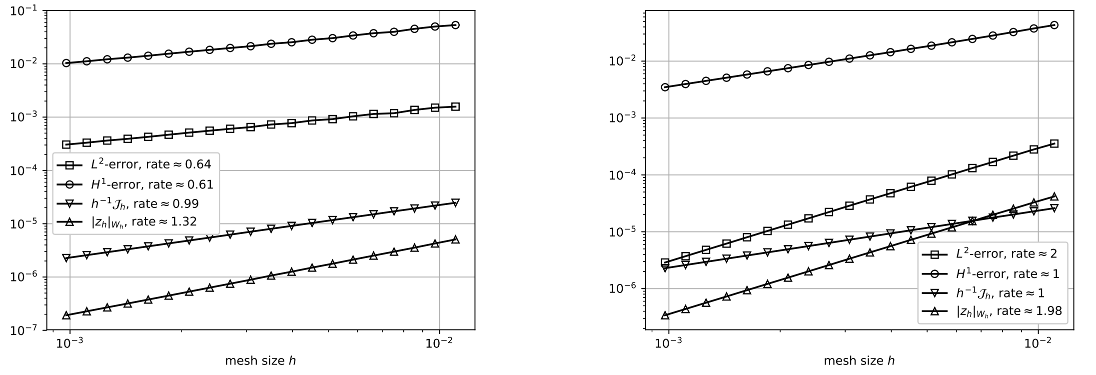
We will give results for two kinds of solutions: a Gaussian bump, that is independent of the wave number, centred on the top side of the unit square, Ω given in Example 4.11, and a variation of the well-known Hamadard solution given in Example 4.12.

Example 4.11. *Let the Gaussian bump*

$$u(x,y) = \exp\left(-\frac{(x-0.5)^2}{2\sigma_x} - \frac{(y-1)^2}{2\sigma_y}\right), \quad \sigma_x = 0.01,\, \sigma_y = 0.1,$$

be a non-homogeneous solution of (4.1) *with* $f = -\Delta u - k^2 u$ *and* $u_\omega = u|_\omega$.

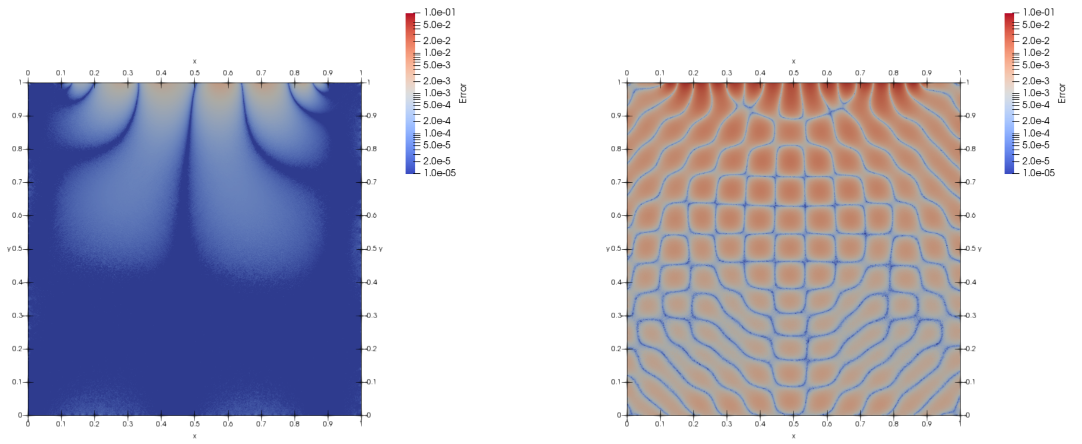
Figure 4.3a shows that for Example 4.11, when $k = 10$, the numerical results strongly agree with the convergence rates expected from Theorems 4.3 and 4.4, respectively, and Lemma 4.2, i.e. sub-linear convergence for the relative error in the L^2- and H^1-norms, and quadratic convergence for $\mathcal{J}_h(u_h, u_h)$.



(a) $k = 10$. (b) $k = 50$.

Figure 4.3: Convergence in B for Example 4.11 in the convex direction (4.14).

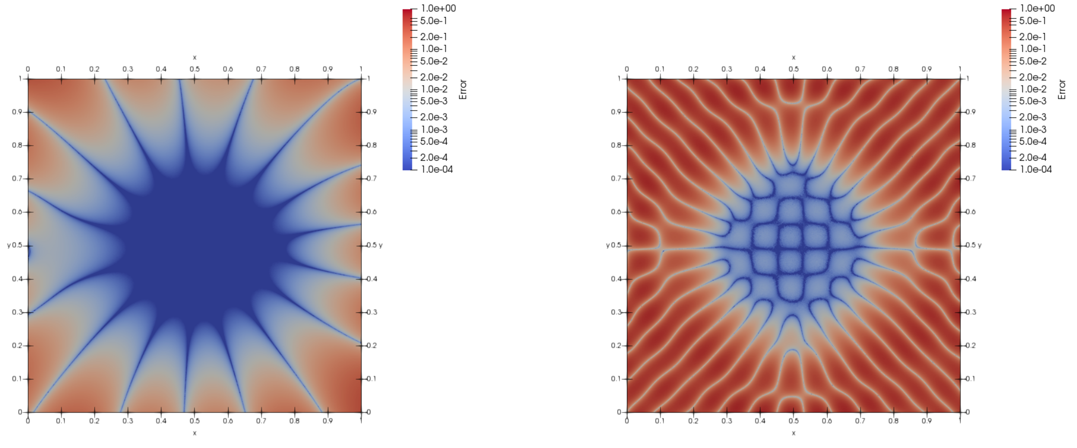
Although in Figure 4.3b we do obtain smaller errors and better than expected convergence rates when $k = 50$, various numerical experiments indicate that the behaviour of this example when increasing the wave number k is rather an exception. For oscillatory solutions, such as those in Example 4.12, with fixed n, or the homogeneous solution $u(x, y) = \sin(kx/\sqrt{2})\cos(ky/\sqrt{2})$, we have noticed that the stability deteriorates when increasing the wave number as shown in Figures 4.4 and 4.5 for unstructured meshes with 512 elements on a side.



(a) $k = 10$. (b) $k = 50$.

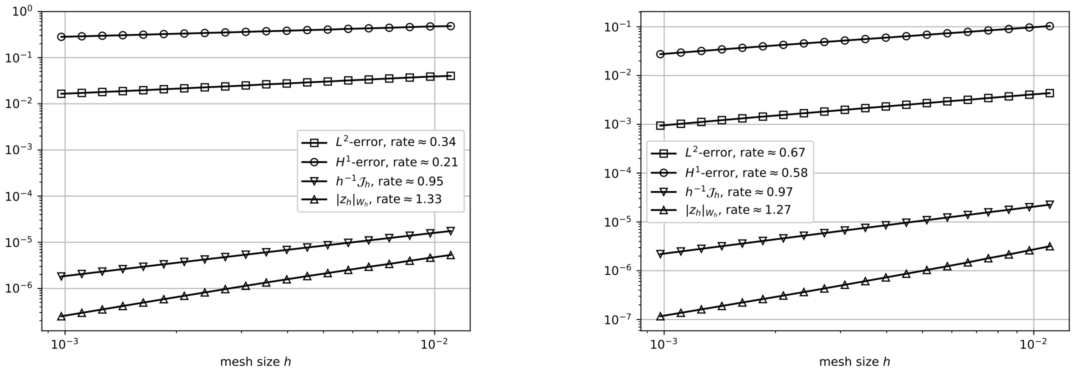
Figure 4.4: Absolute errors for $u(x, y) = \sin(kx/\sqrt{2})\cos(ky/\sqrt{2})$ in the convex direction (4.14). Mesh size $h \approx 0.0025$.

.



(a) $k = 10$. (b) $k = 50$.

Figure 4.5: Absolute errors for $u(x, y) = \sin(kx/\sqrt{2})\cos(ky/\sqrt{2})$ in the non-convex direction (4.16). Mesh size $h \approx 0.0025$.



(a) Non-convex direction (4.15). (b) Non-convex direction (4.16).

Figure 4.6: Convergence in B for Example 4.11, $k = 10$.

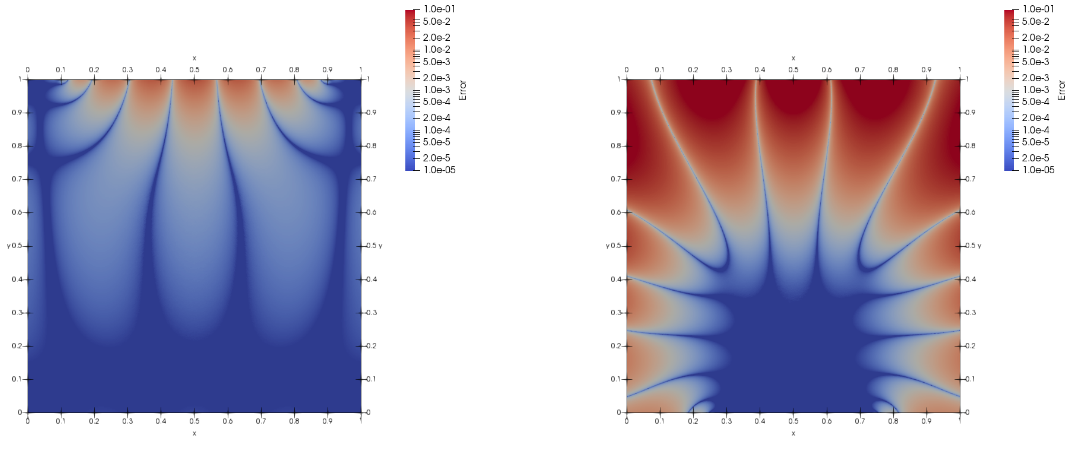
In the non-convex setting we let

$$\omega = (0.25, 0.75) \times (0, 0.5), \quad B = (0.125, 0.875) \times (0, 0.95), \tag{4.15}$$

and the concentric discs

$$\omega = B((0.5, 0.5), 0.25), \quad B = B((0.5, 0.5), 0.45), \tag{4.16}$$

respectively shown in Figures 4.1b and 4.1c, and we see in Figure 4.6 that the convergence order decreases, especially for (4.15). In Figure 4.7 we considered an unstructured mesh with 512 elements on a side and we notice that the stability strongly deteriorates when one continues the solution outside the convex hull of ω.



(a) Convex direction (4.14). (b) Non-convex direction (4.15).

Figure 4.7: Absolute errors in Ω for Example 4.11, $k = 10$. Mesh size $h \approx 0.0025$.

Let us recall from Chapter 1 that the stability estimates for the unique continuation problem are closely related to those for the severely ill-posed Cauchy problem. It is thus of interest to consider the following variation of the well-known Hadamard's Example 1.11.

Example 4.12. *Let $n \in \mathbb{N}$ and consider the Cauchy problem*

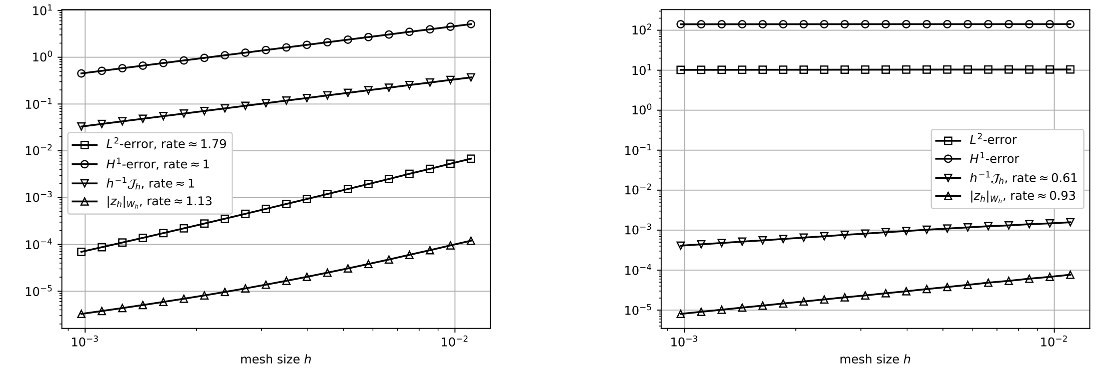
$$\begin{cases} \Delta u + k^2 u = 0 & \textit{in } \Omega = (0,\pi) \times (0,1), \\ u(x,0) = 0 & \textit{for } x \in [0,\pi], \\ u_y(x,0) = \sin(nx) & \textit{for } x \in [0,\pi], \end{cases}$$

whose solution for $n > k$ is given by $u(x,y) = \frac{1}{\sqrt{n^2-k^2}} \sin(nx)\sinh(\sqrt{n^2-k^2}y)$, for $n = k$ by $u(x,y) = \sin(kx)y$, and for $n < k$ by $u(x,y) = \frac{1}{\sqrt{k^2-n^2}} \sin(nx)\sin(\sqrt{k^2-n^2}y)$.

It can be seen in Figure 4.8a that the convergence rates are better than predicted for the convex setting

$$\omega = \Omega \setminus [\tfrac{\pi}{4}, \tfrac{3\pi}{4}] \times [0, 0.25], \quad B = \Omega \setminus [\tfrac{\pi}{4}, \tfrac{3\pi}{4}] \times [0, 0.95], \tag{4.17}$$

i.e. sub-linear convergence for the relative error in the L^2- and H^1-norms. We observe quadratic convergence for the jump stabiliser $\mathcal{J}_h(u_h, u_h)$, although one can notice that its values visibly increase compared to Example 4.11.



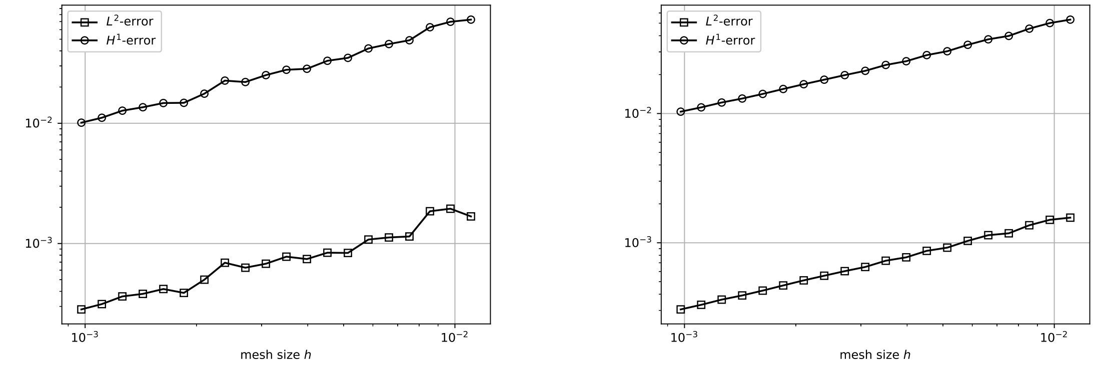
(a) Convex direction (4.17). (b) Non-convex direction (4.18).

Figure 4.8: Convergence in B for Example 4.12, $k = 10$, $n = 12$.

When continuing the solution in the non-convex direction, the stability strongly deteriorates and the numerical approximation doesn't seem to reach the convergence regime, as it can be seen in Figure 4.8b for the non-convex setting

$$\omega = \left(\tfrac{\pi}{4}, \tfrac{3\pi}{4}\right) \times (0, 0.5), \quad B = \left(\tfrac{\pi}{8}, \tfrac{7\pi}{8}\right) \times (0, 0.95). \tag{4.18}$$

Data perturbations. We exemplify data perturbations by polluting f and u_ω in (4.1) with uniformly distributed values in $[-h, h]$, respectively $[-h^2, h^2]$, on every node of the mesh. In agreement with Theorems 4.7 and 4.8, it can be seen in Figure 4.9 that the perturbations are visible for an $\mathcal{O}(h)$ amplitude, but not for an $\mathcal{O}(h^2)$ one.



(a) Perturbation $\mathcal{O}(h)$. (b) Perturbation $\mathcal{O}(h^2)$.

Figure 4.9: Convergence in B when perturbing f and u_ω in Example 4.11 for (4.14), $k = 10$.

We conclude by emphasising the role of geometry in the convergence of the method. For unique continuation inside the convex hull of the data set, the continuum stability estimate is robust in the wave number (or the stability constant grows linearly in the wave number, depending on the norms), as discussed in Section 2.4, and we observe that the discrete solution converges at a sub-linear rate that reflects the continuum Hölder stability. When the solution is approximated outside the convex hull of the data set, the continuum stability deteriorates since the stability constant might grow very fast in the wave number, as shown in Example 2.10. In this case, we notice from the numerical experiments that the convergence rate decreases and, in some cases, it might be hard to observe any convergence unless very fine meshes are considered.

Chapter 5

Diffusion-dominated problems

In this chapter based on [22] we consider the unique continuation problem for the convection–diffusion equation

$$\mathfrak{L}u := -\mu\Delta u + \beta \cdot \nabla u = f \quad \text{in } \Omega, \tag{5.1}$$

where $\Omega \subset \mathbb{R}^n$ is open, bounded and connected, $\mu > 0$ is the diffusion coefficient and $\beta \in [W^{1,\infty}(\Omega)]^n$ is the convective velocity field. We assume that there exists a solution $u \in H^2(\Omega)$ satisfying (5.1). For an open and connected subset $\omega \subset \Omega$, define the perturbed restriction $\tilde{u}_\omega := u|_\omega + \delta u$, where $\delta u \in L^2(\omega)$ is an unknown function modelling measurement noise. Here the coefficients μ and β are assumed to be known. Hence, given $f \in L^2(\Omega)$ and $\tilde{u}_\omega \in L^2(\omega)$, the perturbed unique continuation problem we will consider consists in finding $u \in H^2(\Omega)$ such that

$$\begin{cases} -\mu\Delta u + \beta \cdot \nabla u = -f & \text{in } \Omega, \\ \qquad\qquad\qquad u = \tilde{u}_\omega & \text{in } \omega. \end{cases}$$

The aim is to design a finite element method for unique continuation with weakly consistent regularisation following the approach described in Chapter 3. In the present analysis we consider the regime where diffusion dominates and in the following Chapter 6 we treat the one with dominating convective transport. To make this more precise we recall the Péclet number associated to a given length scale l by

$$\text{Pe}(l) := \frac{|\beta| l}{\mu},$$

for a suitable norm $|\cdot|$ for β. If h denotes the characteristic length scale of the computation, we define the diffusive regime by $\text{Pe}(h) < 1$ and the convective regime by $\text{Pe}(h) > 1$. It is known that the character of the system changes drastically in the two regimes and we therefore need to apply different concepts of stability in the two cases. In this chapter we assume that the Péclet number is small and we use an approach similar to that used in Chapter 4 for the Helmholtz equation – we combine conditional stability estimates for the physical problem with optimal numerical stability obtained using a bespoke weakly consistent stabilising term. For high Péclet numbers on the other hand, we prove in Chapter 6 weighted estimates directly on the discrete solution, that reflect the anisotropic character of the convection–diffusion problem.

In the case of optimal control problems subject to convection-diffusion problems that are well-posed, there are several works in the literature on stabilised finite element methods. In [27] the authors considered stabilisation using a Galerkin least squares approach in the Lagrangian. Symmetric stabilisation in the form of local projection stabilisation was proposed in [6] and using penalty on the gradient jumps in [37, 61]. The key difference between the well-posed case and the ill-posed case that we consider herein is that we can not use the stability of neither the forward nor the backward equations. Crucial instead is the convergence of the weakly consistent stabilising terms and the stability of the continuous problem. Such considerations lead to results both in the case of high and low Péclet numbers, but the different stability properties in the two regimes lead to a different analysis for each case.

5.1 Discretisation

As in Section 3.2, let V_h denote the space of piecewise affine finite element functions defined on a conforming computational mesh $\mathcal{T}_h = \{K\}$. $\mathcal{T}_h$ consists of triangular elements K with diameter h_K and the global mesh size is given by $h = \max_{K \in \mathcal{T}_h} h_K$. We assume that the family $\{\mathcal{T}_h\}$ is quasi-uniform. The interior faces of the triangulation will be denoted by $\mathcal{F}_i$, the jump of a quantity across a face F by $[\![\cdot]\!]_F$, and the unit normal by n.

Let $\beta \in [W^{1,\infty}(\Omega)]^n$ and adopt the shorthand notation

$$|\beta| := \|\beta\|_{[L^\infty(\Omega)]^n}.$$

We consider the diffusion-dominated regime given by the low mesh Péclet number

$$\mathrm{Pe}(h) := \frac{|\beta| h}{\mu} < 1. \tag{5.2}$$

We recall from Section 2.3 that the constant of the stability estimate depends exponentially on $\tilde{\text{Pe}} = 1 + \frac{|\beta|}{\mu}$ and we will also assume that $\tilde{\text{Pe}}$ is small. We will denote by C a generic positive constant independent of the mesh size and the Péclet number. Let $\pi_h : L^2(\Omega) \mapsto V_h$ denote the standard L^2-projection on V_h, which for $k = 1, 2$ and $m = 0, k-1$ satisfies

$$\|\pi_h u\|_{H^m(\Omega)} \leq C\|u\|_{H^m(\Omega)}, \quad u \in H^m(\Omega),$$
$$\|u - \pi_h u\|_{H^m(\Omega)} \leq Ch^{k-m}\|u\|_{H^k(\Omega)}, \quad u \in H^k(\Omega),$$

see Appendix A. For convenience we recall the standard inner products with the induced norms

$$(v_h, w_h)_\Omega := \int_\Omega v_h w_h \,\mathrm{d}x,$$
$$\langle v_h, w_h\rangle_{\partial\Omega} := \int_{\partial\Omega} v_h w_h \,\mathrm{d}s,$$

and introduce the following bilinear form used in the weak formulation of (5.1)

$$a_h(v_h, w_h) := (\beta \cdot \nabla v_h, w_h)_\Omega + (\mu \nabla v_h, \nabla w_h)_\Omega - \langle \mu \nabla v_h \cdot n, w_h\rangle_{\partial\Omega}.$$

As discussed in Chapter 3, the discrete regularisation will be driven by interior penalty, but here we rescale it using the physical parameters μ and β, and define

$$\mathcal{J}_h(v_h, w_h) := \gamma \sum_{F \in \mathcal{F}_i} \int_F h(\mu + |\beta| h) [\![\nabla v_h \cdot n]\!]_F [\![\nabla w_h \cdot n]\!]_F \,\mathrm{d}s.$$

We consider the scaled inner product

$$s_\omega(v_h, w_h) := ((\mu + |\beta| h) v_h, w_h)_\omega,$$

and then sum

$$s(v_h, w_h) := \mathcal{J}_h(v_h, w_h) + s_\omega(v_h, w_h).$$

For the dual stabilisers we define

$$s^*(v_h, w_h) := \gamma^* \Big(\big\langle (\mu h^{-1} + |\beta|) v_h, w_h \big\rangle_{\partial\Omega} + (\mu \nabla v_h, \nabla w_h)_\Omega + \mathcal{J}_h(v_h, w_h) \Big).$$

The parameters γ and γ^* in $\mathcal{J}_h$ and s^*, respectively, are fixed at the implementation level and we emphasize that our analysis covers the choice $\gamma = 1 = \gamma^*$ to avoid the proliferation of constants.

Notice that here the interior penalty $\mathcal{J}_h$ and the inner product on ω are scaled by a factor of $\mu + |\beta|h$. After this scaling, the Poincaré-type inequality Lemma 3.2 becomes

$$\|(\mu^{\frac{1}{2}}h + |\beta|^{\frac{1}{2}}h^{\frac{3}{2}})v_h\|_{H^1(\Omega)} \leq C\gamma^{-\frac{1}{2}}s(v_h, v_h)^{\frac{1}{2}}, \quad \forall v_h \in V_h, \tag{5.3}$$

and Lemma 3.3 gives the weak consistency inequality

$$\mathcal{J}_h(\pi_h u, \pi_h u) \leq C\gamma(\mu + |\beta|h)h^2|u|^2_{H^2(\Omega)}, \quad \forall u \in H^2(\Omega). \tag{5.4}$$

We can then use the general framework in Chapter 3 to write the finite element method for unique continuation as follows. Consider a discrete Lagrange multiplier $z_h \in V_h$ and look for the saddle point of the functional

$$\begin{aligned} L_h(u_h, z_h) := \tfrac{1}{2}s_\omega(u_h - \tilde{u}_\omega, u_h - \tilde{u}_\omega) + a_h(u_h, z_h) - (f, z_h)_\Omega \\ + \tfrac{1}{2}\mathcal{J}_h(u_h, u_h) - \tfrac{1}{2}s^*(z_h, z_h), \end{aligned}$$

where we recall that $\tilde{u}_\omega = u|_\omega + \delta u$ and $u \in H^2(\Omega)$ is a solution to (5.1). The (primal) stabiliser used for the discrete solution u_h is the interior penalty $\mathcal{J}_h$, and the (dual) stabiliser used for z_h is given by s^*. The optimality conditions for L_h lead to the following discrete problem: find $(u_h, z_h) \in [V_h]^2$ such that

$$\begin{cases} a_h(u_h, w_h) - s^*(z_h, w_h) &= (f, w_h)_\Omega. \\ a_h(v_h, z_h) + s(u_h, v_h) &= s_\omega(\tilde{u}_\omega, v_h). \end{cases} \quad \forall (v_h, w_h) \in [V_h]^2, \tag{5.5}$$

Notice that here the parameters γ and γ^* are included in s and s^*. We observe that by the ill-posed character of the problem, only the stabilisation operators $\mathcal{J}_h$ and s^* provide some stability to the discrete system, and the corresponding system matrix is expected to be ill-conditioned. To quantify this effect we first prove an upper bound on the condition number.

Proposition 5.1. *The finite element formulation* (5.5) *has a unique solution* $(u_h, z_h) \in [V_h]^2$ *and the Euclidean condition number* $\mathcal{K}_2$ *of the system matrix satisfies*

$$\mathcal{K}_2 \leq Ch^{-4}.$$

Proof. We write (5.5) as the linear system $A[(u_h, z_h), (v_h, w_h)] = (f, w_h)_\Omega + s_\omega(\tilde{u}_\omega, v_h)$, for all $(v_h, w_h) \in [V_h]^2$, where

$$A[(u_h, z_h), (v_h, w_h)] := a_h(u_h, w_h) - s^*(z_h, w_h) + a_h(v_h, z_h) + s(u_h, v_h).$$

Since $A[(u_h, z_h), (u_h, -z_h)] = s(u_h, u_h) + s^*(z_h, z_h)$, using (5.3) the following inf-sup condition holds

$$\Psi_h := \inf_{(u_h,z_h)\in[V_h]^2} \sup_{(v_h,w_h)\in[V_h]^2} \frac{A[(u_h, z_h), (v_h, w_h)]}{\|(u_h, z_h)\|_{L^2(\Omega)}\|(v_h, w_h)\|_{L^2(\Omega)}} \geq C\mu(1 + \mathrm{Pe}(h))h^2.$$

This provides the existence of a unique solution for the linear system. We use [33, Theorem 3.1] to estimate the condition number by

$$\mathcal{K}_2 \leq C\frac{\Upsilon_h}{\Psi_h}, \tag{5.6}$$

where

$$\Upsilon_h := \sup_{(u_h,z_h)\in[V_h]^2} \sup_{(v_h,w_h)\in[V_h]^2} \frac{A[(u_h, z_h), (v_h, w_h)]}{\|(u_h, z_h)\|_{L^2(\Omega)}\|(v_h, w_h)\|_{L^2(\Omega)}}.$$

We recall the discrete inverse inequality (A.1)

$$\|\nabla v_h\|_{L^2(K)} \leq Ch^{-1}\|v_h\|_{L^2(K)}, \quad \forall v_h \in \mathbb{P}_1(K). \tag{5.7}$$

We also recall the continuous trace inequality (A.2)

$$\|v\|_{L^2(\partial K)} \leq C(h^{-\frac{1}{2}}\|v\|_{L^2(K)} + h^{\frac{1}{2}}\|\nabla v\|_{L^2(K)}), \quad \forall v \in H^1(K), \tag{5.8}$$

and the discrete one (A.3)

$$\|\nabla v_h \cdot n\|_{L^2(\partial K)} \leq Ch^{-\frac{1}{2}}\|\nabla v_h\|_{L^2(K)}, \quad \forall v_h \in \mathbb{P}_1(K). \tag{5.9}$$

Using the Cauchy-Schwarz inequality together with (5.9) and (5.7) we get

$$\begin{aligned}\mathcal{J}_h(u_h, v_h) &= \gamma\mu(1 + \mathrm{Pe}(h)) \sum_{F\in\mathcal{F}_i} \int_F h[\![\nabla u_h \cdot n]\!]_F [\![\nabla v_h \cdot n]\!]_F \, \mathrm{d}s \\ &\leq C\mu(1 + \mathrm{Pe}(h))h^{-2}\|u_h\|_{L^2(\Omega)}\|v_h\|_{L^2(\Omega)},\end{aligned}$$

hence

$$s(u_h, v_h) \leq C\mu(1 + \mathrm{Pe}(h))h^{-2}\|u_h\|_{L^2(\Omega)}\|v_h\|_{L^2(\Omega)}.$$

Combining this with the Cauchy-Schwarz inequality and the inequalities (5.7) and (5.8), we obtain

$$-s^*(z_h, w_h) \leq C\mu(1 + \mathrm{Pe}(h))h^{-2}\|z_h\|_{L^2(\Omega)}\|w_h\|_{L^2(\Omega)}.$$

Again due to the Cauchy-Schwarz inequality, and trace and inverse inequalities, we have

$$\begin{aligned} a_h(u_h, w_h) &= (\beta \cdot \nabla u_h, w_h)_\Omega + \mu \sum_{F \in \mathcal{F}_i} \int_F h [\![\nabla u_h \cdot n]\!]_F w_h \, \mathrm{d}s \\ &\leq C\mu(1 + \mathrm{Pe}(h)) h^{-2} \|u_h\|_{L^2(\Omega)} \|w_h\|_{L^2(\Omega)}, \end{aligned}$$

Collecting the above estimates we have $\Upsilon_h \leq C\mu(1 + \mathrm{Pe}(h))h^{-2}$, and we conclude by (5.6). □

5.2 Error estimates

The error analysis proceeds in two main steps:

- First we prove that the stabilising terms and the data fitting term must vanish at an optimal rate for smooth solutions, with constant independent of the physical stability (Proposition 5.4).
- Then we show that the residual of the PDE is bounded by the stabilising terms and the data fitting term. Using this result together with the first step and the continuous stability estimates in Section 2.3, we prove L^2- and H^1-convergence results (Theorems 5.6 and 5.7).

To quantify stabilisation and data fitting for $(v_h, w_h) \in [V_h]^2$ we introduce the norm

$$\|(v_h, w_h)\|_s^2 := s(v_h, v_h) + s^*(w_h, w_h).$$

We also define the "continuity norm" on $H^{\frac{3}{2}+\epsilon}(\Omega)$, for any $\epsilon > 0$,

$$\|v\|_\sharp := |\beta|^{\frac{1}{2}} \|h^{-\frac{1}{2}} v\|_\Omega + \mu^{\frac{1}{2}} \|\nabla v\|_\Omega + \mu^{\frac{1}{2}} h^{\frac{1}{2}} \|\nabla v \cdot n\|_{\partial\Omega},$$

which scales as a discrete H^1-norm. Using standard approximation properties and the trace inequality (5.8), we have

$$\|u - \pi_h u\|_\sharp \leq C(\mu^{\frac{1}{2}} h + |\beta|^{\frac{1}{2}} h^{\frac{3}{2}}) |u|_{H^2(\Omega)}.$$

Using (5.4) and interpolation

$$\begin{aligned} \|(u - \pi_h u, 0)\|_s^2 = s(u - \pi_h u, u - \pi_h u) &= \mathcal{J}_h(\pi_h u, \pi_h u) + s_\omega(u - \pi_h u, u - \pi_h u) \\ &\leq C(\mu h^2 + |\beta| h^3) |u|_{H^2(\Omega)}^2, \end{aligned}$$

where we used that $\mathcal{J}_h(u, v_h) = 0$, since $u \in H^2(\Omega)$. Hence it follows that for $u \in H^2(\Omega)$

$$\|(u - \pi_h u, 0)\|_s + \|u - \pi_h u\|_\sharp \leq C(\mu^{\frac{1}{2}} h + |\beta|^{\frac{1}{2}} h^{\frac{3}{2}})|u|_{H^2(\Omega)}. \tag{5.10}$$

Observe that, when $\mathrm{Pe}(h) < 1$, the first term dominates and the estimate is $\mathcal{O}(h)$, whereas when $\mathrm{Pe}(h) > 1$ the bound is $\mathcal{O}(h^{\frac{3}{2}})$. We note in passing that the same estimates hold for the nodal interpolant.

Lemma 5.2 (Consistency)**.** *Assume that* $u \in H^2(\Omega)$ *is a solution to* (5.1) *and let* $(u_h, z_h) \in [V_h]^2$ *be the solution to* (5.5)*, then*

$$a_h(\pi_h u - u_h, w_h) + s^*(z_h, w_h) = a_h(\pi_h u - u, w_h),$$

and

$$-a_h(v_h, z_h) + s(\pi_h u - u_h, v_h) = \mathcal{J}_h(\pi_h u - u, v_h) + s_\omega(\pi_h u - \tilde{u}_\omega, v_h),$$

for all $(v_h, w_h) \in [V_h]^2$*.*

Proof. The first claim follows from the definition of a_h, since

$$a_h(u_h, w_h) - s^*(z_h, w_h) = (f, w_h)_\Omega = (\beta \cdot \nabla u - \mu \Delta u, w_h)_\Omega = a_h(u, w_h),$$

where in the last equality we integrated by parts. The second claim follows similarly from

$$a_h(v_h, z_h) + s(u_h, v_h) = s_\omega(\tilde{u}_\omega, v_h),$$

leading to

$$\begin{aligned} -a_h(v_h, z_h) + s(\pi_h u - u_h, v_h) &= s(\pi_h u, v_h) - s_\omega(\tilde{U}_\omega, v_h) \\ &= \mathcal{J}_h(\pi_h u - u, v_h) + s_\omega(\pi_h u - \tilde{u}_\omega, v_h). \end{aligned}$$

□

Lemma 5.3 (Continuity)**.** *Assume the low Péclet regime* (5.2) *and that* $|\beta|_{1,\infty} \leq C|\beta|$*. Let* $v \in H^2(\Omega)$ *and* $w_h \in V_h$*, then*

$$a_h(v, w_h) \leq C\|v\|_\sharp \|(0, w_h)\|_s.$$

Proof. Writing out the terms of a_h and integrating by parts in the advective term leads to

$$a_h(v, w_h) = -(v, \beta\cdot\nabla w_h)_\Omega - (v\nabla\cdot\beta, w_h)_\Omega + \langle v\beta\cdot n, w_h\rangle_{\partial\Omega} + (\mu\nabla v, \nabla w_h)_\Omega - \langle \mu\nabla v\cdot n, w_h\rangle_{\partial\Omega}\,.$$

Using the Cauchy-Schwarz inequality and the trace inequality (5.8) for v, we see that

$$\langle v\beta\cdot n, w_h\rangle_{\partial\Omega} + (\mu\nabla v, \nabla w_h)_\Omega - \langle \mu\nabla v\cdot n, w_h\rangle_{\partial\Omega} \leq C\|v\|_\sharp \|(0, w_h)\|_s.$$

By the Cauchy-Schwarz inequality and a Poincaré inequality for w_h we bound

$$-(v\nabla\cdot\beta, w_h)_\Omega \leq C|\beta|_{1,\infty}\|v\|_\Omega\|w_h\|_\Omega \leq C\frac{|\beta|_{1,\infty}}{|\beta|}\,\mathrm{Pe}(h)^{\frac{1}{2}}\|v\|_\sharp\|(0, w_h)\|_s.$$

Under the assumption $|\beta|_{1,\infty}\leq C|\beta|$, we get

$$-(v\nabla\cdot\beta, w_h)_\Omega \leq C\,\mathrm{Pe}(h)^{\frac{1}{2}}\|v\|_\sharp\|(0, w_h)\|_s.$$

We bound the remaining term by

$$-(v, \beta\cdot\nabla w_h)_\Omega \leq |\beta|^{\frac{1}{2}}h^{\frac{1}{2}}\|v\|_\sharp\|\nabla w_h\|_\Omega \leq C\,\mathrm{Pe}(h)^{\frac{1}{2}}\|v\|_\sharp\|(0, w_h)\|_s.$$

Finally, exploiting the low Péclet regime $\mathrm{Pe}(h) < 1$, we obtain the conclusion. □

Proposition 5.4 (Convergence of regularisation)**.** *Assume the low Péclet regime* (5.2) *and that* $|\beta|_{1,\infty}\leq C|\beta|$. *Assume that* $u \in H^2(\Omega)$ *is a solution to* (5.1) *and let* $(u_h, z_h) \in [V_h]^2$ *be the solution to* (5.5)*, then*

$$\|(\pi_h u - u_h, z_h)\|_s \leq C(\mu^{\frac{1}{2}}h + |\beta|^{\frac{1}{2}}h^{\frac{3}{2}})(|u|_{H^2(\Omega)} + h^{-1}\|\delta u\|_\omega).$$

Proof. Denoting $e_h = \pi_h u - u_h$, it holds by definition that

$$\|(e_h, z_h)\|_s^2 = a_h(e_h, z_h) + s^*(z_h, z_h) - a_h(e_h, z_h) + s(e_h, e_h).$$

Using both claims in Lemma 5.2 we may write

$$\|(e_h, z_h)\|_s^2 = a_h(\pi_h u - u, z_h) + \mathcal{J}_h(\pi_h u - u, e_h) + s_\omega(\pi_h u - \tilde{u}_\omega, e_h).$$

Lemma 5.3 gives the bound

$$a_h(\pi_h u - u, z_h) \leq C\|\pi_h u - u\|_\sharp\|(0, z_h)\|_s.$$

The other terms are simply bounded using the Cauchy-Schwarz inequality as follows

$$\mathcal{J}_h(\pi_h u - u, e_h) + s_\omega(\pi_h u - \tilde{u}_\omega, e_h) \leq (\|(\pi_h u - u, 0)\|_s + (\mu^{\frac{1}{2}} + |\beta|^{\frac{1}{2}} h^{\frac{1}{2}})\|\delta u\|_\omega)\|(e_h, 0)\|_s.$$

Collecting the above bounds we have

$$\|(e_h, z_h)\|_s^2 \leq C(\|\pi_h u - u\|_\sharp + \|(\pi_h u - u, 0)\|_s + (\mu^{\frac{1}{2}} + |\beta|^{\frac{1}{2}} h^{\frac{1}{2}})\|\delta u\|_\omega)\|(e_h, z_h)\|_s,$$

and the claim follows by applying the approximation (5.10). □

Lemma 5.5 (Covergence of the convective term)**.** *Assume the low Péclet regime* (5.2) *and that* $|\beta|_{1,\infty} \leq C|\beta|$*. Let* $u \in H^2(\Omega)$ *be a solution to* (5.1)*,* $(u_h, z_h) \in [V_h]^2$ *the solution to* (5.5) *and* $w \in H_0^1(\Omega)$*, then*

$$(\beta \cdot \nabla u_h, w - \pi_h w)_\Omega \leq C(\mu + |\beta|)(h\|u\|_{H^2(\Omega)} + \|\delta u\|_\omega)\|w\|_{H^1(\Omega)},$$

Proof. Denote by $\beta_h \in [V_h]^n$ a piecewise linear approximation of β that is L^∞-stable and for which

$$\|\beta - \beta_h\|_{0,\infty} \leq Ch|\beta|_{1,\infty},$$

and recall the approximation estimate in [13, Theorem 2.2]

$$\inf_{x_h \in V_h} \|h^{\frac{1}{2}}(\beta_h \cdot \nabla u_h - x_h)\|_\Omega \leq C\left(\sum_{F \in \mathcal{F}_i} \|h[\![\beta_h \cdot \nabla u_h]\!]\|_F^2\right)^{\frac{1}{2}} \leq C|\beta|^{\frac{1}{2}} \mathcal{J}_h(u_h, u_h)^{\frac{1}{2}}. \quad (5.11)$$

We also use Proposition 5.4 and the jump inequality (5.4) to estimate

$$\begin{aligned}\mathcal{J}_h(u_h, u_h)^{\frac{1}{2}} &\leq \mathcal{J}_h(u_h - \pi_h u, u_h - \pi_h u)^{\frac{1}{2}} + \mathcal{J}_h(\pi_h u, \pi_h u)^{\frac{1}{2}} \\ &\leq C(\mu^{\frac{1}{2}} h + |\beta|^{\frac{1}{2}} h^{\frac{3}{2}})(|u|_{H^2(\Omega)} + h^{-1}\|\delta u\|_\omega) + C(\mu^{\frac{1}{2}} + |\beta|^{\frac{1}{2}} h^{\frac{1}{2}})h|u|_{H^2(\Omega)},\end{aligned}$$

obtaining

$$\mathcal{J}_h(u_h, u_h)^{\frac{1}{2}} \leq C(\mu^{\frac{1}{2}} h + |\beta|^{\frac{1}{2}} h^{\frac{3}{2}})(|u|_{H^2(\Omega)} + h^{-1}\|\delta u\|_\omega). \quad (5.12)$$

We now write

$$(\beta \cdot \nabla u_h, w - \pi_h w)_\Omega = (\beta_h \cdot \nabla u_h, w - \pi_h w)_\Omega + ((\beta - \beta_h) \cdot \nabla u_h, w - \pi_h w)_\Omega,$$

and using orthogonality, (5.11), (5.12), interpolation and (5.2), we bound the first term by

$$\begin{aligned}(\beta_h \cdot \nabla u_h, w - \pi_h w)_\Omega &\leq C|\beta|^{\frac{1}{2}} h^{-\frac{1}{2}} \mathcal{J}_h(u_h, u_h)^{\frac{1}{2}} h \|w\|_{H^1(\Omega)} \\ &\leq C|\beta|^{\frac{1}{2}} h^{\frac{1}{2}} (\mu^{\frac{1}{2}} + |\beta|^{\frac{1}{2}} h^{\frac{1}{2}})(h|u|_{H^2(\Omega)} + \|\delta u\|_\omega)\|w\|_{H^1(\Omega)} \\ &\leq C(\mu + |\beta| h)(h|u|_{H^2(\Omega)} + \|\delta u\|_\omega)\|w\|_{H^1(\Omega)}.\end{aligned}$$

We now use the Poincaré-type inequality (5.3) and interpolation to bound the second term

$$\begin{aligned}((\beta - \beta_h) \cdot \nabla u_h, w - \pi_h w)_\Omega &\leq C h^2 |\beta|_{1,\infty} \|\nabla u_h\|_\Omega \|w\|_{H^1(\Omega)} \\ &\leq C h |\beta|_{1,\infty} (\mu^{\frac{1}{2}} + |\beta|^{\frac{1}{2}} h^{\frac{1}{2}})^{-1} s(u_h, u_h)^{\frac{1}{2}} \|w\|_{H^1(\Omega)} \\ &\leq C h |\beta|_{1,\infty} (h|u|_{H^2(\Omega)} + \|u\|_\Omega + \|\delta u\|_\omega)\|w\|_{H^1(\Omega)} \\ &\leq C h |\beta|_{1,\infty} (\|u\|_{H^2(\Omega)} + \|\delta u\|_\omega)\|w\|_{H^1(\Omega)}\end{aligned}$$

since due to Proposition 5.4 and inequality (5.4)

$$\begin{aligned}s(u_h, u_h)^{\frac{1}{2}} &\leq s(u_h - \pi_h u, u_h - \pi_h u)^{\frac{1}{2}} + \mathcal{J}_h(\pi_h u, \pi_h u)^{\frac{1}{2}} + s_\omega(\pi_h u, \pi_h u)^{\frac{1}{2}} \\ &\leq C(\mu^{\frac{1}{2}} + |\beta|^{\frac{1}{2}} h^{\frac{1}{2}})(h|u|_{H^2(\Omega)} + \|\delta u\|_\omega + \|u\|_\Omega).\end{aligned}$$

Under the assumption $|\beta|_{1,\infty} \leq C|\beta|$, we collect the above bounds to get

$$(\beta \cdot \nabla u_h, w - \pi_h w)_\Omega \leq C(\mu + |\beta|)(h\|u\|_{H^2(\Omega)} + \|\delta u\|_\omega)\|w\|_{H^1(\Omega)}.$$

□

We now combine these results with the conditional stability estimates from Section 2.3 to obtain error bounds for the discrete solution. For this purpose, we consider an open bounded set $B \subset \Omega$ that contains the data region ω such that $B \setminus \omega$ does not touch the boundary of Ω. Then the estimates in Lemma 2.7 and Corollary 2.9 hold true by a covering argument, see e.g. [52], and we obtain local error estimates in B.

Theorem 5.6 (L^2-error estimate)**.** *Assume the low Péclet regime* (5.2) *and that* $|\beta|_{1,\infty} \leq C|\beta|$. *Consider* $\omega \subset B \subset \Omega$ *such that* $\overline{B \setminus \omega} \subset \Omega$. *Assume that* $u \in H^2(\Omega)$ *is a solution to* (5.1) *and let* $(u_h, z_h) \in [V_h]^2$ *the solution to* (5.5)*, then there exists* $\kappa \in (0,1)$ *such that*

$$\|u - u_h\|_{L^2(B)} \leq C h^\kappa e^{C\tilde{\mathrm{Pe}}^2} (\|u\|_{H^2(\Omega)} + h^{-1}\|\delta u\|_\omega),$$

where $\tilde{\mathrm{Pe}} = 1 + |\beta|/\mu$.

Proof. Let us consider the residual defined by $\langle r, w\rangle = a(u_h, w) - \langle f, w\rangle$, for $w \in H_0^1(\Omega)$. Using (5.5) we obtain

$$\begin{aligned}\langle r, w\rangle &= a(u_h, w - \pi_h w) - \langle f, w - \pi_h w\rangle + a(u_h, \pi_h w) - \langle f, \pi_h w\rangle \\ &= a(u_h, w - \pi_h w) - \langle f, w - \pi_h w\rangle + s^*(z_h, \pi_h w).\end{aligned}$$

We split the first term in the right-hand side into convective and non-convective terms, and for the latter we integrate by parts on each element K and use Cauchy-Schwarz followed by the trace inequality (5.8) to get

$$\begin{aligned}&(\mu\nabla u_h, \nabla(w - \pi_h w))_\Omega - \langle \mu\nabla u_h \cdot n, w - \pi_h w\rangle_{\partial\Omega} \\ &= \sum_{F\in\mathcal{F}_i} \int_F \mu[\![\nabla u_h \cdot n]\!]_F (w - \pi_h w)\,\mathrm{d}s \\ &\leq C\mu(\mu + |\beta|h)^{-\frac{1}{2}} \mathcal{J}_h(u_h, u_h)^{\frac{1}{2}} (h^{-1}\|w - \pi_h w\|_{L^2(\Omega)} + \|w - \pi_h w\|_{H^1(\Omega)}).\end{aligned}$$

Using (5.12) and interpolation we obtain

$$(\mu\nabla u_h, \nabla(w - \pi_h w))_\Omega - \langle \mu\nabla u_h \cdot n, w - \pi_h w\rangle_{\partial\Omega} \leq C\mu(h|u|_{H^2(\Omega)} + \|\delta u\|_\omega)\|w\|_{H^1(\Omega)}.$$

We bound the convective term in $a(u_h, w - \pi_h w)$ by Lemma 5.5, hence obtaining

$$a(u_h, w - \pi_h w) \leq C(\mu + |\beta|)(h\|u\|_{H^2(\Omega)} + \|\delta u\|_\omega)\|w\|_{H^1(\Omega)}.$$

The next term in the residual is bounded by

$$\langle f, w - \pi_h w\rangle \leq \|f\|_{L^2(\Omega)}\|w - \pi_h w\|_{L^2(\Omega)} \leq Ch\|f\|_{L^2(\Omega)}\|w\|_{H^1(\Omega)}.$$

The last term left to bound from the residual is

$$s^*(z_h, \pi_h w) \leq \|(0, z_h)\|_s \|(0, \pi_h w)\|_s,$$

and using (5.9) for the jump term, together with the H^1-stability of π_h, we see that

$$\begin{aligned}\|(0, \pi_h w)\|_s &\leq C(\mu^{\frac{1}{2}}\|\nabla(\pi_h w)\|_\Omega + (\mu^{\frac{1}{2}} + |\beta|^{\frac{1}{2}} h^{\frac{1}{2}})\|\nabla(\pi_h w)\|_\Omega + (\mu h^{-1} + |\beta|)^{\frac{1}{2}}\|\pi_h w\|_{\partial\Omega}) \\ &\leq C(\mu^{\frac{1}{2}} + |\beta|^{\frac{1}{2}} h^{\frac{1}{2}})\|w\|_{H^1(\Omega)},\end{aligned}$$

where for the boundary term we used that $w|_{\partial\Omega} = 0$ together with interpolation and (5.8).

Bounding $\|(0, z_h)\|_s$ by Proposition 5.4, we get

$$s^*(z_h, \pi_h w) \le C(\mu + |\beta| h)(h|u|_{H^2(\Omega)} + \|\delta u\|_\omega)\|w\|_{H^1(\Omega)}.$$

Collecting the above estimates we bound the residual norm by

$$\begin{aligned}\|r\|_{H^{-1}(\Omega)} &\le C(\mu + |\beta|)(h\|u\|_{H^2(\Omega)} + \|\delta u\|_\omega) + Ch\|f\|_{L^2(\Omega)} \\ &\le C(\mu + |\beta|)(h\|u\|_{H^2(\Omega)} + \|\delta u\|_\omega).\end{aligned}$$

We now use the stability estimate in Lemma 2.7 to write

$$\|u - u_h\|_{L^2(B)} \le Ce^{C\tilde{\mathrm{Pe}}^2} \left(\|u - u_h\|_{L^2(\omega)} + \frac{1}{\mu}\|r\|_{H^{-1}(\Omega)}\right)^\kappa \|u - u_h\|_{L^2(\Omega)}^{1-\kappa}.$$

By Proposition 5.4 we have

$$\begin{aligned}\|u - u_h\|_{L^2(\omega)} &\le \|u - \pi_h u\|_{L^2(\omega)} + \|u_h - \pi_h u\|_{L^2(\omega)} \\ &\le Ch^2|u|_{H^2(\Omega)} + Ch|u|_{H^2(\Omega)} + C\|\delta u\|_\omega. \\ &\le C(h|u|_{H^2(\Omega)} + \|\delta u\|_\omega).\end{aligned}$$

Using (5.3) and Proposition 5.4 again, we bound

$$\begin{aligned}\|u - u_h\|_{L^2(\Omega)} &\le \|u - \pi_h u\|_{L^2(\Omega)} + \|u_h - \pi_h u\|_{L^2(\Omega)} \\ &\le Ch^2|u|_{H^2(\Omega)} + C(\mu^{\frac{1}{2}} h + |\beta|^{\frac{1}{2}} h^{\frac{3}{2}})^{-1} s(u_h - \pi_h u, u_h - \pi_h u)^{\frac{1}{2}} \\ &\le C(|u|_{H^2(\Omega)} + h^{-1}\|\delta u\|_\omega).\end{aligned}$$

Hence we conclude by

$$\begin{aligned}\|u - u_h\|_{L^2(B)} &\le Ce^{C\tilde{\mathrm{Pe}}^2} \left(h\|u\|_{H^2(\Omega)} + \|\delta u\|_\omega\right)^\kappa \left(|u|_{H^2(\Omega)} + h^{-1}\|\delta u\|_\omega\right)^{1-\kappa} \\ &\le Ce^{C\tilde{\mathrm{Pe}}^2} h^\kappa (\|u\|_{H^2(\Omega)} + h^{-1}\|\delta u\|_\omega),\end{aligned}$$

where we have absorbed the $\tilde{\mathrm{Pe}} = 1 + |\beta|/\mu$ factor into the exponential factor $e^{C\tilde{\mathrm{Pe}}^2}$. □

Theorem 5.7 (H^1-error estimate)**.** *Assume the low Péclet regime* (5.2) *and that* $|\beta|_{1,\infty} \le C|\beta|$ *and* $\operatorname{ess\,sup}_\Omega \nabla \cdot \beta \le 0$. *Consider* $\omega \subset B \subset \Omega$ *such that* $\overline{B \setminus \omega} \subset \Omega$. *Assume that* $u \in H^2(\Omega)$ *is a solution to* (5.1) *and let* $(u_h, z_h) \in [V_h]^2$ *the solution to* (5.5), *then there* *exists* $\kappa \in (0,1)$ *such that*

$$\|u-u_h\|_{H^1(B)} \leq Ch^{\kappa} e^{C\tilde{\mathrm{Pe}}^2}(\|u\|_{H^2(\Omega)} + h^{-1}\|\delta u\|_{\omega}),$$

where $\tilde{\mathrm{Pe}} = 1 + |\beta|/\mu$.

Proof. Letting $e_h = u - u_h$, we combine the proof of Theorem 5.6 with the stability estimate in Corollary 2.9 to obtain

$$\begin{aligned}\|e_h\|_{H^1(B)} &\leq Ce^{C\tilde{\mathrm{Pe}}^2}\left(\|e_h\|_{L^2(\omega)} + \tfrac{1}{\mu}\|r\|_{H^{-1}(\Omega)}\right)^{\kappa}\left(\|e_h\|_{L^2(\Omega)} + \tfrac{1}{\mu}\|r\|_{H^{-1}(\Omega)}\right)^{1-\kappa} \\ &\leq Ce^{C\tilde{\mathrm{Pe}}^2} h^{\kappa}(\|u\|_{H^2(\Omega)} + h^{-1}\|\delta u\|_{\omega}).\end{aligned}$$

□

5.3 Numerical examples

We illustrate the theoretical results with some numerical examples. The implementation of the stabilised FEM (5.5) has been carried out in FreeFem++ [36] on uniform triangulations with alternating left and right diagonals. The mesh size is taken as the inverse square root of the number of nodes. The parameters in $\mathcal{J}_h$ and s^* are set to $\gamma = 10^{-5}$ and $\gamma^* = 1$. We also rescale the boundary term in s^* by a factor of 50, drawing on results from different numerical experiments. In this section we denote $e_h = \pi_h u - u_h$.

We consider Ω to be the unit square and the exact solution with global unit L^2-norm

$$u(x,y) = 30x(1-x)y(1-y).$$

We take the diffusion coefficient $\mu = 1$ and investigate two cases for the convection field: the coercive case of the constant field

$$\beta_c = (1,0),$$

and the case

$$\beta_{nc} = 100(x+y, y-x),$$

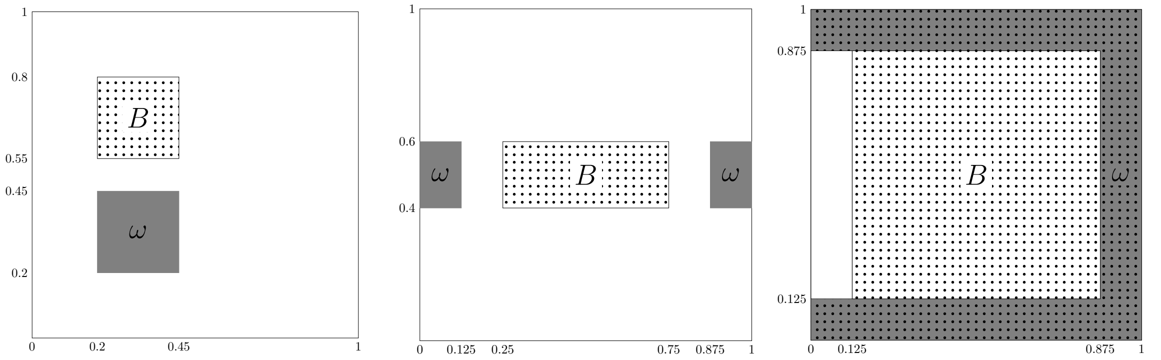
plotted in Figure 5.2, for which $\nabla \cdot \beta = 200$ and $\|\beta\|_{0,\infty} = 200$. In this case the boundary value problem in Example 3.1 is strongly non-coercive with a medium high Péclet number. This example was also considered in [14] for numerical experiments on a non-coercive convection–diffusion equation with Cauchy data.

We consider the following domains for unique continuation, shown in Figure 5.1,

$$\omega = (0.2, 0.45) \times (0.2, 0.45), \quad B = (0.2, 0.45) \times (0.55, 0.8), \tag{5.13}$$

$$\omega = (0, 0.125) \times (0.4, 0.6) \cup (0.875, 1) \times (0.4, 0.6), \quad B = (0.25, 0.75) \times (0.4, 0.6), \tag{5.14}$$

$$\omega = \Omega \setminus [0, 0.875] \times [0.125, 0.875], \quad B = \Omega \setminus [0, 0.125] \times [0.125, 0.875]. \tag{5.15}$$



(a) Domains in (5.13). (b) Domains in (5.14). (c) Domains in (5.15).

Figure 5.1: Computational domains. Data set ω (grey) and error measurement regions B (dotted).

The condition number upper bound in Proposition 5.1 is illustrated for a particular case in Figure 5.2, where we plot the condition number $\mathcal{K}_2$ versus the mesh size h, together with reference dotted lines proportional to h^{-3} and h^{-4}. For five meshes with 2^N elements on each side, $N = 3, \ldots, 7$, the approximate rates for $\mathcal{K}_2$ are -3.03, -3.16, -3.2, -3.34.

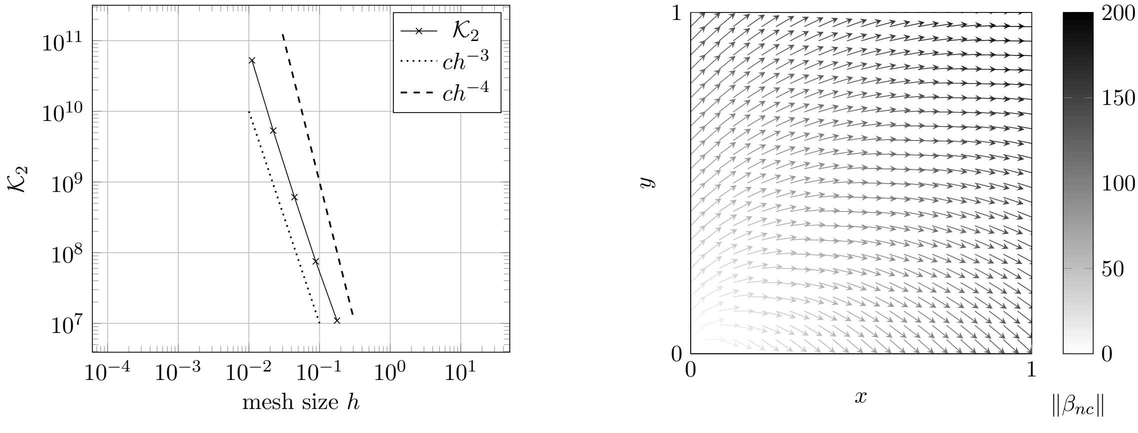
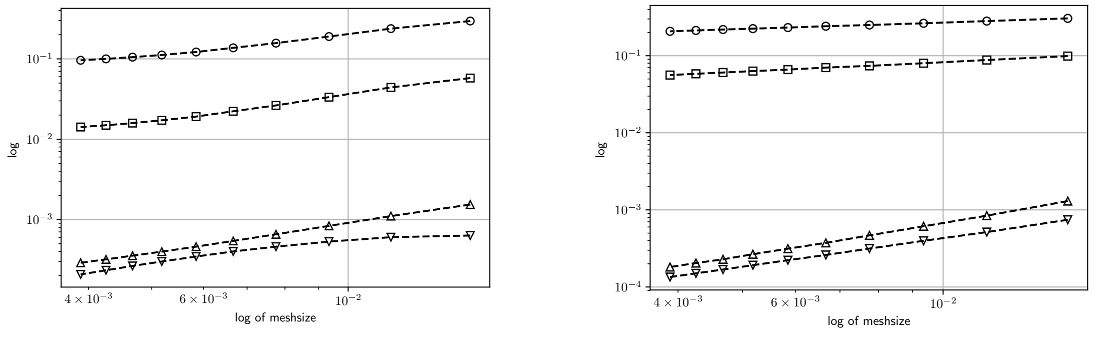


Figure 5.2: Left: condition number $\mathcal{K}_2$ for domains (5.13), $\beta = \beta_c$. Right: convection field β_{nc}.

The results in Figure 5.3 for the domains (5.13) strongly agree with the convergence rates expected from Theorems 5.6 and 5.7 for the relative errors in B computed in the L^2- and H^1-norms, and with the rates for $\|(e_h, z_h)\|_s$ given in Proposition 5.4.

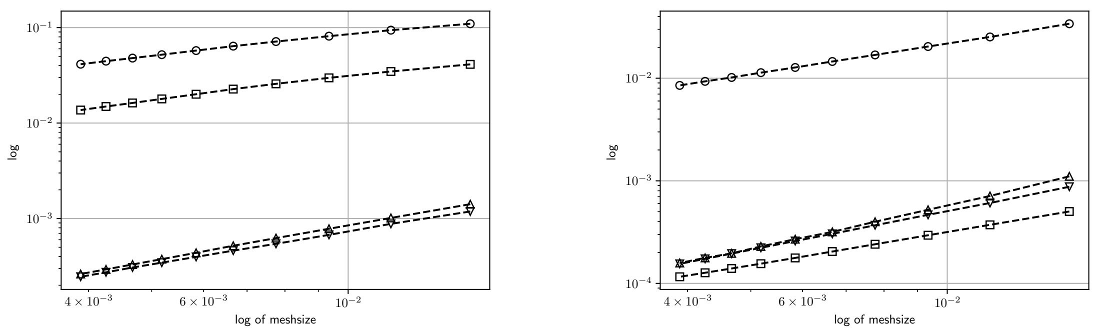
The numerical approximation improves when considering the setting in (5.14), in which data is given both downstream and upstream, as reported in Figure 5.4. The convergence is almost linear and the size of the errors is considerably reduced in the non-coercive case.



(a) Circles: H^1-error, rate ≈ 0.45; Squares: L^2-error, rate ≈ 0.56; Up triangles: $s(e_h, e_h)^{\frac{1}{2}}$, rate ≈ 1.1; Down triangles: $s^*(z_h, z_h)^{\frac{1}{2}}$, rate ≈ 1.33.

(b) Circles: H^1-error, rate ≈ 0.29; Squares: L^2-error, rate ≈ 0.42; Up triangles: $s(e_h, e_h)^{\frac{1}{2}}$, rate ≈ 1.32; Down triangles: $s^*(z_h, z_h)^{\frac{1}{2}}$, rate ≈ 1.34.

Figure 5.3: Convergence for domains (5.13). Left: $\beta = \beta_c$. Right: $\beta = \beta_{nc}$.



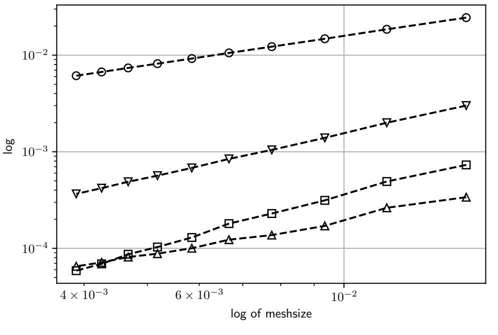
(a) Circles: H^1-error, rate ≈ 0.8; Squares: L^2-error, rate ≈ 0.94; Up triangles: $s(e_h, e_h)^{\frac{1}{2}}$, rate ≈ 1.24; Down triangles: $s^*(z_h, z_h)^{\frac{1}{2}}$, rate ≈ 1.2.

(b) Circles: H^1-error, rate ≈ 1.02; Squares: L^2-error, rate ≈ 1.07; Up triangles: $s(e_h, e_h)^{\frac{1}{2}}$, rate ≈ 1.3; Down triangles: $s^*(z_h, z_h)^{\frac{1}{2}}$, rate ≈ 1.25.

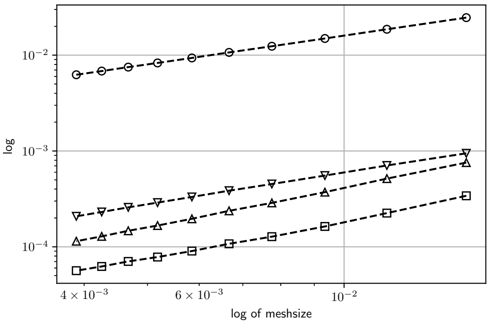
Figure 5.4: Convergence for domains (5.14). Left: $\beta = \beta_c$. Right: $\beta = \beta_{nc}$.

Comparing the geometries in (5.13) and (5.14) we also expect to see different effects of the two convective fields β_c and β_{nc}. Notice that for both geometries the horizontal magnitude of β_{nc} is greater than that of β_c. In (5.13) the solution is continued in the crosswind direction for both β_c and β_{nc}, and a stronger convective field is not expected to improve the reconstruction. On the other side, in (5.14) information is propagated both downstream and upstream, and a stronger convective field can improve the resolution, despite the increase in the Péclet number. Indeed, we can see in Figure 5.3 that for the geometry in (5.13) the numerical approximation is better for β_c than for β_{nc}, while Figure 5.4 shows better results for β_{nc} than for β_c in the case of (5.14), especially for the L^2-error.

The resolution increases all the more when data is given near a big part of the boundary $\partial\Omega$, as for the computational domains (5.15) considered in Figure 5.5. In this configuration of the set ω, for both convective fields β_c and β_{nc}, the L^2-errors decrease below 10^{-4} with superlinear rates on the same meshes considered in Figures 5.3 and 5.4.



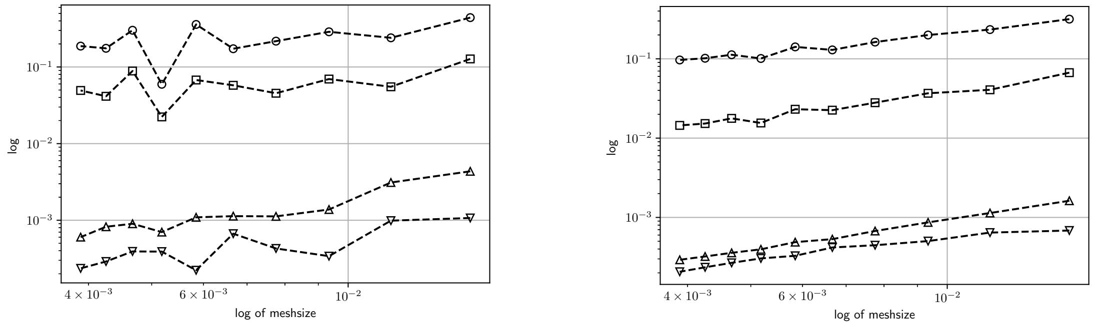
(a) Circles: H^1-error, rate $\approx$ 1; Squares: L^2-error, rate $\approx$ 1.81; Up triangles: $s(e_h, e_h)^{\frac{1}{2}}$, rate $\approx$ 1.04; Down triangles: $s^*(z_h, z_h)^{\frac{1}{2}}$, rate $\approx$ 1.52.



(b) Circles: H^1-error, rate $\approx$ 1; Squares: L^2-error, rate $\approx$ 1.13; Up triangles: $s(e_h, e_h)^{\frac{1}{2}}$, rate $\approx$ 1.30; Down triangles: $s^*(z_h, z_h)^{\frac{1}{2}}$, rate $\approx$ 1.16.

Figure 5.5: Convergence for domains (5.15). Left: $\beta = \beta_c$. Right: $\beta = \beta_{nc}$.

Data perturbations. To exemplify the noisy data $\tilde{U}_\omega = u|_\omega + \delta u$, we perturb the restriction of u to ω on every node of the mesh with uniformly distributed values in $[-h^{\frac{1}{2}}, h^{\frac{1}{2}}]$, respectively $[-h, h]$. Recall that by the error estimates in Section 5.2 the contribution of the perturbation δu is bounded by $h^{-1}\|\delta u\|_\omega$. It can be seen in Figure 5.6 that the perturbations are strongly visible for an $\mathcal{O}(h^{\frac{1}{2}})$ amplitude, but not for an $\mathcal{O}(h)$ one.



(a) Noise amplitude $\mathcal{O}(h^{\frac{1}{2}})$.

(b) Noise amplitude $\mathcal{O}(h)$.

Figure 5.6: Convergence for perturbed $\tilde{u}_\omega$ in domains (5.13), $\beta = \beta_c$.

Chapter 6

Convection-dominated problems

In this chapter we consider the unique continuation problem for the convection–diffusion equation

$$\mathfrak{L}u := -\mu\Delta u + \beta \cdot \nabla u = f \quad \text{in } \Omega \subset \mathbb{R}^n, \tag{6.1}$$

when convection dominates, that is, $0 < \mu \ll |\beta|$. We assume that $\Omega \subset \mathbb{R}^n$ is open, bounded and connected, and there exists a solution $u \in H^2(\Omega)$ to (6.1). The problem under study is to approximate the solution u given the source f in Ω and the perturbed restriction $\tilde{u}_\omega = u|_\omega + \delta u$ of the solution to an open subset $\omega \subset \Omega$. The perturbation δu is taken in $L^2(\omega)$. This chapter is based on [23].

We have seen in Chapter 2 (Lemma 2.7) that for an open bounded set $B \subset \Omega$ that contains the data region ω such that $B \setminus \omega$ does not touch the boundary of Ω, and for $u \in H^1(\Omega)$, the following conditional stability estimate holds for $\mu > 0$ and $\beta \in L^\infty(\Omega)^n$,

$$\|u\|_{L^2(B)} \leq C_{st} \left(\|u\|_{L^2(\omega)} + \tfrac{1}{\mu} \|\mathfrak{L}u\|_{H^{-1}(\Omega)} \right)^\kappa \|u\|_{L^2(\Omega)}^{1-\kappa}, \tag{6.2}$$

where the Hölder exponent $\kappa \in (0,1)$ depends only on the geometric setting. In the case of simple geometric configurations, e.g. when ω, B, Ω are three concentric balls, the exponent $\kappa \in (0,1)$ can be given explicitly, see Remark 2.8. The stability constant C_{st} is given explicitly in terms of the physical parameters

$$C_{st} = C_1 \exp\left(C_2 (1 + \tfrac{|\beta|}{\mu})^2 \right), \quad |\beta| := \|\beta\|_{L^\infty(\Omega)^n}, \tag{6.3}$$

with constants $C_{1,2} > 0$ depending only on the geometry. Note that the continuum estimate (6.2) is valid in both the diffusion-dominated and convection-dominated regimes, and that the stability constant C_{st} is uniformly bounded when diffusion dominates. However, when convection dominates C_{st} grows exponentially, rendering the stability estimate ineffective in practice.

The prototypical effect of dominating diffusion is shown in Figure 6.1, where the problem is set in the unit square and absolute error contour plots are shown for unique continuation from a centred disc of radius 0.1. One can notice that the errors oscillate and grow in size away from the data region towards the boundary in all directions. The plot is not symmetric due to the presence of the convective term. The exact solution in this example is $u(x, y) = 2\sin(5\pi x)\sin(5\pi y)$ where the factor 2 is taken to make its L^2-norm unitary. For the computation we used an unstructured mesh with 512 elements on a side and mesh size $h \approx 0.0025$.

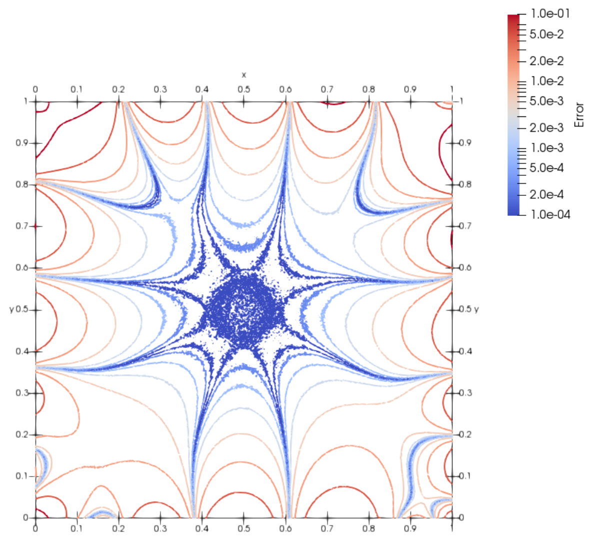


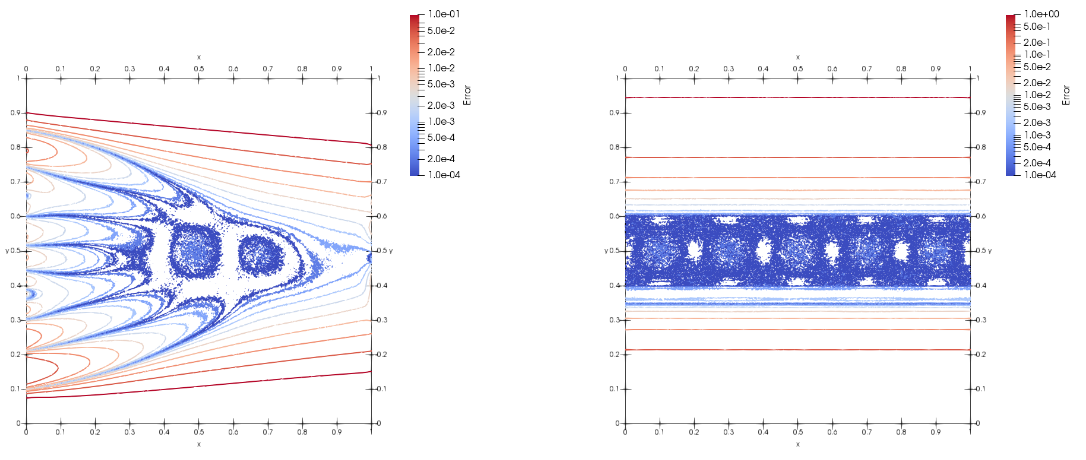
Figure 6.1: Contour plot in the diffusion-dominated case, $\mu = 1$, $\beta = (1, 0)$. Unit square domain, data given in a centred disc of radius 0.1, exact solution $u = 2\sin(5\pi x)\sin(5\pi y)$.

Since the behaviour of the physical system changes fundamentally when convection dominates and

$$\text{Pe}(h) \gg 1,$$

we reconsider the numerical method discussed in Chapter 5 and provide an error analysis that captures and exploits the governing transport phenomenon. To illustrate this, let us consider the example in Figure 6.1 and gradually decrease the diffusion coefficient μ in Figure 6.2. We see a transition from a diffusion-dominated regime towards a convection-dominated regime through an intermediate one. We first observe the intermediate regime in Figure 6.2a with a clear asymmetry between downstream and upstream regions. Downstream, the impact of the convection is stronger and the errors decrease along the characteristics through the data region. Upstream, the impact of the diffusion is stronger and the errors still oscillate. Then in Figure 6.2b we see the convection-dominated regime in which the approximation greatly improves both downstream and upstream from the data region (with no difference between the two directions), but it severely deteriorates in the crosswind direction away from the data region. We aim to obtain sharper local error estimates along the characteristics of the convective field through the data region. Even though the error analysis that we perform herein is different in nature to the one in Chapter 5, the numerical method itself is only slightly changed (see Remark 6.1 below). For the error localisation technique we draw on ideas used for the streamline diffusion method in [46], continuous interior penalty in [17], and non-coercive hyperbolic problems in [15].

From the definition of the Péclet number we see that the regime will also depend on the resolution of the computation besides the physical parameters. We can therefore expect the method to change behaviour as the resolution increases and $\mathrm{Pe}(h)$ decreases. This phenomenon was already observed computationally in [14] and can now be explained theoretically.



(a) $\mu = 10^{-2}$. (b) $\mu = 10^{-6}$.

Figure 6.2: Contour plot when convection becomes dominant, $\beta = (1, 0)$. Unit square domain, data given in a centred disc of radius 0.1, exact solution $u = 2\sin(5\pi x)\sin(5\pi y)$.

To make the presentation as simple as possible we consider a model case in the unit square Ω with constant convection

$$\beta := (\beta_1, 0), \quad \beta_1 \in \mathbb{R},$$

and the solution given in the subset

$$\omega := (0, x) \times (y^-, y^+), \text{ with } x > h \text{ and } y^+ - y^- > h.$$

For the subset $\omega_\beta \subset \Omega$ covered by the characteristics of β that go through ω, we introduce the stability region $\mathring{\omega}_\beta \subset \omega_\beta$ by cutting off a crosswind layer of width $\mathcal{O}(h^{\frac{1}{2}}|\ln(h)|)$ (see Section 6.1.1 for more details). We separate the convection-dominated and diffusion-dominated regimes by introducing a constant $\mathrm{Pe}_{\mathrm{lim}} > 1$ such that

$$\mathrm{Pe}(h) > \mathrm{Pe}_{\mathrm{lim}} > 1.$$

To reduce the number of constants appearing in the analysis, we will write this as $\mathrm{Pe}(h) \gtrsim 1$. As suggested by Figure 6.2, we expect different results for unique continuation downstream vs upstream in an intermediate regime. We prove in Theorem 6.12 weighted error estimates that for $\beta_1 > 0$ essentially take the following form

$$\|u - u_h\|_{L^2(\mathring{\omega}_\beta)} \leq C\Big(|\beta|^{\frac{1}{2}} h^{\frac{3}{2}} |u|_{H^2(\Omega)} + |\beta|^{\frac{1}{2}} h^{-\frac{1}{2}} \|\delta u\|_{L^2(\omega)}\Big), \text{ when } \mathrm{Pe}(h) \gtrsim 1.$$

This is similar to the typical error estimates for piecewise linear stabilised FEMs for convection-dominated well-posed problems, such as local projection stabilisation, dG methods, continuous interior penalty or Galerkin least squares. On general shape-regular meshes these methods have an $\mathcal{O}(h^{\frac{1}{2}})$ gap to the best approximation convergence order. Taking this into account, our result is thus quasi-optimal. For a recent overview of challenges and open problems in the well-posed case, see e.g. [45] and [54].

When going against the characteristics, i.e. $\beta_1 < 0$, we prove in Theorem 6.16 first the pre-asymptotic bound

$$\|u - u_h\|_{L^2(\mathring{\omega}_\beta)} \leq C(|\beta|^{\frac{1}{2}} h |u|_{H^2(\Omega)} + h^{-1} \|\delta u\|_{L^2(\omega)}), \text{ when } 1 \lesssim \mathrm{Pe}(h) < h^{-1},$$

followed by

$$\|u - u_h\|_{L^2(\mathring{\omega}_\beta)} \leq C(|\beta|^{\frac{1}{2}} h^{\frac{3}{2}} |u|_{H^2(\Omega)} + h^{-\frac{1}{2}} \|\delta u\|_{L^2(\omega)}), \text{ when } \mathrm{Pe}(h) > h^{-1}.$$

It follows that when solving the unique continuation problem against the flow, the diffusivity reduces the convergence order in an intermediate regime. Only for very small diffusion coefficients $\mu < |\beta| h^2$ do we get quasi-optimal bounds. This asymmetry of the error distribution for moderate Péclet numbers is clearly visible in the left plot of Figure 6.2.

6.1 Discretisation

We use the notation from Section 3.2 and recall it briefly. Let $V_h \subset H^1(\Omega)$ be the conforming finite element space of piecewise affine $\mathbb{P}_1$ functions defined on a computational mesh $\mathcal{T}_h$ that consists of shape-regular triangular elements K with diameter h_K. The mesh size h is the maximum over h_K and we will assume that $h < 1$. The interior faces of all the elements are collected in the set $\mathcal{F}_i$ and the jump of a quantity across such a face F is denoted by $[\![\cdot]\!]_F$. We denote by n the unit normal.

We recall the standard inner products with the induced norms

$$(v_h, w_h)_\Xi := \int_\Xi v_h w_h \, \mathrm{d}x, \quad \langle v_h, w_h \rangle_{\partial\Xi} := \int_{\partial\Xi} v_h w_h \, \mathrm{d}s,$$

and introduce the bilinear form in the weak formulation of (6.1)

$$a_h(v_h, w_h) := (\beta \cdot \nabla v_h, w_h)_\Omega + (\mu \nabla v_h, \nabla w_h)_\Omega - \langle \mu \nabla v_h \cdot n, w_h \rangle_{\partial\Omega}.$$

Following Chapters 3 and 5, we will use for stabilisation the continuous interior penalty

$$\mathcal{J}_h(v_h, w_h) := \gamma \sum_{F \in \mathcal{F}_i} \int_F h(\mu + |\beta| h) [\![\nabla v_h \cdot n]\!]_F \cdot [\![\nabla w_h \cdot n]\!]_F \, \mathrm{d}s,$$

which acts on the discrete solution penalising the jumps of its normal gradient across interior faces, and

$$s^*(v_h, w_h) := \gamma^* \Big(\big\langle (|\beta| + \mu h^{-1}) v_h, w_h \big\rangle_{\partial\Omega} + (\mu \nabla v_h, \nabla w_h)_\Omega + \mathcal{J}_h(v_h, w_h) \Big),$$

where γ and γ^* are positive constants that can be heuristically chosen at implementation. As discussed before, they do not play a role in the convergence of the method and most of the time we will include them in the generic constant C. For the data term we consider the scaled inner product in the data set ω given by

$$s_\omega(v_h, w_h) := ((|\beta| h^{-1} + \mu h^{-\zeta}) v_h, w_h)_\omega, \quad \zeta \in [0, 2].$$

To this we add the stabilising interior penalty term $\mathcal{J}_h$ to define

$$s(v_h, w_h) := \mathcal{J}_h(v_h, w_h) + s_\omega(v_h, w_h),$$

As discussed in Chapter 3, the idea behind the computational method is to first discretise and then formulate the unique continuation problem as a PDE-constrained optimisation problem with additional stabilising terms. For an approximation $u_h \in V_h$ and a discrete Lagrange multiplier $z_h \in V_h$, we consider the Lagrangian functional

$$L_h(u_h, z_h) := \tfrac{1}{2} s_\omega(u_h - \tilde{u}_\omega, u_h - \tilde{u}_\omega) + a_h(u_h, z_h) - (f, z_h)_\Omega \\ + \tfrac{1}{2} \mathcal{J}_h(u_h, u_h) - \tfrac{1}{2} s^*(z_h, z_h),$$

where the first term is measuring the misfit between u_h and the known perturbed restriction $\tilde{u}_\omega = u|_\omega + \delta u$, the next two terms are imposing the weak form of the PDE (6.1) as a constraint, and the last two terms have stabilising role and act only on the discrete level.

We look for the saddle points of the Lagrangian L_h and analyse their convergence to the exact solution. Using the optimality conditions we obtain the finite element method for unique continuation subject to (6.1), which reads as follows: find $(u_h, z_h) \in [V_h]^2$ such that

$$\left\{ \begin{array}{rcl} a_h(u_h, w_h) - s^*(z_h, w_h) & = & (f, w_h)_\Omega \\ a_h(v_h, z_h) + s(u_h, v_h) & = & s_\omega(\tilde{u}_\omega, v_h) \end{array} \right. , \quad \forall (v_h, w_h) \in [V_h]^2. \tag{6.4}$$

Notice that the exact solution $u \in H^2(\Omega)$ (with noise $\delta u \equiv 0$) and the dual variable $z \equiv 0$ satisfy (6.4) since the gradient of the exact solution has no jumps across interior faces.

Remark 6.1. *The same finite element method* (6.4) *has been proposed in Chapter 5 for the diffusion-dominated case;* $\mathcal{J}_h$ *and* s^* *are exactly the stabilising terms introduced there. However, herein we have increased the penalty coefficient in the data term* s_ω *from* $|\beta| h + \mu$ *to* $|\beta| h^{-1} + \mu h^{-\zeta}$*. We note that the bounds in Chapter 5 hold also for this stronger penalty term, but the sensitivity to perturbations in data increases by a factor of* h^{-1}*.*

Proposition 6.2. *The finite element method* (6.4) *has a unique solution* $(u_h, z_h) \in [V_h]^2$ *and the Euclidean condition number* $\mathcal{K}_2$ *of the system matrix satisfies*

$$\mathcal{K}_2 \leq C h^{-4}.$$

Proof. The proof given in Proposition 5.1 holds verbatim. □

6.1.1 Stability region and weight functions

We will exploit the convective term of the PDE to obtain stability in the zone that connects through characteristics to the data region ω. The objective is to obtain weighted L^2-estimates in this region that are independent of μ (but not of the regularity of the exact solution) with the underlying assumption that $\mu \ll |\beta|$. To this end we first define the

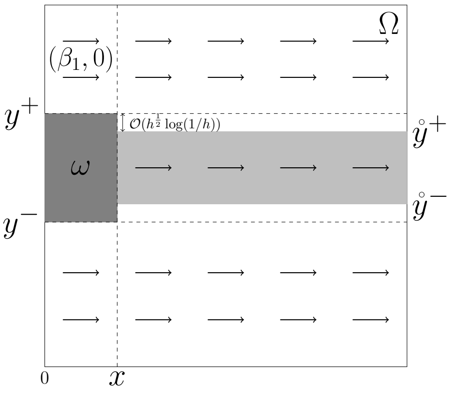


Figure 6.3: Data set ω (grey) and the stability region $\mathring{\omega}_\beta$ (light grey).

subdomain where we can obtain stability (see Figure 6.3 for a sketch) and some weight functions that will be used to define weighted norms. These can be given in explicit form in the simple framework where $\beta = (\beta_1, 0)$ and

$$\omega := (0, x) \times (y^-, y^+), \text{ with } x > h \text{ and } y^+ - y^- > h.$$

Let ω_β denote the closed set of all the points $p \in \bar{\Omega}$ that can be reached through characteristics from ω, i.e. for which there exists $s \in \mathbb{R}$ such that $p + s\beta \in \partial\omega$. Similarly to the classical work [46], we define the stability region $\mathring{\omega}_\beta$ by cutting off a crosswind layer from ω_β, namely

$$\mathring{\omega}_\beta := \left\{ p \in \omega_\beta : \operatorname{dist}(p, \Omega \setminus \omega_\beta) \geq c_\lambda h^{\frac{1}{2}} \ln(1/h) \right\}, \tag{6.5}$$

with the constant c_λ to be made precise in the following. In our setting, we simply have that $\mathring{\omega}_\beta = [0, 1] \times [\mathring{y}^-, \mathring{y}^+]$ for some $\mathring{y}^+ > \mathring{y}^- > 0$.

We will consider different weight functions depending on the direction of the convection field. In the downstream case we let

$$\psi_1(x, y) := e^{-x}, \text{ when } \beta_1 > 0,$$

and in the upstream case

$$\psi_1(x, y) := -e^{-x}, \text{ when } \beta_1 < 0.$$

In both cases we have that $\nabla\psi_1 = (-\psi_1, 0)$. Let then $\psi_2 \in W^{1,\infty}(\Omega)$ be a function satisfying

$$\psi_2 = \begin{cases} 1 & \text{in } \mathring{\omega}_\beta, \\ \mathcal{O}(h^3) & \text{in } \Omega \setminus \omega_\beta, \end{cases} \qquad \beta \cdot \nabla\psi_2 = 0, \quad |\nabla\psi_2| \leq Ch^{-\frac{1}{2}}. \tag{6.6}$$

Such a function can be obtained by taking a positive constant λ that will be specified later and letting

$$\psi_2(x, y) := \begin{cases} \exp\left(\frac{\mathring{y}^+ - y}{\lambda h^{\frac{1}{2}}}\right), & y > \mathring{y}^+ \\ 1, & (x, y) \in \mathring{\omega}_\beta \\ \exp\left(\frac{y - \mathring{y}^-}{\lambda h^{\frac{1}{2}}}\right), & y < \mathring{y}^-. \end{cases}$$

Note that ψ_2 is only piecewise continuously differentiable. For ψ_2 to decrease sufficiently rapidly to $\mathcal{O}(h^3)$ outside of ω_β, we can take

$$\text{dist}(\mathring{\omega}_\beta, \Omega \setminus \omega_\beta) = \min(y^+ - \mathring{y}^+, \mathring{y}^- - y^-) \geq 3\lambda h^{\frac{1}{2}} \ln(\tfrac{1}{h}),$$

which corresponds to $c_\lambda = 3\lambda$ in the definition of $\mathring{\omega}_\beta$ given in (6.5). We thus have that

$$|\nabla\psi_2| \leq \lambda^{-1} h^{-\frac{1}{2}},$$

and in the subsequent proofs the constant λ will be taken large enough.

We now define the weight function $\varphi \in W^{1,\infty}(\Omega)$ that will be used in the weighted norms. For the downstream case we take in Section 6.2.1

$$\varphi := \psi_1\psi_2 \in (0, 1), \text{ when } \beta_1 > 0, \tag{6.7}$$

and for the upstream case in Section 6.2.2,

$$\varphi := \psi_1\psi_2 \in (-1, 0), \text{ when } \beta_1 < 0. \tag{6.8}$$

Using the product rule and the fact that $\beta \cdot \nabla\psi_2 = 0$, it follows that in both cases we have

$$\beta \cdot \nabla\varphi = -|\beta||\varphi|, \tag{6.9}$$

and

$$|\nabla\varphi| \leq (1 + \lambda^{-1} h^{-\frac{1}{2}})|\varphi|. \tag{6.10}$$

We will denote the inflow and outflow boundaries by $\partial\Omega^-$ and $\partial\Omega^+$, i.e. $\beta \cdot n < 0$ on $\partial\Omega^-$ and $\beta \cdot n > 0$ on $\partial\Omega^+$. We will also assume that $\beta \cdot n = 0$ can only hold on the boundary of $\Omega \setminus \omega_\beta$, and that $\mu \leq \mathrm{Pe}_{\lim}^{-1}|\beta \cdot n|h$ when $\beta \cdot n \neq 0$. This is straightforward to verify in the model case of the unit square that we are considering.

We recall for convenience several inequalities that will be used repeatedly: the standard discrete inverse inequality (A.1)

$$\|\nabla v_h\|_{L^2(K)} \leq Ch^{-1}\|v_h\|_{L^2(K)}, \quad \forall v_h \in \mathbb{P}_1(K), \tag{6.11}$$

the continuous trace inequality (A.2)

$$\|v\|_{L^2(\partial K)} \leq C(h^{-\frac{1}{2}}\|v\|_{L^2(K)} + h^{\frac{1}{2}}\|\nabla v\|_{L^2(K)}), \quad \forall v \in H^1(K), \tag{6.12}$$

and the discrete trace inequality (A.3)

$$\|\nabla v_h \cdot n\|_{L^2(\partial K)} \leq Ch^{-\frac{1}{2}}\|\nabla v_h\|_{L^2(K)}, \quad \forall v_h \in \mathbb{P}_1(K). \tag{6.13}$$

As in Chapter 5, we will use the Poincaré-type inequality Lemma 3.2 which now reads as

$$\|(\mu^{\frac{1}{2}}h + |\beta|^{\frac{1}{2}}h^{\frac{3}{2}})v_h\|_{H^1(\Omega)} \leq C\gamma^{-\frac{1}{2}}s(v_h, v_h)^{\frac{1}{2}}, \quad \forall v_h \in V_h, \tag{6.14}$$

and Lemma 3.3 that gives

$$\mathcal{J}_h(\pi_h u, \pi_h u)^{\frac{1}{2}} \leq C\gamma^{\frac{1}{2}}(\mu^{\frac{1}{2}}h + |\beta|^{\frac{1}{2}}h^{\frac{3}{2}})|u|_{H^2(\Omega)}, \quad \forall u \in H^2(\Omega). \tag{6.15}$$

We also recall that for a Lipschitz domain K – and hence for any element $K \in \mathcal{T}_h$ – a function φ is Lipschitz continuous if and only if $\varphi \in W^{1,\infty}(K)$. This follows from the proof in [34, Theorem 4, p. 294] where the extension operator in the third step of the proof is replaced by the extension operator in [57, Theorem 5, p. 181]. This equivalence holds for more general domains satisfying the minimal smoothness property in [57, p. 189]. The proof in [34, Theorem 4, p. 294] also shows that the mean value theorem holds and for any $x, y \in K$,

$$|\varphi(x) - \varphi(y)| \leq C_{ex}h_K|\varphi|_{W^{1,\infty}(K)}, \tag{6.16}$$

where the constant $C_{ex} > 0$ is the norm of the extension operator used. We adopt the notation $\|\varphi\|_{\infty,K} := \|\varphi\|_{L^\infty(K)}$ and $|\varphi|_{W^{1,\infty}(K)} := \|\nabla\varphi\|_{L^\infty(K)}$.

Lemma 6.3. *For all $v_h \in V_h$ and $K \in \mathcal{T}_h$, the following inequalities hold*

$$\|\varphi\|_{\infty,K}\|v_h\|_K \le C\|v_h\varphi\|_K, \tag{6.17}$$

$$\|v_h\varphi\|_{\partial K} \le Ch^{-\frac{1}{2}}\|v_h\varphi\|_K, \tag{6.18}$$

assuming that $(h + \lambda^{-1}h^{\frac{1}{2}})$ is small enough.

Proof. Let $x^* \in K$ be such that $|\varphi(x^*)|= \|\varphi\|_{\infty,K}$. Using the triangle inequality we may write

$$\|\varphi\|_{\infty,K}\|v_h\|_K\le \|\varphi v_h\|_K+\|(\varphi(x^*) - \varphi)v_h\|_K.$$

By the mean value theorem (6.16) we have that

$$|\varphi(x^*) - \varphi|\le C_{ex}h|\varphi|_{W^{1,\infty}(K)},$$

and by (6.10) together with the assumption that $C_{ex}(h + \lambda^{-1}h^{\frac{1}{2}}) < \frac{1}{2}$ we get

$$C_{ex}h|\varphi|_{W^{1,\infty}(K)}\le C_{ex}(h + \lambda^{-1}h^{\frac{1}{2}})\|\varphi\|_{\infty,K}\le \frac{1}{2}\|\varphi\|_{\infty,K}.$$

It follows that

$$\|\varphi\|_{\infty,K}\|v_h\|_K\le \|\varphi v_h\|_K+\tfrac{1}{2}\|\varphi\|_{\infty,K}\|v_h\|_K,$$

from which the claim (6.17) is immediate. Considering now (6.18), using the standard element-wise trace inequality (6.12) we have

$$\|h^{\frac{1}{2}}v_h\varphi\|_{\partial K}\le C(\|v_h\varphi\|_K+h\|\nabla(v_h\varphi)\|_K).$$

We bound the gradient term using (6.10) and the inverse inequality (6.11),

$$\begin{aligned} h\|\nabla(v_h\varphi)\|_K &\le h\|v_h\nabla\varphi\|_K+h\|\varphi\nabla v_h\|_K \\ &\le (h + \lambda^{-1}h^{\frac{1}{2}})\|\varphi\|_{\infty,K}\|v_h\|_K+C\|\varphi\|_{\infty,K}\|v_h\|_K. \end{aligned}$$

We conclude by collecting the terms and using (6.17). □

6.1.2 Discrete commutator property

We denote by i_h the Lagrange nodal interpolant. We herein consider a Lipschitz weight function and prove the following super approximation result, also known as the discrete commutator property. This result will be essential to derive local weighted estimates and is similar to the one proven in [8] for smooth compactly supported weight functions.

Lemma 6.4. *Let $v_h \in V_h$ and $K \in \mathcal{T}_h$. Then for any weight function $\varphi \in W^{1,\infty}(K)$, there holds*

$$\|v_h\varphi - i_h(v_h\varphi)\|_K + h\|\nabla(v_h\varphi - i_h(v_h\varphi))\|_K \leq Ch|\varphi|_{W^{1,\infty}(K)}\|v_h\|_K.$$

Proof. We will first show the L^2-norm estimate

$$\|v_h\varphi - i_h(v_h\varphi)\|_K \leq Ch|\varphi|_{W^{1,\infty}(K)}\|v_h\|_K.$$

Let $x^* \in K$ be such that $|\varphi(x^*)| = \|\varphi\|_{\infty,K}$ and let $R_\varphi = \varphi - \varphi(x^*)$. Note that

$$\|(1 - i_h)(v_h\varphi)\|_K = \|(1 - i_h)(v_h R_\varphi)\|_K.$$

Observe that $i_h(v_h\varphi) = i_h(v_h i_h\varphi)$ and therefore

$$\|(1 - i_h)(v_h R_\varphi)\|_K = \|v_h R_\varphi - i_h(v_h i_h R_\varphi)\|_K.$$

By the triangle inequality

$$\|i_h(v_h i_h R_\varphi) - v_h R_\varphi\|_K \leq \|i_h(v_h i_h R_\varphi) - v_h i_h R_\varphi\|_K + \|v_h(i_h R_\varphi - R_\varphi)\|_K.$$

For the first term, since $v_h i_h R_\varphi \in H^1(K)$ we have by standard interpolation that

$$\|i_h(v_h i_h R_\varphi) - v_h i_h R_\varphi\|_K \leq Ch\|\nabla(v_h i_h R_\varphi)\|_K$$

and then

$$\|\nabla(v_h i_h R_\varphi)\|_K \leq |i_h R_\varphi|_{W^{1,\infty}(K)}\|v_h\|_K + \|i_h R_\varphi\|_{\infty,K}\|\nabla v_h\|_K.$$

By inserting $\nabla\varphi$ and φ, respectively, and using the interpolation estimate (A.4) and the mean value theorem (6.16) we have the following approximation

$$h|i_h R_\varphi|_{W^{1,\infty}(K)} + \|i_h R_\varphi\|_{\infty,K} \leq Ch|\varphi|_{W^{1,\infty}(K)}.$$

Combined with the previous estimate and the inverse inequality (6.11) this gives that

$$\|\nabla(v_h i_h R_\varphi)\|_K \leq C|\varphi|_{W^{1,\infty}(K)}\|v_h\|_K. \tag{6.19}$$

For the second term, using again interpolation (A.4) we have

$$\|v_h(i_hR_\varphi - R_\varphi)\|_K \leq \|i_hR_\varphi - R_\varphi\|_{\infty,K}\|v_h\|_K \leq Ch|\varphi|_{W^{1,\infty}(K)}\|v_h\|_K,$$

and thus we have shown that

$$\|v_h\varphi - i_h(v_h\varphi)\|_K \leq Ch|\varphi|_{W^{1,\infty}(K)}\|v_h\|_K.$$

The approximation estimate for the gradient follows by the same arguments. Indeed,

$$\begin{aligned}\|\nabla(1-i_h)(v_h\varphi)\|_K &= \|\nabla(1-i_h)(v_hR_\varphi)\|_K = \|\nabla(v_hR_\varphi) - \nabla i_h(v_h i_hR_\varphi)\|_K \\ &\leq \|\nabla(v_hR_\varphi) - \nabla(v_h i_hR_\varphi)\|_K + \|\nabla(v_h i_hR_\varphi) - \nabla i_h(v_h i_hR_\varphi)\|_K.\end{aligned}$$

We first use interpolation and the inverse inequality (6.11) to get

$$\begin{aligned}\|\nabla(v_h(R_\varphi - i_hR_\varphi))\|_K &\leq \|v_h\nabla(R_\varphi - i_hR_\varphi)\|_K + \|(R_\varphi - i_hR_\varphi)\nabla v_h\|_K \\ &\leq C|\varphi|_{W^{1,\infty}(K)}\|v_h\|_K.\end{aligned}$$

Then we use an inverse inequality followed by interpolation and (6.19) to obtain

$$\|\nabla(v_h i_hR_\varphi) - \nabla i_h(v_h i_hR_\varphi)\|_K \leq C\|\nabla(v_h i_hR_\varphi)\|_K \leq C|\varphi|_{W^{1,\infty}(K)}\|v_h\|_K.$$

□

6.2 Error estimates

We recall the following consistency result that holds exactly as in the diffusion-dominated case, see Lemma 5.2.

Lemma 6.5 (Consistency). *Assume that $u \in H^2(\Omega)$ is a solution to (6.1) and let $(u_h, z_h) \in [V_h]^2$ be the solution to (6.4), then*

$$a_h(\pi_h u - u_h, w_h) + s^*(z_h, w_h) = a_h(\pi_h u - u, w_h),$$

and

$$-a_h(v_h, z_h) + s(\pi_h u - u_h, v_h) = s_\Omega(\pi_h u - u, v_h) + s_\omega(\pi_h u - \tilde{u}_\omega, v_h),$$

for all $(v_h, w_h) \in [V_h]^2$.

We now introduce the stabilisation norm on $[V_h]^2$ by combining the primal and dual stabilisers

$$\|(v_h, w_h)\|_s^2 := s(v_h, v_h) + s^*(w_h, w_h),$$

and the "continuity norm" defined on $H^{\frac{3}{2}+\varepsilon}(\Omega)$, for any $\varepsilon > 0$,

$$\|v\|_\sharp := \||\beta|^{\frac{1}{2}} h^{-\frac{1}{2}} v\|_\Omega + \||\beta|^{\frac{1}{2}} h^{\frac{1}{2}} \nabla v\|_\Omega + \|h^{\frac{1}{2}} \mu^{\frac{1}{2}} \nabla v \cdot n\|_{\partial\Omega}.$$

From the jump inequality (6.15), standard approximation bounds for π_h and the trace inequality (6.12), it follows that for $u \in H^2(\Omega)$

$$\|(u - \pi_h u, 0)\|_s + \|u - \pi_h u\|_\sharp \leq C(\mu^{\frac{1}{2}} h + |\beta|^{\frac{1}{2}} h^{\frac{3}{2}})|u|_{H^2(\Omega)}. \tag{6.20}$$

In the high Péclet regime, $\mu^{\frac{1}{2}} h \leq |\beta|^{\frac{1}{2}} h^{\frac{3}{2}}$ and the second term in the right-hand side dominates. We also define the orthogonal space

$$V_h^\perp := \{v \in H^2(\Omega) : (v, w_h)_\Omega = 0, \quad \forall w_h \in V_h\}.$$

Lemma 6.6 (Continuity). *Let $v \in V_h^\perp$ and $w_h \in V_h$, then*

$$a_h(v, w_h) \leq C\|v\|_\sharp \|(0, w_h)\|_s.$$

Proof. Integrating by parts in the convective term of a_h and using $\nabla \cdot \beta = 0$ leads to

$$a_h(v, w_h) = -(v, \beta \cdot \nabla w_h)_\Omega + \langle v\beta \cdot n, w_h\rangle_{\partial\Omega} + (\mu \nabla v, \nabla w_h)_\Omega - \langle \mu \nabla v \cdot n, w_h\rangle_{\partial\Omega}.$$

For the first term we recall the discrete approximation estimate that holds for any piecewise linear β, see e.g. [13, Theorem 2.2],

$$\inf_{x_h \in V_h} \|h^{\frac{1}{2}}(\beta \cdot \nabla w_h - x_h)\|_\Omega \leq C \left(\sum_{F \in \mathcal{F}_i} \|h [\![\beta \cdot \nabla w_h]\!]\|_F^2 \right)^{\frac{1}{2}} \leq C|\beta|^{\frac{1}{2}} \gamma^{-\frac{1}{2}} \mathcal{J}_h(w_h, w_h)^{\frac{1}{2}} \tag{6.21}$$

and use orthogonality to obtain

$$-(v, \beta \cdot \nabla w_h)_\Omega \leq \|h^{-\frac{1}{2}} v\|_\Omega \inf_{x_h \in V_h} \|h^{\frac{1}{2}}(\beta \cdot \nabla w_h - x_h)\|_\Omega \leq C\|v\|_\sharp \|(0, w_h)\|_s.$$

For the remaining terms, applying the Cauchy-Schwarz inequality we see that

$$\langle v\beta \cdot n, w_h\rangle_{\partial\Omega} + (\mu\nabla v, \nabla w_h)_\Omega - \langle \mu\nabla v \cdot n, w_h\rangle_{\partial\Omega} \leq C\|v\|_\sharp \|(0, w_h)\|_s.$$

□

The above continuity estimate holds for any divergence-free piecewise linear velocity field β. For a general velocity field $\beta \in W^{1,\infty}(\Omega)$ we can use a similar argument by considering its piecewise linear approximation as in Lemma 5.5. The continuity constant would be proportional to $\mathrm{Pe}(h)^{\frac{1}{2}}|\beta|_{1,\infty}/\|\beta\|_\infty$. Assuming that β is divergence-free, the constant becomes $h\,\mathrm{Pe}(h)^{\frac{1}{2}}|\beta|_{1,\infty}/\|\beta\|_\infty$.

We now prove optimal convergence for the stabilising and data terms.

Proposition 6.7. *(Convergence of regularisation). Assume that $u \in H^2(\Omega)$ is a solution to (6.1) and let $(u_h, z_h) \in [V_h]^2$ be the solution to (6.4), then there holds*

$$\|(\pi_h u - u_h, z_h)\|_s \leq C(\mu^{\frac{1}{2}} h + |\beta|^{\frac{1}{2}} h^{\frac{3}{2}})(|u|_{H^2(\Omega)} + h^{-2}\|\delta u\|_\omega).$$

Proof. Denoting $e_h = \pi_h u - u_h$ we have that

$$\|(e_h, z_h)\|_s^2 = a_h(e_h, z_h) + s^*(z_h, z_h) - a_h(e_h, z_h) + s(e_h, e_h).$$

Using both claims in Lemma 6.5 we may write

$$\|(e_h, z_h)\|_s^2 = a_h(\pi_h u - u, z_h) + \mathcal{J}_h(\pi_h u - u, e_h) + s_\omega(\pi_h u - \tilde{u}_\omega, e_h).$$

Since $\pi_h u - u \in V_h^\perp$ we have by Lemma 6.6 that

$$a_h(\pi_h u - u, z_h) \leq C\|\pi_h u - u\|_\sharp \|(0, z_h)\|_s.$$

We bound the other terms using the Cauchy-Schwarz inequality

$$\mathcal{J}_h(\pi_h u - u, e_h) + \omega(\pi_h u - \tilde{u}_\omega, e_h) \leq \Big(\|(\pi_h u - u, 0)\|_s + (|\beta| h^{-1} + \mu h^{-\zeta})^{\frac{1}{2}} \|\delta u\|_\omega\Big)\|(e_h, 0)\|_s.$$

Collecting the above bounds we have

$$\|(e_h, z_h)\|_s^2 \leq C\Big(\|\pi_h u - u\|_\sharp + \|(\pi_h u - u, 0)\|_s + (|\beta| h^{-1} + \mu h^{-\zeta})^{\frac{1}{2}} \|\delta u\|_\omega\Big)\|(e_h, z_h)\|_s$$

and the claim follows by applying the approximation inequality (6.20). □

Remark 6.8. *Compared to the result in Proposition 5.4 for the diffusion-dominated, the sensitivity to data perturbations has increased by a factor of h^{-1}. This is due to the stronger penalty in the data term s_ω (c.f. Remark 6.1).*

6.2.1 Downstream estimates

In this case we consider $\beta = (\beta_1, 0)$ with $\beta_1 > 0$ and the data set

$$\omega = (0, x) \times (y^-, y^+)$$

touching part of the inflow boundary $\partial\Omega^-$. We aim to obtain control of the following weighted triple norm defined on V_h,

$$|||v_h|||_\varphi^2 := \||\beta|^{\frac{1}{2}} v_h \varphi^{\frac{1}{2}}\|_\Omega^2 + \|\mu^{\frac{1}{2}} \nabla v_h \varphi^{\frac{1}{2}}\|_\Omega^2 + \||\beta \cdot n|^{\frac{1}{2}} v_h \varphi^{\frac{1}{2}}\|_{\partial\Omega^+}^2, \tag{6.22}$$

where φ is given by (6.7). Since $\varphi \in (0, 1)$, we will often use that $\|\cdot\varphi\|_\Omega \le \|\cdot\varphi^{\frac{1}{2}}\|_\Omega$. We first consider $v_h\varphi$ as a test function in the weak bilinear form a_h and obtain the following bound.

Lemma 6.9. *There exist $\alpha > 0$ and $h_0 > 0$ such that for all $h < h_0$ and $v_h \in V_h$ we have*

$$\alpha |||v_h|||_\varphi^2 \le a_h(v_h, v_h\varphi) + C\|(v_h, 0)\|_s^2.$$

Proof. We start with the convective term. Since $\nabla \cdot \beta = 0$, the divergence theorem leads to

$$2(\beta \cdot \nabla v_h, v_h\varphi)_\Omega = \langle v_h \beta \cdot n, v_h\varphi\rangle_{\partial\Omega} - (v_h, v_h \beta \cdot \nabla\varphi)_\Omega.$$

Then combining with (6.9) we have that

$$(\beta \cdot \nabla v_h, v_h\varphi)_\Omega = \tfrac{1}{2}\left(\langle v_h \beta \cdot n, v_h\varphi\rangle_{\partial\Omega} + |\beta| \|v_h \varphi^{\frac{1}{2}}\|_\Omega^2\right).$$

We split the boundary term into inflow and outflow

$$\langle v_h \beta \cdot n, v_h\varphi\rangle_{\partial\Omega} = -\||\beta \cdot n|^{\frac{1}{2}} v_h \varphi^{\frac{1}{2}}\|_{\partial\Omega^-}^2 + \||\beta \cdot n|^{\frac{1}{2}} v_h \varphi^{\frac{1}{2}}\|_{\partial\Omega^+}^2,$$

and write

$$\frac{1}{2}(\||\beta \cdot n|^{\frac{1}{2}} v_h \varphi^{\frac{1}{2}}\|_{\partial\Omega^+}^2 + |\beta| \|v_h \varphi^{\frac{1}{2}}\|_\Omega^2) = (\beta \cdot \nabla v_h, v_h\varphi)_\Omega + \frac{1}{2}\||\beta \cdot n|^{\frac{1}{2}} v_h \varphi^{\frac{1}{2}}\|_{\partial\Omega^-}^2.$$

Splitting now the inflow boundary with respect to the closed set ω_β and using the discrete trace inequality (6.12) in ω, and the weight decay (6.6) together with a standard global trace inequality for H^1 functions outside, we have that

$$\begin{aligned} \| |\beta \cdot n|^{\frac{1}{2}} v_h \varphi^{\frac{1}{2}} \|_{\partial\Omega^-} &\leq C|\beta|^{\frac{1}{2}} (\|v_h \varphi^{\frac{1}{2}}\|_{\partial\Omega^- \cap \omega_\beta} + \|v_h \varphi^{\frac{1}{2}}\|_{\partial\Omega^- \setminus \omega_\beta}) \\ &\leq C|\beta|^{\frac{1}{2}} h^{-\frac{1}{2}} \|v_h\|_\omega + C|\beta|^{\frac{1}{2}} h^{\frac{3}{2}} \|v_h\|_{H^1(\Omega)} \\ &\leq C\gamma^{-\frac{1}{2}} \|(v_h, 0)\|_s, \end{aligned} \tag{6.23}$$

where in the last step we used the Poincaré-type inequality (6.14). Hence we obtain control over the convective terms in the triple weighted norm

$$\tfrac{1}{2}(\| |\beta \cdot n|^{\frac{1}{2}} v_h \varphi^{\frac{1}{2}} \|^2_{\partial\Omega^+} + |\beta| \|v_h \varphi^{\frac{1}{2}}\|^2_\Omega) \leq (\beta \cdot \nabla v_h, v_h \varphi)_\Omega + C\gamma^{-1} \|(v_h, 0)\|^2_s. \tag{6.24}$$

Let us consider the terms in $a_h(v_h, v_h\varphi)$ corresponding to the diffusion operator, starting with

$$(\mu \nabla v_h, \nabla(v_h \varphi))_\Omega = \|\mu^{\frac{1}{2}} \nabla v_h \varphi^{\frac{1}{2}}\|^2_\Omega + (\mu \nabla v_h, v_h \nabla \varphi)_\Omega,$$

which we rearrange as

$$\|\mu^{\frac{1}{2}} \nabla v_h \varphi^{\frac{1}{2}}\|^2_\Omega = (\mu \nabla v_h, \nabla(v_h \varphi))_\Omega - (\mu \nabla v_h, v_h \nabla \varphi)_\Omega.$$

Bounding $\nabla\varphi$ by (6.10) and using Cauchy-Schwarz together with $\mu \leq |\beta| h$ we have that

$$\begin{aligned} |(\mu \nabla v_h, v_h \nabla \varphi)_\Omega| &\leq \mu(|\nabla v_h \cdot \nabla \varphi|, v_h)_\Omega \\ &\leq \mu(1 + \lambda^{-1} h^{-\frac{1}{2}})(|\nabla v_h| \varphi^{\frac{1}{2}}, v_h \varphi^{\frac{1}{2}})_\Omega \\ &\leq \tfrac{1}{3} \|\mu^{\frac{1}{2}} \nabla v_h \varphi^{\frac{1}{2}}\|^2_\Omega + C(h + \lambda^{-2}) |\beta| \|v_h \varphi^{\frac{1}{2}}\|^2_\Omega. \end{aligned}$$

We split the boundary term $\langle \mu \nabla v_h \cdot n, v_h \varphi \rangle_{\partial\Omega}$ into inflow and outflow again. Similarly to (6.23) we have that

$$\langle \mu \nabla v_h \cdot n, v_h \varphi \rangle_{\partial\Omega^-} \leq Ch\gamma^{-1} \|(v_h, 0)\|^2_s.$$

For the outflow boundary term we use Cauchy-Schwarz and a trace inequality to obtain

$$\begin{aligned} \langle \mu \nabla v_h \cdot n, v_h \varphi \rangle_{\partial\Omega^+} &\leq \|\mu^{\frac{1}{2}} \nabla v_h \cdot n \varphi^{\frac{1}{2}}\|_{\partial\Omega^+} \|\mu^{\frac{1}{2}} v_h \varphi^{\frac{1}{2}}\|_{\partial\Omega^+} \\ &\leq Ch^{-\frac{1}{2}} \|\mu^{\frac{1}{2}} \nabla v_h \varphi^{\frac{1}{2}}\|_\Omega \mathrm{Pe}_{\lim}^{-\frac{1}{2}} h^{\frac{1}{2}} \| |\beta \cdot n|^{\frac{1}{2}} v_h \varphi^{\frac{1}{2}} \|_{\partial\Omega^+} \\ &\leq \tfrac{1}{3} \|\mu^{\frac{1}{2}} \nabla v_h \varphi^{\frac{1}{2}}\|^2_\Omega + \mathrm{Pe}_{\lim}^{-1} \| |\beta \cdot n|^{\frac{1}{2}} v_h \varphi^{\frac{1}{2}} \|^2_{\partial\Omega^+}. \end{aligned}$$

We denote the part of the boundary where $\beta \cdot n = 0$ by $\partial\Omega^0$ and use the assumption that

$\partial\Omega^0$ is away from ω_β, meaning that the weight function φ is $\mathcal{O}(h^3)$ there. Together with the trace inequalities (5.9) and (5.8), the Poincaré-type inequality (6.14) gives that

$$\langle \mu\nabla v_h \cdot n, v_h\varphi\rangle_{\partial\Omega^0} \leq C\mu h^2 \|\nabla v_h\|_\Omega \|v_h\|_\Omega \leq C\gamma^{-1}\|(v_h,0)\|_s^2.$$

Collecting the above bounds we obtain that

$$\begin{aligned}\tfrac{1}{3}\|\mu^{\frac{1}{2}}\nabla v_h\varphi^{\frac{1}{2}}\|_\Omega^2 \leq{} & (\mu\nabla v_h, \nabla(v_h\varphi))_\Omega - \langle \mu\nabla v_h\cdot n, v_h\varphi\rangle_{\partial\Omega} \\ & + C(h+\lambda^{-2}+\mathrm{Pe}_{\mathrm{lim}}^{-1})(\||\beta|^{\frac{1}{2}}v_h\varphi^{\frac{1}{2}}\|_\Omega^2 + \||\beta\cdot n|^{\frac{1}{2}}v_h\varphi^{\frac{1}{2}}\|_{\partial\Omega^+}^2) + C\gamma^{-1}\|(v_h,0)\|_s^2.\end{aligned}$$

We conclude by combining this with (6.24) and assuming that h is small enough, and λ and $\mathrm{Pe}_{\mathrm{lim}}$ are large enough (thus absorbing the convective terms from the rhs into the lhs). □

Now we refine the control over the triple norm $|||v_h|||_\varphi$ by taking the projection $\pi_h(v_h\varphi) \in V_h$ as a test function.

Corollary 6.10. *There exist $\alpha > 0$ and $h_0 > 0$ such that for all $h < h_0$ and $v_h \in V_h$ we have*

$$\alpha|||v_h|||_\varphi^2 \leq a_h(v_h, \pi_h(v_h\varphi)) + C\|(v_h,0)\|_s^2.$$

Proof. Since

$$a_h(v_h, \pi_h(v_h\varphi)) = a_h(v_h, (\pi_h - 1)(v_h\varphi)) + a_h(v_h, v_h\varphi),$$

we must bound $a_h(v_h, (\pi_h - 1)(v_h\varphi))$ in a suitable way. Writing out the terms we have

$$\begin{aligned}a_h(v_h, (\pi_h - 1)(v_h\varphi)) ={} & (\beta\cdot\nabla v_h, (\pi_h-1)(v_h\varphi))_\Omega + (\mu\nabla v_h, \nabla(\pi_h-1)(v_h\varphi))_\Omega \\ & - \langle \mu\nabla v_h\cdot n, (\pi_h-1)(v_h\varphi)\rangle_{\partial\Omega} = I + II + III.\end{aligned}$$

Let us consider the convection term first, and use orthogonality combined with (6.21)

$$\begin{aligned}I = (\beta\cdot\nabla v_h, (\pi_h-1)(v_h\varphi))_\Omega &\leq C|\beta|^{\frac{1}{2}}\gamma^{-\frac{1}{2}}\|(v_h,0)\|_s h^{-\frac{1}{2}}\|(\pi_h-1)(v_h\varphi)\|_\Omega \\ &\leq C|\beta|^{\frac{1}{2}}\gamma^{-\frac{1}{2}}\|(v_h,0)\|_s h^{-\frac{1}{2}}\|(i_h-1)(v_h\varphi)\|_\Omega.\end{aligned}$$

Integrating by parts and using that $\Delta v_h = 0$ on every element K we obtain by the trace inequality (6.12) and the assumption $\mathrm{Pe}(h) > 1$ that

$$\begin{aligned} II + III &= \sum_{F \in \mathcal{F}_i} \int_F \mu [\![\nabla v_h \cdot n]\!] (\pi_h - 1)(v_h \varphi) \, \mathrm{d}s \\ &\leq C |\beta|^{\frac{1}{2}} \gamma^{-\frac{1}{2}} \mathcal{J}_h(v_h, v_h)^{\frac{1}{2}} (h^{-\frac{1}{2}} \|(\pi_h - 1)(v_h\varphi)\|_\Omega + h^{\frac{1}{2}} \|\nabla(\pi_h - 1)(v_h \varphi)\|_\Omega). \end{aligned}$$

Notice that $i_h(v_h\varphi) = \pi_h(i_h(v_h\varphi))$ and the stability of the projection gives

$$\|\nabla(\pi_h - i_h)(v_h\varphi)\|_\Omega = \|\nabla \pi_h (1 - i_h)(v_h\varphi)\|_\Omega \leq C \|\nabla(1 - i_h)(v_h\varphi)\|_\Omega, \tag{6.25}$$

and hence

$$\begin{aligned} h^{\frac{1}{2}} \|\nabla(\pi_h - 1)(v_h\varphi)\|_\Omega &\leq h^{\frac{1}{2}} (\|\nabla(\pi_h - i_h)(v_h\varphi)\|_\Omega + \|\nabla(i_h - 1)(v_h\varphi)\|_\Omega) \\ &\leq C h^{\frac{1}{2}} \|\nabla(i_h - 1)(v_h\varphi)\|_\Omega. \end{aligned} \tag{6.26}$$

Since

$$h^{-\frac{1}{2}} \|(\pi_h - 1)(v_h\varphi)\|_\Omega \leq C h^{-\frac{1}{2}} \|(i_h - 1)(v_h\varphi)\|_\Omega,$$

collecting the contributions above we see that

$$I + II + III \leq C |\beta|^{\frac{1}{2}} \gamma^{-\frac{1}{2}} \|(v_h, 0)\|_s \underbrace{\Big(h^{-\frac{1}{2}} \|(i_h - 1)(v_h\varphi)\|_\Omega + h^{\frac{1}{2}} \|\nabla(i_h - 1)(v_h\varphi)\|_\Omega \Big)}_{IV},$$

and hence

$$a_h(v_h, (\pi_h - 1)(v_h\varphi)) = I + II + III \leq C\gamma^{-1} \|(v_h, 0)\|_s^2 + |\beta| (IV)^2.$$

The discrete commutator property Lemma 6.4 together with the φ-bounds (6.10) and (6.17) give that

$$IV \leq C h^{\frac{1}{2}} \|\nabla \varphi\|_{\infty,\Omega} \|v_h\|_\Omega \leq C(h^{\frac{1}{2}} + \lambda^{-1}) \|v_h \varphi\|_\Omega. \tag{6.27}$$

Since $\varphi \in (0, 1)$ and $\varphi < \varphi^{\frac{1}{2}}$, it follows that for h small enough and λ large enough, given some $\alpha > 0$ we have

$$|\beta| IV^2 \leq \tfrac{\alpha}{2} |\!|\!| v_h |\!|\!|_\varphi^2. \tag{6.28}$$

Collecting the estimates for $a_h(v_h, (\pi_h - 1)(v_h\varphi))$ and using Lemma 6.9 we see that

$$a_h(v_h, \pi_h(v_h\varphi)) = a_h(v_h, (\pi_h - 1)(v_h\varphi)) + a_h(v_h, v_h\varphi) \geq \tfrac{\alpha}{2} |\!|\!| v_h |\!|\!|_\varphi^2 - C\gamma^{-1} \|(v_h, 0)\|_s^2,$$

from which we conclude by renaming $\alpha/2$ as α. □

Lemma 6.11. *For all $v_h \in V_h$ there holds*

$$\|(0, \pi_h(v_h\varphi))\|_s^2 \leq C(|||v_h|||_\varphi^2 + \|(v_h, 0)\|_s^2).$$

Proof. First note that by triangle inequalities we have that up to a constant

$$\begin{aligned}\|(0, \pi_h(v_h\varphi))\|_s \leq\; & \|\mu^{\frac{1}{2}}\nabla(\pi_h - 1)(v_h\varphi))\|_\Omega + \|\mu^{\frac{1}{2}}\nabla(v_h\varphi))\|_\Omega \\ & + (|\beta| + \mu h^{-1})^{\frac{1}{2}}(\|(\pi_h - 1)(v_h\varphi)\|_{\partial\Omega} + \|v_h\varphi\|_{\partial\Omega}) \\ & + \mathcal{J}_h(\pi_h(v_h\varphi), \pi_h(v_h\varphi))^{\frac{1}{2}}.\end{aligned}$$

We bound these terms line by line. Using (6.26), (6.27), (6.10) and (6.17) we bound the first line by

$$|\beta|^{\frac{1}{2}} h^{\frac{1}{2}} \|\nabla(i_h - 1)(v_h\varphi)\|_\Omega + \|\mu^{\frac{1}{2}}\nabla v_h\varphi\|_\Omega + 2|\beta|^{\frac{1}{2}}(h^{\frac{1}{2}} + \lambda^{-1})\|v_h\varphi\|_\Omega \leq C|||v_h|||_\varphi.$$

For the second line, using the trace inequality (6.12) and again the discrete commutator property through the bounds (6.26) and (6.27) we have that

$$(|\beta| + \mu h^{-1})^{\frac{1}{2}} \|(\pi_h - 1)(v_h\varphi)\|_{\partial\Omega} \leq C|\beta|^{\frac{1}{2}} \|v_h\varphi^{\frac{1}{2}}\|_\Omega$$

Splitting the boundary into inflow and outflow, we use the trivial bound $\varphi \leq \varphi^{\frac{1}{2}}$, and by (6.23) for the inflow term we have that

$$\begin{aligned}(|\beta| + \mu h^{-1})^{\frac{1}{2}} \|v_h\varphi\|_{\partial\Omega} &\leq C|\beta|^{\frac{1}{2}} \|v_h\varphi^{\frac{1}{2}}\|_{\partial\Omega^-} + C\||\beta\cdot n|^{\frac{1}{2}} v_h\varphi^{\frac{1}{2}}\|_{\partial\Omega^+} \\ &\leq C\|(v_h, 0)\|_s + C|||v_h|||_\varphi.\end{aligned}$$

For the contribution of the jump term, we insert i_h and bound

$$\begin{aligned}\mathcal{J}_h(\pi_h(v_h\varphi), \pi_h(v_h\varphi))^{\frac{1}{2}} \leq\; & \mathcal{J}_h((\pi_h - i_h)(v_h\varphi), (\pi_h - i_h)(v_h\varphi))^{\frac{1}{2}} \\ & + \mathcal{J}_h((i_h - 1)(v_h\varphi), (i_h - 1)(v_h\varphi))^{\frac{1}{2}} \\ & + \mathcal{J}_h(v_h\varphi, v_h\varphi)^{\frac{1}{2}}.\end{aligned} \tag{6.29}$$

We first observe that using (6.13) and (6.25), we can bound the first term by

$$\begin{aligned} \mathcal{J}_h((\pi_h - i_h)(v_h\varphi), (\pi_h - i_h)(v_h\varphi))^{\frac{1}{2}} &\leq |\beta|^{\frac{1}{2}} h^{\frac{1}{2}} \|\nabla(\pi_h - i_h)(v_h\varphi)\|_\Omega \\ &\leq |\beta|^{\frac{1}{2}} h^{\frac{1}{2}} \|\nabla(i_h - 1)(v_h\varphi)\|_\Omega \\ &\leq C|\beta|^{\frac{1}{2}} h^{\frac{1}{2}} \|\nabla\varphi\|_{\infty,\Omega} \|v_h\|_\Omega \\ &\leq C|\beta|^{\frac{1}{2}} (h^{\frac{1}{2}} + \lambda^{-1}) \|v_h\varphi\|_\Omega, \end{aligned}$$

where for the last two inequalities we used the discrete commutator property Lemma 6.4 together with the φ-bounds (6.10) and (6.17). Since φ is Lipschitz continuous on K, $\varphi|_F$ is also Lipschitz continuous, and so $\varphi|_F \in W^{1,\infty}(F)$. The restriction of the nodal interpolant on K onto F gives the nodal interpolant on F, hence applying Lemma 6.4 to F instead of K we have the discrete commutator estimate

$$h\|n \cdot \nabla(i_h - 1)(v_h\varphi)\|_F \leq Ch|\varphi|_{W^{1,\infty}(K)} \|v_h\|_F \leq C(h^{\frac{1}{2}} + \lambda^{-1}) \|v_h\varphi\|_K,$$

where in the last step we used (6.10) and (6.17) together with a discrete trace inequality. After summation we have that

$$\mathcal{J}_h((i_h - 1)(v_h\varphi), (i_h - 1)(v_h\varphi))^{\frac{1}{2}} \leq C|\!|\!| v_h |\!|\!|_\varphi.$$

Finally we use the trivial bound (since $|\varphi| < 1$)

$$\mathcal{J}_h(v_h\varphi, v_h\varphi)^{\frac{1}{2}} \leq \mathcal{J}_h(v_h, v_h)^{\frac{1}{2}}.$$

We conclude the proof by summing up the above contributions. □

We can now prove in the downstream case $\beta = (\beta_1, 0)$, $\beta_1 > 0$, the following error estimate showing that in the zone $\mathring{\omega}_\beta$ where we have stability, the convergence in the L^2-norm is of order $\mathcal{O}(h^{\frac{3}{2}})$ on unstructured meshes, which is known to be optimal. We also obtain that in this region the convergence in the H^1-seminorm is $\mathcal{O}(h)$.

Theorem 6.12. *Assume that $u \in H^2(\Omega)$ is a solution to (6.1) and let $(u_h, z_h) \in [V_h]^2$ be the solution to (6.4). Then there exists $h_0 > 0$ such that for all $h < h_0$ with $\mathrm{Pe}(h) \gtrsim 1$ there holds*

$$|\!|\!| u - u_h |\!|\!|_\varphi \leq C(|\beta|^{\frac{1}{2}} h^{\frac{3}{2}} |u|_{H^2(\Omega)} + |\beta|^{\frac{1}{2}} h^{-\frac{1}{2}} \|\delta u\|_\omega).$$

Proof. Let $e_h = \pi_h u - u_h \in V_h$, then $u - u_h = u - \pi_h u + e_h$. By Corollary 6.10 there exists

$\alpha > 0$ such that

$$\alpha |||e_h|||_\varphi^2 \le a_h(e_h, \pi_h(e_h\varphi)) + C\gamma^{-1}\|(e_h, 0)\|_s^2.$$

By Cauchy-Schwarz combined with Lemma 6.11 and Young's inequality

$$-s^*(z_h, \pi_h(e_h\varphi)) \le C\varepsilon_1^{-1}\|(0, z_h)\|_s^2 + \varepsilon_1(|||e_h|||_\varphi^2 + \|(e_h, 0)\|_s^2),$$

for some $0 < \varepsilon_1 < \alpha/2$, hence

$$\tfrac{\alpha}{2}|||e_h|||_\varphi^2 \le a_h(e_h, \pi_h(e_h\varphi)) + s^*(z_h, \pi_h(e_h\varphi)) + C\varepsilon_1^{-1}\|(0, z_h)\|_s^2 + C\|(e_h, 0)\|_s^2.$$

Applying the first equality of the consistency Lemma 6.5 we obtain

$$\frac{\alpha}{2}|||e_h|||_\varphi^2 \le a_h(\pi_h u - u, \pi_h(e_h\varphi)) + C\varepsilon_1^{-1}\|(0, z_h)\|_s^2 + C\|(e_h, 0)\|_s^2. \tag{6.30}$$

Since $\pi_h u - u \in V_h^\perp$ we may apply Lemma 6.6 to bound

$$a_h(\pi_h u - u, \pi_h(e_h\varphi)) \le C\|\pi_h u - u\|_\sharp \|(0, \pi_h(e_h\varphi))\|_s.$$

From Lemma 6.11 and Young's inequality we thus have that for some $\varepsilon_2 > 0$,

$$a_h(\pi_h u - u, \pi_h(e_h\varphi)) \le C((1 + \varepsilon_2^{-1})\|\pi_h u - u\|_\sharp^2 + \|(e_h, 0)\|_s^2 + \varepsilon_2|||e_h|||_\varphi^2).$$

Taking $\varepsilon_2 < \alpha/4$ and combining the above bound with (6.30) we see that

$$\tfrac{\alpha}{4}|||e_h|||_\varphi^2 \le C((1 + \varepsilon_2^{-1})\|\pi_h u - u\|_\sharp^2 + (1 + \varepsilon_1^{-1})\|(e_h, z_h)\|_s^2).$$

Since $\varepsilon_{1,2}$ are independent of h we can absorb them in the generic constant C and using the approximation inequality (6.20) together with Proposition 6.7, we conclude that

$$\begin{aligned} |||e_h|||_\varphi &\le C(\mu^{\frac{1}{2}}h + |\beta|^{\frac{1}{2}}h^{\frac{3}{2}})(|u|_{H^2(\Omega)} + h^{-2}\|\delta u\|_\omega) \\ &\le C(|\beta|^{\frac{1}{2}}h^{\frac{3}{2}}|u|_{H^2(\Omega)} + |\beta|^{\frac{1}{2}}h^{-\frac{1}{2}}\|\delta u\|_\omega), \end{aligned}$$

where we used that $\mathrm{Pe}(h) > 1$. □

6.2.2 Upstream estimates

In this case we consider $\beta = (\beta_1, 0)$ with $\beta_1 < 0$ and the data set

$$\omega = (0, x) \times (y^-, y^+)$$

touching part of the outflow boundary $\partial\Omega^+$. We must choose the weight function differently and this time we take a negative φ given by (6.8)

$$\varphi := \psi_1\psi_2 \in (-1, 0).$$

It seems that in this case we can not simultaneously get control of the L^2-norm and the weighted H^1-norm and we have to sacrifice the latter since it is not uniform in μ. We now take the weighted triple norm to be

$$|||v_h|||_\varphi^2 := \||\beta|^{\frac{1}{2}} v_h|\varphi|^{\frac{1}{2}}\|_\Omega^2 + \||\beta \cdot n|^{\frac{1}{2}} v_h|\varphi|^{\frac{1}{2}}\|_{\partial\Omega^-}^2, \tag{6.31}$$

and rederive the results obtained in Section 6.2.1, aiming for a local error estimate. Since $\varphi \in (-1, 0)$, we will use that $\|\cdot\varphi\|_\Omega \leq \|\cdot|\varphi|^{\frac{1}{2}}\|_\Omega$.

We start with an analogue of Lemma 6.9 by taking $v_h\varphi$ as a test function in the weak bilinear form a_h and notice that since $\varphi < 0$ we now have that

$$\frac{1}{2}\left(\||\beta \cdot n|^{\frac{1}{2}} v_h|\varphi|^{\frac{1}{2}}\|_{\partial\Omega^-}^2 + |\beta|\|v_h|\varphi|^{\frac{1}{2}}\|_\Omega^2\right) = (\beta \cdot \nabla v_h, v_h\varphi)_\Omega + \frac{1}{2}\||\beta \cdot n|^{\frac{1}{2}} v_h|\varphi|^{\frac{1}{2}}\|_{\partial\Omega^+}^2.$$

Arguing as previously in (6.23) but now for the outflow boundary, we obtain the bound

$$\begin{aligned}\||\beta \cdot n|^{\frac{1}{2}} v_h|\varphi|^{\frac{1}{2}}\|_{\partial\Omega^+} &\leq C|\beta|^{\frac{1}{2}}(\|v_h|\varphi|^{\frac{1}{2}}\|_{\partial\Omega^+\cap\omega_\beta} + \|v_h|\varphi|^{\frac{1}{2}}\|_{\partial\Omega^+\setminus\omega_\beta}) \\ &\leq C|\beta|^{\frac{1}{2}}h^{-\frac{1}{2}}\|v_h\|_\omega + C|\beta|^{\frac{1}{2}}h^{\frac{3}{2}}\|v_h\|_{H^1(\Omega)} \\ &\leq C\gamma^{-\frac{1}{2}}\|(v_h, 0)\|_s,\end{aligned} \tag{6.32}$$

and thus

$$\frac{1}{2}\left(|\beta|\|v_h|\varphi|^{\frac{1}{2}}\|_\Omega^2 + \||\beta \cdot n|^{\frac{1}{2}} v_h|\varphi|^{\frac{1}{2}}\|_{\partial\Omega^-}^2\right) \leq (\beta \cdot \nabla v_h, v_h\varphi)_\Omega + C\gamma^{-1}\|(v_h, 0)\|_s^2. \tag{6.33}$$

For the diffusion term we no longer have any positive contribution due to the change in sign of the weight function φ, since now

$$(\mu\nabla v_h, \nabla(v_h\varphi))_\Omega = -\|\mu^{\frac{1}{2}}\nabla v_h|\varphi|^{\frac{1}{2}}\|_\Omega^2 + (\mu\nabla v_h, v_h\nabla\varphi)_\Omega.$$

We must therefore control this entirely using the stabilisation. Integrating by parts and using the weighted trace inequality (6.18)

$$\begin{aligned}(\mu\nabla v_h, \nabla(v_h\varphi))_\Omega - \langle \mu\nabla v_h \cdot n, v_h\varphi\rangle_{\partial\Omega} &= \sum_{F\in\mathcal{F}_i}\int_F \mu[\![\nabla v_h\cdot n]\!]\, v_h\varphi \,\mathrm{d}s \\ &\leq C\gamma^{-\frac{1}{2}}\mathcal{J}_h(v_h,v_h)^{\frac{1}{2}}\mu^{\frac{1}{2}}h^{-1}\|v_h\varphi\|_\Omega \\ &\leq C\gamma^{-\frac{1}{2}}\mathcal{J}_h(v_h,v_h)^{\frac{1}{2}}\mu^{\frac{1}{2}}h^{-1}\|v_h|\varphi|^{\frac{1}{2}}\|_\Omega.\end{aligned}$$

To bound this by the triple norm we can simply use that $|\varphi|< 1$ and $\mu \leq |\beta|h$, giving that $\mu^{\frac{1}{2}}h^{-1}|\varphi|^{\frac{1}{2}}\leq |\beta|^{\frac{1}{2}}h^{-\frac{1}{2}}$. Hence we have that for some $\varepsilon > 0$,

$$|(\mu\nabla v_h, \nabla(v_h\varphi))_\Omega - \langle \mu\nabla v_h \cdot n, v_h\varphi\rangle_{\partial\Omega}| \leq C\varepsilon^{-1}\gamma^{-1}h^{-1}\mathcal{J}_h(v_h,v_h) + C\varepsilon|\!|\!|v_h|\!|\!|_\varphi^2.$$

However, when $\mathrm{Pe}(h)h > 1$ one can obtain a better estimate due to $\mu^{\frac{1}{2}}h^{-1}|\varphi|^{\frac{1}{2}}\leq |\beta|^{\frac{1}{2}}$, which gives that

$$|(\mu\nabla v_h, \nabla(v_h\varphi))_\Omega - \langle \mu\nabla v_h \cdot n, v_h\varphi\rangle_{\partial\Omega}| \leq C\varepsilon^{-1}\gamma^{-1}\mathcal{J}_h(v_h,v_h) + C\varepsilon|\!|\!|v_h|\!|\!|_\varphi^2.$$

Summing these contributions we obtain the following result corresponding to Lemma 6.9.

Lemma 6.13. *There exists $\alpha > 0$ such that for all $v_h \in V_h$ we have*

$$\alpha|\!|\!|v_h|\!|\!|_\varphi^2 \leq a_h(v_h, v_h\varphi) + Ch^{-1}\|(v_h,0)\|_s^2, \text{ when } 1 \lesssim \mathrm{Pe}(h) < h^{-1},$$

and

$$\alpha|\!|\!|v_h|\!|\!|_\varphi^2 \leq a_h(v_h, v_h\varphi) + C\|(v_h,0)\|_s^2, \text{ when } \mathrm{Pe}(h) > h^{-1}.$$

Again, we can refine the control over the triple norm $|\!|\!|v_h|\!|\!|_\varphi$ by taking the projection $\pi_h(v_h\varphi) \in V_h$ as a test function and we obtain corresponding results.

Corollary 6.14. *There exists $\alpha > 0$ such that for all $v_h \in V_h$ we have*

$$\alpha|\!|\!|v_h|\!|\!|_\varphi^2 \leq a_h(v_h, \pi_h(v_h\varphi)) + Ch^{-1}\|(v_h,0)\|_s^2, \text{ when } 1 \lesssim \mathrm{Pe}(h) < h^{-1},$$

and

$$\alpha|\!|\!|v_h|\!|\!|_\varphi^2 \leq a_h(v_h, \pi_h(v_h\varphi)) + C\|(v_h,0)\|_s^2, \text{ when } \mathrm{Pe}(h) > h^{-1}.$$

Proof. The argument in the proof of Corollary 6.10 remains valid with the remark that we now use the inequality $|\varphi|< |\varphi|^{\frac{1}{2}}$. □

Lemma 6.15. *For all $v_h \in V_h$ there holds*

$$\|(0, \pi_h(v_h\varphi))\|_s^2 \leq C(h^{-1}|||v_h|||_\varphi^2 + \|(v_h, 0)\|_s^2), \text{ when } 1 \lesssim \mathrm{Pe}(h) < h^{-1},$$

and

$$\|(0, \pi_h(v_h\varphi))\|_s^2 \leq C(|||v_h|||_\varphi^2 + \|(v_h, 0)\|_s^2), \text{ when } \mathrm{Pe}(h) > h^{-1}.$$

Proof. We follow the proof of Lemma 6.11 and we focus on the bounds that are now different. As before, by the triangle inequality we have that up to a constant

$$\begin{aligned}\|(0, \pi_h(v_h\varphi))\|_s \leq & \|\mu^{\frac{1}{2}}\nabla(\pi_h - 1)(v_h\varphi))\|_\Omega + \|\mu^{\frac{1}{2}} v_h \nabla\varphi\|_\Omega + \|\mu^{\frac{1}{2}}\nabla v_h\varphi\|_\Omega \\ & + (|\beta| + \mu h^{-1})^{\frac{1}{2}}(\|(\pi_h - 1)(v_h\varphi)\|_{\partial\Omega} + \|v_h\varphi\|_{\partial\Omega}) \\ & + \mathcal{J}_h(\pi_h(v_h\varphi), \pi_h(v_h\varphi))^{\frac{1}{2}}.\end{aligned}$$

The first two terms can be bounded by $C|||v_h|||_\varphi$ as previously. For the third one, we can use the inverse inequality (6.11) and (6.17) to obtain

$$\|\mu^{\frac{1}{2}}\nabla v_h\varphi\|_\Omega \leq C\mu^{\frac{1}{2}} h^{-1}\|\varphi\|_{\infty,\Omega}\|v_h\|_\Omega \leq C\mu^{\frac{1}{2}} h^{-1}\|v_h\varphi\|_\Omega \leq C\mu^{\frac{1}{2}} h^{-1}\|v_h|\varphi|^{\frac{1}{2}}\|_\Omega.$$

Hence we have that

$$\|\mu^{\frac{1}{2}}\nabla v_h\varphi\|_\Omega \leq Ch^{-\frac{1}{2}}|||v_h|||_\varphi, \text{ when } 1 \lesssim \mathrm{Pe}(h) < h^{-1},$$

and

$$\|\mu^{\frac{1}{2}}\nabla v_h\varphi\|_\Omega \leq C|||v_h|||_\varphi, \text{ when } \mathrm{Pe}(h) > h^{-1}.$$

Arguing as previously, we can bound the second line by $C|||v_h|||_\varphi$ using (6.32) instead of (6.23). We conclude the proof by recalling the estimate (6.29) for the jump term and the subsequent bounds. □

We now prove the weighted error estimate in the upstream case $\beta = (\beta_1, 0)$, $\beta_1 < 0$, showing that in the stability region $\mathring{\omega}_\beta$ we have quasi-optimal convergence for high Péclet numbers and a reduction of the convergence order by $\mathcal{O}(h^{\frac{1}{2}})$ in an intermediate regime.

Theorem 6.16. *Assume that $u \in H^2(\Omega)$ is a solution to (6.1) and let $(u_h, z_h) \in [V_h]^2$ be the solution to (6.4), then there holds*

$$|||u - u_h|||_\varphi \leq C(|\beta|^{\frac{1}{2}} h|u|_{H^2(\Omega)} + |\beta|^{\frac{1}{2}} h^{-1}\|\delta u\|_\omega), \text{ when } 1 \lesssim \mathrm{Pe}(h) < h^{-1},$$

and

$$|||u - u_h|||_\varphi \leq C(|\beta|^{\frac{1}{2}} h^{\frac{3}{2}} |u|_{H^2(\Omega)} + |\beta|^{\frac{1}{2}} h^{-\frac{1}{2}} \|\delta u\|_\omega), \ \textit{when} \ \mathrm{Pe}(h) > h^{-1}.$$

Proof. We combine Lemma 6.13, Corollary 6.14 and Lemma 6.15 as in the proof of Theorem 6.12 and note that the argument holds verbatim when $\mathrm{Pe}(h) > h^{-1}$. Observe that when $1 \lesssim \mathrm{Pe}(h) < h^{-1}$ we similarly obtain for some $\alpha > 0$ and $0 < \varepsilon_1 < \alpha/2$,

$$\tfrac{\alpha}{2} |||e_h|||_\varphi^2 \leq a_h(\pi_h u - u, \pi_h(e_h\varphi)) + C\varepsilon_1^{-1} \|(0, z_h)\|_s^2 + Ch^{-1} \|(e_h, 0)\|_s^2. \tag{6.34}$$

Since $\pi_h u - u \in V_h^\perp$ we may apply Lemma 6.6 to bound

$$a_h(\pi_h u - u, \pi_h(e_h\varphi)) \leq C \|\pi_h u - u\|_\sharp \|(0, \pi_h(e_h\varphi))\|_s.$$

From Lemma 6.15 and Young's inequality we thus have that for some $\varepsilon_2 > 0$,

$$a_h(\pi_h u - u, \pi_h(e_h\varphi)) \leq C((1 + \varepsilon_2^{-1} h^{-1}) \|\pi_h u - u\|_\sharp^2 + \|(e_h, 0)\|_s^2 + \varepsilon_2 |||e_h|||_\varphi^2).$$

Taking $\varepsilon_2 < \alpha/4$ and combining the above bound with (6.34) we see that

$$\tfrac{\alpha}{4} |||e_h|||_\varphi^2 \leq C((1 + \varepsilon_2^{-1} h^{-1}) \|\pi_h u - u\|_\sharp^2 + \varepsilon_1^{-1} h^{-1} \|(e_h, z_h)\|_s^2).$$

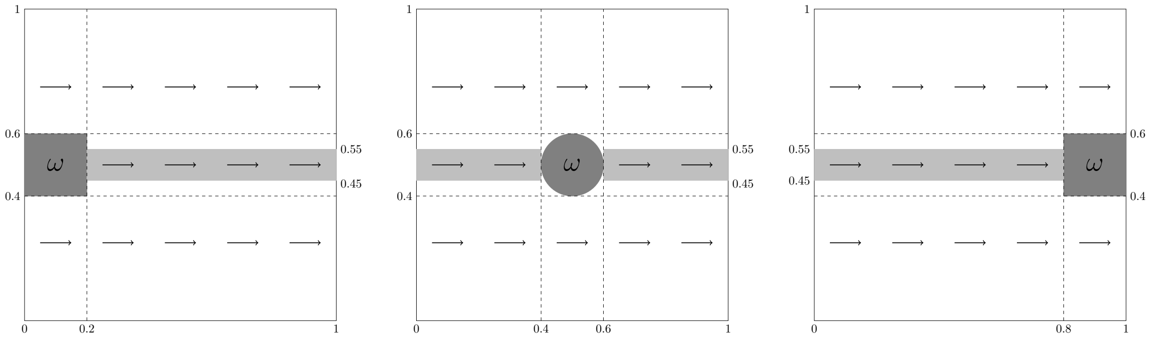
Since $\varepsilon_{1,2}$ are independent of h we can absorb them in the generic constant C and conclude the proof by using the approximation inequality (6.20) and Proposition 6.7 to obtain that

$$\begin{aligned} |||e_h|||_\varphi &\leq C(\mu^{\frac{1}{2}} h^{\frac{1}{2}} + |\beta|^{\frac{1}{2}} h)(|u|_{H^2(\Omega)} + h^{-2} \|\delta u\|_\omega) \\ &\leq C(|\beta|^{\frac{1}{2}} h |u|_{H^2(\Omega)} + |\beta|^{\frac{1}{2}} h^{-1} \|\delta u\|_\omega), \end{aligned}$$

when $1 \lesssim \mathrm{Pe}(h) < h^{-1}$. □

6.3 Numerical examples

We let Ω be the unit square and illustrate the performance of the numerical method (6.4) for different locations of the data domain ω and different regions of interest where we measure the approximation error. The computational domains are given in Figure 6.4 and the implementation is done using FEniCS [2]. In all the examples below we have used uniform triangulations with alternating left and right diagonals. In the definition of $\mathcal{J}_h$ and s^* we have taken the parameters $\gamma = 10^{-5}$ and $\gamma^* = 1$, and $\zeta = 2$ for s_ω. The effect of different combinations of γ and γ^* on the L^2-errors is shown in Figures 6.5 and 6.6 when data is given in a centred disc. Similar results are obtained when the data set is near the inflow/outflow boundary. Notice that our choice is empirically close to being optimal both locally and globally.

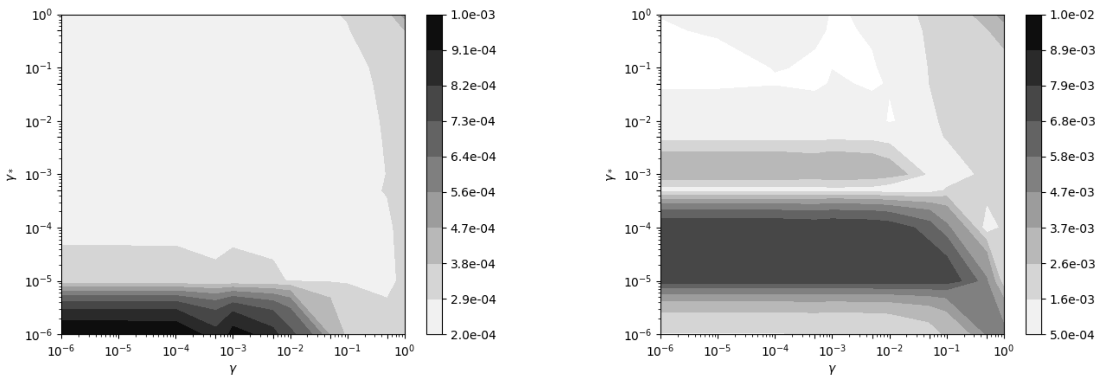


(a) $\omega = (0, 0.2) \times (0.4, 0.6)$, region $(0.2, 1) \times (0.45, 0.55)$ downstream.

(b) $\omega = B((0.5, 0.5), 0.1)$, regions both downstream and upstream.

(c) $\omega = (0.8, 1) \times (0.4, 0.6)$, region $(0, 0.8) \times (0.45, 0.55)$ upstream.

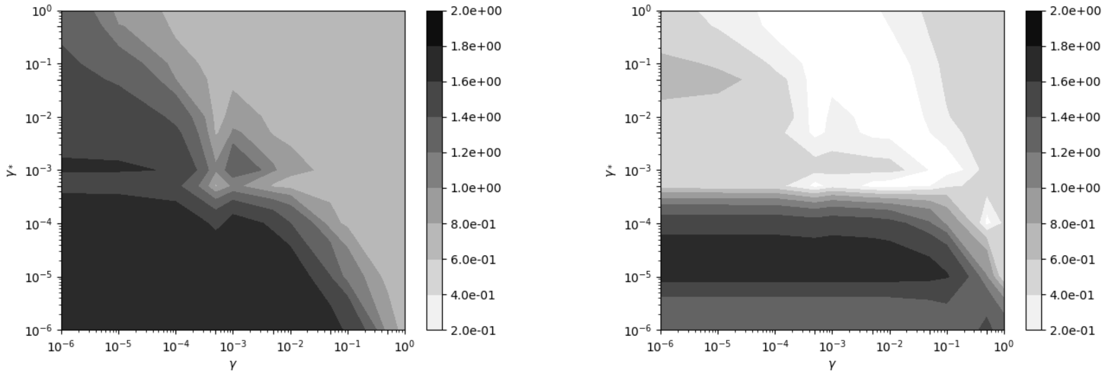
Figure 6.4: Data set ω (grey) and error measurement regions (light grey).



(a) $\mu = 10^{-3}$.

(b) $\mu = 10^{-2}$.

Figure 6.5: Varying the stabilisation parameters γ and γ^*. Absolute L^2-errors downstream, computational domains in Figure 6.4b. $\beta = (1, 0)$, exact solution $u = 2\sin(5\pi x)\sin(5\pi y)$. Similar results in the upstream case.

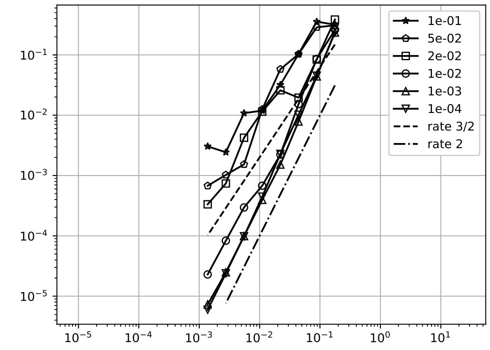


(a) $\mu = 10^{-3}$. (b) $\mu = 10^{-2}$.

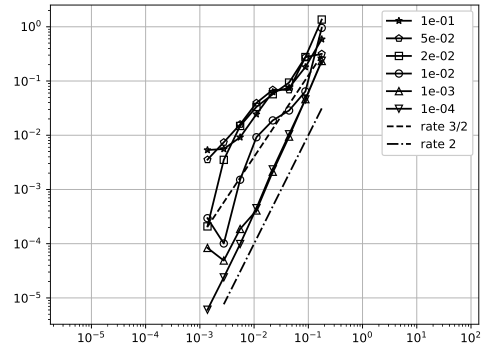
Figure 6.6: Varying the stabilisation parameters γ and γ^*. Absolute L^2-errors globally, computational domains in Figure 6.4b. $\beta = (1, 0)$, exact solution $u = 2\sin(5\pi x)\sin(5\pi y)$.

We first show convergence plots both downstream and upstream from the data set when varying the diffusion coefficient μ and keeping the convection field β fixed. As in the case of well-posed convection-dominated problems, the observed L^2-convergence order is typically $\mathcal{O}(h^2)$, surpassing by $\mathcal{O}(h^{\frac{1}{2}})$ the weighted error estimates proven for general meshes.

Data set near the inflow/outflow boundary. We consider the data set ω near the inflow and outflow boundaries of Ω, as assumed in Section 6.1.1. We observe in Figure 6.7 that as diffusion is reduced the convergence order for the L^2-errors increases, culminating with quadratic convergence when convection dominates. Confirming the theoretical analysis in Section 6.2.2, we note the presence of an intermediate regime for Péclet numbers in which the upstream convergence orders are reduced and the upstream errors are typically larger. This can also be seen in Figure 6.9 where we consider the diffusion coefficient $\mu = 10^{-2}$ and an interior data set. The errors in the H^1-seminorm are given in Figure 6.8 which shows almost linear convergence, corresponding to an $\mathcal{O}(h^{\frac{3}{2}})$ convergence order for the gradient term in the triple norm (6.22).

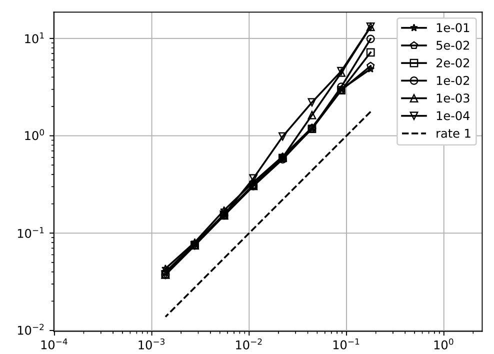


(a) Computational domains in Figure 6.4a.

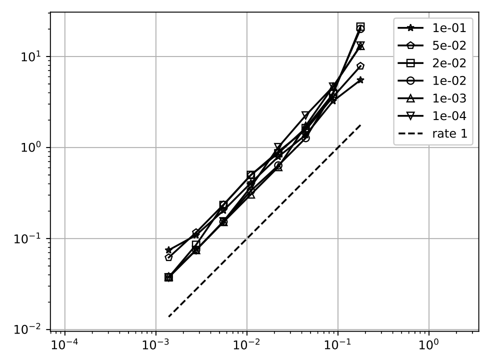


(b) Computational domains in Figure 6.4c.

Figure 6.7: Absolute L^2-errors against mesh size h when varying the diffusion coefficient μ for fixed $\beta = (1, 0)$, exact solution $u = 2\sin(5\pi x)\sin(5\pi y)$.

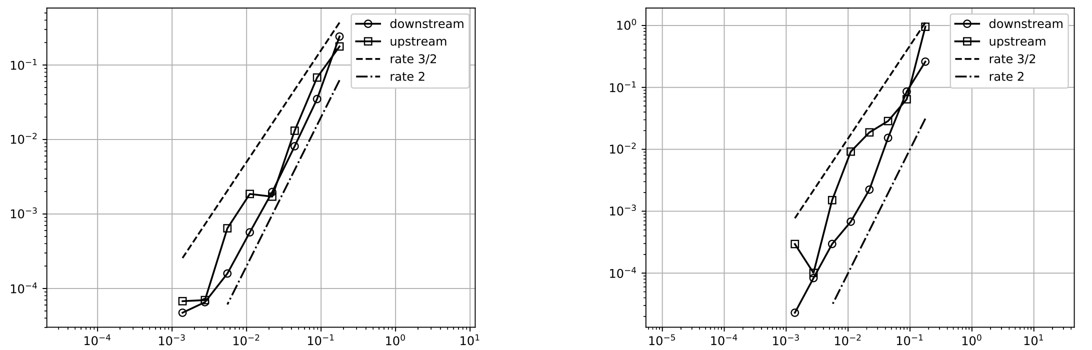


(a) Computational domains in Figure 6.4a.



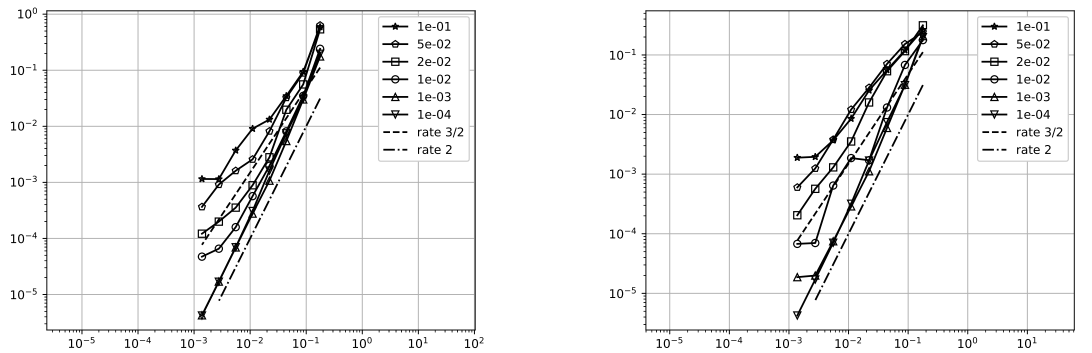
(b) Computational domains in Figure 6.4c.

Figure 6.8: H^1-errors against mesh size h when varying the diffusion coefficient μ for fixed $\beta = (1, 0)$, exact solution $u = 2\sin(5\pi x)\sin(5\pi y)$.



(a) Domains in Figure 6.4b. (b) Domains in Figure 6.4a and Figure 6.4c.

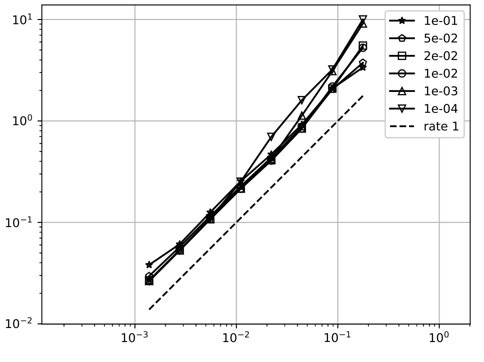
Figure 6.9: L^2-errors against mesh size h, downstream vs upstream for $\mu = 10^{-2}$, $\beta = (1, 0)$, exact solution $u = 2\sin(5\pi x)\sin(5\pi y)$.



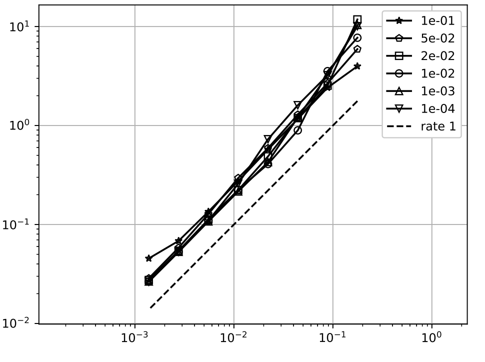
(a) Downstream. (b) Upstream.

Figure 6.10: L^2-errors against mesh size h, computational domains in Figure 6.4b. Varying the diffusion coefficient μ for fixed $\beta = (1, 0)$, exact solution $u = 2\sin(5\pi x)\sin(5\pi y)$.

Interior data set. Next we consider the setting of the example discussed in Figure 6.2, where data is given in the centre of the domain. We give the convergence of the L^2-errors in Figure 6.10 with the caveat that this location of the data set ω is not rigorously covered by the theoretical analysis of the previous sections. Nonetheless, the experiments are in agreement with the proven results. Notice that the L^2-convergence is faster as μ decreases and for high Péclet numbers (above 10) one has optimal quadratic convergence both downstream and upstream, with the distinction that in the upstream case the convergence order is reduced in an intermediate regime, in agreement with the theoretical results. Also, as expected from the error estimates proven in Chapter 5, when diffusion is moderately small one can see the transition towards the diffusion-dominated regime as the mesh gets refined – the convergence changes from almost quadratic to sub-linear as the Péclet number decreases below 1. Figure 6.11 shows almost linear convergence in the H^1-seminorm which corresponds to order $\mathcal{O}(h^{\frac{3}{2}})$ convergence for the gradient term in the triple norm (6.22). We also remark almost no distinction between upstream and downstream for this example, probably because the gradient term is controlled by the L^2-norm for small enough μ.



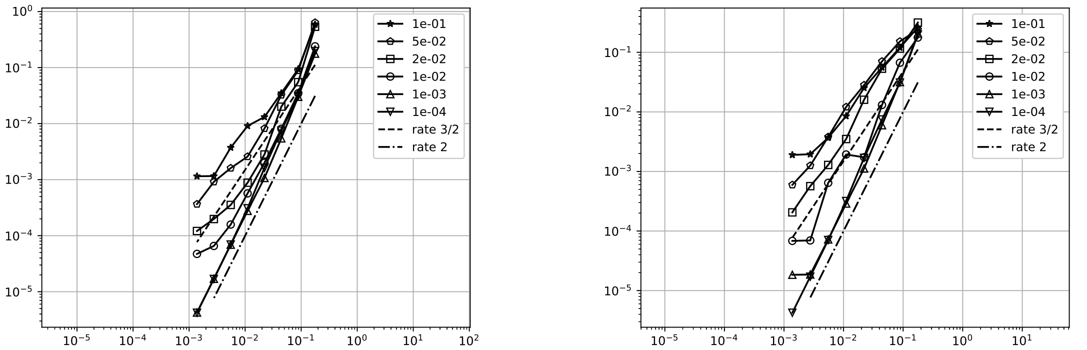
(a) Downstream.



(b) Upstream.

Figure 6.11: H^1-errors against mesh size h, computational domains in Figure 6.4b. Varying the diffusion coefficient μ for fixed $\beta = (1, 0)$, exact solution $u = 2\sin(5\pi x)\sin(5\pi y)$.

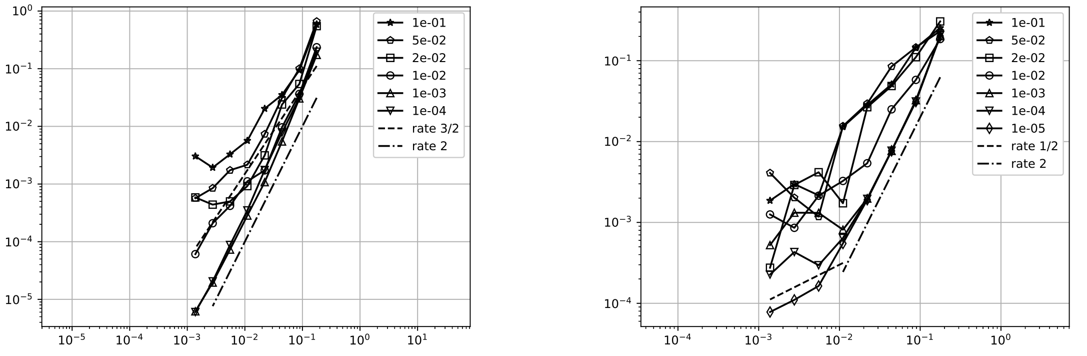
Data perturbations. We demonstrate the effect of data perturbations $\tilde{u}_\omega = u|_\omega + \delta u$ in a downstream vs upstream setting by polluting the restriction of u to each node of the mesh in ω with uniformly distributed values in $[-h^2, h^2]$, $[-h, h]$ and $[-h^{\frac{1}{2}}, h^{\frac{1}{2}}]$, respectively. Comparing first Figures 6.10 and 6.12 we see that perturbations of amplitude $\mathcal{O}(h^2)$ have no effect on the L^2-errors, as expected.



(a) Downstream, perturbation $\mathcal{O}(h^2)$. (b) Upstream, perturbation $\mathcal{O}(h^2)$.

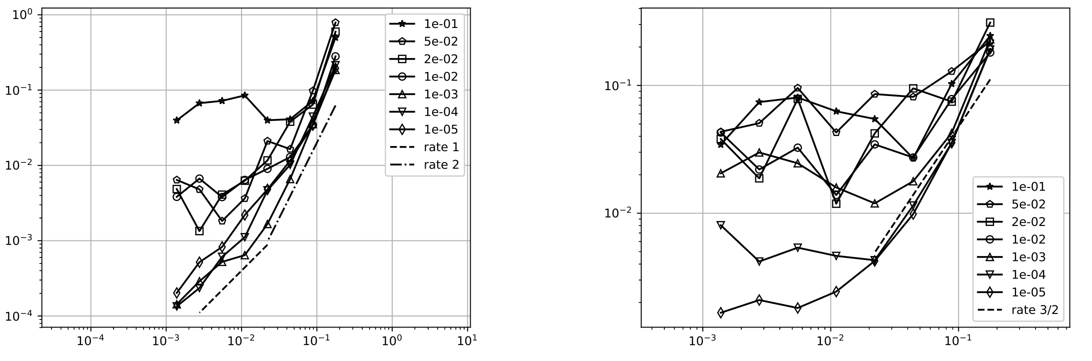
Figure 6.12: L^2-errors against mesh size h for perturbations in data, computational domains in Figure 6.4b. Varying the diffusion coefficient μ for fixed $\beta = (1, 0)$, exact solution $u = 2\sin(5\pi x)\sin(5\pi y)$.

An $\mathcal{O}(h)$ noise amplitude exhibits in Figure 6.13 the difference – proven in Theorems 6.12 and 6.16 – between the downstream and upstream scenarios. In the upstream case the noise has a strong effect for moderate Péclet numbers and the errors stagnate. Only for high Péclet numbers one has convergence of order $\mathcal{O}(h^{\frac{1}{2}})$. In the downstream case one observes lower errors, faster convergence and almost no noise effect for high Péclet numbers. The difference is also very clear for perturbations of amplitude $\mathcal{O}(h^{\frac{1}{2}})$ shown in Figure 6.14. In the upstream case the errors stagnate and there seems to be no convergence, while in the downstream case the errors still convergence for high Péclet numbers.



(a) Downstream, perturbation $\mathcal{O}(h)$. (b) Upstream, perturbation $\mathcal{O}(h)$.

Figure 6.13: L^2-errors against mesh size h for perturbations in data, computational domains in Figure 6.4b. Varying the diffusion coefficient μ for fixed $\beta = (1, 0)$, exact solution $u = 2\sin(5\pi x)\sin(5\pi y)$.



(a) Downstream, perturbation $\mathcal{O}(h^{\frac{1}{2}})$.

(b) Upstream, perturbation $\mathcal{O}(h^{\frac{1}{2}})$.

Figure 6.14: L^2-errors against mesh size h for perturbations in data, computational domains in Figure 6.4b. Varying the diffusion coefficient μ for fixed $\beta = (1, 0)$, exact solution $u = 2\sin(5\pi x)\sin(5\pi y)$.

Variable convective field. We note that this case is not rigorously covered by the theoretical analysis presented above. We briefly mention that a potential way of extending the analysis to this case could be to assume that β has no closed curves and vanishes nowhere, and construct the weight functions and the stability region in terms of the η function in [3, Assumption (H3)], for which $\beta \cdot \nabla\eta \geq C\|\beta\|_{L^\infty(\Omega)^n}$. We consider a unit vector field β_{var}, shown in Figure 6.15, that rotates around the point $(-0.1, -0.1)$ and is given by

$$\beta_{var}(x, y) := \frac{1}{\sqrt{(x+0.1)^2 + (y+0.1)^2}}(y + 0.1, -x - 0.1). \tag{6.35}$$

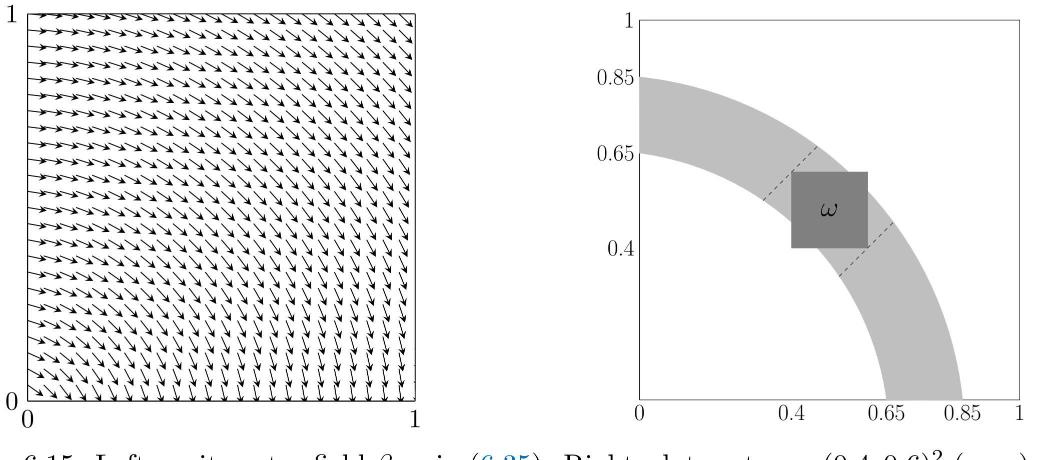
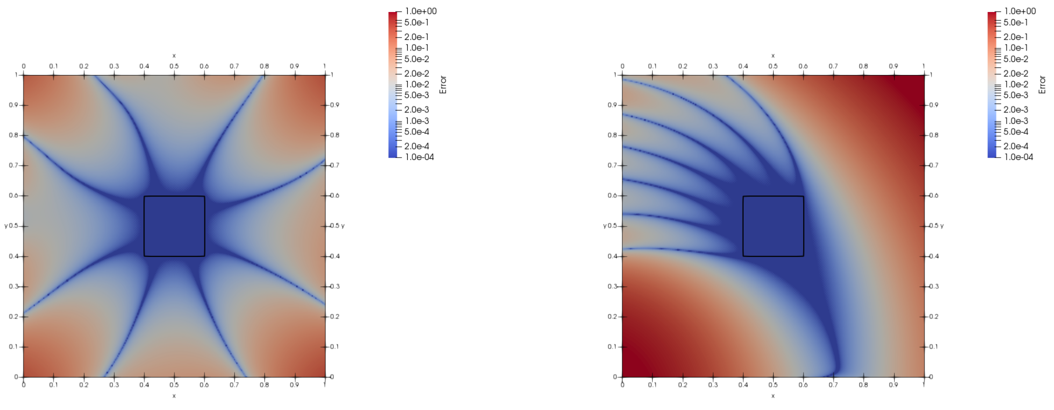


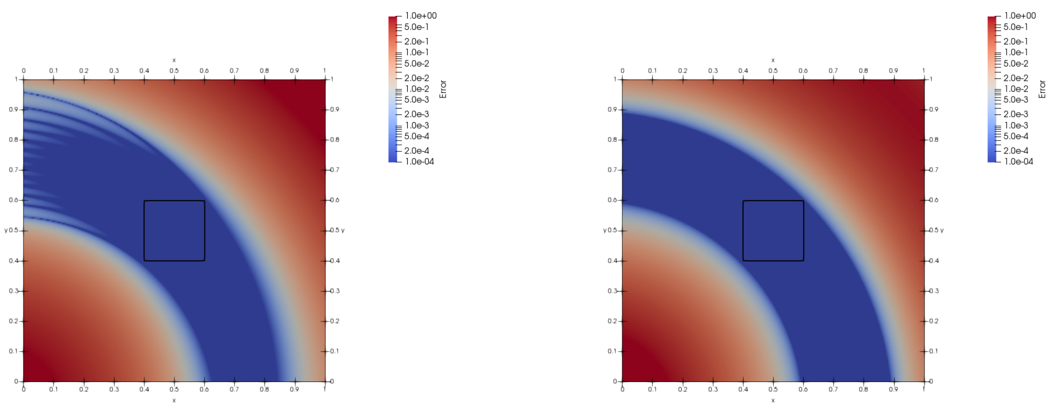
Figure 6.15: Left: unit vector field β_{var} in (6.35). Right: data set $\omega = (0.4, 0.6)^2$ (grey) and error measurement regions (light grey).

We consider an interior data set $\omega = (0.4, 0.6)^2$ and the exact solution with unit L^2-norm $u = 30x(1-x)y(1-y)$. Without changing β_{var}, we decrease the diffusion coefficient μ and present the absolute errors in Figures 6.16 and 6.17. We first see in Figure 6.16a that the errors oscillate away from the data set when diffusion dominates. Then in Figure 6.16b we notice the stronger impact of convection with a clear distinction between downstream and upstream regions, similarly to the introductory example in Figure 6.2a. This intermediate regime can still be observed in Figure 6.17a, while Figure 6.17b shows the convection-dominated regime and a clear stability region with downstream–upstream symmetry.



(a) $\mu = 1$. (b) $\mu = 10^{-2}$.

Figure 6.16: Absolute error for variable β in (6.35) and $u = 30x(1-x)y(1-y)$. Data given in the outlined square $(0.4, 0.6)^2$ and mesh size $h \approx 0.005$.

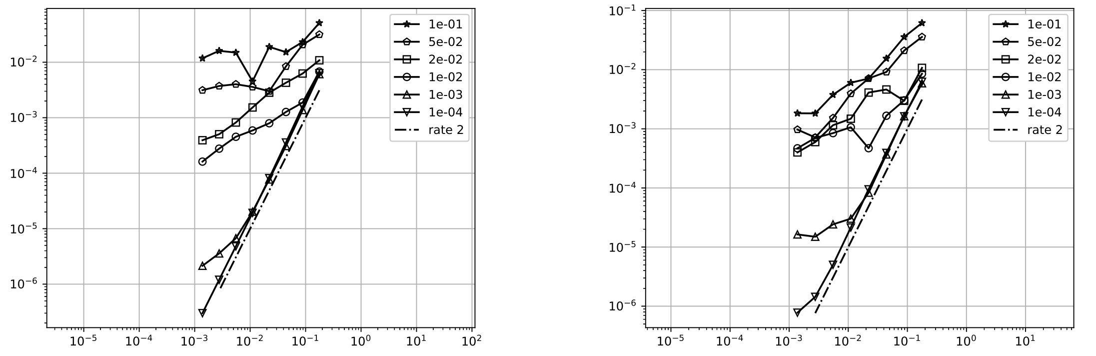


(a) $\mu = 10^{-3}$. (b) $\mu = 10^{-6}$.

Figure 6.17: Absolute error for variable β in (6.35) and $u = 30x(1-x)y(1-y)$. Data given in the outlined square $(0.4, 0.6)^2$ and mesh size $h \approx 0.005$.

We report in Figure 6.18 the L^2-convergence measured in the downstream and upstream parts of the region shown in Figure 6.15. We notice a clear separation between a diffusion-dominated regime with sub-linear convergence and a convection-dominated regime with quadratic convergence. As expected, the regimes will also depend on the resolution of the computation and we observe for $\mu = 10^{-3}$ that, as the mesh gets refined and the Péclet number decreases, the convergence of the method changes from quadratic to sub-linear. We also see that a smaller diffusion coefficient is needed in the upstream case to obtain optimal convergence.

The annular downstream region is taken to be $\{(x, y) \in (0,1)^2 : 0.75^2 + 0.1^2 < (x + 0.1)^2 + (y + 0.1)^2 < 0.95^2 + 0.1^2 \text{ and } x - y < 0.2\}$ and the upstream one $\{(x, y) \in (0,1)^2 : 0.75^2 + 0.1^2 < (x + 0.1)^2 + (y + 0.1)^2 < 0.95^2 + 0.1^2 \text{ and } x - y < -0.2\}$.



(a) Downstream. (b) Upstream.

Figure 6.18: L^2-errors against mesh size h. Varying the diffusion coefficient μ for fixed β in (6.35), exact solution $u = 30x(1 - x)y(1 - y)$ and computational domains in Figure 6.15.

Conclusions

In this thesis we have considered the unique continuation problem for second order elliptic differential operators, focusing on the Helmholtz and convection–diffusion equations. After discussing the ill-posedness of such problems in Chapter 1, conditional stability estimates that are explicit in the coefficients of the partial differential equation (PDE) have been derived in Chapter 2. A methodology for designing numerical methods for such problems has been presented in Chapter 3, by first discretising the problem written in the form of PDE-constrained optimisation, and then regularising on the discrete level using techniques from stabilised finite element methods (FEMs). Based on continuous interior penalty stabilisation, piecewise linear FEMs have been proposed in Chapters 4 and 5 for Helmholtz and diffusion-dominated problems. The error estimates are explicit in the physical parameters and the convergence order matches the continuum stability of the problems. Convection-dominated problems have been analysed in Chapter 6 where the numerical method was proven to converge with quasi-optimal rates in local weighted norms.

In Chapters 4 and 5, continuum stability estimates were used for the approximation error and bounding the residual in terms of the stabilisation was a key step in the error analysis. Due to the a posteriori nature of these estimates, one direction for future work could be to explore adaptive methods.

Since unique continuation is ill-posed and stability is only provided through the discrete weakly consistent stabilisation, the ill-conditioning of the linear system becomes a problem for fine meshes. A possible future area of development consists in devising preconditioning techniques and efficient solvers for such systems.

Unique continuation problems for PDEs with coefficients that jump across an internal interface could also be considered based on the methodology presented in this thesis. Developing fitted and unfitted stabilised FEMs for this kind of ill-posed interface problems would be another important extension.

Appendix A

Finite element inequalities

We collect here some fundamental inequalities regarding finite element functions and their approximation properties. Consider a domain $\Omega \subset \mathbb{R}^n$ and let $\{\mathcal{T}_h\}$ be a shape-regular family of triangulations $\mathcal{T}_h = \{K\}$ with elements K having maximal diameter h. For a positive integer k we denote by $\mathbb{P}_k$ the set of polynomials of degree at most k.

Inverse and trace inequalities

We recall some inverse and trace inequalities, see e.g. [28, Section 1.4.3], starting with the following inverse inequality

$$\|\nabla v_h\|_{L^2(K)} \leq Ch^{-1}\|v_h\|_{L^2(K)}, \quad \forall v_h \in \mathbb{P}_k(K). \tag{A.1}$$

We also recall the continuous trace inequality

$$\|v\|_{L^2(\partial K)} \leq C(h^{-\frac{1}{2}}\|v\|_{L^2(K)} + h^{\frac{1}{2}}\|\nabla v\|_{L^2(K)}), \quad \forall v \in H^1(K), \tag{A.2}$$

and the discrete trace inequality

$$\|\nabla v_h \cdot n\|_{L^2(\partial K)} \leq Ch^{-\frac{1}{2}}\|\nabla v_h\|_{L^2(K)}, \quad \forall v_h \in \mathbb{P}_1(K). \tag{A.3}$$

Approximation inequalities

We consider the conforming piecewise affine finite element space

$$V_h := \left\{u \in C(\bar{\Omega}) : u|_K \in \mathbb{P}_1(K), K \in \mathcal{T}_h\right\}$$

and recall some approximation inequalities, see e.g. [32, Chapter 1]. Let $\pi_h : L^2(\Omega) \to V_h$ be the L^2-projection that satisfies the orthogonality

$$(u - \pi_h u, v)_{L^2(\Omega)} = 0, \quad \forall u \in L^2(\Omega), \forall v \in V_h.$$

The L^2-projection is stable in the L^2-norm

$$\|\pi_h u\|_{L^2(\Omega)} \le \|u\|_{L^2(\Omega)}, \quad \forall u \in L^2(\Omega),$$

and, if the family $\{\mathcal{T}_h\}$ is quasi-uniform, it is also stable in the H^1-norm

$$\|\pi_h u\|_{H^1(\Omega)} \le C\|u\|_{H^1(\Omega)}, \quad \forall u \in H^1(\Omega).$$

see e.g. [32, Lemma 1.131]. The following approximation estimate holds

$$\|u - \pi_h u\|_{L^2(\Omega)} + h\|\nabla(u - \pi_h u)\|_{L^2(\Omega)} \le Ch^m |u|_{H^m(\Omega)}, \quad \forall u \in H^m(\Omega), \, m = 1, 2.$$

We also recall the Scott-Zhang [56] interpolant $\pi_{sz} : H^1(\Omega) \to V_h$ which preserves vanishing Dirichlet boundary conditions. It is stable in both the L^2- and the H^1-norm and enjoys the same approximation error estimate

$$\|u - \pi_{sz} u\|_{L^2(\Omega)} + h\|\nabla(u - \pi_{sz} u)\|_{L^2(\Omega)} \le Ch^m |u|_{H^m(\Omega)}, \quad \forall u \in H^m(\Omega), \, m = 1, 2.$$

Let i_h be the nodal Lagrange interpolant on V_h. For any function $v \in W^{1,\infty}(K)$ the following approximation holds

$$\|v - i_h v\|_{\infty,K} + h\|\nabla(v - i_h v)\|_{\infty,K} \le Ch\|\nabla v\|_{\infty,K}, \tag{A.4}$$

see e.g. [32, Theorem 1.103].

Appendix B

Pseudodifferential operators

We briefly recall herein the definition of semiclassical pseudodifferential operators and semiclassical Sobolev spaces. We then discuss the composition rule of two such operators, which is also called symbol calculus, and some estimates that are used in the proof of Lemmas 2.7 and 2.15. This presentation is based on [62, Chapter 4] and [51, Section 2], to which we refer the reader for more details.

We shall use the following standard notation. For $\xi \in \mathbb{R}^n$ we set $\langle \xi \rangle = (1+|\xi|^2)^{\frac{1}{2}}$, and for a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ let $|\alpha| = \alpha_1 + \dots \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$. Let also $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$, $D = \frac{1}{i}\partial$ and $D^\alpha = \frac{1}{i^{|\alpha|}}\partial^\alpha$. The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is the set of rapidly decreasing C^∞ functions and its dual $\mathcal{S}'(\mathbb{R}^n)$ is the set of tempered distributions. The semiclassical parameter $\hbar > 0$ can be taken arbitrarily small and we assume that $\hbar \in (0,1)$.

The semiclassical Fourier transform is a rescaled version of the standard Fourier transform. It is given by

$$\mathcal{F}_\hbar \varphi(\xi) := \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar} x \cdot \xi} \varphi(x) \, \mathrm{d}x$$

and its inverse is

$$\mathcal{F}_\hbar^{-1} \psi(x) := \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} e^{\frac{i}{\hbar} x \cdot \xi} \psi(\xi) \, \mathrm{d}\xi.$$

The following properties hold: $\mathcal{F}_\hbar((\hbar D_x)^\alpha \varphi) = \xi^\alpha \mathcal{F}_\hbar \varphi$ and $(\hbar D_\xi)^\alpha \mathcal{F}_\hbar \varphi(\xi) = \mathcal{F}_\hbar((-x)^\alpha \varphi)$.

Symbol classes

For $m \in \mathbb{R}$ the symbol class S^m consists of functions $a(x,\xi,\hbar) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that for all multi-indices $\alpha, \tilde{\alpha} \in \mathbb{N}^n$ there exists a constant $C_{\alpha,\tilde{\alpha}} > 0$ uniform in $\hbar \in (0,1)$

such that

$$|\partial_x^\alpha \partial_\xi^{\tilde{\alpha}} a(x,\xi,\hbar)| \leq C_{\alpha,\tilde{\alpha}} \langle \xi \rangle^{m-|\tilde{\alpha}|}, \quad x \in \mathbb{R}^n,\, \xi \in \mathbb{R}^n.$$

Symbols in S^m thus behave roughly as polynomials of degree m. We write that $a \in \hbar^N S^m$ if

$$|\partial_x^\alpha \partial_\xi^{\tilde{\alpha}} a(x,\xi,\hbar)| \leq C_{\alpha,\tilde{\alpha}} \hbar^N \langle \xi \rangle^{m-|\tilde{\alpha}|}, \quad x \in \mathbb{R}^n,\, \xi \in \mathbb{R}^n.$$

Lemma B.1 (Asymptotic series)**.** *Let $m \in \mathbb{R}$ and the symbols $a_j \in S^{m-j}$ for $j = 0, 1, \ldots$ Then there exists a symbol $a \in S^m$ such that $a \sim \sum_{j=0}^{\infty} \hbar^j a_j$, that is for every $N \in \mathbb{N}$,*

$$a - \sum_{j=0}^{N} \hbar^j a_j \in \hbar^{N+1} S^{m-N-1}.$$

The symbol a is unique up to $\hbar^\infty S^{-\infty}$, in the sense that the difference of two such symbols is in $\hbar^N S^{-M}$ for all $N, M \in \mathbb{R}$. The principal symbol of a is given by a_0.

Operators

Using these symbol classes we can define semiclassical pseudodifferential operators (ψDOs). For a symbol $a \in S^m$ we define the corresponding semiclassical ψDO of order m, $\mathrm{Op}(a) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$,

$$\mathrm{Op}(a)u(x) := \tfrac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{\hbar}(x-y)\cdot\xi} a(x,\xi,\hbar) u(y) \,\mathrm{d}y\mathrm{d}\xi.$$

This is also called quantization of the symbol. $\mathrm{Op}(a)$ can be extended to $\mathcal{S}'(\mathbb{R}^n)$ and $\mathrm{Op}(a) : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ continuously. Note that $\mathrm{Op}(a)u(x) = \mathcal{F}_\hbar^{-1}(a(x,\cdot)\mathcal{F}_\hbar u(\cdot))$ and that the operator corresponding to the symbol $a(x,\xi) = \sum_{|\alpha|\leq N} a_\alpha(x)\xi^\alpha$ is

$$\mathrm{Op}(a)u = \sum_{|\alpha|\leq N} a_\alpha(x)(\hbar D)^\alpha u.$$

Notice that each derivative of this operator scales with $\hbar$.

For the present paper the most important example is the second order differential operator $-\hbar^2\Delta + \hbar^2 \sum_{j=1}^n \beta_j(x)\partial_j$. Its symbol is given by $a(x,\xi,\hbar) = |\xi|^2 + i\hbar \sum_{j=1}^n \beta_j(x)\xi_j$, and its principal symbol is $a_0(x,\xi,\hbar) = |\xi|^2$.

Semiclassical Sobolev spaces

For $s \in \mathbb{R}$ the semiclassical Sobolev spaces $H^s_{\mathrm{scl}}(\mathbb{R}^n)$ are algebraically equal to the standard Sobolev spaces $H^s(\mathbb{R}^n)$ but are endowed with different norms

$$\|u\|_{H^s_{\mathrm{scl}}(\mathbb{R}^n)} = \|J^s u\|_{L^2(\mathbb{R}^n)},$$

where the semiclassical Bessel potential is defined by $J^s = \mathrm{Op}(\langle\xi\rangle^s)$. Informally,

$$J^s = (1 - \hbar^2\Delta)^{s/2}, \quad s \in \mathbb{R}.$$

For example, $\|u\|^2_{H^1_{\mathrm{scl}}(\mathbb{R}^n)} = \|u\|^2_{L^2(\mathbb{R}^n)} + \|\hbar\nabla u\|^2_{L^2(\mathbb{R}^n)}$. A semiclassical ψDO of order m is continuous from $H^s_{\mathrm{scl}}(\mathbb{R}^n)$ to $H^{s-m}_{\mathrm{scl}}(\mathbb{R}^n)$.

Composition

Composition of semiclassical ψDOs can be analysed using the following calculus.

Theorem B.2 (Symbol calculus)**.** *Let $a \in S^m$ and $b \in S^{m'}$. Then* $\mathrm{Op}(a)\circ\mathrm{Op}(b) = \mathrm{Op}(a\#b)$ *for a certain $a\#b \in S^{m+m'}$ that admits the following asymptotic series*

$$a\#b(x,\xi,\hbar) \sim \sum_\alpha \frac{\hbar^{|\alpha|}i^{|\alpha|}}{\alpha!} D^\alpha_\xi a(x,\xi,\hbar) D^\alpha_x b(x,\xi,\hbar).$$

The commutator and disjoint support estimates (2.14) and (2.13) follow, respectively, from the following.

Corollary B.3 (Commutator and disjoint support)**.** *Let $a \in S^m$ and $b \in S^{m'}$. Then*

1. *$a\#b - b\#a \in \hbar S^{m+m'-1}$.*

2. *If* $\mathrm{supp}(a) \cap \mathrm{supp}(b) = \emptyset$*, then $a\#b \in \hbar^\infty S^{-\infty}$, i.e. $a\#b \in \hbar^N S^{-M}$ for all $N, M \in \mathbb{R}$.*

Proof. (1) The principal symbol of $a\#b$, that is the first term in its asymptotic series, is ab. The second term is $\frac{\hbar}{i}\sum_{j=1}^n \partial_{\xi_j} a(x,\xi,\hbar)\partial_{x_j} b(x,\xi,\hbar)$. We thus have that the principal symbol of the commutator $[\mathrm{Op}(a), \mathrm{Op}(b)] = \mathrm{Op}(a\#b - b\#a)$ is given by

$$\frac{\hbar}{i}\sum_{j=1}^n (\partial_{\xi_j} a \partial_{x_j} b - \partial_{x_j} a \partial_{\xi_j} b) \in \hbar S^{m+m'-1}.$$

(2) If $\mathrm{supp}(a) \cap \mathrm{supp}(b) = \emptyset$, then each term in the asymptotic series of $a\#b$ vanishes. □

References

[1] G. Alessandrini, L. Rondi, E. Rosset, and S. Vessella. The stability for the Cauchy problem for elliptic equations. *Inverse Problems*, 25:123004, 2009.

[2] M. Alnæs, J. Blechta, J. Hake, A. Johansson, B. Kehlet, A. Logg, C. Richardson, J. Ring, M. E. Rognes, and G. N. Wells. The FEniCS project version 1.5. *Archive of Numerical Software*, 3(100), 2015.

[3] B. Ayuso and L. D. Marini. Discontinuous Galerkin methods for advection-diffusion-reaction problems. *SIAM J. Numer. Anal.*, 47(2):1391–1420, 2009.

[4] I. M. Babuška and S. A. Sauter. Is the pollution effect of the FEM avoidable for the Helmholtz equation considering high wave numbers? *SIAM J. Numer. Anal.*, 34(6):2392–2423, 1997.

[5] D. Baskin, E. A. Spence, and J. Wunsch. Sharp high-frequency estimates for the Helmholtz equation and applications to boundary integral equations. *SIAM J. Math. Anal.*, 48(1):229–267, 2016.

[6] R. Becker and B. Vexler. Optimal control of the convection-diffusion equation using stabilized finite element methods. *Numer. Math.*, 106(3):349–367, 2007.

[7] F. B. Belgacem. Why is the Cauchy problem severely ill-posed? *Inverse Problems*, 23(2):823, 2007.

[8] S. Bertoluzza. The discrete commutator property of approximation spaces. *C. R. Acad. Sci. Paris Sér. I Math.*, 329(12):1097–1102, 1999.

[9] M. Boulakia, E. Burman, M. A. Fernández, and C. Voisembert. Data assimilation finite element method for the linearized Navier-Stokes equations in the low Reynolds regime. *Inverse Problems*, 36:085003, 2020.

[10] L. Bourgeois. Convergence rates for the quasi-reversibility method to solve the Cauchy problem for Laplace's equation. *Inverse problems*, 22(2):413, 2006.

[11] L. Bourgeois and L. Chesnel. On quasi-reversibility solutions to the Cauchy problem for the Laplace equation: regularity and error estimates. *ESAIM Math. Model. Numer. Anal.*, 54(2):493–529, 2020.

[12] A. N. Brooks and T. J. Hughes. Streamline upwind/Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations. *Comput. Methods Appl. Mech. Engrg.*, 32(1-3):199–259, 1982.

[13] E. Burman. A unified analysis for conforming and nonconforming stabilized finite element methods using interior penalty. *SIAM J. Numer. Anal.*, 43(5):2012–2033, 2005.

[14] E. Burman. Stabilized finite element methods for nonsymmetric, noncoercive, and ill-posed problems. Part I: elliptic equations. *SIAM J. Sci. Comput.*, 35(6):A2752–A2780, 2013.

[15] E. Burman. Stabilized finite element methods for nonsymmetric, noncoercive, and ill-posed problems. Part II: hyperbolic equations. *SIAM J. Sci. Comput.*, 36(4):A1911–A1936, 2014.

[16] E. Burman, A. Feizmohammadi, and L. Oksanen. A finite element data assimilation method for the wave equation. *Math. Comp.*, 89(324):1681–1709, 2020.

[17] E. Burman, J. Guzmán, and D. Leykekhman. Weighted error estimates of the continuous interior penalty method for singularly perturbed problems. *IMA J. Numer. Anal.*, 29(2):284–314, 2009.

[18] E. Burman and P. Hansbo. Edge stabilization for Galerkin approximations of convection–diffusion–reaction problems. *Comput. Methods Appl. Mech. Engrg.*, 193(15-16):1437–1453, 2004.

[19] E. Burman and P. Hansbo. Stabilized nonconforming finite element methods for data assimilation in incompressible flows. *Math. Comp.*, 87(311):1029–1050, 2018.

[20] E. Burman, P. Hansbo, and M. G. Larson. Solving ill-posed control problems by stabilized finite element methods: an alternative to Tikhonov regularization. *Inverse Problems*, 34:035004, 2018.

[21] E. Burman, M. Nechita, and L. Oksanen. Unique continuation for the Helmholtz equation using stabilized finite element methods. *J. Math. Pures Appl.*, 129:1–22, 2019.

[22] E. Burman, M. Nechita, and L. Oksanen. A stabilized finite element method for inverse problems subject to the convection–diffusion equation. I: diffusion-dominated regime. *Numer. Math.*, 144(451–477), 2020.

[23] E. Burman, M. Nechita, and L. Oksanen. A stabilized finite element method for inverse problems subject to the convection-diffusion equation. II: convection-dominated regime. *arXiv preprint arXiv:2006.13201*, 2020.

[24] E. Burman and L. Oksanen. Data assimilation for the heat equation using stabilized finite element methods. *Numer. Math.*, 139(3):505–528, 2018.

[25] E. Burman, H. Wu, and L. Zhu. Linear continuous interior penalty finite element method for Helmholtz equation with high wave number: one-dimensional analysis. *Numer. Methods Partial Differential Equations*, 32(5):1378–1410, 2016.

[26] T. Carleman. Sur un problème d'unicité pour les systèmes d'équations aux dérivées partielles à deux variables indépendantes. *Ark. Mat. Astr. Fys.*, 26B(17):1–9, 1939.

[27] L. Dede' and A. Quarteroni. Optimal control and numerical adaptivity for advection-diffusion equations. *ESAIM Math. Model. Numer. Anal.*, 39(5):1019–1040, 2005.

[28] D. A. Di Pietro and A. Ern. *Mathematical aspects of discontinuous Galerkin methods*, volume 69. Springer Science & Business Media, 2011.

[29] D. Dos Santos Ferreira, C. E. Kenig, M. Salo, and G. Uhlmann. Limiting Carleman weights and anisotropic inverse problems. *Invent. Math.*, 178(1):119–171, 2009.

[30] J. Douglas and T. Dupont. Interior penalty procedures for elliptic and parabolic Galerkin methods. In *Computing methods in applied sciences*, pages 207–216. Springer, 1976.

[31] H. W. Engl, M. Hanke, and A. Neubauer. *Regularization of inverse problems*, volume 375. Springer, 1996.

[32] A. Ern and J.-L. Guermond. *Theory and practice of finite elements*, volume 159 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2004.

[33] A. Ern and J.-L. Guermond. Evaluation of the condition number in linear systems arising in finite element approximations. *ESAIM Math. Model. Numer. Anal.*, 40(1):29–48, 2006.

[34] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, 2010.

[35] J. Hadamard. *Lectures on Cauchy's Problem in Linear Partial Differential Equations.* Yale University Press, 1923.

[36] F. Hecht. New development in FreeFem++. *J. Numer. Math.*, 20(3-4):251–265, 2012.

[37] M. Hinze, N. Yan, and Z. Zhou. Variational discretization for optimal control governed by convection dominated diffusion equations. *J. Comput. Math.*, 27(2-3):237–253, 2009.

[38] L. Hörmander. *The analysis of linear partial differential operators*, volume III. Springer-Verlag, 1985.

[39] L. Hörmander. *The analysis of linear partial differential operators*, volume IV. Springer-Verlag, 1985.

[40] T. Hrycak and V. Isakov. Increased stability in the continuation of solutions to the Helmholtz equation. *Inverse Problems*, 20(3):697–712, 2004.

[41] F. Ihlenburg and I. Babuška. Finite element solution of the Helmholtz equation with high wave number. I. The h-version of the FEM. *Comput. Math. Appl.*, 30(9):9–37, 1995.

[42] F. Ihlenburg and I. Babuška. Finite element solution of the Helmholtz equation with high wave number. II. The h-p version of the FEM. *SIAM J. Numer. Anal.*, 34(1):315–358, 1997.

[43] V. Isakov. *Inverse problems for partial differential equations*, volume 127 of *Applied Mathematical Sciences*. Springer, 3rd edition, 2017.

[44] F. John. Continuous dependence on data for solutions of partial differential equations with a prescribed bound. *Comm. Pure Appl. Math.*, 13:551–585, 1960.

[45] V. John, P. Knobloch, and J. Novo. Finite elements for scalar convection-dominated equations and incompressible flow problems: a never ending story? *Comput. Vis. Sci.*, 19(5-6):47–63, 2018.

[46] C. Johnson, U. Nävert, and J. Pitkäranta. Finite element methods for linear hyperbolic problems. *Comput. Methods Appl. Mech. Engrg.*, 45:285–312, 1984.

[47] M. V. Klibanov and F. Santosa. A computational quasi-reversibility method for Cauchy problems for Laplace's equation. *SIAM J. Appl. Math.*, 51(6):1653–1675, 1991.

[48] M. V. Klibanov and A. A. Timonov. *Carleman estimates for coefficient inverse problems and numerical applications*, volume 46. Walter de Gruyter, 2012.

[49] R. Lattes and J. Lions. *Méthode de quasi-réversibilité et applications*, volume 15 of *Travaux et Recherches Mathématiques*. Dunod, Paris, 1967.

[50] M. M. Lavrent'ev, V. G. Romanov, and S. P. Shishatskiĭ. *Ill-posed problems of mathematical physics and analysis*, volume 64 of *Translations of Mathematical Monographs*. American Mathematical Society, 1986.

[51] J. Le Rousseau and G. Lebeau. On Carleman estimates for elliptic and parabolic operators. Applications to unique continuation and control of parabolic equations. *ESAIM Control Optim. Calc. Var.*, 18(3):712–747, 2012.

[52] E. Malinnikova and S. Vessella. Quantitative uniqueness for elliptic equations with singular lower order terms. *Math. Ann.*, 353(4):1157–1181, 2012.

[53] J. M. Melenk and S. Sauter. Wavenumber explicit convergence analysis for Galerkin discretizations of the Helmholtz equation. *SIAM J. Numer. Anal.*, 49(3):1210–1243, 2011.

[54] H.-G. Roos and M. Stynes. Some open questions in the numerical analysis of singularly perturbed differential equations. *Comput. Methods Appl. Math.*, 15(4):531–550, 2015.

[55] H.-G. Roos, M. Stynes, and L. Tobiska. *Robust numerical methods for singularly perturbed differential equations: convection-diffusion-reaction and flow problems*, volume 24. Springer Science & Business Media, 2008.

[56] L. R. Scott and S. Zhang. Finite element interpolation of nonsmooth functions satisfying boundary conditions. *Math. Comp.*, 54(190):483–493, 1990.

[57] E. M. Stein. *Singular integrals and differentiability properties of functions*, volume 30 of *Princeton Mathematical Series*. Princeton University Press, 1970.

[58] D. Tataru. Unique continuation problems for partial differential equations. In C. B. Croke, M. S. Vogelius, G. Uhlmann, and I. Lasiecka, editors, *Geometric Methods in Inverse Problems and PDE Control*, pages 239–255. Springer, New York, NY, 2004.

[59] L. N. Trefethen. Quantifying the ill-conditioning of analytic continuation. *BIT*, 2020.

[60] H. Wu. Pre-asymptotic error analysis of CIP-FEM and FEM for the Helmholtz equation with high wave number. Part I: linear version. *IMA J. Numer. Anal.*, 34(3):1266–1288, 2014.

[61] N. Yan and Z. Zhou. A priori and a posteriori error analysis of edge stabilization Galerkin method for the optimal control problem governed by convection-dominated diffusion equation. *J. Comput. Appl. Math.*, 223(1):198–217, 2009.

[62] M. Zworski. *Semiclassical analysis*, volume 138 of *Graduate Studies in Mathematics*. American Mathematical Society, 2012.