Halfway to Rota's basis conjecture

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Abstract

In 1989, Rota made the following conjecture. Given n bases $B_1, \ldots, B_n$ in an n-dimensional vector space V, one can always find n disjoint bases of V, each containing exactly one element from each B_i (we call such bases *transversal bases*). Rota's basis conjecture remains wide open despite its apparent simplicity and the efforts of many researchers (for example, the conjecture was recently the subject of the collaborative "Polymath" project). In this paper we prove that one can always find $(1/2 - o(1))n$ disjoint transversal bases, improving on the previous best bound of $\Omega(n/\log n)$. Our results also apply to the more general setting of matroids.

1   Introduction

Given bases $B_1, \ldots, B_n$ in an n-dimensional vector space V, a *transversal basis* is a basis of V containing a single distinguished vector from each of $B_1, \ldots, B_n$. Two transversal bases are said to be *disjoint* if their distinguished vectors from B_i are distinct, for each i (here "distinguished" means that two copies of the same vector appearing in two B_is are considered distinct). In 1989, Rota conjectured (see [23, Conjecture 4]) that for any vector space V over a characteristic-zero field, and any choice of $B_1, \ldots, B_n$, one can always find n pairwise disjoint transversal bases.

Despite the apparent simplicity of this conjecture, it remains wide open, and has surprising connections to apparently unrelated subjects. Specifically, it was discovered by Huang and Rota [23] that there are implications between Rota's basis conjecture, the Alon–Tarsi conjecture [2] concerning enumeration of even and odd Latin squares, and a certain conjecture concerning the supersymmetric bracket algebra.

Rota also observed that an analogous conjecture could be made in the much more general setting of *matroids*, which are objects that abstract the combinatorial properties of linear independence in vector spaces. Specifically, a finite matroid $M = (E, \mathcal{I})$ consists of a finite ground set E (whose elements may be thought of as vectors in a vector space), and a collection $\mathcal{I}$ of subsets of E, called independent sets. The defining properties of a matroid are that:

- the empty set is independent (that is, $\emptyset \in \mathcal{I}$);

- subsets of independent sets are independent (that is, if $A' \subseteq A \subseteq E$ and $A \in \mathcal{I}$, then $A' \in \mathcal{I}$);

- if A and B are independent sets, and $|A| > |B|$, then an independent set can be constructed by adding an element of A to B (that is, there is $a \in A \backslash B$ such that $B \cup \{a\} \in \mathcal{I}$). This final property is called the *augmentation property*.

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Observe that any finite set of elements in a vector space (over any field) naturally gives rise to a matroid, though not all matroids arise this way. A *basis* in a matroid M is a maximal independent set. By the augmentation property, all bases have the same size, and this common size is called the *rank* of M. The definition of a transversal basis generalises in the obvious way to matroids, and the natural matroid generalisation of Rota's basis conjecture is that for any rank-n matroid and any bases $B_1, \ldots, B_n$, there are n disjoint transversal bases.

Although Rota's basis conjecture remains open, various special cases have been proved. Several of these have come from the connection between Rota's basis conjecture and the Alon–Tarsi conjecture, which has since been simplified by Onn [26]. Specifically, due to work by Drisko [15] and Glynn [21] on the Alon–Tarsi conjecture, Rota's original conjecture for vector spaces over a characteristic-zero field is now known to be true whenever the dimension n is of the form $p \pm 1$, for p a prime. Wild [30] proved Rota's basis conjecture for so-called "strongly base-orderable" matroids, and used this to prove the conjecture for certain classes of matroids arising from graphs. Geelen and Humphries proved the conjecture for "paving" matroids [18], and Cheung [8] computationally proved that the conjecture holds for matroids of rank at most 4.

Various authors have also proposed variations and weakenings of Rota's basis conjecture. For example, Aharoni and Berger [1] showed that in any matroid one can cover the set of all the elements in $B_1, \ldots, B_n$ by at most $2n$ "partial" transversals, and Bollen and Draisma [4] considered an "online" version of Rota's basis conjecture, where the bases B_i are revealed one-by-one. In 2017, Rota's basis conjecture received renewed interest when it was chosen as the twelfth "Polymath" project, in which amateur and professional mathematicians from around the world collaborated on the problem. Some of the fruits of the project were a small improvement to Aharoni and Berger's theorem, and improved understanding of the online version of Rota's basis conjecture [28]. See [9] for Timothy Chow's proposal of the project, see [10, 11, 12] for blog posts where much of the discussion took place, and see [13] for the Polymath wiki summarising most of what is known about Rota's basis conjecture.

One particularly natural direction to attack Rota's problem is to try to find lower bounds on the number of disjoint transversal bases. Rota's basis conjecture asks for n disjoint transversal bases, but it is not completely obvious that even two disjoint transversal bases must exist! Wild [30] proved some lower bounds for certain matroids arising from graphs, but the first nontrivial bound for general matroids was by Geelen and Webb [19], who used a generalisation of Hall's theorem due to Rado [29] to prove that there must be $\Omega(\sqrt{n})$ disjoint transversal bases. Recently, this was improved by Dong and Geelen [14], who used a beautiful probabilistic argument to prove the existence of $\Omega(n/\log n)$ disjoint transversal bases. In this paper we improve this substantially and obtain the first linear bound.

Theorem 1.1. *For any $\varepsilon > 0$, the following holds for sufficiently large n. Given bases $B_1, \ldots, B_n$ of a rank-n matroid, there are at least $(1/2 - \varepsilon)n$ disjoint transversal bases.*

Of course, since matroids generalise vector spaces, this also implies the same result for bases in an n-dimensional vector space. We also remark that for the weaker fact that there exist $\Omega(n)$ disjoint transversal bases, our methods give a simpler proof; see Remark 2.13.

In contrast to the previous work by Dong, Geelen and Webb, our approach is to show how to build a collection of transversal bases in an iterative fashion (reminiscent of augmenting path arguments in matching problems). It is tempting to imagine a future path to Rota's basis conjecture (at least in the case of vector spaces) using such an approach: by improving on our arguments, perhaps introducing some randomness, it might be possible to iteratively build a collection of $(1 - o(1))n$ transversal bases, and then it might be possible to use some sort of "template" or "absorber" structure to finish the job. This was precisely the approach taken in Keevash's celebrated proof of the existence of designs [24]. Actually, it has been observed by participants of the Polymath project (see [10])

that Rota's basis conjecture and the existence of designs conjecture both seem to fall into a common category of problems which are not quite "structured" enough for purely algebraic methods, but too structured for probabilistic methods.

Notation. We will frequently want to denote the result of adding and removing single elements from a set. For a set S and some $x \notin S$, $y \in S$, we write $S + x$ to mean $S \cup \{x\}$, and we write $S - y$ to mean $S \setminus \{y\}$.

2 Finding many disjoint transversal bases

In this section we prove Theorem 1.1. It is convenient to think of $B_1, \ldots, B_n$ as "colour classes".

Definition 2.1. Let $U = \{(x,c) : x \in B_c, 1 \le c \le n\}$ be the set of all coloured elements that appear in one of $B_1, \ldots, B_n$. For $S \subseteq U$, let $\pi(S) = \{x : (x,c) \in S \text{ for some } c\}$ be its set of matroid elements. We say that a subset of elements of U is a *rainbow independent set* (RIS for short) if all its matroid elements are distinct and form an independent set, and all their colours are distinct.

Note that an RIS with size n corresponds to a transversal basis. We remark that RISs are sometimes also known as *partial transversals*. Note that two transversal bases are disjoint if and only if their corresponding RISs are disjoint as subsets of U.

Let $f = (1 - \varepsilon)n/2$. The basic idea is to start with a collection of f empty RISs (which are trivially disjoint), and iteratively enlarge the RISs in this collection, maintaining disjointness, until we have many disjoint transversal bases.

Let $\mathcal{S}$ be a collection of f disjoint RISs. We define the *volume* $\sum_{S \in \mathcal{S}} |S|$ of $\mathcal{S}$ to be the total number of elements in the RISs in $\mathcal{S}$. We will show how to modify $\mathcal{S}$ to increase its volume. We let $F = \bigcup_{S \in \mathcal{S}} S$ be the set of all currently used elements. One should think of F as being the set of all elements which we cannot add to any $S \in \mathcal{S}$ without violating the disjointness of RISs in $\mathcal{S}$.

We stress that in the following two subsections we fix a collection $\mathcal{S}$ and define F as above. All our definitions and claims are with respect to these F and $\mathcal{S}$. We will show that under certain conditions the size of $\mathcal{S}$ can be increased, at which point one needs to restart the argument from the beginning with a new $\mathcal{S}$ (and a new F). This is made precise in Section 2.3.

Remark. *We remark that it is actually possible to reduce to the case where each B_c is disjoint, by making duplicate copies of all elements that appear in multiple B_c. So, instead of working with the universe U of element/colour pairs, one can alternatively think of U as being a collection of n^2 different matroid elements (each of which has a colour associated with it).*

2.1 Simple swaps

Our objective is to increase the volume of $\mathcal{S}$. If an RIS $S \in \mathcal{S}$ is missing a colour c and there is $x \in B_c$ independent to the elements of S, such that $(x,c) \notin F$, then we can add (x,c) to S to create a larger RIS, increasing the volume of $\mathcal{S}$. We will want much more freedom than this: we also want to consider those elements that can be added to S after making a small change to S. This motivates the following definition.

Definition 2.2. Consider an RIS S and a colour b that does not appear in S. Say an element $(x,c) \in U$ (possibly $(x,c) \in F$) is (S,b)-*addable* if either

- $S + (x,c)$ is an RIS, or;

- There is $(x',c) \in S$ and $(y,b) \notin F$ such that $S - (x',c) + (y,b) + (x,c)$ is an RIS.

In the second case we say that y is a *witness* for the (S,b)-addability of (x,c). For $(x',c) \in S$ and $(y,b) \notin F$ when $S - (x',c) + (y,b)$ is an RIS we say it is the result of applying a *simple swap* to S.

If for some RIS $S \in \mathcal{S}$ missing a colour b there is an (S,b)-addable element $(x,c) \notin F$, then we can increase the volume of $\mathcal{S}$ by adding (x,c) to S, possibly after applying a simple swap to S. Note that we do not require $S \in \mathcal{S}$ for the definition of (S,b)-addability, though in practice we will only ever consider S that are either in $\mathcal{S}$ or slight modifications of RISs in $\mathcal{S}$.

Our next objective is to show that for any S missing a colour b, either there is an (S,b)-addable element that is not in F (which would allow us to increase the volume of $\mathcal{S}$, as above), or else there are *many* (S,b)-addable elements (which must therefore be in F). Although this will not allow us to immediately increase the volume of $\mathcal{S}$, it will allow us to transfer an element to S from some other $S' \in \mathcal{S}$, and this freedom to perform local modifications will be very useful.

Towards this end, we study which elements of S can be used in a simple swap.

Definition 2.3. Consider an RIS S and consider a colour b that does not appear on S. We say that a colour c appearing on S is (S,b)-*swappable* if there is a simple swap yielding an RIS $S + (y,b) - (x',c)$, with $(y,b) \notin F$ and $(x',c) \in S$. (For $S + (y,b) - (x',c)$ to be an RIS, we just need $\pi(S) + y - x'$ to be an independent set in our matroid.) We say that y is a witness for the (S,b)-swappability of c.

(Basically, a colour is (S,b)-swappable if we can replace it with a b-coloured element which is not in F). For a colour c we denote by $F_c = \{x \in B_c : (x,c) \in F\}$ the set of matroid elements which appear in $\mathcal{S}$ with colour c.

Lemma 2.4. *For a nonempty RIS S and a colour b not appearing in S, either there is an (S,b)-addable element $(y,b) \notin F$ or there are at least $n - |F_b|$ colours which are (S,b)-swappable.*

Proof. For the purpose of contradiction, suppose that there is no (S,b)-addable element $(y,b) \notin F$, and that there are fewer than $n - |F_b|$ colours which are (S,b)-swappable. Let $S' \subseteq S$ be the set of all elements of S which have an (S,b)-swappable colour, so $|S'| < n - |F_b|$. Also $|S'| < |S|$ because otherwise we would have $|S| < n - |F_b|$, so by the augmentation property there would be $y \in B_b \setminus F_b$ such that $S + (y,b)$ is an RIS (meaning that $(y,b) \notin F$ would be (S,b)-addable). Repeating this argument for S' in place of S, there is $y \in B_b \setminus F_b$ such that $S' + (y,b)$ is an RIS. By repeatedly using the augmentation property, we can add $|S - S'| - 1$ elements of $S - S'$ to $S' + (y,b)$. This gives an RIS of size $|S|$ of the form $S + (y,b) - (x',c)$ for some $(x',c) \in S - S'$. But this means c is (S,b)-swappable, so $(x',c) \in S'$ by the definition of S'. This is a contradiction. $\square$

Now we show that all elements of an (S,b)-swappable colour which are independent to $\pi(S)$ are (S,b)-addable, unless there is an (S,b)-addable element not in F. (Recall that $\pi(S)$ is the set of matroid elements in S, without colour data.)

Lemma 2.5. *Consider an RIS S with no element of a colour b and consider a colour c that is (S,b)-swappable with witness y. Either $S + (y,b)$ is an RIS (thus, $(y,b) \notin F$ is (S,b)-addable), or otherwise for any $x \in B_c$ independent of $\pi(S)$, (x,c) is (S,b)-addable with witness (y,b).*

Proof. Let (x',c) be the element with colour c in S. Consider some $x \in B_c$ independent to $\pi(S)$. Let $I = \pi(S) + x$ and $J = \pi(S) + y - x'$. By the augmentation property, there is an element of $I \setminus J$ that is independent of J; this element is either x' or x. In the former case $S + (y,b)$ is an RIS. In the latter case, $S + (y,b) - (x',c) + (x,c)$ is an RIS, showing that (x,c) is (S,b)-addable. $\square$

The following lemma gives a good illustration of how to use the ideas developed in this section to find many addable elements. It will be very useful later on.

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Lemma 2.6. *Let $S \in \mathcal{S}$ and let b be a colour which does not appear in S. Then either we can increase the volume of $\mathcal{S}$ or there are at least $(n - |S|)(n - f)$ elements that are (S, b)-addable.*

Proof. If there is an element $(y, b) \notin F$ which is (S, b)-addable, then we can directly add this element to S (making a simple swap if necessary), increasing the volume of $\mathcal{S}$. Otherwise, observe that $|F_b| \leq |\mathcal{S}| = f$, so by Lemma 2.4 there are at least $n - f$ colours that are (S, b)-swappable. For each such colour c, by the augmentation property, there are at least $n - |S|$ elements $x \in B_c$ independent to all the elements of S, each of which is (S, b)-addable by Lemma 2.5. That is to say, there are at least $(n - |S|)(n - f)$ elements which are (S, b)-addable, as claimed. $\square$

In our proof of Theorem 1.1 we also make use of the following lemma. In the course of our arguments, when we need to find many addable elements with a given colour, it will allow us to ensure that these elements are actually distinct.

Lemma 2.7. *Let S be an RIS. Then for each B_b, we can find an injection $\phi_b : S \to B_b$ such that for all $(x, c) \in S$, $\phi_b((x, c))$ is independent of $\pi(S - (x, c))$.*

Proof. Consider the bipartite graph G where the first part consists of the elements of S and the second part consists of the elements of B_b, with an edge between $(x, c) \in S$ and $y \in B_b$ if y is independent of $\pi(S - (x, c))$. We use Hall's theorem to show that there is a matching in this bipartite graph covering S. Indeed, consider some $W \subseteq S$. By the augmentation property, there are at least $|W|$ elements $y \in B_b$ such that $\pi(S - W) + y$ is an independent set, and again using the augmentation property, each of these can be extended to an independent set of the form $\pi(S) + y - x$ for some $(x, c) \in W$. That is to say, W has at least $|W|$ neighbours in G. $\square$

We thank the anonymous referees for pointing out that Lemma 2.7 also follows from a result due to Brualdi [5].

2.2 Cascading swaps

Informally speaking, for any $S_0 \in \mathcal{S}$ which is not a transversal basis, we have shown that either we can directly augment S_0, or there are many elements $(x_1, c_1) \in U$ with which we can augment S_0 after performing a simple swap. It's possible that each such (x_1, c_1) already appears in some other $S_1 \in \mathcal{S}$, but if this occurs we need not give up: we can transfer (x_1, c_1) from S_1 to S_0 and then continue to look for elements $(x_2, c_2) \in U$ with which we can augment $S_1 - (x_1, c_1)$ (again, possibly with a swap). We can iterate this idea, looking for sequences

$$S_1, \ldots, S_\ell \in \mathcal{S}, \quad (x_1, c_1) \in S_1, (x_2, c_2) \in S_2, \ldots, (x_\ell, c_\ell) \in S_\ell, (x_{\ell+1}, c_{\ell+1}) \notin \bigcup_{S \in \mathcal{S}} S$$

such that, after a sequence of simple swaps, each (x_i, c_i) is transferred from S_i to S_{i-1}, and then $(x_{\ell+1}, c_{\ell+1})$ can be added to S_ℓ. (We also need to ensure that the simple swaps we perform preserve disjointness of RISs in $\mathcal{S}$.) This transformation has the net effect of adding an element to S_0 and keeping the size of all other $S \in \mathcal{S}$ constant, thus increasing the volume of $\mathcal{S}$.

Crucially, because of the freedom afforded by simple swaps, each time we expand our search to consider longer cascades, our number of options for $(x_{\ell+1}, c_{\ell+1})$ increases. For sufficiently large ℓ, the number of options will be so great that there must be suitable $(x_{\ell+1}, c_{\ell+1})$ not appearing in any RIS in $\mathcal{S}$. In order to keep this analysis tractable, we will only consider transformations that cascade along a single sequence of RISs $S_0, \ldots, S_\ell$; we will iteratively construct this sequence of RISs in such a way that there are many possibilities $(x_i, c_i) \in S_i$ relative to the number of possibilities $(x_{i-1}, c_{i-1}) \in S_{i-1}$ in the previous step. The next definition makes precise the cascades that we consider.

Definition 2.8. Consider a sequence of distinct RISs $S_0, \ldots, S_{\ell-1} \in \mathcal{S}$. Say an element $(x_\ell, c_\ell) \notin S_0, \ldots, S_{\ell-1}$ is *cascade-addable with respect to* $S_0, \ldots, S_{\ell-1}$ if there is a colour c_0 and sequences

$$(x_1, c_1), \ldots, (x_{\ell-1}, c_{\ell-1}) \in U, \qquad y_0 \in B_{c_0}, \ldots, y_{\ell-1} \in B_{c_{\ell-1}},$$

such that the following hold.

- For each $1 \le i \le \ell - 1$, we have $(x_i, c_i) \in S_i$;

- c_0 does not appear in S_0, and (x_1, c_1) is (S_0, c_0)-addable with witness y_0;

- for each $0 \le i \le \ell - 1$, (x_{i+1}, c_{i+1}) is $(S_i - (x_i, c_i), c_i)$-addable with witness y_i;

- the colours $c_0, \ldots, c_\ell$ are distinct.

We call $c_0, c_1, \ldots, c_{\ell-1}$ a *sequence of colours freeing* (x_ℓ, c_ℓ).

We write $Q(S_0, \ldots, S_{\ell-1})$ for the set of all elements outside $S_0, \ldots, S_{\ell-1}$ which are cascade-addable with respect to $S_0, \ldots, S_{\ell-1}$.

We remark that if $\ell = 1$ then most of the conditions in the above definition become vacuous and an element being cascade-addable with respect to S_0 is equivalent to it being (S_0, c_0)-addable with a witness, for some colour c_0. Observe that if an element (x_ℓ, c_ℓ) is cascade-addable then we can transfer it into $S_{\ell-1}$, as the final step in a cascading sequence of simple swaps and transfers. The following lemma makes this precise.

Lemma 2.9. *Suppose that (x_ℓ, c_ℓ) is cascade-addable with respect to $S_0, \ldots, S_{\ell-1}$ and $c_0, c_1, \ldots, c_{\ell-1}$ is a sequence of colours freeing (x_ℓ, c_ℓ). Then there are $S'_0 \ldots S'_{\ell-1} \subseteq S_0 \cup \cdots \cup S_{\ell-1} \cup B_{c_0} \cup \cdots \cup B_{c_{\ell-1}}$ such that replacing $S_0, \ldots, S_{\ell-1}$ with $S'_0, \ldots, S'_{\ell-1}$ in $\mathcal{S}$ results in a family $\mathcal{S}'$ of disjoint RISs of the same total volume as $\mathcal{S}$, in such a way that $S'_{\ell-1} + (x_\ell, c_\ell)$ is an RIS.*

Proof. Let $(x_1, c_1), \ldots, (x_{\ell-1}, c_{\ell-1}) \in U, y_0 \in B_{c_0}, \ldots, y_{\ell-1} \in B_{c_{\ell-1}}$ be as in the definition of cascade-addability. For each $i = 0, \ldots, \ell - 1$, let (x'_i, c_{i+1}) be the colour c_{i+1} element of S_i (which exists, because, from cascade-addability, (x_{i+1}, c_{i+1}) is $(S_i - (x_i, c_i), c_i)$-addable *with a witness*). For each $i = 1, \ldots, \ell - 2$, let $S'_i = S_i - (x_i, c_i) - (x'_i, c_{i+1}) + (y_i, c_i) + (x_{i+1}, c_{i+1})$. Let $S'_0 = S_0 - (x'_0, c_1) + (y_0, c_0) + (x_1, c_1)$ and $S'_{\ell-1} = S_{\ell-1} - (x_{\ell-1}, c_{\ell-1}) - (x'_{\ell-1}, c_\ell) + (y_{\ell-1}, c_{\ell-1})$. Let $\mathcal{S}'$ be the family formed by replacing $S_0, \ldots, S_{\ell-1}$ with $S'_0, \ldots, S'_{\ell-1}$ in $\mathcal{S}$. It is easy to check that $\mathcal{S}'$ has the same total volume as $\mathcal{S}$, so it remains to check that it is a family of disjoint RISs.

For $i = 1, \ldots, \ell - 2$, S'_i is an RIS because it comes from $S_i - (x_i, c_i)$ by making the change in the definition of (x_{i+1}, c_{i+1}) being $(S_i - (x_i, c_i), c_i)$-addable with witness y_i (and addability always produces an RIS by definition). Similarly $S'_{\ell-1} + (x_\ell, c_\ell)$ is an RIS. To see that S'_0 is an RIS we use that (x_1, c_1) is (S_0, c_0)-addable with witness y_0, and that c_0 does not appear in S_0, both of which come from the definition of cascade-addability.

It remains to show that the RISs $S'_0, \ldots, S'_{\ell-1}$ are disjoint from each other and the other RISs in $\mathcal{S}$. The elements (y_i, c_i) occur in only one RIS S'_i because they come from outside F (since they are addability witnesses), and because their colours $c_0, \ldots, c_{\ell-1}$ are distinct (from the definition of cascade-addability). The elements (x_i, c_i) occur in only one RIS because they get removed from S_i and added to S_{i-1}. $\square$

The following lemma lets us build longer cascades.

Lemma 2.10. *Suppose that $(x_\ell, c_\ell) \in S_\ell$ is cascade-addable with respect to $S_0, \ldots, S_{\ell-1}$ and $c_0, c_1, \ldots, c_{\ell-1}$ is a sequence of colours freeing (x_ℓ, c_ℓ). If (x, c) is $(S_\ell - (x_\ell, c_\ell), c_\ell)$-addable with a witness then either $(x, c) \in S_0 \cup \cdots \cup S_\ell \cup B_{c_0} \cup \cdots \cup B_{c_\ell}$ or (x, c) is cascade-addable with respect to $S_0, \ldots, S_\ell$.*

Proof. Suppose that $(x, c) \notin S_0, \ldots, S_\ell, B_{c_0}, \ldots, B_{c_\ell}$. For the definition of (x, c) being cascade-addable, all the conditions not involving (x, c) and (x_ℓ, c_ℓ) hold as a consequence of $(x_\ell, c_\ell) \in S_\ell$ being cascade-addable with respect to $S_0, \ldots, S_{\ell-1}$. It remains to check the conditions that $(x, c) \notin S_0, \ldots, S_\ell$ and that each of $c_0, \ldots, c_\ell, c$ are distinct, both of which hold as a consequence of our assumption $(x, c) \notin S_0, \ldots, S_\ell, B_{c_0}, \ldots, B_{c_\ell}$. $\square$

In the next lemma, we essentially show that given $S_0, \ldots, S_{\ell-1}$, it is possible to choose S_ℓ in such a way that the number of cascade-addable elements increases.

Lemma 2.11. *Consider a sequence of distinct RISs $S_0, \ldots, S_{\ell-1} \in \mathcal{S}$ with $1 \leq \ell < f = |\mathcal{S}|$. Then either we can modify $\mathcal{S}$ to increase its volume, or we can choose $S_\ell \neq S_0, \ldots, S_{\ell-1}$ from $\mathcal{S}$ such that*

$$|Q(S_0, \ldots, S_\ell)| \geq \frac{|Q(S_0, \ldots, S_{\ell-1})|}{f - \ell} \cdot (n - f - \ell) - (\ell + 1)n. \tag{1}$$

Proof. If $Q(S_0, \ldots, S_{\ell-1})$ contains an element (x, c) not in any $S \in \mathcal{S}$, then we can increase the volume of $\mathcal{S}$ with a cascading sequence of simple swaps and transfers (using Lemma 2.9, noting that if $(x_\ell, c_\ell) \notin F$, then we can add (x_ℓ, c_ℓ) to $S'_{\ell-1}$ in that lemma to get a larger family of RISs).

Otherwise, all the elements of $Q(S_0, \ldots, S_{\ell-1})$ belong to some RIS $S \in \mathcal{S} \setminus \{S_0, \ldots S_{\ell-1}\}$ (since $Q(S_0, \ldots, S_{\ell-1})$ is defined to not contain any elements from $S_0, \ldots, S_{\ell-1}$). Choose $S_\ell \in \mathcal{S} \setminus \{S_0, \ldots S_{\ell-1}\}$ containing maximally many elements of $Q(S_0, \ldots, S_{\ell-1})$. Since the $f - \ell$ RISs $S \in \mathcal{S} \setminus \{S_0, \ldots S_{\ell-1}\}$ collectively contain all elements of $Q(S_0, \ldots, S_{\ell-1})$, our chosen RIS S_ℓ must contain a proportion of at least $1/(f - \ell)$ of the elements of $Q(S_0, \ldots, S_{\ell-1})$. In other words, if we let $Q = S_\ell \cap Q(S_0, \ldots, S_{\ell-1})$, we have

$$|Q| \geq \frac{|Q(S_0, \ldots, S_{\ell-1})|}{f - \ell}. \tag{2}$$

Apply Lemma 2.7 to S_ℓ to obtain an injection ϕ_b, for every colour b. Fix some $(x_\ell, c_\ell) \in Q$ and a sequence of colours $c_0, \ldots, c_{\ell-1}$ freeing (x_ℓ, c_ℓ). We prove a sequence of claims about how many elements are swappable/addable with respect to $(S_\ell - (x_\ell, c_\ell), c_\ell)$, assuming we cannot increase the size of $\mathcal{S}$.

Claim. *There are at least $n - f$ colours which are $(S_\ell - (x_\ell, c_\ell), c_\ell)$-swappable.*

Proof. By Lemma 2.4, either there is an $(S_\ell - (x_\ell, c_\ell), c_\ell)$-addable element $(y, c_\ell) \notin F$, or there are at least $n - |F_{c_\ell}| \geq n - f$ colours which are $(S_\ell - (x_\ell, c_\ell), c_\ell)$-swappable. In the former case, we can increase the volume of $\mathcal{S}$, by a cascading sequence of swaps and transfers (first consider $\mathcal{S}'$ from Lemma 2.9, then move (x_ℓ, c_ℓ) from S_ℓ to $S'_{\ell-1}$, then add (y, c_ℓ) to $S_\ell - (x_\ell, c_\ell)$). $\square$

Claim. *There are at least $n - f$ colours c for which $(\phi_c((x_\ell, c_\ell)), c)$ is $(S_\ell - (x_\ell, c_\ell), c_\ell)$-addable.*

Proof. Let c be a colour which is $(S_\ell - (x_\ell, c_\ell), c_\ell)$-swappable with witness y, as in the previous claim. If y is independent to $\pi(S_\ell - (x_\ell, c_\ell))$, we can increase the volume of $\mathcal{S}$ by adding it to S_ℓ after a cascading sequence of swaps and transfers (first consider $\mathcal{S}'$ from Lemma 2.9, then move (x_ℓ, c_ℓ) from S_ℓ to $S'_{\ell-1}$, then add (y, c_ℓ) to $S_\ell - (x_\ell, c_\ell)$). Otherwise, by Lemma 2.5 applied with $b = c_\ell$, $S = S_\ell - (x_\ell, c_\ell)$, the element $(\phi_c((x_\ell, c_\ell)), c)$ is $(S_\ell - (x_\ell, c_\ell), c_\ell)$-addable. Here we are using that $(\phi_c((x_\ell, c_\ell)), c)$ is independent from $(S_\ell - (x_\ell, c_\ell), c_\ell)$ (which comes from the definition of ϕ_c in Lemma 2.7). $\square$

Claim. *There are at least $n - f - \ell$ colours $c \notin \{c_0, \ldots, c_{\ell-1}\}$ for which $(\phi_c((x_\ell, c_\ell)), c)$ is $(S_\ell - (x_\ell, c_\ell), c_\ell)$-addable.*

Proof. This ensues from the previous claim and the fact that the only requirement on c, besides addability, is that it is different from the ℓ colours in $\{c_0, \ldots, c_{\ell-1}\}$. □

We now prove the following:

$$|Q(S_0, \ldots, S_\ell)| \geq |Q|(n - \ell - f) - (\ell + 1)n. \tag{3}$$

From the last claim, we have $|Q|(n - \ell - f)$ elements of the form $(\phi_c((x_\ell, c_\ell)), c)$ which are all $(S_\ell - (x_\ell, c_\ell), c_\ell)$-addable, with c outside a sequence of colours freeing (x_ℓ, c_ℓ). Notice that these $(\phi_c((x_\ell, c_\ell)), c)$ are all distinct because ϕ_c is an injection. By Lemma 2.10, each of these is cascade-addable with respect to $S_0, \ldots, S_\ell$, unless it appears in one of $S_0, \ldots, S_\ell$. The total number of elements in $S_0, \ldots, S_\ell$ is at most $(\ell + 1)n$, so we have found $|Q|(n - \ell - f) - (\ell + 1)n$ cascade-addable elements with respect to $S_0, \ldots, S_\ell$, as required by (3).

The lemma immediately follows by combining (2) and (3). □

Now, we want to iteratively apply Lemma 2.11 starting from some $S_0 \in \mathcal{S}$, to obtain a sequence $S_0, S_1, \ldots, S_h \in \mathcal{S}$. There are two ways this process can stop: either we find a way to increase the volume of $\mathcal{S}$, in which case we are done, or else we run out of RISs in $\mathcal{S}$ (that is, $h = f-1$). We want to show that this latter possibility cannot occur by deducing from (1) that the $|Q(S_0, \ldots, S_\ell)|$ increase in size at an exponential rate: after logarithmically many steps there will be so many cascade-addable elements that they cannot all be contained in the RISs in $\mathcal{S}$, and it must be possible to increase the volume of $\mathcal{S}$.

A slight snag with this plan is that (1) only yields an exponentially growing recurrence if the "initial term" is rather large. To be precise, let C (depending on ε) be sufficiently large such that

$$C(1 + \varepsilon/2)^{\ell-1} \frac{1}{1 - \varepsilon} - \ell - 1 \geq C(1 + \varepsilon/2)^\ell \tag{4}$$

for all $\ell \geq 1$.

Lemma 2.12. *For $S_0, \ldots, S_h$ as above, suppose that $|Q(S_0)| \geq Cn$ or $|Q(S_0, S_1)| \geq Cn$. Then, for $0 < \ell \leq \min\{h, \varepsilon n/2\}$, we have*

$$|Q(S_0, \ldots, S_\ell)| \geq C(1 + \varepsilon/2)^{\ell-1} n.$$

Proof. We first establish a technical inequality. Recall that $f = (1 - \varepsilon)n/2$, so

$$\frac{n - f - \ell}{f - \ell} \geq \frac{n - (1 - \varepsilon)n/2 - n\varepsilon/2}{(1 - \varepsilon)n/2} = \frac{1}{1 - \varepsilon}. \tag{5}$$

Now, let $Q_\ell = Q(S_0, \ldots, S_\ell)$. We proceed by induction. First observe that if $|Q_0| \geq Cn$ then (1), (5) and (4) for $\ell = 1$ imply $|Q_1| \geq Cn(n - f - 1)/(f - 1) - 2n \geq (C/(1 - \varepsilon) - 2)n \geq Cn$, giving us the base case. If $|Q_\ell| \geq C(1 + \varepsilon/2)^{\ell-1} n$, then once again using (1), (5) and (4), we obtain

$$\begin{aligned}
|Q_{\ell+1}| &\geq \frac{C(1 + \varepsilon/2)^{\ell-1} n}{f - \ell} \cdot (n - f - \ell) - (\ell + 1)n \\
&= \left(C(1 + \varepsilon/2)^{\ell-1} \frac{(n - f - \ell)}{f - \ell} - \ell - 1 \right) n \\
&\geq \left(C(1 + \varepsilon/2)^{\ell-1} \frac{1}{1 - \varepsilon} - \ell - 1 \right) n \\
&\geq C(1 + \varepsilon/2)^\ell n. \qquad \qquad \square
\end{aligned}$$

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If we could choose S_0, S_1 such that $|Q(S_0)| \geq Cn$ or $|Q(S_0, S_1)| \geq Cn$, then Lemma 2.12 would imply that during the construction of $S_1, \ldots, S_h$ we never run out of RISs in $\mathcal{S}$ (that is, $h < f - 1$). Indeed, otherwise $Q(S_0, \ldots, S_{\varepsilon n/2})$ would have size exponential in n, which is impossible. Therefore, the process must stop at some point when we find a way to increase the volume of $\mathcal{S}$. Provided we can again find suitable S_0, S_1 we can then repeat the arguments in this section, further increasing the volume of $\mathcal{S}$. After repeating these arguments enough times we will have obtained $f = (1 - \varepsilon)n/2 \geq (1/2 - \varepsilon)n$ disjoint transversal bases, completing the proof of Theorem 1.1.

There may not exist suitable $S_0, S_1 \in \mathcal{S}$, but in the next section we will show that if at least $\varepsilon n/2$ of the RISs in S are not transversal bases, then it is possible to modify $\mathcal{S}$ without changing its volume, in such a way that suitable S_0, S_1 exist.

Remark 2.13. With the results we have proved so far, we can already find linearly many disjoint transversal bases. Indeed, if S_0 is not a transversal basis (missing a colour b, say), and the volume of $\mathcal{S}$ cannot be increased by adding an element to S_0 (possibly after a simple swap), then Lemma 2.6 implies that there are at least $n - f$ elements which are (S_0, b)-addable, meaning that $|Q(S_0)| \geq n - f$. Take for example $\varepsilon = 4/5$, meaning that $f \leq n/10$ and $|Q(S_0)| \geq 9n/10$. We can check that (4) holds for all $\ell \geq 1$ if $C = 9/10$. That is to say, as long as we have not yet completed $\mathcal{S}$ to a collection of disjoint transversal bases, we can keep increasing its volume without the considerations in the next section. This proves already that it is possible to find linearly many disjoint transversal bases.

Remark 2.14. It is not hard to add a term $(n - |S_\ell|)(n - f)$ to the right hand side of the inequality given by Lemma 2.11 by considering also cascades along the sequence $S_0, \ldots, S_{\ell-1}$ of length strictly less than ℓ. However, since this increase is only significant when $|S_\ell|$ is not close to n, which may never be the case, we omit it from our argument for the sake of readability.

2.3 Increasing the number of initial addable elements

Consider a collection $\mathcal{S}$ of $f = (1 - \varepsilon)n/2$ disjoint RISs, at least $\varepsilon n/2$ of which are not transversal bases. Recall the choice of C from the previous section, and let $D = 2C + 4$, so that $D(n - f - 1) - 2n \geq Cn$ for large n. We prove the following (for large n).

Lemma 2.15. *We can modify $\mathcal{S}$ in such a way that at least one of the following holds.*

   (a) The volume of $\mathcal{S}$ increases;

   (b) the volume of $\mathcal{S}$ does not change, and there is now $S_0 \in \mathcal{S}$ missing at least D colours;

   (c) the volume of $\mathcal{S}$ does not change, and there are now distinct $S_0, S_1 \in \mathcal{S}$ such that S_1 contains at least D elements that are (S_0, b)-addable, for some colour b.

This suffices for our proof of Theorem 1.1; indeed, if S_0 is missing at least D colours, then by Lemma 2.6, either we can increase the volume of $\mathcal{S}$ or there are at least $D(n - f) \geq Cn$ elements which are (S_0, b)-addable for every b not appearing in S_0, meaning that $|Q(S_0)| \geq Cn$. If S_1 contains at least D elements that are (S_0, b)-addable, then in the proof of Lemma 2.11 with $\ell = 1$ we have $|Q| \geq D$ so either we can increase the volume of $\mathcal{S}$ or $|Q(S_0, S_1)| \geq D(n - f - 1) - 2n \geq Cn$ (recall (3)).

Before proceeding to the proof of Lemma 2.15, we first observe that using Lemma 2.4 we can modify $\mathcal{S}$ to ensure that every $S \in \mathcal{S}$ that is not a transversal basis can be assigned a distinct missing colour $b(S)$. To see this, we iteratively apply the following lemma to $\mathcal{S}$.

Lemma 2.16. *Consider $f \leq n/2$ and let $\mathcal{S} = \{S_1, \ldots, S_f\}$ be a collection of disjoint RISs. We can either increase the size of $\mathcal{S}$ or we can modify $\mathcal{S}$ in such a way that the size of each S_i remains the same, and in such a way that that there is a choice of disjoint colours $\{b_1, \ldots, b_f\}$ for which any S_i that is not a transversal basis has no element of colour b_i.*

Proof. Suppose for some i that we found distinct colours $b_1, \ldots, b_{i-1}$ such that, for all S_j which are not transversal bases, no element of S_j is of colour b_j. If S_i is a transversal basis we choose an arbitrary unused colour as b_i. Otherwise there is a colour, say c, not appearing in S_i. Then by Lemma 2.4 either we can increase the size of $\mathcal{S}$ or there are at least $n - f \geq n/2$ colours which are (S_i, c)-swappable. At least one of these colours does not appear in $\{b_1, \ldots, b_{i-1}\}$, since $i - 1 < f \leq n/2$. Let b be such a colour and set $b_i = b$. By performing a simple swap, we transform S_i into a new RIS, still disjoint to all other $S_j \in \mathcal{S}$ and missing the colour b. $\square$

Now we prove Lemma 2.15.

Proof of Lemma 2.15. Recall that we are assuming there are at least $\varepsilon n/2$ RISs in $\mathcal{S}$ that are not transversal bases. Let E be the largest integer such that there are at least $M_E = \left(\varepsilon/\left(4D^2\right)\right)^E n$ RISs in $\mathcal{S}$ missing at least E colours. We may assume $1 \leq E < D$. By Lemma 2.16 we may assume that each $S \in \mathcal{S}$ which is not a transversal basis has a distinct missing colour $b(S)$. We describe a procedure that modifies $\mathcal{S}$ to increase E.

We create an auxiliary digraph G on the vertex set $\mathcal{S}$ as follows. For every $S_0 \in \mathcal{S}$ missing at least E colours, put an arc to S_0 from every $S_1 \in \mathcal{S}$ such that S_1 contains at least $E + 1$ elements that are $(S_0, b(S_0))$-addable.

Say an $(E+1)$-*out-star* in a digraph is a set of $E + 1$ arcs directed away from a single vertex. Our goal is to prove that there are M_{E+1} vertex-disjoint $(E + 1)$-out-stars. To see why this suffices, consider an $(E + 1)$-out-star (with centre S_1, say). We show how to transfer $E + 1$ elements from S_1 to its out-neighbours, the end result of which is that S_1 is then missing $E + 1$ colours. We will then be able to repeat this process for each of our out-stars.

For each of the $E + 1$ out-neighbours S_0 of S_1 there are at least $E + 1$ elements of S_1 which are $(S_0, b(S_0))$-addable. Therefore, for each such S_0 we can make a specific choice of such an $(S_0, b(S_0))$-addable element, in such a way that each of these $E + 1$ choices are *distinct*. For each S_0 we can then transfer the chosen element from S_1 to S_0, possibly with a simple swap. These simple swaps will not create any conflicts, because any addability witness for any element in S_0 is in a colour unique to that S_0 (by the property from Lemma 2.16). After this operation, S_i is now missing at least $E + 1$ colours.

It will be a relatively straightforward matter to find our desired out-stars by studying the digraph G. First we show that G must have many edges.

Claim. *In the above auxiliary digraph, we may assume that every $S_0 \in \mathcal{S}$ missing at least E colours has in-degree at least $\varepsilon n/D$.*

Proof. By Lemma 2.6 we can assume that there are at least $E(n - f)$ elements which are $(S_0, b(S_0))$-addable. All these elements appear in various $S \in \mathcal{S}$ (otherwise we can increase the volume of $\mathcal{S}$).

Let $N^-(S_0)$ be the set of all S_1 such that there is an arc from S_1 to S_0 in G (so $|N^-(S_0)|$ is the indegree of S_0). By definition, every $S \notin N^-(S_0)$ has at most E elements which are $(S_0, b(S_0))$-addable. Moreover, observe that every $S \in \mathcal{S}$ has fewer than D elements that are $(S_0, b(S_0))$-addable, or else (c) trivially occurs. It follows that

$$D|N^-(S_0)| + E(f - |N^-(S_0)|) \geq E(n - f),$$

so

$$|N^-(S_0)| \geq \frac{E((n - f) - f)}{D - E} \geq \frac{\varepsilon n}{D},$$

as desired. $\square$

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We have proved that G has at least $M_E \varepsilon n / D$ edges. Now we finish the proof by showing how to find our desired out-stars.

Claim. *G has at least M_{E+1} vertex-disjoint $(E + 1)$-out-stars.*

Proof. We can find these out-stars in a greedy fashion. Suppose that we have already found t vertex-disjoint $(E + 1)$-out-stars, for some $t < M_{E+1}$. We show that there must be an additional $(E + 1)$-out-star disjoint to these. Let G' be obtained from G by deleting all vertices in the out-stars we have found so far. Each of these out-stars has $E + 2$ vertices, so the number of arcs in G' is at least

$$
M_E \frac{\varepsilon n}{D} - t(E + 2) \cdot 2f > M_E \frac{\varepsilon n}{D} - M_{E+1}(E + 2) \cdot 2f
$$
$$
= M_E \frac{\varepsilon n}{D} - \frac{M_E \varepsilon}{2D^2} \cdot (E + 2)f
$$
$$
\geq M_E \varepsilon \left(\frac{n}{D} - \frac{f}{D} \right)
$$
$$
\geq M_E \varepsilon \cdot \frac{f}{D} \geq (E + 1)f,
$$

where the last inequality holds for sufficiently large n, using the fact that M_E is linear in n. This means that G' (having at most f vertices) has a vertex with outdegree at least $E + 1$, which means G' contains an $(E + 1)$-out-star disjoint to the out-stars we have found so far. □

□

3   Concluding remarks

In this paper we proved that that given bases $B_1, \ldots, B_n$ in a matroid, we can find $(1/2 - o(1))n$ disjoint transversal bases. Although our methods do not extend past $n/2$, we do not think that there is a fundamental obstacle preventing related methods from going further. Indeed, by tracking the possible cascades of swaps more carefully, it might be possible to find $(1 - o(1))n$ disjoint transversal bases, or at least to find $(1 - o(1))n$ disjoint partial transversals each of size $(1 - o(1))n$. Although we cannot completely rule out the possibility that a full proof of Rota's basis conjecture could be obtained in this way, we imagine that more ingredients will be required. We are hopeful that ideas used to prove existence of designs (see [24, 20]) could be relevant, at least in the case of vector spaces.

Also, we remark that Rota's basis conjecture is reminiscent of some other problems concerning rainbow structures in graphs (actually, for a graphic matroid, Rota's basis conjecture can be interpreted as a conjecture about rainbow spanning forests in edge-coloured multigraphs). The closest one to Rota's basis conjecture seems to be the Brualdi–Hollingsworth conjecture [6], which posits that for every $(n - 1)$-edge-colouring of the complete graph K_n, the edges can be decomposed into rainbow spanning trees. This conjecture has recently seen some exciting progress (see for example [22, 27, 3, 25]). We wonder if some of the ideas developed for the study of rainbow structures could be profitably applied to Rota's basis conjecture.

We also mention the following strengthening of Rota's basis conjecture due to Kahn (see [23]). This is simultaneously a strengthening of the Dinitz conjecture [16] on list-colouring of $K_{n,n}$, solved by Galvin [17].

Conjecture 3.1. *Given a rank-n matroid and bases $B_{i,j}$ for each $1 \leq i, j \leq n$, there exist representatives $b_{i,j} \in B_{i,j}$ such that each of the sets $\{b_{1,j}, \ldots, b_{n,j}\}$ and $\{b_{i,1}, \ldots, b_{i,n}\}$ are bases.*

The methods developed in this paper are also suitable for studying Conjecture 3.1. In particular, the argument used to prove Theorem 1.1 can readily be modified to show the following natural partial result towards Kahn's conjecture.

Theorem 3.2. *For any $\varepsilon > 0$ the following holds for sufficiently large n. Given a rank-n matroid and bases $B_{i,j}$ for each $1 \leq i \leq n$ and $1 \leq j \leq f = (1-\varepsilon)n/2$, there exist representatives $b_{i,j} \in B_{i,j}$ and $L \subseteq \{1,\ldots,f\}$ such that each $\{b_{i,j} : i \in L\}$ is independent, and such that $\{b_{i,1},\ldots,b_{i,n}\}$ is a basis for any $i \in L$ and $|L| \geq (1/2 - \varepsilon)n$.*

Note that if we are in the setting of Conjecture 3.1 where bases are given for all $1 \leq i, j \leq n$ then the above theorem allows us to choose roughly which rows we would like to find our bases in.

Note also that if, for each fixed j, the bases $B_{1,j},\ldots,B_{n,j}$ are all equal, then Kahn's conjecture reduces to Rota's basis conjecture. This observation also shows that Theorem 3.2 implies Theorem 1.1.

It is not hard to adapt the proof of Theorem 1.1 to prove Theorem 3.2. However, since it would require repeating most of the argument, we omit the details here. For interested readers we present the details in a companion note, which we will not publish but will make available on the arXiv [7].

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