Partitioning a graph into monochromatic connected subgraphs

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Abstract

We show that every 2-edge-coloured graph on n vertices with minimum degree $\geq \frac{2n-5}{3}$ can be partitioned into two monochromatic connected subgraphs, provided n is sufficiently large. This minimum degree condition is tight and the result proves a conjecture of Bal and DeBiasio. We also make progress on another conjecture of Bal and DeBiasio on covering graphs with large minimum degree with monochromatic components of distinct colours.

1 Introduction

It is an old observation of Erdős and Rado that every 2-edge-colouring of the complete graph contains a monochromatic spanning tree. While this fact is easy enough to prove (one line with induction) its discovery opened up a new avenue in graph Ramsey theory; the study of large, sparse structures that appear in every 2-edge-colouring of the complete graph.

A, now classical, example appears in a seminal paper of Erdős, Gyárfás and Pyber [7], that for any r-edge-colouring of K_n (the complete graph on n vertices) the vertices can be covered by $O(r^2 \log r)$ vertex-disjoint, monochromatic cycles. We note that throughout the paper, when we say that the vertices of a graph are covered (or partitioned) by a collection of subgraphs, we mean that the vertices are covered by the *vertex sets* of these subgraphs.

Gyárfás, Ruszinkó, Sárközy and Szemerédi [9] improved the above result by showing that if the edges of the complete graph are r-coloured then the vertices can be partitioned into $O(r \log r)$ monochromatic cycles. In the other direction, Pokrovskiy [14] showed that one needs strictly more than r cycles, disproving a conjecture of Erdős, Gyárfás and Pyber [7]. Conlon and Stein [4] showed similar results for colourings where every vertex is incident with at most r distinct colours. The question of whether one

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can partition an r-coloured graph into $O(r)$ monochromatic cycles remains an enticing open problem in this area.

In a different direction, Erdős, Gyárfás and Pyber [7] conjectured that the vertices of an r-coloured complete graph may be partitioned into at most $r-1$ monochromatic connected subgraphs. This conjecture is known to be tight when $r-1$ is a prime power and n is sufficiently large, due to a well-known construction which requires the existence of an affine plane of an appropriate order. Haxell and Kohayakawa [10] proved a slightly weaker result, showing that one can partition an r-coloured complete graph on n vertices into r monochromatic subgraphs, for sufficiently large n.

Interestingly, this problem is closely related to a well-known conjecture of Ryser on packing and covering edges in r-partite, r-uniform hypergraphs. This link was first noted by Gyárfás [8] in 1997 and leads to the following natural formulation of the conjecture of Ryser, published in [11], where $\alpha(G)$ is the size of the largest independent set in the graph G.

Conjecture 1. *The vertex set of an r-coloured graph G can be covered by at most $(r-1)\alpha(G)$ monochromatic connected subgraphs.*

In this form, it is clear that Ryser's conjecture implies the *covering version* of the aforementioned conjecture of Erdős, Gyárfás and Pyber about monochromatic connected subgraphs. Although not much is known about Ryser's conjecture in general, a few special cases are understood. The case $r=2$ is equivalent to König's classical theorem (see [5], for example), while the case $r=3$ was proved by Aharoni [1] in 2001, who built on the earlier advances of Aharoni and Haxell [2]. Beyond this, there are only a few other cases where the conjecture is known to hold: when G is a complete graph and when $r \leq 5$, cumulatively proved by Gyárfás [8] ($r=3$), Duchet [6] and Tuza [16] ($r=4,5$).

Following Schelp [15], who suggested several variants of Ramsey-type problems (e.g. determining the length of the longest monochromatic path in a 2-coloured graphs), we consider variants of the above problems for graphs with large minimum degree. Our first main result proves a conjecture of Bal and DeBiasio [3] on partitioning the vertices of a 2-coloured graph with large minimum degree. Recall that $\delta(G)$ denotes the minimum degree of the graph G.

Theorem 2. *There exists an integer n_0 such that every 2-edge-coloured graph G on $n \geq n_0$ vertices and with minimum degree at least $\frac{2n-5}{3}$ can be partitioned into two monochromatic connected subgraphs.*

This result is seen to be sharp by a construction of Bal and DeBiasio [3] which we generalise in Section 4. We also note that Theorem 2 generalises the result of Haxell and Kohayakawa [10] to all graphs with sufficiently large minimum degree - in the case of two colours. One can think of this result as saying that $\frac{2n-5}{3}$ is the minimum degree 'threshold' that guarantees a partition of every 2-colouring into two monochromatic connected subgraphs. It is therefore natural to ask what minimum degree condition on a graph G guarantees a partition into t monochromatic connected subgraphs, no matter how the graph is 2-coloured. We conjecture the following.

Conjecture 3. *For every t there exists n_0, such that for every 2-colouring of a graph G on $n \geq n_0$ vertices with $\delta(G) \geq \frac{2n-2t-1}{t+1}$ there exists a partition of the vertex set into at most t monochromatic connected subgraphs.*

We support this conjecture by observing an analogous result for *covers* of the vertices by monochromatic components.

Proposition 4. *Let t be integer and let G be a 2-coloured graph on n vertices with $\delta(G) \geq \frac{2n-2t-1}{t+1}$. Then the vertices of G can be covered by at most t monochromatic components.*

We also give a construction, showing that the inequality in this proposition (and therefore the conjecture) cannot be improved.

Bal and DeBiasio [3] also considered the problem of covering coloured graphs with monochromatic components of distinct colours. In particular, they conjectured the following.

Conjecture 5. *Let G be an r-coloured graph on n vertices with $\delta(G) \geq (1-1/2^r)n$. Then the vertices can be covered by monochromatic components of distinct colours.*

Again, Bal and DeBiasio provided examples showing that if true, the bound $(1-2^{-r})n$ is best possible. We prove Conjecture 5 for $r = 2, 3$.

Theorem 6. *Let G be a 3-coloured graph on n vertices with $\delta(G) \geq 7n/8$. Then the vertices of G can be covered by monochromatic components of distinct colours.*

We conclude the introduction with a description of the notation that we shall use in this paper. We prove Theorem 2 in Section 2, and prove Theorem 6 in Section 3. We conclude the paper in Section 4 with some final remarks and open problems and give a proof of Proposition 4.

1.1 Notation

By an *r-coloured* graph, we mean a graph whose edges are coloured with r colours. When a graph is 2-coloured we call the colours *red* and *blue*; and when it is 3-coloured, we call the colours *red*, *blue* and *yellow*.

For a set of vertices W, we denote by $N_r(W)$ the set of vertices in $V(G) \setminus W$ that are adjacent to a vertex in W by a red edge. If $x \in V(G)$ is a vertex, we define $d_r(x) = |N_r(\{x\})|$ which we refer to as the *red degree* of x. We say that y is a *red neighbour* of x if xy is a red edge. By a *red component* of a graph G, we mean the *vertex set* $C \subseteq V(G)$ of a component of the red graph. We denote the red component that contains x by $C_r(x)$. A *red set* $U \subseteq V(G)$ is a set of vertices for which the red edges induced by U form a connected graph.

All the above definitions and notation, that were defined for red, also works for blue or yellow; e.g. $d_b(x)$ and $d_y(x)$ are the blue and yellow degrees of x, respectively, and a blue set is a set of vertices that is connected in blue.

2 Partitioning into monochromatic connected subgraphs

In this section we prove Theorem 2.

Theorem 2. *There exists an integer n_0 such that every 2-edge-coloured graph G on $n \geq n_0$ vertices and with minimum degree at least $\frac{2n-5}{3}$ can be partitioned into two monochromatic connected subgraphs.*

We note that the minimum degree condition in this theorem cannot be improved; this can be seen by taking $t = 2$ in Example 21, described in Section 4.

Proof of Theorem 2. Throughout this proof, we assume that the number of vertices n is sufficiently large. Suppose, for a contradiction, that the vertices of G cannot be partitioned into two monochromatic sets.

Claim 7. *There is a blue component of order at most $(n+2)/6$.*

Proof of Claim 7. We may assume that there are at least three red components and at least three blue components, as otherwise the vertices may be partitioned into two red sets or two blue sets (recall that a *red set* is defined to be a set of vertices that is connnected in red, and similarly for blue), contradicting our assumption. Let R be a red component of smallest order, so $|R| \leq n/3$.

Let us assume first that $|R| \leq (n-5)/3$. Since every vertex in R sends at least $(2n-5)/3-(|R|-1) > (n-|R|)/2$ blue edges outside of R, every two vertices in R have a common blue neighbour outside of R. Hence, R is contained in a blue component of order at least $|R| + (2n-5)/3 - (|R|-1) \geq (2n-2)/3$. Since there are at least three blue components, there is a blue component of order at most $(n-(2n-2)/3)/2 = (n+2)/6$.

We now assume that $(n-4)/3 \leq |R| \leq n/3$. If every two vertices in R have a common blue neighbour, then, again, R is contained in a blue component of order at least $(2n-2)/3$ and as before there is a blue component of order at most $(n+2)/6$. Otherwise, there exist two vertices $u, v \in R$ whose blue neighbourhoods do not intersect. But both u and v have at least $(n-5)/3$ blue neighbours outside of R, and therefore every vertex in $R \setminus \{u, v\}$ has a common blue neighbour with either u or w. It follows that there are two blue components (namely, the components $C_b(u)$ and $C_b(w)$) whose union has order at least $|R| + 2(n-5)/3 > n-5$, hence there is a blue component of order at most 4. □

Claim 8. *There is a red set U of size at most $27 \log n$ such that $|N_r(U)| \geq 2n/3 - 27 \log n$.*

Proof of Claim 8. By the previous claim, there is a blue component B of order at most $(n+2)/6$. Note that every vertex in B has at least $(2n-5)/3 - |B|$ red neighbours in $V(G) \setminus B$. Fix a vertex $u \in B$ and let T be the set of red neighbours of u outside B. Every $w \in B$ has at least the following number of red neighbours in N.

$$2 \cdot ((2n-5)/3 - |B|) - (n - |B|) = (n-10)/3 - |B| \geq (n-22)/6.$$

Now let U' be a random subset of T where each vertex $w \in T$ belongs to U', independently, with probability $13\log n/n$. Let I_w be the event that w (where $w \in B$) does not have a red neighbour in U'. We bound

$$\mathbb{P}\Big(\bigcup_{w\in B} I_w\Big) \le |B| \cdot \mathbb{P}(I_w) \le n \cdot \left(1 - \frac{13\log n}{n}\right)^{\frac{n-22}{6}} \le n \cdot e^{-2\log n} < 1/2.$$

Note that since $\mathbb{E}(|U'|) \le 13\log n$, we have $\mathbb{P}(|U'| \ge 26\log n) \le 1/2$, by Markov's inequality. Therefore, there is a choice of $U' \subseteq T$ such that $|U'| \le 26\log n$ and every vertex in B is joined by a red edge to some vertex in U'. We choose $U = U' \cup \{u\}$. Note that

$$\begin{aligned} N_r(U' \cup \{u\})| &\ge |T \setminus U'| + |B \setminus \{u\}| \\ &\ge ((2n-5)/3 - |B| - 26\log n) + (|B| - 1) \\ &= 2n/3 - 27\log n. \end{aligned}$$

Hence, the set $U = U' \cup \{u\}$ satisfies the requirements of Claim 8. $\square$

Let U be a red set as in Claim 8 and let $N = N_r(U)$. Now choose a maximal sequence of distinct vertices $x_1, \dots, x_t \in V \setminus (N \cup U)$ so that x_i has at least $\log n$ red neighbours in the set $N \cup \{x_1, \dots, x_{i-1}\}$, for every $i \in [t]$. Then put $\overline{N} = N \cup \{x_1, \dots, x_t\}$ and write $W = V(G) \setminus \big(U \cup \overline{N}\big)$. Note that every vertex in W has at most $\log n$ red neighbours in $\overline{N}$.

Claim 9. $\big|\overline{N}\big| \le 2n/3 + 3\log n + 4$.

Proof of Claim 9. For a contradiction, suppose that $\big|\overline{N}\big| > 2n/3 + 3\log n + 4$. We shall deduce that the vertices can be partitioned into a red set and a blue set, a contradiction.

To define the partition, fix $w \in W$ and let $X = N_b(w) \cap \overline{N}$. Let S be a random subset of X, obtained by taking each vertex of X independently with probability $1/2$. We claim that, with positive probability, $(U \cup \overline{N}) \setminus S$ is red *and* $W \cup S$ is blue.

To bound the probability that $W \cup S$ is blue, we consider the probability that every vertex in W is joined by a blue edge to S (an event which would imply that $W \cup S$ is blue). For every $x, y \in V$ we have $|N(x) \cap N(y)| \ge n/3 - 10/3$, hence $|N(x) \cap N(y) \cap \overline{N}| \ge 3\log n$. Since every vertex in W has at most $\log n$ red neighbours in $\overline{N}$, we have $|N_b(x) \cap N_b(y) \cap \overline{N}| \ge \log n$. Therefore the probability that a given $x \in W$ has no blue neighbours in S is at most $2^{-\log n} = 1/n$. Thus, the expected number of vertices in W with no edges to S is smaller than $1/2$ (note that $|W| \le n/3$). Hence, $\mathbb{P}(W \cup S \textit{ is blue}) > 1/2$.

We now estimate the probability that $(U \cup \overline{N}) \setminus S$ is red. First note that as $N = N_r(U)$, we have that $U \cup N'$ is red for any subset $N' \subseteq N$. So it remains to show that the vertices of $\{x_1, \dots, x_t\} \setminus S$ can be joined, via a red path, to $U \cup (N \setminus S)$, with sufficiently high probability. For $i \in [t]$, let E_i be the event that vertex x_i is joined by a red edge to $(N \cup \{x_1, \dots, x_{i-1}\}) \setminus S$. Note that if the event

$E = \bigcap_i^t E_i$ holds, $(U \cup \overline{N}) \setminus S$ is red. Now, to estimate $\mathbb{P}(E_i)$, for $i \in [t]$, note that each vertex x_i has at least $\log n$ forward neighbours, and the probability that one of these vertices is deleted is at most $1/2$. Thus $\mathbb{P}(E_i) \geq 1 - 2^{-\log n} = 1 - 1/n$, therefore $\mathbb{P}\big((U \cup \overline{N}) \setminus S \text{ is red}\big) \geq \mathbb{P}(E) > 1/2$, where the second inequality holds since $t < n/2$.

Thus, with positive probability, $W \cup S$ is blue and $(U \cup \overline{N}) \setminus S$ is red. In particular, the vertices can be partitioned into a blue set and a red one, a contradiction. □

Claim 10. *There is a vertex of blue degree at most* $90 \log n$*.*

Proof of Claim 10. By definition of $\overline{N}$ and since $|\overline{N}| \geq 2n/3 - 27\log n$, every vertex in W has at least $n/3 - 29 \log n$ blue neighbours in $\overline{N}$.

Fix a vertex $w \in W$. If there is a vertex $v \in W$ with $|N_b(v) \cap N_b(w) \cap \overline{N}| < \log n$, then the blue components containing v and w (at most 2) cover all vertices of W and all but at most $62 \log n$ vertices of $\overline{N}$ (as $|\overline{N}| \leq 2n/3 + 3\log n + 4$, by the previous claim). Since $|U| \leq 27 \log n$, it follows that these two components cover all but at most $90 \log n$ vertices. Recall that there are at least three blue components, hence there is a component of order at most $90 \log n$, and any vertex in that component has blue degree at most $90 \log n$.

Otherwise, every vertex $v \in W$ satisfies $|N_b(v) \cap N_b(w) \cap \overline{N}| \geq \log n$. As in Claim 9, let S be an uniformly random subset of $N_b(w) \cap \overline{N}$; we find that, with positive probability, $\big(U \cup \overline{N}\big) \setminus S$ is red and $W \cup S$ is blue, so the vertices can be partitioned into a red set and a blue set, contradicting our assumption. □

Let r be a vertex of blue degree at most $90 \log n$, which exists by the previous claim. By symmetry, there is a vertex b of red degree at most $90 \log n$. Then $d_r(r), d_b(b) \geq 2n/3 - 90 \log n - 2$. Write $A_1 = N_b(b) \setminus N_r(r)$, $A_2 = N_b(b) \cap N_r(r)$ and $A_3 = N_r(r) \setminus N_b(b)$. Then $|A_2| \geq n/3 - 180 \log n - 4$ and $|A_1|, |A_3| \leq n/3 + 90 \log n + 2$.

Claim 11. *There is a vertex with no blue neighbours in* A_1*, no red neighbours in* A_3*, and at most* $2 \log n$ *neighbours in* A_2*.*

Proof of Claim 11. Suppose that the statement does not hold. Let $\{B, R\}$ be a random partition of A_2, obtained by putting vertices in B, independently, with probability $1/2$. Then, with positive probability, every vertex in G has a blue neighbour in $A_1 \cup B \subseteq N_b(b)$ or a red neighbour in $A_3 \cup R \subseteq N_r(r)$. We thus obtain a partition of the vertices into a red set and a blue set, a contradiction. □

Let x be a vertex with no blue neighbours in A_1, no red neighbours in A_3, and at most $2 \log n$ neighbours in A_2 (its existence is guaranteed by the previous claim). Then $|A_2| \leq n/3 + 3 \log n$, so $|A_1|, |A_3| \geq n/3 - 95 \log n$. Furthermore, x has at least $n/3 - 100 \log n$ red neighbours in A_1 and at least $n/3 - 100 \log n$ blue neighbours in A_3. Write $A_1' = A_1 \cap N_r(x)$, $A_2' = A_2 \setminus N(x)$, and $A_3' = A_3 \cap N_b(x)$ (so $|A_1'|, |A_3'| \geq n/3 - 100 \log n$ and $|A_2'| \geq n/3 - 190 \log n$).

Claim 12. *The vertices x and b are in distinct blue components; similarly, x and r are in distinct red components.*

Proof of Claim 12. Suppose that x and b are in the same blue component. Then there is a blue path P from $\{x\} \cup A'_3$ to $\{b\} \cup A'_1 \cup A'_2$. We may assume that the inner vertices of P are outside of $A'_1 \cup A'_2 \cup A'_3 \cup \{x, b\}$. Hence, $|P| \leq 400 \log n$.

Now, let $\{B, R\}$ be a random partition of $(A'_2 \cup A'_3) \setminus V(P)$, obtained by putting each vertex in B, independently, with probability $1/2$. It is easy to see that, with positive probability, every vertex in G has a red neighbour in R or a blue neighbour in B, from which it can be deduced that there is a partition of the vertices into a red set and a blue set, a contradiction. Indeed, note that $P \cup \{x, b\} \cup B$ is a blue set and $\{r\} \cup R$ is a red set. Thus, we have that b and x are in distinct blue components; by symmetry, r and x are in different red components. □

Note that $|C_b(b)|, |C_r(r)| \geq 2n/3 - 91 \log n$ and $|C_b(x)|, |C_r(x)| \geq n/3 - 100 \log n$. Recall that there are at least three blue components. Hence, there is a vertex r_1 which is not in $C_b(b)$ or in $C_b(x)$. It follows that $d_b(r_1)$ is at most $191 \log n$, hence it has red degree at least $2n/3 - 192 \log n$, so $r_1 \in C_r(r)$. Similarly, there is a vertex b_1 which is not in $C_r(r)$ or in $C_r(x)$, and therefore it must belong to $C_b(b)$. We claim that the set $X = \{b_1, r_1, x\}$ is independent. We cannot have r_1x or b_1x, for this either contradicts the choice of $r_1 \notin C_b(x)$ and $b_1 \notin C_r(x)$ or it contradicts the statement of Claim 12. If we had r_1b_1 and this edge was coloured red then $b_1 \in C_r(r)$ which is a contradiction, by definition of b_1. If r_1b_1 is coloured blue then we arrive at the contradiction $r_1 \in C_b(b)$. Thus X is independent. So finally, by the minimum degree condition, there must be a vertex w that is adjacent to all three vertices in X. Indeed, if no such w exists, then the number of edges between X and $V(G) \setminus X$ is at most $2(n-3) < 3(2n-5)/3$, a contradiction. Without loss of generality, w sends two red edges into X, implying that two of these vertices in X belong to the same red component, a contradiction. This completes our proof of Theorem 2. □

3 Covering with monochromatic components of distinct colours

In this section we verify Conjecture 5 for $r \in \{2, 3\}$. Most of the difficulty is in the proof for $r = 3$, but we include a short proof for $r = 2$ for completeness. Actually, the $r = 2$ case (for n large) already follows from a difficult result of Letzter [12], who showed that when $\delta(G) \geq 3n/4$, the vertices can be partitioned into two monochromatic *cycles* of different colours, for every 2-colouring of G. Before turning to the proofs, we mention the following construction of Bal and DeBiasio [3], which shows that the minimum degree condition in Conjecture 5 cannot be improved.

Example 13. *Let $n \geq 2^r$; we shall define a graph on vertex set $[n]$ as follows. Partition $[n]$, as equally* *as possible, into 2^r sets which are indexed by the sequences $s \in \{0,1\}^r$. We write*

$$[n] = \bigcup_{s \in \{0,1\}^r} A(s)$$

and define the following, where $\mathbb{1} = (1, \ldots, 1)$.

$$E = [n]^{(2)} \setminus \bigcup_{s \in \{0,1\}^r} \{xy : x \in A(s), y \in A(\mathbb{1} - s)\}.$$

In other words, we include all edges in the graph except for the edges between parts of the partition corresponding to antipodal elements of $\{0,1\}^r$. Now, colour all edges xy, where $x \in A(s), y \in A(s')$, by the first coordinate on which s, s' agree; e.g. the edge between $(0,1,0,0)$ and $(1,0,0,1)$ is coloured 3.

We now show that G cannot be covered by components of distinct colours. Suppose that it can, and note that the i-coloured components are of the form $\bigcup_{s \in S_i} A(s)$ where S_i is a set of elements that agree on their i-th coordinate; denote this coordinate by a_i. It follows that the vertices of $A((1-a_1, \ldots, 1-a_r))$ are not covered by any of these components, a contradiction.

We now prove Conjecture 5 for $r = 2$.

Lemma 14. *Let G be a 2-coloured graph with $\delta(G) \geq 3n/4$. Then the vertices of G can be covered by a red component and a blue component.*

Proof. We first show that there is a monochromatic component of order greater than $n/2$. If G is red connected we are done. Hence, there exists a red component R with $|R| \leq n/2$. Then, any two vertices $u, w \in R$ have a common blue neighbour, as $|N_b(u) \cap N_b(w) \cap \overline{R}| \geq 2 \cdot (3n/4 - (|R|-1)) - (n - |R|) > 0$. So $R \subseteq C_b(u)$ and $C_b(u)$ is a blue component of order at least $3n/4$, as required.

Without loss of generality, there is a red component R of order larger than $n/2$. Note that there is a vertex x which is not in R (otherwise we are done), and $|N_b(x) \cap R| = |N(x) \cap R| > n/4$, as x does not send red edges to R. In particular, $|C_b(x) \cap R| > n/4$. It follows that every vertex sends at least one edge to $C_b(x) \cap R$ and thus the components R and $C_b(x)$ cover the whole graph. □

We now turn to prove Theorem 6, which is the case of three colours in Conjecture 5.

Theorem 6. *Let G be a 3-coloured graph on n vertices with $\delta(G) \geq 7n/8$. Then the vertices of G can be covered by monochromatic components of distinct colours.*

Proof. We begin with a series of preparatory claims (Claims 15 to 17).

Claim 15. *If there are three monochromatic components of distinct colours whose intersection has order at least $n/8$, then the vertices can be covered by monochromatic components of distinct colours.*

Proof of Claim 15. Suppose that R, B and Y are red, blue and yellow components, whose intersection $U = R \cap B \cap Y$ has size at least $n/8$. Then, by the minimum degree condition, every vertex not in U has a neighbour in U, implying that every vertex in the graph belongs to at least one of R, B and Y, as required. □

Claim 16. *If there are two monochromatic components of distinct colours whose intersection has order at least $n/4$, then the vertices of G may be covered by monochromatic components of distinct colours.*

Proof of Claim 16. Suppose that R and B are red and blue components whose intersection $U = R \cap B$ has size at least $n/4$. We show that one of the following holds.

1. $R \cup B = V(G)$;
2. there is a yellow component whose intersection with $R \cap B$ has size at least $n/8$.

Suppose that the first assertion does not hold. Then there is a vertex $u \notin R \cup B$. By the minimum degree condition, u sends at least $n/8$ edges to $R \cap B$, but these edges cannot be red or blue (because $u \notin R \cup B$), hence they are yellow, so by picking Y to be the yellow component containing u, the second assertion holds. If the first assertion holds, we are done immediately; otherwise, we are done by Claim 15. □

Claim 17. *If there is a monochromatic component of order at least $n/2$, then the vertices can be covered by three monochromatic components of distinct colours.*

Proof of Claim 17. As in the proof of Claim 16, we show that one of the following assertions holds, where R is a red component of order at least $n/2$.

1. $R = V(G)$;
2. there are monochromatic components B and Y in colours blue and yellow respectively, such that $R \cup B \cup Y = V(G)$;
3. there are monochromatic components B and Y in colours blue and yellow respectively, such that $|R \cap B \cap Y| \geq n/8$.

Suppose that $R \neq V(G)$ and let $u \notin R$. Consider the blue and yellow components, B and Y, containing u. By the minimum degree condition, u sends at least $|R| - n/8$ edges to R, none of which are red. So $|(B \cup Y) \cap R| \geq |R| - n/8$. Suppose that R, B and Y do not cover the whole graph. Let $w \notin R \cup B \cup Y$, and denote the blue and yellow components containing w by B' and Y'. By the same argument as before, $|(B' \cap Y') \cap R| \geq |R| - n/8$, which implies the following.

$$|(B \cup Y) \cap (B' \cup Y') \cap R| \geq |R| - n/4 \geq n/4.$$

Since $B \cap B' = \emptyset$ and $Y \cap Y' = \emptyset$, either $|B \cap Y' \cap R| \geq n/8$ or $|B' \cap Y \cap R| \geq n/8$. This completes the proof that one of the above assertions holds. If one of the first two assertions holds, we are done immediately; and if the third assertion holds, Claim 17 follows from Claim 15. $\square$

Henceforth, we assume G cannot be covered by monochromatic components of distinct colours.

Claim 18. *There are two monochromatic components of distinct colours of order at least* $3n/8$.

Proof of Claim 18. We will show that for every pair of colours there is a monochromatic component of order at least $3n/8$ in one of the two colours; the claim easily follows from this fact. Let the two colours be red and blue. Since G is not connected in yellow, we may find a partition $\{X, Y\}$ of the vertices of G such that no $X - Y$ edges are yellow. Without loss of generality, at least half the edges between X and Y are red; set $H = G_r[X, Y]$, denote $d(x) = d_H(x)$ for any vertex x, and given an edge xy in H, set $s(xy) = d(x) + d(y)$. We will show that there is an edge xy with $s(xy) \geq 3n/8$; note that this would imply the existence of a red component of order at least $3n/8$, as required. Put $e = e(H)$ and, without loss of generality, we assume that $|X| \leq |Y|$. We have

$$
\begin{aligned}
\frac{1}{e} \sum_{xy \in E(H)} s(xy) &= \frac{1}{e} \sum_{xy \in E(H)} (d(x) + d(y)) \\
&= \frac{1}{e} \left(\sum_{x \in X} d(x)^2 + \sum_{y \in Y} d(y)^2 \right) \\
&\geq \frac{1}{e} \left(\frac{\left(\sum_{x \in X} d(x)\right)^2}{|X|} + \frac{\left(\sum_{y \in Y} d(y)\right)^2}{|Y|} \right) \\
&= e \left(\frac{1}{|X|} + \frac{1}{|Y|} \right) \\
&\geq \frac{1}{2} |X| \left(|Y| - n/8\right) \left(\frac{1}{|X|} + \frac{1}{|Y|} \right) \\
&= \frac{1}{2} \left(|Y| - n/8 + |X| - \frac{|X| \cdot n/8}{n - |X|} \right) \\
&\geq 3n/8.
\end{aligned}
$$

Indeed, the first inequality follows from the Cauchy-Schwarz inequality; the second follows from the minimum degree condition and the assumption that red is the majority colour between X and Y; and the last inequality follows since $|X| + |Y| = n$ and the expression $\frac{|X|}{n - |X|}$ is maximised when $|X| = n/2$ (as we have the constraint $|X| \leq n/2$).

This chain of inequalities shows that the average value of $s(xy)$ is at least $3n/8$; in particular, there is a red component of order at least $3n/8$, as required. $\square$

We remark that the idea of double counting $s(xy)$ as in the proof of the previous claim originated in a paper by Liu, Morris and Prince [13].

By the previous claim, we may assume that R and B are red and blue components of order at least $3n/8$.

Claim 19. *Either $|R \setminus B| < n/4$ or $|R \setminus B| < n/4$.*

Proof of Claim 19. Assume that $|R \setminus B| \geq n/4$ and $|R \setminus B| \geq n/4$. Note that every edge between the disjoint sets $R \setminus B$ and $B \setminus R$ is yellow. Furthermore, any two vertices in $B \setminus R$ have a common neighbour in $R \setminus B$, and vice versa. Therefore $B \triangle R$ is contained in a yellow component; in particular, there exists a yellow components of order at least $n/2$, a contradiction, by Claim 17. $\square$

By the previous claim, we may assume that $|B \setminus R| < n/4$. Hence, $|B \cup R| = |R| + |B \setminus R| < n/2 + n/4 = 3n/4$, by Claim 17. Therefore, the set $W = V(G) \setminus (R \cup B)$ has size larger than $n/4$. Since all edges between $R \cap B$ and W are yellow, it follows that every two vertices in $R \cap B$ have a common yellow neighbour in W and hence $R \cap B$ is contained in a yellow component. Thus, Claim 15 implies that $|R \cap B| < n/8$. It follows that $|B| = |B \cap R| + |B \setminus R| < 3n/8$, in contradiction with the choice of B. This completes the proof of Theorem 6. $\square$

4 Concluding remarks

We conclude by noting a few directions for future research. Of course, the conjecture of Bal and DeBiasio for $r > 3$ remains open.

Conjecture 5. *Let G be an r-coloured graph on n vertices with $\delta(G) \geq (1 - 1/2^r)n$. Then the vertices can be covered by monochromatic components of distinct colours.*

Bal and Debiasio also conjectured a r-colour analogue of (what is now) our Theorem 2.

Conjecture 20. *Let G be an r-coloured graph on n vertices with $\delta(G) \geq \frac{r(n-r-1)+1}{r+1}$. Then the vertices of G can be covered by at most r monochromatic components.*

Finally, we recall our conjecture for the best minimum degree condition that guarantees a partition of every 2-coloured graph into t-monochromatic components.

Conjecture 3. *For every t there exists n_0, such that for every 2-colouring of a graph G on $n \geq n_0$ vertices with $\delta(G) \geq \frac{2n-2t-1}{t+1}$ there exists a partition of the vertex set into at most t monochromatic connected subgraphs.*

To motivate this conjecture, we prove Proposition 4, a weaker version of Conjecture 3, where instead of partitioning the vertices into t monochromatic components, we aim only to *cover* the vertices with t monochromatic components.

Proof of Proposition 4. We use the link with König's Theorem first noted by Gyárfás [8]. Let G be a 2-coloured graph with minimum degree at least $\frac{2n-2t-1}{t+1}$. Let $\mathcal{R}$ be the collection of red components (some of which may be singletons, if there are vertices that are not incident with any red edges), and let $\mathcal{B}$ be the collection of blue components. Define an auxiliary bipartite graph $H = (\mathcal{R}, \mathcal{B}, E)$, where for $R \in \mathcal{R}$ and $B \in \mathcal{B}$, we have $RB \in E$ if and only if $R \cap B \neq \emptyset$.

We claim that there is no matching of size larger than t. Indeed, suppose that $\{R_1B_1, \ldots, R_{t+1}B_{t+1}\}$ is a matching of size $t+1$. Let $u_i \in R_i \cap B_i$, for $i \in [t+1]$ and $U = \{u_1, \ldots, u_{t+1}\}$. Then the vertices of U are in distinct red and blue components. In particular, U is independent, so the number of edges between U and $V(G) \setminus U$ is at least $2n - 2t - 1$. On the other hand, no vertex sends more than one red edge into U (and similarly for blue), so every vertex not in U sends at most two edges into U. It follows that the number of edges between U and $V(G) \setminus U$ is at most $2(n - t - 1) < 2n - 2t - 1$, a contradiction.

By König's theorem, which states that in bipartite graphs, the size of a minimum cover equals the size of a maximum matching, it follows that there is a cover W of size at most t; write $W = \{C_1, \ldots, C_t\}$. We claim that $V(G) = C_1 \cup \ldots \cup C_t$. Indeed, consider a vertex u and denote its red and blue components by R and B, respectively. Then $R \cap B \neq \emptyset$, hence RB is an edge in H, so either R or B is in W, which implies that $u \in C_1 \cup \ldots \cup C_t$, as required. In other words, the vertices of G can be covered by at most t monochromatic components. □

Finally, we note that the restriction on the minimum degree in Proposition 4 (and therefore Conjecture 3) cannot be improved. The special case of this example, where $t = 2$, appears in [3] and shows that the minimum degree condition in Theorem 2 is best possible.

Example 21. *Let U be a set of size $n \geq t+1$, and let $\{X, A_1, \ldots, A_{t+1}\}$ be a partition of U, where $|X| = t+1$ and the sizes of $A_1, A_2, \ldots, A_{t+1}$ are as equal as possible; write $X = \{x_1, \ldots, x_{t+1}\}$. We define a 2-coloured graph G on vertex set U as follows.*

- *the sets A_i are cliques, and we colour them arbitrarily;*
- *we add all possible edges between A_i and A_{i+1}, where $i \in [t]$, and colour them red if i is odd, and blue otherwise;*
- *we add all edges between x_i and $A_i \cup A_{i+1}$, for $i \in [t+1]$ (addition is taken modulo $t+1$). We colour these edges red if i is in $[t]$ and i is odd; and blue if i is in $[t]$ and i is even. Finally, we colour the edges from x_{t+1} to A_1 blue, and colour the edges from x_{t+1} to A_{t+1} red if t is even and blue if t is odd.*

An easy calculation shows that G has minimum degree[1] $\lceil (2n-2t-1)/(t+1) \rceil - 1$, *and that no two*

[1]In fact, we need to be a bit more careful here. Write $n = a(t+1) + r$, where a and r are integers and $0 \leq r \leq t$. We consider two cases: $r < \lceil (t+1)/2 \rceil$ and $r \geq \lceil (t+1)/2 \rceil$. In the former case, it is easy to see that $\delta(G) = \lceil (2n-2t-1)/(t+1) \rceil - 1$. In the latter case, note that exactly r of the sets A_i have size a, and the rest have size $a-1$. Then, again, one can check that $\delta(G) = \lceil (2n-2t-1)/(t+1) \rceil - 1$ if $|A_i| = a$ for every odd $i \in [t+1]$ (which is possible as $r \geq (t+1)/2$).

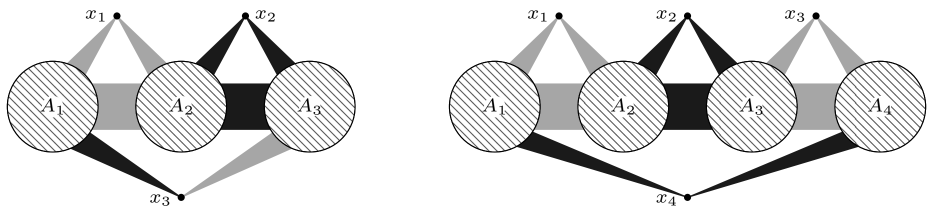

Figure 1: an illustration of Example 21 for $t = 2$ and $t = 3$ (here grey represents red and black represents blue).

vertices in X belong to the same monochromatic component; in particular, the vertices of G cannot be covered by at most t monochromatic components.

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