Annals of Global Analysis and Geometry (2020) 57:489–517
https://doi.org/10.1007/s10455-020-09710-z

S^1-quotient of *Spin*(7)-structures

Udhav Fowdar[1]

Received: 3 December 2019 / Accepted: 5 March 2020 / Published online: 20 March 2020

Abstract
If a *Spin*(7)-manifold N^8 admits a free S^1 action preserving the fundamental 4-form, then the quotient space M^7 is naturally endowed with a G_2-structure. We derive equations relating the intrinsic torsion of the *Spin*(7)-structure to that of the G_2-structure together with the additional data of a Higgs field and the curvature of the S^1-bundle; this can be interpreted as a Gibbons–Hawking-type ansatz for *Spin*(7)-structures. In particular, we show that if N is a *Spin*(7)-manifold, then M cannot have holonomy contained in G_2 unless the N is in fact a Calabi–Yau fourfold and M is the product of a Calabi–Yau threefold and an interval. By inverting this construction, we give examples of $SU(4)$ holonomy metrics starting from torsion-free $SU(3)$-structures. We also derive a new formula for the Ricci curvature of *Spin*(7)-structures in terms of the torsion forms. We then describe this S^1-quotient construction in detail on the Bryant–Salamon *Spin*(7) metric on the spinor bundle of S^4 and on flat $\mathbb{R}^8$.

Keywords Differential geometry · Exceptional holonomy · S^1-quotient · G_2-structures · *Spin*(7)-structures

Mathematics Subject Classification 53C10 · 53C29

1 Introduction

In 1955, Berger classified the possible holonomy groups of irreducible, nonsymmetric, simply connected Riemannian manifolds [6]. The classification included the two exceptional cases of holonomy groups: G_2 and *Spin*(7), of which no examples were known at the time. It is only in 1987 that Bryant proved the existence of local examples in [9], and subsequently, explicit complete non-compact examples were constructed by Bryant and Salamon in [11]. There are by now many known examples of holonomy G_2 and *Spin*(7) metrics cf. [3, 8, 19, 20, 30], yet very few explicitly known ones. In [2], Apostolov and Salamon studied the S^1-reduction of G_2 -manifolds and investigated the situation when the quotient is a Kähler manifold. By inverting their construction, they were able to give several local examples of holonomy G_2 metrics starting from a Kähler threefold with additional data. Motivated by their work, in this article

✉ Udhav Fowdar
udhav.fowdar.12@ucl.ac.uk

1 Department of Mathematics, University College London, Gower Street, London WC1E 6BT, UK

we shall carry out the analogous construction in the *Spin*(7) setting, but more generally, we shall look at S^1-invariant *Spin*(7)-structures which are not necessarily torsion free. The situation when N is a *Spin*(7)-manifold has also been studied by Foscolo in [19]. One motivation for studying the non-torsion-free cases lies in the fact that they also have interesting geometric properties; for instance, balanced *Spin*(7)-structures admit harmonic spinors [28] and compact locally conformally parallel are fibred by nearly parallel G_2 manifolds [29]. A further motivation is that *Spin*(7)-structures have only two torsion classes and thus have only four types, whereas G_2-structures have four classes, thus allowing for a more refined decomposition of the *Spin*(7) torsion classes. The outline for the rest of this article is as follows.

In Sect. 2, we give a brief introduction to G_2 and *Spin*(7)-structures and set up some notation. The reader will find proofs of the mentioned facts in the standard references [9, 30, 34].

In Sect. 3, we describe the quotient of *Spin*(7)-structures which are invariant under a free circle action. The foundational result is Proposition 3.2, which gives explicit expressions relating the torsion of the *Spin*(7)-structure on the 8-manifold N to the torsion of the quotient G_2-structure on M together with a positive function s and the curvature of the S^1 bundle. The key observation is that this construction is reversible. In the subsequent subsections, we specialise to the three cases when the *Spin*(7)-structure is torsion free, locally conformally parallel and balanced. In the torsion-free situation, we show that quotient manifold cannot have holonomy *equal to* G_2 unless N is a Calabi–Yau fourfold and M is the Riemannian product of a Calabi–Yau threefold and a circle. We also give explicit expressions for the $SU(4)$-structure in terms of the data on the quotient manifold, see Theorem 3.6. In the locally conformally parallel situation, we show that M has vanishing Λ^3_{27} torsion component, and furthermore, if the Λ^3_1 torsion component is nonzero, then $N = M \times S^1$, see Theorem 3.7. In the balanced situation, we show that the existence of an invariant *Spin*(7)-structure is equivalent to the existence of a suitable section of Λ^2_{14} of the quotient space, see Theorem 3.9. We provide several examples to illustrate each case.

In Sect. 4, we derive formulae for the Ricci and scalar curvatures of *Spin*(7)-structures in terms of the torsion forms à la Bryant cf. [10], see Proposition 4.1. As a corollary, under our free S^1 action hypothesis, we show that the Λ^2_7 component of the curvature form corresponds to the mean curvature vector of the circle fibres.

In the last two sections, we demonstrate how our construction can be applied to the Bryant–Salamon *Spin*(7)-structure on the (negative) spinor bundle of S^4 and on the flat *Spin*(7)-structure on $\mathbb{R}^8$. In the former case, the quotient space is the anti-self-dual bundle of S^4, and in the latter, it is the cone on $\mathbb{CP}^3$. We interpret the quotient of the spinor bundle as a fibrewise reverse Gibbons–Hawking ansatz. In both cases, we also study the $SU(3)$-structure on the link $\mathbb{CP}^3$.

2 Preliminaries

A G_2-structure on a 7-manifold M^7 is given by a 3-form φ that can be identified at each point $p \in M^7$ with the standard one on $\mathbb{R}^7$:

$$\varphi_0 = \mathrm{d}x_{123} + \mathrm{d}x_{145} + \mathrm{d}x_{167} + \mathrm{d}x_{246} - \mathrm{d}x_{257} - \mathrm{d}x_{347} - \mathrm{d}x_{356} \tag{2.1}$$

where $\mathrm{d}x_{ijk}$ denotes $\mathrm{d}x_i \wedge \mathrm{d}x_j \wedge \mathrm{d}x_k$. More abstractly, it can equivalently be defined as a reduction of the structure group of the frame bundle of M from $GL(7,\mathbb{R})$ to G_2, but we shall use the former more concrete definition. The reason for this nomenclature is the fact that the subgroup of $GL(7,\mathbb{R})$ which stabilises φ_0 is isomorphic to the Lie group G_2. Since G_2 is a subgroup of $SO(7)$ [9], it follows that φ defines a Riemannian metric g_φ and volume form vol_φ on M^7. Explicitly, these are given by

$$\frac{1}{6}\,\iota_X\varphi \wedge \iota_Y\varphi \wedge \varphi = g_\varphi(X, Y)\, vol_\varphi.$$

Thus, φ also defines a Hodge star operator $*_\varphi$. It is known that a 7-manifold admits a G_2 -structure if and only if its first and second Stiefel–Whitney classes vanish [33], so there is a plethora of examples. One of the main motivations for studying this structure is that if φ is parallel with respect to the Levi-Civita connection ∇^{g_φ} (which is a first-order condition), then it has holonomy contained in G_2 and the metric is Ricci-flat. Such a manifold is called a G_2-manifold. Note that in contrast, the Ricci-flat system of equations is second order. The fact that φ is parallel implies the reduction of the holonomy group from $SO(7)$ to (a subgroup of) G_2, and conversely, a holonomy G_2 metric implies the existence of such a 3-form. A useful alternative way to verify the parallel condition is given by the following theorem:

Theorem 2.1 ([17]) $\nabla^{g_\varphi}\varphi = 0$ *if and only if* $\mathrm{d}\varphi = 0$ *and* $\mathrm{d} *_\varphi \varphi = 0$.

The failure of the reduction of the holonomy group to G_2 is measured by the intrinsic torsion. Abstractly, given a general H-structure for a subgroup $H \subset O(n)$ the intrinsic torsion is defined as a section of the associated bundle to $\mathbb{R}^n \otimes \mathfrak{h}^\perp$ where $\mathfrak{so}(n) = \mathfrak{h} \oplus \mathfrak{h}^\perp$ and $\perp$ denotes the orthogonal complement with respect to the Killing form. We shall only give a brief description here, but more details can be found in [10, 34]. The space of differential forms on M^7 can be decomposed as G_2-modules as follows:

$$\begin{aligned}
\Lambda^1 &= \Lambda^1_7\\
\Lambda^2 &= \Lambda^2_7 \oplus \Lambda^2_{14}\\
\Lambda^3 &= \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}
\end{aligned}$$

where the subscript denotes the dimension of the irreducible module. Using the Hodge star operator, we get the corresponding splitting for Λ^4, Λ^5 and Λ^6. The intrinsic torsion is given by $\dim(\mathbb{R}^7 \otimes \mathfrak{g}_2^\perp) = 49$ equations and can be described using the equations

$$\mathrm{d}\varphi = \tau_0 *_\varphi \varphi + 3\,\tau_1 \wedge \varphi + *_\varphi \tau_3 \tag{2.2}$$

$$\mathrm{d} *_\varphi \varphi = 4\,\tau_1 \wedge *_\varphi \varphi + \tau_2 \wedge \varphi \tag{2.3}$$

where $\tau_0 \in \Omega^0$, $\tau_1 \in \Omega^1_7$, $\tau_2 \in \Omega^2_{14}$ and $\tau_3 \in \Omega^4_{27}$. Here, we are denoting by Ω^i_j the space of smooth sections of Λ^i_j. The fact that τ_1 arises in both equations can be proved using the following:

Lemma 2.2 ([10]) *Given* $\alpha \in \Lambda^1_7(M)$ *and* $\beta \in \Lambda^2_7(M)$, *we have*

(1) $2 *_\varphi (\beta \wedge *_\varphi \varphi) \wedge *_\varphi \varphi = 3\beta \wedge \varphi$
(2) $*_\varphi \alpha = -\frac{1}{4} *_\varphi (\alpha \wedge \varphi) \wedge \varphi = \frac{1}{3} *_\varphi (\alpha \wedge *_\varphi \varphi) \wedge *_\varphi \varphi$.

In contrast to the non-torsion-free case, manifolds with holonomy *equal to* G_2 are much harder to find. G_2-structures for which φ is (co-)closed are usually called (co-)calibrated. Another notion we shall need is that of a G_2-instanton.

Definition 2.3 A G_2-instanton on a G_2 manifold (M^7, φ) is a connection 1-form A on a principal G-bundle whose curvature form F_A satisfies

$$F_A \wedge *_\varphi \varphi = 0,$$

Equivalently, F_A belongs to Ω^2_{14}, as an ad(P)-valued 2-form on M^7.

Instantons are solutions of the Yang–Mills equations and as such play an important role in studying topological properties of M^7. There is a similar geometric structure to G_2-structures in dimension eight, again related to exceptional holonomy.

A *Spin*(7)-structure on an 8-manifold N^8 is given by a 4-form Φ that can be identified at each point $q \in N^8$ with the standard one on $\mathbb{R}^8$:

$$\Phi_0 = \mathrm{d}x_0 \wedge \varphi_0 + *_{\varphi_0} \varphi_0 \tag{2.4}$$

where we have augmented the G_2 module $\mathbb{R}^7$ by $\mathbb{R}$ with coordinate x_0. The subgroup of $GL(8, \mathbb{R})$ which stabilises Φ_0 is isomorphic to *Spin*(7) cf. [11, 34]. From this definition, it is clear that G_2 arises as a subgroup of *Spin*(7). Since *Spin*(7) is a subgroup of *SO*(8), it follows that Φ defines a metric g_Φ, volume form vol_Φ and Hodge star $*_\Phi$. Explicitly, the volume form is given by

$$\mathrm{vol}_\Phi = \frac{1}{14} \Phi \wedge \Phi,$$

but the expression for g_Φ is much more complicated than in the G_2 case cf. [31, section 4.3]. An 8-manifold admits a *Spin*(7)-structure if and only if, in addition to having zero first and second Stiefel–Whitney classes, either of the following holds

$$p_1(N)^2 - 4p_2(N) \pm 8\chi(N) = 0$$

cf. [25, 33], noting that the '8' factor is accidentally omitted in the former. If Φ is parallel with respect to the Levi-Civita connection ∇^{g_Φ}, then the metric g_Φ has holonomy contained in *Spin*(7) and the metric is Ricci-flat. Such a manifold is called a *Spin*(7)-manifold. Just as in the G_2 situation, we have the following alternative formulation of the torsion-free condition:

Theorem 2.4 [16] $\nabla^{g_\Phi}\Phi = 0$ *if and only if* $d\Phi = 0$.

The space of differential forms on N^8 can be decomposed as *Spin*(7)-modules as follows:

$$\begin{aligned} \Lambda^1 &= \Lambda^1_8 \\ \Lambda^2 &= \Lambda^2_7 \oplus \Lambda^2_{21} \\ \Lambda^3 &= \Lambda^3_8 \oplus \Lambda^3_{48} \\ \Lambda^4 &= \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27} \oplus \Lambda^4_{35}. \end{aligned}$$

We shall write $\Lambda^i_j(M^7)$ or $\Lambda^i_j(N^8)$ if there is any possible ambiguity. There is also an injection map $i : S^2 \hookrightarrow \Lambda^4$ which restricts to an isomorphism of $Spin(7)$-modules

$$\begin{aligned} \mathsf{i} : \langle g_\Phi \rangle \oplus S^2_0 &\to \Lambda^4_1 \oplus \Lambda^4_{35} \\ a \circ b &\mapsto a \wedge *_\Phi (b \wedge \Phi) + b \wedge *_\Phi (a \wedge \Phi) \end{aligned}$$

where S^2_0 denotes the space of traceless symmetric (0, 2)-tensors. Note that $\mathsf{i}(g_\Phi) = 8\Phi$. We denote by j the inverse map extended to Λ^4 as the zero map on $\Lambda^4_7 \oplus \Lambda^4_{27}$. Similarly, the intrinsic torsion is given by $\dim(\mathbb{R}^8 \otimes \mathfrak{spin}(7)^\perp) = 56$ equations and is completely determined by the exterior derivative of Φ in view of Theorem 2.4. This can be written as

$$d\Phi = T^1_8 \wedge \Phi + T^5_{48} \tag{2.5}$$

where T^5_{48} is defined by the condition $*_\Phi T^5_{48} \wedge \Phi = 0$. If T^1_8 vanishes, the $Spin(7)$-structure is called balanced, if T^5_{48} vanishes, it is locally conformally parallel, and if both are zero, then it is torsion free.

In this article, we shall often use the suggestive notation κ^l_m for an l-form to mean that $\kappa^l_m \in \Omega^l_m$ or write $(\kappa)^l_m$ for the Ω^l_m-component of an l-form κ. Having set up our convention, we now proceed to describe the S^1-reduction of $Spin(7)$-structures.

3 The quotient construction

Given an 8-manifold N^8 endowed with a $Spin(7)$-structure Φ which is invariant under a free circle action generated by a vector field X, the quotient manifold M^7 inherits a natural G_2 -structure $\varphi := \iota_X \Phi$. We can write the $Spin(7)$ form as

$$\Phi = \eta \wedge \varphi + s^{4/3} *_\varphi \varphi \tag{3.1}$$

where $s = \|X\|^{-1}_\Phi$, $\eta(\cdot) = s^2 g_\Phi(X, \cdot)$ and $*_\varphi$ is the Hodge star induced by φ on M. The proof for this expression is analogous to that of Lemma 3.1. The assumption that the action is free, i.e. X is nowhere vanishing, implies that s is a well-defined strictly positive function. The metrics and volume forms of M and N are related by

$$g_\Phi = s^{-2}\eta^2 + s^{2/3} g_\varphi \tag{3.2}$$

$$\mathrm{vol}_\Phi = s^{4/3} \eta \wedge vol_\varphi. \tag{3.3}$$

In this setup, η can be viewed as a connection 1-form on the S^1-bundle N over M and $d\eta$ is its curvature, which by Chern–Weil theory defines a section in $\Omega^2(M, \mathbb{Z})$. We denote by $(d\eta)^2_7$ and $(d\eta)^2_{14}$ its two components. Under the inclusion $G_2 \hookrightarrow Spin(7)$, we may decompose the torsion forms of (2.5) further as

$$\begin{aligned} T^1_8 &= f \cdot \eta + T^1_7 \\ T^5_{48} &= T^5_7 + T^5_{14} + \eta \wedge (T^4_7 + T^4_{27}) \end{aligned}$$

where f is (the pullback of) a function on M^7 and all the differential forms on the right-hand side, aside from η, are basic. Note that $56 = 8 + 48 = (1 + 7) + (7 + 14 + 7 + 27) = 49 + 7$ where 56 and 49 are the dimensions of the space of intrinsic torsions of $Spin(7)$- and G_2-structures. This simple dimension count confirms the absence of any T^4_1 term. Moreover, this says that the intrinsic torsion of Φ is determined by that of φ together with a section of a rank 7 vector bundle. In order to relate the intrinsic torsion of the *Spin*(7)-structure to that of the G_2-structure, we first need to relate their Hodge star operators.

Lemma 3.1 *Given $\alpha \in \Lambda^2_7(Y)$, $\beta \in \Lambda^2_{14}(Y)$, $\gamma \in \Lambda^1_7(Y)$ and using the same notation for their pullbacks to N^8, we have*

(1) $*_\Phi (\alpha \wedge \varphi) = -2s^{-2}\eta \wedge \alpha$
(2) $*_\Phi (\beta \wedge \varphi) = s^{-2}\eta \wedge \beta$
(3) $*_\Phi \gamma = -s^{2/3}\eta \wedge *_\varphi \gamma$
(4) $*_\Phi \eta = s^{10/3}\,\mathrm{vol}_\varphi$
(5) $*_\Phi (\eta \wedge \alpha) = \frac{1}{2}s^2\alpha \wedge \varphi$
(6) $*_\Phi (\eta \wedge \beta) = -s^2\beta \wedge \varphi$
(7) $*_\Phi (\eta \wedge \gamma) = s^{8/3} *_\varphi \gamma$

Proof This is a straightforward computation using 3.2, 3.3 and the characterisation of Λ^2_7 and Λ^2_{14} as having eigenvalues +2 and −1 under wedging with φ and taking the Hodge star [9]. We prove (1) as an example. Since we only need to prove the above formula holds at each point, we may pick coordinates at a point $q \in N$ such that $\eta = \mathrm{d}x_0$ and $\varphi = \varphi_0$. For any given $\vartheta \in \Omega^2(Y)$, we then have

$$*_\Phi (\vartheta \wedge \varphi) = -s^{-2}\eta \wedge *_\varphi (\vartheta \wedge \varphi).$$

If $\vartheta = \alpha$ from [9], we have $*_\varphi (\alpha \wedge \varphi) = 2\alpha$, which completes the proof of (1). □

Proposition 3.2 *The intrinsic torsion of the Spin*(7)*-structure and G_2-structure is related by*

(1) $f = -s^{-4/3}\tau_0$
(2) $7T^1_7 = 24\tau_1 + 3s^{-4/3}\mathrm{d}(s^{4/3}) + 2s^{-4/3} *_\varphi ((\mathrm{d}\eta)^2_7 \wedge *_\varphi \varphi)$
(3) $7T^5_7 = 4(\mathrm{d}\eta)^2_7 \wedge \varphi + 4\mathrm{d}(s^{4/3}) \wedge *_\varphi \varphi + 4s^{4/3}\tau_1 \wedge *_\varphi \varphi$
(4) $T^5_{14} = (\mathrm{d}\eta)^2_{14} \wedge \varphi + s^{4/3}\tau_2 \wedge \varphi$
(5) $T^4_{27} = - *_\varphi \tau_3$
(6) T^4_7 *and* T^5_7 *are G_2-equivalent up to a factor of $s^{-4/3}$; explicitly, the composition*

$$L : \Lambda^5_7 \xrightarrow{*} \Lambda^2_7 \xrightarrow{\wedge *\varphi} \Lambda^6_7 \xrightarrow{*} \Lambda^1_7 \xrightarrow{\wedge\varphi} \Lambda^4_7$$

is a bundle isomorphism and $L(7T^5_7) = 4s^{-4/3}T^4_7$.
Moreover, the occurrence of τ_1 in both (2) *and* (3) *shows that*

$$T^5_7 - \frac{1}{6}s^{4/3}T^1_7 \wedge *_\varphi \varphi = \frac{1}{2}(\mathrm{d}(s^{4/3}) \wedge *_\varphi \varphi + (\mathrm{d}\eta)^2_7 \wedge \varphi) \tag{3.4}$$

and

$$3\tau_1 \wedge *_\varphi \varphi = T^1_7 \wedge *_\varphi \varphi - \frac{3}{4}s^{-4/3}T^5_7; \tag{3.5}$$

in other words, any one of the three 7-dimensional Spin(7) *torsion components determines the other two.*

Proof Using Lemmas 2.2 and 3.1, we compute

$$*_\Phi \, \mathrm{d}\Phi = s^{-2}\eta \wedge (\mathrm{d}\eta)^2_{14} - 2s^{-2}\eta \wedge (\mathrm{d}\eta)^2_7 - 3s^{2/3} *_\varphi (\tau^1 \wedge \varphi) - \tau_0\, s^{2/3}\varphi - s^{2/3}\tau_3 \\ - s^{-2} *_\varphi (\mathrm{d}(s^{4/3}) \wedge *_\varphi \varphi) \wedge \eta + s^{-2/3}\tau_2 \wedge \eta - 4s^{-2/3}\eta \wedge *_\varphi (\tau^1 \wedge *_\varphi \varphi).$$

It now suffices to use the identity $7 *_\Phi T^1_8 = *_\Phi (\mathrm{d}\Phi) \wedge \Phi$ cf. [32] and compare terms. □

Remark 3.3 Note that the above construction can also be extended to non-free S^1 actions by working on the complement of the fixed point locus. The fixed point locus then corresponds to the region where s blows up. We shall in fact see an example of this below when we look at the Bryant–Salamon *Spin*(7) metric.

Equipped with the above proposition, we can now proceed to study the quotient of different types of *Spin*(7)-structures.

3.1 The torsion-free quotient

Theorem 3.4 *Assuming (N^8, Φ) is a Spin(7)-manifold, the quotient G_2-structure φ is calibrated and the curvature is determined by*

$$(\mathrm{d}\eta)^2_7 \wedge *_\varphi \varphi = -\frac{3}{2} *_\varphi \mathrm{d}(s^{4/3}) \tag{3.6}$$

and

$$(\mathrm{d}\eta)^2_{14} = -s^{4/3}\tau_2. \tag{3.7}$$

Proof This follows directly from Proposition 3.2. From (1), (3.5) and (5), we see that τ_0, τ_1 and τ_3 must vanish. The curvature equations follow from (3.4) and (4). □

The above equations have also been described as a Gibbons–Hawking-type ansatz for *Spin*(7)-manifolds in [19], where the author studies ‘adiabatic limits’ of the equations to produce new complete non-compact *Spin*(7)-manifolds. The pair (3.6) and (3.7) generally constitutes a complicated system of PDEs. A strategy for solving this system and hence constructing *Spin*(7) metrics on the total space involves taking a formal limit of the equations as the size of the circle fibres tends to zero and thus allowing for the system to degenerate to the torsion-free G_2 equations. One then employs analytical techniques to perturb the latter equations to construct solutions to the original system. This limiting procedure of shrinking the fibres is referred to as the ‘adiabatic limit’ following a related strategy outlined in [15] in the context of $K3$-fibred G_2 manifolds.

Remark 3.5

(1) First, we note that in our setting if (N, Φ) has holonomy *equal to Spin*(7), then it is necessarily non-compact. This follows essentially from the Cheeger–Gromoll splitting theorem which asserts that if (N, Φ) is compact and Ricci-flat, then its universal cover is isometric to $\mathbb{R}^k \times P^{8-k}$ where P is a simply connected Riemannian manifold and $\mathbb{R}^k$

carries the flat metric. Under the hypothesis that it admits a free isometric S^1 action, it follows that $k \geq 1$ which together with Berger's classification of holonomy groups implies that (the identity component of) the holonomy group of N must be a subgroup of G_2.

(2) If the size of the circle orbits is constant, i.e. s is constant, then τ_2 is proportional to $\mathrm{d}\eta$, so in particular τ_2 is closed. But from equation (4.35) of [10],

$$\mathrm{d}\tau_2 = \frac{1}{7}\|\tau_2\|^2_\varphi + (\mathrm{d}\tau_2)^3_{27}$$

and hence $\tau_2 = 0$, i.e. $N^8 = S^1 \times M^7$ is a Riemannian product.

If we now further demand that (M^7, φ) is also torsion free, then this forces the connection to be a G_2-anti-instanton, i.e. $\mathrm{d}\eta \in \Lambda^2_7$, compared with Definition 2.3. Since ds is closed, $\nabla \mathrm{d}s \in S^2(T^*Y) \cong \Lambda^3_1 \oplus \Lambda^3_{27}$ cf. [11, 34], but we also have

$$\mathrm{d}\eta \wedge *_\varphi \varphi = -\frac{3}{2} *_\varphi \mathrm{d}(s^{4/3}),$$

i.e. $\mathrm{d}\eta$ and $\mathrm{d}(s^{4/3})$ are G_2-equivalent; therefore, the two components of $\nabla \mathrm{d}s$ are completely determined by the Λ^3_1 and Λ^3_{27} components of $\mathrm{d}\mathrm{d}\eta = 0 \in \Lambda^3$. Hence, $\mathrm{d}s$ is a covariantly constant 1-form as such $Hol \subsetneq G_2$ [11, Theorem 4]. Thus, we have proven the following:

Theorem 3.6 *If (N^8, Φ) is a torsion-free $Spin(7)$-structure which is invariant under a free S^1 action such that the quotient structure has holonomy contained in G_2, then $M^7 = Z^6 \times \mathbb{R}^+$ where $(Z^6, h, \omega, \Omega := \Omega^+ + i\Omega^-)$ is a Calabi–Yau threefold. Furthermore, (N^8, Φ) is a Calabi–Yau threefold and is given by $\Phi = \frac{1}{2}\hat{\omega}^2 + Re(\hat{\Omega})$ where*

$$\hat{\omega} = s^{2/3}\omega + \eta \wedge \mathrm{d}(s^{2/3}) \tag{3.8}$$

$$\hat{\Omega} = \Omega \wedge \left(-\eta - i\frac{2}{3}s^{5/3}\mathrm{d}s\right) \tag{3.9}$$

$$\hat{h} = s^{2/3}h + s^{-2}\eta^2 + \left(\frac{2}{3}s^{2/3}\mathrm{d}s\right)^2 \tag{3.10}$$

defines the $SU(4)$-structure and s is the coordinate on the $\mathbb{R}^+$ factor. The curvature form is $\mathrm{d}\eta = -\omega$, corresponding to a G_2-anti-instanton, and the product G_2-structure is given by

$$\varphi = \frac{2}{3}s^{1/3}\mathrm{d}s \wedge \omega + \Omega^+$$
$$*_\varphi \varphi = \frac{1}{2}\omega^2 - \frac{2}{3}s^{1/3}\mathrm{d}s \wedge \Omega^-.$$

Moreover, this construction is reversible, i.e. starting from a CY threefold (Z^6, h, ω, Ω), we can choose a connection form η satisfying $\mathrm{d}\eta = -\omega$ on the bundle defined by $[-\omega] \in H^2(Z^6, \mathbb{Z})$ together with a positive function s and thus define an irreducible CY fourfold $(N^8, \hat{h}, \hat{\omega}, \hat{\Omega})$ by (3.8) ,(3.9) and (3.10).

The above theorem in fact recovers the so-called *Calabi model space*. More precisely, given a compact CY manifold together with an ample line bundle L, the Calabi ansatz gives a way of defining a new CY metric on an open set of L. Although incomplete, the Calabi model space provides a good approximation for the asymptotic behaviour of the complete Tian–Yau metrics and has recently been employed in [26] to study new degenerations of hyperKähler metrics on $K3$ surfaces. Observe that taking (Z^6, h) to be the Riemannian product of a hyper-Kähler metric obtained by the Gibbons–Hawking ansatz and a flat torus $\mathbb{T}^2$, we get infinitely many holonomy $SU(4)$ metrics. We now give a simple example to illustrate this construction. The metric below has also been described in [21] as a solution to the Hitchin flow starting from a 7-nilmanifold endowed with a cocalibrated G_2-structure.

Example Consider $\mathbb{T}^6$, with coordinates $\theta_i \in [0, 2\pi)$, endowed with the flat CY-structure

$$\begin{aligned}\omega &= \mathrm{e}^{12} + \mathrm{e}^{34} + \mathrm{e}56,\\ \Omega &= (\mathrm{e}^1 + i\mathrm{e}2) \wedge (\mathrm{e}^3 + i\mathrm{e}^4) \wedge (\mathrm{e}^5 + i\mathrm{e}^6),\end{aligned}$$

where $\mathrm{e}^i = d\theta_i$. $[-\omega] \in H^2(\mathbb{T}^6, \mathbb{Z})$ defines a non-trivial S^1-bundle diffeomorphic to the nil-manifold P with nilpotent Lie algebra $(0, 0, 0, 0, 0, 0, 12 + 34 + 56)$ where we are using Salamon's notation cf. [35]. The connection form is given by

$$\eta = \mathrm{d}\theta_7 + \theta_2\mathrm{e}^1 + \theta_4\mathrm{e}^3 + \theta_6\mathrm{e}^5,$$

where θ_7 denotes the coordinate of the S^1 fibre. Writing $s = r^3$, the CY metric on $P \times \mathbb{R}^+$ can be written as

$$\hat{h} = r^2 g_{\mathbb{T}^6} + r^{-6}(\mathrm{d}\theta_7 + \theta_2\mathrm{e}^1 + \theta_4\mathrm{e}^3 + \theta_6\mathrm{e}^5)^2 + 4r^8\mathrm{d}r^2.$$

Using MAPLE, we have been able to verify that indeed the matrix of curvature 2-form has rank 15 everywhere, confirming that the holonomy is *equal* to $SU(4)$. If we set $\rho = \frac{2}{5}r^5$, then the metric can be written as

$$\hat{h} = \left(\frac{5}{2}\,\rho\right)^{2/5} g_{\mathbb{T}^6} + \left(\frac{5}{2}\,\rho\right)^{-6/5}(\mathrm{d}\theta_7 + \theta_2\mathrm{e}^1 + \theta_4\mathrm{e}^3 + \theta_6\mathrm{e}^5)^2 + \mathrm{d}\rho^2,$$

and in this form, we can easily show that the volume growth $\sim \rho^{8/5}$ and the curvature tensor $|\mathrm{Rm}| \sim \rho^{-2}$ as $\rho \to \infty$. This metric is in fact incomplete at the end $\rho \to 0$ and complete at the end $\rho \to \infty$. By way of comparing with the approach in [21], the $SU(4)$ holonomy metric can also be obtained by evolving the cocalibrated G_2-structure on P given by

$$\varphi = \eta \wedge \omega + Re(\Omega),$$

in the notation of Theorem 3.6. Our approach however avoids the problem of having to solve the Hitchin flow evolution equations, and moreover, it explains why one only obtains $SU(4)$ holonomy metrics rather than $Spin(7)$ ones.

As we have just seen, one cannot obtain a holonomy G_2 metric from a $Spin(7)$-manifold via this construction. This suggests to study instead the geometric structure of the quotient calibrated G_2-structure. We shall do so in detail for the Bryant–Salamon $Spin(7)$ metric in Sect. 5.3.

3.2 The locally conformally parallel quotient

Theorem 3.7 *If* (N^8, Φ) *is a locally conformally parallel* $Spin(7)$*-structure which is* S^1 *-invariant, then at least one of the following holds:*

(1) $N^8 \simeq M^7 \times S^1$ *and the* G_2*-structure on* M *has* $\tau_3 = 0$ *in the notation of* 2.2, *or*
(2) (M^7, φ) *is locally conformally calibrated, i.e.* τ_0 *and* τ_3 *are both zero, and hence,* τ_1 *is closed.*

Proof Since $T^5_{48} = 0$, we know that T^5_7, T^5_{14},T^4_7 and T^4_{27} all vanish. From Proposition 3.2, it follows that $\tau_0 = -s^{4/3}f$, $\tau_1 = -s^{-4/3}(\mathrm{d}(s^{4/3}) + \frac{2}{3} *_\varphi ((\mathrm{d}\eta)^2_7 \wedge *_\varphi \varphi))$, $\tau_2 = -s^{-4/3}(\mathrm{d}\eta)^2_{14}$ and $\tau^4_{27} = 0$. From Proposition 3.2, we also get

$$T^1_7 = -3s^{-4/3}\mathrm{d}(s^{4/3}) - 2s^{-4/3} *_\varphi ((\mathrm{d}\eta)^2_7 \wedge *_\varphi \varphi)$$

Furthermore, differentiating $\mathrm{d}d\Phi = T^1_8 \wedge \Phi$ we have

$$\mathrm{d}T^1_8 \wedge \Phi = 0.$$

As wedging with Φ defines an isomorphism of Λ^2 and Λ^6, it follows that T^1_8 is closed. Since $\mathcal{L}_X\Phi = 0$, we have

$$\mathrm{d}(\iota_X \mathrm{d}\Phi) = 0$$

and this shows that

$$\mathcal{L}_X T^1_8 \wedge \Phi = \mathrm{d}(\iota_X \mathrm{d}\Phi) = 0.$$

Thus, $f = T^1_8(X)$ is constant and if nonzero, then

$$\mathrm{d}\eta = -\frac{1}{f}\mathrm{d}T^1_7.$$

Since the latter is exact, the Chern class is zero and the bundle is topologically trivial, i.e. $N^8 \simeq M^7 \times S^1$. Otherwise if $f = 0$, then $\tau_0 = 0$. □

In [29, Theorem B], Ivanov et al. prove that any compact locally conformally parallel $Spin(7)$-structure fibres over an S^1 and each fibre is endowed with a nearly parallel G_2-structure, i.e. the only nonzero torsion form is τ_0. Thus, it follows from Proposition 3.2 that one can construct many such examples by taking $N^8 = M^7 \times S^1$ where M^7 is a nearly parallel G_2-manifold and endows N^8 with the product $Spin(7)$-structure. In particular, these examples cover case (1) above where the S^1 is only acting on the second factor. We also point out that aside from the fact that the cone metric on a nearly parallel G_2 manifold has holonomy contained in $Spin(7)$, there exists another Einstein metric, with instead positive scalar curvature, on $(0, \pi) \times M^7$ given by the sine-cone construction:

$$g_{sc} := \mathrm{d}t^2 + \sin(t)^2 g_{M^7}.$$

The latter metric however does not seem to have been studied in detail in the literature. The fact that g_{sc} is Einstein is easily deduced since its Riemannian cone is Ricci-flat. Let us now show how situation (2) can arise. The reader might find it helpful to compare the following example with Sect. 6.

Example As above, let $N^8 = S^7 \times S^1$, where S^7 is given the nearly parallel G_2-structure induced by restricting Φ_0 to $S^7 \hookrightarrow \mathbb{R}^8$. The induced G_2-structure φ_{S^7} satisfies

$$\mathrm{d}\varphi_{S^7} = 4 \ *_{S^7} \varphi_{S^7}$$

and defines the standard round metric on S^7. Consider any free S^1 action, generated by a unit vector field X, on S^7 preserving φ_{S^7}. We can then write

$$\varphi_{S^7} = \eta \wedge \omega + \Omega^+ \text{ and } *_{S^7} \varphi_{S^7} = \frac{1}{2}\omega \wedge \omega - \eta \wedge \Omega^-$$

cf. [2]. The intrinsic torsion of the quotient G_2-structure on $M^7 = \mathbb{C}P^3 \times S^1$, with coordinate θ on the circle, is then given by

$$\mathrm{d}\varphi = 3\Big(-\frac{4}{3}\,\mathrm{d}\theta\Big) \wedge \varphi,$$
$$\mathrm{d} *_\varphi \varphi = 4\Big(-\frac{4}{3}\,\mathrm{d}\theta\Big) \wedge *_\varphi \varphi - \Big(\frac{2}{3}\omega + \mathrm{d}\eta\Big) \wedge \varphi,$$

confirming that indeed τ_0 and τ_3 vanish, but τ_1 and τ_2 do not, cf. (2.2) and (2.3).

3.3 The balanced quotient

Since $T_8^1 = 0$, from Proposition 3.2 (1) we have $\tau_0 = 0$ and (2) gives

$$\tau_1 = -\frac{1}{24}(3s^{-4/3}\mathrm{d}(s^{4/3}) + 2s^{-4/3} *_\varphi ((\mathrm{d}\eta)^2_7 \wedge *_\varphi \varphi)). \tag{3.11}$$

Remark 3.8 Differentiating the balanced condition $*_\Phi (\mathrm{d}\Phi) \wedge \Phi = 0$, we get

$$\|\mathrm{d}\Phi\|^2_\Phi vol_\Phi = -(\mathrm{d} *_\Phi \mathrm{d}\Phi) \wedge \Phi = (\Delta_\Phi \Phi) \wedge \Phi.$$

In particular, this shows that $d\Phi = 0$, i.e. Φ is torsion free iff

$$\Delta_\Phi \Phi \wedge \Phi = 0$$

which is a single scalar PDE.

It is well known that a *Spin*(7)-structure can be equivalently characterised by the existence of a non-vanishing spinor ψ, instead of the 4-form Φ. Following Theorem 2.4, the induced metric has holonomy contained in *Spin*(7) if and only if the spinor is parallel. From this perspective, the action of the Dirac operator D on the spinor was shown to be completely determined by the torsion form T_8^1 cf. [28, (7.21)]. As a consequence, it follows that balanced *Spin*(7)-structures are characterised by the fact that they admit harmonic spinors, i.e. $D\psi = 0$.

In [4], the authors construct many such examples on nilmanifolds by adopting a spinorial point of view. We instead here describe, via a few simple examples, a construction of balanced $Spin(7)$-structures starting from suitable G_2-structures. Henceforth, we shall restrict to the case when $s=1$ so that (3.11) can be equivalently written as

$$(\mathrm{d}\eta)^2_7 = -4 *_\varphi (\tau_1 \wedge *_\varphi \varphi). \tag{3.12}$$

Theorem 3.9 *(N^8, Φ) is a free S^1-invariant balanced $Spin(7)$-structure if and only if the G_2-structure (M^7, φ) has $\tau_0 = 0$ and admits a section $\lambda \in \Omega^2_{14}$ such that*

$$\in H^2(M, \mathbb{Z}),$$

or equivalently,

$$\{\kappa + 4 *_\varphi (\tau_1 \wedge *_\varphi \varphi) \mid [\kappa] \in H^2(M, \mathbb{Z})\} \cap \Omega^2_{14} \neq \emptyset. \tag{3.13}$$

Moreover, the $Spin(7)$-structure on the total space can be written as

$$\Phi = \eta \wedge \varphi + *_\varphi \varphi \tag{3.14}$$

where the connection form η satisfies $\mathrm{d}\eta = \lambda - 4 *_\varphi (\tau_1 \wedge *_\varphi \varphi)$, i.e.

$$(\mathrm{d}\eta)^2_7 = -4 *_\varphi (\tau_1 \wedge *_\varphi \varphi) \ \text{ and } (\mathrm{d}\eta)^2_{14} = \lambda.$$

Proof The *if* statement is clear since given λ we can always choose a connection η with $\mathrm{d}\eta = \lambda - 4 *_\varphi (\tau_1 \wedge *_\varphi \varphi)$. Then, define Φ by (3.14). The *only if* statement follows by setting $\lambda = (\mathrm{d}\eta)^2_{14}$. □

The reader might find such a theorem of little practical use in general. However, as we shall illustrate below via concrete examples, when M^7 is a nilmanifold, Theorem 3.9 provides a systematic way of constructing balanced $Spin(7)$-structures.

Example Let $M^7 = B^5 \times \mathbb{T}^2$, where B is a nilmanifold with an orthonormal coframing given by e^i for $i = 0, \dots, 4$ and satisfying

$$\begin{aligned} \mathrm{d}\mathrm{e}^i &= 0, \quad \text{for } i \neq 4 \\ \mathrm{d}\mathrm{e}^4 &= \mathrm{e}^{02} + \mathrm{e}^{31}, \end{aligned}$$

and for the flat $\mathbb{T}^2$ by e^6 and e^7. The G_2-structure defined by

$$\varphi = \mathrm{e}^{137} + \mathrm{e}^{104} + \mathrm{e}^{162} + \mathrm{e}^{306} + \mathrm{e}^{324} - \mathrm{e}^{702} - \mathrm{e}^{746}$$

has $\tau_0 = 0$. Hence from (3.12), to construct a balanced $Spin(7)$-structure we need to find a connection η whose Λ^2_7-curvature component satisfies

$$\begin{aligned} (\mathrm{d}\eta)^2_7 &= -4 *_\varphi (\tau_1 \wedge *_\varphi \varphi) \\ &= \frac{2}{3}(\mathrm{e}^{03} + \mathrm{e}^{12} - \mathrm{e}^{47}). \end{aligned}$$

Choosing $(\mathrm{d}\eta)^2_{14}$ to be either of the following 2-forms in Ω^2_{14}:

$$\frac{1}{3}(\mathrm{e}^{03} + \mathrm{e}^{12} + 2\mathrm{e}^{47}),$$
$$\frac{2}{3}(2\mathrm{e}^{12} - \mathrm{e}^{03} + \mathrm{e}^{47})$$

gives connections with curvature forms $\mathrm{e}^{03} + \mathrm{e}^{12}$ and $2\mathrm{e}^{12}$, respectively, and thus, we obtain two distinct balanced *Spin*(7)-structures. Denoting η by e^5, the *Spin*(7) form can once again be written in the standard form (2.4). This construction shows that given a balanced *Spin*(7)-structure on an S^1-bundle, we can modify the Λ^2_{14}-component of the curvature form while keeping its Λ^2_7-component, already determined by τ_1, unchanged to construct a new balanced structure.

Suppose that we have fixed $\mathrm{d}\eta = \mathrm{de}^5 = 2\mathrm{e}^{12}$. We can now take the S^1-quotient with respect to the Killing vector field e_4. In other words, the total space can be viewed as a different circle bundle with the new connection form $\tilde{\eta} := \mathrm{e}^4$. We can repeat the above procedure with the new G_2-structure $\tilde{\varphi} := e_4 \lrcorner \Phi$, explicitly given by

$$\tilde{\varphi} = \mathrm{e}^{501} + \mathrm{e}^{523} + \mathrm{e}^{567} + \mathrm{e}^{026} + \mathrm{e}^{073} - \mathrm{e}^{127} - \mathrm{e}^{136},$$

which of course has $\tilde{\tau}_0 = 0$. Once again to construct a balanced *Spin*(7)-structure, we need a connection 1-form ξ satisfying

$$\begin{aligned}(\mathrm{d}\xi)^2_7 &= (\mathrm{d}\tilde{\eta})^2_7\\ &= \frac{2}{3}(\mathrm{e}^{02} + \mathrm{e}^{31} - \mathrm{e}^{57}).\end{aligned}$$

If we choose

$$(\mathrm{d}\xi)^2_{14} = (\mathrm{d}\tilde{\eta})^2_{14} + \mathrm{e}^{51} + 2\mathrm{e}^{26} + \mathrm{e}^{37},$$

then $\mathrm{d}\xi = \mathrm{e}^{02} + \mathrm{e}^{31} + \mathrm{e}^{51} + 2\mathrm{e}^{26} + \mathrm{e}^{37}$ indeed defines an element in $H^2(\tilde{M}, \mathbb{Z})$. Thus, this gives yet another balanced *Spin*(7)-structure. These three examples were found in [4] denoted by $\mathcal{N}_{6,22}$, $\mathcal{N}_{6,23}$ and $\mathcal{N}_{6,24}$, by instead using the spinorial approach described above and computing the Dirac operator.

The above examples in fact illustrate a new procedure for constructing balanced *Spin*(7)-structures on nilmanifolds: starting from an S^1-invariant balanced *Spin*(7)-structure on a nilmanifold, we know that the quotient G_2-structure φ has $\tau_0 = 0$. Given that the de Rham complex of the quotient nilmanifold P^7 is completely determined by the Chevalley–Eilenberg complex of the associated nilpotent Lie algebra, it is relatively straightforward to compute the set (3.13), via say Maple. Thus, by choosing distinct λs we can classify all invariant balanced *Spin*(7)-structures on *different* nilmanifolds which arise as circle bundles over (P^7, φ). A general classification however appears to be quite hard. Closed G_2-structures on nilpotent Lie algebras, hence with $\tau_0 = 0$, were classified in [13]. Although a classification of 7-dimensional nilpotent Lie algebras is known cf. [24], those admitting G_2-structures with only vanishing τ_0 are still unknown.

Having encountered several examples of *Spin*(7)-structures, it seems worth making a brief digression from our main example and derives some curvature formulae of *Spin*(7)-structures in terms of the torsion forms, rather than the metric, that the reader might find quite practical in specific examples.

4 Ricci and Scalar curvatures

In this section, we derive formulae for the Ricci and scalar curvatures of $Spin(7)$-structures in terms of the torsion forms. As a corollary, we show that under the free S^1 action hypothesis and that the circle orbits have constant size, $(\mathrm{d}\eta)^2_7$ can be interpreted as the mean curvature vector of the circle orbits.

Formulae for the Ricci and scalar curvatures of G_2-structures in terms of the torsion forms seem to have first appeared in [10, (4.28), (4.30)] and for the $Spin(7)$ case in [28, (1.5), (7.20)]. The approach taken in each paper to derive the curvature formulae differs greatly. While Ivanov uses the equivalent description of $Spin(7)$-structures as corresponding to the existence of certain parallel spinors, Bryant uses a more representational theoretic argument. In [28], however, it is not obvious from the Ricci formula that it is a symmetric tensor and moreover the presence of a term involving the covariant derivative of the torsion form makes explicit computations quite hard. We instead adapt the technique outlined in [10, Remark 10] to the $Spin(7)$ setting and derive an alternative formula.

Proposition 4.1 *The Ricci and scalar curvatures of a* $Spin(7)$*-structure* (N, Φ) *are given by*

$$
\begin{aligned}
Ric(g_\Phi) = {} & \left(\frac{5}{8}\,\delta T^1_8 + \frac{3}{8}\,\|T^1_8\|^2_\Phi - \frac{2}{7}\,\|T^5_{48}\|^2_\Phi\right) g_\Phi \\
& + \mathrm{j}\left(-3\,\delta(T^1_8 \wedge \Phi) + 4\,\delta T^5_{48} - 2\,(T^1_8 \wedge *_\Phi\, T^5_{48}) - \frac{9}{4} *_\Phi (T^1_8 \wedge \Phi) \wedge T^1_8)\right) \\
& + \frac{1}{2}\, g_\Phi(\cdot \lrcorner *_\Phi T^5_{48}, \cdot \lrcorner *_\Phi T^5_{48}) \\
Scal(g_\Phi) = {} & \frac{7}{2}\,\delta T^1_8 + \frac{21}{8}\,\|T^1_8\|^2_\Phi - \frac{1}{2}\,\|T^5_{48}\|^2_\Phi
\end{aligned}
$$

where $\delta := - *_\Phi \mathrm{d} *_\Phi$ *is the codifferential of* Φ.

Proof Following Bryant's argument in [10] for the G_2 case, we first define the two $Spin(7)$-modules V_1 and V_2 by

$$(\mathfrak{gl}(8,\mathbb{R})/\mathfrak{so}(7)) \otimes S^k(\mathbb{R}^8) = V_k \oplus (\mathbb{R}^8 \otimes S^{k+1}(\mathbb{R}^8)),$$

where $S^k(\mathbb{R}^8)$ denotes the k *th* symmetric power. We shall refer to these modules to also mean the corresponding associated vector bundles on N. Representing irreducible $Spin(7)$-modules by the highest weight vector, we have the following decomposition:

$$
\begin{aligned}
V_1 &= V_{0,0,1} \oplus V_{1,0,1},\\
V_2 &= V_{0,0,0} \oplus V_{1,0,0} \oplus V_{0,1,0} \oplus V_{1,1,0} \oplus V_{2,0,0} \oplus V_{0,2,0} \oplus 2V_{0,0,2} \oplus V_{1,0,2},\\
S^2(V_1) &= 2V_{0,0,0} \oplus V_{1,0,0} \oplus V_{0,1,0} \oplus 2V_{1,1,0} \oplus 2V_{2,0,0} \oplus V_{0,2,0} \oplus 4V_{0,0,2} \oplus 2V_{1,0,2} \oplus V_{2,0,2}.
\end{aligned}
$$

It is known that the second-order term of the scalar curvature values in the trivial component of V_2 of which there is only one. This is spanned by δT^1_8. The first-order terms are at most quadratic in sections of V_1 of which there are only two components. These are just the norm squared of the torsion forms: $\|T^1_8\|^2_\Phi$ and $\|T^5_{48}\|^2_\Phi$. So the scalar curvature can be expressed in terms of these three terms and to determine the coefficients it suffices to test it on a few examples. A similar argument applies for the traceless part of the Ricci tensor. The second-order terms correspond to sections of the module $V_{0,0,2} \cong S^2_0(\mathbb{R}^8)$ in V_2, and there are exactly two of those. These are spanned by the projections of $\delta(T^1_8 \wedge \Phi)$ and δT^5_{48}. For the first-order terms, they are given by sections of the module $V_{0,0,2}$ in $S^2(V_1)$. There are in fact four of those: one quadratic in T^1_8, two quadratic in T^5_{48} and one mixed term. All but one quadratic term in T^5_{48} appear in the Ricci formula. Again to determine the coefficients, it suffices to test the formula on a few examples. This can be done quite easily using MAPLE. □

From the results of Sect. 3, we have the following lemma:

Lemma 4.2 *In the S^1-invariant setting, δT^1_8, $\|T^1_8\|^2_\Phi$ and $\|T^5_{48}\|^2_\Phi$ are given in terms of the data (M^7, φ, η, s) by*

$$\delta T^1_8 = \frac{1}{7} s^{-4/3}\delta_\varphi(24 s^{2/3}\tau_1 + 4 s^{-1/3}\mathrm{d}s + 2 s^{-2/3} *_\varphi ((\mathrm{d}\eta)^2_7 \wedge *_\varphi \varphi))) \tag{4.1}$$

$$\|T^1_8\|^2_\Phi = s^{-2/3}\tau_0^2 + \frac{1}{49} s^{-2/3}\|24\tau_1 + 4 s^{-1}\mathrm{d}s + 2 s^{-4/3} *_\varphi ((\mathrm{d}\eta)^2_7 \wedge *_\varphi \varphi)\|^2_\varphi \tag{4.2}$$

$$\begin{aligned}\|T^5_{48}\|^2_\Phi = {} & s^{-2/3}\|\tau_3\|^2_\varphi + s^{-4/3}\|s^{-1}(\mathrm{d}\eta)^2_{14} + s^{1/3}\tau_2\|^2_\varphi \\ & + s^{-10/3}\|\frac{8}{7}(\mathrm{d}\eta)^2_7 + \frac{4}{7} *_\varphi (\mathrm{d}(s^{4/3}) \wedge *_\varphi \varphi) + \frac{4}{7} s^{4/3} *_\varphi (\tau_1 \wedge *_\varphi \varphi)\|^2_\varphi \\ & + 4\|\frac{3}{7} s^{2/3}\tau_1 + \frac{2}{7} s^{-2/3} *_\varphi ((\mathrm{d}\eta)^2_7 \wedge *_\varphi \varphi) + \frac{3}{7} s^{-2/3}\mathrm{d}(s^{4/3})\|^2_\varphi\end{aligned} \tag{4.3}$$

where δ_φ is the codifferential of φ acting on k-forms by $\delta_\varphi = (-1)^k *_\varphi \mathrm{d} *_\varphi$.

Proof This is a straightforward albeit long computation using the expressions for the torsion forms of the *Spin*(7)-structure from Proposition 3.2. □

Of course, these formulae are far from practical to compute the scalar curvature, but nonetheless in the case of Riemannian submersions, they do simplify considerably.

Corollary 4.3 *In the case of a Riemannian submersion, i.e. $s = 1$,*

$$\begin{aligned} Scal(g_\Phi) = {} & Scal(g_\varphi) - \frac{1}{2}\|\mathrm{d}\eta\|^2_\varphi - g_\varphi((\mathrm{d}\eta)^2_{14}, \tau_2) + \delta_\varphi(*_\varphi ((\mathrm{d}\eta)^2_7 \wedge *_\varphi \varphi)) \\ & + 4 g_\varphi(*_\varphi \tau_1, (\mathrm{d}\eta)^2_7 \wedge *_\varphi \varphi).\end{aligned}$$

Proof This follows by combining the above lemma with our formula for scalar curvature and the one in the G_2 case from [10, (4.28)]. □

Remark 4.4 Comparing the above formula with the general formula for scalar curvatures in Riemannian submersions cf. [7, (9.37)], we can geometrically interpret the anti-instanton part of the curvature form:

$$*_\varphi ((\mathrm{d}\eta)^2_7 \wedge *_\varphi \varphi)$$

as the dual with respect to g_φ of the 'mean' curvature vector of the S^1 fibres. For an immersed submanifold of codimension greater than one, the mean curvature is defined by a normal vector, rather than a scalar, cf. [7, (1.73)]. In our present situation, the 'mean' curvature of the circle fibres is determined by a vector in the rank 7 normal bundle, which we can identify with $(\mathrm{d}\eta)^2_7$. Therefore, it vanishes if and only if the circles are geodesics. Of course, the word 'mean' here is redundant as our submanifolds are only one dimensional.

We now turn to our main example, namely the S^1 quotient, of the Bryant–Salamon metric.

5 S^1-quotient of the Spinor bundle of S^4

Let us first outline our general strategy to performing the quotient construction. Recall that the fibres of the spinor bundle of S^4 are diffeomorphic to $\mathbb{R}^4 \simeq \mathbb{C}^2$. We shall consider the action of the diagonal $U(1)$ in $SU(2)$ on the fibres. This fibrewise quotient can be interpreted as a reverse Gibbons–Hawking (GH) ansatz. We begin by giving a brief overview of the GH construction in Sect. 5.1 and describe it in detail for the Hopf map by viewing our quotient construction as a fibrewise Hopf fibration in Sect. 5.2. Extending this to the total space, we construct the quotient G_2-structure on the anti-self-dual bundle of S^4, see Sect. 5.3. From the results of Sect. 3.1 we know that the quotient G_2-structure cannot be torsion free, but on the other hand, it is well known that the anti-self-dual bundle of S^4 also admits a holonomy G_2 metric cf. [11]. Motivated by the fact that both of these G_2 -structures are asymptotic to a cone metric on $\mathbb{CP}^3$, we study the induced $SU(3)$-structures. In Sect. 5.4, we give explicit formulae for the $SU(3)$-structures on the link and show that in both cases the induced almost complex structure corresponds to the nearly Kähler one.

5.1 The Gibbons–Hawking ansatz

Since we shall use the Gibbons–Hawking ansatz in the next section, we quickly describe the general construction. In essence, it provides a local construction of hyperKähler metrics starting from a 3-manifold together with a connection form on an S^1-bundle and a harmonic function. We begin by recalling the definition of a hyperKähler manifold.

Definition 5.1 An oriented Riemannian manifold (M^{4n}, g) is called hyperKähler (HK) if it admits a triple of closed non-degenerate 2-forms ω_1, ω_2 and ω_3 satisfying the compatibility conditions

$$\frac{1}{2}\,\omega_i \wedge \omega_j = \delta_{ij}\,\mathrm{dvol}_g.$$

Let U be an open subset of $\mathbb{R}^3$ with the standard Euclidean metric g_0 and M^4 a principal S^1 bundle on U generated by a vector field X normalised to have period 2π. Suppose we are also given a connection 1-form η on M^4 such that $\eta(X) = 1$ (using the natural identification $\mathfrak{u}(1) \cong \mathbb{R}$). For a positive harmonic function f on U satisfying $*_{g_0}\, df = d\eta$, the metric

$$g_{M^4} = f\pi^* g_0 + f^{-1}\eta \otimes \eta,$$

and the anti-self-dual (ASD) 2-forms

$$\omega_1 = \eta \wedge \mathrm{d}x_1 - f\mathrm{d}x_2 \wedge \mathrm{d}x_3$$
$$\omega_2 = \eta \wedge \mathrm{d}x_2 - f\mathrm{d}x_3 \wedge \mathrm{d}x_1$$
$$\omega_3 = \eta \wedge \mathrm{d}x_3 - f\mathrm{d}x_1 \wedge \mathrm{d}x_2$$

define a HK structure on M^4. By construction, the triple of symplectic forms are closed:

$$\begin{aligned}\mathrm{d}\omega_1 &= (*\ \mathrm{d}f) \wedge \mathrm{d}x_1 - \mathrm{d}f \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3\\ &= \partial_1 f \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_1 - \partial_1 f \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3\\ &= 0\end{aligned}$$

and likewise for ω_2 and ω_3. The compatible almost complex structures are defined by $g(J_i v, w) = \omega_i(v, w)$. The closedness of ω_i is equivalent to $\nabla^{g_{M^4}} J_i = 0$, i.e. J_i are indeed complex structures and thus $Hol(g) \subseteq Sp(1)$.

Note that setting $U = \mathbb{R}^3$, $\alpha = \mathrm{d}x_0$ and f constant gives a flat HK cylinder. More interestingly, the projection map $\pi : \mathbb{R}^4 - \{0\} \to \mathbb{R}^3 - \{0\}$ given in quaternionic coordinates by $\pi(p) = \frac{1}{2}\bar{p}ip$ is the moment map of the Hopf bundle, where the S^1 action is generated by left multiplication by $-i$. It turns out that this map can be smoothly extended to the origin whenever f is a suitable harmonic function. Moreover, one can recover the flat HK metric on $\mathbb{R}^4$, which we shall describe explicitly in the next subsection.

5.2 S^1-quotient of a fixed fibre of the spinor bundle

We begin by reminding the reader of the construction of the Bryant–Salamon $Spin(7)$-manifold. Given S^4 with the standard round metric and orientation, we denote by $P \simeq SO(5)$ the total space of the $SO(4)$-structure. Since $H^2(S^4, \mathbb{R}) = 0$, in particular the second Stiefel–Whitney class vanishes; hence, it is a spin manifold, so we can lift P to its double cover $\tilde{P}$. The spin group can be described explicitly via the well-known isomorphism

$$Spin(4) \cong Sp(1)_+ \times Sp(1)_- \cong SU(2)_+ \times SU(2)_-$$

where the $\pm$ subscripts distinguish the two copies of $SU(2)$. Taking the standard representation of $SU(2)_-$ on $\mathbb{C}^2_-$, we construct the (negative) spinor bundle $V_- := \tilde{P} \times_{SU(2)_-} \mathbb{C}^2_-$ as an associated bundle.

There is also an action of $SU(2)$ on the fibres of V_- which can be described as follows. If we ignore the complex structure, the fibres of V_- are simply $\mathbb{R}^4$ and its complexification is isomorphic to $\mathbb{C}^2_- \otimes \mathbb{C}^2$. The desired $SU(2)$ action is the standard action on $\mathbb{C}^2$ and is well defined on the realification of $V_- \otimes \mathbb{C}$. In the description of the Bryant–Salamon construction in [11], this action on the fibre can also be viewed as a global $Sp(1)$ action (acting on the right) on $\mathbb{H}$ in

$$\tilde{P} \times \mathbb{H} \xrightarrow{/Sp(1)_-} V_-,$$

thus commuting with the left action of $Sp(1)_-$ and hence passes to the quotient. Having now justified the existence of this $SU(2)$ action, we fix a point, $p \in S^4$ and describe the action of an $S^1 \hookrightarrow SU(2)$ on the fibre of V_-. This will enable us to describe a fibrewise HK quotient and then reconstruct the $\mathbb{R}^4$ fibre using the Gibbons–Hawking ansatz with harmonic function $f = 1/2R$ where R denotes the radius in $\mathbb{R}^3 - \{0\}$ as described in the previous section. Note that topologically, the base manifold is just the anti-self-dual bundle of S^4 which we denote by $\Lambda^2_- S^4$. This is due to the fact that the quotient construction reduces the

$Sp(1)_-$ action on the $\mathbb{R}^4_-$ fibre to an action of $SO(3)_-$ on $\mathbb{R}^3 = \mathbb{R}^4/S^1$, as we shall see below, and the associated bundle construction for this representation is $\Lambda^2_- S^4$ cf. [34].

Let (x_1, x_2, x_3, x_4) denote the coordinates on the fibre, so that we may write the fibre metric as

$$g = \sum_{i=1}^{4} \mathrm{d}x_i \otimes \mathrm{d}x_i,$$

i.e. g denotes the restriction of the Bryant–Salamon metric g_Φ to the vertical space. Denoting by r the radius function in the fibre, i.e. $r^2 = \sum_{i=1}^4 x_i^2$, we have $rdr = \sum_{i=1}^4 x_i \mathrm{d}x_i$. We make the identifications $\mathbb{R}^4 \cong \mathbb{C}^2 \cong \mathbb{H}$ by

$$(x_1, x_2, x_3, x_4) = (x_1 + ix_2, x_3 + ix_4) = x_1 + ix_2 + jx_3 - kx_4$$

Consider now the $U(1)$ action on $\mathbb{R}^4 \cong \mathbb{C}^2$ given by

$$\mathrm{e}^{i\theta}(z_1, z_2) = (\mathrm{e}^{-i\theta} z_1, \mathrm{e}^{+i\theta} z_2)$$

or equivalently by left multiplication by $-i$ on $\mathbb{H}$. Note that this S^1 is just the diagonal torus of $SU(2)$. The Killing vector field X generating this action is given by

$$X = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} - x_4 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_4}.$$

and thus $\|X\|_g = r$. We also endow the fibre with a HK structure given by the triple

$$\begin{aligned}
\gamma_1 &= \mathrm{d}x_1 \wedge \mathrm{d}x_2 - \mathrm{d}x_3 \wedge (-\mathrm{d}x_4)\\
\gamma_2 &= \mathrm{d}x_1 \wedge \mathrm{d}x_3 - (-\mathrm{d}x_4) \wedge \mathrm{d}x_2\\
\gamma_3 &= \mathrm{d}x_1 \wedge (-\mathrm{d}x_4) - \mathrm{d}x_2 \wedge \mathrm{d}x_3
\end{aligned}$$

They can be extended to a local orthonormal basis of the bundle $\Lambda^2_- S^4$, but the resulting forms will *not* be closed. The spin bundle *does* have a global HK structure, but arising from $SU(2)_+$ and since we have already fixed one of its complex structures, this HK structure is not relevant. In view of our quotient construction, we define

$$\eta := r^{-2} g_\Phi(X, \cdot) = r^{-2}(x_2 \mathrm{d}x_1 - x_1 \mathrm{d}x_2 - x_4 \mathrm{d}x_3 + x_3 \mathrm{d}x_4),$$

i.e η is a connection 1-form on V_-. The map

$$\begin{aligned}
\mu :\ & \mathbb{R}^4 \to \mathbb{R}^3\\
(x_1, x_2, x_3, x_4) &\mapsto (\mu_1, \mu_2, \mu_3)
\end{aligned}$$

where

$$\begin{aligned}
\mu_1 &= \frac{1}{2}(x_1^2 + x_2^2 - x_3^2 - x_4^2)\\
\mu_2 &= x_1 x_4 + x_2 x_3\\
\mu_3 &= x_1 x_3 - x_2 x_4
\end{aligned}$$

is the HK moment map for the $U(1)$ action. By identifying $\mathbb{R}^3$ with $\mathrm{Im}(\mathbb{H})$, μ can also be expressed using quaternions as:

$$\mu(q) = \frac{1}{2}\overline{q}\, i\, q, \qquad q = x_1 + x_2 i + x_3 j - x_4 k,$$

making the S^1-invariance clear. Thus, μ induces a diffeomorphism

$$\mathbb{R}^4/U(1) \simeq \mathbb{R}^3.$$

Note that strictly speaking this action is not free, but nonetheless the construction can be carried out on $\mathbb{R}^4 - \{0\}$ and can be extended smoothly to the origin. A direct computation gives

$$\begin{aligned}
\gamma_3 &= \mathrm{d}x_{32} + \mathrm{d}x_{41}\\
&= r^{-2}\Big((x_2\mathrm{d}x_1 - x_1\mathrm{d}x_2 - x_4\mathrm{d}x_3 + x_3\mathrm{d}x_4) \wedge (x_1\mathrm{d}x_3 + x_3\mathrm{d}x_1 - x_2\mathrm{d}x_4 - x_4\mathrm{d}x_2)\\
&\quad - (x_1\mathrm{d}x_1 + x_2\mathrm{d}x_2 + x_3\mathrm{d}x_3 + x_4\mathrm{d}x_4) \wedge (x_1\mathrm{d}x_4 + x_4\mathrm{d}x_1 + x_2\mathrm{d}x_3 + x_3\mathrm{d}x_2)\Big)\\
&= \eta \wedge \mathrm{d}\mu_3 - f\,\mathrm{d}\mu_1 \wedge \mathrm{d}\mu_2.
\end{aligned}$$

where $f = \frac{1}{2R}$ and $R = \sqrt{\mu_1^2 + \mu_3^2 + \mu_3^2}$ is the radius on $\mathbb{R}^3$. Similarly, we obtain

$$\begin{aligned}
\gamma_1 &= \eta \wedge \mathrm{d}\mu_1 - f\,\mathrm{d}\mu_2 \wedge \mathrm{d}\mu_3\\
\gamma_2 &= \eta \wedge \mathrm{d}\mu_2 - f\,\mathrm{d}\mu_3 \wedge \mathrm{d}\mu_1
\end{aligned}$$

This confirms that η is the connection form that features in the Gibbons–Hawking ansatz with

$$g_{\mathbb{R}^4} = f^{-1}\eta \otimes \eta + f\,\pi^* g_U,$$

where $g_U = \mathrm{d}\mu_1^2 + \mathrm{d}\mu_2^2 + \mathrm{d}\mu_3^2$ is the Euclidean metric on $\mathbb{R}^3$ with volume form $vol_{\mathbb{R}^3} = \mathrm{d}\mu_{123}$. Using $R^2 = \sum_{i=1}^3 \mu_i^2 = \frac{1}{4}r^4$, we can directly verify that

$$*_{\mathbb{R}^3}\, \mathrm{d}f = \mathrm{d}\eta.$$

Having described the GH ansatz for the Euclidean space, we proceed to our main example.

5.3 S^1-quotient of the Bryant–Salamon cone metric

We shall now take the quotient of Bryant–Salamon metric by applying the above construction to each $\mathbb{R}^4$ fibre. The conical Bryant–Salamon *Spin*(7) 4-form is given (pointwise) in our notation by

$$\Phi = 16r^{-8/5}\mathrm{d}x_{1234} + 20r^{2/5}\sum \gamma_i \wedge \epsilon_i + 25r^{12/5}\mathrm{dvol}_{S^4},$$

where $\{\epsilon_i\}$ is a local basis of ASD forms on S^4 and dvol_{S^4} is the (pullback of) the volume form. The *Spin* (7) metric is then given by

$$g_\Phi = 4r^{-4/5}\sum_{i=1}^{4} \mathrm{d}x_i^{\,2} + 5r^{6/5}g_{s^4},$$

and so the 1-forms $\mathrm{d}\mu_i$ (or rather, $\pi^*\mathrm{d}\mu_i = \mathrm{d}(\mu_i \circ \pi)$) have norm

$$\|\mathrm{d}\mu_i\|^2_\Phi = \frac{1}{4} r^{14/5}.$$

On the other hand, from (3.1) we compute

$$s^{-2} = g_\Phi(X, X) = 4r^{6/5},$$

so $s = r^{-3/5}$. We know that the G_2 metric g_φ satisfies

$$g_\varphi = s^{-2/3}(g_\Phi - s^{-2}\eta^2) = r^{2/5}(g_\Phi - 4r^{6/5}\eta^2).$$

Considering the volume form of the fibre of the quotient, we have

$$\begin{aligned} -r^{-2}\mathrm{d}\mu_{123} &= -x_3\mathrm{d}x_{123} - x_4\mathrm{d}x_{124} + x_1\mathrm{d}x_{134} + x_2\mathrm{d}x_{234} \\ &= X \lrcorner \mathrm{d}x_{1234}. \end{aligned}$$

Defining $\mathrm{d}\nu_i = \iota_X\gamma_i$, we have that $d\nu_{123} = -\mathrm{d}\mu_{123}$. Putting all together, we have

$$\begin{aligned} X\lrcorner\Phi &= X\lrcorner\, 16r^{-8/5}\mathrm{d}x_{1234} + 20r^{2/5}\sum (X\lrcorner\gamma_i)\wedge\epsilon_i \\ &= 2^{11/5}(R^{-9/5}\mathrm{d}\nu_{123} + 5R^{1/5}\sum_{i=1}^{3}\mathrm{d}\nu_i\wedge\epsilon_i). \end{aligned}$$

We can now extend this pointwise construction to the whole of V^-. From our construction, the induced G_2-structure on the quotient is given (after rescaling) by

$$\varphi_{GH} = \frac{1}{6}R^{-9/5}\,\beta + 5R^{1/5}\,\mathrm{d}\tau$$

We are here using the globally well-defined forms defined in [34, pg 94] (see also 'Appendix') where τ is tautological 2-form on the ASD bundle and $\frac{1}{6}\beta$ is the volume form of the fibre which was pointwise denoted by dx_{1234}. By contrast, the holonomy G_2 form is given by

$$\varphi_{BS} = \frac{1}{6}R^{-3/2}\,\beta + 2R^{1/2}\,\mathrm{d}\tau.$$

Since the Bryant–Salamon metric on $\mathbb{R}^+\times\mathbb{CP}^3$ is just the cone metric on $\mathbb{CP}^3$ endowed with its nearly Kähler (NK) structure, we may also write it as

$$g_{BS} = \mathrm{d}t^2 + t^2\left(\frac{1}{2}g_{S^4} + \frac{1}{4}\hat{g}_{S^2}\right)$$

where t denotes the coordinate of $\mathbb{R}^+$ and $g_{NK} := \frac{1}{2}g_{S^4} + \frac{1}{4}\hat{g}_{S^2}$ is the NK metric (up to homothety). Here, we are interpreting g_{NK} as a metric on the twistor space of S^4 where g_{S^4} denotes the pullback of the round metric and $\hat{g}_{S^2}$ the metric on the S^2 fibres (see 'Appendix' for more details). Comparing φ_{BS} with φ_{GH} and using our expression for g_{BS}, we can perform a pointwise computation as above and show that

$$g_{GH} = \mathrm{d}t^2 + \frac{8}{5}t^2\left(\frac{1}{2}g_{S^4} + \frac{1}{10}\hat{g}_{S^2}\right).$$

The quotient metric is thus the cone metric on the twistor space of S^4 but with 'smaller' S^2 fibres. In order to gain better understanding of the geometric structure on the $\mathbb{CP}^3$, we look at the induced $SU(3)$-structure.

5.4 Remarks on the induced $SU(3)$-structure on $\mathbb{CP}^3$

We remind the reader that an $SU(3)$-structure on a 6-manifold consists of a non-degenerate 2-form ω and a pair of 3-forms $\Omega^{\pm}$ satisfying the compatibility conditions

$$\omega \wedge \Omega^{\pm} = 0 \text{ and } \frac{2}{3}\omega^3 = \Omega^+ \wedge \Omega^-.$$

The relevance of this here comes the fact that oriented hypersurfaces in G_2-structures naturally inherit such a structure. If $\mathbf{n}$ denotes the unit normal to a hypersurface Q^6, then the forms are given by:

$$\begin{aligned} \omega &= \mathbf{n} \lrcorner \varphi \\ \Omega^+ &= \varphi\big|_{Q^6} \\ \Omega^- &= -\mathbf{n} \lrcorner *_\varphi \varphi. \end{aligned}$$

It is known that the NK structure on $\mathbb{C}P^3$ satisfies

$$\mathrm{d}\omega_{NK} = 3\ \Omega^+_{NK} \text{ and } \mathrm{d}\Omega^-_{NK} = -2\ \omega^2_{NK}.$$

In contrast, the $SU(3)$-structure $(\omega_{GH}, \Omega^+_{GH}, \Omega^-_{GH})$ on the link (for $t = 1$) of the quotient G_2 -structure satisfies

$$\begin{aligned} \mathrm{d}\omega_{GH} =& 3\ \Omega^+_{GH} \\ \mathrm{d}\Omega^-_{GH} =& -2\ \omega^2_{GH} - \frac{1}{5}\left(\frac{1}{5}\,\sigma - \tau\right) \wedge \omega_{GH} \end{aligned}$$

The proof is a straightforward computation using the formulae in 'Appendix'. Two things worth noting are that $\Omega^+_{GH} = \frac{32}{25}\,\Omega^+_{NK} = \frac{8}{25}\,d\tau$, so in particular both define the same almost complex structure and the extra-torsion component $\frac{1}{5}\sigma - \tau$ lives in $[\Lambda^{1,1}_0]$. Using the formulae from [5, Thm 3.4–3.6], we can confirm directly that this metric is not Einstein which is consistent with the canonical variation approach [7, pg. 258] which asserts that there are only two Einstein metrics in this family: the Fubini–Study metric and the NK one. Moreover, it was also shown in [20] that in fact there is no other cohomogeneity one NK structure on $\mathbb{C}P^3$. Nonetheless, the scalar curvature of g_{GH} is still constant and positive:

$$\begin{aligned} Scal(g_{GH}) &= 30 - \frac{1}{2} \cdot \|\frac{1}{5}\left(\frac{1}{5}\sigma - \tau\right)\|^2_{g_{GH}} \\ &= 30 - \frac{1}{2} \cdot \frac{3}{8} \\ &= \frac{477}{16} > 0. \end{aligned}$$

It is also worth pointing out that this $SU(3)$-structure is half-flat cf. [12, 18] and as such can be evolved by the Hitchin flow to construct a torsion-free G_2-structure. The resulting metric belongs to the general class of metrics of the form

$$g = \mathrm{d}t^2 + a(t)^2 \hat{g}_{S^2} + b(t)^2 g_{S^4},$$

which were considered in [14, Sect. 5B]. It was also shown, after suitable normalisations, that the Bryant–Salamon metrics are the only solutions to this system.

Remark 5.2 Observe that, as in the GH ansatz for the Hopf map, this construction extends to the smooth Bryant–Salamon $Spin(7)$ metric with the same circle action but which now has as fixed point locus an S^4 corresponding to the zero section of the spinor bundle. Extending the above construction to the smooth metric simply amounts to replacing R by $R+1$ in the expressions φ_{BS} and φ_{GH}. Thus, we obtain a closed G_2-structure on all of $\Lambda^2_- S^4$.

6 S^1-quotient of flat *Spin*(7) metric

We now consider a simpler situation that of the S^1-reduction of the flat $Spin(7)$-structure $\Phi_0 = \frac{1}{8}(-\mathrm{d}\alpha_1^2 + \mathrm{d}\alpha_2^2 + \mathrm{d}\alpha_3^2)$ on $\mathbb{R}^8$ where

$$\begin{aligned}
\alpha_1 &= -x_1\mathrm{d}x_0 + x_0\mathrm{d}x_1 + x_3\mathrm{d}x_2 - x_2\mathrm{d}x_3 - x_5\mathrm{d}x_4 + x_5\mathrm{d}x_4 + x_7\mathrm{d}x_6 - x_6\mathrm{d}x_7,\\
\alpha_2 &= -x_2\mathrm{d}x_0 + x_0\mathrm{d}x_2 + x_1\mathrm{d}x_3 - x_3\mathrm{d}x_1 - x_6\mathrm{d}x_4 + x_4\mathrm{d}x_6 + x_5\mathrm{d}x_7 - x_7\mathrm{d}x_5,\\
\alpha_3 &= -x_3\mathrm{d}x_0 + x_0\mathrm{d}x_3 + x_2\mathrm{d}x_1 - x_1\mathrm{d}x_2 - x_7\mathrm{d}x_4 + x_4\mathrm{d}x_7 + x_6\mathrm{d}x_5 - x_5\mathrm{d}x_6.
\end{aligned}$$

This explicit construction was motivated by the work of Acharya, Bryant and Salamon [1] where they investigate the S^1-reduction of the conical G_2 metric on $\mathbb{R}^+ \times \mathbb{C}P^3$. We can identify $\mathbb{R}^8$ with coordinates $(x_0, x_1, \dots, x_7)$ with $\mathbb{H}^2$ by $(x_0 + ix_1 + jx_2 + kx_3, x_4 + ix_5 + jx_6 + kx_7)$. There are natural actions given by $Sp(2)$ acting by left multiplication and $Sp(1)$ acting by multiplication on the right. The 1-forms α_i are simply the dual of the S^1 actions given by right multiplication by the imaginary quaternions. We consider the S^1 action generated by the vector field

$$X = -x_1\partial_0 + x_0\partial_1 - x_3\partial_2 + x_2\partial_3 - x_5\partial_4 + x_4\partial_5 - x_7\partial_6 + x_6\partial_7$$

given by a diagonal $U(1) \subset Sp(2)$. A simple computation shows that

$$\mathrm{d}(X \lrcorner \mathrm{d}\alpha_i \wedge \mathrm{d}\alpha_i) = 0 \text{ for } i = 1, 2, 3$$

from which it follows that $\mathcal{L}_X \Phi_0 = 0$. Thus, we get a closed G_2-structure on the quotient space $\mathbb{R}^+ \times \mathbb{C}P^3$ given by $\varphi = \iota_X \Phi$ from 3.1. Noting that Φ_0 is also invariant by the right S^1 action generated by the vector field

$$Y = -x_1\partial_0 + x_0\partial_1 + x_3\partial_2 - x_2\partial_3 - x_5\partial_4 + x_4\partial_5 + x_7\partial_6 - x_6\partial_7,$$

i.e $\mathcal{L}_Y \Phi_0 = 0$, and that both S^1 actions commute, we can take the (topological) $\mathbb{T}^2$ reduction to the 6-manifold $\mathbb{R}^3 \oplus \mathbb{R}^3 - \{0\}$. More concretely, we can split $\mathbb{R}^8 = \mathbb{R}^4 \oplus \mathbb{R}^4$ with coordinates x_0, x_1, x_4, x_5 on the first factor and x_2, x_3, x_6, x_7 on the second and we consider the equivalent $\mathbb{T}^2$ action given by the vector fields $\frac{1}{2}(X+Y)$ and $\frac{1}{2}(X-Y)$, each acting non-trivially on only one $\mathbb{R}^4$ factor. Using the HK moment maps as in the previous section, we get coordinates u_i and v_i on $\mathbb{R}^3 \oplus \mathbb{R}^3 - \{0\}$ given by

$$\begin{aligned} u_1 &= x_0^2 + x_1^2 - x_4^2 - x_5^2 & v_1 &= x_2^2 + x_3^2 - x_6^2 - x_7^2 \\ u_2 &= 2\,(x_0x_4 + x_1x_5) & v_2 &= 2\,(x_2x_6 + x_3x_7) \\ u_3 &= 2\,(x_0x_5 - x_1x_4) & v_3 &= 2\,(x_2x_7 - x_3x_6). \end{aligned}$$

These coordinates can now be pulled back to $\mathbb{R}^+ \times \mathbb{C}P^3$ and will allow us to give an explicit expression for φ. From this point of view, we have the S^1-bundle:

$$\mathbb{R}^+ \times \mathbb{C}P^3 \xrightarrow{/S^1} \mathbb{R}^3 \oplus \mathbb{R}^3 - \{0\}.$$

Following the Apostolov–Salamon construction [2], we can write

$$\varphi = \xi \wedge \omega + H^{3/2}\,\Omega^+ \tag{6.1}$$

$$*_\varphi\,\varphi = \frac{1}{2}\,H^2\omega^2 - \xi \wedge H^{1/2}\,\Omega^- \tag{6.2}$$

where $H := \|Y\|_\varphi^{-1}$, ξ is the connection 1-form defined by

$$\xi(\cdot) := H^2\, g_\varphi(Y, \cdot)$$

and $(\omega, \Omega^+, \Omega^-)$ is the $SU(3)$-structure induced on $\mathbb{R}^3 \oplus \mathbb{R}^3 - \{0\}$. We now give coordinate expressions for the aforementioned differential forms.

Proposition 6.1 *In the above notation, the closed G_2-structure on $\mathbb{R}^+ \times \mathbb{C}P^3$ given by $\varphi = \iota_X \Phi_0$ can be expressed as*

$$\varphi = \xi \wedge \frac{1}{2}\sum_{i=1}^{3} \mathrm{d}v_i \wedge \mathrm{d}u_i + \frac{1}{8}\left(\frac{1}{u}\big(\mathrm{d}u_{123} - \{\mathrm{d}\boldsymbol{v}, \mathrm{d}\boldsymbol{u}, \mathrm{d}\boldsymbol{u}\}\big) + \frac{1}{v}\big(\mathrm{d}v_{123} - \{\mathrm{d}\boldsymbol{v}, \mathrm{d}\boldsymbol{v}, \mathrm{d}\boldsymbol{u}\}\big)\right),$$

where $\{\mathrm{d}\boldsymbol{v}, \mathrm{d}\boldsymbol{v}, \mathrm{d}\boldsymbol{u}\}$ denotes

$$\mathrm{d}v_1 \wedge \mathrm{d}v_2 \wedge \mathrm{d}u_3 + \mathrm{d}v_2 \wedge \mathrm{d}v_3 \wedge \mathrm{d}u_1 + \mathrm{d}v_3 \wedge \mathrm{d}v_1 \wedge \mathrm{d}u_2,$$

similarly for $\{\mathrm{d}\boldsymbol{v}, \mathrm{d}\boldsymbol{u}, \mathrm{d}\boldsymbol{u}\}$. Moreover, we have

$$\begin{aligned} H^{\frac{1}{2}}\Omega^- &= \frac{1}{4R^{\frac{2}{3}}}\left(\{\mathrm{d}\boldsymbol{v}, \mathrm{d}\boldsymbol{v}, \mathrm{d}\boldsymbol{u}\} - \{\mathrm{d}\boldsymbol{u}, \mathrm{d}\boldsymbol{u}, \mathrm{d}\boldsymbol{v}\} + \frac{u}{v}\mathrm{d}v_{123} - \frac{v}{u}\mathrm{d}u_{123}\right), \\ H &= \frac{R^{2/3}}{2\,u^{1/2}\,v^{1/2}}, \end{aligned}$$

where $R^2 := x_0^2 + \cdots + x_7^2$, $u^2 := u_1^2 + u_2^2 + u_3^2 = (x_0^2 + x_1^2 + x_4^2 + x_5^2)^2$ and likewise for v. The curvature of the S^1-bundle over $\mathbb{R}^3 \oplus \mathbb{R}^3 - \{0\}$ is given by

$$\mathrm{d}\xi = -\frac{\{\boldsymbol{v}, \mathrm{d}\boldsymbol{v}, \mathrm{d}\boldsymbol{v}\}}{4(v_1^2 + v_2^2 + v_3^2)^{3/2}} + \frac{\{\boldsymbol{u}, \mathrm{d}\boldsymbol{u}, \mathrm{d}\boldsymbol{u}\}}{4(u_1^2 + u_2^2 + u_3^2)^{3/2}},$$

where $\{\boldsymbol{v}, \mathrm{d}\boldsymbol{v}, \mathrm{d}\boldsymbol{v}\}$ denotes

$$v_1\mathrm{d}v_2 \wedge \mathrm{d}v_3 + v_2\mathrm{d}v_3 \wedge \mathrm{d}v_1 + v_3\mathrm{d}v_1 \wedge \mathrm{d}v_2,$$

and likewise for $\{\boldsymbol{u}, \mathrm{d}\boldsymbol{u}, \mathrm{d}\boldsymbol{u}\}$.

The proof is a long computation which was carried out with the help of MAPLE. One can directly verify the above formulae hold using the definitions of u_i, v_i and expressing them in terms of x_i. The reader might find it interesting to compare our expressions to those in [1] for the torsion-free G_2 quotient.

In [27], Hitchin shows that an $SU(3)$-structure is completely determined by the pair (ω, Ω^+). Note that here Ω^+ can easily be read off from the expressions for φ and H in Proposition 6.1 and formula (6.1). Thus, we can explicitly compute the induced complex structure and metric on $\mathbb{R}^3 \oplus \mathbb{R}^3 - \{0\}$.

Proposition 6.2 *The metric induced by (ω, Ω^+) on $\mathbb{R}^3 \oplus \mathbb{R}^3 - \{0\}$ is given by*

$$g_\omega = \frac{1}{2}\left(\frac{v^{1/2}}{u^{1/2}} (\mathrm{d}u_1^2 + \mathrm{d}u_2^2 + \mathrm{d}u_3^2) + \frac{u^{1/2}}{v^{1/2}} (\mathrm{d}v_1^2 + \mathrm{d}v_2^2 + \mathrm{d}v_3^2) \right)$$

and the almost complex structure J by

$$J(u^{1/2}\partial_{u_i}) = v^{1/2}\partial_{v_i}, \text{ for } i = 1, 2, 3.$$

Note that since φ is closed from (2.3), we have that

$$\mathrm{d} *_\varphi \varphi = \tau_2 \wedge \varphi.$$

We shall now derive an explicit expression for the torsion of the G_2-structure φ. Under the inclusion $SU(3) \hookrightarrow G_2$, we can write the torsion form as

$$\tau_2 = \xi \wedge \tau_v + \tau_h$$

where τ_v and τ_h are basic 1-form and 2-form, respectively, i.e. they are (pullback of) forms on $\mathbb{R}^3 \oplus \mathbb{R}^3 - \{0\}$. It is not hard to show that $\tau_h \in [\Lambda^{2,0}] \oplus [[\Lambda_0^{1,1}]]$ and that the $[\Lambda^{2,0}]$-component of τ_h is $SU(3)$-equivalent to τ_v. We compute τ_h and τ_v as

$$\begin{aligned} \tau_h \cdot \left(3uv \cdot R^{8/3}\right) = & -u \cdot \left(\frac{1}{2}(\{\boldsymbol{u}, \mathrm{d}\boldsymbol{v}, \mathrm{d}\boldsymbol{v}\} + \{\boldsymbol{v}, \mathrm{d}\boldsymbol{v}, \mathrm{d}\boldsymbol{v}\}) + \frac{3u}{2v}\{\boldsymbol{v}, \mathrm{d}\boldsymbol{v}, \mathrm{d}\boldsymbol{v}\} \right) \\ & - v \cdot \left(\frac{1}{2}(\{\boldsymbol{v}, \mathrm{d}\boldsymbol{u}, \mathrm{d}\boldsymbol{u}\} + \{\boldsymbol{u}, \mathrm{dd}\boldsymbol{u}, \mathrm{d}\boldsymbol{u}\}) + \frac{3v}{2u}\{\boldsymbol{u}, \mathrm{d}\boldsymbol{u}, \mathrm{d}\boldsymbol{u}\} \right) \\ & - \frac{1}{2}(u\{\boldsymbol{v}, \mathrm{d}\boldsymbol{v}, \mathrm{d}\boldsymbol{u}\} + v\{\boldsymbol{u}, \mathrm{d}\boldsymbol{v}, \mathrm{d}\boldsymbol{u}\}) \\ & - \frac{1}{2}(v\{\boldsymbol{u}, \mathrm{d}\boldsymbol{u}, \mathrm{d}\boldsymbol{v}\} + u\{\boldsymbol{v}, \mathrm{d}\boldsymbol{u}, \mathrm{d}\boldsymbol{v}\}) \end{aligned}$$

and

$$\begin{aligned} \tau_v \cdot (\frac{3}{2} \cdot R^{8/3}) &= \sum_{i=1}^{3} \left(\frac{1}{v}(vu_i - 3uv_i)\, \mathrm{d}v_i - \frac{1}{u}(uv_i - 3vu_i)\, \mathrm{d}u_i \right) \\ &= \left(\boldsymbol{u} \cdot \mathrm{d}\boldsymbol{v} - \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{u} - 3(u\mathrm{d}v - v\mathrm{d}u) \right) \end{aligned}$$

where $\boldsymbol{u} \cdot \mathrm{d}\boldsymbol{v}$ denotes $\sum_{i=1}^{3} u_i \mathrm{d}v_i$ and likewise for $\boldsymbol{v} \cdot \mathrm{d}\boldsymbol{u}$. From these expressions, one can show that the $[\Lambda_0^{1,1}]$-component of τ_h is nonzero, i.e. J is non-integrable.

Remark 6.3

- If we restrict the $Spin(7)$ 4-form Φ_0 on $\mathbb{R}^8$ to S^7, we get a G_2 4-form $*_{S^7}\,\varphi_{S^7}$ and the flat metric restricts to give the standard round metric. Since the cone metric is just the flat metric again, this means that this cocalibrated G_2- structure is inducing the round metric. This statement is in agreement with the fact that with the round metric S^7 is a 3-Sasakian manifold. Note that in contrast, the squashed Einstein metric on S^7 has exactly one Killing spinor, so the cone metric has holonomy *equal* to $Spin(7)$ [22, 23]. We can now take the S^1-quotient with respect to any free S^1 action preserving the round nearly parallel G_2-structure. Since this quotient is also a Riemannian submersion (as the size of the circle orbits is constant), the quotient metric is just the Fubini–Study metric. However by contracting the 4-form with the vector field generated by the S^1, we get the (negative) imaginary part of a (3, 0) form on the $\mathbb{C}P^3$. The latter induces an almost complex structure compatible with the Fubini–Study metric but which definitely cannot be the integrable one; otherwise, this contradicts the fact that the canonical bundle of $\mathbb{C}P^3$ with the Fubini–Study complex structure is non-trivial. The above closed G_2-structure is then just the Riemannian cone on this $\mathbb{C}P^3$. More explicitly, we can write the flat metric on $\mathbb{R}^8$ as

$$g_{\mathbb{R}^8} = \mathrm{d}R^2 + R^2(\eta^2 + g_{FS}) = R^2\eta^2 + R^{-2/3}g_{\varphi}$$

 where η is just the connection form of the S^1 action for the Fubini–Study quotient as above and $s = \|X\|_{\Phi}^{-1} = R^{-1}$. Thus, the metrics of Proposition 6.1 can also be expressed as

$$g_{\varphi} = R^{2/3}\cdot \mathrm{d}R^2 + R^{8/3}g_{FS} = \mathrm{d}r^2 + \frac{16}{9}r^2 g_{FS}$$
$$g_{\omega} = 2(u\cdot v)^{1/2}\,(\mathrm{d}R^2 + R^2\, g_{FS} - \frac{4\,u\cdot v}{R^2}\xi^2).$$

 Note that by construction, the latter metric is invariant under the vector field Y and thus passes to the quotient $(\mathbb{R}^+ \times \mathbb{C}P^3)/S^1$.
- Observe that one can also view this construction as a $\mathbb{T}^2$-quotient of a $Spin(7)$-structure to a 6-manifold endowed with an $SU(3)$-structure $(\omega, \Omega^+, \Omega^-)$ given by

$$\Phi = \eta\wedge\xi\wedge\omega + H^{3/2}\eta\wedge\Omega^+ + \frac{1}{2}s^{4/3}H^2\omega\wedge\omega - s^{4/3}H^{1/2}\xi\wedge\Omega^-$$

 and the metrics are related by

$$g_{\Phi} = s^{-2}\eta^2 + s^{2/3}H^{-2}\xi^2 + s^{2/3}Hg_{\omega}.$$

 This quotient construction under the assumption that the 6-manifold is Kähler is the current work in progress by the author.

Acknowledgements The author would like to thank his supervisors Jason Lotay and Simon Salamon for their constant support and many helpful discussions that led to this article. This work was supported by the Engineering and Physical Sciences Research Council [EP/L015234/1] and the EPSRC Centre for Doctoral Training in Geometry and Number Theory (The London School of Geometry and Number Theory), University College London.

Appendix

For the convenience of the reader and to make this article self-contained, we describe the construction of the Bryant–Salamon metrics on the anti-self-dual bundle of S^4. We shall follow the approach described in [34]. The reader will find proofs of the assertions made therein.

Considering S^4 embedded in $\mathbb{R}^5$ with coordinates $x_1, \dots, x_5$, we may choose the following local orthonormal frame

$$v_1 = \frac{1}{R}\begin{pmatrix} x_2 \\ -x_1 \\ x_4 \\ -x_3 \\ 0 \end{pmatrix},\ v_2 = \frac{1}{R}\begin{pmatrix} -x_3 \\ x_4 \\ x_1 \\ -x_2 \\ 0 \end{pmatrix},\ v_3 = \frac{1}{R}\begin{pmatrix} x_4 \\ x_3 \\ x_2 \\ -x_1 \\ 0 \end{pmatrix},\ v_4 = \frac{1}{\sqrt{-1+\frac{1}{x_5^2}}}\begin{pmatrix} -x_1 \\ -x_2 \\ -x_3 \\ -x_4 \\ -x_5 + \frac{1}{x_5} \end{pmatrix},$$

where $R^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$. Denoting by e^i the corresponding coframe, we compute the following:

$$\begin{aligned}
\mathrm{d}\mathrm{e}^1 &= \frac{2}{R}\mathrm{e}^{23} + \frac{\sqrt{1-R^2}}{R}\mathrm{e}^{14} \\
\mathrm{d}\mathrm{e}^2 &= \frac{2}{R}\mathrm{e}^{31} + \frac{\sqrt{1-R^2}}{R}\mathrm{e}^{24} \\
\mathrm{d}\mathrm{e}^3 &= \frac{2}{R}\mathrm{e}^{12} + \frac{\sqrt{1-R^2}}{R}\mathrm{e}^{34} \\
\mathrm{d}\mathrm{e}^4 &= 0
\end{aligned}$$

In the Cartan moving frame setting, the structure equations are given by $\mathrm{d}\mathbf{e} = -\omega \wedge \mathbf{e}$ and $F = \mathrm{d}\omega + \omega \wedge \omega \in \Lambda^2 \otimes \mathfrak{so}(4)$ where ω is the Levi-Civita connection form and F the curvature. We compute them as

$$\omega = -\begin{pmatrix} 0 & -\frac{1}{R}\mathrm{e}^3 & \frac{1}{R}\mathrm{e}^2 & \frac{\sqrt{1-R^2}}{R}\mathrm{e}^1 \\ \cdot & 0 & -\frac{1}{R}\mathrm{e}^1 & \frac{\sqrt{1-R^2}}{R}\mathrm{e}^2 \\ \cdot & \cdot & 0 & \frac{\sqrt{1-R^2}}{R}\mathrm{e}^3 \\ \cdot & \cdot & \cdot & 0 \end{pmatrix} \text{ and } F = \begin{pmatrix} 0 & \mathrm{e}^{12} & \mathrm{e}^{13} & \mathrm{e}^{14} \\ \cdot & 0 & \mathrm{e}^{23} & \mathrm{e}^{24} \\ \cdot & \cdot & 0 & \mathrm{e}^{34} \\ \cdot & \cdot & \cdot & 0 \end{pmatrix}$$

Here, we are only writing the upper triangular entries since the matrices are skew-symmetric. The second equation confirms that the round metric has constant curvature and that the scalar curvature is 12. We can define a local orthonormal basis of the anti-self-dual bundle by $c^1 := \mathrm{e}^{12} - \mathrm{e}^{34}$, $c^2 := \mathrm{e}^{13} - \mathrm{e}^{42}$ and $c^3 := \mathrm{e}^{14} - \mathrm{e}^{23}$. ω induces a connection on this bundle given by

$$\nabla c^i = \psi^i_j \otimes c^i$$

Since the connection is torsion free, we can compute ψ^i_j by

$$\begin{aligned} \mathrm{d}c^1 &= \psi^1_2 \wedge c^2 + \psi^1_3 \wedge c^3 \\ \mathrm{d}c^2 &= \psi^2_1 \wedge c^1 + \psi^2_3 \wedge c^3 \\ \mathrm{d}c^3 &= \psi^3_1 \wedge c^1 + \psi^3_2 \wedge c^2 \end{aligned}$$

where $\psi^2_1 = \frac{\sqrt{1-R^2}+1}{R}\mathrm{e}^1$, $\psi^1_3 = \frac{\sqrt{1-R^2}+1}{R}\mathrm{e}^2$, $\psi^2_3 = \frac{\sqrt{1-R^2}+1}{R}\mathrm{e}^3$ and $\psi^i_j = -\psi^j_i$. These forms can all be pulled back to the total space of the ASD bundle which we denote by the same letter. We introduce fibre coordinates (a_1, a_2, a_3) with respect to the coordinate system defined by c^i. We can define vertical 1-forms by

$$b^i = \mathrm{d}a_i + a_j \psi^j_i,$$

i.e. they vanish on horizontal vectors. Together with the pull back of the e^i, they give an absolute parallelism of the ASD bundle. The following forms are all $SO(4)$-invariant and are hence globally well defined on the total space:

$$\begin{aligned} \rho &= a_1 a_1 + a_2 a_2 + a_3 a_3 \\ \sigma &= 2\,(a_1 b^2 b^3 + a_2 b^3 b^1 + a_3 b^1 b^2) \\ \alpha &= a_1 b^2 c^3 + a_2 b^3 c^1 + a_3 b^1 c^2 - a_1 b^3 c^2 - a_2 b^1 c^3 - a_3 b^2 c^1 \\ \tau &= a_1 c^1 + a_2 c^2 + a_3 c^3 \\ \beta &= 6\, b^{123}. \end{aligned}$$

The unit ($\rho = 1$) sphere bundle is diffeomorphic to $\mathbb{C}P^3$, and restricting the above forms, we have

$$\begin{aligned} g_{FS} &= \frac{1}{2}((\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + (\mathrm{e}^3)^2 + (\mathrm{e}^4)^2) + \frac{1}{2}((b^1)^2 + (b^2)^2 + (b^3)^2)\Big|_{S^2} \\ \omega_{FS} &= \frac{1}{2}\tau - \frac{1}{4}\sigma \\ g_{NK} &= \frac{1}{2}((\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + (\mathrm{e}^3)^2 + (\mathrm{e}^4)^2) + \frac{1}{4}((b^1)^2 + (b^2)^2 + (b^3)^2)\Big|_{S^2} \\ \omega_{NK} &= \frac{1}{2}\tau + \frac{1}{8}\sigma \\ \Omega_{NK} &= \frac{1}{4}(\mathrm{d}\tau - i\alpha) \end{aligned}$$

The subscript *FS* refers to the Fubini–Study metric and *NK* to the nearly Kähler one. Our choice of scaling was made to fit the conventions of Sect. 5.1. The Bryant–Salamon form is then given by

$$\varphi_{BS} = u^2 v\mathrm{d}\tau + \frac{1}{6}v^3\beta$$

where $u = (2\rho + 1)^{1/4}$ and $v = (2\rho + 1)^{-1/4}$.

References

1. Acharya, B.S., Bryant, R.L., Salamon, S.: A circle quotient of a G_2 cone. *arXiv e-prints*, page arXiv:1910.09518 (2019)
2. Apostolov, V., Salamon, S.: Kähler reduction of metrics with holonomy G_2. Commun. Math. Phys. **246**(1), 43–61 (2004)
3. Bazaikin, Y.V., Bogoyavlenskaya, O.A.: Complete riemannian metrics with holonomy group G_2 on deformations of cones over $S^3 \times S^3$. Math. Notes **93**(5), 643–653 (2013)
4. Bazzoni, G., Martin-Merchan, L., Munoz, V.: Spin-harmonic structures and nilmanifolds. *arXiv e-prints*, page arXiv:1904.01462 (2019)
5. Bedulli, L., Vezzoni, L.: The Ricci tensor of SU(3)-manifolds. J. Geom. Phys. **57**(4), 1125–1146 (2007)
6. Berger, M.: Sur les groupes d'holonomie homogènes de variétés à connexion affine et des variétés Riemanniennes. Bulletin de la Société Mathématique de France **83**, 279–330 (1955)
7. Besse, A.L.: Einstein Manifolds. Springer, Berlin (2008)
8. Brandhuber, A., Gomis, J., Gubser, S.S., Gukov, S.: Gauge theory at large N and new G_2 holonomy metrics. Nucl. Phys. B **611**(1), 179–204 (2001)
9. Bryant, R.L.: Metrics with exceptional holonomy. Ann. Math. **126**(3), 525–576 (1987)
10. Bryant, R.L.: Some remarks on G_2-structures. In: Proceedings of Gökova Geometry–Topology Conference 2005, edited by. International Press (2006)
11. Bryant, R.L., Salamon, S.M.: On the construction of some complete metrics with exceptional holonomy. Duke Math. J. **58**(3), 829–850 (1989)
12. Chiossi, S., Salamon, S.: The intrinsic torsion of $SU(3)$ and G_2 structures. In: Differential Geometry, Valencia 2001, pp 115–133. World Scientific, Singapore (2002)
13. Conti, D., Fernández, M.: Nilmanifolds with a calibrated G_2-structure. Differ. Geom. Appl. **29**(4), 493–506 (2011)
14. Cvetič, M., Gibbons, G.W., Lü, H., Pope, C.N.: Cohomogeneity one manifolds of $Spin(7)$ and G_2 holonomy. Phys. Rev. D **65**, 106004 (2002)
15. Donaldson, S.: Adiabatic Limits of Co-associative Kovalev–Lefschetz Fibrations, vol. 324, pp. 1–29. Birkhäuser, Cham (2017)
16. Fernández, M.: A classification of Riemannian manifolds with structure group Spin(7). Annali di Matematica Pura ed Applicata **143**(1), 101–122 (1986)
17. Fernández, M., Gray, A.: Riemannian manifolds with structure group G_2. Annali di Matematica Pura ed Applicata **132**(1), 19–45 (1982)
18. Fino, A., Raffero, A.: Coupled SU(3)-structures and supersymmetry. Symmetry **7**(2), 625–650 (2015)
19. Foscolo, L.: Complete non-compact Spin(7) manifolds from self-dual Einstein 4-orbifolds. *arXiv e-prints*, page arXiv:1901.04074 (2019)
20. Foscolo, L., Haskins, M.: New G_2-holonomy cones and exotic nearly Kähler structures on S^6 and $S^3 \times S^3$. Ann. Math. **185**(1), 59–130 (2017)
21. Freibert, M.: $SU(4)$-holonomy via the left-invariant hypo and Hitchin flow. Annali di Matematica Pura ed Applicata **197**, 1051–1084 (2016)
22. Friedrich, Th, Kath, I., Moroianu, A., Semmelmann, U.: On nearly parallel G_2-structures. J. Geom. Phys. **23**(3), 259–286 (1997)
23. Friedrich, T., Kath, I.: 7-dimensional compact Riemannian manifolds with Killing spinors. Commun. Math. Phys. **133**(3), 543–561 (1990)
24. Gong, M.-P.: Classification of Nilpotent Lie Algebras of Dimension 7 (over Algebraically Closed Field and $\mathbb{R}$). *PhD thesis UWSpace* (1998)
25. Gray, A., Green, P.S.: Sphere transitive structures and the triality automorphism. Pac. J. Math. **34**(1), 83–96 (1970)
26. Hein, H.-J., Sun, S., Viaclovsky, J., Zhang, R.: Nilpotent structures and collapsing Ricci-flat metrics on K3 surfaces. *arXiv e-prints*, arXiv:1807.09367 (2018)
27. Hitchin, N.: The geometry of three-forms in six and seven dimensions. J. Differ. Geom. **55**(3), 547–576 (2000)
28. Ivanov, S.: Connection with torsion, parallel spinors and geometry of Spin(7) manifolds. Math. Res. Lett. **11**(2), 171–186 (2004)
29. Ivanov, S., Parton, M., Piccinni, P.: Locally conformal parallel G_2 and $Spin(7)$ manifolds. Math. Res. Lett. **13**(2), 167–177 (2006)
30. Joyce, D.: Riemannian Holonomy Groups and Calibrated Geometry, vol. 12. Oxford University Press, Oxford (2007)

31. Karigiannis, S.: Deformations of G_2 and Spin(7) Structures on Manifolds. *arXiv Mathematics e-prints*, arXiv:math/0301218 (2003)
32. Karigiannis, S.: Flows of Spin(7)-structures. *arXiv e-prints*, arXiv:0709.4594 (2007)
33. Lawson, H.B., Michelsohn, M.-L.: Spin Geometry (PMS-38). Princeton University Press, Princeton (1989)
34. Salamon, S.: Riemannian Geometry and Holonomy Groups. Longman Scientific & Technical, Harlow (1989)
35. Salamon, S.: Complex structures on nilpotent Lie algebras. J. Pure Appl. Algebra **157**(2), 311–333 (2001)