Calculating Ramsey numbers by partitioning coloured graphs

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Abstract

In this paper we prove a new result about partitioning coloured complete graphs and use it to determine certain Ramsey Numbers exactly. The partitioning theorem we prove is that for $k \geq 1$, in every edge colouring of K_n with the colours red and blue, it is possible to cover all the vertices with k disjoint red paths and a disjoint blue balanced complete $(k+1)$-partite graph. When the colouring of K_n is connected in red, we prove a stronger result—that it is possible to cover all the vertices with k red paths and a blue balanced complete $(k+2)$-partite graph.

Using these results we determine the Ramsey Number of a path, P_n, versus a balanced complete t-partite graph on tm vertices, K_m^t, whenever $m \equiv 1 \pmod{n-1}$. We show that in this case $R(P_n, K_m^t) = (t-1)(n-1) + t(m-1) + 1$, generalizing a result of Erdős who proved the $m = 1$ case of this result. We also determine the Ramsey Number of a path P_n versus the power of a path P_n^t. We show that $R(P_n, P_n^t) = t(n-1) + \left\lfloor \frac{n}{t+1} \right\rfloor$, solving a conjecture of Allen, Brightwell, and Skokan.

1 Introduction

Ramsey Theory is a branch of mathematics concerned with finding ordered substructures in a mathematical structure which may, in principle, be highly disordered. An early example of a result in Ramsey Theory is a theorem due to Van der Waerden [18], which says that for any k and $r \geq 1$ there is a number $W(k,r)$, such that any colouring of the numbers $1, 2, \ldots, W(k,r)$ with r colours contains a monochromatic k-term arithmetic progression. A special case of a theorem due to Ramsey [16] says that for every n, there exists a number $R(n)$, such that every 2-edge-coloured complete graph on more than $R(n)$ vertices contains a monochromatic complete graph on n vertices. The number $R(n)$ is called a *Ramsey number.*

A central definition in Ramsey Theory is the *generalized Ramsey number* $R(G)$ of a graph G: the minimum n for which every 2-edge-colouring of K_n contains a monochromatic copy of G. For a pair of graphs G and H the Ramsey number of G versus H, $R(G,H)$, is defined to be the minimum n for which every 2-edge-colouring of K_n with the colours red and blue contains either a red copy of G or a blue copy of H. Although there have been many results which give good bounds on Ramsey numbers of graphs [6], the exact value of the Ramsey number $R(G,H)$ is only known when G and H each belong to one of a few families of graphs.

One of the first Ramsey numbers to be determined exactly was the Ramsey number of the path.

Theorem 1.1 (Gerencsér and Gyárfás, [5]). *For $m \leq n$ we have that*

$$R(P_n, P_m) = n + \left\lfloor \frac{m}{2} \right\rfloor - 1.$$

In the same paper where Gerencsér and Gyárfás proved Theorem 1.1, they also proved the following.

Theorem 1.2 (Gerencsér and Gyárfás, [5]). *The vertices of every 2-edge-coloured complete graph can be covered by two vertex-disjoint monochromatic paths of different colours.*

Throughout this paper we will prove several "partitioning theorems" of the form "the vertices of every 2-edge-coloured complete graph, K_n, can be covered by certain disjoint monochromatic subgraphs". In all the theorems that we prove, some of the subgraphs partitioning K_n are allowed to be empty (so, for example in Theorem 1.2 it would be allowed to cover the whole graph by just one monochromatic path). Whenever we have a 2-edge-coloured graph, the colours will always be called "red" and "blue".

The proof of Theorem 1.2 is so short that it was originally published in a footnote of [5]. Indeed to see that the theorem holds, simply find a red path R in K_n and a disjoint blue path B in K_n such that $|R| + |B|$ is as large as possible. Let r and b be endpoints of R and B respectively. If there is a vertex $x \notin R \cup B$, then it is easy to see that the triangle $\{x, r, b\}$ contains either a red path between x and r or a blue path between x and b. This path can be joined to R or B to obtain a new partition into two disjoint monochromatic paths of different colours (after perhaps deleting r or b from R or B). This contradicts the maximality of $|R| + |B|$.

Any result about covering the vertices of edge-coloured graphs by a small number of monochromatic subgraphs will imply a Ramsey-type result as a corollary. For example Theorem 1.2 implies the bound $R(P_n, P_m) \leq n + m - 1$. Indeed Theorem 1.2 shows that every 2-edge-coloured K_{n+m-1} can be covered by a red path R and a disjoint blue path B. Clearly these paths cannot cover all the vertices unless $|R| \geq n$ or $|B| \geq m$. This is the main technique we shall use to bound Ramsey numbers in this paper.

Although Theorem 1.2 originated as a technique to bound Ramsey Numbers, it subsequently gave birth to the area of partitioning edge-coloured complete graphs into monochromatic subgraphs. There have been many further results and conjectures in this area, many of which generalise Theorem 1.2. One particularly relevant conjecture which attempts to generalize Theorem 1.2 is the following.

Conjecture 1.3 (Gyárfás, [8]). *The vertices of every r-edge-coloured complete graph can be covered with r vertex-disjoint monochromatic paths.*

Although Theorems 1.1 and 1.2 have both led to many generalizations, there have not been many further attempts to use results about partitioning coloured graphs in order to bound Ramsey Numbers. A notable exception is the following result of Gyárfás and Lehel.

Theorem 1.4 (Gyárfás & Lehel, [7, 9]). *Suppose that the edges of $K_{n,n}$ are coloured with two colours such that one of the vertex classes of $K_{n,n}$ is contained in a monochromatic connected component. Then there exist two disjoint monochromatic paths with different colours which cover all, except possibly one, of the vertices of $K_{n,n}$.*

Gyárfás and Lehel used this result to determine the bipartite Ramsey Number of a path i.e. the smallest n for which every 2-edge-coloured $K_{n,n}$ contains a red copy of P_i or a blue P_j, for fixed integers i and j. Recently Theorem 1.4 was used by the author in the proof of the $r = 3$ case of Conjecture 1.3 [15].

In this paper we prove a new theorem about partitioning 2-edge-coloured complete graphs, and use it to determine certain Ramsey Numbers exactly. Our starting point will be the following lemma which was used by the author in the proof of the $r = 3$ case of Conjecture 1.3 [15].

Recall that a complete bipartite graph is called *balanced* if both of its parts have the same order.

Lemma 1.5. *Suppose that the edges of K_n are coloured with two colours. Then K_n can be covered by a red path and a disjoint blue balanced complete bipartite graph.*

Lemma 1.5 immediately implies the bound $R(P_n, K_{m,m}) \leq n + 2m - 2$. It turns out that when $m \equiv 1 \pmod{n-1}$, this bound is best possible. The following theorem was proved by Häggkvist.

Theorem 1.6 (Häggkvist, [10]). *If $m, \ell \equiv 1 \pmod{n-1}$, then we have*

$$R(P_n, K_{m,\ell}) = n + m + \ell - 2.$$

The lower bound on Theorem 1.6 comes from considering a colouring of $K_{n+m+\ell-3}$ consisting of $1 + (m + \ell - 2)/(n - 1)$ red copies of K_{n-1} and all other edges are coloured blue. The condition $m, \ell \equiv 1 \pmod{n-1}$ ensures that the number $1 + \frac{(m+\ell-2)}{(n-1)}$ is an integer.

The main theorem about partitioning coloured graphs that we will prove in this paper is a generalization of Lemma 1.5. Recall that the balanced complete k-partite graph, K_m^k, is the graph whose vertices can be partitioned into k sets $A_1, \dots, A_k$ such that $|A_1| = \cdots = |A_k| = m$ for all i, and there is an edge between $a_i \in A_i$ and $a_j \in A_j$ if, and only if, $i \neq j$. We will prove the following.

Theorem 1.7. *Let $k \geq 1$. Suppose that the edges of K_n are coloured with two colours. Then K_n can be covered by k disjoint red paths and a disjoint blue balanced complete $(k+1)$-partite graph.*

As a corollary of Theorem 1.7 we obtain that for all m satisfying $m \equiv 1 \pmod{n-1}$ we have $R(P_n, K_m^t) = (t-1)(n-1) + t(m-1) + 1$. This generalizes a result of Erdős who showed that $R(P_n, K_t) = (t-1)(n-1) + 1$ (see [4, 13]).

Instead of proving Theorem 1.7 directly, we will actually prove a strengthening of it, and then deduce Theorem 1.7 as a corollary. The strengthening that we prove is the following.

Theorem 1.8. *Let $k \geq 2$. Suppose that the edges of K_n are coloured with two colours, such that the red subgraph is connected. Then K_n can be covered by a red tree T with at most k leaves and a disjoint blue balanced complete $(k+1)$-partite graph.*

In the above theorem "red subgraph" means the subgraph of K_n consisting of all the red edges. It is not immediately clear that Theorem 1.8 implies Theorem 1.7. Notice that every tree with k leaves can be covered by $k-1$ vertex-disjoint paths. Therefore Theorem 1.8 has the following corollary.

Corollary 1.9. *Let $k \geq 1$. Suppose that the edges of K_n are coloured two colours, such that the red subgraph is connected. Then K_n can be covered by k disjoint red paths and a disjoint blue balanced complete $(k+2)$-partite graph.*

Corollary 1.9 shows that when the colouring of K_n is connected in red, then the conclusion of Theorem 1.7 can actually be strengthened—we can use one less red path in the covering of K_n.

Theorem 1.7 is easy to deduce from Corollary 1.9.

Proof of Theorem 1.7, assuming Corollary 1.9. For $k = 1$, Theorem 1.7 is just Lemma 1.5. This lemma was originally proven in [15], and this proof is also reproduced in Section 2. We shall therefore assume that $k \geq 2$.

Suppose that we have an arbitrary 2-edge-colouring of K_n. We add an extra vertex v to the graph and add red edges between v and all other vertices. The resulting colouring of K_{n+1} is connected in red. Therefore we can apply Corollary 1.9 to $K_n + v$ in order to cover it by $k-1$ disjoint red paths and a disjoint blue balanced complete $(k+1)$-partite graph H. Since all the edges containing v are red, the vertex v cannot be in H. Therefore, v must be contained in one of the red paths. Deleting v from K_{n+1} will split this path into two red paths. Therefore, deleting v gives a partition of K_n into k disjoint red paths and a blue balanced complete $(k+1)$-partite graph as required. □

A well known remark of Erdős and Rado says that any 2-edge-coloured complete graph is connected in one of the colours. Therefore Theorem 1.8 implies that every 2-edge-coloured complete graph can be covered by a monochromatic path and a monochromatic balanced complete tripartite graph (where we have no control over which colour each graph has).

The tth power of a path of order n is the graph with vertex set $\{1, \ldots, n\}$ and ij an edge whenever $1 \leq |i-j| \leq t$. It is easy to see that K_m^t contains a copy of P_{tm}^{t-1}. Therefore, Theorem 1.7 and Corollary 1.9 imply the following.

Corollary 1.10. *Let $k \geq 1$. Suppose that the edges of K_n are coloured with two colours.*

- *K_n can be covered with k disjoint red paths and a disjoint blue kth power of a path.*
- *If K_n is connected in red, then K_n can be covered with k disjoint red paths and a disjoint blue $(k+1)$th power of a path.*

The first part of this corollary may be seen as a generalization of Theorem 1.2. We are also able to use Corollary 1.10 and Theorem 1.1 to determine the Ramsey numbers of a path on n vertices versus a power of a path on n vertices.

Theorem 1.11. *For all $k \geq 1$ and $n \geq k+1$, we have*

$$R(P_n, P_n^k) = (n-1)k + \left\lfloor \frac{n}{k+1} \right\rfloor .$$

Theorem 1.11 solves a conjecture of Allen, Brightwell, and Skokan who asked for the value of $R(P_n, P_n^k)$ in [1].

The structure of this paper is as follows. In Section 2 we define some notation and prove certain weakenings of Theorem 1.8. These weakenings serve to illustrate the main ideas used in the proof of Theorem 1.8 and hopefully aid the reader in understanding that theorem. In addition the results we prove in Section 2 will be strong enough to imply Corollary 1.10. This means that it is possible to prove Theorem 1.11 without using the full strength of Theorem 1.8. In Section 3 we prove Theorem 1.8. In Section 4 we prove Theorem 1.11 and also determine $R(P_n, K_m^t)$ whenever $m \equiv 1 \pmod{n-1}$. In Section 5 we discuss some further problems which may be approachable using the techniques presented in this paper.

2 Preliminaries

For a non-empty path P, it will be convenient to distinguish between the two endpoints of P saying that one endpoint is the "start" of P and the other is the "end" of P. Thus we will often say things like "Let P be a path from u to v". Let P be a path from a to b in a graph G and Q a path from c to d in G. If P and Q are vertex disjoint and bc is an edge in G, then we define $P + Q$ to be the unique path from a to d formed by joining P

and Q with the edge bc. If P is a path and Q is a subpath of P sharing an endpoint with P, then $P - Q$ will denote the subpath of P with vertex set $V(P) \setminus V(Q)$.

If $H_1, \dots, H_k$, are subgraphs of G satisfying $V(G) = V(H_1) \cup \dots \cup V(H_k)$, we say that $H_1, \dots, H_k$ *cover* G. If $H_1, \dots, H_k$ are also vertex-disjoint, we say that $H_1, \dots, H_k$ *partition* G.

A 2-edge-colouring of a graph G is an arbitrary assignment of colours "red" and "blue" to the edges of G. Since, in this paper, we are one interested in edge-coloured graphs we will sometimes abbreviate "2-edge-coloured" to "2-coloured". If a graph G is edge-coloured we define the *red subgraph* or *red colour class* of G to be the subgraph of G with vertex set $V(G)$ and edge set consisting of all the red edges of G. We say that G is *connected in* red, if the red colour class is a connected graph. Similar definitions are made for the colour blue as well. For a vertex v in a coloured graph G, $N_r(v)$ denotes the set of vertices in G to which v is connected by a red neighbour, and $N_b(v)$ denotes the set of vertices in G to which v is connected by a blue neighbour.

For all other notation, we refer to [3].

2.1 Proof outline

In this section we go through some of the ideas which go into the proofs of Theorems 1.8 and 1.11.

Outline of Theorem 1.8

The proof of Theorem 1.8 could be summarized as follows "first we find a partition of our graph into a tree T and a complete (but not necessarily balanced) multipartite graph H which in some way extremal and then we show that in order to satisfy the extremality conditions H must in fact be balanced". In order to illustrate this strategy, we give the proof of Lemma 1.5 here.

Proof of Lemma 1.5. Notice that a graph with no edges is a complete bipartite graph (with one of the parts empty). Therefore, any 2-edge-coloured K_n certainly has a partition into a red path and a blue complete bipartite graph (by assigning all of K_n to be one of the parts of the complete bipartite graph). Partition K_n into a red path P and a blue complete bipartite graph $B(X, Y)$ with parts X and Y such that the following hold.

(i) $\max(|X|, |Y|)$ is as small as possible.

(ii) $|P|$ is as small as possible, whilst keeping (i) true.

We are done if $|X| = |Y|$ holds. Therefore, without loss of generality, suppose that we have $|X| < |Y|$.

Suppose that $P = \emptyset$. Then let y be any vertex in Y, $P' = \{y\}$, $Y' = Y - y$, and $X' = X$. This new partition of K_n satisfies $\max(|Y'|, |X'|) < |Y| = \max(|X|, |Y|)$, contradicting the minimality of the original partition in (i).

Now, suppose that P is non-empty. Let p be an end vertex of P. If there is a red edge py for $y \in Y$, then letting $P' = P + y$ and $Y' = Y - y$ gives a partition of K_n into a red path, P', and the complete bipartite graph $B(X, Y')$ with parts X and Y'. However we have $\max(|Y'|, |X|) < |Y| = \max(|X|, |Y|)$, contradicting the minimality of the original partition in (i).

If all the edges between p and Y are blue, then note that letting $P' = P - p$ and $X' = X + p$ gives a partition of K_n into a red path and a complete bipartite graph $B(X', Y)$ with parts X' and Y. We have that $\max(|X'|, |Y|) = |Y| = \max(|X|, |Y|)$ and $|P'| < |P|$, contradicting the minimality of the original partition in (ii). $\square$

The proof of Theorem 1.8 is similar to the above proof. The overall strategy will be again to consider a partition into a red tree T and blue complete multipartite graph which satisfy extremality conditions similar to (i) and (ii). The main difficulty is that in the proof of Theorem 1.8, the extremality conditions are much more complicated than in the above proof, and so it takes a lot longer to prove that the multipartite graph is balanced.

Next we present another weakening of Theorem 1.8, which can replace it in some applications. Given a 2-coloured K_n, and a set $S \subseteq K_n$, let $c(S)$ be the order of the largest red component of $K_n[S]$. The following is a weakening of Theorem 1.8.

Theorem 2.1. *Let $k \geq 2$. Suppose that the edges of K_n are coloured with the colours red and blue, such that the red subgraph is connected. Then K_n can be covered by a red tree with at most k leaves and a disjoint set S satisfying $c(S) \leq |S|/(k+1)$.*

Notice that Theorem 2.1 is indeed a weakening of Theorem 1.8. To see this, simply note that if we have a set $S \subseteq V(K_n)$ such that the induced colouring of K_n on S contains a spanning blue balanced $(k+2)$-partite graph, then S satisfies $c(S) \leq |S|/(k+2)$.

Proof of Theorem 2.1. We partition K_n into a red tree T and a set S with the following properties.

(i) T has at most k leaves.

(ii) $c(S)$ is as small as possible, whilst keeping (i) true.

(iii) The number of red components in S of order $c(S)$ is as small as possible, whilst keeping (i) and (ii) true.

(iv) $|T|$ is as small as possible, whilst keeping (i) – (iii) true.

Let S^+ be the subset of S formed by taking the union of the red components of order $c(S)$ in S. Let S^- be $S \setminus S^+$. Let $v_1, \ldots, v_\ell$ be the leaves of T. By assumption (i), we have $\ell \leq k$.

Suppose that for some i, v_i has a red neighbour $u \in S^+$. Then we can let T' be the red tree formed from T by adding the vertex u and the edge $v_i u$, and $S' = V(K_n) \setminus V(T')$. Notice that T' still has at most k leaves. Since S' is a subset of S, we must have $c(S') = c(S)$ (by minimality of $c(S)$ in (ii)). But since u was in a red component of order $c(S)$, S' must have one less component of order $c(S)$ than S had. This contradicts minimality of the original partition in (iii).

Suppose that $\ell < k$. By connectedness of the red colour class of K_n there is a red edge between some $v \in T$ and $u \in S^+$. Let T' be the red tree formed from T by adding the vertex u and the edge vu, and $S' = V(K_n) \setminus V(T')$. Then T' has $\ell + 1 \leq k$ leaves, and so satisfies (i). As before, since S' is a subset of S, we must have $c(S') = c(S)$. But since u was in a red component of order $c(S)$, S' must have one less component of order $c(S)$ than S had, contradicting the minimality of the original partition in (iii).

For the remainder of the proof, we can suppose that $\ell = k$, and the vertices $v_1, \dots, v_k$ do not have any red neighbours in S^+. For a leaf v_i, let $\overline{N}(v_i)$ be the red connected component containing v_i in the induced graph on $S^- \cup \{v_i\}$

Suppose that $\overline{N}(v_i) \cap \overline{N}(v_j) \neq \emptyset$ for some $i \neq j$. Then there must be a red path P between v_i and v_j contained in $S^- \cup \{v_i, v_j\}$. Let T_1 be the graph formed by adding the path P to the tree T. Notice that T_1 is a red graph with $k - 2$ leaves and exactly one cycle. By connectedness of the red colour class of K_n there is a red edge between some $v \in T_1$ and $u \in S^+$. Let T_2 be the graph formed by adding the vertex u and the edge uv to T_1. Notice that T_2 is a red graph with between 1 and $k - 1$ leaves and exactly one cycle. Therefore T_2 contains an edge xy which is contained on the cycle and the vertex x has degree at least 3. Let T_3 be T_2 minus the edge xy and $S' = V(K_n) \setminus V(T_2)$. Now T_3 is a red tree with at most k leaves. As before $S' \subset S$ and (ii) implies that we must have $c(S') = c(S)$. As before this contradicts (iii) since the vertex u which we removed from S^+ was contained in a red component of order $c(S)$.

Suppose that $\overline{N}(v_i) \cap \overline{N}(v_j) = \emptyset$ for all $i \neq j$. If there is some i for which $\overline{N}(v_i) < c(S)$, then letting $T' = T - v_i$ and $S' = V(K_n) \setminus T'$ gives a new tree satisfying (i) – (iii), but with $|T'| < |T|$, contradicting the minimality of the original partition in (iv). Therefore we can suppose that $\overline{N}(v_i) \geq c(S)$ for all i. Let $I = \{v_i : \overline{N}(v_i) = c(S)\}$, $T' = T \setminus I$, and $S' = S \cup I$. Notice that by the definition of I and S', $c(S') = c(S)$. Recall that we have $\overline{N}(v_i) - v_i \subseteq S^-$ for all i which implies

$$|S| \geq |S^+| + \sum_{i=1}^{k} |\overline{N}(v_i) - v_i| \geq c(S) + \sum_{i=1}^{k} (|\overline{N}(v_i)| - 1) \geq (k+1)c(S) - |I|.$$

This is equivalent to $|S'| \geq (k+1)c(S)$ which, using $c(S') = c(S)$, gives $c(S') \leq |S'|/(k+1)$ and so the partition into T' and S' satisfies the conditions of the theorem. $\square$

Most of the steps of the above proof reoccur in the proof of Theorem 1.8. Notice that the partition into T and S satisfying (i) – (iv) did not necessarily satisfy the conditions of Theorem 2.1—indeed in the last theorem of the proof we had to define a new partition into $T' = T \setminus I$ and $S' = S \cup I$ in order to prove the theorem. A similar idea will be employed in the proof of Theorem 1.8—in the proof of that theorem we will first proceed similarly to how we did in Thereom 2.1 in order to partition K_n into a red tree and a blue multipartite graph which is "reasonably" balanced. Then we will use an auxiliary lemma (Lemma 3.1) in order to turn the reasonably balanced multipartite graph into a balanced one.

Deducing Corollary 1.10 from Theorem 2.1

Here we show that Theorem 2.1 implies Corollary 1.10. This allows one to prove Theorem 1.11 without resorting to the full strength of Theorem 1.8. Notice that to deduce Corollary 1.10 from Theorem 2.1, it is sufficient to prove the following proposition.

Proposition 2.2. *Let K_n be a 2-edge-coloured complete graph. Suppose that K_n contains a set S which satisfies $c(S) \leq |S|/(k+2)$. Then S contains a spanning blue $(k+1)$th power of a path.*

Indeed combining Proposition 2.2 with Theorem 2.1 we obtain that every 2-edge-coloured complete graph which is connected in red can be covered by a red tree T with at most k leaves and a blue $(k+1)$th power of a path. Since every tree with at most k leaves can be partitioned into $k-1$ disjoint paths, this implies part (ii) of Corollary 1.10. For $k \geq 2$, part (i) of Corollary 1.10 follows from part (ii) in exactly the same way as we deduced Theorem 1.7 from Corollary 1.9 in the introduction. Indeed, to prove part (i) of Corollary 1.10, we start with an arbitrary colouring of K_n. We add a vertex v to the graph and add red edges between v and all other vertices. The resulting colouring of K_{n+1} is connected in red. Therefore we can apply part (ii) of Corollary 1.10 to $K_n + v$ in order to cover it by $k-1$ disjoint red paths and a disjoint blue kth power of a path P. Since all the edges containing v are red, the vertex v cannot be in P (unless $|P| \leq 1$). Therefore, removing v gives a partition of K_n into k disjoint red paths a blue kth power of a path as required.

It remains to verify Proposition 2.2. One way of doing this is to notice that if $c(S) \leq |S|/(k+2)$, then the induced blue subgraph of K_n on S must have minimal degree at least $\frac{k+1}{k+2}|S|$. A conjecture of Seymour says that all graphs with minimal degree $\frac{k}{k+1}|S|$ contain a kth power of Hamiltonian cycle [17]. Seymour's Conjecture has been proven for graphs with sufficiently large order by Komlós, Sárközy, and Szemerédi [11]. Seymour's Conjecture readily implies Proposition 2.2. However given that our set S has a very specific structure, it is not hard to prove that it contains a spanning blue $(k+1)$th power of a path without using the full strength of Komlós, Sárközy, and Szemerédi's result. One way of doing this is by induction on the number of vertices of S. We omit the details, because Corollary 1.10 follows much more readily from the stronger Theorem 1.8.

Outline of Theorem 1.11

Theorem 1.11 is proved by a case distinction depending on how large the largest red component of a 2-edge-coloured K_n is.

Notice that if a complete graph on $(n-1)k + \left\lfloor \frac{n}{k+1} \right\rfloor$ vertices contains a red component C with more than $(n-1)k$ vertices, then part (ii) of Corollary 1.11 implies that C contains either a red P_n or a blue P_n^k. Indeed Corollary 1.11 gives us a partition of C into $k-1$ red paths and a blue kth power of a path—by the Pigeonhole Principle, one of these must have at least n vertices.

When the largest red component C has size less than $(n-1)k$, but not too small, then a similar strategy can be used to find a blue P_n^k. In this case we apply part (ii) of

Corollary 1.11 to the largest red component $C \subseteq K_n$ in order to find a long blue jth power of a path P in C for suitably chosen j. Then, we apply part (i) of Corollary 1.11 to $K_n \setminus C$ in order to find a blue long $(k-j)$ power of a path Q in the complement of C. Using the fact that all the edges between P and Q must be blue (since C is a red component), it is possible to join P and Q in order to obtain a kth power of a path on n vertices.

Finally, when all red components are very small, then the strategy changes. In this case we apply Theorem 1.1 to the graph several times in order to find several long blue paths in the graph. Since the red components of K_n are all small, these paths can be constructed to have no red edges between them. This will allow us to join them together into a blue P_n^k.

3 Partitioning coloured complete graphs

In this section we prove Theorem 1.8. The proof has many ideas in common with the proofs of Lemma 1.5 and Theorem 2.1.

We begin by proving an intermediate lemma. The following lemma will allow us to take a partition of K_n into a red tree T and a blue multipartite graph H which is "reasonably balanced", and output a partition of K_n into a red tree and a blue balanced complete $(k+1)$-partite graph as required.

Lemma 3.1. *Suppose that we have a 2-edge-coloured complete graph K_n containing $k+1$ sets $A_0, \dots A_k$, k sets $B_1, \dots B_k$, and k sets $N_1, \dots, N_k$ such that the following hold.*

(i) *The sets $A_0, \dots A_k, B_1, \dots B_k$ partition $V(K_n)$.*

(ii) *For all $1 \le i < j \le k$ all the edges between any of the sets A_0, A_i, B_i, A_j, and B_j are blue.*

(iii) *For all i, every red connected component of B_i intersects N_i.*

(iv) *$|A_0| \ge |A_i|$ for all $i \ge 1$.*

(v) *$|A_i| + |B_i| \ge |A_0|$ for all $i \ge 1$.*

(vi) *For all $i \ge 1$ either $|B_i| \le 2 \min_{t=1}^{k} |B_t|$ or $|A_i| + |B_i| \le |A_0| + \min_{t=1}^{k} |B_t|$ holds.*

Then, there is a partition of K_n into k red paths $P_1, \dots, P_k$ and a blue balanced $k+1$ partite graph. In addition, for each i, the path P_i is either empty or starts in N_i.

Proof. The proof is by induction on the quantity $\sum_{t=1}^{k} |B_t|$.

First we prove the base case of the induction, i.e. we prove the lemma when $\sum_{t=1}^{k} |B_t| = 0$. In this case $B_i = \emptyset$ for all i, and so conditions (iv) and (v) imply that $|A_i| = |A_0|$ for all i. Therefore, by (ii), K_n contains a spanning blue complete $(k+1)$-partite graph with parts $A_0, \dots, A_k$. We can take $P_1 = \dots = P_k = \emptyset$ to obtain the required partition.

We now prove the induction step. Suppose that the lemma holds for all 2-edge-coloured complete graphs K'_n containing sets $A'_0, \dots A'_k$, $B'_1, \dots B'_k$, and $N'_1, \dots, N'_k$ as in the statement of the lemma but satisfying $\sum_{t=1}^{k} |B'_t| < \sum_{t=1}^{k} |B_t|$. We will show that the lemma holds for K_n as well.

First we show that if there is a partition of K_n satisfying (i) – (vi), then the sets $A_0, \dots, A_k$ and $B_1, \dots, B_k$ can be relabeled to obtain a partition satisfying (i) – (vi) and also the following

$$|A_0| \geq |A_1| \geq \cdots \geq |A_k|, \tag{1}$$

$$|B_1| \leq \cdots \leq |B_k|. \tag{2}$$

The following claim guarantees this.

Claim 3.2. *Let σ be a permutation of $(0, 1, \dots, k)$ ensuring that $|A_{\sigma(0)}| \geq |A_{\sigma(1)}| \geq \cdots \geq |A_{\sigma(k)}|$ holds. Let τ be a permutation of $(1, \dots, k)$ ensuring that $|B_{\tau(1)}| \leq \cdots \leq |B_{\tau(k)}|$ holds. Let $A'_i = A_{\sigma(i)}$, $B'_i = B_{\tau(i)}$, and $N'_i = N_{\tau(i)}$. Then the sets A'_i, B'_i, and N'_i satisfy (i) – (vi).*

Proof. Notice that N'_i, A'_i and B'_i satisfy (i) – (iii) trivially.

Since the sets A_i satisfy (iv), we can assume that $\sigma(0) = 0$. This ensures that the sets $A_{\sigma(i)}$ satisfy (iv).

For (v), note that if for some $j \geq 1$, $|A_{\sigma(j)}| + |B_{\tau(j)}| < |A_0|$, then we also have $|A_{\sigma(x)}| + |B_{\tau(y)}| < |A_0|$ for all $x \geq j$ and $y \leq j$. However, the Pigeonhole Principle implies that $\sigma(x) = \tau(y)$ for some $x \geq j$ and $y \leq j$, contradicting the fact that A_i and B_i satisfy (v) for all i.

Suppose that (vi) fails to hold. Then for some j, $|B_{\tau(j)}| > 2 \min_{t=1}^{k} |B_t|$ and $|A_{\sigma(j)}| + |B_{\tau(j)}| > |A_0| + \min_{t=1}^{k} |B_t|$ both hold. If we have $|A_{\tau(i)}| \geq |A_{\sigma(j)}|$ for some $i \geq j$, then $|B_{\tau(i)}| \geq |B_{\tau(j)}| > 2 \min_{t=1}^{k} |B_t|$ and $|A_{\tau(i)}| + |B_{\tau(i)}| \geq |A_{\sigma(j)}| + |B_{\tau(j)}| > |A_0| + \min_{t=1}^{k} |B_t|$ both hold, contradicting the fact that A_i and B_i satisfy (vi) for all i. Therefore, we can assume that $|A_{\tau(i)}| < |A_{\sigma(j)}|$ for all $i \geq j$. This, together with $|A_{\sigma(0)}| \geq |A_{\sigma(1)}| \geq \cdots \geq |A_{\sigma(k)}|$ implies that $\{\tau(j), \tau(j+1), \dots, \tau(k)\} \subseteq \{\sigma(j+1), \sigma(j+2), \dots, \sigma(k)\}$, contradicting τ being injective. $\square$

By the above claim, without loss of generality we may assume that the A_is and B_is satisfy (1) and (2).

Notice that the lemma holds trivially if we have the following.

$$|A_0| = |A_1| + |B_1| = |A_2| + |B_2| = \cdots = |A_k| + |B_k|. \tag{3}$$

Indeed, if (3) holds, then K_n contains a spanning blue complete $(k+1)$-partite graph with parts $A_0, A_1 \cup B_1 \dots, A_k \cup B_k$, and so taking $P_1 = \cdots = P_k = \emptyset$ gives the required partition.

Therefore, we can assume that (3) fails to hold, so there is some j such that $|A_j| + |B_j| > |A_0|$. In addition, we can assume that j is as large as possible, and so $|A_i| + |B_i| = |A_0|$ for all $i > j$.

First we deal with the case when $|B_j| \leq 1$. Notice that in this case (2) implies that $|B_i| \leq 1$ for all $i \leq j$. Using the fact that $|A_i| + |B_i| > |A_0|$ for these i, (iv) implies that $|B_i| = 1$ and $|A_i| = |A_0|$ for all $i \leq j$. For $i \leq j$ let P_i be the single vertex in B_i, and for $i > j$ let $P_i = \emptyset$. This ensures that $K_n \setminus (P_1, \dots, P_k)$ is a balanced complete k-partite graph with classes $A_1, \dots, A_j, A_{j+1} \cup B_{j+1}, \dots, A_k \cup B_k$, giving the required partition of K_n.

For the remainder of the proof, we assume that $|B_j| \geq 2$. We split into two cases depending on whether B_j is connected in red or not.

Case 1: Suppose that B_j is connected in red. Let v be a vertex in $B_j \cap N_j$ (which exists by (iii)). Let $K'_n = K_n - v$, $B'_j = B_j - v$, $N'_j = N_r(v)$ and $A'_i = A_i, B'_i = B_i, N'_i = N_i$ for all other i. We show that the graph K'_n with the sets A'_i, B'_i, and N'_i satisfies (i) – (vi).

Conditions (i), (ii), and (iv) hold trivially for the new sets as a consequence of them holding for the original sets A_* and B_*. Condition (iii) holds trivially whenever $i \neq j$, and holds for $i = j$ as a consequence of B_j being connected in red.

To prove (v), it is sufficient to show that $|A'_j| + |B'_j| \geq |A'_0|$. This is equivalent to $|A_j| + |B_j - v| \geq |A_0|$, which holds since $|A_j| + |B_j| > |A_0|$.

We now prove (vi). Note that by (2) we have $\min_{t=1}^k |B'_t| = \min(|B_1|, |B'_j|)$. If $\min_{t=1}^k |B'_t| = |B_1|$ holds, then (vi) is satisfied for the new sets $A'_0, \dots, A'_k, B'_0, \dots, B'_k$ as a consequence of it being satisfied for the original sets $A_0, \dots, A_k, B_0, \dots, B_k$. Now, suppose that we have $\min_{t=1}^k |B'_t| = |B'_j|$. For $i > j$, we have $|A'_i| + |B'_i| = |A'_0|$ by choice of j, which implies that (vi) holds for these i. If $i \leq j$, then we have $|B_i| \leq |B_j|$ which together with $|B_j| \geq 2$ implies that $B'_i \leq 2|B_j| - 2 = 2|B'_j|$ holds.

Therefore, the graph K'_n with the sets A'_i, B'_i,and N'_i satisfies (i) – (vi). We also have $\sum_{t=1}^k |B'_t| = \sum_{t=1}^k |B_t| - 1$, and so, by induction K'_n can be partitioned into k red paths $P'_1, \dots, P'_k$ starting in $N'_1, \dots, N'_k$ respectively and a blue balanced $(k+1)$-partite graph H. Since P'_j starts in $N'_j = N_r(v)$, we have the required partition of K_n into k paths $P'_1, \dots, v + P'_j, \dots P'_k$ and a blue balanced $(k+1)$-partite graph H.

Case 2: Suppose that B_j is disconnected in red. We will find a new partition of K_n into sets $A'_0, \dots, A'_k$ and $B'_1, \dots, B'_k$, which together with $N_1, \dots, N_k$ satisfy (i) – (vi). We will also have $\sum_{t=1}^k |B'_t| < \sum_{t=1}^k |B_t|$ which implies the lemma by induction.

Let B_j^- be the smallest red component of B_j and let $B_j^+ = B_j \setminus B_j^-$. Notice that since $|B_j^-| + |B_j^+| = |B_j|$, we have $2|B_j^-| \leq |B_j|$ and $|B_j| \leq 2|B_j^+|$. There are two subcases depending on whether we have $|A_j| + |B_j^-| \leq |A_0|$ or not.

Case 2.1: Suppose that we have $|A_j| + |B_j^-| \leq |A_0|$. Let $B'_j = B_j^+$ and $A'_j = A_j \cup B_j^-$, and $A'_i = A_i$, $B'_i = B_i$ for all other i. As before, conditions (i) – (iii) hold trivially.

To prove (iv), it is sufficient to show that $|A'_0| \geq |A'_j|$ which is true since we are assuming that $|A_j| + |B_j^-| \leq |A_0|$.

To prove (v), it is sufficient to show that $|A'_j| + |B'_j| \geq |A'_0|$ which holds since we have $|A'_j| + |B'_j| = |A_j| + |B_j| \geq |A_0|$.

To prove (vi), first note that we have $\min_{t=1}^k |B'_t| = \min(B_1, B'_j)$. If $\min_{t=1}^k |B'_t| = |B_1|$ holds, then (vi) is satisfied for the new sets $A'_0, \dots, A'_k, B'_0, \dots, B'_k$ as a consequence of it being satisfied for the original sets $A_0, \dots, A_k, B_0, \dots, B_k$. Now, suppose that we have $\min_{t=1}^k |B'_t| = |B'_j|$. For $i > j$, we have $|A'_i| + |B'_i| = |A'_0|$ which implies that (vi) holds for these i. If $i \leq j$, then we have $|B_i| \leq |B_j|$ which together with $|B_j| \leq 2|B_j^+|$ implies that $B_i' \leq 2|B_j'|$ holds.

Notice that we have $\sum_{t=1}^k |B_t'| < \sum_{t=1}^k |B_t|$, and so the lemma holds by induction.

Case 2.2: Suppose that we have $|A_j| + |B_j^-| > |A_0|$.

We claim that in this case $|B_j| \leq 2|B_1|$ holds. Indeed by (vi) and (2), we have that either $|B_j| \leq 2|B_1|$ holds, or we have $|A_j| + |B_j| \leq |A_0| + |B_1|$. Adding $|A_j| + |B_j| \leq |A_0| + |B_1|$ to $|A_j| + |B_j^-| > |A_0|$ gives $|B_j^+| < |B_1|$. This, together with $|B_j| \leq 2|B_j^+|$ implies that $|B_j| \leq 2|B_1|$ always holds.

There are two cases, depending on whether we have $j = k$ or not.

Case 2.2.1: Suppose that $j \neq k$. Let $B_j' = B_j^+$, $A_{j+1}' = A_{j+1} \cup B_j^-$, and $A_i' = A_i$, $B_i' = B_i$ for all other i. As before, conditions (i) – (iii) hold trivially.

To prove (iv), it is sufficient to show that $|A_0'| \geq |A_{j+1}'|$, which holds as a consequence of $|A_{j+1}| + |B_{j+1}| = |A_0|$ and (2).

To prove (v), it is sufficient show that $|A_j'| + |B_j'| \geq |A_0'|$, which holds as a consequence of $|B_j^+| \geq |B_j^-|$ and $|A_j| + |B_j^-| > |A_0|$.

We now prove (vi). For $i \geq j+2$, note that we have $|A_i'| + |B_i'| = |A_0'|$ which implies that (vi) holds for these i. For $i \leq j$, (vi) holds since using $|B_i'| \leq |B_j| \leq 2|B_j^+| = 2|B_j'|$ and $|B_j| \leq 2|B_1|$ we get that $|B_i'| \leq 2\min(|B_1|, |B_j'|) = 2\min_{t=1}^k |B_t'|$. For $i = j+1$, we have $|A_{j+1}'| + |B_{j+1}'| \leq |A_0'| + \min_{t=1}^k |B_t'|$ as a consequence of $|A_{j+1}'| + |B_{j+1}'| = |A_0| + |B_j^-|$, $|B_j^-| \leq \frac{1}{2}|B_j|$, and $|B_j| \leq \min(2|B_1|, 2|B_j'|) = 2\min_{t=1}^k |B_t'|$.

Notice that we have $\sum_{t=1}^k |B_t'| < \sum_{t=1}^k |B_t|$, and so the lemma holds by induction.

Case 2.2.2: Suppose that $j = k$. Let $B_k' = B_k^+$, $A_k' = A_0$, $A_0' = A_k \cup B_k^-$, and $A_i' = A_i, B_i' = B_i$ for all other i. As before, conditions (i) – (iii) hold trivially.

Since $|A_0| \geq |A_i'|$ for all $i \geq 1$, to prove (iv), it is sufficient to show that $|A_0'| \geq |A_0|$. This holds since we assumed that $|A_k| + |B_k^-| > |A_0|$.

To prove (v), we have to show that $|A_i| + |B_i| \geq |A_k| + |B_k^-|$ for all $i < k$ and also that $|A_0| + |B_k^+| \geq |A_k| + |B_k^-|$. We know that for all i we have $|B_k^-| \leq \frac{1}{2}|B_k| \leq |B_1| \leq |B_i|$ which, combined with (1), implies that we have $|A_i| + |B_i| \geq |A_k| + |B_k^-|$. We also know that $|B_k^+| \geq |B_k^-|$ which, combined with (1), implies that we have $|A_0| + |B_k^+| \geq |A_k| + |B_k^-|$.

To prove (vi), note that we have $\min_{t=1}^k |B_t'| = \min(B_1, B_k')$. Suppose that $\min_{t=1}^k |B_t'| = |B_k'|$ holds. Then we have $|B_i'| \leq 2|B_k'|$ for all i as a consequence of (2) and $2|B_k'| \geq |B_k|$. Suppose that $\min_{t=1}^k |B_t'| = |B_1'|$ holds. Then for $i < k$, (vi) is satisfied for the new sets $A_0', \ldots, A_k', B_0', \ldots, B_k'$ as a consequence of it being satisfied for the original sets $A_0, \ldots, A_k, B_0, \ldots, B_k$ and $|A_0'| \geq |A_0|$. For $i = k$, (vi) holds since we have $|B_k'| \leq |B_k| \leq 2|B_1|$.

Notice that we have $\sum_{t=1}^k |B_t'| < \sum_{t=1}^k |B_t|$, and so the lemma holds by induction. □

We now use the above lemma to prove Theorem 1.8. The proof has many similarities to that of Theorem 2.1.

Proof of Theorem 1.8. We will partition K_n into a red tree T, and sets $A_0, A_1, \ldots, A_k$ and $B_1, \ldots, B_k$ with certain properties. For convenience we will define $A = A_0 \cup A_1 \cup \cdots \cup A_k$ and $B = B_1 \cup \cdots \cup B_k$. The tree T will have l leaves which will be called $v_1, v_2, \ldots, v_l$. For a set $S \subseteq K_n$, let $c(S)$ be the order of the largest red component of $K_n[S]$. Define $f(S)$ to be the number of red components contained in S of order $c(A \cup B)$. The tree T, and sets $A_0, A_1, \dots, A_k$ and $B_1, \dots, B_k$ are chosen to satisfy the following.

(I) For $1 \le i < j \le k$, all the edges between A_0,A_i, A_j, B_i, and B_j are blue.

(II) T has l leaves $v_1, \dots, v_l$, where $l \le k$. For $i = 1, \dots, l$, the leaf v_i, is joined to every red component of B_i by a red edge.

(III) $c(A \cup B)$ is as small as possible, whilst keeping (I) – (II) true.

(IV) $\sum_{t=1}^{k} |f(B_t) - \frac{1}{2}|$ is as small as possible, whilst keeping (I) – (III) true.

(V) $f(A)$ is as small as possible, whilst keeping (I) – (IV) true.

(VI) $|T|$ is as small as possible, whilst keeping (I) – (V) true.

(VII) $|\{i \in \{1, \dots, k\} : |B_i| \ge c(A \cup B)\}|$ is as large as possible, whilst keeping (I) – (VI) true.

(VIII) $\sum_{\{t:|B_t|<c(A\cup B)\}} |B_t|$ is as large as possible, whilst keeping (I) – (VII) true.

(IX) $\sum_{t=1}^{k} |B_t|$ is as small as possible, whilst keeping (I) – (VIII) true.

(X) $\max_{t=1}^{k} |A_t|$ is as small as possible, whilst keeping (I) – (IX) true.

(XI) $|\{i \in \{1, \dots, k\} : |A_i| = \max_{t=1}^{k} |A_t|\}|$ is as small as possible, whilst keeping (I) – (X) true.

In order to prove Theorem 1.8 we will show that the partition of $A \cup B$ into A_i and B_i satisfies conditions (i), (ii), (iv), (v), and (vi) of Lemma 3.1. Then, Lemma 3.1 will easily imply the theorem.

Without loss of generality, we may assume that the A_is are labelled such that we have

$$|A_0| \ge |A_1| \ge \dots \ge |A_k|. \tag{4}$$

We begin by proving a sequence of claims.

Claim 3.3. *For each i, $f(B_i)$ is either 0 or 1.*

Proof. Suppose that $f(B_i) \ge 2$. Let C be a red component in B_i of order $c(A \cup B)$. Let $B_i' = B_i \setminus C$, $A_0' = A_0 \cup C$, $T' = T$ and $A_j' = A_j$, $B_j' = B_j$ for other j. It is easy to see that the new partition satisfies (I) – (III). We have that $f(B_i') = f(B_i) - 1$, which combined with $f(B_i) \ge 2$ implies that $|f(B_i') - \frac{1}{2}| < |f(B_i) - \frac{1}{2}|$ contradicting the minimality of the original partition in (IV). □

Claim 3.4. *If we have $f(B_i) = 1$ for some i, then we also have $|B_i| = c(A \cup B)$.*

Proof. Suppose that $f(B_i) = 1$ and $|B_i| > c(A \cup B)$ both hold. Then B_i contains some non-empty red connected component C of order strictly less than $c(A \cup B)$. Let $T' = T$, $A'_0 = A \cup C$, $B'_i = B_i \setminus C$, and $A'_t = \emptyset$, $B'_t = B_t$ for all other t.

It is easy to see that the new partition satisfies (I) – (VIII). However $|B'_i| < |B_i|$ and $|B'_t| = |B_t|$ for $t \neq i$ which contradicts the minimality of the original partition in (IX). □

Claim 3.5. *We have that $f(A) \geq 1$.*

Proof. Suppose that we have $f(A) = 0$. Then all the red components of order $c(A \cup B)$ of $A \cup B$ must be contained in B. Let $I = \{i : c(B_i) = 1\}$, and for each $i \in I$, let C_i be a red component of order $c(A \cup B)$ contained in B_i. By Claim 3.3 any red component of $A \cup B$ or order $c(A \cup B)$ must be one of the C_is. For $i \in \{1, \ldots, l\} \cap I$, condition (II) implies that v_i has a red neighbour u_i in C_i. For $i \in \{l+1, \ldots, k\} \cap I$, by red-connectedness of K_n and condition (I), there is a red edge $u_i w_i$ between $u_i \in C_i$ and some $w_i \in T$.

Let $A'_0 = A \cup B \setminus \{u_i : i \in I\}$ and $A'_j = B'_j = \emptyset$ for $j \geq 1$. Let T' be the tree with vertex set $V(T) \cup \{u_i : i \in I\}$ formed from T by joining u_i to v_i for $i \in \{1, \ldots, l\} \cap I$ and u_i to w_i for $i \in \{l+1, \ldots, k\} \cap I$.

Clearly the new partition satisfies (I) and (II). However since each of the largest components of $A \cup B$ lost a vertex, we must have $c(K_n \setminus T') < c(A \cup B)$ contradicting the minimality of the original partition in (III). □

Claim 3.6. *If $i > l$, then $f(B_i) = 1$ holds.*

Proof. Suppose that $f(B_i) = 0$ for some $i > l$.

By Claim 3.5, there is a red component C of order $c(A \cup B)$ in A. Let $T' = T$, $A'_0 = A \setminus C$, $B'_i = B_i \cup C$, and $A'_t = \emptyset$, $B'_t = B_t$ for all other t.

It is easy to see that the new partition satisfies (I) – (IV). However we have $f(A) = f(A) - 1$ contradicting the minimality of the original partition in (V). □

Claim 3.7. *For every i, we have $|A_0| \leq |A_i| + c(A \cup B)$.*

Proof. Suppose that for some i we have $|A_0| > |A_i| + c(A \cup B)$. Let C be any red component of A_0. We have $|C| \leq c(A \cup B)$. Let $A'_0 = A_0 \setminus C$, $A'_i = A_i \cup C$, $T' = T$ and $A'_j = A_j$, $B'_j = B_j$ otherwise. It is easy to see that the new partition will satisfy (I) – (IX). Notice that $|A_0| > |A_i| + c(A \cup B)$ ensures that we have $|A'_0|, |A'_i| < |A_0|$. This implies that $\max_{t=0}^k |A'_t| \leq \max_{t=0}^k |A_t|$, which, by minimality of the original partition in (X) implies that $\max_{t=0}^k |A'_t| = \max_{t=0}^k |A_t|$. But $|A'_0|, |A'_i| < |A_0| = \max_{t=0}^k |A_t|$ implies that the quantity $|\{j \in \{1, \ldots, k\} : |A'_j| = \max_{t=1}^k |A'_t|\}|$ must be smaller than it was in the original partition, contradicting (XI). □

Claim 3.8. *For every i, we have $|B_i| \geq c(A \cup B)$.*

Proof. Suppose that $|B_i| < c(A \cup B)$ for some i. Notice that this implies that $f(B_i) = 0$. Thus, by Claim 3.6, we have that $i \leq l$.

First suppose that we have $N_r(v_i) \cap A \neq \emptyset$. Let C be a red component of A which intersects $N_r(v_i)$. Let $T' = T$, $B'_i = B_i \cup C$, and $A'_t = A_t \setminus C$, $B'_t = B_t$, for all other t.

The new partition satisfies (I) trivially. By choice of C, the new partition satisfies (II). It is easy to see that $c(A'_t), c(B'_t) \leq c(A \cup B)$ for every t which implies that (III) holds for the new partition. Since $f(B_i) = 0$ holds, we have that $f(B'_i) \leq 1$ and hence $|f(B'_i) - \frac{1}{2}| = |f(B_i) - \frac{1}{2}|$ which implies that (IV) holds for the new partition.

It is easy to see that $f(A'_t) \leq f(A_t)$ for all t, which implies that (V) holds for the new partition. Since $T' = T$, (VI) holds for the new partition.

We have that $|B'_t| \geq |B_t|$ for all t. Therefore, so as not to contradict the maximality of the original partition in (VII), we must have $|B'_i| < c(A \cup B)$. However since $|B'_i| > |B_i|$, this contradicts maximality of the original partition in (VIII).

For the remainder of the proof of this claim, we may assume that we have $N_r(v_i) \subseteq B$. Recall that (II) implies that $N_r(v_i) \cap B_i \neq \emptyset$. There are two cases depending on whether the red neighbours of v_i are all contained B_i or not.

Case 1: Suppose that $N_r(v_i) \subseteq B_i$.

Let $T' = T - v_i$, $B'_i = B_i + v_i$, and $A'_j = A_j$, $B'_j = B_j$ for all other j. The resulting partition satisfies (I) since $N_r(v_i) \subseteq B_i$. Condition (II) implies that $B_i + v_i$ is connected in red. This, together with the fact that the neighbour of v_i in T is connected to B'_i by a red edge implies that condition (II) holds for the new partition. The only red component of the new partition which was not a red component of the old partition is $B_i \cup v$, which is of order at most $c(A \cup B)$ since $|B_i| < c(A \cup B)$. This implies that (III) is satisfied. Since $f(B_i) = 0$, we must have $f(B'_i) = 0$ or 1, which means that $|f(B'_i) - \frac{1}{2}| = |f(B_i) - \frac{1}{2}|$ and hence the new partition satisfies (IV). The new partition satisfies (V) since we have $A'_0 \cup \cdots \cup A'_k = A$. However $|T'| = |T| - 1$, contradicting the minimality of the original tree T in (VI).

Case 2: Suppose that $N_r(v_i) \cap B_j \neq \emptyset$ for some $j \neq i$. Let C be a red component of B_j which intersects $N_r(v_i)$. Recall that Claim 3.4 implies that either $c(B_j) < c(A \cup B)$ or $|B_j| = c(A \cup B)$ holds. In either case we obtain $c(B_j \setminus C) < c(A \cup B)$.

There are two subcases, depending on whether $j \leq l$ holds.

Case 2.1: Suppose that $j > l$. By Claim 3.5 there is a red component $C_A \subseteq A$ of order $c(A \cup B)$. Let $B'_i = B_i \cup C$, $B'_j = (B_j \cup C_A) \setminus C$, $T' = T$ and $A'_t = A_t \setminus C_A$, $B'_t = B_t$ for all other t.

The resulting partition trivially satisfies (I). Condition (II) follows from the fact that v_i is connected to C by a red edge. We have $A'_0 \cup \cdots \cup A'_k \cup B'_1 \cup \cdots \cup B'_k = A \cup B$ which implies that the new partition satisfies (III). Using $|B_i|, c(B_j \setminus C) < c(A \cup B)$ we obtain that $f(B'_i), f(B'_j) \leq 1$ and $f(B'_t) = f(B_t)$ otherwise. This implies that $\sum_{t=1}^{k} |f(B'_t) - \frac{1}{2}| = \sum_{t=1}^{k} |f(B_t) - \frac{1}{2}|$, and so the new partition satisfies (IV). However, we have $f(A'_0 \cup \cdots \cup A'_k) = f(A) - 1$, contradicting the minimality of the original partition in (V).

Case 2.2: Suppose that $j \leq l$. Since $i \neq j$, this implies that we have $l \geq 2$.

Let u_i be a red neighbour of v_i in C. By (II), v_j has a red neighbour u_j in C. There must be a red path P between u_i and u_j contained in C.

Notice that joining T and P using the edges $u_i v_i$ and $u_j v_j$ produces a graph T_1 which has $l - 2$ leaves and exactly one cycle (which passes through P). By Claim 3.5 A contains a red component C_A of order $c(A \cup B)$. By red-connectedness of K_n, there must be some edge xv'_j between $x \in T$ and a vertex $v'_j \in C_A$.

We construct a tree T' and sets A'_t and B'_t as follows.

(a) Suppose that $x \neq v_t$ for any $t \in \{1, \ldots, l\} \setminus \{i, j\}$. In this case we let T_2 be the graph with vertices $V(T_1) \cup \{v'_j\}$, formed from T_1 by adding the edge xv'_j. Notice that T_2 has $l-1$ leaves and exactly one cycle. Therefore, the cycle in T_2 must contain a vertex y of degree at least 3. Let v'_i be a neighbour of y on the cycle. We let T' be the tree formed from T_2 by removing the edge yv'_i. The leaves of T' are $\{v_1, \ldots, v_l\} \setminus \{v_i, v_j\}$, v'_j and possibly v'_i (depending on whether the degree of v'_i in T_2 is 2 or not.)

We also let $A'_0 = (A \cup B_i \cup B_j) \setminus (P \cup v'_j)$, $B'_i = B'_j = \emptyset$, and $A_t = \emptyset$, $B'_t = B_t$, $v'_t = v_t$ for $t \neq i, j$.

(b) Suppose that $x = v_s$ for some $s \in \{1, \ldots, l\} \setminus \{i, j\}$ and $f(B_s) = 1$. In this case, Claim 3.4 implies that B_s is connected. Let v'_s be a neighbour of x in B_s. Let T_2 be the graph with vertices $V(T_1) \cup \{v'_j, v'_s\}$, formed from T_1 by adding the edges xv'_j and xv'_s. As before T_2 has $l-1$ leaves and exactly one cycle, which contains a vertex y of degree at least 3. Let v'_i be a neighbour of y on the cycle. We let T' be the tree formed from T_2 by removing the edge yv'_i. The leaves of T' are $\{v_1, \ldots, v_l\} \setminus \{v_i, v_j, v_s\}$, v'_j, v'_s and possibly v'_i (depending on whether the degree of v'_i in T_2 is 2 or not.)

We also let $A'_0 = (A \cup B_i \cup B_j) \setminus (P \cup v'_j)'_j$, $B'_i = B'_j = \emptyset$, $B'_s = B_s - v'_s$ and $A_t = \emptyset$, $B'_t = B_t$, $v'_t = v_t$ for $t \neq i, j, s$.

(c) Suppose that $x = v_s$ for some $s \in \{1, \ldots, l\} \setminus \{i, j\}$ and $f(B_s) = 0$. Let T_2 be the graph with vertices $V(T_1) \cup \{v'_j\}$, formed from T_1 by adding the edge xv'_j. Then T_2 has $l-2$ leaves and exactly one cycle, which contains a vertex y of degree at least 3. Let v'_i be a neighbour of y on the cycle. We let T' be the tree formed from T_2 by removing the edge yv'_i. The leaves of T' are $\{v_1, \ldots, v_l\} \setminus \{v_i, v_j, v_s\}$, v'_j and possibly v'_i (depending on whether the degree of v'_i in T_2 is 2 or not.)

We also let $A'_0 = (A \cup B_i \cup B_j \cup B_s) \setminus (P \cup v'_j)$, $B'_i = B'_j = B'_s = \emptyset$, and $A_t = \emptyset$, $B'_t = B_t$, $v'_t = v_t$ for $t \neq i, j, s$.

Clearly the new partition satisfies (I). Notice that all the leaves of T' are vertices v'_t for various t. It is easy to see that for all t for which v'_t is defined, v'_t is connected to all the red components of B'_t, so the new partition satisfies (II).

Since $A'_0 \cup \cdots \cup A'_k \cup B'_1 \cup \cdots \cup B'_k \subseteq A \cup B$, we must have $c(A'_0 \cup \cdots \cup A'_k \cup B'_1 \cup \cdots \cup B'_k) \leq c(A \cup B)$ and hence the new partition satisfies (III). Since for all t, we have $B'_t \subseteq B_t$, the new partition satisfies (IV). Recall that $c(B_j \setminus C) < c(A \cup B)$, which combined with the fact that P is non-empty, $P \subseteq C$, and $|C| \leq c(A \cup B)$ implies that $c(B_j \setminus P) < c(A \cup B)$. This, combined with the fact that $c(B_i) < c(A \cup B)$ (and, in case (c), $c(B_s) < c(A \cup B)$) implies that the red components of size $c(A \cup B)$ in $A'_0 \cup \cdots \cup A'_k$ are exactly those of A, minus C_A. Therefore we have $f(A'_0 \cup \cdots \cup A'_k) = f(A) - 1$, contradicting the minimality of the original partition in (V). $\square$

Claim 3.9. *For every i, we have $|B_i| \leq 2c(A \cup B)$.*

Proof. Suppose that $B_i > 2c(A \cup B)$. Combining this with Claim 3.3, means that there is a red component, C, in B_i satisfying $|C| < c(A \cup B)$. Let $B'_i = B_i \setminus C$, $A'_0 = A_0 \cup C$, and $A'_t = A_t$, $B'_t = B_t$, $T' = T$ otherwise.

The new partition satisfies (I) – (II) trivially. It is easy to see that $c(A'_t) = c(A_t)$ and $c(B'_t) = c(B_t)$ for every t which implies that (III) holds for the new partition. Also we have $f(A'_t) = f(A_t)$ and $f(B'_t) = f(B_t)$ for every t which implies that (IV) – (V) hold for the new partition. Since $T' = T$, (VI) holds for the new partition. Since $|B'_t| = |B_t|$ for $t \neq i$ and $|B'_i| \geq c(A \cup B)$, the new partition satisfies (VII) and (VIII).

However, we have that $|B'_i| < |B_i|$ which contradicts minimality of the original partition in (IX). □

We now prove the theorem.

For each $i = 1, \ldots, k$ we define a set $N_i \subseteq A \cup B$. If $i \leq l$, let $N_i = N_r(v_i)$. If $i > l$, let $N_i = \bigcup_{v \in T} N_r(v)$.

We will show that the graph $K_n \setminus T$, together with the sets $A_0, \ldots, A_k$, $B_1, \ldots, B_k$, and $N_1, \ldots, N_k$ satisfies conditions (i) – (vi) of Lemma 3.1.

Condition (i) follows from the definition of $A_0, \ldots, A_k$, and $B_1, \ldots, B_k$. Condition (ii) follows immediately from (I). Condition (iii) follows from (II) whenever $i \leq l$ and from red-connectedness of K_n whenever $i > l$. Condition (iv) follows from the fact that we are assuming (4).

Combining Claims 3.7 and 3.8 implies that we have $|B_i| + |A_i| \geq c(A \cup B) + |A_i| \geq |A_0|$ for all i. This proves condition (v) of Lemma 3.1.

Combining Claims 3.8 and 3.9 implies that we have $2|B_i| \geq 2c(A \cup B) \geq |B_j|$ for all i and j . This proves condition (vi) of Lemma 3.1.

Therefore, the graph $K_n \setminus T$, together with the sets $A_0, \ldots, A_k$, $B_1, \ldots, B_k$, and $N_1, \ldots, N_k$ satisfies all the conditions of Lemma 3.1. By Lemma 3.1, $K_n \setminus T$ can be partitioned into paths $P_1, \ldots, P_k$ starting in $N_1, \ldots, N_k$ and a balanced complete $(k+1)$-partite graph H. For each i, the path P_i can be joined to T by a red edge in order to obtain the required partition of K_n into a tree with at most k leaves on the vertex set $T \cup P_1 \cup \cdots \cup P_k$ and a balanced complete $(k+1)$-partite graph H. □

4 Ramsey Numbers

In this section, we use the results of the previous section to determine the the value of the Ramsey number of a path versus certain other graphs.

First we determine $R(P_n, K^t_m)$ whenever $m \equiv 1 \pmod{n-1}$.

Theorem 4.1. *If $m \equiv 1 \pmod{n-1}$ then we have*

$$R(P_n, K^t_m) = (t-1)(n-1) + t(m-1) + 1.$$

Proof. For the upper bound, apply Theorem 1.7 to the given 2-edge-coloured complete graph on $(t-1)(n-1) + t(m-1) + 1$ vertices. This gives us $t-1$ red paths and a blue balanced complete t-partite graph which cover all the vertices of $K_{(t-1)(n-1)+t(m-1)+1}$. By the Pigeonhole Principle either one of the paths has order at least n or the complete t-partite graph has order at least $t(m-1)+1$. Since the complete t-partite graph is balanced, if it has order $\geq t(m-1)+1$, then it must have at least tm vertices.

For the lower bound, consider a colouring of the complete graph on $(t-1)(n-1)+t(m-1)$ vertices consisting of $(t-1)+t(m-1)/(n-1)$ disjoint red copies of K_{n-1} and all other edges coloured blue. The condition $m \equiv 1 \pmod{n-1}$ ensures that we can do this. Since all the red components of the resulting graph have order at most $n-1$, the graph contains no red P_n. The graph contains no a blue K^t_m, since every vertex class of such a graph would have to intersect at least $(m-1)/(n-1)+1$ of the red copies of K_{n-1} and there are only $(t-1)+t(m-1)/(n-1)$ of these. □

In the remainder of this section we will prove Theorem 1.11. An outline of this proof is given in Section 2. First we will use Theorem 1.7 and Corollary 1.9 to find upper bounds on $R(P_n, P^t_m)$.

Lemma 4.2. *The following statements are true.*

(a) $R(P_n, P^t_m) \leq (n-2)t + m$ *for all* n, m *and* $t \geq 1$.

(b) Suppose that $t \geq 2$ *and* $n, m \geq 1$. *Every* 2*-edge-coloured complete graph on* $(n-1)(t-1)+m$ *vertices which is connected in red contains either a red* P_n *or a blue* P^t_m.

Proof. For part (a), notice that by Theorem 1.7, we can partition a 2-edge-coloured $K_{(n-2)t+m}$ into t red paths $P_1, \ldots, P_t$ and a blue tth power of a path P^t. Suppose that there are no red paths of order n in $K_{(n-2)t+m}$. Suppose that i of the paths $P_1, \ldots, P_t$ are of order $n-1$. Without loss of generality we may assume that these are the paths $P_1, \ldots, P_i$. We have $|P^t| + (n-2)(t-i) + (n-1)i \geq |P^t| + |P_1| + \cdots + |P_t| = (n-2)t + m$ which implies $i + |P^t| \geq m$. For each j, let v_j be one of the endpoint of P_j. Notice that since there are no red paths of order n in $K_{(n-2)t+m}$, all the edges in $\{v_1, \ldots, v_i, p\}$ are blue for any $p \in P^t$. This allows us to extend P^t by adding i extra vertices $v_1, \ldots, v_i$ to obtain a tth power of a path of order m.

Part (b) follows immediately from Corollary 1.9 and the fact that a balanced complete t-partite graph contains a spanning $(t-1)$th power of a path. □

It is worth noticing that the above lemma could also be deduced from Theorem 2.1 and Proposition 2.2.

The following simple lemma allows us to join powers of paths together.

Lemma 4.3. *Let* G *be a graph. Suppose that* G *contains a* $(k-i)$*th power of a path,* P^{k-i}*, and a disjoint* $(i-1)$*th power of a path,* Q^{i-1}*, such that the following hold.*

(i) *All the edges between* P^{k-i} *and* Q^{i-1} *are present.*

(ii) $|P^{k-i}| = (k-i+1)\left\lfloor \frac{n}{k+1} \right\rfloor + r_P$ *for some* $r_P \leq k-i+1$.

(iii) $|Q^{i-1}| = i\left\lfloor \frac{n}{k+1} \right\rfloor + r_Q$ *for some* $r_Q \leq i$.

(iv) $|P^{k-i}| + |Q^{i-1}| \geq n$.

Then G contains a kth power of a path on n vertices.

Proof. Let $p_1, \ldots, p_{|P^{k-i}|}$ be the vertices of P^{k-i} and $q_1, \ldots, q_{|Q^{i-1}|}$ be the vertices of Q^{i-1}. It is easy to see that the following sequence of vertices is a kth power of a path on at least n vertices.

$$\begin{aligned}
&q_1, \ldots q_{r_Q}\\
&p_1, \ldots, p_{k-i+1}, q_{r_Q+1}, \ldots, q_{r_Q+i}\\
&p_{k-i+2}, \ldots, p_{2(k-i+1)}, q_{r_Q+i+1}, \ldots, q_{r_Q+2i}\\
&\qquad\qquad\vdots\\
&p_{(k-i+1)\left(\left\lfloor \frac{n}{k+1} \right\rfloor-1\right)+1}, \ldots, p_{(k-i+1)\left\lfloor \frac{n}{k+1} \right\rfloor}, q_{r_Q+\left(\left\lfloor \frac{n}{k+1} \right\rfloor-1\right)i+1}, \ldots, q_{r_Q+\left\lfloor \frac{n}{k+1} \right\rfloor i}\\
&p_{(k-i+1)\left\lfloor \frac{n}{k+1} \right\rfloor+1}, \ldots, p_{(k-i+1)\left\lfloor \frac{n}{k+1} \right\rfloor+r_P}
\end{aligned}$$

□

We are now ready to prove Theorem 1.11.

Proof of Theorem 1.11. For the lower bound $R(P_n, P_n^k) \geq (n-1)k + \left\lfloor \frac{n}{k+1} \right\rfloor$, consider a colouring of $K_{(n-1)k+\left\lfloor \frac{n}{k+1} \right\rfloor-1}$ consisting of k disjoint red copies of K_{n-1} and one disjoint red copy of $K_{\left\lfloor \frac{n}{k+1} \right\rfloor-1}$. All edges outside of these are blue. It is easy to see that when $n \geq k+1$, this colouring contains neither a red path on n vertices nor a blue P_n^k.

It remains to prove the upper bound $R(P_n, P_n^k) \leq (n-1)k + \left\lfloor \frac{n}{k+1} \right\rfloor$. Let K be a 2-edge-coloured complete graph on $(n-1)k + \left\lfloor \frac{n}{k+1} \right\rfloor$ vertices. Suppose that K does not contain any red paths of order n. We will find a blue copy of P_n^k.

Let C be the largest red component of K. The following claim will give us three cases to consider.

Claim 4.4. *One of the following always holds.*

(i) $|C| \geq 2(n-1) - (k-2)\left\lfloor \frac{n}{k+1} \right\rfloor + 1$.

(ii) *There is a set B, such that all the edges between B and $V(K) \setminus B$ are blue and also*

$$n + \left\lfloor \frac{n}{k+1} \right\rfloor \leq |B| \leq 2(n-1) - (k-2)\left\lfloor \frac{n}{k+1} \right\rfloor.$$

(iii) *The vertices of K can be partitioned into k disjoint sets $B_1, \ldots, B_k$ such that for $i \neq j$ all the edges between B_i and B_j are blue and we have*

$$|B_1| \geq |B_2| \geq \cdots \geq |B_k| \geq \left\lceil \frac{n}{k+1} \right\rceil.$$

Proof. Suppose that neither (i) nor (ii) hold.

This implies that all the red components in K have order at most $n + \lfloor \frac{n}{k+1} \rfloor - 1$. Let B be a subset of $V(K)$ such that the following hold.

(a) All the edges between B and $V(K) \setminus B$ are blue.

(b) $|B| \leq n - 1 + \lfloor \frac{n}{k+1} \rfloor$.

(c) $|B|$ is as large as possible, whilst keeping (a) and (b) true.

Suppose that there is a red component C' in $V(K) \setminus B$ of order at most $\lceil \frac{n}{k+1} \rceil - 1$. Let $B' = B \cup C'$. Notice that $n \geq k \lfloor \frac{n}{k+1} \rfloor + \lceil \frac{n}{k+1} \rceil$ holds for all integers $n, k \geq 0$. This implies that we have $|B'| = |B| + |C'| \leq 2(n-1) - (k-2) \lfloor \frac{n}{k+1} \rfloor$ which implies that B' satisfies (ii) (since otherwise B' would be a set satisfying (a) and (b) of larger order than B).

Suppose that all the red components in $V(K) \setminus B$ have order at least $\lceil \frac{n}{k+1} \rceil$. Since $n \geq 2$, we have

$$|V(K) \setminus B| \geq (n-1)(k-1) > (k-2)\left(n - 1 + \left\lfloor \frac{n}{k+1} \right\rfloor\right). \tag{5}$$

Using the fact that all red components of K have order at most $n - 1 + \lfloor \frac{n}{k+1} \rfloor$, (5) implies that $V(K) \setminus B$ must have at least $k-1$ components. Therefore $V(K) \setminus B$ can be partitioned into $k-1$ sets $B_2, \ldots, B_k$ which, together with $B_1 = B$, satisfy (iii). □

We distinguish three cases, depending on which part of Claim 4.4 holds.

Case 1: If part (i) of Claim 4.4 holds, then there must be some $i \leq k-2$, such that we have

$$(k-i)(n-1) - i\left\lfloor \frac{n}{k+1} \right\rfloor + 1 \leq |C| \leq (k-i+1)(n-1) - (i-1)\left\lfloor \frac{n}{k+1} \right\rfloor. \tag{6}$$

Combining $(k-i)(n-1) - i\lfloor \frac{n}{k+1} \rfloor + 1 \leq |C|$ with part (b) of Lemma 4.2 shows that C must contain a blue $(k-i)$th power of a path, P^{k-i}, on $n - i\lfloor \frac{n}{k+1} \rfloor$ vertices. If $i = 0$, then P^{k-i} is a copy of P_n^k, and so the theorem holds. Therefore, we can assume that $i \geq 1$.

Notice that (6) implies that we have $|V(K) \setminus C| \geq (i-1)(n-1) + i\lfloor \frac{n}{k+1} \rfloor$. Combining this with part (a) of Lemma 4.2 shows that $V(K) \setminus C$ must contain a blue $(i-1)$th power of a path, Q^{i-1}, on $i\lfloor \frac{n}{k+1} \rfloor + i - 1$ vertices.

Notice that $|P^{k-i}| = n - i\lfloor \frac{n}{k+1} \rfloor \geq (k-i+1)\lfloor \frac{n}{k+1} \rfloor$. Let $\hat{P}^{k-i}$ be a subpath of P^{k-i} of length $\min\left(n - i\lfloor \frac{n}{k+1} \rfloor, (k-i+1)\lfloor \frac{n}{k+1} \rfloor + k - i + 1\right)$. Since all the edges between C and $V(K) \setminus C$ are blue, we can apply Lemma 4.3 to $\hat{P}^{k-i}$ and Q^{i-1} with $r_P = \min(n \pmod{k+1}, k-i+1)$ and $r_Q = i-1$ in order to find a blue kth power of a path on n vertices in G.

Case 2: Suppose that there is some set $B \subseteq V(K)$ such that all the edges between B and $V(K) \setminus B$ are blue and also

$$n + \left\lfloor \frac{n}{k+1} \right\rfloor \leq |B| \leq 2(n-1) - (k-2)\left\lfloor \frac{n}{k+1} \right\rfloor.$$

Apply Theorem 1.1 to B in order to find a blue path, P, of order $2\left\lfloor\frac{n}{k+1}\right\rfloor+2$ in B.

Notice that we have $|V(K)\setminus B|\geq(k-2)(n-1)+(k-1)\left\lfloor\frac{n}{k+1}\right\rfloor$. Part (a) of Lemma 4.2 shows that $V(K)\setminus B$ must contain a blue $(k-2)$nd power of a path, Q^{k-2} , on $(k-1)\left\lfloor\frac{n}{k+1}\right\rfloor+k-2$ vertices.

Since all the edges between B and $V(K)\setminus B$ are blue we can apply Lemma 4.3 to P and Q^{k-2} with $i=k-1$, $r_P=2$, and $r_Q=k-2$ in order to find a blue kth power of a path of order n vertices in G.

Case 3: Suppose that the vertices of K can be arranged into disjoint sets $B_1,\ldots,B_k$ such that for $i\neq j$ all the edges between B_i and B_j are blue and we have

$$|B_1|\geq|B_2|\geq\cdots\geq|B_k|\geq\left\lceil\frac{n}{k+1}\right\rceil.$$

Let t be the maximum index for which $|B_t|>n-1$. Notice that $|K|\geq k(n-1)+\left\lfloor\frac{n}{k+1}\right\rfloor$ implies that we have $|B_1|+\cdots+|B_t|-t(n-1)\geq\left\lfloor\frac{n}{k+1}\right\rfloor$. Therefore, for $i\leq t$, we can choose numbers x_i satisfying $0\leq x_i\leq|B_i|-n+1$ for all i and also $x_1+\cdots+x_t=\left\lfloor\frac{n}{k+1}\right\rfloor$.

For each $i=1\ldots k$ we define a path R_i. For $i>t$, we set $R_i=\emptyset$. For $i\leq t$ we have $|B_i|\geq n-1+x_i$, which combined with Theorem 1.1, implies that B_i contains a blue path R_i of order $2x_i+1$. Let $r_{i,0},r_{i,1},\ldots,r_{i,2x_i}$ be the vertex sequence of R_i. For each $i\in\{1,\ldots,t\}$ and $j\neq i$ choose a set $A_{i,j}$ of vertices in B_j satisfying $|A_{i,j}|=x_i$. Note that for $j>t$, the inequality $|B_j|\geq\left\lceil\frac{n}{k+1}\right\rceil$ implies that we have

$$|A_{1,j}|+\cdots+|A_{t,j}|=\left\lfloor\frac{n}{k+1}\right\rfloor\leq|B_j|. \tag{7}$$

For $j\leq t$, the inequalities $|B_j|\geq n$ and $x_j\leq\left\lfloor\frac{n}{k+1}\right\rfloor$ imply that we have

$$|A_{1,j}|+\cdots+|A_{j-1,j}|+|R_j|+|A_{j+1,j}|+\cdots+|A_{t,j}|=\left\lfloor\frac{n}{k+1}\right\rfloor+x_j+1\leq|B_j|. \tag{8}$$

Now, (7) and (8) imply that we can choose the sets $A_{i,j}$, such that $A_{i,j}$ and $A_{i',j}$ are disjoint for $i\neq i'$. In addition, for every $j\leq t$, (8) implies that we can choose the sets $A_{i,j}$ to be disjoint from R_j. Let $a_{i,j,1},\ldots,a_{i,j,x_i}$ be the vertices of $A_{i,j}$. If $n\not\equiv 0\pmod{k+1}$, then the inequalities in both (7) and (8) must be strict, and so there must be at least one vertex contained in B_i outside of $R_i\cup A_{1,i}\cup\cdots\cup A_{t,i}$. Let b_i be this vertex.

For $i=1,\ldots,t$ and $j=1,\ldots,x_i$, we will define blue paths $P_{i,j}$ of order $k+1$ as follows. If $i=1$ and $j\in\{1,\ldots,x_1-1\}$, then $P_{i,j}$ has the following vertex sequence.

$$P_{1,j}=r_{1,2j-1},r_{1,2j},a_{1,2,j},a_{1,3,j},\ldots,a_{1,k,j}.$$

If $i=1$ and $j=x_1$, then $P_{i,j}$ has the following vertex sequence.

$$P_{1,x_1}=r_{1,2x_1-1},r_{1,2x_1},r_{2,0},a_{1,3,x_1},\ldots,a_{1,k,x_1}.$$

If $i \in \{2, \ldots, t-1\}$ and $j \in \{1, \ldots, x_i - 1\}$, then $P_{i,j}$ has the following vertex sequence.

$$P_{i,j} = r_{i,2j-1}, a_{i,1,j}, a_{i,2,j}, \ldots, a_{i,i-1,j}, r_{i,2j}, a_{i,i+1,j}, a_{i,i+2,j}, \ldots, a_{i,k,j}.$$

If $i \in \{2, \ldots, t-1\}$ and $j = x_i$, then $P_{i,j}$ has the following vertex sequence.

$$P_{i,x_i} = r_{i,2x_i-1}, a_{i,1,x_i}, a_{i,2,x_i}, \ldots, a_{i,i-1,x_i}, r_{i,2x_i}, r_{i+1,0}, a_{i,i+2,x_i}, \ldots, a_{i,k,x_i}.$$

If $i = t$ and $j \in \{1, \ldots, x_t\}$, then $P_{i,j}$ has the following vertex sequence.

$$P_{t,j} = r_{t,2j-1}, a_{t,1,j}, a_{t,2,j}, \ldots, a_{t,t-1,j}, r_{t,2j}.$$

If $n \not\equiv 0 \pmod{k+1}$, we also define a path P_0 of order k with vertex sequence

$$P_0 = r_{1,0}, b_2, b_3, \ldots, b_k.$$

If $n \equiv 0 \pmod{k+1}$, let $P_0 = \emptyset$.

Notice that the paths $P_{i,j}$ and $P_{i',j'}$ are disjoint for $(i,j) \neq (i',j')$. Similarly P_0 is disjoint from all the paths $P_{i,j}$. We have the following

$$|P_0| + \sum_{i=1}^{k} \sum_{j=1}^{x_i} |P_{i,j}| = |P_0| + (k+1)(x_1 + \cdots + x_k) = |P_0| + (k+1)\left\lfloor \frac{n}{k+1} \right\rfloor \geq n. \tag{9}$$

We claim that the following path is in fact a blue kth power of a path.

$$P = \begin{cases} P_0 + \\ P_{1,1} + P_{1,2} + \cdots + P_{1,x_1} + \\ P_{2,1} + P_{2,2} + \cdots + P_{2,x_2} + \\ \vdots \\ P_{t,1} + P_{t,2} + \cdots + P_{t,x_t}. \end{cases}$$

To see that P is a kth power of a path one needs to check that any pair of vertices a, b at distance at most k along P are connected by a blue edge. It is easy to check that for any such a and b, either $a \in B_i$ and $b \in B_j$ for some $i \neq j$ or a and b are consecutive vertices along P_0 or $P_{i,j}$ for some i, j. In either case ab is blue implying that P is a blue kth power of a path.

The identity (9) shows that $|P| \geq n$, completing the proof. $\square$

5 Remarks

In this section we discuss some further directions one might take with the results presented in this paper.

- It would be interesting to see if there are any other Ramsey numbers which can be determined using the techniques we used in this paper.

  If G is a graph of (vertex)-chromatic number $\chi(G)$, then $\sigma(G)$ is defined to be the smallest possible order of a colour class in a proper $\chi(G)$-vertex colouring of G. Generalising a construction of Chvatal and Harary, Burr [2] showed that if H is a graph and G is a connected graph and satisfying $|G| \geq \sigma(H)$, then we have

  $$R(G,H) \geq (\chi(H)-1)(|G|-1) + \sigma(H) \tag{10}$$

  This identity comes from considering a colouring consisting of $\chi(H)-1$ red copies of $K_{|G|-1}$ and one red copy of $K_{\sigma(H)-1}$. Notice that for a kth power of a path, we have $\chi(P_n^k) = k+1$ and $\sigma(P_n^k) = \left\lfloor \frac{n}{k+1} \right\rfloor$. Therefore, Theorem 1.11 shows that (10) is best possible when $G = P_n$ and $H = P_n^k$.

  It is an interesting question to find other pairs of graphs for which equality holds in (10) (see [1, 12]). Allen, Brightwell, and Skokan conjectured that when G is a path, then equality holds in (10) for any graph H satisfying $|G| \geq \chi(H)|H|$.

  Conjecture 5.1 (Allen, Brightwell, and Skokan)**.** *For every graph H, $R(P_n, H) = (\chi(H)-1)(n-1) + \sigma(H)$ whenever $n \geq \chi(H)|H|$.*

  It is easy to see that in order to prove Conjecture 5.1, it is sufficient to prove it only in the case when H is a (not necessarily balanced) complete multipartite graph.

  The techniques used in this paper look like they may be useful in approaching Conjecture 5.1. One reason for this is that several parts of the proof of Theorem 1.11 would have worked if we were looking for the Ramsey number of a path versus a balanced complete multipartite graph instead of a power of a path.

- Recall that Lemma 1.5 only implies part of Häggkvist's result (Theorem 1.6). However, it is easy to prove an "unbalanced" version of Lemma 1.5 which implies Theorem 1.6.

  Lemma 5.2. *Suppose that the edges of K_n are coloured with 2 colours and we have an integer t satisfying $0 \leq t \leq n$. Then there is a partition of K_n into a red path and a blue copy of $K_{m,m+t}$ for some integer m.*

  The proof of this lemma is nearly identical to the one we gave of Lemma 1.5 in the Section 2. Indeed, the only modification that needs to be made is that we need to add the condition "$\big||X|-|Y|\big| \geq t$" on the sets X and Y in the proof of Lemma 1.5.

- It would be interesting to see whether Theorems 1.7 and 1.8 have any applications in the area of partitioning coloured complete graphs. In particular, given that Lemma 1.5 played an important role in the proof of the $r=3$ case of Conjecture 1.3 in [15], it is possible that Theorems 1.7 and 1.8 may help with that conjecture.

Classically, results about partitioning coloured graphs would partition a graph into monochromatic subgraphs which all have the same structure. For example Theorems 1.2 and 1.4 partition graphs into monochromatic paths. Lemma 1.5 and Theorem 1.8 stand out from these since they partition a 2-edge-coloured complete graph into two monochromatic subgraphs which have very different structure. It would be interesting to find other natural results along the same lines. Some results about partitioning a 2-edge-coloured complete graph into a monochromatic cycle and a monochromatic graph with high minimum degree will appear in [14].

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