

Stability Analysis of Piecewise Affine Systems with Multi-Model Predictive Control

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Abstract

We propose an input-output stability analysis for closed-loop systems of piece-wise affine models under unstructured uncertainty and controlled by multi-model linear MPC with input constraints. Integral quadratic constraints (IQCs) are employed to assess the robustness of MPC under uncertainty. We efficiently create a model pool, by performing linearisation on selected transient points. All the possible uncertainties and nonlinearities (including the controller) can be introduced in the framework, assuming that they admit the appropriate IQCs, whilst the dissipation inequality can provide necessary conditions for stability through the incorporation of IQCs. We demonstrate the existence of static multipliers, which can reduce the conservatism of the stability analysis significantly. The proposed methodology is demonstrated through two illustrative case studies.

Key words: Unstructured uncertainty, piecewise affine, model predictive control, robust stability,

1 Introduction

Model predictive control (MPC) is a powerful technique that largely relies on receding horizon-based optimization of an objective function to compute the optimum trajectories of manipulated variables and outputs. Linear MPC has been widely used in a number of industries (Rawlings et al., 2017) due to its relative simplicity and robustness (Heath et al., 2006). Nonlinear MPC (Rawlings et al., 2017) is more appropriate for handling complex processes with underlying nonlinear dynamics. Nevertheless, computations for nonlinear MPC may become prohibitively slow, making it difficult to handle the process model in real time. Piecewise affine (PWA) models (Bemporad and Morari, 1999) can accurately represent the underlying nonlinear dynamic system, but their use in MPC can jeopardize computational performance as the resulting mixed-integer programming problem is NP-complete (Borrelli et al., 2017). In this work a multi-model approach (Du and Johansen, 2015; Bonis et al., 2014) is employed where one model is used per optimization (or per sample) to avoid

mixed integer computations. The resulting controller uses the same model for the whole horizon (although it might not always be admissible) with minimum computational cost. In this work, stability conditions are established for ensuring input-to-output stability within the IQC (Megretski and Rantzer, 1997) and dissipativity (Brogliato et al., 2007) framework. Mayne et al. (2000) have presented a survey of stability and optimality conditions for MPC; however the main focus is on analysis using state terminal constraints and terminal cost for state feedback stability, which can only provide local stability at the expense of additional complexity. Lazar et al. (2006) employed a Piecewise Quadratic (PWQ) Lyapunov function (Johansson and Rantzer, 1998) for a class of PWA MPC problems, proposing sufficient conditions for asymptotic stability with terminal constraints and cost. PWQ Lyapunov functions have been developed in (Wei et al., 2018; Qiu et al., 2018) for the synthesis of output-feedback controllers and for the design of reliable static output feedback control for uncertain discrete-time PWA, respectively. Løvaas et al. (2008) have proposed a class of output robust model predictive control with all the MPC policies (within this class) satisfying a robust stability test when unstructured uncertainties are present. Alternatively, simple output feedback linear MPC with only input constraints has been shown to guarantee input-output stability (Heath et al., 2005; Heath and Li, 2010) under

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structured or unstructured uncertainties. However, to the best of the authors' knowledge, there is no systematic framework for analyzing the input-output stability of feedback interconnections with PWA systems and multi-model MPC under unstructured uncertainty. A major challenge is to appropriately handle nonlinear and uncertain components. The theory of integral quadratic constraints (IQCs) can be used to conveniently model these components to construct a generic global stability analysis framework. Here, we propose the use of IQCs to perform input-output stability analysis for such feedback interconnected systems. IQCs (Megretski and Rantzer (1997)) have been widely used for input-to-output stability taking advantage of appropriate input-output properties as well as to perform stability and robustness analysis of dynamic systems in the frequency (D'Amato et al., 2001; Heath and Li, 2010) or in the time domain. Jönsson and Rantzer (2000) have proposed a unified framework for IQC stability analysis based on efficient computation of *multipliers*. Recently, Fetzner and Scherer (2017) proposed a comprehensive stability analysis for the case of slope-restricted nonlinearities in discrete time. Pfifer and Seiler (2015) propose a framework for stability analysis of linear parameter varying (LPV) models introducing IQC multipliers through J-spectral factorization, bringing together the frequency IQC stability with the dissipation approach significantly reducing conservatism. Similarly, Carrasco and Seiler (2018) prove the equivalence between IQC and graph-separation stability, if a doubly-hard factorization is applied. Time domain frameworks in contrast, are not restricted to linear time invariant systems, permitting further generalization. Robustness analysis (Pfifer and Seiler, 2015) and robust synthesis (Wang et al., 2016) of LPV systems using time domain IQCs have recently been developed.

1.1 Contributions

The main contribution of this work is to construct a general framework for the analysis of input-to-output stability of PWA systems for multi-model MPC under unstructured uncertainty. The MPC as well as the uncertainties arising due to model mismatches are handled by appropriate IQCs. Four methodologies are proposed for stability analysis (i) single parametrization (SP) (ii) conic combination (CC)(iii) static multipliers for box constraints (SS)(iv) static multipliers combined with a PWQ function (PWQ).

1.2 Assumptions

- (i) Only constraints on the actuators are applied.
- (ii) The (potentially nonlinear) system is: (a) open-loop stable; (b) both controllable and observable.
- (iii) All considered uncertainties and nonlinearities admit suitable IQCs

1.3 Structure of the paper

Relevant notation and the problem statement and associated definitions are given in Sections 2 and 3, respectively. Time domain IQCs are formulated in Section 4. Stability theorems using time-domain IQCs combined with the dissipation inequality for PWA models under unstructured uncertainties are introduced in Section 5. The developed methodologies are applied to two illustrative case studies in Section 6. Finally, conclusions and future work are given in Section 7.

2 Notation

Let (\mathbb{Z}_+) \mathbb{Z} be the set of (positive) integer numbers including 0. l_2^m is the Hilbert space of all square integrable and Lebesgue measurable functions of size m , $f : \mathbb{Z}_+ \rightarrow \mathbb{R}^m$ and l^m the space of all real-valued sequences. The truncation of the function $f = f(t)$ at T , $f_T(t)$, is defined as:

$$f_T(t) = \begin{cases} f(t) , & \forall t \leq T \\ 0 , & \forall t > T , \end{cases} \quad (1)$$

with $f \in l^m$ if $f_T(t) \in l_2^m$ for all $T > 0$. \mathbf{RH}_∞ stands for the set of rational transfer functions without poles outside the unit circle. A^* is the complex conjugate transpose of complex matrix A . G^* is the l_2 -adjoint operator of $G \in \mathbf{RH}_\infty$. The inner product $\langle f, g \rangle$ is defined as $\sum_{k=0}^{\infty} f(k)^T g(k) = \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{f}(e^{j\omega}) \hat{g}(e^{j\omega}) d\omega$, \hat{f} and \hat{g} denoting the Fourier transforms of f and g , respectively. The l_2 norm $\|f\|_2$ is defined as $\sqrt{\langle f, f \rangle}$, while $\|f\|_1$ is $\sum_{k=0}^{\infty} |f(k)|$. G_i is the i^{th} mode of PWA system, G . The size of signal x is n_x . For diagonal blocks the notation

$$\text{diag} \text{ is used, e.g. } \text{diag}(A, B) = \begin{bmatrix} A & \\ & B \end{bmatrix}.$$

3 Problem Statement

In this work, the robust stability of PWA systems under unstructured uncertainty for multi-model MPC is studied. Only one model at each sampling time is employed, to reduce computational costs as well as the difficulty of including uncertainty into the optimization problem. The stability is analyzed using dissipation inequalities, whereas the uncertainties (Δ) and the controller (ϕ) are included in the analysis using IQCs.

3.1 Piece-wise Affine Models

The physical system is approximated by a PWA model under unstructured uncertainty (Δ) (Fig. 1) given in (2) for every $i \in \mathbf{M} \subset \mathbb{Z}_+$, where \mathbf{M} is the set of indices that

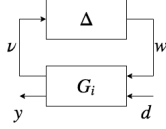


Fig. 1. PWA model under unstructured uncertainty

defines the pool of linear sub-models. For each region Ω_i where h_i holds, the dynamics evolve as:

$$\begin{bmatrix} x(k+1) \\ \nu(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} A_i & B_i^1 & B_i^2 & f_i \\ C_i^1 & D_i^{11} & D_i^{21} & g_i^1 \\ C_i^2 & D_i^{12} & D_i^{22} & g_i^2 \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \\ d(k) \\ 1 \end{bmatrix} \quad (2)$$

$$\begin{aligned} w(k) &= \Delta(\nu(k)) \\ \Omega_i &: \{h_i(x(k), u(k)) \leq \zeta_i\}. \end{aligned}$$

Vectors $x(k) \in \mathbb{R}^{n_x}$, $u(k) \in \mathbb{R}^{n_u}$ and $y(k) \in \mathbb{R}^{n_y}$ are the state, input and measured output at time $k \in \mathbb{Z}_+$, respectively; w represents the uncertainty and $\Delta : l^{n_w} \rightarrow l^{n_w}$ is a causal nonlinear (*possibly unknown*) map. The matrix A_i is Hurwitz, the superscripts in B^m correspond to the coefficients of the m^{th} input, e.g. B^1 corresponds to input 1, in this case w . Furthermore, D requires an additional superscript to represent the corresponding output, e.g. D^{12} corresponds to output 1 (ν) and input 2 (d). The interconnection is well posed if for each $d \in l^{n_d}$ and $y \in l^{n_y}$ there exists a unique $v \in l^{n_v}$ such that the map from (d, y) to (v, w) is causal (Megretski and Rantzer, 1997). The vectors g_i (also corresponding to inputs 1 and 2) and f_i represent an arbitrary set of constant real valued vectors. The set of arbitrary inequalities, h_i , have upper bound ζ_i . Less conservative results can be obtained if a set of polyhedral inequalities is assumed instead: ($h_i := H_i^x x(k) + H_i^u u(k)$). Collection $\{\Omega_i\}$ is the set of (not necessarily closed or bounded) polyhedra, with $\cup \Omega_i = \mathbb{R}^n$, assuming that the regions, i , are not overlapping. The model i changes with respect to $x(k)$ and $u(k)$, and also with respect to time, k . Hence, the corresponding subscripts will be denoted as $i(k)$ instead of $i(x(k), u(k))$. In this work, different strategies for stability analysis of PWA models are considered using new IQC multipliers constructed for multi-linear MPC including uncertain/nonlinear components.

3.2 Integral quadratic constraints

IQCs replace difficult to identify/analyze components with quadratic constraints satisfied by the inputs and outputs of those components (Fetzer and Scherer, 2017). Let Π be a bounded self-adjoint operator; then inequality (3) defines a general IQC in the frequency domain, and it is deemed that ‘‘uncertainty $w = \Delta(\nu)$ admits

IQC’’, defined by multiplier Π ($\Delta \in IQC(\Pi)$), when

$$\int_{-\pi}^{\pi} \begin{bmatrix} \hat{\nu}(e^{j\omega}) \\ \hat{w}(e^{j\omega}) \end{bmatrix}^* \Pi(e^{j\omega}) \begin{bmatrix} \hat{\nu}(e^{j\omega}) \\ \hat{w}(e^{j\omega}) \end{bmatrix} \geq 0. \quad (3)$$

It is more convenient here to use time domain analysis as nonlinear systems can be thus handled in a more natural way. Multiplier Π can be factorized as $\Psi^* M \Psi$ and applying Parseval’s theorem (Zhou and Doyle, 1998) with $r(k) := \Psi \begin{bmatrix} \nu(k)^T & w(k)^T \end{bmatrix}^T$ (see Fig. 2), inequality (3) is transformed to the soft-IQC inequality (4)

$$\sum_{k=0}^{\infty} r(k)^T M r(k) \geq 0. \quad (4)$$

The theory of dissipation, however, requires a finite

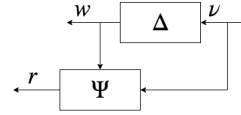


Fig. 2. Graphical representation of the connection between the auxiliary system Ψ and uncertainty Δ

time horizon. Hence, *hard* IQCs (Megretski and Rantzer (1997)) are necessary forcing the quadratic inequality constraints to hold for every finite time horizon, T :

$$\sum_{k=0}^T r(k)^T M r(k) \geq 0. \quad (5)$$

For nonlinearities varying in time, we define IQCs with a multiplier M_i :

$$\sum_{k=0}^T r(k)^T M_{i(k)} r(k) \geq 0. \quad (6)$$

Here, we will form time domain hard-IQCs directly using the KKT conditions.

3.3 Dissipation inequality

The robustness of the interconnection between the dynamic system and its uncertainties (or nonlinearities) is analyzed using the extended system G_i^s (Fig. 3) where $x^s := \begin{bmatrix} x \\ \psi \end{bmatrix}$ is the state space vector and ψ the states of Ψ .

$$\begin{aligned} \psi(k+1) &= A_\psi \psi(k) + B_\psi^1 w(k) + B_\psi^2 v(k) \\ r(k) &= C_\psi \psi(k) + D_\psi^{11} w(k) + D_\psi^{12} v(k) \end{aligned} \quad (7)$$

The particular values of $A_\psi, B_\psi^1, B_\psi^2, C_\psi, D_\psi^{11}$ and D_ψ^{12}

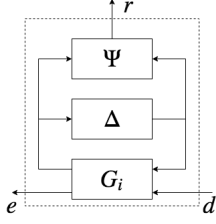


Fig. 3. Extended system G_i^s

depend on the factorization of the multiplier M (see (Carrasco and Seiler, 2018)). For time-invariant multipliers, we have $D_\psi^{11} = [1 \ 0]^T$ and $D_\psi^{12} = [0 \ 1]^T$. The extended system can now be constructed:

$$\begin{aligned} x^s(k+1) &= A_i^s x^s(k) + B_i^{s1} w(k) + B_i^{s2} d(k) \\ r(k) &= C_i^{s1} x^s(k) + D_i^{s11} w(k) + D_i^{s12} d(k) \\ e(k) &= C_i^{s2} x^s(k) + D_i^{s21} w(k) + D_i^{s22} d(k). \end{aligned} \quad (8)$$

Different strategies can be implemented for MPC, which will affect the structure of the problem. Written in state space as in (2), the matrices in (8) are as follows:

$$A_i^s = \begin{bmatrix} A_i & 0 \\ B_\psi^1 & A_\psi \end{bmatrix} \quad (9a)$$

$$B_i^{s1} = \begin{bmatrix} B_i^1 \\ B_{\psi 1} D_i^{11} + B_\psi^2 \end{bmatrix}, B_i^{s2} = \begin{bmatrix} B_i^2 & f_i \\ B_\psi^1 D_i^{12} \end{bmatrix} \quad (9b)$$

$$C_i^{s1} = [D_\psi^{11} C_i^1 \ C_\psi], C_i^{s2} = [C_i^2 \ 0] \quad (9c)$$

$$D_i^{s11} = D_\psi^{11} D_i^{11} + D_\psi^{12}, D_i^{s12} = D_\psi^{11} [D_i^{12} \ g_i] \quad (9d)$$

$$D_i^{s21} = D_i^{21}, D_i^{s22} = D_i^{22}. \quad (9e)$$

The induced controller gain, l_2 , from d to e (Fig. 3) is defined using the interconnection between G_i and Δ :

$$\|G_i, \Delta\| = \sup_{d \neq 0, d \in l_2} \frac{\|e\|}{\|d\|} \quad (10)$$

where Δ is now $\text{diag}(\Delta_1, \dots, \Delta_m, \dots, \Delta_N)$ with $m \in [1, \dots, N]$ and N being the number of all the uncertainties and nonlinearities in the closed-loop, assumed to satisfy time-domain hard IQCs (5 and 6). Storage functions can be used for stability analysis of PWA problems and their type will affect the conservatism of the stability results (Johansson and Rantzer, 1998). A common storage function can be employed, $V(x) = x^T P x$, or a piecewise quadratic function, $V(x) = x^T P_i x$, with P (or P_i) a symmetric positive definite matrix. Although the latter may reduce the conservatism of stability estimates, its construction requires significant computational time that may lead to computationally intractable problems.

3.4 Model predictive control

The MPC controller here exploits a multi-model scheme. The control law consists of only input constraints (guaranteed feasibility) and it can be described according to equation (11) for every possible model i .

$$\begin{aligned} u^*(k) &= \arg \min_U \sum_{t=0}^{N_{out}} l_x(\tilde{x}(t), u(t)) + \\ &\sum_{t=0}^{N_{in}} l_u(\tilde{x}(t), u(t)) + F(x(N_{out} + 1)) \\ &s. t. \tilde{x}(0) = x(k) \\ &\tilde{x}(t+1) = A_i \tilde{x}(t) + B_i^1 u(t) + f_i \\ &L_i U \leq b, \end{aligned} \quad (11)$$

where

$$\begin{aligned} l_x &= (\tilde{x}(t) - x_r)^T Q (\tilde{x}(t) - x_r) \\ l_u &= (u(t) - u_r(t))^T R (u(t) - u_r(t)) \end{aligned} \quad (12a)$$

$$F(x(N_{out}+1)) = (\tilde{x}(N_{out}+1) - x_r)^T P_r (\tilde{x}(N_{out}+1) - x_r)^2, \quad (12b)$$

with \tilde{x}, u^* and x_r being the states, the control actions (resulting from optimization) and the vector of setpoints, respectively; $b \geq 0$, P_r is the solution of the Discrete Algebraic Riccati Equation, and $U = [u(0)^T, \dots, u(N_{in} - 1)^T]^T$ the future input actions ($u \in \mathbb{R}^{n_u}$) for the control horizon, N_{in} . Only the first control action is applied at each sampling time. This formulation employs the same model for the whole horizon, reducing the computational cost. It is, however, crucial to guarantee the input-to-output stability of the closed-loop, as using the same model may not be appropriate. Equation (11) can be easily transformed to:

$$\begin{aligned} u^*(k) &= \phi(\xi) = \arg \min_U U^T H_i U - U^T \xi(k) \\ &s. t. L_i U \leq b, \end{aligned} \quad (13)$$

where $\xi(k) = F_i x(k) + D_i$. Hessian, H_i , and F_i , D_i can be easily deduced from (Maciejowski, 2002). MPC corresponds to nonlinearity ϕ and block uncertainties can be used as in (Jönsson and Rantzer, 2000), $\Delta_{tot} = \text{diag}(\phi, \Delta)$.

4 Integral Quadratic Constraints for MPC

To perform input-output stability analysis, we propose to compute IQCs, defined as in (6) through the construction of three different types of multipliers, for multi-model MPC (13) using its KKT conditions.

4.1 Sector-bounded MPC

For the case of linear MPC, IQCs can be calculated in the frequency domain using the KKT conditions (Heath et al., 2006). Here, we will construct IQCs in the time-domain using appropriate inequalities. Nonlinearity $\phi(\xi)$ is sector-bounded in the sense that there exists some $S > 0$ such that $\phi(\xi)^T S^{-1} \phi(\xi) - \phi(\xi)^T \xi \leq 0$.

Lemma 1 *The nonlinearity $u^* = \phi(\xi)$ (13) belongs to the sector $[0, H_i^{-1}]$ for each Ω_i .*

PROOF. From the KKT conditions of (13) we have:

$$H_i u^* - \xi + L_i^T \lambda = 0 \quad (14a)$$

$$\tilde{\lambda}_j (L_{ij} u^* - b_j) = 0 \quad (14b)$$

$$\tilde{\lambda}_j \geq 0, \quad (14c)$$

with $\tilde{\lambda}_j$ the Lagrange multipliers. Multiplying (14a) by u^{*T} and assuming $b_j \geq 0$, with $u^* = \phi(\xi)$ leads to:

$$\phi(\xi)^T H_i \phi(\xi) - \phi(\xi)^T \xi \leq 0. \quad (15)$$

A different sub-model can be used at every sampling time with only input constraints. Therefore, a new type of IQC multiplier can be introduced:

Lemma 2 (Petsagkourakis et al., 2017) *The nonlinearity $\phi : \mathbb{R}_{n_\xi}^n \rightarrow \mathbb{R}^{n_{u^*}}$ (13) admits the time-domain hard*

IQC (6) with $M_i = \begin{bmatrix} O & I \\ I & -2H_{i(k)} \end{bmatrix}$, $r = [\xi^T \ \phi(\xi)^T]^T$ and Ψ equal to the identity matrix, for every $\xi \in l^{n_\xi}$.

Here, KKT conditions guarantee that IQC using SP multipliers, hold for any time T . The IQCs from Lemma 2 will give conservative stability results. Hence, a conic combination of the optimality conditions (14) can be utilized to reduce the conservatism, by incorporating more degrees of freedom:

Lemma 3 *The nonlinearity $\phi : \mathbb{R}_{n_\xi}^n \rightarrow \mathbb{R}^{n_{u^*}}$ (13) admits the time-domain hard IQC (6) for $\lambda_i \geq 0$ with*

$$M_i = \begin{bmatrix} O & I \\ I & -2\lambda_i H_{i(k)} \end{bmatrix} \text{ for every } \xi \in l^{n_\xi}.$$

PROOF. For every time interval k that a model i is employed the following holds:

$$\sum_{k=T_1}^{T_2} \begin{bmatrix} \xi(k) \\ \phi(\xi)(k) \end{bmatrix}^T \begin{bmatrix} O & I \\ I & -2H_{i(k)} \end{bmatrix} \begin{bmatrix} \xi(k) \\ \phi(\xi)(k) \end{bmatrix} \geq 0 \quad (16)$$

Employing a conic combination completes the proof. \square

These new IQC multipliers will be termed *conic combination (CC)* multipliers. This simple transformation increases the degrees of freedom of the stability analysis, allowing a more accurate estimation of the stable region.

4.2 Multipliers for box and stage constraints

Here we develop more general, less conservative IQC multipliers for multi-model problems with a tighter class of constraints, i.e. box and stage constraints. A special structure of fixed constraints has been exploited by Heath and Li (2010) for the case of linear MPC, where the existence of multipliers in the frequency domain has been demonstrated, reducing the conservatism of the analysis. We extend these results to prove the existence of static multipliers in the time domain for the multi-model case. For each Ω_i an equivalent structure can be found where the controller $u^* = \phi(\xi)$ (13) is written as parallel saturation functions together with a feedback. Let $\psi_c : \mathbb{R}^{N_U} \rightarrow \mathbb{R}^{N_U}$ be the following quadratic program:

$$u^* = \psi_c(z) = \arg \min_U \frac{1}{2} U^T U - U^T z \quad (17)$$

s.t. $LU \leq b$.

If we define $z = \xi + (I - H_i)u^*$, then the feedback $u^* = \phi(\xi)$ (13) is equivalent to $u^* = \psi_c(z)$. The structure of ψ_c is depicted in Fig. 4 for each sub-model i with N_U being the size of signal u^* . The constraints in (17)

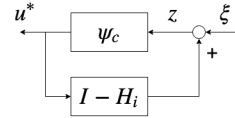


Fig. 4. Structure of ψ_c

have a specific structure for our case. For the staged/box constraints $L \in \mathbb{R}^{N_L \times N_U}$ and $b \in \mathbb{R}^{N_L}$ can be written as

$$L^T = [L_0^T, \dots, L_{N_L-1}^T] \quad (18)$$

$$b^T = [b_0^T, \dots, b_{N_L-1}^T],$$

with

$$L_i L_j^T = 0, \forall i \neq j = 0, \dots, N_L - 1. \quad (19)$$

For box constraints this structure can be simply seen as:

$$L_i = [0, \dots, \tilde{L}_i, \dots, 0], \quad (20)$$

with $\tilde{L}_i = [1, -1]^T$ and $b_i = [\bar{b}_i, -\underline{b}_i]$ with $\underline{b}_i \leq 0 \leq \bar{b}_i$. Also, $L^0 L^{0T} = I$ and $L^c L^T = 0$. The rows of L^0 form an orthonormal basis spanned by the rows of L . Exploiting

the orthogonality of L_j , we can break ψ_c into several QPs. u^* can be written as:

$$u^*(k) = \sum_{j=0}^{N_L} v_j(k), \quad (21)$$

where j refers to the j^{th} sub-QP of the main QP instead of the QP of sub-model i . Then for each j :

$$v_j = \arg \min_U \frac{1}{2} U^T U - U^T z \quad (22a)$$

$$\text{s.t. } L_j U \leq b_j$$

$$L_j^c U = 0$$

$$\forall j = 0, \dots, N_L - 1$$

$$v_{N_L} = -L^{cT} L^c z. \quad (22b)$$

Each v_j can be written as

$$v_j(z) = L_j^{0T} \theta_j(L_j^0 z) \quad (23)$$

with $\theta_j : \mathbb{R}^{n_j} \rightarrow \mathbb{R}^{n_j}$ being the quadratic program:

$$\theta_j(p) = \arg \min_q \frac{1}{2} q^T q - q^T p \quad (24)$$

$$\text{s.t. } L_j L_j^{0T} q \leq b_j,$$

where $p = L_j^0 z$ and q the optimization variables. It follows immediately from the KKT conditions of (24) that θ_j is sector-bounded if $b_j \geq 0$ (Heath and Wills, 2007):

$$\theta_j^T \theta_j - \theta_j^T L_j^0 z \leq 0. \quad (25)$$

The main result of this section can then be given:

Lemma 4 *The controller output $u^* = \phi : \mathbf{R}^{n_\varepsilon} \rightarrow \mathbf{R}^{n_u}$ (13) for static constraints, admits the IQC:*

$$\sum_{k=0}^T \begin{bmatrix} \xi(k) \\ \phi(\xi)(k) \end{bmatrix}^T M_i^\phi \begin{bmatrix} \xi(k) \\ \phi(\xi)(k) \end{bmatrix} \geq 0 \quad (26)$$

where

$$M_i^\phi = \begin{bmatrix} O & K_{i(k)} \\ K_{i(k)} & -K_{i(k)} H_{i(k)} - H_{i(k)} K_{i(k)}. \end{bmatrix} \quad (27)$$

Multipiers $K_i = \text{diag}(\kappa_{0i}, \dots, \kappa_{(N_U-1)i})$, with $\kappa_{ji} \geq 0$ can be computed for each sub-model $i \in M$, for the case of box and staged constraints (18)-(20).

PROOF.

For each model $i \in \mathbf{M}$, ψ_c admits the following time-domain IQC using a conic combination and (24):

$$\sum_{j=0}^{N_U-1} \kappa_{ji} \sum_{k=T_1(i)}^{T_2(i)} \begin{bmatrix} z(k) \\ \psi_c(x)(k) \end{bmatrix}^T M_j^{\psi_c} \begin{bmatrix} z(k) \\ \psi_c(x)(k) \end{bmatrix} \geq 0 \quad (28)$$

with $M_j^{\psi_c} = \begin{bmatrix} L_j^0 \\ L_j^0 \end{bmatrix}^T \begin{bmatrix} O & I \\ I & -2I \end{bmatrix} \begin{bmatrix} L_j^0 \\ L_j^0 \end{bmatrix}$. Because of the orthogonality of L_j^0 , it follows immediately for the time interval $[T_1(i), T_2(i)]$ that:

$$\sum_{k=T_1(i)}^{T_2(i)} \begin{bmatrix} \xi(k) \\ \phi(\xi)(k) \end{bmatrix}^T M_i^\phi \begin{bmatrix} \xi(k) \\ \phi(\xi)(k) \end{bmatrix} \geq 0 \quad (29)$$

with M_i^ϕ given by (27). The summation of (29) from 0 to T gives (26). \square

In section 6 the multipliers developed are compared in terms of conservatism. Next, IQC multipliers are employed in stability theorems through the dissipation inequality.

5 Stability Analysis

This section will provide the main conditions for achieving input-output stability. First a general parametrized linear matrix inequality (LMI) is given, and then it is adapted according to the needs of each theorem to prove input-output stability using the dissipation inequality (Brogliato et al., 2007). Define LMI as:

$$\begin{aligned} LMI(\Lambda_i, \lambda_k, \gamma, P_x, P_y) := & \begin{bmatrix} A_i^{sT} P_y A_i^s - P_x & A_i^{sT} P_y B_i^{s1} & A_i^{sT} P_y B_i^{s2} \\ B_i^{s1T} P_y A_i^s & B_i^{s1T} P_y B_i^{s1} & B_i^{s1T} P_y B_i^{s2} \\ B_i^{s2T} P_y A_i^s & B_i^{s2T} P_y B_i^{s1} & B_i^{s2T} P_y B_i^{s2} - \gamma^2 I \end{bmatrix} + \\ & + \begin{bmatrix} C_i^{s2T} \\ D_i^{s21T} \\ D_i^{s22T} \end{bmatrix} \begin{bmatrix} C_i^{s2T} \\ D_i^{s21T} \\ D_i^{s22T} \end{bmatrix}^T + \\ & + \begin{bmatrix} C_i^{s1T} \\ D_i^{s11T} \\ D_i^{s12T} \end{bmatrix} \begin{bmatrix} \Lambda_i & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_N M_N \end{bmatrix} \begin{bmatrix} C_i^{s1T} \\ D_i^{s11T} \\ D_i^{s12T} \end{bmatrix}^T < 0 \end{aligned} \quad (30)$$

Here, P_x, P_y , are symmetric positive matrices and Λ_i is symmetric with $i \in \mathbf{M}$, λ_k and γ are non-negative and $k \in \{2, \dots, N\}$ for N nonlinearities.

5.1 Conic combination-common storage function

Sufficient conditions are provided for the closed-loop stability using CC multipliers (Lemma 3). The IQCs for each MPC hold for arbitrary time, as long as we use static multipliers derived through the KKT conditions. We can use similar arguments for every memoryless nonlinearity with static multipliers.

Theorem 5 Let $G_i \in \mathbf{RH}_\infty^{(n_e+n_w) \times (n_w+n_d)}$ be a stable system and $\Delta_m : \mathcal{L}^{n_{vm}} \rightarrow \mathcal{L}^{n_{wm}}$ a bounded, causal operator containing every nonlinearity. The interconnection is well-posed and Δ_m satisfies IQC with multiplier M_m and the controller multiple IQCs given by Lemma 3. Then $\|(G_i, \Delta)\| < \gamma$ if $\Lambda_i = \lambda_1^i M_1^i$ and there exists a symmetric matrix $P_x = P_y = P \geq 0$ and non-negative γ, λ_1^i and $\lambda = [\lambda_2, \dots, \lambda_N]$ such that $LMI(\Lambda_i, \lambda, \gamma, P) < 0$ holds.

PROOF. Multiplying the LMI with $[x^s T, w^T, d^T]$ from the left and right respectively we get (for positive δ):

$$\begin{aligned} & \lambda_1^i r^c(k)^T M_1^i r^c(k) + \sum_{j=2}^N \lambda_j r(k)^T M_j r(k) + \\ & + \Delta V(k) + e(k)^T e(k) \leq (\gamma^2 - \delta) d(k)^T d(k). \end{aligned} \quad (31)$$

Summing from $k = T_1$ to T_2 , with $x^s(0) = 0$, $[T_1 T_2]$ being the interval in which a model is employed, we have:

$$\begin{aligned} & \lambda_1^i \sum_{k=T_1}^{T_2} r^c(k)^T M_1^i r^c(k) + \sum_{j=2}^N \lambda_j \sum_{k=T_1}^{T_2} r(k)^T M_j r(k) \\ & + V(T_2 + 1) - V(T_1) + \sum_{k=T_1}^{T_2} e(k)^T e(k) \\ & \leq (\gamma^2 - \delta) \sum_{k=T_1}^{T_2} d(k)^T d(k) \end{aligned} \quad (32)$$

Summation of (32) over all intervals, with positive definite storage function and IQC given by Lemma 3, yields:

$$\sum_{k=0}^T e(k)^T e(k) < \gamma^2 \sum_{k=0}^T d(k)^T d(k) \quad (33)$$

from which follows that $\|e\| < \gamma \|d\|$ \square

The degrees of freedom include parameters λ_i . The conditions for single parametrization IQC follow directly:

Corollary 6 The stability conditions for the single parametrization case are given by modifying the constraints of Theorem 5: $\Lambda_i = \lambda_1 M_1^i$

Hence, conic combination can provide less conservative results, since from Corollary 6 additional constraints are provided for stability, reducing the degrees of freedom.

5.2 Stability analysis for box/staged constraints

For systems with box and staged constraints the existence of IQC multipliers, K_i , was proven in section 4.2. We can then easily modify Theorem 5 to provide stability conditions using SS multipliers.

Theorem 7 Let $G_i \in \mathbf{RH}_\infty^{(n_e+n_w) \times (n_w+n_d)}$ be a stable system and $\Delta_m : \mathcal{L}^{n_{vm}} \rightarrow \mathcal{L}^{n_{wm}}$ a bounded, causal operator containing every nonlinearity. The interconnection is well-posed and Δ_m satisfies IQC with multipliers M_m and the controller IQCs with multipliers M_i^ϕ (Lemma 4). Then $\|(G_i, \Delta)\| < \gamma$ if $\Lambda_i = M_i^\phi$ and there exists a symmetric matrix $P_x = P_y = P \geq 0$, non-negative $\gamma, \lambda = [\lambda_2, \dots, \lambda_N]$, and $K = \text{diag}(K_1, \dots, K_M)$ such that $LMI(\lambda, \gamma, P, K) < 0$ holds.

PROOF. Similar to Theorem 7 using (26). \square

Staged and box constraints are assumed for the controller (13). The resulting diagonal static multipliers can reduce conservatism even further by increasing the degrees of freedom of the stability analysis. This is an extension of the conic combination case, where the scalar parameters λ are substituted by diagonal matrices.

5.3 Stability analysis using PWQ storage function

The common storage function may yield over-conservative results for PWA systems. Also, finding a single common storage function for all sub-models is quite hard, which can be overcome by using different storage functions for h_i being polyhedral. Alternatively, a PWQ function can be employed at a significant computational cost. Theorem 8 provides the sufficient stability conditions for SS multipliers and PWQ storage function:

Theorem 8 Let $G_i \in \mathbf{RH}_\infty^{(n_e+n_w) \times (n_w+n_d)}$ be a stable system and $\Delta_m : \mathcal{L}^{n_{vm}} \rightarrow \mathcal{L}^{n_{wm}}$ a bounded, causal operator containing every nonlinearity. The interconnection is well-posed and Δ_m satisfies IQC with multipliers M_m and the controller IQCs with multipliers M_i^ϕ (Lemma 4). Then $\|(G_i, \Delta)\| < \gamma$ if $\Lambda_i = M_i^\phi$ and there exists a symmetric matrix $P_y = P^j > 0, P_x = P_i > 0$, non-negative $\gamma, \lambda = [\lambda_2, \dots, \lambda_N]$, and $K = \text{diag}(K_1, \dots, K_M)$ such that $LMI(\lambda, \gamma, P^j, P_i, K)$ holds for all i, j .

PROOF. The difference with Theorem 7 is the use of a PWQ storage function. Here we have a fixed model for

each time interval yielding the following inequality:

$$\begin{aligned}
& \sum_{k=0}^T (V^{i(k+1)}(k+1) - V^{i(k)}(k)) = \\
& \sum_{k=0}^{T_1} (V^{i(k+1)}(k+1) - V^{i(k)}(k)) + \\
& \sum_{k=T_1+1}^{T_2} (V^{i(k+1)}(k+1) - V^{i(k)}(k)) + \dots + \\
& \sum_{k=T_n+1}^T (V^{i(k+1)}(k+1) - V^{i(k)}(k)) = \\
& V^{i(T+1)}(T+1) - V^{i(0)}(0) \geq 0,
\end{aligned} \tag{34}$$

with $V^{i(0)}(0) = 0$. Hence, only $\sum_{k=0}^T (V^{j(k)}(k+1) - V^{i(k)}(k)) \geq 0$ is needed with $j(k)$ be $i(k+1)$. \square

6 Applications

The proposed IQC multipliers are tested for two case studies: First for a numerical example, where the stability region can be numerically predicted, then for a larger more complex case where the stability of a PDE-based tubular reactor (Xie et al., 2015) is considered.

6.1 Numerical Example

The system's dynamics are given by:

$$\begin{aligned}
x(k+1) = & \\
& \begin{cases} \begin{bmatrix} -0.85 & & & \\ & 0.85 & & \\ & & 0.52 & \\ & & & 0.26 \end{bmatrix} x(k) + \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} u(k), \\ \text{if } V_1 x \leq b_1 \\ \begin{bmatrix} -0.1 & 0.3 & 0.1 & 0.05 \\ 0.3 & 0.15 & 0.4 & 0.2 \\ 0.1 & 0.4 & 0.4 & 0.1 \\ 0.05 & 0.2 & 0.1 & 0.3 \end{bmatrix} x(k) + \begin{bmatrix} 1 & .5 & 0 & 0 \\ 0 & 0 & .1 & 1 \end{bmatrix} u(k) + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \\ \text{otherwise} \end{cases}
\end{aligned} \tag{35}$$

where $V_1 = \begin{bmatrix} -0.92 & 0.33 & -0.60 & 1.48 \end{bmatrix}$ and $b_1 = 0.35$. Here, it is assumed that all states can be measured. The design parameters in (12a) are: $Q = I$, $R = rI$, where I the identity matrix and P is the solution of the Riccati equation. Matrices H_i , F_i and D_i can be easily constructed (see Maciejowski (2002)). Boxed constraints are applied, namely $-0.1 \leq u(k) \leq 0.1$. The objective is to compute the maximum positive constant gain $\tilde{\kappa}$ that can

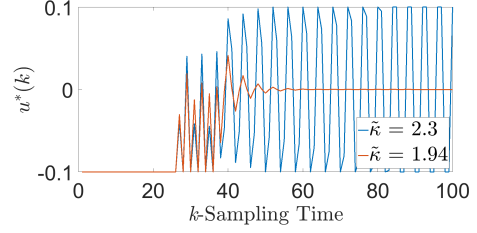


Fig. 5. Stable & unstable regions for the numerical example.

be applied on the plant's output implementing different values for r . The results are shown in Table 1 and 2. As expected, small values of r correspond to aggressive controller behaviour. Hence, the maximum value of $\tilde{\kappa}$ increases for larger values of r (Table 1). Next, r is kept

Table 1

Maximum $\tilde{\kappa}$ for which stability is guaranteed for $N = 3$

IQC	$r = 0.1$	$r = 0.3$	$r = 0.5$
SP	0.97	1.07	1.16
CC	0.98	1.07	1.16
SS	1.92	2.09	2.25
PWQ	1.94	2.1	2.25

constant and the stability is tested for various values of N . The controller uses one model per optimization for the whole horizon, which requires less computational time than the MIQP controller at the cost of increased uncertainty. As a result for larger horizons the actual stability region should decrease. The results confirm this. As can be seen in Table 2, the maximum $\tilde{\kappa}$ decreases as the horizon becomes larger. SP and CC multipliers

Table 2

Maximum $\tilde{\kappa}$ for which stability is guaranteed for $r = 0.1$

IQC	$N = 3$	$N = 4$	$N = 5$
SP	0.97	0.77	0.67
CC	0.98	0.78	0.67
SS	1.92	1.78	1.64
PWQ	1.94	1.84	1.68

for this case study give almost the same maximum $\tilde{\kappa}$ for all N . Using SS multipliers has a significant impact on the conservatism of our results. This is related to the additional degrees of freedom added to the analysis. Additionally, employing a PWQ storage function does not change vastly the conservatism. To validate further these results, a simulation is conducted for $r = 0.1$, $N = 3$ and $\tilde{\kappa} = 2.2$. In Fig. 5 the input $u^*(k)$ applied is depicted. For $\tilde{\kappa}$ outside of our predicted region, the closed-loop is unstable whereas for the maximum $\tilde{\kappa}$ computed is stable.

6.2 Tubular reactor

To further illustrate the features of the proposed analysis, we apply it to a tubular reactor, where an irreversible

exothermic reaction takes place. The system's dynamics are given by the following dimensionless equations:

$$\begin{aligned}\frac{\partial c}{\partial t} &= \frac{1}{Pe_1} \frac{\partial^2 c}{\partial y^2} - \frac{\partial c}{\partial y} - Da c e^{\gamma_1 T/(1+T)} \\ \frac{\partial T}{\partial t} &= \frac{1}{Pe_2} \frac{\partial^2 T}{\partial y^2} - \frac{\partial T}{\partial y} - B Da c e^{\gamma_1 T/(1+T)} \\ &\quad + b(T - T_w)\end{aligned}\quad (36)$$

Here c and T are the dimensionless concentration and temperature, respectively. T_w is the temperature of the cooling zones on the reactor jacket, separated in 8 different sectors, representing the problem's degrees of freedom, depicted in Fig. 6. The system parameters are $Pe_1 = Pe_2=7$, $Da=0.1$, $B=2$ $b = 1$, and $\gamma_1=10$. Neumann boundary conditions are used:

$$\begin{aligned}\frac{\partial c}{\partial y}\Big|_{y=0} &= -Pe_1 c, \quad \frac{\partial T}{\partial y}\Big|_{y=0} = -Pe_2 T \\ \frac{\partial c}{\partial y}\Big|_{y=L} &= 0, \quad \frac{\partial T}{\partial y}\Big|_{y=L} = 0\end{aligned}\quad (37)$$

The PDE-based model was discretized in 16 finite elements. To construct the model pool, 180 trajectories were collected over a range of cooling temperatures, T_w . Principal component analysis was employed to reduce the data-set size and to handle noisy data. K-means (Hastie et al., 2009) was then applied to identify the clusters and their centroids. Finally, linearization around the closest data-point of each centroid was used to construct the system Jacobian. The model pool consisted of 18 affine sub-models. It was assumed that only 10 out of 16 points along the length of the reactor can be measured. The model error was assumed to be

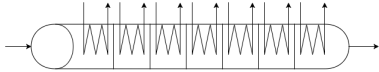


Fig. 6. Tubular reactor with 8 cooling zones

norm-bounded with $b^2 = 0.01$ and the MPC had the same design parameter, r , and prediction and control horizons, $N_{out}=3$ and $N_{in}=2$, respectively, as the previous application. The input variables (8 cooling temperatures) had upper and lower bounds $-1 \leq T_{wi} \leq 1$ for $i = 1 \dots N_{in}$, hence the method from section 4.2 could be implemented. This case is more computationally intensive as it comprises 32 states and 18 models with 8 manipulated variables for each control horizon. The inherent computational intensity of the PWQ storage function produced an intractable computational problem. Thus, only a common storage function was employed. Stability analysis was carried out, with the same objective as in the previous application. Here too, as shown in Table 3, SS multipliers produced a substantially less conservative stability estimate. Hence, SS multipliers can be confidently used with a common storage function to obtain realistic stability estimates

for moderately-sized distributed parameter systems. For validation purposes we show the closed-loop per-

Table 3

Minimum r for which stability is guaranteed

IQC Multiplier	r_{limit}
SP	0.70
CC	0.25
SS	0.01
PWQ	-

formance of the tubular reactor in Fig. 7, for $r = 0.01$. Despite the small value of r , the closed-loop system is stable. The semi-definite programming problems were

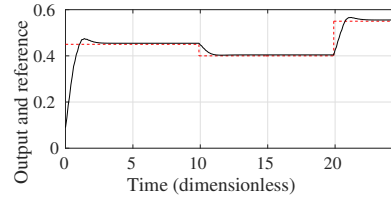


Fig. 7. closed-loop performance

solved using MATLAB with YALMIP (Löfberg, 2004) and MOSEK ApS (2015) on a computer with 3.40GHz Intel Core i5-3570 CPU processor and 8 GB of memory.

7 Conclusions and Future Work

This paper focuses on the development of a robust stability analysis methodology for PWA models under unstructured uncertainty and multi-model-based MPC. A systematic framework was developed to account for uncertainties such as model error. Sufficient conditions were presented using three different types of IQC multipliers in conjunction with common and PWQ storage functions. It was shown, through two illustrative case studies, that SS multipliers significantly reduce conservatism in the prediction of stability boundaries. For the first example with two sub-models and four states the SP multipliers with a common storage function required 2 cpu-sec per each calculation, and 1 cpu-min when the PWQ storage function was employed. For the tubular reactor with 18 sub-models and 32 states, ~ 40 cpu-min are required for the SS multipliers with a common storage function. The available computer memory was not enough to solve the problem with the PWQ storage function. Therefore, the difference is substantial when the number of states increases and additional work needs to be performed in regards to the handling of large-scale systems. In related work, model order reduction is employed (Theodoropoulos, 2011; Petsagkourakis et al., 2018) to describe the infinite dimensional system with a finite reduced one. This is the first time that IQCs have been used beyond the scope of linear MPC and we believe this is a significant step towards their use for the analysis of complex nonlinear systems.

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