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RAMSEY GOODNESS OF CYCLES*

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Abstract. Given a pair of graphs G and H, the Ramsey number $R(G,H)$ is the smallest N such that every red-blue coloring of the edges of the complete graph K_N contains a red copy of G or a blue copy of H. If a graph G is connected, it is well known and easy to show that $R(G,H) \geq (|G|-1)(\chi(H)-1)+\sigma(H)$, where $\chi(H)$ is the chromatic number of H and $\sigma(H)$ is the size of the smallest color class in a $\chi(H)$-coloring of H. A graph G is called H-good if $R(G,H)=(|G|-1)(\chi(H)-1)+\sigma(H)$. The notion of Ramsey goodness was introduced by Burr and Erdős in 1983 and has been extensively studied since then. In this paper we show that if $n \geq 10^{60}|H|$ and $\sigma(H) \geq \chi(H)^{22}$, then the n-vertex cycle C_n is H-good. For graphs H with high $\chi(H)$ and $\sigma(H)$, this proves in a strong form a conjecture of Allen, Brightwell, and Skokan.

Key words. Ramsey theory, cycles, expanders

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1. Introduction.

A celebrated theorem of Ramsey from 1930 says that for every n, there is a number $R(n)$ such that any 2-edge-coloring of a complete graph on $R(n)$ vertices contains a monochromatic complete subgraph on n vertices. Estimating $R(n)$ is a very difficult problem and one of the central problems in combinatorics. For a pair of graphs G and H, we can define the *Ramsey number* $R(G,H)$ to be the smallest integer N such that any red-blue edge coloring of the complete graph on N vertices contains a red copy of G or a blue copy of H. As a corollary of Ramsey's theorem, $R(G,H)$ is finite, since we always have $R(G,H) \leq R(\max(|G|,|H|))$.

Although in general determining $R(G,H)$ is very difficult, for some pairs of graphs G and H, their Ramsey number can be computed exactly. For example, Erdős [11] in 1947 showed that the Ramsey number of an n-vertex path versus a complete graph of order m satisfies $R(P_n,K_m)=(n-1)(m-1)+1$. The construction showing that this is tight comes from considering a 2-edge-coloring of $K_N, N=(n-1)(m-1)$ consisting of $m-1$ disjoint red cliques of size $n-1$ with all the edges between them blue. It is easy to check that this coloring has no red P_n or blue K_m. Chvátal and Harary observed that the same construction serves as a lower bound for $R(G,H)$ where G is any connected graph on n vertices and H is an m-partite graph. Let $\chi(H)$ be the *chromatic number* of H, i.e., the smallest number of colors needed to color the vertices of H so that no pair of adjacent vertices have the same color, and let $\sigma(H)$ be the the size of the smallest color class in a $\chi(H)$-coloring of H. Refining the above construction, Burr [5] obtained the following lower bound for the Ramsey number of a pair of graphs.

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LEMMA 1.1 (Burr [5]). *Letting H be a graph and G a connected graph with $|G| \geq \sigma(H)$, we have*

$$R(G,H) \geq (|G|-1)(\chi(H)-1)+\sigma(H). \tag{1}$$

To prove this bound, consider a 2-edge-coloring of complete graph on $N = (|G|-1)(\chi(H)-1)+\sigma(H)-1$ vertices consisting of $\chi(H)-1$ disjoint red cliques of size $|G|-1$ as well as one disjoint red clique of size $\sigma(H)-1$. This coloring has no red G because all red connected components have size $\leq |G|-1$, and there is no blue H since the partition of this H induced by red cliques would give a coloring of H by $\chi(H)$ colors with one color class smaller than $\sigma(H)$, contradicting the definition of $\sigma(H)$.

The bound in Lemma 1.1 is very general but for some graphs is quite far from the truth. For example, Erdős [11] showed that $R(K_n, K_n) \geq \Omega(2^{n/2})$, which is much larger than the quadratic bound we get from (1). However, there are many known pairs of graphs (such as when G is a path and H is a clique) for which $R(G,H) = (|G|-1)(\chi(H)-1)+\sigma(H)$. If this is a case we say that *G is H-good*. The notion of Ramsey goodness was introduced by Burr and Erdős [6] in 1983 and was extensively studied since then.

A lot of early research on Ramsey goodness focused on proving that a particular pair of graphs is good. For example, Gerencsér and Gyárfás [15] showed that for $n \geq m$ the path P_n is P_m-good. Chvátal showed that any tree T is K_m-good [8]. For more recent progress on Ramsey goodness see [1, 9, 13, 18, 19] and their references.

The problem of Ramsey goodness of cycles goes back to the work of Bondy and Erdős [4], who proved that the cycle C_n is K_m-good when $n \geq m^2-2$. Motivated by their result, Erdős et al. conjectured as follows.

CONJECTURE 1.2 (Erdős et al. [12]). *If $n \geq m \geq 3$, then $R(C_n, K_m) = (n-1)(m-1)+1$.*

Over the years, this problem has attracted a lot of attention. After several improvements, Nikiforov [18] showed that conjecture holds for $n \geq 4m+2$. In addition several authors proved it for small m (see [7] and the references therein). Very recently, Keevash, Long, and Skokan [16] showed that $R(C_n, K_m) = (n-1)(m-1)+1$ for $n \geq \Omega(\frac{\log m}{\log\log m})$. This proves the conjecture for large m.

In this paper we investigate Ramsey goodness of an n-vertex cycle versus a general graph H. When n is sufficiently large as a function of $|H|$, Burr and Erdős [6] proved more than 30 years ago that C_n is H-good. Recently Allen, Brightwell, and Skokan [1] conjectured that the cycle is H-good already when its length is linear in the order of H.

CONJECTURE 1.3 (Allen, Brightwell, and Skokan [1]). *For $n \geq \chi(H)|H|$ we have $R(C_n, H) = (n-1)(\chi(H)-1)+\sigma(H)$.*

There has been some work (see, e.g., [20] and its references) showing that the path P_n is H-good. Since $R(P_n, H)$ is always at most $R(C_n, H)$, a weakening of the above conjecture is to show that P_n is H-good for $n \geq \chi(H)|H|$. This was achieved by the authors of this paper in [21].

In this paper, we prove the following result.

THEOREM 1.4. *For $n \geq 10^{60} m_k$ and $m_k \geq m_{k-1} \geq \cdots \geq m_1$ satisfying $m_i \geq i^{22}$, we have $R(C_n, K_{m_1,\ldots,m_k}) = (n-1)(k-1)+m_1$.*

Here $K_{m_1,\ldots,m_k}$ is a complete multipartite with k parts of sizes $m_1,\ldots,m_k$. Notice that the vertices of a k-chromatic graph H can be partitioned into k independent sets

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of sizes $m_1, \dots, m_k$ with $\sigma(H) = m_1 \leq m_2 \leq \cdots \leq m_k$. This is equivalent to H being a subgraph of $K_{m_1,\dots,m_k}$. Therefore Theorem 1.4 implies the following.

Corollary 1.5. *Suppose that we have numbers n, and a graph H with $n \geq 10^{60}|H|$ and $\sigma(H) \geq \chi(H)^{22}$. Then $R(C_n, H) = (n-1)(\chi(H)-1) + \sigma(H)$.*

For graphs H with large $\chi(H)$ and $\sigma(H)$, the above theorem proves Conjecture 1.3 in a very strong form—it shows that in this case, the condition "$n \geq \chi(H)|H|$" is unnecessary, and $n \geq 10^{60}|H|$ suffices. For certain graphs H, Theorem 1.4 shows that C_n is H-good in a range which is even better than "$n \geq 10^{60}|H|$." For example, if H is balanced (i.e., if $|H| = \sigma(H)\chi(H)$), then Theorem 1.4 implies that C_n is H-good as long as $n \geq 10^{60}|H|/\chi(H)$.

1.1. Proof sketch. Here we give an informal sketch of the proof of Theorem 1.4. For simplicity we talk just about the balanced case of the theorem, i.e., the proof of $R(C_n, K_m^k) = (k-1)(n-1) + m$.

Let $R(C_{\geq n}, K_{m_1,\dots,m_k})$ denote the smallest number N such that in every coloring of K_N by the colors red and blue there is a red cycle of length *at least* n or a blue $K_{m_1,\dots,m_k}$. In [21] the following theorem is proved.

Theorem 1.6. *Given integers $m_1 \leq m_2 \leq \cdots \leq m_k$ and $n \geq 3m_k + 5m_{k-1}$, we have*

$$R(C_{\geq n}, K_{m_1,\dots,m_k}) = (k-1)(n-1) + m_1.$$

Notice that the above theorem is essentially a version of Theorem 1.4, except that it produces a red cycle of length *at least* n rather than one of length *exactly* n. The proof of our main theorem uses many ideas from the proof of Theorem 1.6. Because of this it may help readers to familiarize themselves with the very short proof of that theorem in [21]. It can be summarized as follows: If K_N is colored so that there is no blue $K_{m_1,\dots,m_k}$, then we use induction to find a large red subgraph G in K_N which is an expander. Then we use the famous Pósa rotation-extension technique to find a long red cycle in G.

To prove Theorem 1.4 we use a similar strategy, except that we build a red cycle of length at least n to also contain a special red subgraph called a *gadget*. Informally a gadget is a path between two special vertices x and y which has many chords. Because of these chords, the gadget has the property that it has paths between x and y of many different lengths. A consequence of this is that if we can find a cycle C of length *at least* n which contains a suitable gadget, then C also contains a cycle of length *exactly* n. Thus the proof of Theorem 1.4 naturally splits into two parts. The first part is to show that a large graph with no blue K_m^k contains a gadget (see section 2). The second part is to build a cycle of length at least n containing a gadget we found (see section 3).

To find a gadget in a graph with no blue K_m^k, we make heavy use of expanders. It turns out that if K_N has no blue K_m^k, then it contains a large red subgraph G with good expansion properties (see Lemma 2.5). Once we have an expander, we prove several lemmas which find various structures inside expanders such as trees (Lemma 2.8), paths (Lemma 2.9), and cycles (Lemma 2.13). We then put these structures together to build a gadget (Lemma 2.2). We remark that the gadgets that we use are very similar to *absorbers* introduced by Montgomery in [17] during the study of spanning trees in random graphs.

After constructing gadgets, the proof of Theorem 1.4 has three main ingredients—Lemmas 3.7, 3.9, and 3.14.

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The first ingredient, Lemma 3.7, should be thought of as a version of the $k = 2$ case of Theorem 1.4. Since the full proof of Theorem 1.4 is inductive, Lemma 3.7 serves as the initial case of the induction. The proof of this lemma is quite similar to the proof of Theorem 1.6 in [21], with one extra ingredient—namely gadgets.

The second ingredient, Lemma 3.9, should be thought of as a strengthening of Theorem 1.4 in the case when the red subgraph of K_N is highly connected. In this case it turns out that the Ramsey number can be lowered significantly (to $n+0.07kn$). The proof of this lemma again uses gadgets.

The third ingredient, Lemma 3.14, should be thought of as a stability version of Theorem 1.4. It says that for N close to $R(C_n, K_m^k)$, if we have a 2-colored K_N with no red C_n or blue K_m^k, then the coloring on K_N must be close to the extremal coloring. Specifically it shows that most of the graph can be partitioned into $k-1$ large sets $A_1, \dots, A_{k-1}$ with only blue edges between them. Once we have this structure, Theorem 1.4 is fairly easy to prove—since $A_1, \dots, A_{k-1}$ only have blue edges between them, they cannot contain a blue K_m^2 (or else the whole graph would contain a blue K_m^k). Then we apply the a version of the $k = 2$ case of Theorem 1.4 to one of the sets A_i to obtain a red C_n (specifically we apply Lemma 3.7 which serves as the "initial case" of the induction.)

1.2. Notation. For a graph G, the set of vertices is denoted by $V(G)$ and the set of edges by $E(G)$. We will often identify a graph with its vertex set, for example, we will use $|G|$ to mean the number of vertices of G. Throughout this paper the *order* of a path P, denoted $|P|$, is the number of vertices it has. The *length* of P is the number of edges P has, which is $|P| - 1$. Similarly, for a cycle C, both the order and length of C are defined to be $|C|$, the number of vertices of C. If $P = p_1, p_2, \dots, p_t$ is a path, then p_1 and p_t are called the *endpoints* of P, and $p_2, \dots, p_{t-1}$ are called the *internal vertices* of P. We will say things like "P is internally contained in S" or "P is internally disjoint from S" to mean that the internal vertices of P are contained in S or disjoint from S. For a graph G and two vertices $x, y \in G$ we let $d_G(x, y)$ be the length of the shortest path in G between x and y.

Recall that a forest is a graph with no cycles, and a tree is a connected graph with no cycles. A *rooted tree* is a tree with a designated vertex called the root. In tree T with root r, we call $T \setminus \{r\}$ the *internal* vertices of T. We think of the edges in a rooted tree as being directed away from the root. Then for a vertex v, the out-neighbors of v are called the *children* of v, and the in-neighbor of v is the *parent* of v. The *depth* of a rooted tree is the maximum distance of a vertex from the root. A binary tree is a tree of maximum degree 3. Notice that for any m, there is a rooted binary tree of depth $\lceil \log m \rceil$ and order m.

Recall that for a vertex v in a graph G $N_G(v)$ denotes the neighborhood of v in G—the set of vertices with edges going to v. The degree of a vertex in G is denoted by $d_G(v) = |N_G(v)|$, and the maximum degree taken over all vertices of the graph is denoted by $\Delta(G)$. For a set of vertices S in a graph G we let $N_G(S) = \bigcup_{s \in S} N_G(S)$ denote the set of neighbors in G of vertices of S. For $U \subseteq G$, we let $N_U(S) = N_G(S) \cap U = \{u \in U : us \text{ is an edge for some } s \in S\}$. When there is no ambiguity in what the underlying graph is, we will abbreviate $N_G(S)$ to $N(S)$.

The *complement* of a graph G, denoted $\overline{G}$, is the graph on $V(G)$ with $xy \in E(\overline{G}) \iff xy \notin G$. Notice that $R(H, K) \leq R$ is equivalent to saying that in any graph on G on R vertices either G contains H or $\overline{G}$ contains K. We let K_m^k denote the complete multipartite graph with k parts of size m. With this notation, K_m^1 means a set of k vertices (with no edges.) Notice that we have $R(K_m^1, G) \leq m$ for any graph G.

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Throughout the paper "log" always means "$\log_2$," the binary log. In this paper we will omit floor and ceiling signs where they are not essential.

2. Gadgets. In this section we construct gadgets which are one of the main technical tools which we use in this paper. A gadget is a graph containing paths of several different lengths between a designated pair of vertices a and b.

DEFINITION 2.1. *A k-gadget is a graph J containing two vertices a and b such that J has a to b paths of orders $|J|$ and $|J| - k$.*

The vertices a and b are called the *endpoints* of the k-gadget. We will often identify a k-gadget J with the path of order $|J|$ contained in it. A $(\leq k)$-gadget is a graph J with two vertices a and b with a to b paths of lengths $|J|, |J|-1, \dots, |J|-k$. In other words a $(\leq k)$-gadget is simultaneously a k'-gadget for $k' = 1, 2, \dots k$.

An example of a k gadget is a cycle with $k+2$ vertices with a and b a pair of adjacent vertices. Then a to b paths of orders $k+2$ and 2 can be obtained by going around the cycle in different directions. For our purposes we will construct more complicated gadgets. The reason for this is that short cycles do not necessarily exist in graphs whose complements are K_m^k-free.

The main goal of this section is to prove the following lemma.

LEMMA 2.2. *There exists a constant $N_1 = 10^7$ so that the following holds for any $\lambda, \mu, k, m \in \mathbb{N}$ with $m \geq k^3$, $\lambda \geq 2\mu \geq 10^9$, and $\mu m \geq 4100(\lambda m)^{\frac{3}{4}}$.*

Let G be a graph with $|G| \geq (N_1 \lambda \mu k)m$ and with $\overline{G}$ K_m^k-free. Then G contains a $(\leq \lambda m)$-gadget J of order $(\lambda + \mu)m$ with endpoints a and b as well as an internally disjoint a – b path Q of order μm.

The above lemma could be be rephrased as a Ramsey-type statement. If we let $\mathcal{J}_{t,n}$ be the family of all $\leq t$ gadgets on n vertices, then Lemma 2.2 implies that $R(\mathcal{J}_{\lambda m, (\lambda+\mu)m}, K_m^k) \leq (N_1 \lambda \mu k)m$.

Notice that Lemma 2.2 also finds a path Q between the two endpoints of the gadget it produces. This path should be thought of as a technical tool which we will later use to join gadgets together.

The structure of this section is as follows. In section 2.1 we introduce expanders and give their basic properties. In section 2.2 we give a variant of a result of Friedman and Pippenger about embedding trees into expanders. In section 2.3 we prove some lemmas about embedding paths and cycles into expanders. In section 2.4 we prove Lemma 2.2. In section 2.5 we prove some additional properties of gadgets which we will need.

2.1. Expanders. We'll use the following notion of expansion.

DEFINITION 2.3. *For a graph G and $W \subseteq V(G)$, we say that G (Δ, β, m)-expands into W if the following hold.*
(i) $|N_W(S)| \geq \Delta |S|$ *for* $S \subseteq V(G)$ *with* $|S| < m$.
(ii) $|N_G(S) \cup S| \geq |S| + \beta m$ *for* $S \subseteq V(G)$ *with* $m \leq |S| \leq |G|/2$.

The following easy observation shows how we can change the parameters Δ and β while maintaining expansion.

Observation 2.4. Suppose that G (Δ, β, m)-expands into W.
(i) If $W' \supseteq W$, $\Delta' \leq \Delta$, and $\beta' \leq \beta$, then G (Δ', β', m)-expands into W'.
(ii) If $W \subseteq U \subseteq V(G)$ with $|V(G) \setminus U| \leq tm$, then $G[U]$ $(\Delta, \beta - t, m)$-expands into W.

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The following lemma shows that graphs whose complement is K_m^k-free contain large subgraphs which expand well.

LEMMA 2.5. *For all β, m, M, $\Delta \geq 1$ with $\beta+2 < M/4$ and $3\Delta < \beta$ the following holds.*

Let G be a graph with $\overline{G}$ K_m^k-free and $|G| \geq \max(m, M(k-1.5)m)$. Then there exists an integer k' and an induced subgraph $H \subseteq G$ such that the following hold:

- *$\overline{H}$ is $K_m^{k'}$-free.*
- *$M(k'-1.5)m - m \leq |H| \leq M(k'-1.5)m$. Also we have $|H| \geq m$.*
- *H (Δ, β, m)-expands into $V(H)$.*

Proof. The proof is by induction on k. The initial case is when $k = 1$, which holds vacuously since any graph with m vertices contains a copy of K_m^1 (by definition K_m^1 is just any set of m vertices).

Assume that for $k \geq 2$ we have a graph G as in the statement of the lemma, and the result holds for all $\hat{k} < k$. Without loss of generality, we may assume that $|G| = M(k-1.5)m$ (by possibly passing to a subgraph of G of this order.)

Suppose that there is a set S with $m \leq |S| \leq |G|/2$ such that $|N_G(S) \cup S| < |S| + (\beta+1)m$. Let $T = V(G) \setminus (N_G(S) \cup S)$. Using $|S| \leq |G|/2$, $|N_G(S) \cup S| < |S| + (\beta+1)m$, $|G| \geq M(2-1.5)m$, and $\beta+2 < M/4$ we obtain that $|T| \geq m$. We also have that $|S \cup T| = |G| - |N_G(S) \setminus S| \geq M(k-1.5)m - (\beta+1)m$. Choose s and t maximum integers for which $|S| \geq M(s-1.5)m$ and $|T| \geq M(t-1.5)m$. We certainly have $s, t \geq 1$. From the maximality of s and t, we have $|S \cup T| = |S| + |T| < M(s+1-1.5)m + M(t+1-1.5)m = M(s+t-1.5)m + Mm/2$. Combining this with $|S \cup T| \geq M(k-1.5)m - (\beta+1)m$, we get $k - (\beta+1)/M \leq s+t+1/2$. Together with $(\beta+1)/M + 1/2 < 1$ and the integrality of k, s, and t this gives $s+t \geq k$. Let $s' \in [1, s]$ and $t' \in [1, t]$ be arbitrary integers with $s' + t' = k$. This ensures $s', t' \leq k-1$. Since $\overline{G}$ is K_m^k-free and there are no edges between S and T, we have that either $\overline{G[S]}$ is $K_m^{s'}$-free or $\overline{G[T]}$ is $K_m^{t'}$-free. By induction either S or T contains a subgraph with the required properties.

Now suppose that for every set S with $m \leq |S| \leq |G|/2$ we have $|N_G(S) \cup S| \geq |S| + (\beta+1)m$. Let S be the largest set of vertices in G with $|S| \leq 2m$ for which $|N_G(S) \setminus S| < (\Delta+1)|S|$. We have that $|N_G(S) \cup S| \leq (\Delta+2)|S| < |S| + (\beta+1)m$. By our assumption we have that $|S| < m$.

Let $G' = G \setminus S$. We claim that this graph satisfies the conditions of the lemma with $k' = k$. Certainly $\overline{G'}$ is $K_m^{k'}$-free. Also since $k \geq 2$, we have $|G'| \geq |G| - m = M(k'-1.5)m - m \geq m$. Suppose that we have $S' \subseteq V(G')$ with $|S| \leq |G'|/2 \leq |G|/2$. If $|S'| \geq m$, then $|N_{G'}(S') \cup S'| \geq |N_G(S') \cup S'| - |S| \geq |S'| + (\beta+1)m - |S| \geq |S'| + \beta m$. If $|S'| \leq m$, then by maximality of S, we have $|N_G(S' \cup S) \setminus (S' \cup S)| \geq (\Delta+1)|S \cup S'|$, which implies that $|N_{G'}(S')| \geq |N_{G'}(S' \cup S)| - |N_{G'}(S)| \geq |N_G(S' \cup S) \setminus (S' \cup S)| - |N_G(S) \setminus S| \geq (\Delta+1)(|S'| + |S|) - (\Delta+1)|S| \geq \Delta|S'|$, proving the lemma. ∎

Notice that in the above lemma, we can always take $k' \geq 2$, since no graph with $|H| \geq m$ has $\overline{H}$ is K_m^1-free. The following lemma shows that expanders have good connectivity properties.

LEMMA 2.6. *Suppose that G (Δ, β, m)-expands into $W \subseteq V(G)$. Suppose that we have three disjoint sets of vertices $A, B, C \subseteq V(G)$ with $(\Delta-2)|A| \geq |C \cap W|$, $(\Delta-2)|B| \geq |C \cap W|$, and $\beta m \geq 2|C|$.*

Then there is an A to B path P in G, avoiding C, and with $|P| \leq 8\log m + 2|G|/\beta m$.

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Proof. Let $a = 4\log m$ and $b = |G|/\beta m$. With this notation, it is sufficient to find an A to B path of length $\leq 2a + 2b$.

Set $A^0 = A$ and $A^{i+1} = \big(N_G(A^i) \cup A^i\big) \setminus C$ for each i. Using the definition of "(Δ, β, m)-expands" and $(\Delta - 2)|A| \geq |C \cap W|$ we have that $|A^{i+1}| \geq 2|A^i|$ whenever $|A_i| < m$, which implies that $|A^a| \geq m$. Using the definition of "(Δ, β, m)-expands" and $\beta m \geq 2|C|$ we have that $|A^{i+1}| \geq |A^i| + \beta m/2$ whenever $m \leq |A_i| \leq |G|/2$. Combining this with $|A^a| \geq m$ gives $|A^{a+b}| > b\beta m/2 \geq |G|/2$.

Similarly, letting $B^0 = B$ and $B^{i+1} = \big(N_G(B^i) \cup B^i\big) \setminus C$ we have $|B^{a+b}| > |G|/2$. Therefore A^{a+b} and B^{a+b} intersect, giving us the required path. ∎

2.2. Embedding trees. We'll need a version of a theorem of Friedman and Pippenger [14] about embedding trees into expanding graphs. The following lemma is proved in [2]

LEMMA 2.7 (see [2, Lemma 5.2]). *Suppose that we have Δ, M, m, and n such that $9\Delta m < M$. Let $X = \{x_1, \ldots, x_t\}$ be a set of vertices in a graph G on n vertices. Suppose that we have rooted trees $T(x_1), \ldots, T(x_t)$ satisfying $\sum_{i=1}^t |T(x_i)| \leq M$ and $\Delta\big(T(x_i)\big) \leq \Delta$ for all i. Suppose that for all $S \subseteq V(G)$ with $m \leq |S| \leq 2m$ we have $|N(S)| \geq M + 10\Delta m$, and for $S \subseteq V(G)$ with $|S| \leq m$ we have*

$$|N(S) \setminus X| \geq 4\Delta|S \setminus X| + \sum_{x \in S\cap X} \Big(d_{root}\big(T(x)\big) + \Delta\Big). \tag{2}$$

Then we can find disjoint copies of the trees $T(x_1), \ldots, T(x_t)$ in G such that for each i, $T(x_i)$ is rooted at x_i. In addition for all $S \subseteq V(G)$ with $|S| \leq m$, we have

$$|N(S) \setminus \big(T(x_1) \cup \cdots \cup T(x_t)\big)| \geq \Delta|S|. \tag{3}$$

The following version of the above lemma will be easier to apply.

LEMMA 2.8. *Suppose that we have a graph G and a set $W \subseteq V(G)$ such that G $(4\Delta, \beta, m)$-expands into W with $20\Delta \leq \beta$. Let $X = \{x_1, \ldots, x_t\} = G \setminus W$.*

Then for any family of rooted trees $\{T(x_1), \ldots, T(x_t)\}$ with $\Delta(T(x_i)) \leq \Delta$ and $\sum_{i=1}^t |T(x_i)| \leq (\beta - 10\Delta)m$ we can find disjoint copies of $T(x_1), \ldots, T(x_t)$ in G with $T(x_i)$ rooted at x_i such that G (Δ, β, m)-expands into $W \setminus \big(T(x_1) \cup \cdots \cup T(x_t)\big)$.

Proof. By setting $M = (\beta - 10\Delta)m$, we see that the assumptions of Lemma 2.7 hold for the family of trees $T(x_1), \ldots, T(x_t)$. This allows us to embed the trees $T(x_1), \ldots, T(x_t)$ such that $|N_G(S) \setminus (T(x_1) \cup \cdots \cup T(x_t))| \geq \Delta|S|$ holds for all $S \subseteq V(G)$ with $|S| \leq m$. This shows that part (i) holds of the definition of G (Δ, β, m)-expanding into $G \setminus \big(T(x_1) \cup \cdots \cup T(x_t)\big) = W \setminus \big(T(x_1) \cup \cdots \cup T(x_t)\big)$. Part (ii) also holds as a consequence of G $(4\Delta, \beta, m)$-expanding into W. ∎

2.3. Embedding paths and cycles. In this section we prove several lemmas about embedding paths and cycles into expanders. They will be the building blocks for the gadgets which we construct in the next section.

The following lemma allows us to connect prescribed vertices together by short paths.

LEMMA 2.9. *Let G be a graph, $\beta, m, t \in \mathbb{N}$, and fix $\ell = 4|G|/\beta m + 10\log \beta m$. Suppose that G $(16, \beta, m)$-expands into $W \subseteq V(G)$ with $(\beta - 80)m \geq 4\ell t^2 + |G \setminus W|$. Suppose that we have pairs of vertices $x_1, y_1, x_2, y_2, \ldots, x_t, y_t \in G \setminus W$.*

Then there are vertex-disjoint paths $P_1, \ldots, P_t$ in G with P_i going from x_i to y_i and $|P_i| \leq \ell$.

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Proof. Let $X = G \setminus W$ and list the vertices of X as $(x_1, y_1, x_2, y_2, \dots, x_t, y_t, z_1, \dots, z_r)$ for $r = |X| - 2t$. We assign a tree $T(v)$ to each $v \in X$ as follows. For $i = 1, \dots, t$ the trees $T(x_i)$ and $T(y_i)$ are both rooted binary trees with $t\ell$ vertices of depth $\le \lceil \log t\ell \rceil$. For vertices z_i we let $T(z_i)$ be the tree consisting of a single vertex. Notice that $\Delta(T(v)) < 4$ for all v, G $(16, \beta, m)$-expands into W, $20 \cdot 4 \le \beta$, and that $\sum_{v \in X} |T(v)| \le 2t^2\ell + |G \setminus W| \le (\beta - 10 \cdot 4)m$. Therefore we can apply Lemma 2.8 to G with $\Delta = 4$ in order to find disjoint copies of $T(v)$ rooted at all $v \in X$ such that G $(4, \beta, m)$-expands into $W' = W \setminus \bigcup_{v \in X} T(v)$.

For $i = 1, \dots, t$ let $A_i = V(T(x_i))$ and $B_i = V(T(y_i))$. Notice that to prove the lemma it is sufficient to find vertex-disjoint paths Q_i from A_i to B_i internally inside W' of length $\le \ell - 2\lceil \log t\ell \rceil - 1$. Indeed once we have such paths, we can join Q_i to the paths P_{x_i} in $T(x_i)$ and P_{y_i} in $T(y_i)$ from the endpoints of Q_i to x_i and y_i, respectively, in order to obtain P_i (since A_i and B_i are binary trees of depth $\le \lceil \log t\ell \rceil$, we know that $e(P_{x_i}), e(P_{y_i}) \le \lceil \log t\ell \rceil$). We will repeatedly apply Lemma 2.6 to G and W' t times in order to find such paths $Q_1, \dots, Q_t$ of length $\le \ell - 2\lceil \log t\ell \rceil - 1$.

Suppose that for some $i \in \{1, \dots, t\}$, we have already found vertex-disjoint paths $Q_1, \dots, Q_{i-1}$, each of length $\le \ell - 2\lceil \log t\ell \rceil - 1$. Let $C = (\bigcup_{j<i} Q_j) \cup (\bigcup_{j \ne i} A_j \cup B_j)$. Notice that we have $2|A_i|, 2|B_i| = 2t\ell \ge |Q_1| + \dots + |Q_{i-1}| \ge |C \cap W'|$. We also have $\beta m \ge 4t^2\ell \ge 2(i-1)\ell + 2t^2\ell \ge 2|C|$. Therefore, by Lemma 2.6, there is a path Q_i from A_i to B_i avoiding C with $|Q_i| \le 8 \log m + 2|G|/\beta m \le \ell - 2\lceil \log t\ell \rceil - 1$ (the last inequality uses $t\ell \le \beta m$). ∎

The following lemma allows us to find a short cycle C in an expander, such that the graph expands outside C.

LEMMA 2.10. *Suppose that we have a nonbipartite graph G which (Δ, β, m)-expands into $W \subseteq G$ with $\Delta \ge 2$.*

Then G contains an odd cycle C with $|C| \le 16 \log m + 4|G|/\beta m$ such that G $(\Delta - 5, \beta, m)$-expands into $W \setminus V(C)$.

Proof. Let C be the shortest odd cycle in G.

CLAIM 2.11. *For any vertices $x, y \in C$ we have $d_C(x, y) = d_G(x, y)$.*

Proof. We certainly have $d_C(x, y) \ge d_G(x, y)$. Suppose for the sake of contradiction that we have $x, y \in C$ with $d_C(x, y) > d_G(x, y)$. Without loss of generality, we may suppose that $d_G(x, y)$ is as small as possible among such pairs of vertices. Let P be a $x - y$ path of length $d_G(x, y)$.

Suppose that $P \cap C$ contains some vertex $z \notin \{x, y\}$. We have $d_G(x, z) < d_G(x, y)$, so by minimality of $d_G(x, y)$ we have that $d_C(x, z) = d_G(x, z) \le d_P(x, z)$. Similarly, we obtain $d_C(z, y) = d_G(z, y) \le d_P(z, y)$. This gives us $d_C(x, y) \le d_C(x, z) + d_C(z, y) \le d_P(x, z) + d_P(z, y) = d_G(x, y)$, contradicting $d_C(x, y) > d_G(x, y)$.

Suppose that $P \cap C = \{x, y\}$. Let Q be the $x - y$ path along C with $|C|$ having the same parity as $|P|$. By replacing Q by P we obtain an odd cycle shorter than C, contradicting the minimality of $|C|$. ∎

From Lemma 2.6 applied with $C = \emptyset$, we have $\mathrm{Diam}(G) \le 8 \log m + 2|G|/\beta m$ and so Claim 2.11 implies that $|C| \le 2\mathrm{Diam}(G) \le 16 \log m + 4|G|/\beta m$.

For any $v \in V(G)$, Claim 2.11 implies that $|N_G(v) \cap C| \le 5$, since otherwise there would be two vertices $x, y \in N_G(v) \cap C$ with $d_C(x, y) \ge 3 > 2 = d_G(x, y)$. Using the fact that G (Δ, β, m)-expands into W we obtain that for any S with $|S| < m$ we have $|N_G(S) \cap (W \setminus C)| \ge |N_G(S) \cap W| - |N_G(S) \cap C| \ge (\Delta - 5)|S|$. This implies that G $(\Delta - 5, \beta m, m)$-expands into $W \setminus V(C)$. ∎

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The same proof also proves the following.

LEMMA 2.12. *Suppose that we have a graph G which (Δ, β, m)-expands into $W \subseteq G$ with $\Delta \geq 2$, and we have two vertices $x, y \in V(G)$.*

Then there is a path P from x to y with $|P| \leq 16 \log m + 4|G|/\beta m$ such that G $(\Delta - 5, \beta, m)$-expands into $W \setminus V(P)$.

To prove Lemma 2.12 one lets P be the shortest x to y path in G. The path P ends up having the required properties by the same argument as in Lemma 2.10.

The following lemma allows us to find a cycle whose length is close to a prescribed value.

LEMMA 2.13. *Suppose that we have a nonbipartite graph G which (Δ, β, m)-expands into G for $\Delta \geq 20$ and $\beta \geq 8\Delta$. Let r be an odd integer with $r \leq m$.*

Then G contains an odd cycle C with $r + 2 \leq |C| \leq r + 16 \log m + 5|G|/\beta m$. In addition there is an induced subgraph graph G' of G such that G' $(\Delta/4 - 7, \beta - 3, m)$-expands into $V(G) \setminus V(C)$, and $C \setminus V(G')$ is a path of order r.

Proof. By Lemma 2.10, G contains an odd cycle C_{odd} such that $|C_{odd}| \leq 16 \log m + 4|G|/\beta m$ such that G $(\Delta - 5, \beta, m)$-expands into $V(G) \setminus V(C_{odd})$. If $|C_{odd}| \geq r + 2$, then the lemma holds with $C = C_{odd}$ and G' a subgraph of G formed by deleting r consecutive vertices on C (here G' $(\Delta/4 - 7, \beta - 3, m)$-expands into $V(G) \setminus V(C)$ using $r \leq m$ and Observation 2.4). Therefore, suppose that $|C_{odd}| \leq r$, and let x, y be two vertices in C_{odd} at distance $\lfloor |C_{odd}|/2 \rfloor$. Notice that this means that there are x to y paths R^+ and R^- in C of orders $|C_{odd}|/2 + 1/2$ and $|C_{odd}|/2 - 1/2$, respectively.

By Lemma 2.8, G contains a path P of order $r - |C_{odd}|/2 + 5/2$ starting with x, with $P \cap C = \{x\}$, such that G $(\Delta/4 - 2, \beta, m)$-expands into $V(G) \setminus (V(C) \cup V(P))$. (For this application, we have $G = G, W = V(G) \setminus V(C_{odd})$, $X = V(C_{odd})$, $\Delta' = \Delta/4 - 2$, $\beta = \beta$, and $m = m$. Let $T(x)$ be a path of order $r - |C_{odd}|/2 + 5/2$, and let $T(x')$ be the single-vertex tree for all $x' \in X \setminus \{x\}$.) Let $z \neq x$ be the other endpoint of P and $W = V(G) \setminus (V(C_{odd}) \cup V(P))$.

Suppose that zy is an edge. Joining R^+ to P gives a cycle C of order $r + 2$ for which the lemma holds with G' a subgraph of G formed by deleting r consecutive vertices on C (here G' $(\Delta/4 - 7, \beta - 3, m)$-expands into $V(G) \setminus V(C)$ using $r \leq m$ and Observation 2.4).

Suppose that zy is a nonedge. Let G_1 be the induced subgraph of G on $(V(G) \setminus (C_{odd} \cup P)) \cup \{z, y\}$. Notice that since $r \leq m$, Observation 2.4(ii) implies that G_1 $(\Delta/4 - 2, \beta - 2, m)$-expands into W. By Lemma 2.12, G_1 contains a z to y path Q of length $\leq 16 \log m + 4|G|/(\beta - 2)m \leq 16 \log m + 5|G|/\beta m$ such that G_1 $(\Delta/4 - 7, \beta - 2, m)$-expands into $W \setminus V(Q)$. Since zy is a nonedge, we have $|Q| \geq 3$.

Notice that $|R^+|$ and $|R^-|$ have different parities. Therefore we obtain an odd cycle C with $|C| \leq r + 16 \log m + 5|G|/\beta m$ by joining Q to P to either R^+ or R^-. We have either $|C| = |R^-| + |P| + |Q| - 3$ or $|C| = |R^+| + |P| + |Q| - 3$. Using $|R^-| = |C_{odd}|/2 - 1/2$, $|R^+| = |C_{odd}|/2 + 1/2$, $|Q| \geq 3$, and $|P| = r - |C_{odd}|/2 + 5/2$ we get $|C| \geq |R^-| + |P| + |Q| - 3 \geq r + 2$ and $|Q| \geq |C| - |R^+| - |P| + 3 = |C| - r$. Since $|Q| \geq |C| - r$, we can choose a set U of $|C| - r$ consecutive vertices on Q. Let G' be the induced subgraph of G on $(V(G_1) \setminus V(Q)) \cup U$ to get G' $(\Delta/4 - 7, \beta - 3, m)$-expanding into $W \setminus V(Q) = V(G) \setminus V(C)$ as required (using Observation 2.4(ii)). ∎

2.4. Constructing gadgets. In this section we construct gadgets in graphs whose complement is K_m^k-free. The overall goal of this section is to prove Lemma 2.2.

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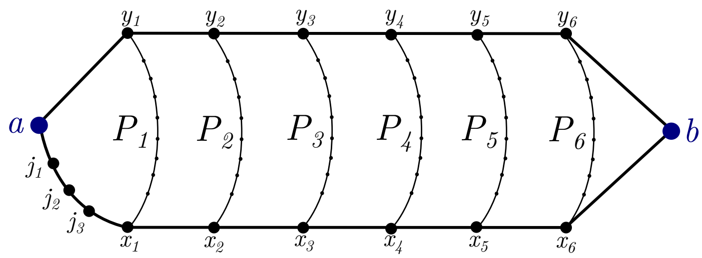

FIG. 1. *A 3-gadget.*

The following lemma shows that odd r-gadgets exist in graphs whose complements are K_m^k-free. It also finds two large binary trees attached to the endpoints of the gadget. These binary trees will later be used to join several gadgets together.

LEMMA 2.14. *Let m, k, and r be integers with $m \geq \max(k^3, 10^9)$, r odd, and $r \leq m$. Let G be a graph with $\overline{G}$ K_m^k-free and $|G| \geq 9100000km$.*

Then G contains an r-gadget J with $|J| \leq r + 2000m^{\frac{2}{3}}$ with endpoints a and b. In addition there are two disjoint binary trees T_a and T_b in G of order m and depth $\leq \lceil \log m \rceil$ with T_a rooted at a and having $T_a \cap J = \{a\}$ and T_b rooted at b and having $T_b \cap J = \{b\}$.

Proof. For this lemma we fix $M = 9000000$, $\Delta = 4000$, and $\beta = 1500000$. See Figure 1 for a diagram of what kind of r-gadget we will find in G.

Apply Lemma 2.5 to G in order to find an integer k' and a subgraph G_1 of G with $(M-2)(k'-1.5)m \leq |G_1| \leq M(k'-1.5)m$ such that G_1 (Δ, β, m)-expands into G_1 and $\overline{G_1}$ is $K_m^{k'}$-free. We have $k' \geq 2$, since $|G_1| \geq m$ implies that $\overline{G_1}$ cannot be K_m^1-free. Notice that since $|G_1| \leq M(k'-1.5)m \leq Mkm \leq Mm^{\frac{4}{3}}$ and $M/\beta \leq 6$, we have $|G_1|/\beta m \leq 6m^{\frac{1}{3}}$. Notice that G_1 is nonbipartite—indeed since $\overline{G_1}$ is $K_m^{k'}$-free, every set of size mk' in G_1 contains an edge which implies that $\alpha(G_1) \leq mk'$ and $\chi(G_1) \geq |G_1|/\alpha(G_1) \geq (M-2)(k'-1.5)/k' > 10000$.

Apply Lemma 2.13 to G_1 in order to find an odd cycle C with vertex sequence $a, j_1, \dots, j_r, x_1, x_2, \dots, x_t, b, y_t, y_{t-1}, \dots, y_1$ such that $t \leq 16 \log m + 5|G_1|/\beta m \leq 50m^{\frac{1}{3}}$. In addition, we obtain a subgraph $G_2 \subseteq G_1$ which $(\Delta/5, \beta - 3, m)$-expands into $W_2 = V(G_1) \setminus C$. Without loss of generality, we may assume that C is labeled so that $\{x_1, x_2, \dots, x_t, b, y_t, y_{y-1}, \dots, y_1, a\} = C \cap G_2$.

Apply Lemma 2.8 to G_2 and W_2 in order to find two binary trees T_a and T_b internally in W_2 of order m and depth $\leq \lceil \log m \rceil$ with $T_a \cap C = \{a\}$ and $T_b \cap C = \{b\}$ (for this application let T_x be a single-vertex tree for $x \in C \setminus \{a, b\}$). From the application of Lemma 2.8 we have that G_2 $(\Delta/20, \beta-3, m)$-expands into $W_2 \setminus (T_a \cup T_b)$. Let $G_3 = G_2 \setminus (T_a \cup T_b)$ and $W_3 = W_2 \setminus (T_a \cup T_b)$. Notice that since G_2 $(\Delta/20, \beta-3, m)$-expands into W_3 and $|T_a \cup T_b| = 2m$, by Obervation 2.4(ii), G_3 $(\Delta/20, \beta - 5, m)$-expands into W_3.

Apply Lemma 2.9 to G_3, W_3, and the set of pairs $x_1, y_1, \dots, x_t, y_t$ in order to find disjoint paths $P_1, \dots, P_t$ in G_3 with P_i joining x_i to y_i and $|P_i| \leq 40m^{\frac{1}{3}}$ (for this application we use $\beta' = \beta - 5$, $t \leq 50m^{\frac{1}{3}}$, $|G_3 \setminus W_3| = |G_3 \cap C| = 2t + 2$, $m \geq 10^9$, and $\ell = 4|G|/\beta' m + 10 \log(\beta' m) \leq 40m^{\frac{1}{3}}$ which ensure that we have $(\beta' - 80)m \geq 4 \cdot 50m^{\frac{1}{3}} \cdot 50m^{\frac{1}{3}} \cdot 40m^{\frac{1}{3}} + 2 \cdot 50m^{\frac{1}{3}} + 2 \geq 4t^2\ell + |G_3 \setminus W_3|$).

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Let $J = C \cup P_1 \cup \cdots \cup P_t$. We will show that J is an r-gadget satisfying all the conditions of the lemma. Notice that the following are both vertex sequences of paths from a to b in J:

$$Q_1 = a, j_1, j_2, \ldots, j_r, x_1, P_1, y_1, y_2, P_2, x_2, x_3, P_3, y_3, \ldots, x_t, P_t, y_t, b,$$

$$Q_2 = a, y_1, P_1, x_1, x_2, P_2, y_2, y_3, P_3, x_3, \ldots, y_t, P_t, x_t, b.$$

We have that $|Q_1| = |J|$ and $|Q_2| = |J| - r$, and so Q_1 and Q_2 qualify as the two paths in the definition of the r-gadget J. Finally we have $|J| \leq r + t \max_{i=1}^{t} |P_i| \leq r + 2000m^{\frac{2}{3}}$. ∎

The following lemma shows that if the complement of a sufficiently large graph is K_m^k-free, then the graph contains a $(\leq t)$-gadget.

LEMMA 2.15. *Let $m, k,$ and r be integers with $m \geq \max(k^3, 10^9)$ and $r \leq \log m$. Let G be a graph with $\overline{G}$ K_m^k-free and $|G| \geq 9500000km$.*

Then G contains a $(\leq 2^r)$-gadget J with $|J| \leq 2^r + 2050 \cdot r \cdot m^{\frac{2}{3}}$ with endpoints a and b. In addition there are two disjoint binary trees T_a and T_b in G of order m and depth $\leq \lceil \log m \rceil$ with T_a rooted at a and having $T_a \cap J = \{a\}$ and T_b rooted at b and having $T_b \cap J = \{b\}$.

Proof. For this lemma we fix $M = 9500000$, $\Delta = 40000$, and $\beta = 1500000$. Apply Lemma 2.5 to G in order to find an integer k' and a subgraph G' of G with $(M-2)(k'-1.5)m \leq |G'| \leq M(k'-1.5)m$ such that G' (Δ, β, m)-expands into G' and $\overline{G'}$ is $K_m^{k'}$-free. Notice that since $|G'| \leq M(k'-1.5)m \leq Mkm \leq Mm^{\frac{4}{3}}$ and $M/\beta \leq 7$, we have $|G'|/\beta m \leq 7m^{\frac{1}{3}}$.

The strategy of the proof of this lemma is to repeatedly apply Lemma 2.14 in order to find 2^i-gadgets for $i \in \{1, \ldots, r\}$ and join all these gadgets together using Lemma 2.6. See Figure 2 for an illustration of what the final $(\leq 2^r)$-gadget looks like.

CLAIM 2.16. *For $s \leq r$, G' contains a $(\leq 2^s)$-gadget J with $|J| \leq 2^s + (s+1)2050m^{\frac{2}{3}}$ with endpoints a and b. In addition there are two disjoint binary trees T_a and T_b in G' of order m and depth $\leq \lceil \log m \rceil$ with T_a rooted at a and having $T_a \cap J = \{a\}$ and T_b rooted at b and having $T_b \cap J = \{b\}$.*

Proof. The proof is by induction on s. The initial case "$s = 0$" follows from Lemma 2.14. Let $s \geq 1$. Suppose that we have a $(\leq 2^{s-1})$-gadget J in G' with $|J| \leq 2^{s-1} + s \cdot 2050 \cdot m^{\frac{2}{3}}$ with endpoints a and b as well as two disjoint binary trees T_a and T_b in G' of order m and depth $\leq \lceil \log m \rceil$ with $T_a \cap J = \{a\}$ and $T_b \cap J = \{b\}$.

The cases "$s = 1$" and "$s \geq 2$" are slightly different. If $s \geq 2$, apply Lemma 2.14 to $G' \setminus (J \cup T_a \cup T_b)$ in order to find a $(2^{s-1}+1)$-gadget J' in $G' \setminus (J \cup T_a \cup T_b)$ with $|J'| \leq 2^{s-1} + 1 + 2000m^{\frac{2}{3}}$ and with endpoints a' and b' as well as two disjoint binary trees T'_a and T'_b of order m and depth $\leq \lceil \log m \rceil$ with $T'_a \cap J' = \{a'\}$ and $T'_b \cap J' = \{b'\}$.

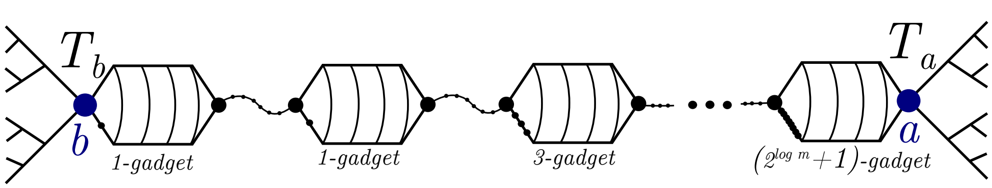

FIG. 2. *Constructing a $(\leq 2^{\log m})$-gadget in the proof of Lemma* 2.15.

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If $s = 1$, we do the same, except we apply Lemma 2.14 to get a 1-gadget J' (rather than a $(2^{1-1}+1)$-gadget which we wouldn't be able to obtain from Lemma 2.14 since $2^{1-1}+1$ is even).

Let $A = T_a$, $B = T'_b$, and $C = J \cup J' \cup T_b \cup T'_a \setminus \{a, b'\}$ to get three sets with $|C| \leq 30000m \leq (\Delta-2)|A|, (\Delta-2)|B|, \beta m/2$. Applying Lemma 2.6 to these three sets gives us a path P from T_a to T'_b avoiding $J \cup J' \cup T_b \cup T'_a \setminus \{a, b'\}$ and satisfying $|P| \leq 8\log m + 2|G|/\beta m \leq 22m^{\frac{1}{3}}$. Notice that since T_a and T'_b are trees with depth $\leq \lceil \log m \rceil$, there are paths P_a and $P_{b'}$ of length $\leq 2\lceil \log m \rceil$ from a and b' to the two endpoints of P. Joining P to P_a and $P_{b'}$ gives a path Q from a to b' of length $\leq 26m^{\frac{1}{3}}$.

We claim that $\hat{J} = J \cup J' \cup Q$ is a $(\leq 2^s)$-gadget in G' with endpoints a' and b. We'll deal with the $s \geq 2$ case first. Let $t \in \{0, \dots, 2^s\}$. We need to find an a' to b path in $\hat{J}$ of order $|\hat{J}| - t$. Since J is a $(\leq 2^{s-1})$-gadget, J contains an a to b path R with $|R| = |J| - (t \bmod 2^{s-1}+1)$. Since J' is a $(2^{s-1}+1)$-gadget, J' contains a' to b' paths R_0 and R_1 with $|R_0| = |J'|$ and $|R_1| = |J'| - 2^{s-1} - 1$. Now, depending on whether $t \geq 2^{s-1}+1$ or not, either RQR_0 or RQR_1 is a path of the required length. If $s = 1$, then a similar argument works. (Since both J and J' are 1-gadgets, we obtain paths Q_0 and Q_1 in J of orders $|J|$ and $|J|-1$ and paths R_0 and R_1 in J' of orders $|J'|$ and $|J'|-1$. Now Q_0QR_0, Q_0QR_1, and Q_1QR_1 are paths of lengths $|\hat{J}|$, $|\hat{J}|-1$, and $|\hat{J}|-2$, respectively.)

Notice that as required by the claim, we have the binary trees T'_a and T_b of order m and depth $\leq \lceil \log m \rceil$ with $T'_a \cap \hat{J} = \{a'\}$ and $T_b \cap \hat{J} = \{b\}$. Finally, we have $|\hat{J}| \leq |J| + |J'| + |Q| \leq (2^{s-1} + s \cdot 2050 \cdot m^{\frac{2}{3}}) + (2^{s-1} + 1 + 2000m^{\frac{2}{3}}) + 26m^{\frac{1}{3}} \leq 2^s + (s+1)2050m^{\frac{2}{3}}$ completing the induction step. ∎

The lemma is immediate from the above claim with $s = r$. ∎

We are now ready to prove Lemma 2.2

Proof of Lemma 2.2. For this lemma we fix $M = N_1 = 10^7$, $\Delta = 40000$, $\beta = 1500000$, and $\tilde{m} = \lambda m$. Notice that $\overline{G}$ is $K^k_{\tilde{m}}$-free. Apply Lemma 2.5 to G with $m = \tilde{m}$ in order to find an integer k' and a subgraph G' of G with $(M-2)(k'-1.5)\tilde{m} \leq |G'| \leq M(k'-1.5)\tilde{m}$ such that G' $(\Delta, \beta, \tilde{m})$-expands into G' and $\overline{G'}$ is $K^{k'}_{\tilde{m}}$-free. Notice that since $|G'| \leq M(k'-1.5)\tilde{m} \leq Mk\tilde{m} \leq M\tilde{m}^{\frac{4}{3}}$ and $M/\beta \leq 7$, we have $|G'|/\beta\tilde{m} \leq 7\tilde{m}^{\frac{1}{3}}$.

Apply Lemma 2.15 twice with $m = \tilde{m}$ and $r = \lceil \log \tilde{m} \rceil$ in order to obtain two disjoint $(\leq \tilde{m})$-gadgets J_1 and J_2 in G' with $|J_1|, |J_2| \leq \tilde{m} + 2050 \cdot \log \tilde{m} \cdot \tilde{m}^{\frac{2}{3}}$. In addition, letting the endpoints of J_i be a_i and b_i we obtain disjoint binary trees T_{a_i} and T_{b_i} of order $\tilde{m}$ and depth $\leq \lceil \log \tilde{m} \rceil$ with $T_{a_i} \cap (J_1 \cup J_2) = \{a_i\}$ and $T_{b_i} \cap (J_1 \cup J_2) = \{b_i\}$. In order to have disjointness, we first apply Lemma 2.15 to the graph G', and then apply Lemma 2.15 to the graph $G' \setminus (J_1 \cup T_{a_1} \cup T_{b_1})$.

Let $A = T_{a_1}$, $B = T_{a_2}$, and $C = J_1 \cup J_2 \cup T_{b_1} \cup T_{b_2} \setminus \{a_1, a_2\}$ to get three sets of vertices with $|C| \leq 30000\tilde{m} \leq (\Delta-2)|A|, (\Delta-2)|B|, \beta\tilde{m}/2$. Applying Lemma 2.6 to these three sets gives us a path P_a from T_{a_1} to T_{a_2} avoiding $J_1 \cup J_2 \cup T_{b_1} \cup T_{b_2} \setminus \{a_1, a_2\}$ and satisfying $|P| \leq 8\log\tilde{m} + 2|G|/\beta\tilde{m} \leq 22\tilde{m}^{\frac{1}{3}}$. Notice that since T_{a_1} and T_{a_2} are trees with depth $\leq \lceil \log \tilde{m} \rceil$, there are paths P_1 and P_2 of length $\leq 2\log\tilde{m}$ from a_1 and a_2 to the two endpoints of P_a. Joining P_a to P_1 and P_2 gives a path Q_a from a_1 to a_2 of order $\leq 26\tilde{m}^{\frac{1}{3}}$. By the same argument we can find a disjoint path Q_b from b_1 to b_2 of order $\leq 26\tilde{m}^{\frac{1}{3}}$ (using $A = T_{b_1}$, $B = T_{b_2}$, and $C = J_1 \cup J_2 \cup T_{b_1} \cup T_{b_2} \cup Q_a \setminus \{b_1, b_2\}$).

Now, we have two $(\leq \lambda m)$-gadgets J_1 and J_2 of order $\leq \lambda m + 2050 \cdot \log \lambda m \cdot (\lambda m)^{\frac{2}{3}} \leq (\lambda + \mu)m$ (using $\mu m \geq 4100(\lambda m)^{\frac{3}{4}}$), as well as two paths Q_a and Q_b between their endpoints with $|Q_a|, |Q_b| \leq 26(\lambda m)^{\frac{1}{3}}$.

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Notice that the following holds:

$$0 \le |J_1 \cup J_2 \cup Q_a \cup Q_b| - (\lambda + 2\mu)m + 2 \le \lambda m.$$

Indeed, the left-hand inequality follows from $|J_1|, |J_2| \ge \lambda m$ and $\lambda \ge 2\mu$, whereas the right-hand inequality comes from $\mu m \ge 4100(\lambda m)^{\frac{3}{4}}$ and $|Q_a|, |Q_b| \le 26(\lambda m)^{\frac{1}{3}}$, $(|J_1| - \lambda m), (|J_2| - \lambda m) < 2050(\lambda m)^{\frac{3}{4}}$.

Therefore, since J_1 is a $(\le \lambda m)$-gadget, there is a path Q_1 from a_1 to b_1 in J_1 of order $|J_1| - \big(|J_1 \cup J_2 \cup Q_a \cup Q_b| - (\lambda + 2\mu)m + 2\big)$. Notice that $|J_2 \cup Q_a \cup Q_b \cup Q_1| = (\lambda + 2\mu)m - 2$ and $|J_2| < (\lambda + \mu)m$. Therefore we can choose two vertices a and b on the path $Q_aQ_1Q_b$ such that the interval Q of $Q_aQ_1Q_b$ from a to b has exactly μm vertices. Let J be J_2 together with the two segments of $Q_aQ_1Q_b$ outside the internal vertices of Q. Noting that connecting paths to the endpoints of a $(\le t)$-gadget produces another $(\le t)$-gadget, we have a $(\le \lambda m)$-gadget J with $|J| = (\lambda + \mu)m$ and an internally disjoint path Q of order μm joining its endpoints. □

2.5. Gadget-cycles. We'll use gadgets by joining many of them into a cycle and then using the property of a $(\le k)$-gadget to shorten the cycle into one of prescribed length. The following definition captures the notion of a cycle containing many gadgets on it.

DEFINITION 2.17. *An (a, b, m)-gadget-cycle C is a set of disjoint gadgets $J_1, \dots, J_t$ together with a set of disjoint paths $Q_1, \dots, Q_t$ with the following properties:*

(i) *J_i has endpoints a_i and b_i. Q_i goes from b_i to $a_{i+1 \pmod t}$. Other than at these vertices, the paths do not intersect the gadgets.*

(ii) *$|J_i| \le m$ for each $i = 1, \dots, t$.*

(iii) *$|\bigcup_{i=1}^t (J_i \cup Q_i)| \ge b$.*

(iv) *There is a number k such that each J_i is a $(\le k)$-gadget with $\left|\bigcup_{i=1}^t (J_i \cup Q_i)\right| - tk \le a$.*

See Figure 3 for a diagram of a gadget-cycle. Notice that if C is an (a, b, m)-gadget-cycle C, then we have $|C| \ge b$. Notice that any (a, b, m)-gadget-cycle C is also an $(a, |C|, m)$-gadget-cycle. If C is a gadget-cycle as in Definition 2.17, we say that it *contains* the gadgets $J_1, \dots, J_t$. If P_i is the path in J_i of order $|J_i|$ for $i = 1, \dots, t$, then we will sometimes identify C with the cycle with vertex sequence $P_1Q_1P_2Q_2 \dots P_tQ_t$.

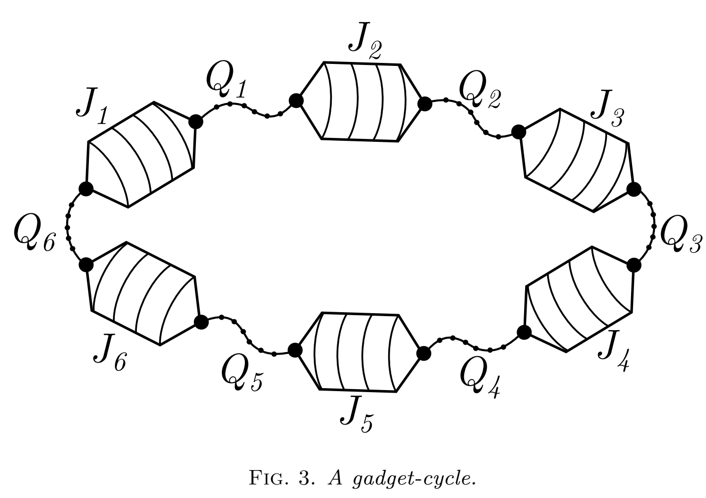

FIG. 3. *A gadget-cycle.*

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The following simple lemma shows that gadget-cycles contain cycles of all lengths between the parameters a and b.

LEMMA 2.18. *For any n with $a \leq n \leq b$, every (a,b,m)-gadget-cycle contains a cycle of length n.*

Proof. Let $J_1,\dots,J_t$ be $(\leq k)$-gadgets and $Q_1,\dots,Q_t$ paths as in the definition of (a,b,m)-gadget-cycle. Choose numbers $k_1,\dots,k_t \in \{1,\dots,k\}$ such that $\left|\bigcup_{i=1}^t (J_i \cup Q_i)\right| - \sum_{k=1}^t k_i = n$ (parts (iii) and (iv) of the definition of "gadget-cycle" ensure that we can do this). Now since each J_i is a $(\leq k)$-gadget, it contains a path P_i between its endpoints of length $|J_i| - k_i$. Now $\bigcup_{i=1}^t P_i \cup Q_i$ is a cycle of length n. ∎

The following lemma allows us to join two gadget-cycles into a larger gadget-cycle.

LEMMA 2.19. *Suppose that we have an (a_1,b_1,m)-gadget-cycle C_1, a vertex-disjoint (a_2,b_2,m)-gadget-cycle C_2, and $r \geq 16$ vertex-disjoint C_1 to C_2 paths P_1, $\dots, P_r$ of length $\leq \ell$. Then for some $i,j \leq r$ there is an (a,b,m)-gadget-cycle C with $V(C) \subseteq C_1 \cup C_2 \cup P_i \cup P_j$, $|C| \geq (1-\frac{2}{\sqrt{r}})(|C_1|+|C_2|) \geq (|C_1|+|C_2|)/2$, and*

$$a = a_1 + a_2 + 4m + 2\ell,$$
$$b = (b_1+b_2)\left(1-\frac{2}{\sqrt{r}}\right).$$

Proof. Without loss of generality, we can suppose that $b_1 = |C_1|$ and $b_2 = |C_2|$. For $i = 1,\dots,r$, by possibly replacing each P_i by a shorter path, we can assume that each P_i is internally outside $C_1 \cup C_2$. Let $c_1^1,\dots,c_1^{|C_1|}$ be the vertex sequence of C_1 and $c_2^1,\dots,c_2^{|C_2|}$ the vertex sequence of C_2. For a path P_i, let $x(P_i) = (s,t)$ where c_1^s and c_2^t are the endpoints of P_i. Notice that $x(P_i) \in [1,|C_1|] \times [1,|C_2|]$ for each i. There must be two paths P_i and P_j with $x(P_i)$ and $x(P_j)$ within L^1 distance $2(|C_1|+|C_2|)/\sqrt{r}$. (Otherwise, the r L^1-balls of radius $(|C_1|+|C_2|)/\sqrt{r}$ would all be disjoint. This gives a contradiction to the total volume of these balls being less than $|C_1| \cdot |C_2|$.)

Let S be the set of $\leq 2(b_1+b_2)/\sqrt{r}$ vertices of C_1 and C_2 between the endpoints of P_i and P_j. Let C be the gadget-cycle on $(C_1 \cup C_2 \cup P_i \cup P_j) \setminus S$ formed by joining C_1 and C_2 with P_i and P_j and discarding the vertices of S. The gadgets of C are all the gadgets of C_1 or C_2 which are completely contained in C. The paths in C are all the other vertices in C.

Notice that there are at most four gadgets in C_1 and C_2 which can intersect C but not be gadgets in C (the only way such a gadget can arise if one of the endpoints of P_i or P_j is contained in it). From this we see that C is an (a,b,m)-gadget-cycle with $a = a_1 + a_2 + 4m + |P_i| + |P_j| \leq a_1 + a_2 + 4m + 2\ell$ and $b = |C_1| + |C_2| + |P_i| + |P_j| - |S| \geq b_1 + b_2 - 2(b_1+b_2)/\sqrt{r}$. We also have $|C| \geq |C_1| + |C_2| + |P_i| + |P_j| - |S| \geq (b_1+b_2)(1-\frac{2}{\sqrt{r}}) = (|C_1|+|C_2|)(1-\frac{2}{\sqrt{r}}) \geq (|C_1|+|C_2|)/2$. ∎

3. Ramsey numbers.

In this section we will prove Theorem 1.4. The only results from the previous section which we will use here are Lemmas 2.2, 2.18, and 2.19. We will also employ Theorem 1.6 in this section. However, it is worth noting that the weaker result $R(C_{\leq n}, K_m^k) \leq O(n)$ would also suffice in all our applications of Theorem 1.6.

The structure of this section is as follows. In section 3.1 we introduce expanders. The expanders which we introduce here are slightly different from the ones we used

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in the previous section. In section 3.2 we prove the special case of Theorem 1.4 when $k = 2$. Since the full proof of Theorem 1.4 is inductive, the "$k = 2$" case will serve as the initial case for our induction. In section 3.3 we prove Theorem 1.4.

3.1. Expanders. We will use the following notion of expansion.

DEFINITION 3.1. *Let $H \subseteq G$ be an induced subgraph of a graph G. We say that H is an (d, m, n)-expander in G if the following hold:*
(i) $|N_H(S)| \geq d|S|$ *for* $S \subseteq V(H)$ *with* $|S| < m$.
(ii) $|N_G(S) \cup S| \geq n$ *for* $S \subseteq V(H)$ *with* $|S| \geq m$.

Notice that if H is an (d, m, n)-expander in G and we have $G' \supseteq G$, $d' \leq d$, $m' \geq m$, $n \geq dm'$, and $n' \leq n$, then H is an (d', m', n')-expander in G'.

The following lemma shows that if the complement of a graph is $K_{m,m}$-free, and large sets expand to n, then the graph contains a large (d, m', n)-expander.

LEMMA 3.2. *Suppose that we have integers n, m, and d with $n > (d+2)(d+3)m$, a graph G, and a set of vertices $U \subseteq G$ with $|U| \geq (d+3)^2m$. Suppose that $\overline{G[U]}$ is $K_{m,m}$-free and that $|N_G(S) \cup S| \geq n$ for every $S \subseteq U$ with $|S| \geq m$.*

Then there is a set $B \subseteq U$ with $|B| < m$ such that $G[U \setminus B]$ is a $(d, (d+2)m, n)$-expander in $G \setminus B$.

Proof. Let B be the largest subset of U with $|B| \leq (d+3)m$ and $|N_U(B) \setminus B| < (d+1)|B|$. Since $|N_U(B) \cup B| \leq (d+2)(d+3)m$ we have that $|U \setminus (N_U(B) \cup B)| \geq m$. Since there are no edges between B and $U \setminus (N_U(B) \cup B)$, the $K_{m,m}$-freeness of $\overline{G[U]}$ implies that $|B| < m$. We show that $G[U \setminus B]$ satisfies (i) and (ii) of the definition of "$(d, (d+2)m, n)$ expander in $G \setminus B$."

To see that (i) holds, let $S \subseteq U \setminus B$ be a subset with $|S| < (d+2)m$. Notice that we have $|N_U(S) \setminus (S \cup B)| \geq (d+1)|S|$ since otherwise $S \cup B$ would be a larger set with $|S \cup B| \leq (d+3)m$ and $|N_U(S \cup B) \setminus (S \cup B)| \leq |N_U(S) \setminus (S \cup B)| + |N_U(B) \setminus B| < (d+1)|S \cup B|$ (contradicting the maximality of B). This shows that $|N_{U \setminus B}(S)| \geq |N_U(S) \setminus (S \cup B)| \geq d|S|$.

To see that (ii) holds, let $S \subseteq U \setminus B$ be a subset with $|S| \geq (d+2)m$. We have $|N_U(B) \cap S| \leq |N_U(B) \setminus B| \leq (d+1)|B| \leq (d+1)m \leq |S| - m$, which implies that $|S \setminus N_U(B)| \geq m$. Therefore, using the assumption of the lemma we get

$$\begin{aligned} |N_{G \setminus B}(S) \cup S| &\geq |N_{G \setminus B}(S \setminus N_U(B)) \cup (S \setminus N_U(B))| \\ &= |N_G(S \setminus N_U(B)) \cup (S \setminus N_U(B))| \\ &\geq n. \end{aligned}$$ ◻

The following lemma shows that expanders are highly connected.

LEMMA 3.3. *Let G be a graph with $\overline{G}$ $K_{m,m}$-free and H a $(d+1, m, n)$-expander in G. Then H is d-connected.*

Proof. Let x, y be two vertices in H and S a set of $d-1$ vertices in $H \setminus \{x, y\}$. To prove the lemma, it is sufficient to find an x to y path avoiding S. Define $N^r_{H \setminus S}(v)$ to be the rth neighbourhood of a vertex $v \in H$, i.e., the set of all vertices in $H \setminus S$ at distance $\leq r$ from v in $H \setminus S$. From the definition of $(d+1, m, n)$-expander, we have that $|N^r_{H \setminus S}(v)| \geq \min(2^r, m)$ for all $v \in H \setminus S$. Therefore we have $|N^{\log m}_{H \setminus S}(x)|$, $|N^{\log m}_{H \setminus S}(y)| \geq m$.

We claim that $N^{\log m+1}_{H \setminus S}(x) \cap N^{\log m+1}_{H \setminus S}(y) \neq \emptyset$. If $N^{\log m}_{H \setminus S}(x) \cap N^{\log m}_{H \setminus S}(y) \neq \emptyset$, then this is obvious. Otherwise by $K_{m,m}$-freeness of $\overline{G}$ there is an edge between $N^{\log m}_{H \setminus S}(x)$

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and $N_{H\setminus S}^{\log m}(y)$ which is equivalent to $N_{H\setminus S}^{\log m+1}(x)\cap N_{H\setminus S}^{\log m+1}(y)\neq\emptyset$. We get an $x-y$ path avoiding S of length $\leq 2\log m+1$ by joining paths from x and y to a vertex in $N_{H\setminus S}^{\log m+1}(x)\cap N_{H\setminus S}^{\log m+1}(y)$. ∎

The same proof as above also gives the following lemma, which shows that any two vertices are connected by a short path in an expander.

LEMMA 3.4. *Let G be a graph with $\overline{G}$ $K_{m,m}$-free and H a $(3,m,n)$-expander in G. Then for any $x,y\in H$, there is an x – y path P in H with $|P|\leq 3\log m$.*

It is also possible to connect given vertices by long paths in an expander.

LEMMA 3.5. *Let G be a graph with $\overline{G}$ $K_{m,m}$-free and H a $(3,m,n)$-expander in G with $|H|\geq 61m$. Then for any $x,y\in H$, there is an x – y path P in H with $10m\leq|P|\leq 12m$.*

Proof. Notice that $H-x-y$ contains a cycle C with $|C|\geq 20m$ (e.g., by Theorem 1.6). By Lemma 3.3 combined with Menger's theorem, there are two disjoint paths P_x and P_y from x and y, respectively, to C. Joining P_x and P_y to the longer segment of C between $P_x\cap C$ and $P_y\cap C$ gives an x to y path P of length $\geq 10m$. If $P>12m$, then by the $K_{m,m}$-freeness of $\overline{G}$, P has a chord whose endpoints are at distance at most $\leq 2m$ on P. By repeatedly shortening P with such chords, we obtain a path of length between $10m$ and $12m$. ∎

3.2. $R(C_n,K_{m_1,m_2})$. The goal of this section is to prove the $k=2$ case of Theorem 1.4. This serves as an initial case of the induction in the full proof of the theorem.

An important tool which we will need is the Pósa rotation-extension technique. Let $P=p_1p_2\dots p_t$ be a path in a graph G. We say that a path Q is a rotation of P if the vertex sequence of Q is $p_1p_2\dots p_{i-1}p_tp_{t-1}\dots p_{i+1}p_i$ for some i. Notice that for Q to be a path, the edge p_tp_{i-1} must be present. We say that a path Q is *derived* from P if there is a sequence of paths $P_0=P,P_1,\dots,P_s=Q$ with P_i being a rotation of P_{i-1} for each i. We say that a vertex x is an *ending vertex* for P if it is the final vertex of some path derived from P. The following lemma from [3] is a variation of a result of Pósa from [22].

LEMMA 3.6. *For $v\in V(G)$, let P be a maximum length path in G starting at v. Let S be the set of ending vertices for P. Then $|N_G(S)|\leq 3|S|$.*

The following lemma could be seen as a strengthening of the statement that "$R(C_n,K_{m,m})\leq n-1+m$"—it says that in a graph whose complement is $K_{m,m}$-free which satisfies certain other conditions, we can connect a given pair of vertices by a path of prescribed length. We will use this lemma at several points in the proof of Theorem 1.4.

LEMMA 3.7. *There is a constant $N_2=2\cdot 10^{49}$ such that the following holds. Let n and m be integers with $n\geq N_2m$ and $m\geq 8$. Let G be a graph with $\overline{G}$ $K_{m,m}$-free and $|N_G(A)\cup A|\geq n$ for every $A\subseteq V(G)$ with $|A|\geq m$. Let x and y be two vertices in G and P an x to y path with $|P|\geq 8m$.*

Then there is an x to y path of order n in G.

Proof. For this lemma we fix $\mu=10^{20}$, $\lambda=10^{21}$, $k=2$ and note that $N_2\geq 2N_1k\lambda\mu$, where N_1 is the constant from Lemma 2.2.

Without loss of generality, we may assume that $|P|\leq 10m$. (Indeed if $|P|>10m$, then by $K_{m,m}$-freeness of $\overline{G}$, P has a chord whose endpoints are at distance $\leq 2m$

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along P. Shortening P with this chord gives a shorter path of length $\geq 8m$. Therefore there is an x to y path with length between $8m$ and $10m$.)

By Lemma 2.2, we see that $G \setminus P$ contains a $(\leq \lambda m)$-gadget J of order $(\lambda + \mu)m$ with endpoints a and b, together with an internally disjoint path Q of order μm from a to b. By $K_{m,m}$-freeness of $\overline{G}$, we can find two disjoint edges from the middle $m+1$ vertices of Q to the middle $m+1$ vertices of P. By deleting the segments of P and Q between these edges, we get two paths P_x and P_y of length $\geq 4m$ going from x and y to a and b. Without loss of generality, we can suppose that a and b are labeled so that P_x connects x to a and P_y connects y to b.

Apply Lemma 3.2 to G with $U = G \setminus (V(P_x) \cup V(P_y) \cup V(J))$ and $d = 4$ in order to find a set $B \subseteq U$ with $|B| < m$ such that the subgraph $H = G \setminus (V(P_x) \cup V(P_y) \cup V(J) \cup B)$ is a $(4, 6m, n)$-expander in $G \setminus B$. Since $|V(P_x) \cup V(P_y) \cup V(J) \cup B| \leq |P| + |Q| + |J| + |B| \leq 10^{22}m$, we have that $|H| \geq m$.

Let P_x have vertex sequence $x = p_0, p_1, p_2, \ldots, p_t$. By $K_{m,m}$-freeness of $\overline{G}$, there is an edge between some $p_r \in \{p_{m+1}, \ldots, p_t\}$ and some vertex $v \in H$. Let R be the longest path in H starting from v and S be the set of ending vertices of P. By maximality of $|R|$, we have that $N_H(S) \subseteq R$. Lemma 3.6 implies that $|N_H(S)| \leq 3|S|$. By property (i) of H being a $(4, 6m, n)$-expander in $G \setminus B$, we have that $|S| \geq 6m$. Therefore by property (ii) of H being a $(4, 6m, n)$-expander in $G \setminus B$ we have $|(N_G(S) \cup S) \cap (R \cup P_x \cup P_y \cup J)| = |N_{G \setminus B}(S) \cup S| \geq n$.

Notice that by $K_{m,m}$-freeness of $\overline{G}$, S has neighbors in $\{p_0, \ldots, p_{r-1}\}$. Let p_i be the last neighbor of S in this set. Let R' be a path derived from R which ends with a neighbor of p_i. Let P' be the x to y path formed by joining $p_0, \ldots, p_i$ to R' to $p_r, p_{r+1}, \ldots, p_t$ to J to P_y. Notice that $(N_G(S) \cup S) \cap (R \cup P_x \cup P_y \cup J) = (N_G(S) \cup S) \cap P'$ (this comes from $(R \cup P_x \cup P_y \cup J) \setminus P' = \{p_{i+1}, \ldots, p_{r-1}\}$ and the fact that there are no edges from S to $\{p_{i+1}, \ldots, p_{r-1}\}$ by the definition of p_i). Together with $|(N_G(S) \cup S) \cap (R \cup P_x \cup P_y \cup J)| \geq n$, this gives $|P'| \geq n$. The path P' is of the form $P'_x J P_y$ for some paths P'_x and P_y. Let P'' be the shortest path with $|P''| \geq n$ and of the form $P''_x J P''_y$ for some paths P''_x and P''_y. Notice that we must have $|P''| \leq n + 5m \leq n + \lambda m$ since otherwise, using $|J| = (\lambda + \mu)m \leq n$ and the $K_{m,m}$-freeness of $\overline{G}$, either P''_x or P''_y has a chord whose endpoints are at distance $\leq 2m$ on P'' (contradicting the minimality of $|P''|$). Now using the property of the $(\leq \lambda m)$-gadget J we can find an a to b path J' in J of order $|J| - (|P''| - n)$. Joining J' to P''_x and P''_y we obtain an x to y path of order n. ∎

From the above lemma it is easy to find $R(C_n, K_{m_1,m_2})$.

Corollary 3.8. *There is a constant $N_2 = 2 \cdot 10^{49}$ such that the following holds. Let n, m_1, m_2 be integers with $m_2 \geq m_1$, $m_2 \geq 8$, and $n \geq N_2 m_2$. Then we have $R(C_n, K_{m_1,m_2}) = n + m_1 - 1$.*

Proof. From Lemma 1.1, we have $R(C_n, K_{m_1,m_2}) \geq n + m_1 - 1$. Therefore it remains to show that $R(C_n, K_{m_1,m_2}) \leq n + m_1 - 1$.

Let G be the red color class of a 2-edge-coloured K_{n+m_1-1}. Suppose that K_{n+m_1-1} contains no blue K_{m_1,m_2}, i.e., that $\overline{G}$ is K_{m_1,m_2}-free.

By Theorem 1.6 there are two adjacent vertices x and y with a path of length $\geq n \geq 8m_2$ between them. Since $\overline{G}$ is K_{m_1,m_2}-free and $|G| = n + m_1 - 1$ we have that $|N_G(A) \cup A| \geq n$ for any $A \subseteq V(G)$ with $|A| \geq m_2$. Also since $m_2 \geq m_1$, $\overline{G}$ is K_{m_2,m_2}-free. Therefore, by Lemma 3.7 applied with $m = m_2$, there is a path of order n from x to y which together with the edge xy gives a cycle of order n in G (and hence a red cycle of order n in the original graph). ∎

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3.3. $R(C_n, K_{m_1,\dots,m_k})$. Here we prove Theorem 1.4. First we need two intermediate lemmas.

Notice that Theorem 1.4 implies that $R(C_n, K_m^k) \le (k-1)(n-1) + m$. The following lemma shows that a much better bound holds as long as the red color class of the 2-colored complete graph is highly connected in a certain sense.

LEMMA 3.9. *There is a constant $N_3 = 10^{56}$ such that the following holds. Suppose that we have m, n, and k satisfying $n \ge N_3 m$ and $m \ge k^{20}$. Let G be a graph with $|G| \ge 0.07kn + n$. Suppose that for any two sets of vertices A, B of order $2m$, there are at least k^{20} disjoint paths from A to B.*

Then either G contains a cycle of length n or $\overline{G}$ contains a copy of K_m^k.

Proof. For this lemma we fix $\lambda = 10^{24}$, $\mu = 10^{21}$, and notice that $N_3 = 10^{49} N_1$, where N_1 is the constant from Lemma 2.2. For $k = 2$, the lemma is weaker than Corollary 3.8, so we will assume that $k \ge 3$. Suppose that we have a graph G as in the lemma with $\overline{G}$ K_m^k-free. We will find a length n cycle in G.

In G, select a maximal collection of disjoint $(\le \lambda m)$-gadgets of order $(\lambda + \mu)m$ together with length μm paths joining their endpoints, i.e., choose disjoint $(\le \lambda m)$-gadgets $J_1, \dots, J_t$ of order $(\lambda + \mu)m$ as well as internally disjoint paths $Q_1, \dots, Q_t$ of order μm with Q_i going between the endpoints of J_i, such that t is as large as possible. Let $U_1 = G \setminus \bigcup_{i=1}^{t} V(J_i) \cup V(Q_i)$. By maximality of t, $G[U_1]$ contains no $(\le \lambda m)$-gadget of order $(\lambda + \mu)m$ with a path of length μm joining its endpoints. By Lemma 2.2, we have that $|U_1| \le (N_1 \lambda \mu k)m$ (since $m \ge k^{20}$, $\lambda = 10^{24}$, and $\mu = 10^{21}$ imply $m \ge k^3$, $\lambda \ge 2\mu$, and $\mu m \ge 4100(\lambda m)^{\frac{3}{4}}$). Using $N_1 \lambda \mu \le 0.005 N_3$ and $n \ge N_3 m$ we get $|U_1| \le (N_1 \lambda \mu k)m \le 0.005kn$. This implies $|J_1 \cup \dots \cup J_t \cup Q_1 \cup \dots \cup Q_t| \ge 0.06kn + n \ge 1.06n$.

Construct an auxiliary graph H on $[t]$ with ij and edge if there are at least 12 disjoint edges from Q_i to Q_j. Using $|J_1 \cup \dots \cup J_t \cup Q_1 \cup \dots \cup Q_t| \ge 0.06kn$ and $n \ge N_3 m$, we have $|H| \ge |G \setminus U_1| / (\lambda + 2\mu)m \ge 900k$. The reason for defining this graph H is that paths in H correspond to gadget-cycles in G. The following claim makes this precise.

CLAIM 3.10. *Let P be a path in H. Then there is an $(a, b, 2\lambda m)$-gadget-cycle contained in $\bigcup_{v \in P} (J_v \cup Q_v)$ where $a = 0.01 \sum_{v \in P} |J_v \cup Q_v|$ and $b = 0.99 \sum_{v \in P} |J_v \cup Q_v|$.*

Proof. Without loss of generality, we may assume vertices are labeled so that P has vertex sequence $1, 2, \dots, |P|$

Notice that it is sufficient to find a gadget-cycle C containing all the gadgets $J_1, \dots, J_{|P|}$ and with $V(C) \subseteq \bigcup_{i=1}^{|P|} V(J_i) \cup V(Q_i)$. Indeed such a gadget-cycle is always an $(a, b, (\lambda + \mu)m)$-gadget-cycle with $a = \sum_{i=1}^{|P|} (|J_i \cup Q_i| - \lambda m)$ and $b = \sum_{i=1}^{|P|} |J_i|$. Using $|J_i| = (\lambda + \mu)m$ and $|Q_i| = \mu m$, we have that $a \le 0.01 \sum_{i=1}^{|P|} |J_i \cup Q_i|$ and $b \ge 0.99 \sum_{i=1}^{|P|} |J_i \cup Q_i|$ (for these we use $\lambda \ge 200\mu$). It remains to show that such a gadget-cycle containing all the gadgets $J_1, \dots, J_{|P|}$ exists.

For each $i = 1, \dots, |P| - 1$, let M_i be the matching of size 12 from Q_i to Q_{i+1} (which exists since $\{i, i+1\}$ is an edge in H). Fix some orientation of Q_i for each i.

Notice that for any two sets of distinct numbers S and T, there are two subsets $S' \subseteq S$ and $T' \subseteq T$ with $|S'| \ge |S|/2 - 1$ and $|T'| \ge |T|/2 - 1$ for which we have either "$s < t$ for all $s \in S', t \in T'$" or "$t < s$ for all $s \in S', t \in T'$." For $i = 1, 2, \dots, |P| - 1$ we apply this repeatedly with $S = Q_i \cap M_{i-1}$ and $T = Q_i \cap M_i$ in order to obtain new matchings $M_1' \subset M_1, \dots, M_{|P|-1}' \subset M_{|P|-1}$ of size 2 with the property that the

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endpoints of M'_{i-1} in Q_i are either all to the left or all to the right of the endpoints of M'_i in Q_i.

Now, for each i, we delete the segment of Q_i between the endpoints of M'_{i-1} and the segment of Q_i between the endpoints of M'_i. Adding the edges of $M_1, \dots, M_{|P|}-1$ to the graph produces the required gadget-cycle containing all the gadgets $J_1, \dots, J_{|P|-1}$. □

We will often use the fact that the gadget-cycle produced by Claim 3.10 has order at least $0.99 \sum_{v \in P} |J_v \cup Q_v|$ (which holds since any (a, b, m)-gadget-cycle has order at least b).

Using the K_m^k-freeness of $\overline{G}$ we obtain that H has small independence number.

Claim 3.11. $\alpha(H) \le k - 1$.

Proof. Suppose for the sake of contradiction that H contains an independent set I of order k. For $i, j \in I$, let $M_{i,j}$ be a maximal matching in G between Q_i and Q_j. From the definition of edges in H we have that $M_{i,j} \le 12$ for $i, j \in I$. For $i \in I$, let $Q'_i = Q_i \setminus \bigcup_{i,j \in I} V(M_{i,j})$. Using $m \ge k^{10}$ we have $|Q_i| = \mu m \ge m + 12k$, which implies that $|Q'_i| \ge m$. By maximality of $M_{i,j}$ there are no edges between Q'_i and Q'_j. But this means that $\overline{G\left[\bigcup_{i \in I} Q'_i\right]}$ contains a copy of K_m^k contradicting the K_m^k-freeness of $\overline{G}$. □

The following is a variant of the well-known fact that a graph can be covered by $\alpha(G)$ vertex-disjoint paths.

Claim 3.12. *There are $k - 1$ vertex-disjoint paths $P_1, \dots, P_{k-1}$ in H with $|H| - |P_1| - \cdots - |P_{k-1}| \le 200k$ and $|P_i| \ge 200$ for $i = 1, \dots, k - 1$.*

Proof. Choose vertex-disjoint paths $Q_1, \dots, Q_t$ in H covering $V(H)$ with t as small as possible. Without loss of generality, suppose that we have $|Q_1| \le |Q_2| \le \cdots \le |Q_t|$. By minimality of t, we have that the starting vertices of $Q_1, \dots, Q_t$ form an independent set (otherwise we could join two of the paths together to obtain a smaller collection of paths). Claim 3.11 implies that $t \le k - 1$.

Let r be the index with $|Q_{r-1}| < 200$ and $|Q_r| \ge 200$ (possibly with $r = 0$.) Let $U_1 = Q_1 \cup \cdots \cup Q_{r-1}$ to obtain a set with $|U_1| \le 200k$. Using $|Q_r \cup \cdots \cup Q_{k-1}| = |H| - |U_1| \ge 900k - 200k = 700k$, it is possible to break some of the paths $Q_r, \dots, Q_{k-1}$ into shorter paths in order to obtain a collection of exactly $k - 1$ paths $P_1, \dots, P_{k-1}$ of orders ≥ 200 (to do this notice that in any collection of $< k - 1$ paths of total order $\ge 700k$, there must be a path of order ≥ 700). □

Let $P_1, \dots, P_{k-1}$ be the paths from the above claim and assume that they are ordered such that $|P_1| \ge |P_2| \ge \cdots \ge |P_{k-1}|$. Let $U_2 = \bigcup_{v \in H \setminus (P_1 \cup \cdots \cup P_{k-1})} J_v \cup Q_v$, and observe that from Claim 3.12 we have that $|U_2| \le 200(\lambda + 2\mu)mk \le 0.005kn$.

Suppose that $\sum_{v \in P_1} |J_v \cup Q_v| \ge 2n$. Then since for each v, $|J_v \cup Q_v| \le (\lambda + 2\mu)m \le n$, there is a path $P \subseteq P_1$ with $3n \ge \sum_{v \in P} |J_v \cup Q_v| \ge 2n$. By Claim 3.10, there is a $\left(0.01 \sum_{v \in P} |J_v \cup Q_v|, 0.99 \sum_{v \in P} |J_v \cup Q_v|, 2\lambda m\right)$-gadget-cycle in G. Notice that we have

$$0.01 \sum_{v \in P} |J_v \cup Q_v| \le 0.01 \cdot 3n \le n \le 0.99 \cdot 2n \le 0.99 \sum_{v \in P} |J_v \cup Q_v|.$$

Lemma 2.18 implies that G contains a cycle of length n.

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Suppose that $\sum_{v\in P_1}|J_v\cup Q_v|\le 2n$. For $i=1,\dots,k-1$, let C_i^g be the gadget-cycle produced out of the path P_i using Claim 3.10. We have

$$C_i^g \text{ is a } \left(0.01\sum_{v\in P_i}|J_v\cup Q_v|, 0.99\sum_{v\in P_i}|J_v\cup Q_v|, 2\lambda m\right)\text{-gadget-cycle.} \tag{4}$$

Let $U_3=\bigcup_{i=1}^{k-1}\left(\bigcup_{v\in P_i}(J_v\cup Q_v)\right)\setminus C_i^g$. Notice that from (4), we have the inequality $|C_i^g|\ge 0.99|\bigcup_{v\in P_i}(J_v\cup Q_v)|$ for $i=1,\dots,k-1$, which together with $\sum_{v\in P_i}|J_v\cup Q_v|\le 2n$ implies that $|U_3|\le 0.02kn$. Let $U=U_1\cup U_2\cup U_3=G\setminus\bigcup_{i=1}^{k-1}C_i^g$ to get a set with $|U|\le 0.03kn$. Notice that as a consequence of (4), $|U|\le 0.03kn\le 0.1|G|$, and $|P_1|\ge\cdots\ge|P_{k-1}|$ we have $|C_1^g|\ge|G|/2k$. Using $|P_i|\ge 200$ and (4) we have that for all i

$$|C_i^g|\ge 0.99\sum_{v\in P_i}|J_v\cup Q_v|\ge 0.99\cdot 200(\lambda m+2\mu m-2)\ge 2m. \tag{5}$$

For a permutation σ of $[k-1]$, we set $S_i^\sigma=\sum_{j=1}^{i}\sum_{v\in P_{\sigma(j)}}|J_v\cup Q_v|$ for $i=1,\dots,k-1$. Notice that $S_{k-1}^\sigma=|G|-|U_1\cup U_2|$ always holds. Using the fact that $|P_i|\ge 200$ for each i, we always have $S_i^\sigma\ge 199(\lambda+2\mu)im$.

Claim 3.13. *There is a sequence of gadget-cycles $D_1,\dots,D_{k-1}$ as well as a permutation σ of $[k-1]$ with the following properties:*
(a) $\sigma(1)=1$.
(b) *For each i we have $D_i\subseteq U\cup C_{\sigma(1)}^g\cup C_{\sigma(2)}^g\cup\cdots\cup C_{\sigma(i)}^g$.*
(c) *For each i we have $|D_i|\ge 2m$.*
(d) *D_i is an $(a_i,b_i,2\lambda m)$-gadget cycle for*

$$a_i=0.01S_i^\sigma+8(i-1)\lambda m+2(i-1)|G|/k^7,$$
$$b_i=0.99(1-2k^{-6})^{i-1}S_i^\sigma.$$

Proof. Set $D_1=C_1^g$ and $\sigma(1)=1$. Now for $i=1$, (a) and (b) hold trivially, (c) comes from (5), and (d) is equivalent to the "$i=1$" case of (4). For $i\ge 2$ we will recursively construct D_i, $\sigma(i)$ from $D_1,\dots,D_{i-1}$, and $\sigma(1),\dots,\sigma(i-1)$. Suppose that we have already constructed $D_1,\dots,D_{i-1}$, and $\sigma(1),\dots,\sigma(i-1)$ satisfying (a)–(d). We construct D_i and $\sigma(i)$ as follows.

By (c), we have $|D_{i-1}|\ge 2m$ and by (5) we have $|\bigcup_{j\in[k-1]\setminus\{\sigma(1),\dots,\sigma(i-1)\}}C_i^g|\ge 2m$. Using the assumption of the lemma, we find at least k^{20} disjoint paths from D_{i-1} to $\bigcup_{j\in[k-1]\setminus\{\sigma(1),\dots,\sigma(i-1)\}}C_i^g$ internally contained outside these sets. Since the paths are all disjoint, there is a subcollection of k^{20-7} of them with length $\le|G|/k^7$. In addition, there is a further subcollection of k^{20-7-1} of them which go from D_{i-1} to C_j^g for some particular j. To get D_i, we apply Lemma 2.19 to this collection of k^{20-7-1} paths, the gadget-cycles $C_1=C_j^g$ and $C_2=D_{i-1}$ and with the parameters $m'=2\lambda m$, $r=k^{20-7-1}$, and $\ell=|G|/k^7$. We set $\sigma(i)=j$.

Now (b) holds as a consequence of "$V(C)\subseteq C_1\cup C_2\cup P_i\cup P_j$" in Lemma 2.19 and (c) holds as a consequence of "$|C|\ge(|C_1|+|C_2|)/2$" in Lemma 2.19.

Recall that $C_1=C_j^g$ is a $(0.01(S_i^\sigma-S_{i-1}^\sigma),0.99(S_i^\sigma-S_{i-1}^\sigma),2\lambda m)$-gadget-cycle by (4) and $C_2=D_{i-1}$ is a $(a_{i-1},b_{i-1},2\lambda m)$-gadget-cycle by (d) holding for D_{i-1}. Thus (d) holds for D_i from the application of Lemma 2.19 together with $m'=2\lambda m$, $r=k^{12}$, $\ell=|G|/k^7$, and "$(b_{i-1}+0.99(S_i^\sigma-S_{i-1}^\sigma))(1-2k^{-6})\ge b_i$." ∎

From here, fix σ to be the permutation from Claim 3.13. Notice that $S_1^\sigma\ge|C_1^g|\ge|G|/2k$ implies $2(i-1)|G|/k^7\le 0.1S_1^\sigma$, while $S_i^\sigma-S_1^\sigma\ge 199(\lambda+2\mu)(i-1)m$ implies

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$8(i-1)\lambda m \le 0.1(S_i^\sigma - S_1^\sigma)$. Combining these gives $8(i-1)\lambda m + 2(i-1)|G|/k^7 \le 0.1 S_i^\sigma$ and hence $a_i \le 0.11 S_i^\sigma$. We also have $b_i \ge 0.99(1-2k^{-6})^k S_i^\sigma \ge 0.99(1-2k^{-6+1}) S_i^\sigma \ge 0.91 S_i^\sigma$ (using $k \ge 3$). Putting these together we have that $a_i \le 0.25 b_i$ for all i.

Since $S_i^\sigma - S_{i-1}^\sigma = \sum_{v \in P_{\sigma(i)}} |J_v \cup Q_v| \le 2n$ for all i, we have that $b_i \le 0.99(1-2k^{-6})^{i-2} S_i^\sigma = b_{i-1} + 0.99(1-2k^{-6})^{i-2}(S_i^\sigma - S_i^\sigma) \le b_{i-1} + 2n$. Also, using $k \ge 3$ we have $S_{k-1}^\sigma = \sum_{s=1}^{k-1} |J_s \cup Q_s| = |G| - |U_1 \cup U_2| \ge (1+0.07k)n - 0.01kn \ge 1.16n$ which implies $b_{k-1} \ge 0.91 \cdot 1.15n \ge n$. Combining these we get that there is some i for which $n \le b_i \le 3n$ and hence $a_i \le 0.25 \cdot 3n \le n$. By Lemma 2.18, D_i contains a cycle of length n. ∎

The following lemma could be seen as a structural statement of the form "If N is close to $R(C_n, K_m^k)$ and K_N is 2-colored without red cycles C_n and blue K_m^k, then the coloring on K_N must be close to the extremal coloring."

Lemma 3.14. *There is a constant $N_3 = 10^{58}$ such that the following holds. Suppose that $n \ge N_3 m$, $m \ge k^{21}$, $k \ge 2$ and G is a graph with $|G| \ge (k-1)n$, G C_n-free, and $\overline{G}$ K_m^k-free. Then $V(G)$ can be partitioned into sets $A_1, \dots, A_{k-1}$ and S such that the following hold:*

(i) $|A_i| \ge m$ *for* $i = 1, \dots, k-1$.
(ii) *There are no edges between* A_i *and* A_j *for* $i \ne j$.
(iii) $\overline{G[A_i]}$ *is* $K_{m,m}$*-free for* $i = 1, \dots, k-1$.
(iv) $|S| \le k^{21}$.

Proof. We construct a sequence of graphs $G_0, G_1, \dots$ recursively as follows. Let $G_0 = G$. If G_i contains three sets A, B, S_i with $|A|, |B| \ge m$, $|S_i| \le k^{20}$, such that A, B, and S_i all lie in the same connected component of G_i and S_i separates A from B, then let $G_{i+1} = G_i \setminus S_i$. Notice that since $\overline{G_0}$ is K_m^k-free, we must have $G_k = G_{k-1}$.

Choose a partition of $V(G_k)$ into sets $A_1, \dots, A_t$ such that for each i, we have $|A_i| \ge m$, there are no edges between A_i and A_j for $i \ne j$, and t is as large as possible. Notice that any A_i with $|A_i| \ge 3m$ must have a connected component C_i of order at least $|A_i| - m + 1$ (otherwise A_i can be split into sets of order $\ge m$ with no edges between them, contradicting the maximality of t). From this we obtain that any A_i with $|A_i| \ge 5m$ must have the property that "for any two subsets $A, B \subseteq A_i$ of order $\ge 2m$, there are at least k^{20} disjoint paths from A to B in A_i." Indeed otherwise, by Menger's theorem there would be a set S' of size $\le k^{20}$ separating $A \cap C_i$ from $B \cap C_i$, contradicting $G_k = G_{k-1}$. Let $S = S_1 \cup \cdots \cup S_{k-1} = V(G) \setminus V(G_k)$ to get a set with $|S| \le k^{21}$.

For $i = 1, \dots, t$, let $x_i = \min(0, |A_i| - n)$.

Claim 3.15. *$\overline{G_k}$ contains K_m^r for $r = t + \lfloor \frac{4}{n} \sum_{i=1}^t x_i \rfloor$.*

Proof. Without loss of generality, suppose that $A_1, \dots, A_t$ are ordered so that $x_1, \dots, x_a \le m$ and $x_{a+1}, \dots, x_t \ge m$ for some integer a.

Using Lemma 3.9, we see that when $x_i \ge 0.25n$, $\overline{G[A_i]}$ contains a K_m^j for $j = 1 + \lceil 4x_i/n \rceil$. (First notice that $\lfloor x_i/0.07n \rfloor \ge 1 + \lceil 4x_i/n \rceil$ for $x_i \ge 0.25n$. This implies that $|A_i| = x_i + n \ge 0.07jn + n$, and so the assumptions of Lemma 3.9 hold for $G[A_i]$ with $k' = j$.) By Corollary 3.8 we know that $\overline{G[A_i]}$ contains a K_m^2 whenever $|A_i| \ge n + m - 1$. This implies that when $m \le x_i < 0.25n$ then $\overline{G[A_i]}$ contains a copy of $K_m^{1+\lceil 4x_i/n \rceil} = K_m^j = K_m^2$. Putting the above observations together, we obtain that $\overline{G[A_{a+1} \cup \cdots \cup A_t]}$ contains a $K_m^{t-a+\sum_{i=a+1}^t \lceil 4x_i/n \rceil}$.

By Corollary 3.8 and the fact that $|A_i| \ge m$, we know that $\overline{G[A_i]}$ contains a K_{m,x_i} whenever $x_i \le m$. Since there are no edges between any of these $K_{m,x_1}, \dots, K_{m,x_a}$,

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their union consists of a K_m^a together with a $K_{x_1,\dots,x_a}$. Using $x_1,\dots,x_a \le m$, we have that $K_{x_1,\dots,x_a}$ contains a $K_m^{\lfloor \sum_{i=1}^a x_i/2m \rfloor}$. Together with K_m^a this gives a copy of $K_m^{a+\lfloor \sum_{i=1}^a x_i/2m \rfloor}$ in $\overline{G[A_1 \cup \cdots \cup A_a]}$. Since $4/n \le 1/2m$, this contains a copy of $K_m^{a+\lfloor \sum_{i=1}^a 4x_i/n \rfloor}$.

Now, we have found a $K_m^{t-a+\sum_{i=a+1}^t \lceil 4x_i/n \rceil}$ and a disjoint $K_m^{a+\lfloor \sum_{i=1}^a 4x_i/n \rfloor}$. Putting these two together, and using $\lfloor x \rfloor + \lceil y \rceil \ge \lfloor x+y \rfloor$, we obtain a $K_m^{t+\lfloor \frac{4}{n} \sum_{i=1}^t x_i \rfloor}$ as required. ∎

From Claim 3.15 and the K_m^k-freeness of $\overline{G}$, we obtain that $t + \lfloor \frac{4}{n} \sum_{i=1}^t x_i \rfloor \le k-1$. We also have that $tn + \sum_{i=1}^t x_i \ge |G| - |S| \ge (k-1)n - k^{21}$. Putting these together, we get $\frac{1}{n} \sum_{i=1}^t x_i \ge \lfloor \frac{4}{n} \sum_{i=1}^t x_i \rfloor - \frac{k^{21}}{n}$. Together with $n > 10k^{21}$, this gives $\sum_{i=1}^t x_i < n/2$. Combined with $tn + \sum_{i=1}^t x_i \ge (k-1)n - k^{21}$ this implies $t - k + 1 \ge -\frac{1}{2} - \frac{k^{21}}{n}$. Since $t-k+1$ is an integer, this implies that $t \ge k-1$. From $t + \lfloor \frac{4}{n} \sum_{i=1}^t x_i \rfloor \le k-1$ we obtain that $t = k-1$. The K_m^k-freeness of $\overline{G}$ implies that each $\overline{G[A_i]}$ is $K_{m,m}$-free, proving the lemma. ∎

We can now prove the main result of this paper.

Proof of Theorem 1.4. From Lemma 1.1 we have that $R(C_n, K_{m_1,\dots,m_k}) \ge (n-1)(k-1) + m_1$. Therefore, it remains to prove the upper bound. Fix $N_3 = 10^{60}$. Let $n, \hat{k}, m_1, \dots, m_{\hat{k}}$ be numbers with $n \ge N_3 m_{\hat{k}}$, $m_{\hat{k}} \ge m_{\hat{k}-1} \ge \cdots \ge m_1$ and $m_i \ge i^{22}$ for $i = 1, \dots, \hat{k}$.

We prove that $R(C_n, K_{m_1,\dots,m_k}) \le (n-1)(k-1)+m_1$ for $k = 2, \dots, \hat{k}$ by induction on k. The initial case is when $k=2$ which comes from Corollary 3.8. Therefore assume that $k \ge 3$ and that we have $R(C_n, K_{m_1,\dots,m_{k-1}}) \le (n-1)(k-2) + m_1$. Let K be a 2-edge-colored complete graph on $(n-1)(k-1)+m_1$ vertices. Suppose, for the sake of contradiction, that K contains neither a red C_n nor a blue $K_{m_1,\dots,m_k}$. Let G be the subgraph consisting of the red edges of K.

Claim 3.16. *$|N_G(W) \cup W| \ge n$ for every $W \subseteq G$ with $|W| \ge m_k$.*

Proof. Suppose that $|N_G(W) \cup W| \le n-1$ for some W with $|W| \ge m_k$. Let $K' = K \setminus (N_G(W) \cup W)$ to get a graph with $|K'| \ge (n-1)(k-2)+m_1$. By induction K' contains either a red C_n or a blue $K_{m_1,\dots,m_{k-1}}$. In the former case, we have a red C_n in K, whereas in the latter case we have a blue $K_{m_1,\dots,m_k}$ formed from the copy of $K_{m_1,\dots,m_{k-1}}$ together with W. ∎

Set $m = m_k$, and notice that $\overline{G}$ contains no blue K_m^k. Apply Lemma 3.14 to G in order to partition it into sets $A_1, \dots, A_{k-1}$ and S satisfying (i)–(iv). Notice that from condition (ii) of Lemma 3.14 and Claim 3.16, we have $|(N_G(W) \cup W) \cap (A_i \cup S)| \ge n$ for any $W \subseteq A_i$ with $|W| \ge m$. Combined with $|S| \le k^{21} \le m$ and $n \ge N_3 m$, this implies that $|A_i| \ge 10^{53} m$ for each i. For $i = 1, \dots, k-1$, apply Lemma 3.2 with $U = A_i$, $G = G[A_i \cup S]$, and $d = 3$ in order to find subsets $H_i \subseteq A_i$ with $|H_i| \ge |A_i| - m$ such that $G[H_i]$ is a $(3, 5m, n)$-expander in $G[H_i \cup S]$. Let $G' = G[H_1 \cup \cdots \cup H_{k-1} \cup S]$.

Suppose that for some i and j, there exist two vertex-disjoint paths from H_i to H_j in G'. Let P_1 and P_2 be two such paths with $|P_1| + |P_2|$ as small as possible. Using Lemma 3.4 we have that $|P_1 \cap H_s|, |P_2 \cap H_s| \le 3 \log 5m$ for all s (since if we had $|P_1 \cap H_s| > 3 \log 5m$, then Lemma 3.4 would give a shorter path in H_s between the first and the last vertex of P_1 in $P_1 \cap H_s$). Together with $m \ge k^{22}$, this implies $|P_1|, |P_2| < 3k \log 5m \le m$. Let p_1^i and p_2^i be the endpoints of P_1 and P_2 in H_i and let p_1^j and p_2^j be the endpoints of P_1 and P_2 in H_j. By Lemma 3.5 applied with

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$m' = 5m$, there is an p_1^i to p_2^i path Q_i in H_i as well as a p_1^j to p_2^j path Q_j in H_j with $50m \le |Q_i|, |Q_j| \le 60m$. Notice that we have $n-62m \le n-|Q_i \cup P_1 \cup P_2|+2 \le n-50m$ and "$|N_{H_j}(W) \cup W| \ge n - |S| \ge n - m$ for $W \subseteq H_j$ with $|W| \ge 5m$." Therefore, we can apply Lemma 3.7 to H_j with $P = Q_j$, $m' = 5m$, and $n' = n - |Q_i \cup P_1 \cup P_2| + 2$ in order find a p_1^j to p_2^j path Q'_j in H_j with $|Q'_j| = n - |Q_i \cup P_1 \cup P_2| + 2$. Joining Q_i to P_1 to Q'_j to P_2 gives a red cycle of length n in K.

Suppose that for all $i \ne j$, there do not exist two vertex-disjoint paths from H_i to H_j in G'. We show that there is a vertex v which separates some H_a from the others.

CLAIM 3.17. *There is a set $A \subseteq V(G')$, a 2-connected subgraph $D \subseteq G'$, a vertex $v \in V(D)$, and an index $a \in \{1, \dots, k-1\}$ such that $H_a \subseteq V(D) \subseteq A$, $A \setminus (S \cup \{v\}) = H_a \setminus \{v\}$, and $N_{G'}(A - v) \subseteq A$.*

Proof. Let $D_1, \dots, D_t$ be the maximal 2-connected subgraphs of G'. By maximality we have that $|D_i \cap D_j| \le 1$ for any $i \ne j$. By Lemma 3.3, H_i is 2-connected for all i, and hence $H_i \subseteq D_j$ for some j. By Menger's theorem we have that each of $D_1, \dots, D_t$ can contain at most one of the sets H_i for $i = 1, \dots, k-1$ (since there do not exist two vertex-disjoint paths between H_i and H_j for distinct i and j).

Let F be an auxiliary graph with $V(F) = \{D_1, \dots, D_t\}$ with D_iD_j an edge whenever $D_i \cap D_j \ne \emptyset$. It is well known that F is a forest (see Proposition 3.11 in [10]). Let T be any subtree of F which contains H_i for some i, and let D_{root} be an arbitrary root of T. There is a vertex $D_b \in T$ such D_b contains H_a for some a, but no descendant of D_b contains H_j for any $j \ne a$. Let $D = D_b$.

If $D_b \ne D_{root}$, then let D_s be the parent of D_b and v the unique vertex in $D_s \cap D_b$. Let A be the set consisting of v plus all the vertices in the connected component of $G' - v$ containing H_a. If $D_b = D_{root}$, then let A be the set consisting of all the vertices in the connected component of G' containing H_a (which is just $\bigcup_{v \in T} D_v$), and let v be an arbitrary vertex in D.

In both of the above cases, $H_a \subseteq V(D) \subseteq A$ and $N_{G'}(A - v) \subseteq A$ are immediate. To see $A \setminus (S \cup \{v\}) = H_a \setminus \{v\}$, recall that $H_1, \dots, H_{k-1}, S$ partitioned $V(G')$ and D_b was chosen so that no descendant of D_b in T contains H_j for any $j \ne a$. ∎

Let A, v, and a be as produced by the above lemma. Notice that $\overline{G[A]}$ is $K_{m+1,m+1}$-free. Indeed given a copy of $K_{m+1,m+1}$ in $\overline{G[A]}$, we have a copy of $K_{m,m}$ in $\overline{G[A]} \setminus \{v\}$. Since from Claim 3.17 there are no edges between this $K_{m,m}$ and $H_t \setminus \{v\}$ for $t \ne a$, we obtain a copy of K_m^k in $\overline{G}$.

Notice that for any $W \subseteq A$ with $|W| \ge 5m + k^{21} + 1$, we have

$$\begin{aligned} |N_A(W) \cup W| &\ge |N_A(W \setminus (S \cup \{v\})) \cup (W \setminus (S \cup \{v\}))| \\ &= |N_A(W \cap (H_a \setminus \{v\})) \cup (W \cap (H_a \setminus \{v\}))| \\ &\ge n \end{aligned}$$

To see the last inequality, recall that from Claim 3.17 we have $A \setminus (S \cup \{v\}) = H_a \setminus \{v\}$, $N_{H_a \cup S}(A - v) \subseteq N_{G'}(A - v) \subseteq A$, that H_a is a $(3, 5m, n)$-expander in $G[H_a \cup S]$, and that $|W \cap (H_a \setminus \{v\})| = |W \setminus (S \cup \{v\})| \ge 5m$.

Let u be any neighbor of v in D. Since $|H_a| \ge 10^{52}m$ and $\overline{H_a}$ is $K_{m,m}$-free, $H_a - v - u$ contains a cycle C with $|C| \ge 100m$ (e.g., by Theorem 1.6). By 2-connectedness of D combined with Menger's theorem, there are two disjoint paths P_u and P_v from u and v, respectively, to C. Joining P_u and P_v to the longer segment of C between $P_u \cap C$ and $P_v \cap C$ gives a u to v path P of length $\ge 50m$. Applying Lemma 3.7 to the graph $G[A]$, the vertices u and v, the path P, and $m' = 5m + k^{21}$

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gives a path of order n from u to v which together with the edge uv forms a cycle of length n in G (and hence a red C_n in K). □

4. Concluding remarks. In Theorem 1.4 we needed two conditions for C_n to be $K_{m_1,\dots,m_k}$-good—we needed $n \geq 10^{60} m_k$ and $m_i \geq i^{22}$.

The first of these conditions "$n \geq 10^{60} m_k$" cannot be removed completely (although the constant 10^{60} can probably be significantly reduced) as there are constructions showing that C_n is not $K_{m_1,\dots,m_k}$-good for $n \leq m_k$. One family of such constructions is to fix a number $r \in \{1, \dots k\}$ and consider a 2-edge-coloring of a complete graph on $(k-r)(n-1)+r(m_r-1)$ vertices consisting of $(k-r)$ red cliques $C_1, \dots, C_{k-r}$ of size $n-1$ and r red cliques $C_{k-r+1}, \dots, C_k$ of size $m_r - 1$. For $n \geq m_r$, this construction has neither red C_n nor blue $K_{m_1,\dots,m_k}$—there is no red C_n since all red components have size $\leq n-1$, and there is no blue $K_{m_1,\dots,m_k}$ since the k parts of $K_{m_1,\dots,m_k}$ have to all be contained in different sets $C_1, \dots, C_k$, but only $k-r$ of these have size bigger than m_r (and so it is impossible to simultaneously embed the $k-r+1$ parts of $K_{m_1,\dots,m_k}$ of sizes $m_r, m_{r+1}, \dots, m_k$). This construction shows that for $n \geq m_r$ we have $R(C_n, K_{m_1,\dots,m_k}) \geq (k-r)(n-1)+r(m_r-1)$. For $r=1$, this is exactly (1). From this bound we obtain that for $m_r \leq n < m_r + \frac{m_r - m_1}{r-1} - 1$, the cycle C_n is not $K_{m_1,\dots,m_k}$-good. By choosing $r = k$, we see that the bound "$n \geq 10^{60} m_k$" in Theorem 1.6 cannot be improved significantly beyond "$n \geq k m_k/(k-1)$."

We conjecture that the second condition "$m_i \geq i^{22}$" in Theorem 1.4 can be omitted completely. Such a result would in particular show that C_n is K_m good, i.e., it would prove particular cases of Conjecture 1.2. Because of this it would likely require different proof techniques from the ones used in this paper (for example, Nikiforov's ideas from [18] showing that C_n is K_m-good for $n \geq 4m+2$ may be helpful).

The gadgets that we use are very similar to *absorbers* introduced by Montgomery in [17] during the study of spanning trees in random graphs. An absorber is a graph A with three special vertices x, y, and v such that A has x to y paths with vertex sets $V(A)$ and $V(A) \setminus \{v\}$. While absorbers have a long history, Montgomery's key insight was that they can be found in very sparse graphs with good expansion properties. The graphs in which we need to find gadgets are also very sparse, and structurally the gadgets that we find are a natural generalization of Montgomery's absorbers. However, the graphs in which we look for gadgets are even sparser than Montgomery's and have weaker expansion properties. Specifically, Montgomery was looking at graphs G in which any small set S satisfies $|N(S)| \geq C|S| \log^4 |G|$, whereas in this paper we consider graphs which only have $|N(S)| \geq C|S|$. The level of expansion at which we find gadgets is optimal up to a constant factor. Since we find gadgets (and as a consequence absorbers) at such a low expansion, our intermediate results are likely to have application in the study of random and pseudorandom graphs.

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REFERENCES

[1] P. Allen, G. Brightwell, and J. Skokan, *Ramsey-goodness and otherwise*, Combinatorica, 33 (2013), pp. 125–160.

[2] I. Balla, A. Pokrovskiy, and B. Sudakov, *Ramsey goodness of bounded degree trees*, Combin. Prob. Comput., 27 (2018), pp. 289–309.

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[3] S. Brandt, H. Broersma, R. Diestel, and M. Kriesell, *Global connectivity and expansion: Long cycles and factors in f-connected graphs*, Combinatorica, 26 (2006), pp. 17–36.
[4] A. Bondy and P. Erdős, *Ramsey numbers for cycles in graphs*, J. Combin. Theory Ser. B, 14 (1973), pp. 46–54.
[5] S. Burr, *Ramsey numbers involving graphs with long suspended paths*, J. London Math. Soc., 24 (1981), pp. 405–413.
[6] S. Burr and P. Erdős, *Generalizations of a Ramsey-theoretic result of Chvátal*, J. Graph Theory, 7 (1983), pp. 39–51.
[7] Y. Chen, T. C. E. Cheng, and Y. Zhang, *The Ramsey numbers $R(C_m, K_7)$ and $R(C_7, K_8)$*, European J. Combin., 29 (2008), pp. 1337–1352.
[8] V. Chvátal, *Tree-complete graph Ramsey number*, J. Graph Theory, 1 (1977), 93.
[9] D. Conlon, J. Fox, C. Lee, and B Sudakov, *Ramsey numbers of cubes versus cliques*, Combinatorica, 136 (2016), pp. 37–70.
[10] R. Diestel, *Graph Theory*, Springer-Verlag, New York, 1997.
[11] P. Erdős, *Some remarks on the theory of graphs*, Bull. Amer. Math. Soc., 53 (1947), pp. 292–294.
[12] P. Erdős, R. Faudree, C. Rousseau, and R. Schelp, *On cycle-complete graph Ramsey numbers*, J. Graph Theory, 2 (1978), pp. 53–64.
[13] G. Fiz Pontiveros, S. Griffiths, R. Morris, D. Saxton, and J. Skokan, *The Ramsey number of the clique and the hypercube*, J. Lond. Math. Soc., 89 (2014), pp. 680–702.
[14] J. Friedman and N. Pippenger, *Expanding graphs contain all small trees*, Combinatorica, 7 (1987), pp. 71–76.
[15] L. Gerencsér and A. Gyárfás, *On Ramsey-type problems*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math, 10 (1967), pp. 167–170.
[16] P. Keevash, E. Long, and J. Skokan, *Cycle-complete Ramsey numbers*, Int. Math. Res. Not. IMRN, (2019), https://doi.org/10.1093/imrn/rnz119.
[17] R. Montgomery, *Spanning trees in random graphs*, Adv. Math., 356 (2019).
[18] V. Nikiforov, *The cycle-complete graph Ramsey numbers*, Combin. Probab. Comput., 14 (2005), pp. 349–370.
[19] V. Nikiforov and C. Rousseau, *Ramsey goodness and beyond*, Combinatorica, 29 (2009), pp. 227–262.
[20] C. Pei and Y. Li, *Ramsey numbers involving a long path*, Discrete Math., 339 (2016), pp. 564–570.
[21] A. Pokrovskiy and B. Sudakov, *Ramsey goodness of paths*, J. Combin. Theory Ser. B, 122 (2017), pp. 384–390.
[22] L. Pósa, *Hamiltonian circuits in random graphs*, Discrete Math., 14 (1976), pp. 359–364.

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