

Local rigidity of Einstein 4-manifolds satisfying a chiral curvature condition

Joel Fine ^{*1}, Kirill Krasnov², and Michael Singer³

¹Université libre de Bruxelles, Belgium

²University of Nottingham, UK

³University College London, UK

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Abstract

Let (M, g) be a compact oriented Einstein 4-manifold. Write R_+ for the part of the curvature operator of g which acts on self-dual 2-forms. We prove that if R_+ is negative definite then g is locally rigid: any other Einstein metric near to g is isometric to it. This is a chiral generalisation of Koiso’s Theorem, which proves local rigidity of Einstein metrics with negative sectional curvature. Our hypotheses are roughly one half of Koiso’s. Our proof uses a new variational description of Einstein 4-manifolds, as critical points of the so-called pure connection action S . The key step in the proof is that when $R_+ < 0$, the Hessian of S is strictly positive modulo gauge.

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1 Introduction

1.1 Statement of the main result

A Riemannian metric g is called *Einstein* if its Ricci curvature is constant:

$$\text{Ric}(g) = \Lambda g, \quad \Lambda \in \mathbb{R} \tag{1.1}$$

If g is Einstein and ϕ is a diffeomorphism, then ϕ^*g is also Einstein. When considered modulo diffeomorphisms however equation (1.1) is elliptic of index zero. This means that naively one might expect Einstein metrics to be isolated modulo diffeomorphisms, or “locally rigid”. It is important to note that there are many situations in which Einstein metrics are not rigid: hyperbolic metrics on surfaces, many Kähler–Einstein metrics, holonomy G_2 metrics, The aim of this article is to give a curvature condition in dimension 4 which ensures this naive expectation of local rigidity is true.

We begin with the definition.

Definition 1.1.

1. Given an Einstein metric g , let D_g denote the linearisation of the map $g \mapsto \text{Ric}(g) - \Lambda g$ at g . An *infinitesimal Einstein deformation* of g is a section h of S^2T^*M for which $D_g(h) = 0$.
2. An Einstein metric g is called *locally rigid* if for any infinitesimal Einstein deformation h of g there is a vector field v such that $h = L_v g$.

Let (M, g) be a locally rigid compact Einstein metric. It can be shown that if \hat{g} is another Einstein metric sufficiently close to g (in an appropriate Hölder space, say) then there is a diffeomorphism ϕ , close to the identity, with $\hat{g} = \phi^*g$. (For a proof see, for example, [2].) This justifies the use of the term “local rigidity” rather than just “infinitesimal rigidity” in the preceding definition.

To state our local rigidity result, we first recall the decomposition of the curvature tensor of an oriented Riemannian 4-manifold (M^4, g) . In this dimension the 2-forms split $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$ as the ± 1 -eigenspaces of the Hodge star. This induces a decomposition of the curvature operator:

$$\text{Rm} = \begin{pmatrix} R_+ & C^* \\ C & R_- \end{pmatrix} \tag{1.2}$$

Here $R_+ \in \text{End}(\Lambda^+)$ and $R_- \in \text{End}(\Lambda^-)$ are self-adjoint, whilst $C \in \text{Hom}(\Lambda^+, \Lambda^-)$. This relates to the standard decomposition into the scalar, trace-free Ricci and Weyl curvatures (denoted R , Ric^0 and W respectively) as follows: $R = 4 \text{Tr}(R_+) = 4 \text{Tr}(R_-)$, $\text{Ric}^0 = C$, and $W = R_+^0 \oplus R_-^0$ (where E^0 denotes the trace-free part of E and where we have used a suitable identification of $\text{Hom}(\Lambda^+, \Lambda^-)$ with symmetric trace-free bilinear forms on TM). In particular, when g is Einstein, $\text{Rm} = R_+ \oplus R_-$.

We now state our main result. In the statement the expression $R_+ < 0$ means that the quadratic form $(R_+(w), w)$ is negative definite.

Theorem 1.2. *Let g be an Einstein metric on a compact orientable 4-manifold, with either $R_+ < 0$ or $R_- < 0$. Then g is locally rigid.*

To put our main result in context, we first describe one situation in which local rigidity has already been established, namely Koiso’s Theorem.

Theorem 1.3 (Koiso [7]). *Let g be an Einstein metric on a compact orientable manifold of dimension at least 3, with negative curvature (i.e. all sectional curvatures of g are negative). Then g is locally rigid.*

Theorem 1.2 is a chiral version of Koiso’s Theorem, in dimension 4, with roughly “one half” the hypotheses. To see this, we relate the sectional curvatures to R_+ and R_- . Let u, v be orthogonal unit tangent vectors, write u^b, v^b for their metric dual

covectors and $\alpha = (u^b \wedge v^b)^+$ and $\beta = (u^b \wedge v^b)^-$. When g is Einstein the sectional curvature in the plane spanned by u and v is

$$\sec(u, v) = \langle R_+(\alpha), \alpha \rangle + \langle R_-(\beta), \beta \rangle$$

By letting u, v vary, we can make α and β take any values in the spheres in Λ^+ and Λ^- . Write λ for the largest eigenvalue of R_+ and μ for the largest eigenvalue of R_- . The requirement that an Einstein metric have negative sectional curvatures is then equivalent to $\lambda + \mu < 0$, whilst the hypothesis in our main result, Theorem 1.2, is that either $\lambda < 0$ or $\mu < 0$.

We remark that *global* rigidity results are much more difficult to establish. The only known results are the following: if (M, g) is a compact hyperbolic or complex-hyperbolic 4-manifold then g is the unique Einstein metric on M up to scale and diffeomorphism. In the hyperbolic case this is due to Besson–Courtois–Gallot [1], whilst the complex-hyperbolic case is due to LeBrun [9].

1.2 Outline of the proof

Consider first Koiso’s Theorem. To prove local rigidity of a negatively curved Einstein metric, Koiso uses a clever Bochner trick. Given an infinitesimal Einstein variation h , it is possible to find a vector field v so that $h - L_v g$ solves an equation of the form $\mathcal{D}(h - L_v g) = 0$ where \mathcal{D} is a Laplace type operator. Koiso finds a Weitzenböck formula $\mathcal{D} = d^*d + B$, where d is an appropriate first order operator. Crucially the remainder term B is positive when the sectional curvatures of g are negative. It follows that $h = L_v g$.

Our proof of local rigidity is different. We use a special variational formulation of 4-dimensional Einstein metrics, due to the second author [8] and independently, albeit in a weaker form, to the first author [3]. A full description of this “pure connection formalism” which is accessible to mathematicians appears in [5]. We give a brief summary here and more details in §2, choosing a description which is better suited to our purposes here than that given in the above references.

Let $E \rightarrow M^4$ be an $\mathrm{SO}(3)$ -vector bundle over M . The following definition is crucial to what follows. It first appeared in this form in [4], but is actually a special case of Weinstein’s “fat connections” [12].

Definition 1.4. An $\mathrm{SO}(3)$ -connection A on E is called *definite* if the curvature $F_A(u, v)$ is non-zero whenever u and v are linearly independent tangent vectors on M .

Examples can be found from Riemannian geometry. Let (M, g) be an oriented Riemannian 4-manifold, take $E = \Lambda^+$ and A to be the Levi-Civita connection of g . Then A is definite precisely when $R_+^2 - C^*C$ is a definite endomorphism of Λ^+ (we use here the notation of the curvature decomposition (1.2); the proof that these metrics give definite connections is in [4]). Notice that Einstein metrics have $C = 0$ and so when R_+ is also non-degenerate (all eigenvalues are non-zero) these metrics give definite connections. Examples include the constant curvature metrics on hyperbolic space or the four-sphere or, more generally, anti-self-dual Einstein metrics with non-zero scalar curvature.

When A is definite it determines in a canonical way a Riemannian metric g_A . The metric tensor g_A is built pointwise out of the curvature F_A . The precise formula for g_A is given below in §2 but it is not important for this introductory discussion. We note however that if A is the Levi-Civita connection on Λ^+ for a Riemannian metric g as in the previous paragraph, then it is *not* in general true that $g_A = g$. This happens precisely when g is Einstein and R_+ is definite, with all eigenvalues nonzero and having the same sign.

Write $S(A) = \frac{\Lambda}{2} \int_M \mathrm{dvol}(g_A)$ where $\mathrm{dvol}(g_A)$ is the volume form of the metric g_A . The key fact we need is:

Theorem 1.5 ([8], see also [5]). *If A is a critical point for S then g_A is an Einstein metric, with R_+ a definite endomorphism of Λ^+ . Conversely, all such Einstein metrics arise this way, with A corresponding to the Levi-Civita connection on Λ^+ .*

We now return to the proof of Theorem 1.2. Let g be an Einstein metric for which $R_+ < 0$. (Note that reversing the orientation of M swaps R_+ and R_- and so when proving Theorem 1.2 it suffices to consider $R_+ < 0$.) The Levi-Civita connection of g on $E = \Lambda^+$ gives a critical point of S . The key to proving Theorem 1.2 is to show that $R_+ < 0$ ensures that the Hessian of S at A is *strictly positive* modulo gauge. So A is an isolated local minimum of S (again, modulo gauge). We then show that this implies the original Einstein metric g is locally rigid.

The article is laid out as follows. In §2 we describe the pure connection formalism, showing how to derive it from the Plebanski formulation of Einstein’s equations. We then use this derivation in §3 to compute the Hessian of the pure connection action. §4 uses this to prove local rigidity of g .

1.3 Remark

We hope this proof highlights an important difference between the pure connection action S and the more classical variational approach to Einstein metrics, given by the Einstein–Hilbert action S_{EH} . At a critical point, the Hessian of S_{EH} is always indefinite, having an *infinite* number of both positive and negative eigenvalues. This makes S_{EH} extremely difficult to analyse from the point of view of the calculus of variations. The pure connection action does not suffer from this problem. Irrespective of curvature assumptions, at a critical point its Hessian is elliptic (modulo gauge), with at most *finitely* many negative eigenvalues [5]. It remains to be seen if techniques from the calculus of variations can be used to greater effect with the pure connection action than has been possible for the Einstein–Hilbert action.

2 From Plebanski to the pure connection action

In [10] Plebanski gave an alternative formulation of Einstein 4-manifolds, as critical points of a certain functional, now called the Plebanski action. By “integrating out” some of the variables of the Plebanski action, one arrives at the pure connection action of [8]. This point of view is central to our computation of the Hessian of the pure connection action in §3.

Before describing Plebanski’s action, we first recall some basic facts about 4-dimensional Riemannian geometry, with an emphasis which, whilst it may seem unusual at first sight, is well suited to our later purposes.

2.1 Some foundational 4-dimensional facts

Let M be an oriented 4-manifold. The wedge product on 2-forms $\Lambda^2 \otimes \Lambda^2 \rightarrow \Lambda^4$ is a non-degenerate symmetric bilinear form on Λ^2 with values in Λ^4 . We interpret this as a fibrewise conformal structure on Λ^2 with signature $(3, 3)$. Given any Riemannian metric g on M , this inner product makes self-dual and anti-self-dual forms orthogonal, is positive definite on Λ_g^+ and negative definite on Λ_g^- . Conversely, given any 3-dimensional sub-bundle $P \subset \Lambda^2$ on which the wedge-product is positive definite, there is a unique conformal class of metrics $[g]$ on M for which $P = \Lambda_g^+$. A Riemannian metric is thus equivalent to the data of a positive definite sub-bundle P together with a choice of volume form.

The abstract isomorphism class of the vector bundle Λ_g^+ is independent of g ; they are all isomorphic to some fixed reference $\text{SO}(3)$ -bundle $E \rightarrow M$. We fix this choice of E once and for all. For any Riemannian metric g on M , there is a (non-unique) vector bundle homomorphism $\Sigma: E \rightarrow \Lambda^2$ which is an isometry onto Λ_g^+ . In fact it is convenient to fix the scale so that unit-length vectors in E are sent to self-dual 2-forms

of length $\sqrt{2}$. The metric g can be recovered from Σ by setting $\Lambda^+(g) = \Sigma(E)$ and taking the volume form $\mu = \frac{1}{2}\Sigma(e) \wedge \Sigma(e)$ where e is any unit-length element of E . This leads us to the following definition.

Definition 2.1. $\Sigma \in \Omega^2(M, E^*)$ is called *wedge-orthogonal* if when thought of as a vector bundle homomorphism $\Sigma: E \rightarrow \Lambda^2$ there is a positive 4-form μ such that for any pair $e_1, e_2 \in E$, we have $\Sigma(e_1) \wedge \Sigma(e_2) = 2 \langle e_1, e_2 \rangle \mu$.

Given a wedge-orthogonal Σ , there is a unique Riemannian metric g_Σ on M for which Σ takes values in $\Lambda_{g_\Sigma}^+$ and with $\text{dvol}(g_\Sigma) = \mu$.

In the Plebanski formulation, we will use wedge-orthogonal 2-forms to parametrise metrics. Next we need to describe the Levi-Civita connection in this picture. For a proof of the following Lemma see Lemmas 2.3 and 2.4 of [3].

Lemma 2.2. *Let $\Sigma \in \Omega^2(M, E^*)$ be wedge-orthogonal. There is a unique metric connection A in E for which $d_A \Sigma = 0$. This connection is the pull-back to E of the Levi-Civita connection in $\Lambda_{g_\Sigma}^+$ via the isomorphism $\Sigma: E \rightarrow \Lambda_{g_\Sigma}^+$.*

The equation $d_A \Sigma = 0$ is a torsion-free condition, see [3] for more details. We are now in a position to recognise Einstein metrics in this set-up. The following result is the translation of the familiar fact that a Riemannian 4-manifold is Einstein if and only if the Levi-Civita connection on Λ^+ is a self-dual instanton.

Lemma 2.3. *Let $\Sigma \in \Omega^2(M, E^*)$ be wedge-orthogonal, with associated metric g_Σ and let A be the unique metric connection in E with $d_A \Sigma = 0$. Then g_Σ is Einstein if and only if F_A is self-dual, i.e. A is an instanton with respect to g_Σ .*

2.2 The Plebanski action

The Plebanski action is a function of three variables, (A, Σ, Ψ) where A is a metric connection on the bundle E , $\Sigma \in \Omega^2(M, E^*)$ and $\Psi \in \Omega^0(M, S_0^2 E)$. Note at this stage these variables are arbitrary; in particular, Σ is *not* assumed to be wedge-orthogonal and is unrelated to A .

To describe the action, it is perhaps simplest to choose a local orthonormal frame e_i for E . Pick an orientation on E and take the local frame also to be oriented (the freedom in choice of orientation will turn out to be slightly subtle—we will return to this point later). We write e^i for the dual frame of E^* and \hat{e}^i for the generator of positive rotations about e_i , giving a local frame of $\mathfrak{so}(E)$. We use the convention that repeated indices are summed over 1, 2, 3.

We write the connection A locally as

$$d_A(e_i) = \epsilon_{ijk} A^j \otimes e^k \quad (2.1)$$

for a triple of 1-forms A^j . Similarly, we write $\Sigma = \Sigma_i \otimes e^i$ for a triple of 2-forms Σ_i and $\Psi = \Psi^{ij} e_i \otimes e_j$ for a symmetric trace-free matrix valued function Ψ_{ij} . We will also need the local expression for the curvature of A . Let

$$F^i = dA^i - \frac{1}{2} \epsilon^i{}_{jk} A^j \wedge A^k \quad (2.2)$$

These are the curvature 2-forms, meaning that $(d_A)^2(e_i) = \epsilon_{ij}{}^k F^j \otimes e_k$.

Definition 2.4. Fix $\Lambda \in \mathbb{R}$. The *Plebanski action* is the functional

$$S_P(A, \Sigma, \Psi) = \int_M \left\{ F^i \wedge \Sigma_i - \frac{1}{2} \left(\Psi^{ij} + \frac{\Lambda}{3} \delta^{ij} \right) \Sigma_i \wedge \Sigma_j \right\}$$

(Note that the integrand does not depend on the choice of local frame e_i and so makes sense globally.)

Write \mathcal{P} for the open set of triples (A, Σ, Ψ) for which $\text{Tr}(\Sigma \wedge \Sigma) > 0$.

Theorem 2.5 (Plebanski [10]). $(A, \Sigma, \Psi) \in \mathcal{P}$ is a critical point of S_P if and only if Σ is wedge-orthogonal and the associated metric g_Σ is Einstein.

Proof. Let $a \in \Omega^1(M, \mathfrak{so}(E))$, $\phi \in \Omega^0(M, S^2 E_0)$ and $\sigma \in \Omega^2(M, E^*)$. We compute the partial derivatives of S_P in these directions.

$$\frac{\partial S_P}{\partial A}(a) = \int_M (d_A a)^i \wedge \Sigma_i = \int_M a^i \wedge (d_A \Sigma)_i \quad (2.3)$$

$$\frac{\partial S_P}{\partial \Psi}(\phi) = -\frac{1}{2} \int_M \phi^{ij} \Sigma_i \wedge \Sigma_j \quad (2.4)$$

$$\frac{\partial S_P}{\partial \Sigma}(\sigma) = \int_M \left\{ F^i \wedge \sigma_i - \left(\Psi^{ij} + \frac{\Lambda}{3} \delta^{ij} \right) \Sigma_i \wedge \sigma_j \right\} \quad (2.5)$$

From this it follows that

$$\frac{\partial S_P}{\partial A} = 0 \iff d_A \Sigma = 0 \quad (2.6)$$

$$\frac{\partial S_P}{\partial \Psi} = 0 \iff \Sigma \text{ is wedge-orthogonal} \quad (2.7)$$

$$\frac{\partial S_P}{\partial \Sigma} = 0 \iff F^i = \left(\Psi^{ij} + \frac{\Lambda}{3} \delta^{ij} \right) \Sigma_j \quad (2.8)$$

By (2.7), Σ determines a metric g_Σ , with $\Lambda^+ = \text{im } \Sigma$, the image of Σ . By (2.6) and Lemma 2.2 the connection A is the pull-back via Σ of the Levi-Civita connection. By (2.8), F_A is self-dual. The result follows from Lemma 2.3. \square

When (A, Σ, Ψ) is a critical point of S_P , equation (2.8) tells us that the Einstein metric g_Σ has $R_+ = \Psi + \frac{\Lambda}{3} \text{Id}$. In other words, the self-dual Weyl curvature W^+ identifies with Ψ via Σ whilst the scalar curvature is given by $R = 4\Lambda$, which gives us the Einstein constant: $\text{Ric}(g_\Sigma) = \Lambda g_\Sigma$. In fact, there is a small subtlety here. Equations (2.6), (2.7) and (2.8) show that $(A, -\Sigma, -\Psi)$ is a critical point for the Plebanski action with Λ replaced by $-\Lambda$. To resolve this ambiguity and determine which sign we should use, note that we fixed an orientation of the bundle E . Meanwhile, the bundle Λ^+ is naturally oriented. We choose the sign of Σ in order that the isomorphism $\Sigma: E \rightarrow \Lambda^+$ is orientation preserving. With this choice of sign, $W^+ = \Psi$ and $\text{Ric}(g_\Sigma) = \Lambda g_\Sigma$.

2.3 The pure connection action

At this stage we must restrict to the case $\Lambda \neq 0$. The idea behind the pure connection formulation is that equations (2.7) and (2.8) can be used to eliminate Ψ and Σ and make the action a function purely of the connection A . To do this, we must begin with a connection A which is *definite* (see Definition 1.4). Write \mathcal{D} for the set of definite connections and $\mathcal{P} = \{(A, \Sigma, \Psi) : \text{Tr}(\Sigma \wedge \Sigma) > 0\}$ for the domain of the Plebanski action. We will define a map $\theta: \mathcal{D} \rightarrow \mathcal{P}$ of the form $\theta(A) = (A, \Sigma_A, \Psi_A)$ with the following crucial property: A is a critical point of $S_P \circ \theta$ if and only if $\theta(A)$ is a critical point of S_P . Once θ has been found, the pure connection action is defined by $S = S_P \circ \theta$.

We first explain how to define Σ_A given $A \in \mathcal{D}$. To satisfy equations (2.7) and (2.8), we must find a wedge-orthogonal 2-form which is also in the span of the curvature forms. Since A is definite,

$$F^i \wedge F^j = Q^{ij} \mu$$

for some positive-definite matrix Q^{ij} and positive 4-form μ . Let Y be a square-root of Q . When $\Lambda > 0$ we take Y to be the positive square root, when $\Lambda < 0$ we take Y to be the negative square root. Scaling μ , scales Q and so Y ; we fix the choice of scale

by requiring that $\text{Tr} Y = \Lambda$. We denote the corresponding volume form μ_A to make clear that it is determined by A . To ease the notation we write $X = Y^{-1}$. Set

$$\Sigma_i = X_{ij} F^j \quad (2.9)$$

The E^* -valued 2-form $\Sigma_i \otimes e^i$ is globally defined and we denote it by $\Sigma_A \in \Omega^2(M, E^*)$ to emphasise that it is determined by A . The point is that the use of X ensures that Σ_A is wedge-orthogonal, so (2.7) is satisfied. Moreover, the Σ_i and the F_i span the same subspace, so it will be possible to satisfy equation (2.8) by the correct choice of Ψ_A . Indeed given that $\Sigma = X F_A$, we have $F_A = Y \Sigma_A$ and so defining Ψ_A via

$$\Psi_A + \frac{\Lambda}{3} \text{Id} = Y \quad (2.10)$$

is the only way to satisfy (2.8).

We now say a word about the sign ambiguity which arose in our discussion of the Plebanski action. The image of Σ_A is a definite sub-bundle of Λ^2 , which we denote by Λ_A^+ . The map $\Sigma_A: E \rightarrow \Lambda_A^+$ is an isomorphism of oriented bundles and it either preserves or reverses orientation. Recall that $\Sigma_A = X F_A$ and the sign of X agrees with that of Λ . So switching the sign of Λ switches the sign of Σ_A . Given a definite connection only *one* of the two signs of Λ is appropriate to consider, that for which the corresponding Σ_A is orientation preserving. This sign can be seen directly from the curvature of A as follows.

Definition 2.6. Given a definite connection A and an orientation on E , the curvature can be interpreted as an isomorphism $E \rightarrow \Lambda_A^+$ of oriented bundles. In our local frame this is the map $e_i \mapsto F^i$. We call A *positive-definite* if this map is orientation preserving and *negative-definite* if this map is orientation reversing. Note that changing the orientation on E changes the sign of this map and so the sign of the definite connection is well-defined.

Positive-definite connections are those for which Σ_A is orientation preserving when $\Lambda > 0$. Negative-definite connections are those for which Σ_A is orientation preserving when $\Lambda < 0$. From now on, given Λ we consider only definite connections with the same sign. When g is Einstein with R_+ definite the Levi-Civita connection on Λ_g^+ is definite and its sign agrees with that of R_+ .

We define $\theta: \mathcal{D} \rightarrow \mathcal{P}$ by $\theta(A) = (A, \Sigma_A, \Psi_A)$. By construction, (A, Σ_A, Ψ_A) automatically solves two of the three Euler–Lagrange equations: $\partial S_P / \partial \Psi = 0 = \partial S_P / \partial \Sigma$. It follows that A is a critical point of $S_P \circ \theta$ if and only if $\theta(A)$ is a critical point of S_P . The remaining Euler–Lagrange equation $d_A \Sigma_A = 0$ is now an equation purely for the definite connection A .

Finally, we can write the action $S_P \circ \theta$ purely in terms of the connection. Using the relationship between Σ_A , Ψ_A and F_A we see that

$$S_P(\theta(A)) = \frac{1}{2} \int_M X_{ij} F^i \wedge F^j = \frac{\Lambda}{2} \int_M \mu_A = \frac{\Lambda}{2} \text{vol}(M, g_A)$$

where $g_A = g_{\Sigma_A}$ denotes the Riemannian metric determined by Σ_A and hence A .

Definition 2.7 (See [8, 5]). Let $\Lambda \neq 0$ and A be a definite connection of the same sign as Λ . We call

$$S(A) = \frac{\Lambda}{2} \text{vol}(M, g_A)$$

the *pure connection action* of A .

We summarise this discussion in the following statement.

Theorem 2.8 (See [8, 5]). *Fix $\Lambda \neq 0$ and let A be a definite connection of the same sign. A is a critical point of the pure connection S action if and only*

$$d_A \Sigma_A = 0$$

When this happens, g_A is an Einstein metric with R_+ definite and of the same sign as Λ . Moreover all Einstein metrics with R_+ definite arise this way.

We close this section with some remarks about the existence of definite connections. In [4] it is shown that if a definite connection exists on M , then $2\chi(M) + 3\tau(M) > 0$. This is “one half” of the Hitchin–Thorpe inequality, a necessary condition for the existence of an Einstein metric [6, 11]. No other obstructions to the existence of definite connections are known. The only known compact examples of *positive* definite connections are on S^4 and $\mathbb{C}\mathbb{P}^2$, given by deforming the corresponding Levi-Civita connections of the standard Einstein metrics there. Meanwhile, there are negative definite connections on manifolds on which it is currently unknown if there are Einstein metrics. See [4, 5] for examples of definite connections and open questions about them.

2.4 The gauge group

Let $\mathcal{G}: E \rightarrow E$ denote the group of all maps which send fibres of E to fibres by linear isometries (i.e. bundle isomorphisms). We call \mathcal{G} the *gauge group*. An element $\gamma \in \mathcal{G}$ covers a diffeomorphism on the base M and this sets up an exact sequence

$$1 \rightarrow \mathcal{G}_0 \rightarrow \mathcal{G} \rightarrow \text{Diff}(M) \rightarrow 1$$

where \mathcal{G}_0 is the subgroup of gauge transformations which cover the identity on M , the “usual” gauge group one sees in traditional gauge theory.

\mathcal{G} acts on the space \mathcal{A} of all connections in E , by pull-back. It preserves the open set $\mathcal{D} \subset \mathcal{A}$ of definite connections. The map $A \mapsto g_A$, sending a definite connection to its corresponding metric, is equivariant with respect to the action of \mathcal{G} on \mathcal{D} and the action of $\text{Diff}(M)$ on metrics. It follows that the pure connection action $S: \mathcal{D} \rightarrow \mathbb{R}$ is \mathcal{G} -invariant.

Given $\eta \in \text{Lie}\mathcal{G}$ we write $L_\eta A$ for the infinitesimal action of η at A . To compute $L_\eta A$ it is convenient to move to the principal bundle description. $P \rightarrow M$ denotes the principal $\text{SO}(3)$ -bundle of frames of E . \mathcal{G} is then the subgroup of diffeomorphisms of P which commute with the principal $\text{SO}(3)$ -action. The Lie algebra $\text{Lie}(\mathcal{G})$ is a subalgebra of vector fields on P . A connection A is an $\text{SO}(3)$ -equivariant connection 1-form with values in $\mathfrak{so}(3)$. In this picture, the infinitesimal action $L_\eta A$ of $\eta \in \text{Lie}(\mathcal{G})$ at A is simply the usual Lie derivative.

We now give a useful formula for $L_\eta A$. Use A to split the vector field η into vertical and horizontal parts. When $\eta \in \text{Lie}(\mathcal{G})$, the vertical part corresponds to a section $\xi \in \Omega^0(M, \mathfrak{so}(E))$; meanwhile, the horizontal part is the A -horizontal lift to P of a vector field v on M . By Cartan’s formula,

$$L_\eta A = d(\iota_\eta A) + \iota_\eta(dA) = d_A(\xi) + \iota_v F_A \quad (2.11)$$

(We consider here a right action of \mathcal{G} on \mathcal{A} , $A \mapsto \gamma^* A$ whereas usually in gauge theory one considers the corresponding left action. This accounts for the difference in sign of the term $d_A(\xi)$ here when compared with standard gauge theory literature.)

3 The Hessian of the pure connection action

In this section we compute the Hessian of the pure connection action S . Recall that \mathcal{D} denotes the set of definite connections (an open set in the affine space of all connections). Given $A \in \mathcal{D}$ which is a critical point of S , the Hessian $D^2 S$ of S at A is a symmetric bilinear form on $T_A \mathcal{D} = \Omega^1(M, \mathfrak{so}(E))$.

Theorem 3.1. *Let A be a critical point of S corresponding to an Einstein metric for which $R_+ < 0$. Then $D^2 S(a, a) \geq 0$ with equality if and only if $a = L_\eta A$ for some $\eta \in \text{Lie}(\mathcal{G})$.*

It turns out to be easiest to compute the Hessian of S by passing via Plebanski action. Recall that $\mathcal{P} = \{(A, \Sigma, \Psi) : \text{Tr}(\Sigma \wedge \Sigma) > 0\}$ denotes the domain of Plebanski action. In the previous section we introduced a map $\theta: \mathcal{D} \rightarrow \mathcal{P}$ given by $\theta(A) = (A, \Sigma_A, \Psi_A)$ where Σ_A and Ψ_A are defined by (2.9) and (2.10). The pure connection

action is the pull-back of Plebanski action by θ . It follows that at a critical point A , the Hessian of the pure connection action is the pull-back by θ of the Hessian of S_P at $\theta(A)$. To compute D^2S we will begin with the Hessian of Plebanski action at an arbitrary critical point (A, Σ, Ψ) (not necessarily of the form $\theta(A)$ for a definite connection A). Then we pull this back to \mathcal{D} via θ . To complete the proof of Theorem 3.1, we will also bring gauge fixing into the picture.

3.1 The Hessian before gauge fixing

We begin with the Hessian of the Plebanski action. Fix a critical point (A, Σ, Ψ) of S_P and choose a tangent direction $t = (a, \sigma, \phi)$. Write $D^2S_P(t, t)$ for the Hessian of S_P in this direction. Differentiating the equations (2.3), (2.4) and (2.5) again with respect to (a, σ, ϕ) (and recalling our conventions (2.1) and (2.2)) we see that $D^2S_P(t, t)$ is equal to

$$\int_M \left\{ -\epsilon_{ij}{}^k a^i \wedge a^j \wedge \Sigma_k + 2((d_A a)^i - \phi^{ij} \Sigma_j) \wedge \sigma_i - \left(\Psi^{ij} + \frac{\Lambda}{3} \delta^{ij} \right) \sigma_i \wedge \sigma_j \right\} \quad (3.1)$$

We now consider the Hessian of the pure connection action S . Let A be a definite connection which is a critical point of S so that $\theta(A) = (A, \Sigma_A, \Psi_A)$ is a critical point of S_P . Given an infinitesimal change a of A , the Hessians of S and S_P are related by

$$D^2S(a, a) = D^2S_P(\theta_*(a), \theta_*(a))$$

Write $\theta_*(a) = (a, \sigma, \phi)$ where σ and ϕ denote the corresponding infinitesimal changes in Σ_A and Ψ_A respectively. To proceed we will need the precise dependence of σ and ϕ on a . To obtain this note that the tangent vector (a, σ, ϕ) preserves the conditions (2.7) and (2.8). Differentiating (2.7) we see that

$$\sigma_i \wedge \Sigma_j + \Sigma_i \wedge \sigma_j - \frac{2}{3}(\sigma_k \wedge \Sigma_k) \delta_{ij} = 0 \quad (3.2)$$

Differentiating (2.8) gives

$$(d_A a)^i - \phi^{ij} \Sigma_j - Y^{ij} \sigma_j = 0 \quad (3.3)$$

(Recall that $Y = \Psi_A + \frac{\Lambda}{3} \text{Id}$.) Rearranging this gives

$$\sigma_i = X_{ij} ((d_A a)^j - \phi^{jk} \Sigma_k) \quad (3.4)$$

(Recall that $X = Y^{-1}$.) From here we can solve for ϕ in terms of $d_A a$ via (3.2). We do this as follows.

Using the local frame Σ^j for Λ^+ we write the self-dual part of σ_i as $\sigma_i^+ = Z_i{}^j \Sigma_j$ for some 3×3 -matrix valued function Z . Similarly, we put $(d_A^+ a)^i = N^{ij} \Sigma_j$, so (3.4) reads

$$Z_i{}^j = X_{ik} (N^{kj} - \phi^{kj}) \quad (3.5)$$

Now an arbitrary 3×3 -matrix has three irreducible parts: the symmetric trace-free component, the skew component and the trace component. Equation (3.2) says that the symmetric trace-free part of Z is zero. This uniquely determines ϕ^{ij} in terms of N^{ij} and X_{ij} . To see this write $L: S_0^2 E \rightarrow S_0^2 E$ for the linear bundle map $L(\phi) = S_0^2(X\phi)$ where S_0^2 denotes the projection operator sending a matrix to the trace-free symmetric part. Then ϕ must solve $L(\phi) = S_0^2(XN)$. This has a unique solution $\phi \in S_0^2 E$ provided L is injective. Suppose then that $L(\psi) = 0$ for some $\psi \in S_0^2 E$ or, in other words, that

$$\psi + X\psi X^{-1} = fX^{-1}$$

for some function f . Taking the trace and using the fact that $\text{Tr} X^{-1} = \Lambda$ we see that $f = 0$. On the one hand the eigenvalues of the similar matrices ψ and $X\psi X^{-1}$ are equal, on the other hand the fact that $\psi = -X\psi X^{-1}$ shows the eigenvalues are opposite. The only conclusion is that $\psi = 0$ and so L is an isomorphism as required.

We can now give the formula for the Hessian of the pure connection action.

Lemma 3.2. *Let $A \in \mathcal{D}$ be a critical point of S and let $a \in T_A \mathcal{D}$. The Hessian of S in the direction a is given by*

$$D^2S(a, a) = \int_M \left[-\epsilon_{ij}{}^k a^i \wedge a^j \wedge \Sigma_k + X_{ij} \left((d_A a)^i - \phi^{ik} \Sigma_k \right) \wedge \left((d_A a)^j - \phi^{jk} \Sigma_k \right) \right]$$

where $\phi \in S_0^2 E$ is the unique solution to $S_0^2(X\phi) = S_0^2(XN)$ where $d_A^+ a = N\Sigma$.

Proof. We have seen that $\theta_*(a) = (a, \sigma, \phi)$ where σ and ϕ satisfy (3.4). We now substitute (3.4) into (3.1) and use the fact that

$$X_{ij} \left(\Psi^{ij} + \frac{\Lambda}{3} \delta^{ij} \right) = X_{ij} Y^{ij} = \delta_i^j$$

This gives the claimed expression for $D^2S(a, a)$. The condition on ϕ is the content of the discussion after (3.5) \square

3.2 The Hessian after gauge fixing

In this section we will explain how the formula of Lemma 3.2 for $D^2S(a, a)$ simplifies when we impose additional gauge fixing conditions on a . Before discussing this however we need a short digression. Given a wedge-orthogonal Σ , we can define a triple of almost complex structures J_i by “raising an index” on Σ_i with g_Σ . (It is for this reason that the metric g_Σ is scaled so that $|\Sigma_i|^2 = 2$.) Equivalently, one can define the action of J_i on 1-forms via the equation

$$J_i(\alpha) = *(\Sigma_i \wedge \alpha) \quad (3.6)$$

One can check that the J_i satisfy the quaternion relations:

$$J_i J_j = \epsilon_{ij}{}^k J_k - \delta_{ij} \quad (3.7)$$

At this point we use crucially that $\Sigma: E \rightarrow \Lambda^+$ is orientation-preserving; if it were orientation-reversing then $-J_i$ would satisfy (3.7).

We will use the freedom to act by gauge transformations to ensure that the infinitesimal change in connection a^i satisfies the following two conditions:

Definition 3.3. Let A be a definite connection which is a critical point of the pure connection action S . Let Σ_A be determined by A via (2.9). We say that $a \in \Omega^1(M, \mathfrak{so}(E))$ is in *horizontal gauge* if

$$\Sigma_i \wedge a^i = 0 \quad (3.8)$$

In terms of the almost complex structures, this is equivalent to $J_i a^i = 0$. Given a in horizontal gauge, we say that it is also in *vertical gauge* if in addition:

$$\epsilon^{ij}{}_k \Sigma_j \wedge (d_A a)^k = 0 \quad (3.9)$$

When a is in both horizontal and vertical gauge we say that a is *fully gauge fixed*.

We briefly explain the names. Loosely speaking, gauge transformations are made up of diffeomorphisms on M and purely “vertical” gauge transformations which preserve each fibre of E . The horizontal gauge fixing condition gives a complement to the infinitesimal action of diffeomorphisms, i.e. of horizontal lifts of vector fields from M . Now for those a which are in horizontal gauge, the second condition (3.9) is equivalent to Coulomb gauge, $d_A^* a = 0$, as the following lemma shows. This then gives a complement to the infinitesimal action of vertical gauge transformations.

Lemma 3.4. *If a is in horizontal gauge and $d_A \Sigma_A = 0$ then*

$$*a^i = \epsilon^{ij}{}_k \Sigma_j \wedge a^k \quad (3.10)$$

$$(d_A^* a)^i = - * \left(\epsilon^{ij}{}_k \Sigma_j \wedge (d_A a)^k \right) \quad (3.11)$$

Proof. In terms of the almost complex structures J_i , the horizontal gauge condition (3.8) reads $J_i a^i = 0$ or, equivalently, $a^i = -\epsilon^{ijk} J_j a_k$. Taking the Hodge star, and using the fact that $*^2 = -1$ on 3-forms, we obtain $*a^i = \epsilon^{ij}_k \Sigma_j \wedge a^k$. Now, since $d_A \Sigma_A = 0$ and $d_A^* = -*d_A*$ equation (3.11) follows from (3.10). \square

Before discussing how to enforce these two conditions, we show the effect they have on the Hessian of the pure connection action.

Proposition 3.5 (The gauge-fixed Hessian). *Let $A \in \mathcal{D}$ be a critical point of the pure connection action S . Let a be an infinitesimal change in A which satisfies both gauge fixing conditions (3.8) and (3.9). Then the Hessian of the pure connection action in this direction is given by*

$$D^2 S(a, a) = \int_M (|a|^2 - X_{ij} \langle (d_A^- a)^i, (d_A^- a)^j \rangle) \mu_A \quad (3.12)$$

In particular, when R_+ is negative definite, so that X is negative definite,

$$D^2 S(a, a) \geq 0$$

with equality if and only if $a = 0$.

Proof. Recall the formula for $D^2 S_A(a, a)$ given in Lemma 3.2 above. When a is in horizontal gauge, the first term in the integrand is

$$-\epsilon_{ij}^k a^i \wedge a^j \wedge \Sigma_k = a^i \wedge *a^i = |a|^2 \mu_A \quad (3.13)$$

We now turn to the second term in the integrand, namely

$$X_{ij} ((d_A a)^i - \phi^{ik} \Sigma_k) \wedge ((d_A a)^j - \phi^{jk} \Sigma_k)$$

Here $\phi \in S_0^2 E$ is determined by a via $S_0^2(X\phi) = S_0^2 XN$ where $(d_A^+ a)^i = N^{ij} \Sigma_j$. The Coulomb gauge condition (3.9) says that the skew-symmetric part of N vanishes. Meanwhile, differentiating the horizontal gauge condition (3.8) and using $d_A \Sigma_A = 0$ gives

$$\Sigma_i \wedge (d_A a)^i = 0 \quad (3.14)$$

which means that the trace of N vanishes. It follows that N is itself a symmetric trace-free matrix and so $\phi = N$. It follows that

$$(d_A a)^i - \phi^{ij} \Sigma^j = (d_A^- a)^i \quad (3.15)$$

is purely anti-self-dual. The second term is thus

$$X_{ij} (d_A^- a)^i \wedge (d_A^- a)^j$$

For anti-self-dual forms, the wedge product is minus the pointwise inner product which completes the proof. \square

3.3 Gauge fixing

We now show that the space of fully gauge fixed a defines a complement to the infinitesimal gauge action. Recall that elements $\eta \in \text{Lie}(\mathcal{G})$ correspond to pairs (v, ξ) where $\xi \in \Omega^0(M, \mathfrak{so}(E))$ and v is a vector field on M . The infinitesimal action of η at A is $L_\eta(A) = \iota_v F_A + d_A \xi$

Let

$$W = \{a \in C^\infty(\Lambda^1 \otimes E) : a \text{ is fully gauge fixed, (3.8) and (3.9) hold}\} \quad (3.16)$$

The goal of this section is to prove the following:

Proposition 3.6. *There is a direct sum decomposition*

$$\Omega^1(M, \mathfrak{so}(E)) = \{L_\eta A : \eta \in \text{Lie}(\mathcal{G})\} \oplus W \quad (3.17)$$

Before giving the proof we first try and lighten the notation a little. Write

$$p: \Lambda^1 \otimes \mathfrak{so}(E) \rightarrow \Lambda^1, \quad p(a) = J_i a^i$$

Recall that the horizontal gauge condition (3.8) is equivalent to $p(a) = 0$. Write

$$q: TM \rightarrow \Lambda^1 \otimes \mathfrak{so}(E), \quad q(v) = \iota_v F_A$$

To prove Proposition 3.6 we must show that for any a there is a unique solution (v, ξ) to the pair of equations

$$p(a + q(v) + d_A \xi) = 0 \quad (3.18)$$

$$d_A^*(a + q(v) + d_A \xi) = 0 \quad (3.19)$$

We will show that this leads to an elliptic equation for ξ . As we will see shortly, the composition $p \circ q$ is invertible. Assuming this for the moment, for any choice of a and ξ , there is a unique choice of v solving (3.18), namely

$$v = -(p \circ q)^{-1} \circ p((d_A \xi) + a) \quad (3.20)$$

Write

$$\Pi = 1 - q \circ (p \circ q)^{-1} \circ p$$

It is simple to check that Π is a projection operator, projecting $\Lambda^1 \otimes E$ onto $\ker(p)$ against $\text{im}(q)$. In other words, $\Pi(a)$ is what one obtains by putting a in horizontal gauge using the infinitesimal action of tangent vectors. Now substituting (3.20) into (3.19) we see that (3.18) and (3.19) are equivalent to the following single equation for ξ :

$$d_A^* \Pi d_A \xi = -d_A^* \Pi a \quad (3.21)$$

We will prove that (3.21) has a unique solution in three steps: by showing that $d_A^* \Pi d_A$ is elliptic; that it has index zero and finally that it is injective.

The symbol of $d_A^* \Pi d_A$ in the direction of the covector α is the map

$$\text{Sym}(\alpha): E \rightarrow E, \quad \text{Sym}(\alpha)(\xi) = \langle \alpha, \Pi(\alpha \otimes \xi) \rangle$$

To compute this we first give a formula for Π . Along the way we show that $p \circ q$ is invertible. Recall that $F^i = Y^{ij} \Sigma_j$ where $Y = \Psi + \frac{\Lambda}{3} \text{Id}$. Our curvature assumption that $R_+ < 0$ is equivalent to $Y < 0$.

Lemma 3.7. *We have $p \circ q(v) = -\Lambda v^b$ where v^b denotes the covector which is metric dual to the tangent vector v . It follows that $(p \circ q)^{-1}(\alpha) = -\Lambda^{-1} \alpha^\#$ where $\alpha^\#$ is the tangent vector metric dual to the covector α .*

Proof. This is a direct calculation, using the quaternion relations (3.7).

$$\begin{aligned} p \circ q(v) &= J_i (\iota_v F^i) \\ &= J_i (Y^{ij} \iota_v \Sigma_j) \\ &= Y^{ij} J_i J_j (v^b) \\ &= Y^{ij} \left(\epsilon_{ij}^k J_k (v^b) - \delta_{ij} v^b \right) \\ &= -\text{Tr}(Y) v^b \\ &= -\Lambda v^b \end{aligned} \quad \square$$

Lemma 3.8. *The projection Π has the following expression in terms of the self-dual curvature matrix Y^{ij} .*

$$(\Pi a)^i = a^i + \Lambda^{-1} \epsilon_{kj}^l Y^{ik} J_l(a^j) - \Lambda^{-1} Y^{ij} a^j$$

Proof. Again, this is direct calculation.

$$\begin{aligned} (\Pi a)^i &= a^i - q \circ (p \circ q)^{-1} \circ p(a)^i \\ &= a^i + \Lambda^{-1} \iota_{(J_j a^j)^\#} F^i \\ &= a^i + \Lambda^{-1} Y^{ik} \iota_{(J_j a^j)^\#} \Sigma^k \\ &= a^i + \Lambda^{-1} Y^{ik} J_k J_j(a^j) \\ &= a^i + \Lambda^{-1} \epsilon_{kj}^l Y^{ik} J^l(a^j) - \Lambda^{-1} Y^{ij} a^j \quad \square \end{aligned}$$

Lemma 3.9. *When R_+ is negative definite, $d_A^* \Pi d_A$ is an elliptic operator on sections of $\mathfrak{so}(E)$.*

Proof. We use the above formula for Π to compute the symbol acting on the covector α . Write $\xi = \xi^i \otimes \hat{e}_i$.

$$\begin{aligned} \text{Sym}(a)(\xi) &= \langle \alpha, \Pi(\alpha \otimes \xi) \rangle \\ &= \xi^i |\alpha|^2 + \Lambda^{-1} \epsilon_{kj}^l \xi^j Y^{ik} \langle \alpha, J_l(\alpha) \rangle - \Lambda^{-1} Y^{ij} \xi^j |\alpha|^2 \\ &= (\xi^i - \Lambda^{-1} Y^{ij} \xi^j) |\alpha|^2 \end{aligned}$$

where in the last line we have used that α and $J_l(\alpha)$ are orthogonal. It follows that, for $\alpha \neq 0$, the kernel of $\text{Sym}(\alpha)$ is exactly the space of eigenvectors of Y with eigenvalue Λ . But $\Lambda = \text{Tr}(Y)$ and all eigenvalues of Y are *strictly* negative, so none can be equal to Λ . It follows that $\text{Sym}(\alpha)$ is invertible and so $d_A^* \Pi d_A$ is elliptic. \square

Lemma 3.10. *When R_+ is negative definite, the index of $d_A^* \Pi d_A$ is zero.*

Proof. If Π were self adjoint then $d_A^* \Pi d_A$ would be also and it would follow immediately that the index vanishes. This is not the case however because, in general, $\ker p$ and $\text{im } q$ are not orthogonal subspaces. To remedy this, consider the path $Y_t^{ij} = t\Psi^{ij} + \frac{\Lambda}{3}\delta^{ij}$ of endomorphisms and the corresponding path $F_t^i = Y_t^{ij}\Sigma^j$ of sections of $\Lambda^2 \otimes \mathfrak{so}(E)$. The forms F_t define a path of maps $q_t: TM \rightarrow \Lambda^1 \otimes \mathfrak{so}(E)$ given by $q_t(v) = \iota_v F_t$. Since $Y = Y_1$ is negative-definite, the same is true for Y_t for all $t \in [0, 1]$. Moreover, $\text{Tr } Y_t = \Lambda$. These were the only properties we used in the above arguments. It follows that $\Pi = 1 - q_t \circ (p \circ q_t)^{-1} \circ p$ is a projection of $\Lambda^1 \otimes \mathfrak{so}(E)$ onto $\ker p$ against $\text{im } q_t$ and, moreover, that $d_A^* \Pi_t d_A$ is a path of elliptic operators. It follows that our operator has the same index as $d_A^* \Pi_0 d_A$. But when $t = 0$, $\text{im } q_0$ is the orthogonal complement of $\ker p$ and so Π_0 is a self-adjoint projection. It follows that the index of $d_A^* \Pi_0 d_A$, and hence of all the $d_A^* \Pi_t d_A$, vanishes for $t \in [0, 1]$. \square

Proposition 3.11. *When R_+ is negative-definite, the map*

$$d_A^* \Pi d_A: C^\infty(E) \rightarrow C^\infty(E)$$

is an isomorphism.

Proof. Now that we know $d_A^* \Pi d_A$ is elliptic of index zero, it suffices to show that it is injective. Suppose then that $d_A^* \Pi d_A \xi = 0$. This means that $\Pi d_A \xi$ is fully gauge fixed. By Proposition 3.5 it follows that either $\Pi d_A \xi = 0$, or the Hessian in this direction is non-zero. Now, by definition of Π , there exists $v \in C^\infty(TM)$ such that $\Pi d_A \xi = d_A \xi + \iota_v F$. In other words, $\Pi d_A \xi$ is pure gauge. Since the pure connection action is gauge invariant, the Hessian vanishes in pure gauge directions. It follows that $\Pi d_A \xi = 0$ or, in other words, that $\iota_v F = -d_A \xi$.

Now the infinitesimal gauge transformation ξ doesn't change the induced metric g_A and so v is a Killing field. But the metric has negative Ricci curvature and so v must vanish¹. It follows that $d_A\xi = 0$. From this we see that $F_A(\xi) = 0$, or in other words that $Y^{ij}\xi^j = 0$. Since Y is invertible we obtain $\xi = 0$ as required. \square

This Proposition, together with the discussion immediately after equations (3.18) and (3.19), completes the proof of Proposition 3.6, which shows that, on the linear level, one can always fully gauge fix a via the infinitesimal action of the gauge group.

3.4 The proof of Theorem 3.1

We now give the proof of Theorem 3.1. Let A be a critical point of the pure connection action S and assume that the corresponding Einstein metric has $R_+ < 0$. Let a be an infinitesimal deformation of A . By Proposition 3.6 we can write $a = b + L_\eta A$ for b fully gauge-fixed and $\eta \in \text{Lie}(\mathcal{G})$. Since S is \mathcal{G} invariant, $D^2S(L_\eta A, \cdot) = 0$. It follows that $D^2S(a, a) = D^2S(b, b)$. Theorem 3.1 now follows from Proposition 3.5.

4 The proof of local rigidity

In this section we show how Theorem 3.1 implies our main result Theorem 1.2. We give the statement again for convenience.

Theorem. *Let g be an Einstein metric on a compact orientable 4-manifold M^4 , with either $R_+ < 0$ or $R_- < 0$. Then g is locally rigid.*

Proof. Let h be an infinitesimal Einstein deformation of g . We must find a vector field v on M for which $L_v g = h$. The idea of the proof is to show that the metric deformation h is induced by a deformation a of the definite connection A corresponding to g and, moreover, with $a \in \ker D^2S$. This will imply that a is pure gauge and so the same will be true of h .

It is certainly not true that an *arbitrary* metric deformation can be achieved by deforming the connection. This can be seen by counting degrees of freedom. The space of connections is modelled on $\Omega^1(M, \mathfrak{so}(E))$ and so has functional dimension 12. The gauge group \mathcal{G} meanwhile has functional dimension 7, so one can only expect to reach a space of metrics with effective functional dimension 5. Meanwhile metrics have functional dimension 10 of which diffeomorphisms account for 4 dimensions, leaving effective functional dimension 6 which is larger. This means we must use in an essential way the fact that h is an infinitesimal Einstein deformation.

Write \mathcal{M} for the space of Riemannian metrics and $G: \mathcal{D} \rightarrow \mathcal{M}$ for the map which sends a definite connection to the corresponding Riemannian metric. This map is equivariant for the action of \mathcal{G} on \mathcal{D} and $\text{Diff}(M)$ on \mathcal{M} . Let $A \in \mathcal{D}$ be the critical point of the pure connection action S with $G(A) = g$. We seek $a \in T_A\mathcal{D}$ such that $D_A G(a) = h$ and with $a \in \ker D^2S$. From here, Theorem 3.1 shows that $a = L_\eta A$ for some $\eta \in \text{Lie}(\mathcal{G})$. Writing $v \in C^\infty(M, TM)$ for the pushforward of η to M , equivariance of G then implies that $h = L_v g$.

To proceed let $\widehat{\Sigma}(t): E \rightarrow \Lambda^2$ be a path of wedge-orthogonal 2-forms with $\widehat{\Sigma}(0)$ corresponding to A via (2.9). Write the corresponding path of metrics as

$$\widehat{g}(t) = g_{\widehat{\Sigma}(t)}$$

and choose $\widehat{\Sigma}(t)$ so that $\widehat{g}'(0) = h$.

Now let $A(t)$ be the pull-back to E of the $\widehat{g}(t)$ -Levi-Civita connection on $\Lambda^+(\widehat{g}(t))$ via $\widehat{\Sigma}(t)$. By construction $A(0) = A$. Each connection $A(t)$ is again definite (at least

¹We mention the simple proof of this standard fact for completeness. If v is Killing, Bochner's formula gives $\frac{1}{2}\Delta|v|^2 = |\nabla v|^2 - \text{Ric}(v, v)$. When $\text{Ric} < 0$ and the manifold is compact, integrating this equality shows v must vanish.

for small t). Write $a = A'(0) \in T_A \mathcal{D}$. $A(t)$ determines a wedge-orthogonal map $\Sigma(t) = \Sigma_{A(t)}$ via 2.9 and a path of metrics $g(t) = g_{A(t)}$.

It is important to note that $g(t)$ need not be equal to the path $\widehat{g}(t)$ we first started with. We will prove that when h is an infinitesimal Einstein deformation, that these paths agree to first order: $g'(0) = \widehat{g}'(0)$. Write $\sigma = \Sigma'(0)$ and $\widehat{\sigma} = \widehat{\Sigma}'(0)$. In fact, we will show that $\sigma = \widehat{\sigma}$, from which it follows that $g'(0) = \widehat{g}'(0)$.

The 2-form $\widehat{\Sigma}(t)$ is wedge-orthogonal. Differentiating this condition with respect to t and evaluating at $t = 0$ we get

$$\widehat{\sigma}_i \wedge \Sigma_j + \Sigma_j \wedge \widehat{\sigma}_i - \frac{2}{3}(\widehat{\sigma}_k \wedge \Sigma_k)\delta_{ij} = 0 \quad (4.1)$$

(Cf. equation (3.2).) Next write

$$F_{A(t)} = B(t) + C(t)$$

where $B(t)$ is $\widehat{g}(t)$ -self-dual and $C(t)$ is $\widehat{g}(t)$ -anti-self-dual. Since $\widehat{g}(0)$ is Einstein and h is an infinitesimal Einstein deformation, $C(t) = O(t^2)$. We write (locally) $B^i = \widehat{Y}^{ij}(t)\widehat{\Sigma}_j(t)$. At $t = 0$, $\widehat{Y}(0) = Y$, the matrix representative for R_+ of the original Einstein metric $\widehat{g}(0) = g$. Symmetries of the Riemann tensor ensure that $\widehat{Y}(t)$ is symmetric for all t . Moreover, since h is an infinitesimal Einstein deformation, $\text{Tr } \widehat{Y} = \Lambda + O(t^2)$. It follows that $\widehat{Y}'(0)$ is trace-free. Differentiating $F_{A(t)}$ with respect to t and evaluating at $t = 0$ we get

$$(\text{d}_{AA})^i = (\widehat{Y}')^{ij}(0)\Sigma_j + Y^{ij}\widehat{\sigma}_j \quad (4.2)$$

Meanwhile, the path $A(t)$ of definite connections determines a path $(\Sigma(t), \Psi(t))$ via (2.9) and (2.10). The tangent (a, σ, ϕ) to the whole path $(A, \Sigma_A, \Psi_A)(t)$ of Plebanski variables at $t = 0$ satisfies the equations (3.2) and (3.3). Moreover, given a , we saw that there are unique σ and ϕ solving (3.2) and (3.3). But these equations are identical to (4.1) and (4.2) satisfied by $\widehat{\sigma}$, with ϕ replaced by $\widehat{Y}'(0)$. It follows that $\sigma = \widehat{\sigma}$ as claimed (and $\phi = \widehat{Y}'(0)$ but we will not use this).

Finally we have to check that the tangent vector $a \in T_A \mathcal{D}$ is in $\ker D^2 S$. This is equivalent to a being tangent to the critical locus of S which is in turn equivalent to $\text{d}_{A(t)}\Sigma(t) = O(t^2)$. Recall that $A(t)$ is the pull-back of the Levi-Civita connection via $\widehat{\Sigma}(t)$. It follows from Lemma 2.2 that $\text{d}_{A(t)}\widehat{\Sigma}(t) = 0$ for all t . But we have just seen that $\widehat{\Sigma}(t) = \Sigma(t) + O(t^2)$ and so $\text{d}_{A(t)}\Sigma(t) = O(t^2)$ and $a \in \ker D^2 S$ as claimed. \square

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