Strategically-Timed Actions in Stochastic Differential Games

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DECLARATION STATEMENT

I, David Mguni confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

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Abstract

Financial systems are rich in interactions amenable to description by stochastic control theory. Optimal stochastic control theory is an elegant mathematical framework in which a controller, profitably alters the dynamics of a stochastic system by exercising costly control inputs. If the system includes more than one agent, the appropriate modelling framework is stochastic differential game theory — a multiplayer generalisation of stochastic control theory.

There are numerous environments in which financial agents incur fixed minimal costs when adjusting their investment positions; trading environments with transaction costs and real options pricing are important examples. The presence of fixed minimal adjustment costs produces adjustment stickiness as agents now enact their investment adjustments over a sequence of discrete points. Despite the fundamental relevance of adjustment stickiness within economic theory, in stochastic differential game theory, the set of players' modifications to the system dynamics is mainly restricted to a continuous class of controls. Under this assumption, players modify their positions through infinitesimally fine adjustments over the problem horizon. This renders such models unsuitable for modelling systems with fixed minimal adjustment costs.

To this end, we present a detailed study of strategic interactions with fixed minimal adjustment costs. We perform a comprehensive study of a new stochastic differential game of impulse control and stopping on a jump-diffusion process and, conduct a detailed investigation of two-player impulse control stochastic differential games. We establish the existence of a value of the games and show that the value is a unique (viscosity) solution to a double obstacle problem which is characterised in terms of a solution to a non-linear partial differential equation (PDE).

The study is contextualised within two new models of investment that tackle a dynamic duopoly investment problem and an optimal liquidity control and lifetime ruin problem. It is then shown that each optimal investment strategy can be recovered from the equilibrium strategies of the corresponding stochastic differential game. Lastly, we introduce a dynamic principal-agent model with a self-interested agent that faces minimally bounded adjustment costs. For this setting, we show for the first time that the principal can sufficiently distort that agent's preferences so that the agent finds it optimal to execute policies that maximise the principal's payoff in the presence of fixed minimal costs.

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Contents

Notation

Throughout the thesis we use the following notation a portion of which is introduced with explanation within the script:

Universal quantifier: $\forall$

Existential Quantifier: $\exists$

Belongs to (set): $\in$

Not in (set): $\notin$

Proper Subset: $\subset$

Subset: $\subseteq$

Maps to: $\to$

Direct Product: $\times$

Identically equivalent to: $\equiv$

The set of real numbers: $\mathbb{R}$

The set of strictly positive real number: $\mathbb{R}_{>0}$

Ordered pair: $(\cdot,\cdot)$

Probability: $\mathbb{P}(\cdot)$

Kronecker-delta symbol: δ_n^m

Union (set): $\cup$

Intersection (set): $\cap$

If and only if (iff) (relational): $\iff$

Temporal Derivative: ∂_s

First, second and n^{th} spatial derivative (resp.): $\partial_{x_i}, \partial^2_{x_i,x_j}, \partial^n_{x_1,\dots x_n}$

$\nabla\phi = (\frac{\partial\phi}{\partial x_1},\dots,\frac{\partial\phi}{\partial x_p})$: The gradient operator acting on some function $\phi \in \mathscr{C}^1([0,T]\times\mathbb{R}^p)$.

$\mathscr{C}^1([a,b],\mathbb{F})$: The set of continuous functions from $\mathbb{R}$ to a field $\mathbb{F}$ over the interval $[a,b]\subseteq\mathbb{R}$ for some bounded open set Ω on $\mathbb{R}^{p+1}$.

$\mathbb{S}(p)$: The set of symmetric $p\times p$ matrices in $\mathbf{GL}(\mathbb{F})$ for some field $\mathbb{F}\subseteq\mathbb{R}^p$.

$\mathbb{L}$: The set of Lebesgue integrable functions.

$\mathbb{L}^2$: The set of Lebesgue square integrable functions.

$B_s(\omega_B) = \omega_B(s)$: The coordinate mapping on $\mathscr{C}(\mathbb{F},\mathbb{G})$for some fields $\mathbb{F},\mathbb{G}\subseteq\mathbb{R}^p$

$\mathscr{F} = (\mathscr{F}_s)_{s\geq t}$: A completed natural filtration generated by the map $s\mapsto B_s$.

$\mathscr{W}_{t,t'}$: The σ-algebra generated by the paths in $\mathscr{C}(\mathbb{F},\mathbb{G})$ up to time t'.

$B(s)\in\mathbb{R}^p$: A p-dimensional standard Brownian motion with state space $S\subset\mathbb{R}^p$.

$\mathscr{C}^d([a,b];U)$: The set of càdlàg functions that map $[a,b]\mapsto U$ for some set $U\subseteq\mathbb{R}^p$.

$|\cdot|$: The Euclidean norm to which $\langle x,y\rangle$ is the associated scalar product acting between two vectors belonging to some finite dimensional space.

$B_r(x)$: The closed ball of radius r and centre x.

Let Ω be a bounded open set on $\mathbb{R}^{p+1}$, then we denote by:

$\bar{\Omega}$: The closure of the set Ω.

$Q(s,x;R) = \{(s',x') \in \mathbb{R}^{p+1} : \max|s'-s|^{\frac{1}{2}}, |x'-x| < R, s' < s\}$.

$\partial\Omega$: the parabolic boundary Ω i.e. the set of points $(s,x) \in \bar{S}$ such that $R > 0, Q(s,x;R) \not\subset \bar{\Omega}$.

$\mathscr{C}^{1,2}([0,T],\Omega) = \{h \in \mathscr{C}^{1,2}(\Omega) : \partial_s h, \partial_{x_i,x_j} h \in \mathscr{C}^1(\Omega)\}$

Sometimes the abbreviations s.th. and w.l.o.g. are used in proofs, meaning such that and without loss of generality.

Introduction

Environments in which agents are required to solve decision problems and in which pertinent information about the future is unknown are ubiquitous in financial systems. Recent increases in computing power and the expansion of algorithmic models in financial markets has led to significant developments in real-time decision-making in financial systems [KL13; Kis13]. These advances have delivered the potential to execute algorithmic decisions that constitute near or exactly optimal investment behaviour. Moreover, current computational methods offer the ability to solve decision problems and then execute the prescribed set of actions [Lit96]. Algorithmic trading, algorithmic mechanism design and applications of algorithmic game theory are some of the many scenarios to which these methods are applied [CJR18; Rou10; NR01; Kis13; KL13].

The ability of these algorithms to execute optimal decisions depends crucially on the accuracy of the models that describe market behaviour. Neglecting important features of financial systems can lead to both poor modelling capabilities and vastly suboptimal solutions [LSCS01]. One such feature is transaction costs. The presence of transaction costs leads to significant changes in the behaviour of market participants as transaction costs induce market rigidities and adjustment stickiness [LMW04]. However, despite their fundamental relevance, at present there remains a number of important financial systems that are analysed using models that have yet to incorporate transaction costs. One set of cases are financial models that describe strategic interactions that occur between market participants, namely multiplayer financial settings.

The work presented in this thesis addresses the challenge of modelling financial systems in the presence of minimally bounded adjustment costs such as transaction costs within strategic multiplayer settings. The thesis performs a detailed investigation of the financial investment problems and the required mathematical formalisms for analysing stochastic systems in which agents face fixed minimal adjustment costs.

Overview

Many financial investment problems are solved using optimal control theory [ZY18; FL19; Pha09]. In classical optimal control, the agent or *controller* has the ability to continuously make infinitesimally fine adjustments for which the associated costs can be made arbitrarily small [Tou13]. This form of control, namely *continuous control* is incompatible with settings that include transaction costs since continuous adjustments would lead the cost of control to explode [Kor99]. Therefore, including fixed minimal control costs requires substituting continuous controls with a form of control that executes discrete, timed actions over the horizon of the problem. In these models known as *impulse control models*, the cost of control is bounded below so that modifying the system dynamics at any point incurs at least, some fixed minimum cost [ØS07; DGW10]. In this setting, the controller alters the system dynamics through a sequence of discrete actions or bursts chosen at times at which the agent chooses to apply their control policy.

Transaction costs or fixed minimal adjustment costs are widespread in economic and financial systems and serve to induce rigidities in economic behaviour [BSS95; LMW04]. The need to achieve increasing performance in these investment scenarios has driven the development of impulse control models for single controller settings [Kor99; Sey09; Azi17]. Given the discrete nature of impulse control, impulse control models represent appropriate modelling frameworks for financial environments with transaction costs, liquidity risks and economic environments in which players face fixed adjustment costs (e.g. 'menu costs') [ARS17; Kor99; JP93]. More generally, impulse control models are useful for describing systems in which the dynamics are modified by sequences of discrete, timed actions.

Why study multiplayer impulse control models?

Financial systems involve many market participants making decisions over time in order to maximise their individual returns. The interdependence of agents' actions and their rewards leads to a strategic interaction between agents. Consequently, performing a systematic analysis of investment scenarios requires modelling strategic effects which are captured in multiplayer frameworks [AEHX11; GM11]. However, despite the importance of transaction costs on systemic behaviour, numerous multiplayer financial investment problems are exclusively described using a continuous class of control which prohibits the inclusion of transaction costs [Zha11]. Unlike single player settings, since in multiplayer settings agents strategically respond to other agents' behaviour, the introduction of features such as transaction costs has a *joint effect* on the collective agent behaviour which in turn, determines system outcomes. Consequently, the introduction of features such as transaction costs in multiplayer models has a profound effect on the behaviour of the system [Ior02].

Game theory is a mathematical framework that is used for making predictions about outcomes in systems with strategically interacting players [MCWG+95]. In these settings, each player reasons about their environment and the actions of other players in order to decide on an action that maximises his or her own reward. In situations that involve self-interested players, the appropriate class of games is known as *non-cooperative games* [OR94]. Stochastic games are a class of non-cooperative games that describe strategic interactions between players that occur over time and have a random component in their dynamics [Sha53; FT91]. For these reasons, stochastic (differential) game theory — a game-theoretic generalisation of stochastic control theory is a central tool for analysing economic and financial systems [Car16].

In order to study multiplayer dynamic systems with minimally bounded adjustment costs, it is necessary to develop the mathematical framework for modelling such systems namely, stochastic differential games to now encompass minimally bounded control costs. A key component of this thesis is therefore dedicated to incorporating fixed and minimally bounded adjustment costs within the framework of stochastic differential games. In particular, we study stochastic differential games in which players modify the system dynamics using discretised actions modelled by impulse controls.

At present, the study of stochastic differential games that incorporate impulse control models is limited to restrictive settings such as zero-sum payoff structures [Yon94; Cos13] and games in which one of the controllers uses continuous controls [Zha11; Yon94]. Zero-sum settings represent extreme strategic settings in which the interests of each players are diametrically opposed. Moreover, current multiplayer control models which use impulse controls are restricted to systems with dynamics that evolve with continuous sample paths [Yon94; Zha11; Cos13]. As a consequence of these restrictions, the need to model complex financial systems in which firm activities alter the size of the market (thus violating the zero-sum structure) is currently unaccounted for. Additionally, incorporating the effects of exogenous market shocks requires extending the analyses to jump-diffusions [Tan03] which have discontinuous sample paths, an analysis which is currently absent.

To this end, the thesis addresses two fundamental issues: first, the thesis addresses the absence of fixed minimal costs within three prominent financial investment problems which, at present are solved using models that admit only continuous controls. In each of the three investment problems we study, the task facing the investor is to strategically modify the dynamics of a financial system with future uncertainty in the presence of other self-interested players. Constructing models of the three investment problems with fixed minimal adjustment costs requires the development of new multiplayer structures that incorporate impulse controls.

Second, the thesis addresses the task of developing variants of the underlying mathematical framework required to analyse the investment problems namely, stochastic differential games with minimally bounded control costs. In addition to incorporating impulse control, we analyse games that also accommodate non zero-sum payoff structures which have dynamics that include jumps. In performing this investigation, the thesis addresses the deficiency of current multiplayer impulse control models in describing various financial scenarios which deviate from idealised settings. This theoretical contribution of the thesis is general and therefore broadly applicable.

The task of introducing game structures that solve the investment problems in the thesis requires a detailed formal treatment to establish important properties of the game. In particular, in order to guarantee the existence of a solution for each game setting, it is necessary to establish the existence of a fixed point or *equilibrium* which constitutes a solution to the game [OR94]. Moreover, in order to give a full systematic treatment for each game setting, it is necessary to characterise the equilibrium strategies for each player.

Each of the three investment problems requires a specific game in order to describe the given scenario. However, each of these games can be viewed as a special case of the game introduced in Chapter 4. In particular, the game analysed in Chapter 4 is a non-cooperative two-player stochastic differential game with a state process that is influenced by impulse controls. This game incorporates the games studied within the thesis as degenerate cases in which one of the player's actions is restricted in a given manner. In particular, we develop the analysis leading to the game in Chapter 4 progressively through the thesis, starting with a special case (a zero-sum stochastic game of impulse control and stopping) in Chapter 2 then leading to the non zero-sum game in which both players use impulse control in Chapter 4. In Chapter 5, we perform an incentive-distortion analysis of a non-cooperative impulse control game. The setting analysed is a degenerate case of the game in Chapter 4 since in this game only one agent can modify the dynamics of the system. However, in this setting, the other agent namely, the principal chooses the transaction costs the other controller must face.

An overview of the investment problems studied in the thesis is as follows:

Investment Problems

- **Dynamic advertising investment in duopoly environments**

  A considerable amount of attention has been dedicated towards modelling the duopoly advertising problem with the purpose of accurately describing optimal advertising investment strategies [PS04; HLL12]. In this setting, two firms use advertising investments to increase their individual market share from which the firm's profits are derived. The problem facing the firm is to find the optimal investment strategy that maximises its cumulative profits.

  Early versions of the problem were formulated as *single-player* optimal control models in which the firm's market share is modelled as a deterministic process [PO78; Set77; FHS94]. Consequently, in the early models of the advertising problem, the influence of competing firms and the effect of future uncertainty derived from market fluctuations and exogenous shocks were neglected (see for example the surveys conducted in [Jør82; Eri95; Set77; FHS94]). To augment modelling accuracy, more recent models include a larger repertoire of descriptive features; this includes an adoption of a (two-player) differential game framework in order to incorporate the effect of competing firms. Following that [PS04] adopts a stochastic differential game approach to model the problem which accounts for future uncertainty and random market fluctuations, the inclusion of which has further increased modelling accuracy [PS04; HLL12].

- **Optimal liquidity control with lifetime ruin**

  The problem of an investor who holds a collection of risky assets and seeks to minimise the probability that they go bankrupt within their lifetime is known as the (probability of) lifetime ruin problem [You04]. Various extensions to the problem have been investigated to solve problems in which the investor, who holds some portfolio of risky assets seeks to both maximise their return and secondly, find the optimal time to exit the market [BS14; BY11; BZ15a]. In this case, the problem faced by the investor is one of minimising the probability of their returns falling below some fixed level whilst adjusting their portfolio so as to maximise their market returns. The problem of lifetime ruin was introduced by [MR00] and studied in depth by [You04]. Since its introduction to the literature, a considerable amount of work has been dedicated to the study of the lifetime ruin problem in addition to a number of variants of the problem which include models with stochastic consumption [BY11], stochastic volatility [BHY11], ambiguity aversion [BZ15a] amongst many other works.

In general, the lifetime ruin problem is tackled using optimal stochastic control models in which the controller seeks both an optimal investment strategy (modelled using continuous controls) and an optimal time to sell all market holdings (discretionary stopping).

- **A dynamic Principal-Agent problem**

  The (dynamic) principal-agent problem analyses whether given a scenario involving a self-interested agent and a principal with misaligned preferences, it is possible for an uninformed principal to sufficiently modify the agent's preferences so that the agent's investment decisions maximise the principal's objectives [Hau19; GH92; MCWG+95].

  Consider the case of a firm that adjusts its production capacity by way of investment in order to maximise its cumulative profits. In this setting, the firm performs its adjustments after making private observations of market demand fluctuations. In the case of a single irreversible firm investment (or the problem of market entry timing), it is widely known that the optimal firm strategy is to delay investment (entry) beyond the point at which the expected returns becomes positive [DP94]. From the consumer's perspective, the firm's decision to delay results in a socially inefficient outcome [KS15]. Similarly, in the case of multiple production capacity decisions, the firm's decisions on production levels (which aim to maximise profit) in general, produce socially inefficient outcomes.

  The principal-agent model firstly seeks to address the question of whether an uninformed *passive* principal is able to commit to contractible transfers of wealth or a *transfer rule* to an informed agent that induces desirable outcomes for the principal. Second, should such a transfer rule exist, the goal of the analysis is to fully characterise it [MCWG+95; PST14]. In [KS15], the single irreversible investment/market entry case was analysed using a principal-agent (mechanism design framework). In [KS15], it is proven that the firm's investment decision process (i.e. the time at which the firm decides to invest or enter the market) can be sufficiently modified to produce socially efficient outcomes by a *posted-price mechanism*.[1]

[1] A posted price mechanism presents each agent with a (possibly different) price, thereafter each agent can choose to either accept or reject the mechanism offer [BKS12].

The motivation for this research is derived from the following observations:

Motivation

(i) In each of the above advertising investment models, the firms' investment modifications are modelled as stochastic differential games in which all players adopt continuous controls. In particular, within these models, it is assumed that the firms are able to conduct infinitesimally small advertising investments each of which incur arbitrarily small costs.

In reality however, advertising investment projects have fixed minimal costs which eliminates the possibility of continual investment since such a strategy would result in immediate firm ruin. The presence of fixed minimal costs produces adjustment stickiness (rigidities) since firms now strategically adjust their investment positions at discrete points over irregular time intervals.

Naturally, the competitive interaction between firms over some time horizon with fixed minimal investment costs gives rise to a non-cooperative stochastic game in which the cost of control is bounded from below.

(ii) Despite the breadth of applications of the lifetime ruin problem and the pervasiveness of transaction costs, current models within the literature have yet to include transaction costs. Indeed, existing models of the problem use stochastic control models with discretionary stopping in which the investor is assumed to modify their investment positions continuously and with costs that can be made arbitrarily small. Similar to advertising investment models, current lifetime ruin models do not produce feasible investment strategies when the investor faces minimally bounded adjustment costs.

The optimal liquidity and lifetime ruin problem involves two distinct and interdependent objectives consequently, the joint problem can delegated to two self-interested, strategically interacting players. This gives rise to a description given by a non-cooperative stochastic differential game with fixed minimal adjustment costs.

(iii) Presently, models that are concerned with sequential investment analysis in which investments incur minimally bounded costs are limited to (at most) entrance and exit problems (see for example, [Zer03]). Moreover, in these settings, no external agents posses the ability to distort the firms' incentives. Consequently, this has left the important case of multiple sequential investments with fixed minimal costs with incentive distortions untreated.

Similar to the lifetime ruin and advertising investment models, the dynamic principal-agent problem involves an agent that performs multiple investment adjustments over time each of which incurs a fixed minimal cost. Since the goal of the principal is to alter the behaviour of the agent (by way of contractible transfers), the problem involves a strategic interaction for which a game-theoretic formalism is the appropriate analytic tool.

To tackle the above problems, we propose a mathematical structure that incorporates minimally bounded adjustment costs within stochastic differential games whose general form appears in Chapter 4. In particular, the framework in Chapter 4 is a stochastic differential game in which two players use impulse controls to modify a system that evolves according to a process with jumps.

For the optimal liquidity control and lifetime ruin problem presented in Chapter 2, the framework involves a two-player zero-sum stochastic differential game in which one of the players alters the system with impulse controls and another chooses a time to terminate the game. Optimal stopping can be viewed as a degenerate case of impulse control in which one of the controller's action sets is restricted to be a single decision to terminate the game (as opposed to choosing a *sequence* of times and intervention magnitudes to alter the process) [DGW10; ØS07]. Consequently, the game of control and stopping introduced in Chapter 2 is a special case of the game studied in Chapter 4. However, the game introduced in Chapter 2 requires an analysis to characterise the stopping criterion for one of the players and secondly, a formal mathematical treatment to establish both the existence and uniqueness of a value of the game which is tackled in Chapter 3.

Similarly, the principal-agent problem in Chapter 5 is a degenerate case of the game in Chapter 4 since in this game only one agent can modify the dynamics of the system. However, the incentive-distortion aspect requires an in-depth treatment which analyses changes to the controller's behaviour under changes to the adjustment costs.

Contribution & Scope

This work makes several theoretical contributions to stochastic differential game theory involving impulse controls resulting in an in-depth analysis of three games. Additionally, the thesis tackles the problem of incorporating minimally bounded adjustment costs within prominent investment problems namely, the optimal liquidity control and lifetime ruin problem, the duopoly advertising investment problem and lastly a dynamic principal-agent problem.

For each of the investment problems, it is firstly demonstrated that the problem can be structured as a type of stochastic differential game with impulse controls and each game is a specific instantiation of the game presented in Chapter 4. We perform a detailed study of each game and investigate the stable solutions or equilibria which describe the behaviour of each player in the game when executing their optimal strategies. The characterisations of the equilibria for each game are then used to give a complete construction of the optimal investment strategies for each of the investment problems. Additionally, for the case of the dynamic principal-agent problem, we demonstrate the existence of a transaction cost selection for the principal that sufficiently distorts the controller's incentives and leads to the principal's objective being maximised.

The theoretical contribution of this research and the corresponding applications to theoretical finance is as follows:

- **Theoretical Framework: Stochastic Differential Games of Control and Stopping with Impulse Controls**

  This research introduces a stochastic differential game in which one of the players modifies the system dynamics using impulse controls and an adversary chooses a process stopping time. We prove that the game admits a value (which proves the existence of an equilibrium point) and that the value is represented by a double obstacle variational inequality characterised by a second order non-linear partial differential equation (PDE) or Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation. We derive the equilibrium characterisations of both saddle point (for zero-sum settings) and Nash (for non zero-sum settings) equilibrium concepts. We show that the value of stochastic differential games of control and stopping is a unique viscosity solution to a HJBI equation.

  The game of control and stopping we introduce can be viewed as an extension of the game of optimal stochastic control and stopping introduced in [MS96] and studied in [KS01; BY11; NZ+15; BHØT13; KZ+08; BH13] in which now, the controller affects the state process using impulse controls instead of continuous controls. Additionally, the analysis extends the investigations in [KS01; BY11; NZ+15; KZ+08; BH13] to accommodate dynamics that evolve according to jump-diffusion processes — generalisations of Itō diffusions that have continuous sample paths [Tan03].

  Our viscosity-theoretic analyses of the game which yields a proof of the existence of a value of the game adds to a vast literature on viscosity-theoretic approaches to optimal stochastic control theory established in [BI08; CIL92; CL83] and has since built on viscosity-theoretic analyses in deterministic settings and others analysed in [Son86; Lio89; Lio83] among others. The concept of viscosity solutions has been exported to differential game settings in order to generalise HJBI equations for non smooth value functions [LS88; ES84]. In these analyses, a viscosity-theoretic approach is used to establish the existence of equilibria in continuous control settings in differential games [Car07; CR09], Dynkin games [BS14; Grü13], games of (continuous) control and stopping [BHØT13] and games of two-sided impulse control [Cos13].

  Our contribution establishes the existence of a value for a new game of stochastic differential game of impulse control and stopping as a viscosity solution to a Hamilton-Jacobi-Bellman-Isaacs variational inequality. Therefore the formal analysis is closely related to the viscosity analysis performed for a game of continuous control and stopping in [BHØT13]. We however, tackle the problem of integrating impulse control within the game of control and stopping which provides a formal treatment for the game and application introduced in Chapter 2.

  * *Investment Problem: The Optimal Liquidity and Capital Injections Problem*

  We apply the results of the investigation of stochastic differential games of control and stopping to solve a new liquidity control and lifetime ruin model with transaction costs. In this setting, the investor is allowed to exit the market with concern for financial ruin. The investor sequentially injects the maximum capital into the firm that their wealth process can tolerate in order to generate firm liquidity for the firm's operations. In order to avoid early lifetime ruin, the investor also exits the market at some optimal time by selling all holdings in the firm.

Our model takes the form of a game in which a single investor has dual objectives; first to maximise their liquidity input and second, to minimise the risk of lifetime ruin defined over a convex risk measure all in the presence of transaction costs. We solve the problem which characterises the optimal investment strategies for the problem.

- **Theoretical Framework: Stochastic Differential Games Involving Impulse Controls**

  We perform a systematic analysis of a stochastic differential game in which two players strategically affect a jump-diffusion process and both players incur fixed minimum costs for their adjustments. For these games, we consider both zero-sum and non zero-sum payoff structures. Our results generalise the analysis in [Cos13] in which a zero-sum game of a controlled classical diffusion process is studied. We give a complete characterisation of the equilibrium conditions for each payoff structure and in doing so, describe both the Nash and saddle point equilibria.

* *Investment Problem: The Duopoly Investment Problem*

  Using the theoretical results derived for the stochastic differential game of two-sided impulse control, we construct a new duopoly investment model of advertising in which each firm incurs at least a fixed minimal cost for each advertising investment. The model we construct now accounts for the fixed costs incurred in advertising projects which is currently neglected in advertising investment modelling. The model provides an analytic description of the dynamic interaction between competing firms in a duopoly environment when advertising investments are used to increase market share. Additionally, in contrast to existing models, the model introduced in this thesis is a non zero-sum structure which allows for market expansion. Secondly, in our model, the system dynamics are by described an underlying diffusion process with jumps. This embeds into the description both future uncertainty and exogenous economic shocks.

- **A Dynamic Principal-Agent Problem with Minimally Bounded Costs**

  Our last contribution is to conduct a detailed study of a dynamic principal-agent problem with minimally bounded adjustment costs. We consider an incentive-distortion problem in which a self-interested agent makes costly purchases that incur some fixed minimal cost the size of which is chosen by a principal. The results generalise the incentive-distortion optimal stopping analysis in [KS15] to cover multiple interventions by the agent. Similarly, our results extend those in [DZ00] beyond single entrance and single exit problems to the case of multiple sequential investments. Our analysis includes a study of the behaviour of impulse control models with varying transaction costs. The results therefore augment the studies presented in [Øks99; ØUZ02; Fra04] which analyse the behaviour of impulse controls systems in the limiting cases when the fixed minimal cost goes to 0.

Our main result demonstrates the existence of a choice of transaction cost that enables the principal to sufficiently distort the agent's preferences so that the agent maximises the principal's payoff. The transaction cost is comprised of two components — a *fixed part* and a *proportional part*. We provide a full characterisation of each quantity and demonstrate the results within applications drawn from economics and finance.

The research presented in this thesis is first to consider incentive-distortions within an impulse control setting therefore leading to a new dynamic principal-agent model that involves impulse control. Additionally, to the author's knowledge, this script presents the first analysis of stochastic differential games of control and stopping in which the controller uses impulse controls to modify the system dynamics wherein the value of the game is both established and characterised. Lastly, to the author's knowledge, the thesis presents for the first time, a non zero-sum stochastic differential game of impulse control on both sides and with a diffusion process that has jumps.

Summary of Contributions

A summary of the contributions of this research is as follows:

- We introduce a new stochastic differential game in which one player modifies a jump-diffusion process with impulse controls and another player decides when to terminate the game. We perform a detailed investigation of the game characterising the minimax equilibrium conditions for zero-sum games and the Nash equilibrium for non zero-sum settings.

- Using the theoretical analysis of the stochastic differential game of impulse control and stopping, we tackle the problem of including transaction costs within a widely applied investment model known as the optimal liquidity and lifetime ruin problem. The resulting model fully characterises the optimal investment strategy when investors are faced with transaction costs and exogenous market shocks.

- Using viscosity theory, we provide a formal proof of the existence of a saddle point equilibrium of the stochastic differential game of impulse control and stopping. We show that the value of the game is a unique viscosity solution to a HJBI equation and show that the problem admits a double obstacle representation. This allows for solutions to be obtained in instances in which the value function may not be everywhere smooth enough to apply Dynkin's formula.

- We extend current results in [Cos13] for a stochastic differential game in which both players apply impulse controls to an Itō diffusion to now accommodate a jump-diffusion dynamics and non zero-sum payoff scenarios. We characterise the Nash equilibrium of the game and in doing so, extend the modelling capabilities of the framework to describe a range of financial settings for which the zero-sum condition does not hold.

- We then apply the theoretical analysis of the stochastic differential game of two-sided impulse control to investigate the duopoly advertising investment problem. We construct a new model that extends existing models e.g. [PS04; Eri95] to now account for the minimum expenditures incurred by firms, the effect of exogenous market shocks and the capabilities of firms to expand the market.

- Lastly, we extend the present study of impulse control to a dynamic principal-agent setting in which the agent incurs minimally bounded costs. We give a precise characterisation of the cost function that the principal is required to impose on a rational agent for the agent to execute the principal's desired policy. We extend the asymptotic analyses in [Øks99; ØUZ02; Fra04] and give a full description of the relationship between an optimal impulse control policy and the cost function components.

The theoretical results of the thesis are accompanied by worked examples to elucidate the workings of the theory in context of investment problems within finance.

Thesis Outline

The thesis is structured into two parts and five chapters. First, we introduce the key mathematical concepts discussed in this research in addition to core results within stochastic differential game theory and impulse control theory. In part I, we conduct a comprehensive theoretical treatment of stochastic differential games with impulse control in which all players have full information about the game. In Part II, we investigate a dynamic principal-agent problem involving impulse controls and perform the first analysis of incentive-distortion in environments with minimally bounded adjustment costs. The five chapters of the thesis are as follows:

- In Chapter 1, we provide some preliminary background for the study of stochastic differential game theory, impulse control theory and the necessary mathematical concepts that underpin the analyses conducted within the thesis.

- In Chapter 2, we introduce the new stochastic differential game of impulse control and stopping and our proposed model of optimal liquidity and lifetime ruin. The contribution of Chapter 2 is encompassed in the following paper: David Mguni, *"Optimal Capital Injections with the Risk of Ruin: A Stochastic Differential Game of Impulse Control and Stopping Approach"* (2018) [Mgu18c].

- In Chapter 3, we revisit the game introduced in Chapter 2, now conducting a formal analysis of the game, addressing the questions of existence and uniqueness of a value for the game using *viscosity theory*. The contribution of Chapter 3 is encompassed in the paper: David Mguni, *"A Viscosity Approach to Stochastic Differential Games of Control and Stopping Involving Impulsive Control of a Jump-Diffusion Process"* (2018) [Mgu18a].

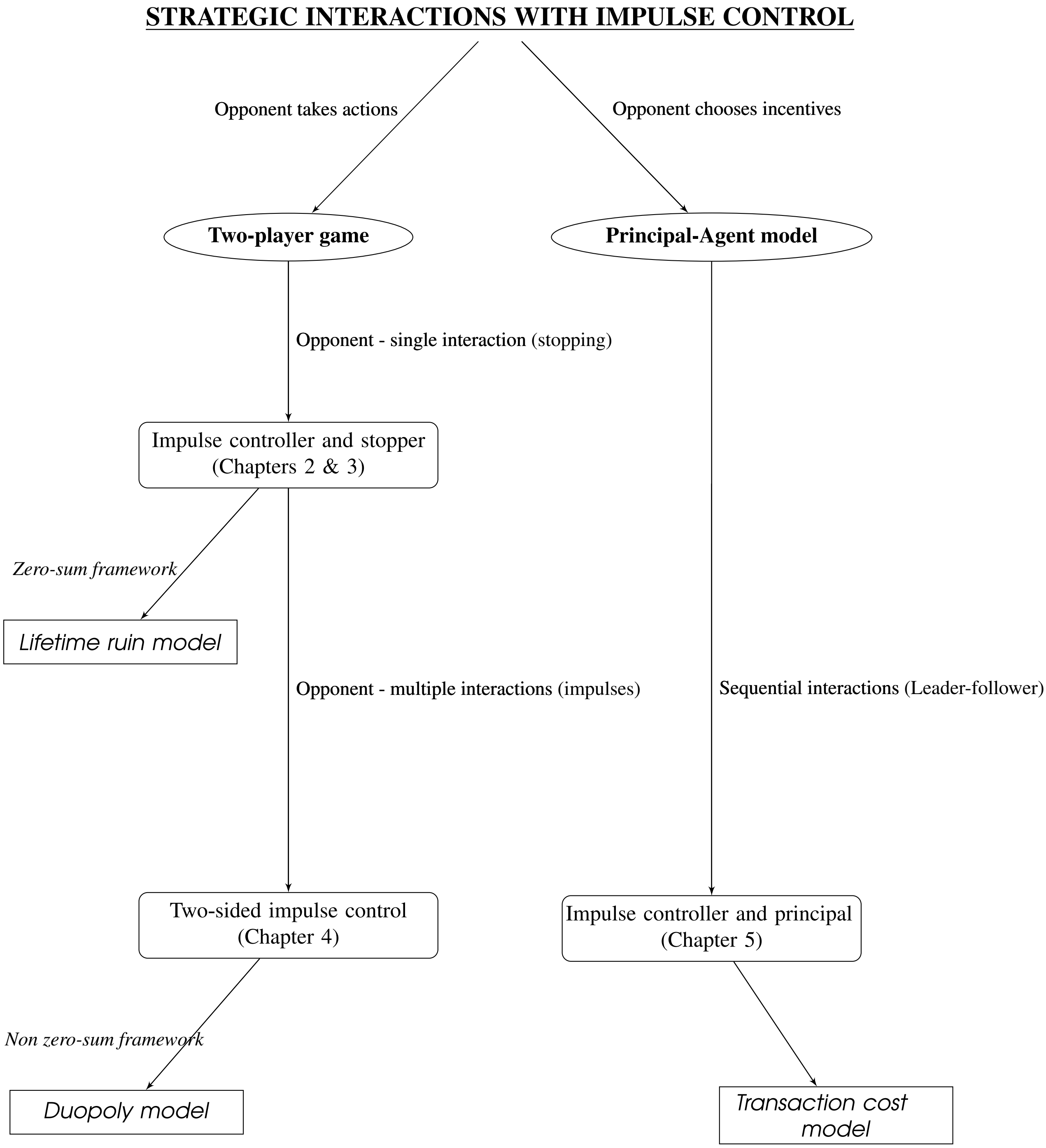

Figure 1: Thesis schematic

- In Chapter 4, we study the stochastic differential game of two-sided impulse control of a jump-diffusion process. We introduce a new model of duopoly advertising investments with fixed minimal costs and solve the problem. The contribution of Chapter 4 is encompassed in the paper: David Mguni, *"Duopoly Investment Problems with Minimally Bounded Adjustment Costs"*, (2018) [Mgu18b].

- In Chapter 5, we perform an in-depth analysis of the effect of transaction costs on the behaviour of a self-interested agent that has minimally bounded adjustment costs. We address the question of which fixed value of the transaction cost the principal must choose to induce a desired behaviour from the agent. The contribution of this chapter is encompassed in the following paper: David Mguni, *"Optimal Selection of Transaction Costs in a Dynamic Principal-Agent Problem"*, (2018) [Mgu18d].

In the last part of the thesis, we give concluding remarks and discuss future studies that relate to the work conducted in this thesis. There is also an Appendix to which parts of the proofs and analyses of Chapters 2 - 5 are relegated. Lastly, there is a bibliography.

Chapter 1

Preliminaries: Stochastic Differential Game Theory and Impulse Control

In this chapter we introduce the underlying principles of stochastic differential game theory and provide a technical description of the key objects that underpin stochastic impulse control models.

In this chapter, we review the key principles of stochastic differential games. We state some of the main results of the subject presented in [Car10; FS89; CR09] for the theory of stochastic differential games with continuous controls. We then introduce some of the necessary concepts to study stochastic differential games that include impulse controls in preparation for the analyses conducted in the thesis.

1.1 Stochastic Game Theory

Stochastic games were introduced by Lloyd Shapley in the seminal paper Stochastic Games [Sha53] and have since had a profound effect on economics and financial modelling. A stochastic game is a model of competitive interactions among self-interested players. In this setting, two players perform actions that jointly manipulate the transitions of a system whose state evolves according to a random process [Sha53]. Assigned to each point of the state space is some reward (or cost). Each player is endowed with a set of controls which when exercised, alters the system transitions. Since each player is self-interested, the players strategically modify the system dynamics through their control inputs in pursuit of their individual objectives.

In the theory of stochastic games, there are various ways of describing the evolution of the game [FT91]. This thesis is concerned with stochastic games in which the evolution of the system is described by a stochastic differential equation. Such games are known as stochastic differential games [FT91; FS89]. In the following section, we review the main methods by which stochastic differential games are solved. We discuss the two equilibrium concepts that we shall study in the games investigated in this thesis namely, Nash and minimax (saddle point) equilibria.

Stochastic Differential Game Theory

Stochastic *differential* game theory is a formulation of stochastic game theory which generalises stochastic control theory to multiplayer systems that evolve according to a stochastic differential equation [Car16; FS89; FT91]. A stochastic differential game is therefore in general, a framework in which two or more players influence a diffusive process which describes the evolution of the system state. While the game is being played, each player observes the other player. The players' payoffs are derived from the values of the system state — in two-player zero-sum games, one of the players seeks to maximise some objective function whilst the other player seeks to minimise the same function.

As in classical game theory, there are numerous equilibrium concepts which describe a solution to the game [OR94]. Consequently, generalising optimal stochastic control to differential game theory introduces various possibilities for a solution definition. Indeed, in stochastic differential game theory there exists a number equilibrium concepts; examples include the Nash equilibrium, the Markov-perfect Nash equilibrium, the saddle point equilibrium and correlated equilibrium, all of which constitute solutions to (differential) games within various settings [MC16; SH69; Meh13]. The saddle point equilibrium represents an appropriate equilibrium concept for two-player zero-sum games. The Nash equilibrium which generalises the saddle point equilibrium to non zero-sum settings, is the most widely applied equilibrium concept and represents stable strategic configurations in general payoff (non zero-sum) non-cooperative settings [MCWG+95].

In stochastic differential games, the goal of each player is to find an optimal sequence of actions or control policy that modifies the system dynamics in a way that maximises an individual payoff criterion in the presence of other players that also affect the system dynamics.

The Dynamics: Canonical Description

Given some feasible set $S \subset \mathbb{R}^p$ $(p \in \mathbb{N})$, the uncontrolled passive state evolves according to a stochastic process $X : [0,T] \times \Omega \to S$, which is a jump-diffusion on $(\mathscr{C}([0,T];\mathbb{R}^p), (\mathscr{F}_{(0,s)_{s\in[0,T]}}, \mathscr{F}, \mathbb{P}_0)$ that is, the state process obeys the following stochastic differential equation $\forall s \in [0,T]$, $\forall (t_0, x_0) \in [0,T] \times S$:

$$dX_s^{t_0,x_0} = \mu(s, X_s^{t_0,x_0})ds + \sigma(s, X_s^{t_0,x_0})dB_s + \int \gamma(X_{s^-}^{t_0,x_0}, z)\tilde{N}(ds,dz),\ X_{t_0}^{t_0,x_0} := x_0,\ \mathbb{P}-\text{a.s.} \tag{1.1}$$

where $t_0 \in [0,T]$ is the start time and $x_0 \in S$ is the initial point of the process. The term B_s is an $m-$dimensional standard Brownian motion, $\tilde{N}(ds,dz) = N(ds,dz) - \nu(dz)ds$ is a compensated Poisson random measure, $N(ds,dz)$ is a jump measure and where $\nu(\cdot) := \mathbb{E}[N(1,\cdot)]$ is a Lévy measure. Both $\tilde{N}$ and B are supported by the filtered probability space and $\mathscr{F}$ is the filtration of the probability space $(\Omega, \mathbb{P}, \mathscr{F} = \{\mathscr{F}_s\}_{s\in[0,T]})$. We assume that N and B are independent.

We assume that the functions $\mu : [0,T] \times S \to S$, $\sigma : [0,T] \times S \to \mathbb{R}^{p\times m}$ and $\gamma : \mathbb{R}^p \times \mathbb{R}^l \to \mathbb{R}^{p\times l}$ are deterministic, measurable functions that are Lipschitz continuous and satisfy a (polynomial) growth condition so as to ensure the existence of (1.1) [IW14]. It can be shown that the jump-diffusion is well defined [SV07].

The generator of X (the uncontrolled process) acting on some function $\phi \in \mathscr{C}^{1,2}(\mathbb{R}^l, \mathbb{R}^p)$ is given by:

$$\mathscr{L}\phi(\cdot,x) = \sum_{i=1}^{p} \mu_i(x) \frac{\partial \phi}{\partial x_i}(\cdot,x) + \frac{1}{2}\sum_{i,j=1}^{p} (\sigma\sigma^T)_{ij}(x) \frac{\partial^2 \phi}{\partial x_i \partial x_j}(\cdot,x) + I\phi(\cdot,x) \tag{1.2}$$

where I is the integro-differential operator defined by:

$$I\phi(\cdot,x) := \sum_{j=1}^{l} \int_{\mathbb{R}^p} \{\phi(\cdot,x+\gamma^j(x,z_j)) - \phi(\cdot,x) - \nabla\phi(\cdot,x)\gamma^j(x,z_j)\}\nu_j(dz_j),\ \forall x \in \mathbb{R}^p. \tag{1.3}$$

The players affect the system dynamics through auxiliary inputs acting on the coefficients of the state process over a time horizon $[0,T]$ which may be infinite. In particular, for any $s \in [0,T]$ and $\omega \in \Omega$, the controls $u = u(s,\omega) \in \mathscr{U}$ exercised by player I and $v(s,\omega) \in \mathscr{V}$ exercised by player II are stochastic processes that modify both the drift and diffusion coefficients. The sets $\mathscr{U}$ and $\mathscr{V}$ define the set of admissible controls for player I player II respectively. In a stochastic differential game, the players use strategies to determine their control policies; the following is a description of a player strategy.

Strategies

In general, in a stochastic differential game, the player who performs an action first employs the use of a *strategy* — a map from the other player's set of controls to the player's own set of controls [BH13]. The use of strategies affords the acting player the ability to increase its rewards since their action is now a function of the other player's latter decisions. An important feature of the players' strategies is that they are non-anticipative — neither player may guess in advance, the future behaviour of other players given their current information. Therefore, one of the players chooses their control and the other player responds by selecting a control according to some strategy. We formalise this condition by constructing *Elliott-Kalton strategies* which were used in the viscosity solution approach to differential games in [FS89]. Elliott-Kalton strategies were introduced in [Rox69; EK72b; Var67; EK72a].

An Elliott-Kalton strategy (also known as Varaiya-Roxin-Elliott-Kalton or non-anticipative strategies) on $[0,T]$ for player I is a measurable map α such that $\alpha : [0,T] \times \mathscr{C}([0,T];S) \to \mathscr{U}$ and for any stopping time $\tau : \Omega \to [0,T]$ and any $v_1, v_2 \in \mathscr{V}$ with $v_1 \equiv v_2$ on $[t,\tau]$ we have that $\alpha(v_1) \equiv \alpha(v_2)$ on $[t,\tau]$. We define the player II Elliott-Kalton strategy $\beta : [0,T] \times \mathscr{C}([0,T];S) \to \mathscr{V}$ analogously. Following the notation in [Car09], we denote the set of all Elliott-Kalton strategies over the time horizon $[0,T]$ for player I (resp., player II) by $\mathscr{A}_{(0,T)}$ (resp., $\mathscr{B}_{(0,T)}$). It can be proven (Lemma 1.1 pg. 5 in [CR09]) that associated to the strategies $\alpha \in \mathscr{A}_{(0,T)}$ and $\beta \in \mathscr{B}_{(0,T)}$ are a unique pair of controls $(u,v) \in \mathscr{U} \times \mathscr{V}$. Therefore, we assume throughout that for all $\alpha \in \mathscr{A}_{(0,T)}$ and $\beta \in \mathscr{B}_{(0,T)}$ there exists a unique set of controls $(u,v) \in \mathscr{U} \times \mathscr{V}$ such that $(u,v) = (\alpha(\cdot, X_\cdot^{t_0,x_0,u,v}, v), \beta(\cdot, X_\cdot^{t_0,x_0,u,v}, u))$, $\forall (t_0,x_0) \in [0,T] \times S$ a.e. on $[0,T]$.

The intuition behind Elliott-Kalton strategies is as follows: suppose player I employs $u_1 \in \mathscr{U}$ and the system follows a path ω and that player II employs the strategy $\beta \in \mathscr{B}_{(0,T)}$ against the control u_1. If in fact player II cannot distinguish between the control u_1 and some other player I control $u_2 \in \mathscr{U}$ then controls u_1 and u_2 induce the same response from the player II strategy, that is to say $\beta(u_1) \equiv \beta(u_2)$. Elliott-Kalton strategies are designed to exclude the possibility of a player exploiting future information of their opponent's control modifications. We carry over this definition throughout the thesis in settings in which the players use discrete (i.e. impulse or stopping) controls.

For the case in which players use continuous controls, the controlled diffusion processes evolves according to an SDE which for all $s \in [0,T]$ is given by:

$$dX_s^{t_0,x_0,\alpha,\beta} = \hat{\mu}(s, X_s^{t_0,x_0,\alpha,\beta}, u, v)ds + \hat{\sigma}(s, X_s^{t_0,x_0,\alpha,\beta}, u, v)dB_s + \int \gamma(X_{s^-}^{t_0,x_0,\alpha,\beta}, z)\tilde{N}(ds,dz),$$
$$X_{t_0}^{t_0,x_0} := x_0,$$

where without loss of generality, we assume that $X_s^{t_0,x_0,\cdot} = x_0$ for any $s \leq t_0$ and where $\hat{\sigma} : [0,T] \times S \times \mathscr{U} \times \mathscr{V} \to \mathbb{R}^{p\times m}$ is the *controlled* diffusion coefficient and $\hat{\mu} : [0,T] \times S \times \mathscr{U} \times \mathscr{V} \to S$ is the *controlled* drift coefficient.

In zero-sum games, the players have opposing interests over some common payoff which represents a cost for player I and a reward for player II. Player I has a gain (or profit) function J which is also the player II cost function. Given two strategies $\alpha \in \mathscr{A}_{(0,T)}$ and $\beta \in \mathscr{B}_{(0,T)}$ (associated to which are the controls $u \in \mathscr{U}$ and $v \in \mathscr{V}$ respectively (c.f. Lemma 1.1 in [CR09]), the payoff function J which player I (resp., player II) seeks to maximise (resp., minimise) is given by the following expression $J[t_0,x_0;u,v] = \mathbb{E}\left[\int_{t_0}^{\tau_S} f(r, X_r^{t_0,x_0,u,v}, u, v)dr + G(X_{\tau_S}^{t_0,x_0u,v})1_{\{\tau_S<\infty\}}\right]$, for any $(t_0,x_0) \in [0,T] \times S$, $\forall u \in \mathscr{U}$, $\forall v \in \mathscr{V}$ where the function $f : [0,T] \times S \times \mathscr{U} \times \mathscr{V} \to \mathbb{R}$ is the *running cost function* and the function $G : S \to \mathbb{R}$ is the terminal payoff or *bequest function*.

Note that when either $\mathscr{U}$ or $\mathscr{V}$ is a singleton, the game is degenerate and collapses into an optimal stochastic control problem with one controller. Indeed, the corresponding single-player optimal stochastic control problem consists of maximising some given objective function J where the objective J is given by:

$$J[t_0,x_0;u] = \mathbb{E}\left[\int_{t_0}^{\tau_S} f(s, X_s^{t_0,x_0,u}, u)ds + G(X_{\tau_S}^{t_0,x_0,u})1_{\{\tau_S<\infty\}}\right], \quad \forall (t_0,x_0) \in [0,T] \times S, \, \forall u \in \mathscr{U}. \tag{1.4}$$

We now define the value functions of the game. As in [FS89], we define the value functions in terms of Elliot-Kalton strategies introduced in [EK72b]:

Definition 1.1

The upper and lower value functions associated to the game are given by the following expressions respectively:

$$V^-(t,x) = \inf_{\alpha\in\mathscr{A}_{(0,T)}} \sup_{\beta\in\mathscr{B}_{(0,T)}} J[t,x;\alpha,\beta]; \tag{1.5}$$

$$V^+(t,x) = \sup_{\beta\in\mathscr{B}_{(0,T)}} \inf_{\alpha\in\mathscr{A}_{(0,T)}} J[t,x;\alpha,\beta], \quad \forall (t,x)\in[0,T]\times S. \tag{1.6}$$

The upper value function V^- represents the least cost outcome that player I can guarantee in the game irrespective of the player II's choice; analogously the lower value function V^+ represents the payoff outcome that player I can guarantee in the game irrespective of the player I's choice. Whenever player I (player II) plays a strategy that guarantees V^- (V^+) it is said that player I (player II) plays a *minimax strategy*.[1]

The intuition of a value of the game is that the outcome of two individual players playing their minimax strategies (derived from individual decision making) coincides. To develop this further, suppose that player I is now required to publicly announce their control $u\in\mathscr{U}$ *in advance* of player II choosing a strategy. Such conditions seem to provide an advantage to player II since they provide player II with foreknowledge over player I's choice. Consequently, player II could choose a strategy $\beta(u)\in\mathscr{B}$ that is a best-response to u in which case player I would incur a cost of V^- (on average). However, this apparent advantage is fictive since player I can commit to a minimax strategy, moreover, when each player's opponent plays their minimax strategy the player cannot profitably deviate from its own strategy. Consequently, applying minimax strategies results in a Nash equilibrium.

In light of the above, the central question in stochastic game theory is whether a value can be unambiguously assigned to the game. Indeed, it is not obvious from the outset that the upper value function and lower value function coincide. We say that the value of the game exists if we can commute the supremum and infimum operators in Definition 1.1 so that the upper value function and lower value functions coincide and we can deduce the existence of a function V with $V\equiv V^- = V^+$ which we refer to as the value of the game.

Lemma 1.2

For any strategies $\alpha\in\mathscr{A}_{(0,T)}$ and $\beta\in\mathscr{B}_{(0,T)}$ we have $-V^-(t,x) = \inf_{\beta\in\mathscr{B}_{(0,T)}}\sup_{\alpha\in\mathscr{A}_{(0,T)}}(-J[t,x;\alpha,\beta])$, $\forall(t,x)\in[0,T]\times S$.

Lemma 1.2 therefore implies that $(-V^-)$ is an upper value game with running payoff as $-f$ and terminal payoff as $-G$. Therefore, the results for V^+ are equivalent results for $-V^-$ and the reverse is also true for V^-. The lemma enables us to focus singularly on either of the value functions since results for the other value function can be derived in an analogous manner.

[1]The term minimax strategy is used for both maximiser's and minimiser's strategies although the term *maxminiser* is also used [OR94].

There is a considerable literature on games in which two players use continuous controls[2] to modify some system dynamics to maximise some performance criterion (e.g. [Car10; FS89; Var67]). Deterministic differential games, that is dynamic games in which the system evolves according a deterministic process were introduced by Isaacs [Isa69]. For this setting, it was shown in [EK72a; ES84] that zero-sum games admit a saddle point or value — a quantity that represents the payoff that each player is guaranteed when both execute their equilibrium strategies. Moreover, the value of the game is completely characterised by a PDE known as the Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation when the solution is interpreted in a *viscosity sense* [ES84]. A viscosity solution is a *weak solution* concept for PDEs for which the solution is no longer required to be differentiable over the domain in which the PDE is defined.

Following this, in [FS89] stochastic differential games with continuous controls and zero-sum payoffs were considered. In this setting, two players exercise continuous controls to modify some diffusive process X over the horizon of the game $T \in \mathbb{R}_{>0} \cup \{\infty\}$. In [FS89] the existence of a value for the game in which player I uses the control $u \in \mathscr{U}$ for whom the objective is to maximise some objective J and a second player, player II who using the control $v \in \mathscr{V}$, seeks to minimise the same quantity is established where $\mathscr{U}$ and $\mathscr{V}$ are admissible control sets for player I and player II respectively.

Building on the achievements of the deterministic cases, the study of stochastic differential game theory has produced significant results and has been applied in various settings within finance and economics [Bro00; PS04]. Stochastic differential game theory underpins theoretical models used to prescribe optimal portfolio strategies in a Black-Scholes market[3] (e.g. [MØ08; Bro00]), descriptions of pursuit-evasion games (e.g. [PY81]) and investment games in competitive advertising (e.g. [Eri95; Jør82; PS04]) amongst others. The study of differential game theory was generalised to games of non-antagonistic interests (non zero-sum payoffs) in [Kon76; Kle93] and in the stochastic case, [BCR04]. In [BCR04; Kon76; Kle93] the relevant equilibrium concept which characterises optimal play by all players is a Nash equilibrium — a strategic configuration in which no player can do better by unilaterally modifying their current strategy. Therefore, the Nash equilibrium characterisation of optimality describes a stable point in strategies in which each player executes a *best-response strategy* against their opponents.

Stochastic Differential Games Involving Impulse Controls

Having reviewed some of the key results in stochastic differential game theory with continuous controls, we now describe the stochastic differential game theory in which the players use impulse controls to modify the state process which is the central framework of the interactions with which the thesis is concerned. Unlike in the case of continuous control where modifications to the system dynamics occur through the state coefficients; in impulse control models the system dynamics are

[2] More accurately, the control process is required to be right continuous with left limits or càdlàg.
[3] A Black-Scholes market consists of a single risky asset e.g. stock and a risk-free asset e.g. a bond.

altered by direct modifications of the state value. Therefore impulse control takes the form of a countable number of discrete interventions or *impulses* that alter the value of the state process by prescribed magnitudes. The problem of finding an optimal policy is augmented since now both an optimal sequence of times to apply the control policy in addition to the optimal control magnitudes must be determined.

The impulse control problem for each player consists of maximising a given payoff function which is dependent on a controlled stochastic process X until some exit time $\tau_S : \Omega \to [0,T]$ over a set of admissible impulse controls. As before, the game is defined over some time horizon $[0,T] \subseteq \mathbb{R}$ which may be of infinite length. A formal description of the setting is as follows: let $S \subset \mathbb{R}^p$ be a given set within which the state process takes its values. The state process $X : [0,T] \times \Omega \to S$ is influenced by a pair of impulse controls $u \in U$ exercised by player I and $v \in V$ exercised by player II respectively which are each stochastic processes that modify the state process directly. The sets U and V define the set of admissible controls for player I and player II respectively. The player I control is given by $u(s) = \sum_{j\geq 1} \xi_j \cdot 1_{\{\tau_j \leq T\}}(s)$ where $\xi_1, \xi_2, \ldots \in \mathscr{Z} \subset S$ are impulses that are executed at times $\{\tau_i\}_{i\in\mathbb{N}}$ where $0 \leq t_0 \leq \tau_1 < \tau_2 < \ldots$. Analogously, the player II control is given by $v(s) = \sum_{m\geq 1} \eta_m \cdot 1_{\{\rho_m \leq T\}}(s)$ where $\eta_1, \eta_2, \ldots \in \mathscr{Z} \subset S$ are impulses that are executed at times $\{\rho_m\}_{m\in\mathbb{N}}$ where $0 \leq t_0 \leq \rho_1 < \rho_2 < \ldots$. We therefore interpret τ_n (resp., ρ_n) as the n^{th} time at which player I (resp., player II) modifies the system dynamics with an impulse intervention ξ_n (resp., η_n).

The result of an impulse intervention executed at a time τ_m^- is to shift the state by a prescribed magnitude. In particular, if an impulse $\zeta \in \mathscr{Z}$ determined by some admissible policy w is applied at a time $\tau : \Omega \to [0,T]$ when the state is $x' = X^{t_0,x_0,\cdot}(\tau^-)$, then the state immediately jumps from $x' = X^{t_0,x_0,\cdot}(\tau^-)$ to $X^{t_0,x_0,\cdot}(\tau) = \Gamma(x',\zeta)$ where $\Gamma : S \times \mathscr{Z} \to S$ is called the impulse response function. We assume that the impulses $\xi_j, \eta_m \in \mathscr{Z}$ are $\mathscr{F}-$measurable for all $m, j \in \mathbb{N}$ and that the times $\{\tau_i\}_{i\in\mathbb{N}}$ and $\{\rho_m\}_{m\in\mathbb{N}}$ are $\mathscr{F}$-measurable stopping times. For notational convenience we use $u = [\tau_j, \xi_j]_{j\geq 1}$ to denote the player I control policy $u = \sum_{j\geq 1} \xi_j \cdot 1_{\{\tau_j \leq T\}}(s)$ and analogously, we use $v = [\rho_m, \eta_m]_{m\geq 1}$ to denote the player II control policy $v = \sum_{m\geq 1} \eta_m \cdot 1_{\{\rho_m \leq T\}}(s)$. For the controls $u \in U$ and $u' \in U$ ($v \in V$ and $v' \in V$), we interpret the notion $u \equiv u'$ ($v \equiv v'$) on $[0,T]$ iff $\mathbb{P}(u = u'$ a.e. on $[0,T]) = 1$ ($\mathbb{P}(v = v'$ a.e. on $[0,T]) = 1$).

In a two player (zero-sum) stochastic differential game, the objective function takes the following form:

$$J[t_0,x_0;u,v] = \mathbb{E}\left[\int_{t_0}^{\tau_S} f(s, X_s^{t_0,x_0,u,v})ds + \sum_{m\geq 1} c(\tau_m,\xi_m)\cdot 1_{\{\tau_m \leq \tau_S\}} - \sum_{l\geq 1} \chi(\rho_l,\eta_l)\cdot 1_{\{\rho_l \leq \tau_S\}} + G(X_{\tau_S}^{t_0,x_0,u,v})1_{\{\tau_S<\infty\}}\right] \quad \forall (t_0,x_0) \in [0,T]\times\mathbb{R}^p, \tag{1.7}$$

where as before, $f : [0,T] \times S \to \mathbb{R}$ is the running cost function and $G : S \to \mathbb{R}$ is the terminal cost function. The functions $c : [0,T] \times \mathscr{Z}$ and $\chi : [0,T] \times \mathscr{Z}$ are the player I intervention cost function and player II intervention cost function respectively so that an intervention $\xi \in \mathscr{Z}$ ($\eta \in \mathscr{Z}$) executed at a time $\tau \in \mathscr{T}$ ($\rho \in \mathscr{T}$) incurs a cost of $c(\tau,\xi)$ ($\chi(\rho,\eta)$) to player I (player II). The function $1_{\{\tau \le \tau_S\}}$ is the Heaviside function which imposes the condition that the controller incurs zero cost for any intervention executed after τ_S.

For zero-sum stochastic differential games in which both players use impulse control, the value of the game V is given by the following:

$$V(t,x) = \sup_{\alpha \in \mathscr{A}} \inf_{v \in \mathscr{V}} J[t,x;\alpha(v),v] = \inf_{\beta \in \mathscr{B}} \sup_{u \in \mathscr{U}} J[t,x;u,\beta(u)], \quad \forall (t,x) \in [0,T] \times \mathbb{R}^p,$$

where $\mathscr{A}$ and $\mathscr{B}$ are given strategy sets for player I and player II respectively.

Lastly, we say that a player I (player II) impulse control policy is admissible if the following holds:

$$\lim_{j \to \infty} \tau_j = \infty \qquad (\lim_{n \to \infty} \rho_n = \infty). \tag{1.8}$$

Note that condition (1.8) guarantees that we are in fact using impulse controls as condition (1.8) prohibits the number of interventions from exploding over some subinterval of the horizon of the problem, while allowing us to still consider infinite horizon problems.

The following definitions are useful within the analyses of impulse control theory:

Definition 1.3

The set of regular impulse controls U consists of all previous processes $u : \Omega \times [0,T] \to U$ with respect to the filtration $\{\mathscr{F}_{(0,s)}\}_{0 \le s \le T}$ for some separable metric space U.

The following object allows us to describe the number of impulse interventions executed over some interval:

Definition 1.4

Denote by $\mathscr{T}_{(t,\tau')}$ the set of all $\mathscr{F}-$measurable stopping times in the interval $[t,\tau']$, where τ' is some stopping time such that $\tau' \le T$. If $\tau' = T$ then we denote by $\mathscr{T} \equiv \mathscr{T}_{(0,T)}$. Let $u = [\tau_j, \xi_j]_{j \in \mathbb{N}}$ be a control policy where $\{\tau_j\}_{j \in \mathbb{N}}$ and $\{\xi_j\}_{j \in \mathbb{N}}$ are $\mathscr{F}_{\tau_j}-$ measurable stopping times and interventions respectively, then we denote by $\mu_{[t,\tau]}(u)$ the number of impulses the controller executes within the interval $[t,\tau]$ under the control policy u for some $\tau \in \mathscr{T}$.

We therefore see that for the player I control policy u (player II control policy v) to be an admissible policy for a game with time horizon $T \in \mathbb{R}_{>0} \cup \{\infty\}$, we must have that given $\mu_{[0,T]}(u) = \sum_{j \ge 1} 1_{\{\tau_j \le T\}}$ ($\mu_{[0,T]}(v) = \sum_{m \ge 1} 1_{\{\rho_m \le T\}}$) then either $\mathbb{E}[\mu_{[0,T]}(u)] < \infty$ ($\mathbb{E}[\mu_{[0,T]}(v)] < \infty$) so that the number of impulses is finite $\mathbb{P}-$a.s. or that $\mu_{[0,T]}(u) = \infty \implies \lim_{j \to \infty} \tau_j = \infty$ ($\mu_{[0,T]}(v) = \infty \implies \lim_{m \to \infty} \rho_m = \infty$).

Definition 1.5

Let $u(s) = \sum_{j\geq 1} \xi_j \cdot 1_{\{\tau_j \leq T\}}(s) \in \mathscr{U}$ be a player I impulse control defined over $[0,T]$, further suppose that $\tau : \Omega \to [0,T]$ and $\tau' : \Omega \to [0,T]$ are two $\mathscr{F}-$measurable stopping times with $\tau \geq s > \tau'$, then we define the restriction $u_{[\tau',\tau]} \in \mathscr{U}$ of the impulse control $u(s)$ to be $u(s) = \sum_{j\geq 1} \xi_{\mu_{]t_0,\tau)}(u)+j} \cdot 1_{\{\tau_{\mu_{]t_0,\tau)}(u)+j} \geq s \geq \tau'\}}(s)$.

We define the restriction for the player II control $v(s) = \sum_{m\geq 1} \eta_m \cdot 1_{\{\rho_m \leq T\}}(s)$; $v \in \mathscr{V}$ over $[0,T]$ analogously that is, given two $\mathscr{F}-$measurable stopping times $\rho : \Omega \to [0,T]$ and $\rho' : \Omega \to [0,T]$ such that $\rho \geq s > \rho'$, we define the restriction $v_{[\rho,\rho']} \in \mathscr{V}$ of the impulse control $v(s)$ to be $v(s) = \sum_{m\geq 1} \eta_{v_{t,\rho}(u)+m} \cdot 1_{\{\rho_{v_{t,\rho}(u)+m} \geq s \geq \rho'\}}$.

Impulse control models serve as an important descriptive formulation of control theory for environments in which continuous and infinitely fast adjustments to positions are not appropriate modelling assumptions [ØS07; BL84]. Impulse control problems provide greater modelling accuracy in financial and economics environments where agents face some fixed minimal costs when adjusting their positions [Kor99]. Optimal portfolio-theoretic problems in which the agent faces adjustment costs and investment problems with fixed transaction costs e.g. [EH88; ØS02] are important examples of such environments. In [EH88], using a single player impulse control model, the question of how an investor may optimally rebalance their portfolio in order to maximise the total utility of consumption is explored using a *verification theorem*. Verification theorems establish a direct link to a characterisation of the value function in terms of a PDE. Single-controller impulse control problems have been solved using *viscosity-theoretic* approaches[4] [TY93; Len89; KMPZ+10; Azi17; Sey09]). We refer the reader to [BL84] as a general reference to impulse control theory and to [VMP07; PS10] for articles on applications. Additionally, matters relating to the application of impulse control models have been surveyed extensively in [Kor99].

The study of impulse control has been extended to differential game-theoretic settings. Deterministic versions of this game were first studied by [Yon94; TY93] — in the model presented in [Yon94] however, impulse controls are restricted to use by one player and the other uses continuous control. Similarly, in [Zha11] stochastic differential games in which one player uses impulse control and the other uses continuous controls were studied. Using a verification argument, the conditions under which the value of game is a solution to a HJBI equation is also shown in [Zha11].

It was shown in [Cos13] that the above game admits a value and the value is a unique viscosity solution to a HJBI equation which is associated to the following double obstacle problem:

$$\max\left\{\min\left[-\frac{\partial V}{\partial s} - \mathscr{L}V - f, \mathscr{M}_{\inf}V - V\right], V - \mathscr{M}_{\sup}V\right\} = 0$$
$$V(\cdot,x) = G(x), \quad \forall x \in S \subset \mathbb{R}^p, \tag{1.9}$$

[4] Viscosity solutions establish a generalisation of classical solutions to PDEs. We defer the discussion on this matter to Chapter 3.

where S is the state space, $\mathscr{L}$ is the local stochastic generator of the diffusion process X and $\mathscr{M}_{\inf}$ and $\mathscr{M}_{\sup}$ are the non-local intervention operators for the player II and player I respectively.

The size of the literature concerning differential games with impulse control remains limited and at present, all analyses of these games have been conducted for diffusion processes without jumps.

Stochastic Control Theory with Discretionary Stopping

Of increasing interest in optimal stochastic control theory are problems in which the controller can select a time to terminate the process in addition to exercising controls to influence the system dynamics. Such problems are known as stochastic control problems with discretionary stopping. The problem involves finding a pair (u,τ) that consists of an admissible control $u \in \mathscr{U}$ where $\mathscr{U}$ is an admissible control set and, a stopping time $\tau \in \mathscr{T}$ which minimise an objective function of the form $J[t_0,x_0;u] = \mathbb{E}\left[\int_{t_0}^{\tau} f(s,X_s^{t_0,x_0,u},u)ds + G(X_\tau^{t_0,x_0,u}) \cdot 1_{\{\tau<\infty\}}\right]$ where, as before, $f:[0,T]\times\mathbb{R}^p \to \mathbb{R}$ and $G:\mathbb{R}^p \to \mathbb{R}$ are a running cost function and terminal cost function respectively.

Optimal stochastic control with discretionary stopping problems can be used to study a wide variety problems within the contexts of finance and economics. One example is the lifetime ruin problem in which an investor who operates in a Black-Scholes market seeks to maximise some utility criterion whilst seeking an optimal time to exit the market according to some risk criterion [BHY11; BZ15a]. Recent interest in optimal control models with discretionary stopping has generated a significant body of literature; particular focus has been placed on models in which a single controller uses continuous controls to modify the system dynamics. Discretionary stopping and stochastic optimal control problems in which the controller exercises modifications through the drift component of the state process (using continuous controls) have been studied by [KO02; KS99; KW00]. Another version of these problems which has attracted significant interest is problems in which the controller acts to modify the system dynamics by finite variations of the state process — such problems have been studied by [DZ94; KOWZ00].

The problem of stochastic optimal control problems with discretionary stopping has also been analysed within game-theoretic settings (see for example [BY11; NZ+15; BHØT13]). Here, the task of controlling the system dynamics and stopping the process is divided between two players. This formulation of the problem generalises the results of single-player models to scenarios that involve competitive interactions among multiple players. Therefore, the game-theoretic approach has the advantage that now the task of controlling the system and terminating the process can be delegated to two players with conflicting objectives.

Stochastic differential games of control and stopping were introduced by [MS96]. Notable papers include [KS01] which studies a game in which the underlying system dynamics are given by a one-dimensional diffusion within a given interval in $\mathbb{R}$ and [KZ+08; BH13]. In [BH13], a multidimensional state space model is studied. Game-theoretic approaches to stochastic optimal control problems with discretionary stopping have been used to analyse the lifetime ruin problem

[BY11; NZ+15; BHØT13]. In [BY11], it is shown that the single investor portfolio problem in a Black-Scholes market in which an investor seeks to both maximise a running reward and minimise the probability of lifetime bankruptcy exhibits a duality with controller-stopper games. Indeed, in [BY11] it is shown that the value function of the investment problem is the convex dual of the value of a controller-stopper game. Similarly, in [BHØT13] an investor portfolio problem with discretionary stopping is analysed by studying a stochastic differential game of control and stopping and proving an equivalence.

In [BHØT13], the value for a game in which the stopper seeks to minimise a convex risk measure defined over a common (zero-sum) payoff objective is characterised in terms of a Hamilton-Jacobi-Bellman Variational Inequality (HJBVI) to which it is proven that the value is a viscosity solution. The inclusion of a convex risk measure, as outlined in [ADEH99; FS02], serves as a means by which risk attitudes of the investor are incorporated into the model. Furthermore, the zero-sum payoff structure of the model implies that the strategies are appropriate for computing optimal control policies in worst-case scenario analyses.

In both the single controller and the game-theoretic approaches to optimal stochastic control with discretionary stopping problems, the focus has been placed on models in which the controller uses continuous controls. However, in [Zer03] a single-controller problem is analysed which the controller's action is to decide an entry and exit time. Therefore the model in [Zer03] can be viewed as regime switching model with switching costs.

Part I

Strategic Interactions with Impulse Control: Two-Player Games

Chapter 2

Stochastic Differential Games of Control and Stopping involving Impulse Controls

In this chapter, we introduce a stochastic differential game of control and stopping in which one of the players modifies the jump-diffusion dynamics using impulse controls and an adversary chooses a stopping time to end the game. We perform an analysis of the game and derive a PDE characterisation of the value of the game. We then apply the results to solve an investment problem for optimal liquidity control with risk-minimisation.

The contribution of this chapter is encompassed in the following paper:

David Mguni, ***"Optimal Capital Injections with the Risk of Ruin: A Stochastic Differential Game of Impulse Control and Stopping Approach"*** **(2018)** [Mgu18c].

Consider an investor that can choose a set of times to inject capital into a firm which increases the firm's market capabilities. Each time the investor performs a capital injection, the investor faces a transaction cost. The investor may exit the market by selling all holdings in the firm, moreover at any point, the firm may go into ruin at which point the investor faces loss of their investments. If the investor seeks to maximise their reward, how should the investor perform their capital injections and at what time should the investor exit the market?

In order to address this question, it is necessary to find both an optimal control process of the investor's capital injections and, an optimal time to exit the market in advance of possible firm ruin. The problem therefore combines two distinct objectives: the first problem, known as an optimal liquidity control addresses the problem of finding an optimal sequence of capital injections. The second objective, known as a lifetime ruin problem is concerned with finding an optimal exit criterion in order to minimise some notion of risk. The combined problem, known as the optimal liquidity control and lifetime ruin problem has been intensely studied within theoretical finance owing to its practical importance [BY11]. Currently however, models that address this problem use continuous controls to model the investment strategies [BHY11; BHØT13; BZ15a]. This limits the scope of application of existing models as transaction costs are prohibited within continuous control descriptions. Moreover, despite remaining unaddressed within the present context, in general the presence of transaction costs produces a vast change in the model prescription for optimal investor behaviour [Kor99].

In this chapter, we tackle the optimal liquidity control and lifetime ruin problem when the investor faces transaction costs. To perform this task, we develop the corresponding mathematical structure required to solve the investment problem. Since in this problem the individual investor objectives are related through interdependencies, a strategy exercised in pursuit of maximising one objective alters the optimality criteria of the other termination objective. Consequently, the problem admits a two-player game in which two players each seek to maximise one of the investor's objectives. In particular, the problem admits a representation as a stochastic differential game of control and stopping. Since we are concerned with settings in which the investor's capital injections are subject to transaction costs, it is necessary to model the optimal investment behaviour using a game in which the controller now faces control costs that are bounded from beneath. This breaks from existing models which assume continuous investment and necessitates solving a new stochastic differential game of control and stopping with minimally bounded control costs, the development and formal analysis of which is the subject of matter of the chapter. In this setting, the controller modifies the system dynamics using impulse controls and an adversary chooses when to stop the game (termination time). In contrast to existing controller and stopper games such as [BHY11; BHØT13; BH13; BZ15a], the game we study involves a controller that uses impulse controls necessitating a markedly different analysis of the game.

Therefore, in this chapter we perform the first study of a stochastic differential game of impulse control and stopping. As part of our analysis, we characterise the optimal strategies in both a zero-sum and a non zero-sum setting. The stochastic differential game introduced in this chapter gives rise to a general mathematical framework for analysing financial investment problems in which both an optimal market exit criterion and an optimal investment strategy must be determined in markets with transaction costs. More generally, the stochastic differential game developed in this chapter is a suitable framework for studying problems in which a controller faces fixed minimal costs when exercising controls while facing the possibility that the process may be terminated by another interested agent.

After performing a comprehensive analytic treatment of the game, we return to the investment problem and use the obtained results to derive a solution to the investment problem. We demonstrate that the optimal investment strategy and optimal exit criterion can be recovered from the equilibrium controls of the stochastic differential game of control and stopping.

Contributions

A summary of the contributions of this chapter is as follows:

- We introduce a new stochastic differential game of control and stopping in which the method of control is impulse controls and for which the dynamics are described by a jump-diffusion process. We perform a detailed investigation of the game beginning with a verification theorem for the zero-sum case (Theorem 2.7). This leads to a full characterisation of the value function of the game and a description of both the minimax equilibrium control and stopping criterion. This extends the analysis of a stochastic differential game of control and stopping in [BHØT13] to accommodate fixed minimal adjustment costs.

- Second, we extend our analysis of the zero-sum case to a stochastic differential game of impulse control and stopping with a non zero-sum payoff structure. In analogy with the zero-sum case, we prove a verification theorem (Theorem 2.12) and characterise the Nash equilibrium control and the corresponding equilibrium stopping criterion for the game.

- Last, we apply the theoretical analysis conducted in the chapter to investigate the optimal liquidity and lifetime ruin problem when the investor faces transaction costs. This results in a new model that accommodates both transaction costs in addition to capturing the effect of exogenous market shocks. The model therefore extends the control settings in [BHY11; BHØT13; BZ15a] to a model that is capable of handling the presence of transaction costs.

The general results of the chapter are accompanied by worked examples to elucidate the workings of the theory in context of investment problems within finance.

The stochastic differential game introduced in this chapter is as an extension of the game of optimal stochastic control and stopping introduced in [MS96] and studied in [KS01; BY11; NZ+15; BHØT13; KZ+08; BH13] in which now, the controller affects the state process using impulse controls instead of continuous controls. Since the analysis of this chapter is performed in a setting that evolves according to jump-diffusion process, the framework we propose allows for a more general set of dynamics than those in [KS01; BY11; NZ+15; KZ+08; BH13] for which the game dynamics are described by Itō diffusions with continuous sample paths. In particular, the jump-diffusion dynamics in the game we study provides additional modelling capabilities for describing systems within finance which are subjected to exogenous shocks [Cha04].

The game in this chapter can be viewed as a particular case of the game we study in Chapter 4 which involves two players that use impulse controls. Unlike the game of Chapter 4, in the current setting, it is necessary to provide a characterisation of the optimal stopping time and establish the existence of a value of the game.

Organisation

The chapter is arranged as follows: first, in Section 2.1 we give a description of the optimal liquidity with lifetime ruin problem that now incorporates transaction costs. In Section 2.2, we give a full statement of the main results and discuss the relevance to the literature concerning stochastic differential games. In Section 2.3, we initiate the study of the game of impulse control and stopping and develop the main set of arguments to characterise the minimax equilibrium of the game. In Section 2.6, we extend the analysis to non zero-sum games. In Section 2.7, we provide some example calculations of the theorems developed in the chapter. Lastly, in Section 2.7.1, we revisit the optimal liquidity control and lifetime ruin problem.

2.1 Risk-Minimising Optimal Liquidity Control with Lifetime Ruin

Overview

The problem of how an investor should inject capital to raise a firm's liquidity process in order to maximise their terminal reward is known as *de Finetti's dividend problem*. De Finetti's dividend problem has been studied intensely within theoretical finance and actuarial science [HJMF18; Loe08; Fin57; KP07]. In this setting, the investor maximises the expected cumulative dividends (or injections) they pay out which their wealth process can tolerate. The central task facing the investor is to identify the optimal sequence of capital injections and the times at which the injections ought to be performed.

The problem of when capital injections should be performed (or dividends should be paid by the firm) is an area of active research within theoretical actuarial science to which a great deal of attention has been focused. In current models within the literature, the optimal capital injections and dividends problem is represented as a single-player impulse control problem in which the controller seeks the optimal sequence of capital injections. In [Kor99], a model in which the firm can seek to raise capital (by issuing new equity) to be injected so as to allow the firm to remain solvent is considered. We refer the reader to [Kor99; Zer03] and references therein for exhaustive discussions.

The lifetime ruin problem is concerned with characterising an exit criterion that allows the investor to minimise their risk of losses due to firm ruin. The problem can be cast as an optimal stopping problem [MR00; MMY06]. In order to include optimal investment behaviour in addition to finding an optimal exit criterion, lifetime ruin problems have recently been extended to include the analysis of optimal investment behaviour within the setting of a risk-minimisation problem [BY11; BZ15a]. Such models however consider only continuous controls to model the investment behaviours which prohibit the inclusion of transaction costs. The combined problem, which we tackle is known as the optimal liquidity and lifetime ruin problem.

We now give a detailed description of the optimal liquidity control and lifetime ruin problem in which the investor faces transaction costs that we address in this chapter.

Problem Description

An investor injects capital into a firm to increase available liquidity for the firm to pursue its market objectives. The investor seeks to maximise their terminal returns by performing the maximal sequence of capital injections at selected times that their wealth process can tolerate. However, the investor also seeks an optimal time to exit the market by selling all firm holdings before firm ruin. As in classical ruin theory, we take firm ruin to mean the first time at which the firm's liquidity process hits zero. In what follows, we begin our description of the problem by explaining how a description of the firm liquidity process is derived.

The Firm Liquidity Process

The firm's liquidity at time $s \leq T$ is described by a stochastic process X over a time horizon T which may be infinite. Suppose that when there are no capital injections, the firm's liquidity process evolves according to a *geometric Lévy process*[1] given by:

$$dX_s^{t_0,x_0} = erX_s^{t_0,x_0}ds + \sigma_f X_s^{t_0,x_0}dB_f(s) + S_f(X_s^{t_0,x_0},s), \qquad X_{t_0}^{t_0,x_0} := x_0; \qquad \mathbb{P}-\text{a.s.}, \tag{2.1}$$

where $t_0 \in [0,T]$ and $x_0 \in \mathbb{R}_{>0}$ are given parameters that describe the start time of the problem and the firm's initial surplus respectively. Without loss of generality we will assume that $X_s^{t_0,x_0,\cdot} = x_0$ for any $s \leq t_0$. The constant $e \in \mathbb{R}_{>0}$ describes the firm's rate of expenditure and $r \in]0,1[$ is the firm's return on capital. The term S_f captures the exogenous shocks in the firm's liquidity process and is given by $S_f(X_s^{t_0,x_0},s) := \int \gamma_f(s,z)X_{s^-}^{t_0,x_0}\tilde{N}_f(ds,dz)$ where $\tilde{N}_f$ is a compensated Poisson random measure and B_f is a $1-$dimensional standard Brownian motion. The constant $\sigma_f > 0$ and the function $\gamma_f : [0,T] \times \mathbb{R} \to \mathbb{R}$ describe the volatility and the jump-amplitude of the firm's liquidity process respectively.

Let us suppose that each time $\tau \leq T$ the investor performs a capital injection, the investor incurs a cost which is discounted by a factor $\delta \in]0,1]$ and is minimally bounded by a transaction cost $\kappa_I \in \mathbb{R}_{>0}$. The cost incurred by the investor for an injection of size $z \in \mathbb{R}_{>0}$ is therefore given by $c(\tau,z) := \exp^{-\delta\tau}(\kappa_I + (1+\lambda)z)$ where the parameter $\lambda \in \mathbb{R}_{>0}$ is the *proportional cost*. Since performing continuous actions would result in immediate bankruptcy, the investor's capital injections must be performed over a discrete sequence of times. The investor therefore performs a sequence of capital injections $\{z_k\}_{k\in\mathbb{N}}$ over the horizon of the problem which are performed over a sequence of *intervention times* $\{\tau_k\}_{k\in\mathbb{N}}$. We denote the investor's control by the double sequence $(\tau,Z) \equiv \sum_{j\in\mathbb{N}} z_j \cdot 1_{\{t_0<\tau_j\leq T\}} \in \Phi$ where $\mathscr{Z}$ is a feasible set of investor capital injections and $\mathscr{T}$ is the set of ($\mathscr{F}-$measurable) intervention times[2] $\Phi \subseteq \mathscr{T} \times \mathscr{Z}$.

Let us denote by $T_s^{(\tau,Z)} := \sum_{m\geq 1} z_m \cdot 1_{\{t_0<\tau_m\leq s\}}$ the investor's capital injections process at time

[1] Geometric Lévy processes are widely used to model financial processes due to their close empirical fit with market data [Cha04].

[2] More specifically, the set $\mathscr{T}$ is the set of stopping times w.r.t. $\{\mathscr{F}_t\}_{t\in[0,T]}$.

$s \leq T$. Since the investor's capital injections are transferred to the firm, the firm's liquidity with capital injections at time $0 < s \leq T$ is given by the following expression:

$$X_s^{t_0,x_0,(\tau,Z)} = x_0 + \int_{t_0}^{s\wedge\rho} erX_r^{t_0,x_0,(\tau,Z)}dr + \int_{t_0}^{s\wedge\rho} \sigma_f X_r^{t_0,x_0,(\tau,Z)}dB_f(r)$$
$$+ T_{s\wedge\rho}^{(\tau,Z)} + \int\int_{t_0}^{s\wedge\rho} \gamma_f(r,z)X_{r-}^{t_0,x_0,(\tau,Z)}\tilde{N}_f(dr,dz), \ \mathbb{P}-\text{a.s.}, \tag{2.2}$$

which indicates that the firm's liquidity process is raised by capital injections performed by the investor where $\rho \in \mathscr{T}$ is a stopping time which will be defined shortly. Although the cost of performing capital injections is deducted from the investor's wealth however, the investor receives a return on capital through some running stream and some terminal reward after liquidating all holdings in the firm. With this, we now discuss the investor's wealth process.

The Investor's Wealth Process

The investor's wealth at time $s \leq T$ is described by a stochastic process Y_s. Denote by $\pi \in [0,1]$, the portion of the investor's wealth invested in risky assets and by $\bar{T}_s^{(\tau,Z)} := \sum_{m\geq 1} \exp^{-\delta\tau_m}[(1+\lambda)z_m + \kappa_I]1_{\{t_0<\tau_m\leq\rho\wedge s\}}$, the total deductions from the investor's wealth process due to the injections. At a time $s \in [0,T]$ the process Y_s is expressed by the following:

$$Y_s^{t_0,x_0,(\tau,Z)} = y_0 + \int_{t_0}^{s\wedge\rho} \Gamma Y_r^{t_0,x_0,(\tau,Z)}dr - \bar{T}_{s\wedge\rho}^{(\tau,Z)} + \int_{t_0}^{s\wedge\rho} \pi\sigma_I Y_r^{t_0,x_0,(\tau,Z)}dB_I(r)$$
$$+ \int\int_{t_0}^{s\wedge\rho} \pi\gamma_I(r,z)Y_{r-}^{t_0,x_0,(\tau,Z)}\tilde{N}_I(dr,dz), \ Y_{t_0}^{t_0,y_0,(\tau,Z)} \equiv y_0, \quad \mathbb{P}-\text{a.s.}, \tag{2.3}$$

where $y_0 \in \mathbb{R}_{>0}$ is the investor's initial wealth where we assume that $Y_s^{t_0,y_0,\cdot} = y_0$ for any $s \leq t_0$. We define the constant $\Gamma := (1-\pi)r_0 + \pi\mu_R$ where $r_0, \mu_R \in \mathbb{R}$ are constants that describe the interest rate and the return on the risky assets respectively. The term $\tilde{N}_I$ is a compensated Poisson random measure and B_I is a standard Brownian motion. The constant $\sigma_I > 0$ and the function $\gamma_I : [0,T] \times \mathbb{R} \to \mathbb{R}$ describe the volatility and the jump-amplitude of the investor's wealth process respectively.

We now explain how the description of the problem facing the investor is derived. First, we describe the elements required to construct the notion of risk of ruin. There are a number of ways to define measures of risk within the financial and actuarial literature; in general, a risk measure describes the proximity of a controller or investor's position to some acceptance region [ADEH99]. In the current setting, in order to take account of risk, it is necessary to derive the optimal stopping criterion through appealing to a measure of risk. One such set of measures of risk is given by *convex risk measures*. Convex risk measures are prevalent within theoretical finance owing to their fulfilment of certain axioms of risk within the context of financial decision-making (see Section A of the Appendix). As in [BHØT13] and in the sense given by [ADEH99; FS02], let θ be a convex risk measure acting on a stochastic process X, then we can write the risk measure associated to the problem as $\theta(X) = \sup_{\mathbb{Q}\in\mathscr{M}_a} \mathbb{E}_{\mathbb{Q}}[-X] - \chi(\mathbb{Q})$ where $\mathscr{M}_a$ is some family of equivalent measures[3] i.e. $\mathbb{Q} \ll \mathbb{P}$ and where $\mathbb{E}_{\mathbb{Q}}$ denotes the expectation w.r.t. the measure $\mathbb{Q} \in \mathscr{M}_a$ and $\chi : \mathscr{M}_a \to \mathbb{R}$ is some convex (penalty) function (see Theorem 2.9 in [Roc16]). Since the investor seeks to minimise the risk of null returns, the investor seeks to exit the market by selling all holdings at a point $\rho \in \mathscr{T}$ that minimises the risk $\theta(X)$ of the investor's returns falling below 0 (after firm ruin) before T where $\mathscr{T}$ is a set of ($\mathscr{F}-$measurable) stopping times. We now observe that since the investor seeks to exit the market in advance of firm ruin, the investor's optimal stopping problem admits the following representation:

$$\inf_{\rho\in\mathscr{T}} \left[\sup_{\mathbb{Q}\in\mathscr{M}_a} \mathbb{E}_{\mathbb{Q}}[-X(\rho)] - \chi(\mathbb{Q}) \right], \tag{2.4}$$

where $X(\rho)$ denotes the process (2.2) stopped at time $\rho \in \mathscr{T}$ [4,5]. The convex risk measure and the subsequent minimax structure appearing in (2.4) are closely related to notions of *robust Bayesian control* and *entropy maximisation control* (see for example, [WN17; Bic19; AGAS18; GD+04; Roc16] for exhaustive discussions). If we now interpret optimality of the stopping time ρ in a sense of risk-minimal w.r.t. the risk measure θ, we can reformulate the problem in (2.4) and the investor's maximisation problem in terms of a decoupled pair of objective functions. Focusing firstly on the investor's capital injection problem, we can write the problem as:

find a strategy $(\hat{\tau},\hat{Z}) \in \Phi$ that maximises the following quantity $\forall (t_0,x_0,y_0) \in [0,T] \times \mathbb{R}_{>0} \times \mathbb{R}_{>0}, \forall \rho \in \mathscr{T}$:

$$J_I^{(1)}(t_0,x_0,y_0,(\tau,Z),\rho) = \mathbb{E}\left[\sum_{m\geq 1} e^{-\delta\tau_m} z_m \cdot 1_{\{t_0<\tau_m\leq \tau_S\wedge\rho\}} + g_2 e^{-\delta(\tau_S\wedge\rho)} Y_{\tau_S\wedge\rho}^{t_0,y_0,(\tau,Z)}\right], \tag{2.5}$$

where $g_2 \in [0,1]$ is a constant that represents the fraction of the firm's wealth returned to the investor upon exit and $\tau_S := \inf\{s \in [0,T] : X_s, Y_s \leq 0\} \wedge T$.

Therefore, (2.5) describes the quantity that the investor seeks to maximise through performing injections to the firm over the horizon of the problem. Lastly, we now turn to describing the component of the investor's goal to exit the market at an optimal time. In light of (2.4), we deduce the following expression which represents the investor's optimal stopping problem: find $\hat{\rho} \in \mathscr{T}$ that minimises the following quantity $\forall (t_0,x_0,y_0) \in [0,T] \times \mathbb{R}_{>0} \times \mathbb{R}_{>0}, \forall (\tau,Z) \in \Phi$:

$$J_I^{(2)}(t_0,x_0,y_0,(\tau,Z),\rho) = -\inf_{\mathbb{Q}\in\mathscr{M}_a} \mathbb{E}_{\mathbb{Q}}\left[e^{-\delta(\tau_S\wedge\rho)}\left(g_1 X_{\tau_S\wedge\rho}^{t_0,x_0,(\tau,Z)} + \lambda_T\right)\right], \tag{2.6}$$

where $g_1 \in]0,1]$ and $\lambda_T \geq 0$ represent the fraction of the firm's liquidity process and some fixed

[3] The measure $\mathbb{Q}$ is said to be equivalent or *absolutely continuous* w.r.t. the measure $\mathbb{P}$ iff the null set of $\mathbb{Q}$ is a proper subset of the null set of $\mathbb{P}$. We denote the equivalence of $\mathbb{Q}$ w.r.t. the measure $\mathbb{P}$ by $\mathbb{Q} \ll \mathbb{P}$.

[4] We observe that the problem in (2.4) can be viewed as a zero-sum game between two players; namely a player that controls the measure $\mathbb{Q}$ which may be viewed as an adverse market and the investor who selects the stopping time $\rho \in \mathscr{T}$. Games of this type are explored in [BH13] and [MØ08].

[5] We shall hereon specialise to the case $\chi \equiv 0$ in which case the risk measure θ is called *coherent*.

amount each received by the investor upon exit respectively. The expressions (2.5) and (2.6) fully express the investor's set of objectives. We can combine the expressions to construct a single problem with an objective function Π given by the following: find an admissible strategy $(\hat{\rho},(\hat{\tau},\hat{Z}))\in\mathscr{T}\times\Phi$ such that $\forall(t_0,x_0,y_0)\in[0,T]\times\mathbb{R}_{>0}\times\mathbb{R}_{>0}$

$$\hat{\rho}\in\arg\inf_{\rho\in\mathscr{T}}\Pi(t_0,x_0,y_0,(\hat{\tau},\hat{Z}),\rho), \tag{2.7}$$

$$(\hat{\tau},\hat{Z})\in\arg\sup_{(\tau,Z)\in\Phi}\Pi(t_0,x_0,y_0,(\tau,Z),\hat{\rho}), \tag{2.8}$$

where $\Pi(t_0,x_0,y_0,(\tau,Z),\rho)=\mathbb{E}\Big[\sup_{\mathbb{Q}\in\mathscr{M}_a}\mathbb{E}_{\mathbb{Q}}\Big[-e^{-\delta(\tau_S\wedge\rho)}\Big(g_1X^{t_0,x_0,(\tau,Z)}_{\tau_S\wedge\rho}+\lambda_T\Big)\Big]+\sum_{m\geq 1}e^{-\delta\tau_m}z_m\cdot 1_{\{t_0<\tau_m\leq\tau_S\wedge\rho\}}+g_2e^{-\delta(\tau_S\wedge\rho)}Y^{t_0,y_0,(\tau,Z)}_{\tau_S\wedge\rho}\Big]$. We now observe that the problem is to find the interdependent set of controls $(\hat{\rho},(\hat{\tau},\hat{Z}))\in\mathscr{T}\times\Phi$. If we think of the two objectives (2.5) and (2.6) as being assigned to two individual players, we recognise the pair of problems (2.5) and (2.6) as jointly representing a stochastic differential game of control and stopping in which the controller modifies the system dynamics using impulse controls. With this interpretation, each of the investor's objectives is delegated to an individual player that seeks to maximise their own objective by playing an optimal response to the other player.

To tackle the investment problem and characterise the optimal investment behaviour, in this chapter we develop the general underlying structure of the investment problem namely, the stochastic differential game of control and stopping.

2.2 Main Results

Computing the optimal controls (best-response strategies) for each player in this game firstly involves obtaining a full characterisation of the value function — a function that quantifies the expected equilibrium payoff for each player. To this end, we prove a *verification theorem* which provides a complete characterisation of the value of the game in terms of a solution to a PDE. Using this characterisation, both the equilibrium payoffs for each player and the equilibrium controls can be computed. Starting with strictly competitive (zero-sum) games, we then generalise the results to accommodate non zero-sum payoff structures. In both cases, we consider a setting in which the underlying system dynamics are governed by a jump-diffusion process which endows the model with the capability of describing numerous dynamics that occur within financial systems.

In particular, for the non zero-sum case, we prove the following result (Theorem 2.12):

let X be a stochastic process that evolves according to (1.1) and suppose $S\subset\mathbb{R}^p$ is some *solvency region*. Let ϕ_i be a smooth test function so that we can take first order temporal derivatives and second order spatial derivatives within the interior of S ($i\in\{1,2\}$); then if ϕ_i satisfy the following quasi-variational inequalities:

$$\begin{cases}\max\{\partial_s\phi_i(y)+\mathscr{L}\phi_i(y)+f_i(y),\phi_i(y)-\mathscr{M}_i\phi_i(y)\} & =0\\ \phi_i(\cdot,x)=G_i(\cdot,x) & \forall x\in S,\forall y\in[0,T]\times S,\end{cases} \tag{2.9}$$

where f_i and G_i are the player i running cost functions and terminal cost functions respectively and where $\mathscr{L}$ is the stochastic generator of the diffusion process X (c.f. (1.2)), then ϕ_i is the player i value function for the non zero-sum game. The verification theorem augments results for the stochastic differential game of control and stopping in [BHØT13] to now accommodate impulse controls and extends the optimal stopping analyses in [ØS05] to a strategic setting which now includes an impulse controller.

The results contained in this chapter are built exclusively under assumptions A.1.1 - A.4 (see Appendix).

2.3 Stochastic Differential Games of Impulse Control and Stopping

We firstly give a description of the setup and the system dynamics.

The Setup

In this game there are two players, player I and player II. Player I influences the state process using impulse controls $u\in\mathscr{U}$ where $u(s)=\sum_{j\geq 1}\xi_j\cdot 1_{\{\tau_j\leq T\}}(s)$ for all $0\leq t_0<s\leq T$. Throughout the horizon of the game, each player incurs a cost which is a function of the value of the state process. Let the set $\mathscr{T}$ be a given family of $\mathscr{F}-$measurable stopping times; at any point in the game $\rho\in\mathscr{T}$, player II can choose to terminate the game at which point the state process is stopped and both players receive a terminal reward (which may be negative).

The evolution of the state process with actions is given by the following $\forall r\in[0,T]$; $\forall(t_0,x_0)\in[0,T]\times S$:

$$\begin{aligned}X_r^{t_0,x_0,u}=x_0+\int_{t_0}^{r\wedge\rho}\mu(s,X_s^{t_0,x_0,u})ds+\int_{t_0}^{r\wedge\rho}\sigma(s,X_s^{t_0,x_0,u})dB_s+\sum_{j\geq 1}\xi_j\cdot 1_{\{\tau_j\leq r\wedge\rho\}}(r)\\ +\int_{t_0}^{r\wedge\rho}\int\gamma(X_{s-}^{t_0,x_0,u},z)\tilde{N}(ds,dz),\ \mathbb{P}-\text{a.s.},\end{aligned} \tag{2.10}$$

and without loss of generality we assume that $X_s^{t_0,x_0,\cdot}=x_0$ for any $s\leq t_0$.

In this chapter and throughout the thesis we use the following notation in some arguments for ease of exposition for any $s\in[0,T-t_0]$, $Y^{y_0,\cdot}(s)\equiv(s+t_0,X^{t_0,x_0,\cdot}(t_0+s))$, $y_0\equiv(t_0,x_0)$ and $\hat{Y}^{y_0,\cdot}(\tau)=Y^{y_0,\cdot}(\tau^-)+\Delta_N Y^{y_0,\cdot}(\tau)$, $\tau\in\mathscr{T}$ where $\Delta_N Y(\tau)$ denotes a jump at time τ due to $\tilde{N}$.

Player I has a cost function which is also the player II gain (or profit) function. The corresponding payoff function is given by the following expression which player I (resp., player II) minimises (resp., maximises) $\forall y_0 \equiv (t_0, x_0) \in [0,T] \times S$:

$$J[y_0; u, \rho] = \mathbb{E}\left[\int_{t_0}^{\tau_s \wedge \rho} f(Y_s^{y_0,u})ds + \sum_{m \geq 1} c(\tau_m, \xi_m) \cdot 1_{\{\tau_m \leq \tau_S \wedge \rho\}} + G(Y_{\tau_S \wedge \rho}^{y_0,u}) 1_{\{\tau_S \wedge \rho < \infty\}}\right], \qquad (2.11)$$

where $\tau_S : \Omega \to [0,T]$ is some random exit time at which point the game is terminated. We assume that the function G satisfies the condition $\lim_{s \to \infty} G(s,x) = 0$ for any $x \in S$. Functions of the form $G(s,x) \equiv e^{-\delta s}\bar{G}(x)$ for some $\delta > 0$ and $\bar{G} : |\bar{G}(x)| < \infty$ satisfy this condition among others. For convenience, we will occasionally use the shorthand $J^{\rho,u}[t,x] \equiv J[t,x;u,\rho]$ for any $(t,x) \in [0,T] \times S$ and $\forall u \in \mathscr{U}$, $\forall \rho \in \mathscr{T}$.

Markov controls are those in which the player uses only information about the current state and duration of the game rather than explicitly incorporating information about the other player's decisions or utilising information on the history of the game. In light of the remarks of Chapter 1 in which it was observed that using strategies may be advantageous, limiting the analysis to Markov controls may incur too strong of a restriction on the abilities of the players to perform optimally. However, the following observation allows us to focus on Markov controls instead of strategies:

Remark 2.1

Under mild conditions, for the game discussed in this chapter which involves a diffusive state process, using Markov controls gives as good performance as an arbitrary $\mathscr{F}$-adapted control (see for example Theorem 11.2.3 in [ØS07]).

Consequently, in the following analysis, we restrict ourselves to Markov controls and hence for player I, the control policy takes the form $u = u(s, \omega) \in \mathscr{U}$. In particular, the player I control can be written in the form $u = \tilde{f}_1(s, X_s)$ for any $s \in [0,T]$ where $\tilde{f}_1 : [0,T] \times S \to U$ and $U \subset \mathbb{R}^p$ and $\tilde{f}_1$ is some measurable map w.r.t. $\mathscr{F}$. Since this form of control does not depend on the history of the process, using Markov controls simplifies the analysis.

We now describe the value functions which are central objects in the analysis of the game:

Value Functions

The upper and lower value functions associated to the game are given by the following expressions:

$$V^-(y) = \inf_{u \in \mathscr{U}} \sup_{\rho \in \mathscr{T}} J[y; u, \rho]; \qquad (2.12)$$

$$V^+(y) = \sup_{\rho \in \mathscr{T}} \inf_{u \in \mathscr{U}} J[y; u, \rho], \qquad \forall y \in [0,T] \times S, \qquad (2.13)$$

where $\mathscr{T}$ is a given family of $\mathscr{F}-$measurable stopping times and $\mathscr{U}$ is the set of player I admissible impulse controls. The value of the game exists if we can commute the supremum and infimum operators in (2.12) and (2.13) where after we can deduce the existence of a function $V \in \mathscr{H}$ such that $V \equiv V^- = V^+$. Moreover, the value (if it exists) represents the expected (equilibrium) payoff that each player obtains when both players enact strategies that are best-responses to the actions of their opponent. In this chapter and beyond, we use the notation $V^\pm$ to mean any element drawn from the set $\{V^+, V^-\}$.

The following definition is a key object in the analysis of impulse control models:

Definition 2.2

For any $\tau \in \mathscr{T}$, we define the [non-local] intervention operator $\mathscr{M} : \mathscr{H} \to \mathscr{H}$ acting at a state $X(\tau^-)$ by the following expression:

$$\mathscr{M}\phi(\tau^-, X(\tau^-)) := \inf_{z \in \mathscr{Z}} [\phi(\tau, \Gamma(X(\tau^-), z)) + c(\tau, z) \cdot 1_{\{\tau \leq T\}}], \tag{2.14}$$

for some function $\phi : [0,T] \times S \to \mathbb{R}$ and $\Gamma : S \times \mathscr{Z} \to S$ is the impulse response function.

Of particular interest is the case when the intervention operator is applied to the value function $\mathscr{M}V(\cdot, x)$ — a quantity which represents the value of a strategy when the controller performs an optimal intervention then behaves optimally thereafter given an immediate optimal intervention taken at a state $x \in S$. The intuition behind (2.14) is as follows: suppose at time τ^- the system is at a state $X(\tau^-)$ and an intervention $z \in \mathscr{Z}$ is applied to the process, then a cost of $c(\tau, z)$ is incurred and the state then jumps from $X(\tau^-)$ to $\Gamma(X(\tau^-), z)$. If the controller acts optimally thereafter, the cost of this strategy, starting at state $\Gamma(X(\tau^-), z)$ is $V(\tau, \Gamma(X(\tau^-), z) + c(\tau, z)$. Lastly, choosing the action that minimises costs leads to $\mathscr{M}V$.

Remark 2.3

We note that whenever it is optimal for the controller to intervene, $\mathscr{M}V = V$ since the value function describes the player payoff under optimal behaviour. However, at any given point an immediate intervention may not be optimal, hence the inequality $\mathscr{M}V(y) \geq V(y)$ holds pointwise for any $y \in [0,T] \times S$.

Heuristic Analysis of The Value Function

We now begin our analysis of the game with the goal of deriving the verification theorems which characterise the equilibrium behaviour. In particular, the verification theorems provide characterisations of the value function in terms of a PDE. Central to the proof of the theorem is Dynkin's formula which leads naturally to the HJBI equation — a PDE that is motivated from an application of a dynamic programming principle.

The derivation of the verification theorem is mainly a technical exercise. Before performing the derivation we demonstrate how the key elements of the theorem can be obtained by studying the complete repertoire of tactics that each player can employ throughout the horizon of the game. By studying the behaviour of each player and relating it back the value function, the origins of properties of the value function that feature within the verification can be elegantly motivated. Moreover, this development additionally provides analytic insight for the HJBI equation and the associated *quasi-integro-variational inequalities* that emerge from the verification theorem as well as providing an intuitive path to the structure of the solution.

Firstly, let us assume that the game admits a value V and that the value function is sufficiently smooth on the interior of S to apply Dynkin's formula so that $V \in \mathscr{C}^{(1,2)}([0,T],S) \cap \mathscr{C}([0,T],\bar{S})$ (i.e. we can take first order temporal derivatives and second order spatial derivatives in S). We assume that the value function obeys the following expression which represents a dynamic programming principle (DPP) for the game:

$$V(y_0) = \inf_{u \in \mathscr{U}} \sup_{\rho \in \mathscr{T}} \mathbb{E}\Bigg[\int_{t_0}^{(t_0+h)\wedge\rho\wedge\tau_S} f(Y_s^{y_0,u})ds + \sum_{j\geq 1} c(\tau_j,\xi_j)\cdot 1_{\{\tau_j \leq (t_0+h)\wedge\rho\wedge\tau_S\}}$$
$$+ G(Y_{\rho\wedge\tau_S}^{y_0,u})\cdot 1_{\{\rho\wedge\tau_S \leq t_0+h\}} + V(Y_{t_0+h}^{y_0,u})\cdot 1_{\{\rho\wedge\tau_S > t_0+h\}}\Bigg]. \tag{2.15}$$

We firstly focus on the optimality conditions for player I; let us therefore fix some player II stopping time $\hat{\rho} \in \mathscr{T}\backslash\{t_0\}$. In view of the set of decisions facing player I, at any given instant, player I is faced with the choice of performing an immediate intervention at some cost or allowing the system to evolve freely. The following analysis is based on studying the cases under which either of these choices are optimal. This in turn generates a set of conditions that underscore the verification theorem.

Case I.1. No Immediate Player I Intervention

We observe that when optimal play involves no immediate player I intervention, we can find a bounded interval $[t_0,s]$ for some sufficiently small $s < (T-t_0)\wedge\tau_1$ in which the process is left to evolve with no interventions by player I (though the process can be terminated by an early choice of player II's stopping criterion, $\hat{\rho}$). In this case, the DPP (2.15) implies the following:

$$V(y_0) = \sup_{\hat{\rho}\in\mathscr{T}} \mathbb{E}\Bigg[\int_{t_0}^{s\wedge\hat{\rho}\wedge\tau_S} f(Y_r^{y_0,u_{0[t_0,s\wedge\hat{\rho}\wedge\tau_S]}})dr + V(Y_s^{y_0,u_{0[t_0,s\wedge\hat{\rho}\wedge\tau_S]}})\cdot 1_{\{s<\hat{\rho}\wedge\tau_S\}}$$
$$+ G(Y_{\hat{\rho}\wedge\tau_S}^{y_0,u_{0[t_0,s\wedge\hat{\rho}\wedge\tau_S]}})\cdot 1_{\{s\geq\hat{\rho}\wedge\tau_S\}}\Bigg], \tag{2.16}$$

where $u_0 \in \mathscr{U}$ is the player I control with no impulses and where we have used the restriction notation (c.f. Definition 1.5).

Using the continuity and boundedness of the term $\partial_{x_i}V$, we can deduce that the stochastic integral term is bounded so that $\mathbb{E}\left[\sum_{i=1}^p\sum_{j=1}^p\int_{t_0}^{s\wedge\hat{\rho}}\partial_{x_i}V(Y_r^{y_0,u_{0[t_0,r\wedge\hat{\rho}]}})\sigma_{ij}dB_r^j\right] = 0$ using the properties of the standard Brownian motion. Given the smoothness assumption on V, we can apply Dynkin's formula for jump-diffusion processes to (2.16) to arrive at a description of the value function in terms of a PDE. To see this, we perform a classical limiting procedure — in particular, let us take the limit as $s \downarrow t_0$ then, using the mean value theorem we find that:

$$0 = \partial_t V(y) + \mathscr{L}V(y) + f(y), \qquad \forall y \in [0,T] \times S, \tag{2.17}$$

$$G(\tau_S \wedge \rho, x) = V(\tau_S \wedge \rho, x), \qquad \forall x \in S, \tag{2.18}$$

where we have commuted the expectation operator with the limit and where the operator $\mathscr{L}$ is the stochastic generator of the diffusion process X (c.f. (1.2)).

The expressions (2.17) - (2.18) describe the evolution of the value function under conditions in which player I does not intervene, therefore the system evolves unaltered.

Case I.2. An Immediate Player I Intervention

We now consider the case when an immediate player I intervention at $\tau_1 = t_0$ is optimal. Let us firstly define an optimal stopping time $\hat{\rho} \in \mathscr{T}$ for player II to terminate the game so that $\sup_{\rho \in \mathscr{T}} J[y; u, \rho] = J[y; u, \hat{\rho}]$ (recall that $\mathscr{T}$ is the set of $\mathscr{F}-$measurable stopping times) for any $y \in [0,T] \times S$. Since an immediate player I intervention is optimal, we can consider some interval after which having *not* performed an intervention would be *suboptimal* for player I. Moreover, since $\tau_1 = t_0$ is an $\mathscr{F}-$measurable stopping time, we know in particular that $\omega; \hat{\tau}_1(\omega) = t_0$ is $\mathscr{F}_{t_0}-$measurable so that we can deduce the following expression $\forall \rho \in \mathscr{T}$, $\forall y_0 \in [0,T] \times S$:

$$V(y_0) \leq \mathbb{E}\left[\int_{t_0}^{s \wedge \rho \wedge \tau_S} f(Y_s^{y_0, \alpha(\rho)}) ds + V(Y_s^{y_0, \alpha(\rho)}) \cdot 1_{\{s < \rho \wedge \tau_S\}} + G(Y_{\rho \wedge \tau_S}^{y_0, \alpha(\rho)}) \cdot 1_{\{\infty > s \geq \rho \wedge \tau_S\}}\right], \tag{2.19}$$

for some $t_0 < s < T - t_0$ and where the inequality arises due to the fact that player I can improve their payoff (decreasing the value of RHS of (2.19)) by performing an intervention in the interval $[t_0, s]$. Since an immediate intervention is optimal for player I, following Remark 2.3, we observe that $\mathscr{M}V(s,x) = V(s,x)\Big|_{s=t_0}$. Let us now define $\hat{u} \in \mathscr{U}$ by $\inf_{u \in \mathscr{U}} J[y; u, \rho] = J[y; \hat{u}, \rho]$ where $\hat{u} = [\hat{\tau}_j, \hat{\xi}_j]_{j \in \mathbb{N}} \in \mathscr{U}$. We now observe that $J[y; \hat{u}, \rho] = \inf_{u \in \mathscr{U}} J[y; u, \rho] \leq \sup_{\rho \in \mathscr{T}} \inf_{u \in \mathscr{U}} J[y; u, \rho] = V(y)$ for any $y \in [0,T] \times S$. Therefore since (2.19) holds for any $\rho \in \mathscr{T}$, we have in particular that:

$$V(y_0) \leq \mathbb{E}\left[\int_{t_0}^{s \wedge \hat{\rho} \wedge \tau_S} f(Y_s^{y_0, \alpha(\hat{\rho})}) ds + V(Y_s^{y_0, \alpha(\hat{\rho})}) \cdot 1_{\{s < \hat{\rho} \wedge \tau_S\}} + G(Y_{\hat{\rho} \wedge \tau_S}^{y_0, \alpha(\hat{\rho})}) \cdot 1_{\{\infty > s \geq \hat{\rho} \wedge \tau_S\}}\right]. \tag{2.20}$$

Therefore, after reapplying the limit procedure to (2.20) as in (2.16) - (2.17), we then deduce that $\partial_t V(y) + \mathscr{L}V(y) + f(y) \geq 0$. Gathering the above cases, we see that it is either optimal for player I to apply an impulse intervention in which case $\mathscr{M}V(y) = V(y)$ and $\partial_t V(y) + \mathscr{L}V(y) + f(y) \geq 0$ or it is optimal for player I to leave the system in which case we have $\mathscr{M}V(y) \geq V(y)$ and $\partial_t V(y) + \mathscr{L}V(y) + f(y) = 0$. After combining these statements, it is straightforward to see that the following expression must hold:

$$\min\left[(\partial_t V(y) + \mathscr{L}V(y) + f(y),\ \mathscr{M}V(y) - V(y))\right] = 0, \quad \forall y \in [0,T] \times S. \tag{2.21}$$

Equation (2.21) describes the obstacle problem facing player I given its impulse control problem. The expression captures the fact that at some given state $x \in S$, the decision facing player I is the choice of whether to perform an immediate intervention or not.

Using this analysis we can characterise the region in which an immediate intervention is *not optimal* which we define to be the continuation region: $D_1 = \{\mathscr{M}V(s,x) \geq V(s,x) | (s,x) \in [0,T] \times S\}$. Conversely for $X(s) \notin D_1$, by definition of D_1, it is optimal to intervene which gives rise to the following inductive definition of the optimal sequence of intervention times: $\hat{\tau}_0 \equiv t_0$ and $\hat{\tau}_{j+1} = \inf\{s > \tau_j; X^{\cdot,\hat{u}_{[t_0,s]}}(s) \notin D_1\} \wedge \tau_S \wedge \rho$.

Analysis of Player II

Having performed a study of the possibilities for player I, we now perform the corresponding analysis for player II. Since the set of decision available to player II consist of either deciding the terminate the game or to allow the game to continue, the derivation of the corresponding expressions for player II is much simpler. We begin by fixing some player I control $\hat{u} = [\hat{\tau}_j, \hat{\xi}_j]_{j \in \mathbb{N}} \in \mathscr{U}$.

Now, it is either optimal for player II to terminate the game straight away or it is optimal for player II to leave the system so that (at most) only player I modifies the system dynamics with impulse interventions. We analyse the two cases individually.

Case II.1. No Immediate Player II Termination

We begin by supposing there exists a continuation region for player II, D_2 — a region in S for which it is not optimal for player II to terminate the process. Let us therefore define the optimal stopping time $\hat{\rho}$ for player II by $\hat{\rho} = \inf\{s \geq t_0; X(s) \notin D_2\}$. We now observe that given some state X, if player II stops the process at some stopping time $\tilde{\rho} \in \mathscr{T}$, a reward of $G(X(\tilde{\rho}))$ is received, moreover, if $X(\tilde{\rho}) \in D_2$ then an immediate stop is suboptimal for player II so that $V(\tilde{\rho},x) \geq G(\tilde{\rho},x)$ since player II can decrease its costs by terminating the process at the stopping time $\hat{\rho}$.

Now, we observe that for some $h < (\hat{\rho} \wedge T) - t_0$ we have that:

$$V(y_0) = \inf_{u \in \mathscr{U}} \mathbb{E}\Bigg[\int_{t_0}^{(t_0+h)\wedge\tau_S} f(Y_s^{y_0,u})ds + \sum_{j\geq 1} c(\tau_j,\xi_j) \cdot 1_{\{\tau_j \leq t_0+h\}} + V(Y_{t_0+h}^{y_0,u}) \cdot 1_{\{t_0+h<\tau_S\}} \\ + G(Y_{\tau_S}^{y_0,u}) \cdot 1_{\{t_0+h\geq\tau_S\}}\Bigg]. \quad (2.22)$$

Given a function $\psi \in \mathscr{C}^{(1,2)}$ the Dynkin formula [Øks13] states that for any finite stopping time $\tau \in \mathscr{T}$ and for any $y_0 \equiv (t_0,x_0) \in [0,T] \times S$ we have:

$$\mathbb{E}\left[\psi(Y_\tau^{y_0,\cdot})\right] = \psi(y_0) + \mathbb{E}\left[\int_{t_0}^{\tau} \mathscr{L}\psi(Y_s^{y_0,\cdot})ds\right], \quad (2.23)$$

where $\mathscr{L}$ is the stochastic generator of the diffusion process (c.f. (1.2)).

After formally applying the Dynkin formula to (2.22) (using the smoothness of V) and by similar reasoning as the above case, we find that the following expressions hold:

(i) $G(y) \geq V(y)$,

(ii) $\partial_t V(y) + \mathscr{L}V(y) + f(y) = 0, \quad \forall y \in [0,T] \times S$.

Case II.2. An Immediate Player II Termination

We now consider the case when it is optimal for player II to terminate the game straight away. Since $\hat{\rho} = t_0$ is an $\mathscr{F}-$measurable stopping time, we know in particular that $\{\omega; \hat{\rho}(\omega) = t_0\}$ is $\mathscr{F}_{t_0}-$measurable. We assume that an immediate player II action is optimal so that $\hat{\rho} = t_0$, in which case it is easy to see that $G(\hat{\rho}, x) = V(\hat{\rho}, x), \forall x \in S$. Putting this together with (i) and (ii) leads to the observation that $\max[\partial_t V(y) + \mathscr{L}V(y) + f(y), G(y) - V(y)] = 0, \quad \forall y \in [0,T] \times S$. Lastly, by definition of J we have that $J^{\rho,u}(\tau_S \wedge \rho, x) = G(\tau_S \wedge \rho, x)$ for any $x \in S$. Since this holds for all $\rho \in \mathscr{T}$, $\forall u \in \mathscr{U}$, we have in particular that $V(\tau_S \wedge \rho, x) = G(\tau_S \wedge \rho, x), \forall x \in S$.

We are now in a position to combine all four possibilities. Indeed, doing so produces the following *double obstacle problem* which is a condition that holds at all points of the game for both players:

$$\min[\max[(\partial_t V + \mathscr{L}V + f, G - V)], V - \mathscr{M}V] = 0,$$
$$V(\rho \wedge \tau_S, x) = G(\rho \wedge \tau_S, x), \ \forall x \in S. \tag{2.24}$$

The above analysis motivates the following pair of definitions:

Definition 2.4

Consider the following optimal stopping problem:

$$V(y_0) = \inf_{\rho \in \mathscr{T}} \mathbb{E}\left[\int_{t_0}^{\tau_S \wedge \rho} f(Y_s^{y_0}) ds + G(Y_{\tau_S \wedge \rho}^{y_0}) 1_{\{\tau_S \wedge \rho < \infty\}}\right]. \tag{2.25}$$

We call the following three relations integro-variational inequalities (IVIs) for the optimal stopping problem (2.25):

$$\mathscr{L}V + f \geq 0 \tag{2.26}$$
$$V \geq G \tag{2.27}$$
$$(G - V)(f - \mathscr{L}V) = 0. \tag{2.28}$$

If the function $V \in \mathscr{H}$ satisfies (2.26) - (2.28) then V is said to be a solution to the IVI for problem (2.25).

Definition 2.5

Consider the impulse control problem (1.7), we call the following three relations quasi-variational inequalities (QVIs) for the impulse control problem (1.7):

$$\mathscr{L}V + f \geq 0 \tag{2.29}$$

$$\mathscr{M}V \geq V \tag{2.30}$$

$$(\mathscr{M}V - V)(\mathscr{L}V - f) = 0. \tag{2.31}$$

If the function V satisfies (2.29) - (2.31) then V is said to be a solution to the QVI for problem (1.7). Correspondingly, if the control $\hat{u} := [\hat{\tau}_j, \hat{\xi}_j]_{j\geq 1} \in \mathscr{U}$ is such that for any $y \in [0,T] \times S$, $V(y) = \inf_{u\in\mathscr{U}} J[y;u] = J[y;\hat{u}]$ so that the control $\hat{u}$ is optimal, then we say that the control $\hat{u}$ is a QVI control policy. From the above analysis, we see that a QVI control policy is that which consists of a strategy of intervening when doing so maximises the future payoff and, intervening by inducing a locally optimal jump. Note that unlike in the case of Definition 2.4, the solution to the QVI has a dependence on the term $\mathscr{M}V$ which is a non-trivial function of V, it is for this reason that the set of relations (2.29) - (2.31) are referred to as *quasi*-variational inequalities.

The following definition is a combination of Definition 2.4 and Definition 2.5 within a game setting:

Definition 2.6

We say that the function $V \in \mathscr{H}$ is a joint solution to the player I QVI problem and the player II IVI problem if V satisfies (2.26) - (2.28) and (2.29) - (2.31).

Definition 2.6 outlines the conditions that are required in the verification theorem for zero-sum stochastic differential games of control and stopping with impulse controls.

We now turn our attention to a formal characterisation of the value function.

2.5.1 A HJBI Equation for Zero-Sum Stochastic Differential Games of Control and Stopping with Impulse Controls

The following theorem provides the conditions under which, if a solution to a HJBI equation which is smooth enough to apply Itō's formula can be found then the solution is a candidate value function of the game. The following verification theorem additionally characterises the conditions under which the value of the game satisfies a HJBI equation and characterises the equilibrium controls for the game. Later, we use the conditions of Theorem 2.7 to derive the optimal investment strategy for the optimal liquidity control and lifetime ruin model.

Theorem 2.7 (Verification theorem for Zero-Sum Stochastic Differential Games of Control and Stopping with Impulse Controls)

Suppose the problem is to find $\phi \in \mathscr{H}$ and $(\hat{u},\hat{\rho}) \in \mathscr{U} \times \mathscr{T}$ such that

$$\phi(y) = \sup_{\rho\in\mathscr{T}}\left(\inf_{u\in\mathscr{U}} J^{(u,\rho)}[y]\right) = \inf_{u\in\mathscr{U}}\left(\sup_{\rho\in\mathscr{T}} J^{(u,\rho)}[y]\right) = J^{(\hat{u},\hat{\rho})}[y], \qquad \forall y \in [0,T]\times S, \tag{2.32}$$

where if $(\hat{u},\hat{\rho}) \in \mathscr{U} \times \mathscr{T}$ exists, it is an equilibrium pair consisting of the best-response control for player I and the best-response stopping time for player II (resp.).

Let τ be some $\mathscr{F}-$measurable stopping time and denote by $\hat{X}(\tau) = X(\tau^-) + \Delta_N X(\tau)$, where $\Delta_N X(\tau)$ denotes a jump at time τ due to $\tilde{N}$. Suppose that the value of the game exists.

Suppose also that there exists a function $\phi \in \mathscr{C}^{1,2}([0,T],S) \cap \mathscr{C}([0,T],\bar{S})$ that satisfies technical conditions (T1) - (T4) (see Appendix) and the following conditions:

(i) $\phi \leq \mathscr{M}\phi$ on S and $\phi \geq G$ on S and the regions D_1 and D_2 are defined by:
$D_1 = \{x \in S; \phi(\cdot,x) < \mathscr{M}\phi(\cdot,x)\}$ and $D_2 = \{x \in S; \phi(\cdot,x) > G(\cdot,x)\}$
where we refer to D_1 (resp., D_2) as the player I (resp., player II) continuation region.

(ii) $\frac{\partial\phi}{\partial s} + \mathscr{L}\phi(\cdot,X^{\cdot,u}(s)) + f(\cdot,X^{\cdot,u}(\cdot)) \geq 0,\ \forall u \in \mathscr{U}$ on $S\backslash\partial D_1$.

(iii) $\frac{\partial\phi}{\partial s} + \mathscr{L}\phi(\cdot,X^{\cdot,\hat{u}}(\cdot)) + f(\cdot,X^{\cdot,\hat{u}}(\cdot)) = 0$ in $D_1 \cap D_2$.

(iv) For $u \in \mathscr{U}$, define $\rho_D = \rho_D^u = \inf\{s > t_0, X^{\cdot,u}(s) \notin D_2\}$ and specifically, $\hat{\rho}_D = \hat{\rho} = \inf\{s > t_0, X^{\cdot,u}(s) \notin D_2\}$.

(v) $X^{\cdot,u}(\tau_S) \in \partial S, \mathbb{P}-$a.s. for $\tau_S < \infty$ and $\phi(s,X^{\cdot,u}(s)) \to G(\tau_S, X^{\cdot,u}(\tau_S \wedge \rho))$ as $s \to \tau_S^- \wedge \rho^-$, $\mathbb{P}-$a.s.,$\forall x \in S$, $\forall u \in \mathscr{U}$.

Put $\hat{\tau}_0 \equiv t_0$ and define $\hat{u} := [\hat{\tau}_j, \hat{\xi}_j]_{j\in\mathbb{N}}$ inductively by:
$\hat{\tau}_{j+1} = \inf\{s > \tau_j; X^{\cdot,\hat{u}_{[t_0,s]}}(s) \notin D_1\} \wedge \tau_S \wedge \hat{\rho}$, then $(\hat{u},\hat{\rho}) \in \mathscr{U} \times \mathscr{T}$ are an equilibrium pair for the game, that is to say that we have:

$$\phi(y) = \inf_{u\in\mathscr{U}}\left(\sup_{\rho\in\mathscr{T}} J^{(u,\rho)}[y]\right) = \sup_{\rho\in\mathscr{T}}\left(\inf_{u\in\mathscr{U}} J^{(u,\rho)}[y]\right) = J^{(\hat{u},\hat{\rho})}[y], \qquad \forall y \in [0,T]\times S. \tag{2.33}$$

Theorem 2.7 provides a first characterisation of the value of a stochastic differential game of impulse control and stopping. The key contribution of the theorem is that it describes the controls that each player is required to adopt in order to respond optimally to the actions of their opponent. Since the theorem principally involves finding a candidate function that satisfies the conditions of the theorem, the theorem provides a practicable method of finding candidate solutions to the value and hence, for the problem to be solved. The characterisation of the value function is expressed in terms of a dynamic programming equation that is, non-linear PDE in (iii). In particular, Theorem

2.7 says that given some solution to the non-linear PDE in (iii), then this solution coincides with the value of the game from which we can calculate the optimal controls for each player. Moreover, Theorem 2.7 provides insight as to the structure of the solution, in particular, provided player II has not terminated the game, player I exercises an impulse control whenever the state process exits the continuation region D_1. Similarly, player II does nothing when the state process remains within the region D_2 and terminates the game at the first hitting time on $S \backslash \partial D_2$.

As we demonstrate in our worked examples, using the results of theorem, we can solve the game by firstly finding candidate solutions to a PDE (HJBI equation) and then target solutions that satisfy the conditions of the theorem. This, in turn, allows us to extract a unique candidate function for the value function. Thereafter the player I equilibrium control and the player II equilibrium stopping criterion can be readily retrieved.

As remarked earlier, Theorem 2.7 imposes a number of conditions on the value function. Many of these conditions can be seen in Definition 2.6, in particular conditions (i) - (iii) of Theorem 2.7 are obtained directly from conditions (2.26) - (2.28) (the integrovariational inequalities) and (2.29) - (2.31) (the quasi-variational inequalities). Specifically, the function ϕ is required to be smooth in the interior i.e $\phi \in \mathscr{C}^{1,2}([0,T],S) \cap \mathscr{C}([0,T],\bar{S})$ for the application of the integro-differential operator $\mathscr{L}$ in (ii) and (iii) to be a well-defined operation; it is also required for application of the Dynkin formula which is a central to the proof of the verification theorem. Condition (v) is required within the proof of the verification theorem when we study the limiting behaviour of the value function close to the termination time and lastly condition (iv) defines the player II stopping criterion.

Theorem 2.7 imposes smoothness on the candidate function which restricts the class of functions in order for the term $\mathscr{L}\phi$ to be well-defined. A second consequence of the conditions (i) - (v) is that they prescribe requirements on the behaviour of the function when the process X crosses the boundary of the continuation region. In general, the conditions necessitate solving a *Stefan problem* — a boundary value problem for which the value function undergoes a phase transition as the value of X passes through the boundary of a continuation region. Demanding that the value function is continuous across the boundary, in some cases, usefully fixes a particular solution of the value function for the game therefore enabling a complete solution to the problem to be obtained. We discuss this in further detail when we solve Example 2.13.

As the following remark explains, the smoothness criterion demanded in the verification theorem in fact imposes a stronger smoothness condition than what is required.

The following remark also applies to the verification theorems 2.7, 2.12 and 4.9 and 4.12:

Remark 2.8

The smoothness condition $V \in \mathscr{C}^{1,2}([0,T],S) \cap \mathscr{C}([0,T],\bar{S})$ is in fact a strong one which is violated in a number of optimal stochastic control problems e.g. Example 9.5, pg. 127 in [ØS07]. In such cases, Dynkin's formula (which is needed to prove the verification theorem) cannot be applied since the term $\mathscr{L}\phi$ is not well-defined. Nonetheless, we can weaken the condition $V \in \mathscr{C}^{1,2}([0,T],S) \cap \mathscr{C}([0,T],\bar{S})$ by considering a wider class of functions for which the aforementioned smoothness criterion may not necessarily hold.

A Sobolev space is an enlargement of the space of functions $\mathscr{C}^{1,2}([0,T],S) \cap \mathscr{C}([0,T],\bar{S})$ that provides a space of weak solutions to PDEs such as the HJBI equations of the verification Theorem 2.7 (see Appendix for a formal definition of a Sobolev space). Indeed, the solutions to such PDEs are naturally encountered within Sobolev spaces which contain functions with well-defined *weak derivatives* — derivatives that are defined in an integral sense. Weak derivatives differ from classical derivatives which, in the case of the latter and for the space of continuous functions, are defined as limits of difference quotients defined in the point-wise sense (for exhaustive discussions on the subject, we refer the reader to [Leo17]). It therefore suffices to consider a class of functions that lie within a Sobolev space, that is $V \in W^{((2,1),k)}_{loc}([0,T],S) \cap \mathscr{C}([0,T],\bar{S}), k \geq 1$ where $W^{(2,k)}_{loc} = \{f \in W^{(k,p)}(U), \forall \text{ compact } U' \in \mathscr{U}\}$ where $W^{(k,p)}(U)$ is the space of $\mathbb{L}^p$ functions with β^{th} weak partial derivatives belonging to $\mathbb{L}^p, \forall |\beta| \leq k$. An immediate consequence of the defining properties of this space is that the function $V \in W^{((2,1),k)}_{loc}([0,T],S) \cap \mathscr{C}([0,T],\bar{S}))$ can be approximated by functions that lie within the set $\mathscr{C}^{1,2}([0,T],S) \cap \mathscr{C}([0,T],\bar{S}))$ — indeed we can find a sequence of functions $\{V^n\}_{n\geq 1} \in \mathscr{C}^{1,2}([0,T],S) \cap \mathscr{C}([0,T],\bar{S}))$ such that $V^n \to V$ uniformly on compact subsets S and, for which we can deduce the existence of some $N > 1$ such that the following statements hold $\forall n,m > N$, $\forall \varepsilon > 0$:

$$
\begin{aligned}
\|\partial_s V^n - \partial_s V^m\|_{\mathbb{L}^2(S)} &< \varepsilon, \\
\|DV^n - DV^m\|_{\mathbb{L}^2(S)} &< \varepsilon, \\
\left\|D^2 V^n - D^2 V^m\right\|_{\mathbb{L}^2(S)} &< \varepsilon.
\end{aligned}
$$

We can therefore deduce that since for the functions $\{V^n\}_{n\geq 1}$ Dynkin's formula holds, we can see (by sending $n \to \infty$) that corresponding results are inherited by V.

Remark 2.8 provides an indication that in fact the smoothness criteria of Theorem 2.7 can be relaxed in some way. Indeed, by replacing the notion of a classical solution to PDEs, we can significantly weaken the smoothness assumptions therefore enabling application of the results to a range of practical problems for which the smoothness criteria of Theorem 2.7 do not hold. In Chapter 3, we perform a more detailed study of this matter when we study the problem using a viscosity-theoretic approach.

Clearly, when the player II control is fixed, that is if $\mathscr{T}$ is either a singleton or the empty set then the game is degenerate and collapses into an optimal stochastic control problem with one player.

The following corollary is manifest and reproduces widely known verification theorems for single controller impulse control (e.g. [LP86]):

Corollary 2.8.1 (Verification Theorem for Optimal Impulse Control)

Let X be a stochastic process that evolves according to (1.1) and suppose that there exists a function $\phi \in \mathscr{C}^{1,2}([0,T],S) \cap \mathscr{C}([0,T],\bar{S})$ that satisfies technical conditions (T1) - (T4) (see Appendix) and the following conditions:

(I) $\phi \leq \mathscr{M}\phi$ on S; define the region D is by:
$D = \{x \in S; \phi(\cdot,x) < \mathscr{M}\phi(\cdot,x)\}$ where D is called the *controller continuation region*.

(II) $\frac{\partial \phi}{\partial s} + \mathscr{L}\phi(\cdot,X^{\cdot,u}(\cdot)) + f(\cdot,X^{\cdot,u}(\cdot)) \geq 0,\ \forall u \in \mathscr{U}$ on $S \backslash \partial D$.

(III) $\frac{\partial \phi}{\partial s} + \mathscr{L}\phi(\cdot,X^{\cdot,\hat{u}}(\cdot)) + f(\cdot,X^{\cdot,\hat{u}}(\cdot)) = 0$ in D.

(IV) $X^{\cdot,u}(\tau_S) \in \partial S$, $\mathbb{P}-$a.s. on $\tau_S < \infty$ and $\phi(s,X^{\cdot,u}(s)) \to G(\tau_S,X^{\cdot,u}(\tau_S))$ as $s \to \tau_S^-$, $\mathbb{P}-$a.s., $\forall u \in \mathscr{U}$.

Put $\hat{\tau}_0 \equiv t_0$ and define $\hat{u} := [\hat{\tau}_j, \hat{\xi}_j]_{j \in \mathbb{N}}$ inductively by:
$\hat{\tau}_{j+1} = \inf\{s > \tau_j; X^{\cdot,\hat{u}_{[t_0,s]}}(s) \notin D_1\} \wedge \tau_S$, then $\hat{u} \in \mathscr{U}$ is an optimal control for the single controller impulse control problem, that is to say that we have:

$$\phi(y) = \inf_{u \in \mathscr{U}} J^u[y] = J^{\hat{u}}[y]; \qquad \forall y \equiv (t_0,x_0) \in [0,T] \times S, \tag{2.34}$$

where

$$J^u[y] = \mathbb{E}\left[\int_{t_0}^{\tau_s} f(Y_s^{y,u})ds + \sum_{m \geq 1} c(\tau_m,\xi_m) \cdot 1_{\{\tau_m \leq \tau_S\}} + G(Y_{\tau_S}^{y,u})1_{\{\tau_S < \infty\}}\right]. \tag{2.35}$$

Before giving the proof of Theorem 2.7, we make the following remark:

Remark 2.9

For the jump-diffusion process considered here, we can automatically conclude that
$\hat{\xi}_k \in \text{argmin}_{z \in \mathscr{Z}}\, \phi(\Gamma(x,z)) + c(\tau_k,z),\ \forall k \in \mathbb{N},\ x \in S$ where $\tau_k \in \mathscr{T}$ is an $\mathscr{F}-$measurable stopping time that exists.

The result follows straightforwardly from the non-emptiness of the set of optimal interventions (Lemma 3.7) the proof of which is deferred until the next chapter.

To prove Theorem 2.7, we firstly require the following result which enables us to perform limiting procedures close to the boundary of the player II continuation region:

Theorem 2.10 ((Approximation Theorem) (Theorem 3.1 in [ØS07]))

Let $\hat{D} \subset S$ be an open set and let us assume that $X(\tau_S) \in \partial S$ and suppose that $\partial\hat{D}$ is a Lipschitz surface. Let $\psi : \bar{S} \to \mathbb{R}$ be a function such that $\psi \in \mathscr{C}^1(S) \cap \mathscr{C}(\bar{S})$ and $\psi \in \mathscr{C}^2(S \backslash \partial\hat{D})$ and suppose the second order derivatives of ψ are locally bounded near $\partial\hat{D}$; then there exists a sequence of functions $\{\psi_m\}_{m=1}^{\infty} \in \mathscr{C}^2(S)\cap\mathscr{C}(\bar{S})$ such that

$$\lim_{m\to\infty}\psi_m \to \psi \text{ pointwise dominatedly in } \bar{S}.$$
$$\lim_{m\to\infty}\frac{\partial\psi_m}{\partial x_i} \to \frac{\partial\psi}{\partial x_i} \text{ pointwise-dominatedly in } S.$$
$$\lim_{m\to\infty}\frac{\partial^2\psi_m}{\partial x_i\partial x_j} \to \frac{\partial^2\psi}{\partial x_i\partial x_j} \text{ and } \lim_{m\to\infty}\mathscr{L}\psi_m \to \mathscr{L}\psi \text{ pointwise dominatedly in } S\backslash\partial\hat{D}.$$

We are now in a position to prove the theorem; some ideas for the proof come from [BHY11; IW14]:

Proof of Theorem 2.7 In order to ease exposition, in the following proof we adopt the following impulse response function $\hat{\Gamma} : \mathscr{T}\times S\times\mathscr{Z} \to \mathscr{T}\times S$ acting on $y' \equiv (\tau, x') \in \mathscr{T}\times S$ where $x' \equiv X^{t_0,x_0,\cdot}(t_0+\tau^-)$ and $\hat{\Gamma}$ is given by:$\hat{\Gamma}(y',\zeta) \equiv (\tau,\Gamma(x',\zeta)) = (\tau, X^{t_0,x_0,\cdot}(\tau)), \quad \forall\xi\in\mathscr{Z}, \forall\tau\in\mathscr{T}$.

We now begin by fixing the player I control $\hat{u}\in\mathscr{U}$ and let us define $\rho_m := \rho\wedge m; m = 1,2\ldots$. By Dynkin's formula for jump-diffusion processes (see for example Theorem 1.24 in [KW00]) we have:

$$\mathbb{E}[\phi(Y^{y_0,\hat{u}}(\hat{\tau}_j))] - \mathbb{E}[\phi(\hat{Y}^{y_0,\hat{u}}(\hat{\tau}_{j+1}))] = -\mathbb{E}\left[\int_{\hat{\tau}_j}^{\hat{\tau}_{j+1}}\frac{\partial\phi}{\partial s} + \mathscr{L}[\phi(Y^{y_0,\hat{u}}(s)]ds\right]. \tag{2.36}$$

Summing (2.36) from $j = 0$ to $j = k$ for some $0 < k < \mu_{[t_0,\rho_m]}(\hat{u}) - 1$ (recall the definition of $\mu_{[t_0,s]}(u)$ from Definition 1.4) and observing that by (iii) we have that -$(\partial_s + \mathscr{L})\phi = f$, we find that:

$$\begin{aligned}&\phi(y_0) + \sum_{j=1}^{k}\mathbb{E}[\phi(Y^{y_0,\hat{u}}(\hat{\tau}_j)) - \phi(\hat{Y}^{y_0,\hat{u}}(\hat{\tau}_j))] - \mathbb{E}[\phi(\hat{Y}^{y_0,\hat{u}}(\hat{\tau}_{k+1}))]\\ &= -\mathbb{E}\left[\int_{t_0}^{\hat{\tau}_{k+1}}\left(\frac{\partial\phi}{\partial s} + \mathscr{L}\phi(Y^{y_0,\hat{u}_{[t_0,s]}}(s))\right)ds\right] = \mathbb{E}\left[\int_{t_0}^{\hat{\tau}_{k+1}} f(Y^{y_0,\hat{u}_{[t_0,s]}}(s))ds\right].\end{aligned} \tag{2.37}$$

Now by definition of the non-local intervention operator $\mathscr{M}$ and by choice of $\hat{\xi}_j \in \mathscr{Z}$, we have that:

$$\phi(Y^{y_0,\hat{u}}(\hat{\tau}_j)) = \phi(\hat{\Gamma}(\hat{Y}^{y_0,\hat{u}}(\hat{\tau}_j^-),\hat{\xi}_j)) = \mathscr{M}\phi(\hat{Y}^{y_0,\hat{u}}(\hat{\tau}_j^-)) - c(\hat{\tau}_j,\hat{\xi}_j)\cdot 1_{\{\hat{\tau}_j\le T\}}, \tag{2.38}$$

(using the fact that $\inf_{z\in\mathscr{Z}}[\phi(\tau',\Gamma(X(\tau'^-),z)) + c(\tau',z)\cdot 1_{\{\tau'\le T\}}] = 0$ whenever $\tau' > \tau_S\wedge\rho$). Hence after deducting $\phi(\hat{Y}^{y_0,\hat{u}}(\hat{\tau}_j^-))$ from both sides we find:

$$\mathscr{M}\phi(\hat{Y}^{y_0,\hat{u}}(\hat{\tau}_j^-)) - \phi(\hat{Y}^{y_0,\hat{u}}(\hat{\tau}_j^-)) - c(\hat{\tau}_j,\hat{\xi}_j)\cdot 1_{\{\hat{\tau}_j\le T\}} = \phi(Y^{y_0,\hat{u}}(\hat{\tau}_j)) - \phi(\hat{Y}^{y_0,\hat{u}}(\hat{\tau}_j^-)), \tag{2.39}$$

and by (v) we readily observe that: $\phi(Y^{y_0,\hat{u}}(\tau_s)) - \phi(\hat{Y}^{y_0,\hat{u}}(\tau_s)) = 0$, hence after plugging (2.39) into

(2.37) we obtain the following:

$$\phi(y_0)+\sum_{j=1}^{k}\mathbb{E}[\mathscr{M}\phi(\hat{Y}^{y_0,\hat{u}}(\hat{\tau}_j^-))-\phi(\hat{Y}^{y_0,\hat{u}}(\hat{\tau}_j^-))]-\mathbb{E}[\phi(\hat{Y}^{y_0,\hat{u}}(\hat{\tau}_{k+1}))]$$
$$=\mathbb{E}\left[\int_{t_0}^{\hat{\tau}_{k+1}}f(Y^{y_0,\hat{u}_{[t_0,s]}}(s))ds+\sum_{j=1}^{k}c(\hat{\tau}_j,\hat{\xi}_j)\cdot 1_{\{\hat{\tau}_j\leq\tau_S\}}\right]. \tag{2.40}$$

Note that our choice of $\hat{\xi}_k\in\mathscr{Z}$ induces equality in (2.40). Since the number of interventions in (2.40) is bounded above by $\mu_{[t_0,\rho_m\wedge\tau_S]}(\hat{u})\wedge m$ for some $m<\infty$ and (2.40) holds for any $k\in\mathbb{N}$, taking the limit as $k\to\infty$ in (2.40) gives:

$$\phi(y_0)+\sum_{j=1}^{\mu_{[t_0,\rho_m\vee\tau_S]}(\hat{u})}\mathbb{E}[\mathscr{M}\phi(\hat{Y}^{y_0,\hat{u}}(\hat{\tau}_j^-))-\phi(\hat{Y}^{y_0,\hat{u}}(\hat{\tau}_j^-))]$$
$$=\mathbb{E}\left[\phi(\hat{Y}^{y_0,\hat{u}}(\rho_m\wedge\tau_S))+\int_{t_0}^{\rho_m\wedge\tau_S}f(Y^{y_0,\hat{u}_{[t_0,s]}})(s))ds+\sum_{j=1}^{\mu_{[t_0,\rho_m\vee\tau_S]}(\hat{u})}c(\hat{\tau}_j,\hat{\xi}_j)\cdot 1_{\{\hat{\tau}_j\leq\rho_m\wedge\tau_S\}}\right]. \tag{2.41}$$

Now $\lim_{m\to\infty}\sum_{j=1}^{\mu_{[t_0,\rho_m\vee\tau_S]}(\hat{u})}\mathbb{E}[\mathscr{M}\phi(\hat{Y}^{y_0,\hat{u}}(\hat{\tau}_j^-))-\phi(\hat{Y}^{y_0,\hat{u}}(\hat{\tau}_j^-))]=0$ since also by (v) we have that $\phi(\hat{Y}^{y_0,\cdot}(\hat{\tau}_j))-\phi(\hat{Y}^{y_0,\cdot}(\hat{\tau}_j^-))=0$, $\mathbb{P}-$a.s. when $\hat{\tau}_j=\tau_S$. Similarly, we have by (v) that $\phi(Y^{y_0,\hat{u}}(s))\to G(Y^{y_0,\hat{u}}(\tau_S\wedge\rho))$ as $s\to\tau_S^-\wedge\rho^-$, $\mathbb{P}-$a.s. Now since $\rho_m\wedge\tau_S\to\rho\wedge\tau_s$ as $m\to\infty$, we can exploit the quasi-left continuity of X (for further details see [Pro05] (Proposition I.2.26 and Proposition I.3.27)) and the continuity properties of f. Indeed, taking the limit as $m\to\infty$ and using the Fatôu lemma and (2.41), we find that:

$$\begin{aligned}\phi(y_0)=&\sum_{j=1}^{\mu_{[t_0,\rho_m\vee\tau_S]}(\hat{u})}\mathbb{E}\Bigg[\mathscr{M}\phi(\hat{Y}^{y_0,\hat{u}}(\hat{\tau}_j^-))-\phi(\hat{Y}^{y_0,\hat{u}}(\hat{\tau}_j^-))]\\&+\mathbb{E}[\phi(\hat{Y}^{y_0,\hat{u}}(\rho_m\wedge\tau_S))]+\int_{t_0}^{\rho_m\wedge\tau_s}f(Y^{y_0,\hat{u}}(s))ds+\sum_{j\geq1}c(\hat{\tau}_j,\hat{\xi}_j)\cdot 1_{\{\hat{\tau}_j\leq\rho_m\wedge\tau_S}\Bigg]\\=&\liminf_{m\to\infty}\mathbb{E}\Bigg[\sum_{j=1}^{\mu_{[t_0,\rho_m\vee\tau_S]}(\hat{u})}\mathbb{E}[\mathscr{M}\phi(\hat{Y}^{y_0,\hat{u}}(\hat{\tau}_j^-))-\phi(\hat{Y}^{y_0,\hat{u}}(\tau_j^-))]\\&+\phi(\hat{Y}^{y_0,\hat{u}}(\rho_m\wedge\tau_S))+\int_{t_0}^{\rho_m\wedge\tau_s}f(Y^{y_0,\hat{u}}(s))ds+\sum_{j\geq1}c(\hat{\tau}_j,\hat{\xi}_j)\cdot 1_{\{\hat{\tau}_j\leq\rho_m\wedge\tau_S\}}\Bigg]\\\geq&\,\mathbb{E}\left[G(\hat{Y}^{y_0,\hat{u}}(\rho\wedge\tau_S))\cdot 1_{\{\rho\wedge\tau_S<\infty\}}+\int_{t_0}^{\rho\wedge\tau_s}f(Y^{y_0,\hat{u}}(s))ds+\sum_{j\geq1}c(\hat{\tau}_j,\hat{\xi}_j)\cdot 1_{\{\hat{\tau}_j\leq\rho\wedge\tau_S\}}\right],\end{aligned}$$

where we have used that $\sum_{j=1}^{\mu_{[t_0,\rho_m\vee\tau_S]}(\hat{u})}c(\hat{\tau}_j,\hat{\xi}_j)=\sum_{j\geq1}c(\hat{\tau}_j,\hat{\xi}_j)\cdot 1_{\{\hat{\tau}_j\leq\rho_m\wedge\tau_S\}}$. Since this holds for all

$\rho \in \mathscr{T}$ we observe that:

$$\phi(y_0) \geq \sup_{\rho \in \mathscr{T}} \mathbb{E}\left[G(\hat{Y}^{y_0,\hat{u}}(\rho \wedge \tau_S)) \cdot 1_{\{\rho \wedge \tau_S < \infty\}} + \int_{t_0}^{\rho \wedge \tau_s} f(Y^{y_0,\hat{u}}(s))ds + \sum_{j \geq 1} c(\hat{\tau}_j, \hat{\xi}_j) \cdot 1_{\{\hat{\tau}_j \leq \rho \wedge \tau_s\}}\right]. \tag{2.42}$$

After which we easily deduce that:

$$\phi(y_0) \geq \inf_{u \in \mathscr{U}} \sup_{\rho \in \mathscr{T}} \mathbb{E}\left[G(\hat{Y}^{y_0,u}(\rho \wedge \tau_S)) \cdot 1_{\{\rho \wedge \tau_S < \infty\}} + \int_{t_0}^{\rho \wedge \tau_s} f(Y^{y_0,u}(s))ds + \sum_{j \geq 1} c(\tau_j, \xi_j) \cdot 1_{\{\tau_j \leq \rho \wedge \tau_s\}}\right]. \tag{2.43}$$

For the second part of the proof, let us fix $\rho' \in \mathscr{T}_{(0,T)}$ as in (iv) and define by $\rho_D := \rho_D^u = \inf\{s > t_0; X^{t_0,x_0,u}(s) \notin D_2\}$. Now we choose a sequence $\{D_{2,m}\}_{m=1}^{\infty}$ of open sets such that the set $\bar{D}_{2,m}$ is compact with $\bar{D}_{2,m} \subset D_{2,m+1}$ and $D_2 = \cup_{m=1}^{\infty} D_{2,m}$ and choose $\rho_D(m) = m \wedge \inf_{s > t_0} X^{\cdot,u}(s) \notin D_{2,m}$. By Dynkin's formula for jump-diffusion processes and (ii) we have:

$$\phi(y_0) + \sum_{j=1}^{k} \mathbb{E}[\phi(Y^{y_0,u}(\tau_j)) - \phi(\hat{Y}^{y_0,u}(\tau_j^-))] - \mathbb{E}[\phi(\hat{Y}^{y_0,u}(\tau_{k+1}^-))] \tag{2.44}$$

$$= -\mathbb{E}\left[\int_{t_0}^{\tau_{k+1}} \frac{\partial \phi}{\partial s} + \mathscr{L}\phi(Y^{y_0,u}(s))ds\right] \leq \mathbb{E}\left[\int_{t_0}^{\tau_{k+1}} f(Y^{y_0,u_{[t_0,s]}}(s))ds\right]. \tag{2.45}$$

Hence,

$$\phi(y_0) + \sum_{j=1}^{k} \mathbb{E}\left[\phi(Y^{y_0,u}(\tau_j)) - \phi(\hat{Y}^{y_0,u}(\tau_j^-))\right] \leq \mathbb{E}\left[\int_{t_0}^{\tau_{k+1}} f(Y^{y_0,u_{[t_0,s]}}(s))ds + \phi(\hat{Y}^{y_0,u}(\tau_{k+1}^-))\right]. \tag{2.46}$$

Now by definition of $\mathscr{M}$ we find that:

$$\phi(Y^{y_0,u}(\tau_j)) = \phi(\hat{\Gamma}(\hat{Y}^{y_0,u}(\tau_j^-), \xi_j)) \geq \mathscr{M}\phi(\hat{Y}^{y_0,u}(\tau_j^-)) - c(\tau_j, \xi_j) \cdot 1_{\{\tau_j \leq \tau_S \wedge \rho\}}. \tag{2.47}$$

(and again using the fact that $\inf_{z \in \mathscr{Z}}[\phi(\tau', \Gamma(X(\tau'^-), z)) + c(\tau', z) \cdot 1_{\{\tau' \leq T\}}] = 0$ whenever $\tau' > \tau_S \wedge \rho$). Subtracting $\phi(\hat{Y}^{y_0,u}(\tau_j^-))$ from both sides of (2.47) and summing and negating, we find that:

$$\sum_{j=1}^{k} \mathbb{E}\left[\phi(Y^{y_0,u}(\tau_j)) - \phi(\hat{Y}^{y_0,u}(\tau_j^-))\right] \geq \sum_{j=1}^{k} \mathbb{E}\left[\mathscr{M}\phi(\hat{Y}^{y_0,u}(\tau_j^-)) - \phi(\hat{Y}^{y_0,u}(\tau_j^-)) - c(\tau_j, \xi_j) \cdot 1_{\{\tau_j \leq \tau_S \wedge \rho\}}\right]. \tag{2.48}$$

Inserting (2.48) into (2.46) gives:

$$\phi(y_0)+\sum_{j=1}^{k}\mathbb{E}[\mathscr{M}\phi(\hat{Y}^{y_0,u}(\tau_j^-))-\phi(\hat{Y}^{y_0,u}(\tau_j^-))]-\mathbb{E}[\phi(\hat{Y}^{y_0,u}(\tau_{k+1}^-))]$$
$$\leq\mathbb{E}\left[\int_{t_0}^{\tau_{k+1}}f(Y^{y_0,u_{[t_0,s]}}(s))ds+\sum_{j=1}^{k}c(\tau_j,\xi_j)\cdot 1_{\{\tau_j\leq\rho\wedge\tau_s\}}\right]. \tag{2.49}$$

Then letting $k\to\infty$ in (2.49) gives:

$$\begin{aligned}\phi(y_0)\leq-\sum_{j=1}^{\mu_{[t_0,\rho_D(m)\vee\tau_S]}(u)}\mathbb{E}[\mathscr{M}\phi(\hat{Y}^{y_0,u}(\tau_j^-))-\phi(\hat{Y}^{y_0,u}(\tau_j^-))]+\mathbb{E}\Bigg[\phi(\hat{Y}^{y_0,u}(\rho_D(m)\wedge\tau_s))\\+\int_{t_0}^{\rho_D(m)\wedge\tau_s}f(Y^{y_0,u}(s))ds+\sum_{j\geq 1}c(\tau_j,\xi_j)\cdot 1_{\{\tau_j\leq\rho\wedge\tau_s\}}\Bigg].\end{aligned} \tag{2.50}$$

Again, using the quasi-left continuity of X we find that: $\lim_{m\to\infty}[\mu_{[t_0,\rho_D(m)\vee\tau_S]}(u)]\equiv\lim_{m\to\infty}[\mu_{[t_0,\rho_D(m)]}(u)\vee\mu_{[t_0,\tau_S]}(u)]=\mu_{[t_0,\rho]}(u)\vee\mu_{[t_0,\tau_S]}(u)$, hence we have that: $\lim_{m\to\infty}\sum_{j=1}^{\mu_{[t_0,\rho_D(m)\vee\tau_S]}(u)}\mathbb{E}[\mathscr{M}\phi(\hat{Y}^{y_0,u}(\tau_j^-))-\phi(\hat{Y}^{y_0,u}(\tau_j^-))]=0$. Moreover, as in the first part of the proof, using the fact that $\rho_D(m)\wedge\tau_S\to\rho_D\wedge\tau_s$ as $m\to\infty$ and using (v) we observe that $\lim_{m\to\infty}\phi(\hat{Y}^{y_0,u}(\rho_D(m)))=\phi(\hat{Y}^{y_0,u}(\rho_D))=G(\hat{Y}^{y_0,u}(\rho_D))$. Hence, by the dominated convergence theorem, after taking the limit $m\to\infty$ in (2.50) we find that:

$$\phi(y_0)\leq\mathbb{E}\left[\int_{t_0}^{\rho_D\wedge\tau_s}f(Y^{y_0,u}(s))ds+\sum_{j\geq 1}c(\tau_j,\xi_j)\cdot 1_{\{\tau_j\leq\rho\wedge\tau_s\}}+G(\hat{Y}^{y_0,u}(\rho_D\wedge\tau_S))\cdot 1_{\{\rho_D\wedge\tau_S<\infty\}}\right]. \tag{2.51}$$

Since this holds for all $u\in\mathscr{U}$ we have that:

$$\phi(y_0)\leq\inf_{u\in\mathscr{U}}\mathbb{E}\left[\int_{t_0}^{\rho_D\wedge\tau_s}f(Y^{y_0,u}(s))ds+\sum_{j\geq 1}c(\tau_j,\xi_j)\cdot 1_{\{\tau_j\leq\rho\wedge\tau_s\}}+G(\hat{Y}^{y_0,u}(\rho_D\wedge\tau_S))\cdot 1_{\{\rho_D\wedge\tau_S<\infty\}}\right], \tag{2.52}$$

from which clearly we have that:

$$\phi(y_0)\leq\sup_{\rho\in\mathscr{T}}\inf_{u\in\mathscr{U}}\mathbb{E}\left[\int_{t_0}^{\rho\wedge\tau_s}f(Y^{y_0,u}(s))ds+\sum_{j\geq 1}c(\tau_j,\xi_j)\cdot 1_{\{\tau_j\leq\rho\wedge\tau_s\}}+G(\hat{Y}^{y_0,u}(\rho\wedge\tau_s))\cdot 1_{\{\rho\wedge\tau_S<\infty\}}\right], \tag{2.53}$$

where we observe that by (2.53) and (2.43) we conclude that:

$$\inf_{u\in\mathscr{U}}\left(\sup_{\rho\in\mathscr{T}}J^{(u,\rho)}[y_0]\right)\leq\phi(y_0)\leq\sup_{\rho\in\mathscr{T}}\left(\inf_{u\in\mathscr{U}}J^{(u,\rho)}[y_0]\right). \tag{2.54}$$

However, for all $u\in\mathscr{U},\rho\in\mathscr{T}$ and $y\in[0,T]\times S$ we have:

$\inf_{u\in\mathscr{U}}(\sup_{\rho\in\mathscr{T}} J^{(u,\rho)}[y]) \geq \sup_{\rho\in\mathscr{T}}(\inf_{u\in\mathscr{U}} J^{(u,\rho)}[y])$. Moreover, choosing $u = \hat{u}$ in (2.54), by (iii) we find equality, hence:

$$\phi(y_0) = \mathbb{E}\left[\int_{t_0}^{\hat{\rho}\wedge\tau_s} f(Y^{y_0,\hat{u}}(s))ds + \sum_{j\geq 1} c(\hat{\tau}_j,\hat{\xi}_j)\cdot 1_{\{\tau_j\leq\hat{\rho}\wedge\tau_s\}} + G(\hat{Y}^{y_0,\hat{u}}(\hat{\rho}\wedge\tau_s))\right], \tag{2.55}$$

from which we find that:

$$\phi(y) = \inf_{u\in\mathscr{U}}\left(\sup_{\rho\in\mathscr{T}} J^{(u,\rho)}[y]\right) = \sup_{\rho\in\mathscr{T}}\left(\inf_{u\in\mathscr{U}} J^{(u,\rho)}[y]\right), \quad \forall y\in[0,T]\times S, \tag{2.56}$$

from which we deduce the result. □

The following result expresses the fact that the state space can be divided into three regions and that the players' actions are governed by which region the state process is within. The result has implications for describing investment behaviour which exhibits stickiness.

Corollary 2.10.1

The sample space splits into three regions in which, when playing their equilibrium strategies, player I applies impulse interventions I_1, a region in which player II stops the game I_2, and a region I_3 in which no action is taken by either player. Moreover, the three regions are characterised by the following expressions:

$$I_1 = \{y\in[0,T]\times S : V(y) = \mathscr{M}V(y),\ \mathscr{L}V(y) + f(y) \geq 0\},$$
$$I_2 = \{y\in[0,T]\times S : V(y) = G(y),\ \mathscr{L}V(y) + f(y) \leq 0\},$$
$$I_3 = \{y\in[0,T]\times S : V(y) < \mathscr{M}V(y),\ V(y) > G(y);\ \mathscr{L}V(y) + f(y) = 0\}.$$

As we have seen, the value associated to the stochastic differential game of control and stopping with impulse controls can be characterised by a HJBI PDE. The above result applies solely to games that have a zero-sum payoff structure. Although zero-sum payoff structures are widely studied, restricting attention to zero-sum games means a large class of games for which the payoff structure is more general are excluded. Indeed, whilst games with zero-sum payoff structures are an important subclass of games, in many economic and financial contexts, the condition is violated and therefore a more general payoff structure is required.

In the next section, we extend the results of the game to a non zero-sum stochastic differential game. We prove a corresponding verification theorem which characterises the value function for the game in the non zero-sum payoff setting.

2.6.1 Non Zero-sum Stochastic Differential Games of Control and Stopping with Impulse Controls

Zero-sum scenarios are frequently encountered within financial and economic interactions and model situations when the interests of participants are highly misaligned [OR94]. There are however, various instances in which players strategically interact within financial and economic environments in pursuit of their own interest but the players' interests are not completely opposed [OR94; MCWG+95].

Examples of non zero-sum economic interactions with the controller-stopper game structure are *real options valuations* [Zer03]. In this setting, a party can acquire the right (but not the obligation) to undertake a business initiative such as expanding capital investment in a project. If another party has the option to terminate the project at some given future point then the decision of how and when to exercise the decision over capital expansion resembles the structure of the game discussed in this chapter with a non zero-sum payoff structure. For further discussions on the topic of real options, we refer the reader to the following [Zer03] and [Tri+96] for an exhaustive discussion of the topic.

To handle non zero-sum scenarios, it is necessary to extend the analysis conducted so far to accommodate an alternative equilibrium concept which does not require the players' interests to be diametrically opposed. In full analogy with the procedure for the zero-sum case, we generalise the zero-sum characterisation of the value function to games with non zero-sum payoff structures. As before, our task is to provide a complete characterisation of the payoff functions when the players are implementing their optimal strategies.

Overview

We start by proving a non zero-sum verification theorem for the game in which both players use impulse controls to modify the state process. As in the zero-sum case, the uncontrolled passive state process $X : [0,T] \times \Omega \to S$ evolves according to a jump-diffusion on $(\mathscr{C}([0,T];\mathbb{R}^p),(\mathscr{F}_{0,s})_{s\in[0,T]},\mathscr{F},\mathbb{P})$.

Since we now wish to study non zero-sum games, we decouple the performance objectives so that we now consider the following pair of payoff functions J_1 and J_2 for player I and player II respectively:

$$J_1^{(u,\rho)}[y_0] = \mathbb{E}\left[\int_{t_0}^{\rho\wedge\tau_S} f_1(Y^{y_0,u}(s))ds - \sum_{j\geq 1} c_1(\tau_j,\xi_j)\cdot 1_{\{\tau_j\leq\rho\wedge\tau_S\}} + G_1(Y^{y_0,u}_{\rho\wedge\tau_S})\cdot 1_{\{\rho\wedge\tau_S<\infty\}}\right], \quad (2.57)$$

$$J_2^{(u,\rho)}[y_0] = \mathbb{E}\left[\int_{t_0}^{\rho\wedge\tau_S} f_2(Y^{y_0,u}(s))ds - \sum_{j\geq 1} c_2(\tau_j,\xi_j)\cdot 1_{\{\tau_j\leq\rho\wedge\tau_S\}} + G_2(Y^{y_0,u}_{\rho\wedge\tau_S})\cdot 1_{\{\rho\wedge\tau_S<\infty\}}\right], \quad (2.58)$$

$$\forall y \in [0,T]\times S,$$

where $\tau_j \in \mathscr{T}; \xi_j \in \mathscr{Z}$ $(j\in\mathbb{N})$ are $\mathscr{F}-$measurable intervention times and $\mathscr{F}-$measurable measurable stopping interventions respectively. The functions c_1 and c_2 and $G_1 : [0,T]\times S \to \mathbb{R}$ and

$G_2 : [0,T] \times S \to \mathbb{R}$ are cost functions and bequest functions for player I and player II respectively.

The function $J_1^{(u,\rho)}[y_0]$ (resp., $J_2^{(u,\rho)}[y_0]$) defines the payoff received by the player I (resp., player II) during the game with beginning at $y \equiv (t_0, x_0) \in [0,T] \times S$ when player I uses the control $u \in \mathscr{U}$ and player II decides to stop the game at time $\rho \in \mathscr{T}$.

In order to discuss the notion of equilibrium in a non zero-sum case, we must introduce a relevant equilibrium concept which generalises the minimax (saddle point) equilibrium to the non zero-sum case.

Definition 2.11 (Nash Equilibrium)

We say that a pair $(\hat{u}, \hat{\rho}) \in \mathscr{U} \times \mathscr{T}$ is a Nash equilibrium of the stochastic differential game if the following statements hold:

(i) $J_1^{(\hat{u},\hat{\rho})}[y] \geq J_1^{(u,\hat{\rho})}[y], \qquad \forall u \in \mathscr{U}, \ \forall y \in [0,T] \times S,$

(ii) $J_2^{(\hat{u},\hat{\rho})}[y] \geq J_2^{(\hat{u},\rho)}[y], \qquad \forall \rho \in \mathscr{T}, \ \forall y \in [0,T] \times S.$

Condition (i) states that given some fixed player II stopping time $\hat{\rho} \in \mathscr{T}$, player I cannot profitably deviate from playing the control policy $\hat{u} \in \mathscr{U}$. Analogously, condition (ii) is the equivalent statement given the player I's control policy is fixed as $\hat{u}$, player II cannot profitably deviate from $\hat{\rho} \in \mathscr{T}$. We therefore see that $(\hat{u}, \hat{\rho}) \in \mathscr{U} \times \mathscr{T}$ is an equilibrium in the sense of a Nash equilibrium since neither player has an incentive to deviate given their opponent plays the equilibrium policy.

Heuristic Analysis of The Value Function

Before characterising the value functions in this setting, as in the zero-sum case, we give a heuristic motivation of the key features of the verification theorem for the game when the payoff structure is non zero-sum. We perform this task by studying the complete repertoire of tactics that each player can employ throughout the horizon of the game.

Suppose firstly that each player's value function is sufficiently smooth on the interior of S to apply Dynkin's formula (i.e. we can take first order temporal derivatives and second order spatial derivatives). Suppose also that the following dynamic programming principle is satisfied for each player's value function:

$$\begin{aligned} V_1(y_0) = \sup_{\alpha \in \mathscr{A}_{(0,T)}} \mathbb{E}\Bigg[\int_{t_0}^{(t_0+h)\wedge\rho} f_1(Y_s^{y_0,\alpha(\rho)}) ds - \sum_{j \geq 1} c_1(\tau_j(\rho), \xi_j(\rho)) \cdot 1_{\{\tau_j(\rho) \leq (t_0+h)\wedge\rho\}} \\ + G_1(Y_{\rho\wedge\tau_S}^{y_0,\alpha(\rho)}) \cdot 1_{\{\rho\wedge\tau_S \leq t_0+h\}} + V_1(Y_{t_0+h\wedge\rho}^{y_0,\alpha(\rho)}) \cdot 1_{\{\rho\wedge\tau_S > t_0+h\}} \Bigg], \\ V_2(y_0) = \sup_{\beta \in \mathscr{B}_{(0,T)}} \mathbb{E}\Bigg[\int_{t_0}^{(t_0+h)\wedge\beta(u)} f_2(Y_s^{y_0,u}) ds - \sum_{j \geq 1} c_2(\tau_j, \xi_j) \cdot 1_{\{\tau_j \leq (t_0+h)\wedge\beta(u)\}} \\ + G_2(Y_{\beta(u)\wedge\tau_S}^{y_0,u}) \cdot 1_{\{\beta(u)\wedge\tau_S \leq t_0+h\}} + V_2(Y_{t_0+h\wedge\beta(u)}^{y_0,u}) \cdot 1_{\{\beta(u)\wedge\tau_S > t_0+h\}} \Bigg], \end{aligned} \tag{2.59}$$

We firstly tackle the optimality conditions for player I hence, we focus only on the function J_1. Let us therefore fix some player II control $\hat{\rho} \in \mathscr{T}$. Suppose then that $\hat{u} \in \mathscr{U}$ is a best-response policy against $J_1[y;\hat{\rho},\cdot]$, $\forall y \in [0,T]\times S$, that is, we identify the control $\hat{u} \in \mathscr{U}$ by $\hat{u} \in \arg\sup_{u\in\mathscr{U}} J_1[y;\hat{\rho},u]$. From (2.59) and via a classical limiting procedure (as in the zero-sum case), we find that the following condition must hold:

$$f_1(y) + \partial_t J_1^{(\hat{\rho},\hat{u})}[y] + \mathscr{L}J_1^{(\hat{\rho},\hat{u})}[y] \geq f_1(y) + \partial_t J_1^{(\hat{\rho},u)}[y] + \mathscr{L}J_1^{(\hat{\rho},u)}[y] \tag{2.60}$$
$$t_0 \leq \hat{\tau}_j < s < \hat{\tau}_{j+1} \leq T,\ \forall u \in \mathscr{U},\ \forall y \in [0,T]\times S.$$

Expression (2.60) is an essential constituent of the verification theorem. To deduce (2.60), firstly we note that for $y_0 \equiv (t_0,x_0) \in [0,T]\times S$:

$$J_1^{(\hat{\rho},u)}[Y_{\tau_j}^{y_0,u}] = \mathbb{E}\left[\int_{\tau_j}^{t'\wedge\hat{\rho}} f_1(Y_s^{y_0,u})ds + J_1^{(u,\hat{\rho})}(Y_{t'}^{y_0,u})\cdot 1_{\{t'<\hat{\rho}\}} + G_1(Y_{\hat{\rho}\wedge\tau_S}^{y_0,u})\cdot 1_{\{t'\geq\hat{\rho}\}}\right], \tag{2.61}$$
$$t_0 \leq \tau_j < t' < \tau_{j+1}\wedge\tau_S, \quad \forall u \in \mathscr{U},$$

where we have used the fact that under the policy u no interventions are executed on the interval $]\tau_j,t']$. We now apply the Dynkin formula for jump-diffusions with which in conjunction with the smoothing theorem we find that:

$$\begin{aligned}&J_1^{(\hat{\rho},u)}[Y_{\tau_j}^{y_0,u}] - \mathbb{E}\left[J_1^{(\hat{\rho},u)}[Y_{\tau_j}^{y_0,u}]\cdot 1_{\{t'<\hat{\rho}\}}\right]\\ &= \mathbb{E}\left[\int_{\tau_j}^{t'\wedge\hat{\rho}} f_1(Y_s^{y_0,u}) + [\partial_t J_1^{(\hat{\rho},u)}(Y_s^{y_0,u}) + \mathscr{L}J_1^{(\hat{\rho},u)}(Y_s^{y_0,u})]\cdot 1_{\{t'<\hat{\rho}\}}ds + G_1(Y_{\hat{\rho}}^{y_0,\hat{u}})\cdot 1_{\{t'\geq\hat{\rho}\}}\right].\end{aligned} \tag{2.62}$$

Since (2.62) holds for all $u \in \mathscr{U}$ and using (2.59) for $u = \hat{u}$, we have that:

$$J_1^{(\hat{\rho},\hat{u})}[Y_{\hat{\tau}_j}^{y_0,\hat{u}}] - \mathbb{E}[J_1^{(\hat{\rho},\hat{u})}[Y_{\hat{\tau}_j}^{y_0,\hat{u}}]\cdot 1_{\{t'<\hat{\rho}\}}] \tag{2.63}$$
$$= \mathbb{E}\left[\int_{\hat{\tau}_j}^{t'\wedge\hat{\rho}} f_1(Y_s^{y_0,\hat{u}}) + \left(\partial_t J_1^{(\hat{\rho},\hat{u})}(Y_s^{y_0,\hat{u}}) + \mathscr{L}J_1^{(\hat{\rho},\hat{u})}(Y_s^{y_0,\hat{u}})\right)\cdot 1_{\{t'<\hat{\rho}\}}ds + G_1(Y_{\hat{\rho}}^{y_0,\hat{u}})\cdot 1_{\{t'\geq\hat{\rho}\}}\right], \tag{2.64}$$

where we have used the fact that under the policy $\hat{u}$ no interventions are executed on the interval $]\hat{\tau}_j,t']$. We now make the following observations:

$$J_1^{(\hat{\rho},u)}[Y_{\tau_j}^{y_0,u}] - \mathbb{E}\left[J_1^{(\hat{\rho},u)}[Y_{\tau_j}^{y_0,u}]\cdot 1_{\{t'<\hat{\rho}\}}\right] = \begin{cases}0, & t'<\hat{\rho}\\ J_1^{(\hat{\rho},u)}[Y_{\tau_j}^{y_0,u}], & t'\geq\hat{\rho}.\end{cases} \quad \forall u \in \mathscr{U}. \tag{2.65}$$

Additionally, using the optimality of the policy $\hat{u}$ against $J_1[y;\hat{\rho},\cdot]$, we have that $J_1^{(\hat{\rho},u)}[Y_{\hat{\tau}_j}^{y_0,u}] \geq J_1^{(\hat{\rho},\hat{u})}[Y_{\hat{\tau}_j}^{y_0,\hat{u}}]$, $\forall u \in \mathscr{U}$. Deducting the terms in expression for $J_1^{(\hat{\rho},u)}[Y_{\hat{\tau}_j}^{y_0,u}] - \mathbb{E}[J_1^{(\hat{\rho},u)}[Y_{\hat{\tau}_j}^{y_0,u}]1_{\{t'<\hat{\rho}\}}]$ in

(2.61) from (2.64) we readily deduce that:

$$0 \leq \mathbb{E}\left[\int_{\hat{\tau}_j}^{t'\wedge\hat{\rho}} (f_1(Y_s^{y_0,\hat{u}}) - f_1(Y_s^{y_0,u}))ds + \int_{\hat{\tau}_j}^{t'\wedge\hat{\rho}} \left(\partial_t J_1^{(\hat{\rho},\hat{u})}[Y_s^{y_0,\hat{u}}] - \partial_t J_1^{(\hat{\rho},u)}[Y_s^{y_0,u}]\right) \cdot 1_{\{t'<\hat{\rho}\}} ds \right.$$
$$+ \int_{\hat{\tau}_j}^{t'\wedge\hat{\rho}} \left(\mathscr{L} J_1^{(\hat{\rho},\hat{u})}[Y_s^{y_0,\hat{u}}] - \mathscr{L} J_1^{(\hat{\rho},u)}[Y_s^{y_0,u}]\right) \cdot 1_{\{t'<\hat{\rho}\}} ds + \sum_{j\geq 1} c_1(\tau_j,\xi_j) \cdot 1_{\{\hat{\tau}_j<\tau_j\leq t'\wedge\hat{\rho}\}}$$
$$\left. + \left(G_1(Y_{\hat{\rho}}^{y_0,\hat{u}}) - G_1(Y_{\hat{\rho}}^{y_0,u})\right) \cdot 1_{\{t'\geq\hat{\rho}\}}\right]. \tag{2.66}$$

Now since (2.66) holds for all $j = 0,1,2,\ldots$ we have in particular for $j = 0$:

$$0 \leq \mathbb{E}\left[\int_{t_0}^{t'\wedge\hat{\rho}} \left(f_1(Y_s^{y_0,\hat{u}}) - f_1(Y_s^{y_0,u})\right) ds + \int_{t_0}^{t'\wedge\hat{\rho}} \left(\partial_t J_1^{(\hat{\rho},\hat{u})}[Y_s^{y_0,\hat{u}}] - \partial_t J_1^{(\hat{\rho},u)}[Y_s^{y_0,u}]\right) \cdot 1_{\{t'<\hat{\rho}\}} ds \right.$$
$$+ \int_{t_0}^{t'\wedge\hat{\rho}} \left(\mathscr{L} J_1^{(\hat{\rho},\hat{u})}[Y_s^{y_0,\hat{u}}] - \mathscr{L} J_1^{(\hat{\rho},u)}[Y_s^{y_0,u}]\right) \cdot 1_{\{t'<\hat{\rho}\}} ds + \sum_{j\geq 1} c_1(\tau_j,\xi_j) \cdot 1_{\{t_0<\tau_j\leq t'\wedge\hat{\rho}\}}$$
$$\left. + \left(G_1(Y_{\hat{\rho}}^{y_0,\hat{u}}) - G_1(Y_{\hat{\rho}}^{y_0,u})\right) \cdot 1_{\{t'\geq\hat{\rho}\}}\right].$$

After taking the limit $t' \downarrow t_0$ we arrive at:

$$f_1(y) + \partial_t J_1^{(\hat{\rho},\hat{u})}[y] + \mathscr{L} J_1^{(\hat{\rho},\hat{u})}[y] \geq f_1(y) + \partial_t J_1^{(\hat{\rho},u)}[y] + \mathscr{L} J_1^{(\hat{\rho},u)}[y], \quad \forall y \in [0,T]\times S, \tag{2.67}$$

as required.

Note also that by similar reasoning as the zero-sum case, we can also deduce that for the pair $(\hat{\rho},\hat{u})$, we have that:

$$f_1(y) + \partial_t J_1^{(\hat{\rho},\hat{u})}[y] + \mathscr{L} J_1^{(\hat{\rho},\hat{u})}[y] \leq 0, \qquad \forall y \in [0,T]\times S, \tag{2.68}$$

where we recall that the inequality arises since it may be optimal for player I to execute an impulse intervention at the initial point. For the player II case (ii), we can straightforwardly adapt the arguments from the zero-sum case. Equations (2.60) and (2.68) are central conditions for equilibrium play and appear in the verification theorem as conditions for equilibrium characterisation.

Having outlined a heuristic argument for the conditions of the verification theorem, we now give a full statement of the theorem. As for the case in Theorem 2.7, the following theorem says that given some pair of solutions to the pair of non-linear PDEs, $i = \{1,2\}$ in (iii), then these solutions coincide with the functions J_i when player i executes their optimal control policy. In particular, as in the zero-sum case, the theorem states that in equilibrium, player I plays a QVI control and player II players an IVI control.

A HJBI Equation for Non Zero-sum Stochastic Differential Games of Control and Stopping with Impulse Controls

Theorem 2.12 (Verification theorem for non zero-sum controller-stopper games with impulse control)

Let $\tau_j, \rho \in \mathscr{T}$ be $\mathscr{F}-$measurable stopping times where $j \in \mathbb{N}$. Suppose that there exist functions $\phi_i \in \mathscr{C}^{1,2}([0,T],S) \cap \mathscr{C}([0,T],\bar{S})$, $i \in \{1,2\}$ such that conditions (T1) - (T4) hold (see Appendix) and additionally:

(i') $\phi_1 \geq \mathscr{M}\phi_1$ on S and $\phi_2 \geq G_2$ on S and the regions D_1 and D_2 are defined by: $D_1 = \{x \in S; \phi_1(\cdot,x) > \mathscr{M}\phi_1(\cdot,x)\}$ and $D_2 = \{x \in S; \phi_2(\cdot,x) > G_2(\cdot,x)\}$ where we refer to D_1 (resp., D_2) as the player I (resp., player II) continuation region.

(ii') $\frac{\partial \phi_1}{\partial s} + \mathscr{L}\phi_1(s,X^{\cdot,u}(s)) + f_1(s,X^{\cdot,u}(s)) \leq \frac{\partial \phi_1}{\partial s} + \mathscr{L}\phi_1(s,X^{\cdot,\hat{u}}(s)) + f_1(s,X^{\cdot,\hat{u}}(s)) \leq 0$ on $S \backslash \partial D_1$ and $\forall u \in \mathscr{U}$.

(iii') $\frac{\partial \phi_i}{\partial s} + \mathscr{L}\phi_i(s,X^{\cdot,\hat{u}}(s)) + f_i(s,X^{\cdot,\hat{u}}(s)) = 0$ in D_1, $i \in \{1,2\}$.

(iv') For $u \in \mathscr{U}$ define $\rho_D = \rho_D^u = \inf\{s > t_0, X^{\cdot,u}(s) \notin D_2\}$ and specifically, $\hat{\rho}_D = \hat{\rho} = \inf\{s > t_0, X^{\cdot,u}(s) \notin D_2\}$.

(v') $X^{\cdot,u}(\tau_S) \in \partial S$, $\mathbb{P}-$a.s. on $\tau_S < \infty$ and $\phi_i(s,X^{\cdot,u}(s)) \to G_i(\tau_S \wedge \rho, X^{\cdot,u}(\tau_S \wedge \rho))$ as $s \to \tau_S^- \wedge \rho^-$, $\mathbb{P}-$a.s., $i \in \{1,2\}$, $\forall u \in \mathscr{U}$.

Put $\hat{\tau}_0 \equiv t_0$ and define $\hat{u} := [\hat{\tau}_j, \hat{\xi}_j]_{j \in \mathbb{N}}$ inductively by $\hat{\tau}_{j+1} = \inf\{s > \tau_j; X^{\cdot,\hat{u}}(s) \notin D_1\} \wedge \tau_S \wedge \hat{\rho}$, then $(\hat{u},\hat{\rho}) \in \mathscr{U} \times \mathscr{T}$ is a Nash equilibrium for the game; that is to say that we have:

$$\phi_1(y) = \sup_{u \in \mathscr{U}} J_1^{(u,\hat{\rho})}[y] = J_1^{(\hat{u},\hat{\rho})}[y], \tag{2.69}$$

$$\phi_2(y) = \sup_{\rho \in \mathscr{T}} J_2^{(\hat{u},\rho)}[y] = J_2^{(\hat{u},\hat{\rho})}[y], \qquad \forall y \in [0,T] \times S. \tag{2.70}$$

Similar to Theorem 2.7, Theorem 2.12 characterises the equilibrium controls for each player in the game. Crucially however, unlike Theorem 2.7, Theorem 2.12 does not impose a zero-sum payoff condition on the game; it therefore applies to a much broader set of economic scenarios.

As with Theorem 2.7, conditions (i') - (iii') of Theorem 2.12 follow from the QVI and IVI conditions motivated in the heuristic analysis. Additionally, the condition $\phi_i \in \mathscr{C}^{1,2}([0,T],S) \cap \mathscr{C}([0,T],\bar{S})$, $i \in \{1,2\}$ is used to allow for the integro-differential operator $\mathscr{L}$ in (ii') and (iii') to be applied in addition to permitting an application of Dynkin's formula which is central to the proof of the theorem. The proof of Theorem 2.12 builds on the zero-sum counterpart (Theorem 2.7). We defer the proof of the theorem to the chapter appendix.

In full analogy to Corollary 2.10.1, we can readily arrive at the following corollary to Theorem 2.12.

Corollary 2.12.1

When each player plays their equilibrium control, the sample space splits into three regions that represent a region in which the controller performs impulse interventions in I_1, a region in which the stopper stops the process I_2 and a region in which no action is taken by either player I_3; moreover the three regions are characterised by the following expressions:

$$I_1 = \{y \in [0,T] \times S : V_1(y) = \mathscr{M}V_1(y),\ \mathscr{L}V_1(y) + f_1(y) \geq 0\},$$
$$I_2 = \{(y \in [0,T] \times S : V_2(y) = G_2(y),\ \mathscr{L}V_2(y) + f_2(y) \geq 0\},$$
$$I_3 = \left\{ y \in [0,T] \times S : V_1(y) < \mathscr{M}V_1(y), V_2(y) < G_1(y);\ \mathscr{L}V_2(y) + f_2(y) = 0, j \in \{1,2\} \right\}.$$

2.7 Examples

In order to demonstrate an application of the theorem, we give a worked example within a financial setting. The first example exemplifies the method by which Theorem 2.7 enables zero-sum stochastic differential games of control and stopping with impulse controls to be solved.

Example 2.13

Consider a system with passive dynamics that are described by a stochastic process X which obeys the following SDE:

$$dX(r) = X(r^-)(\alpha dr + \beta dB(r)), \qquad \forall r \in]0,T], \tag{2.71}$$

where $\alpha, \beta \in \mathbb{R}_{>0}$ are fixed constants, $B(r)$ is a 1-dimensional Brownian motion and $T \in \mathbb{R}_{>0}$ is some finite time horizon. The state process (2.71) is *geometric Brownian motion*. Geometric Brownian motion is widely used to model various financial processes [HB16] and is a particular case of geometric Lévy process (c.f. the optimal liquidity control and lifetime ruin problem) that is restricted to have continuous sample paths.

The state process X is modified by a controller, player I that exercises an impulse control policy $u = [\tau_j, \xi_j] \in \mathscr{U}$. Additionally, at any point $\rho < T$ a second player, player II can choose to stop the process where $\rho \in \mathscr{T}$ is an $\mathscr{F}-$measurable stopping time. The controlled state process therefore evolves according to the following expression $\forall r \in [0,T]$:

$$X(r) = x_0 + \alpha \int_0^{r\wedge\rho} X(s)ds + \beta \int_0^{r\wedge\rho} X(s)dB(s) - \sum_{j\geq 1} (\kappa_1 + (1+\lambda)\xi_j) \cdot 1_{\{\tau_j \leq \rho\wedge r\}},\ \mathbb{P}-\text{a.s.}, \tag{2.72}$$

where $\kappa_1 > 0$ and $\lambda > 0$ are the fixed part and the proportional part of the transaction cost incurred by player I for each intervention (resp.). Player I seeks to choose an admissible impulse control $u = [\tau_j, \xi_j]$ that maximises its reward J where $\{\tau_j\}_{\{j\geq 1\}}$ are intervention times and each $\xi_{j\geq 1} \in \mathscr{Z}$ is an impulse intervention. Player II seeks to choose an $\mathscr{F}-$ measurable stopping time $\rho \in \mathscr{T}$ that minimises the same quantity J which is given by the following expression $\forall (s,x) \in [0,T] \times \mathbb{R}$:

$$J^{\rho,u}[s,x] = \mathbb{E}\left[e^{-\delta(s+\rho)}(X(\rho) - \kappa_2) + \sum_{j\geq 1} e^{-\delta(s+\tau_j)}\xi_j \cdot 1_{\{\tau_j \leq \rho\wedge T\}}\right], \tag{2.73}$$

where $\kappa_2 \in \mathbb{R}_{>0}$ is some fixed constant and $\delta \in]0,1[$ is a discount factor.

An example of a setting for this game is an interaction between a project manager (player I) that seeks to maximise project investments $\{\xi_j\}_{\{j\geq 1\}}$ over some time horizon T, and a second interested party (player II), e.g. a firm owner, that can choose to terminate the project at any point $\rho \leq T$. Whenever the firm owner chooses to terminate the project, they receive a discounted payment of κ_2. The owner however, seeks to terminate the project when the unspent cash flow X is minimal.

The problem is to find a function $\phi \in \mathscr{C}^{1,2}([0,T],\mathbb{R})$ such that

$$\inf_{\rho} \sup_{u} J^{\rho,u}[s,x] = \sup_{u} \inf_{\rho} J^{\rho,u}[s,x] = \phi(s,x), \quad \forall (s,x) \in [0,T] \times \mathbb{R}, \tag{2.74}$$

which by Theorem 2.7 constitutes the value function of the game. By (2.71) and using (1.2), the generator $\mathscr{L}$ for the process X is given by:

$$\mathscr{L}\psi(s,x) = \frac{\partial \psi}{\partial s}(s,x) + \alpha x \frac{\partial \psi}{\partial x}(s,x) + \frac{1}{2}\beta^2 x^2 \frac{\partial \psi}{\partial x^2}(s,x), \tag{2.75}$$

for some test function $\psi \in \mathscr{C}^{1,2}([0,T],\mathbb{R})$.

We wish to firstly derive the functional form of ϕ. Applying (iii) of Theorem 2.7 leads to the *Cauchy-Euler equation* $\mathscr{L}\phi = 0$, (here, $f \equiv 0$ in Theorem 2.7). Following this, we make the following ansatz: $\phi(s,x) = e^{-\delta s}\psi(x)$ where $\psi(x) := ax^c$ for some as yet, undetermined constants $a,c \in \mathbb{R}$. Plugging the ansatz for the function ϕ and using (iii) of Theorem 2.7 into (2.75) immediately gives:

$$-\delta + \alpha c + \frac{1}{2}\beta^2(c-1)c = 0. \tag{2.76}$$

After some manipulation, we deduce that there exist two solutions for c which we denote by c_+ and c_- such that $c_+ > c_-$ with $c_+ > 0$ and $|c_-| > 0$ which are given by the following:

$$c_\pm = -\frac{\alpha - \frac{1}{2}\beta^2}{\beta^2} \pm \frac{1}{\beta^2}\sqrt{(\alpha - \frac{1}{2}\beta^2)^2 + 2\beta^2\delta}. \tag{2.77}$$

We now apply the HJBI equation (iii) of Theorem 2.7 to characterise the function ϕ on the region $D_1 \cap D_2$. Following our ansatz, we observe that by (iii) the following expression for the function ϕ

holds:

$$\phi(s,x) = e^{-\delta s}\psi(x), \qquad \forall (s,x) \in [0,T] \times D_1 \cap D_2, \tag{2.78}$$

$$\psi(x) = (a_1 x^{c_+} + a_2 x^{c_-}), \qquad \forall x \in D_1 \cap D_2, \tag{2.79}$$

where a_1 and a_2 are constants that are yet to be determined and D_1 and D_2 are the continuation regions for player I and player II respectively. In order to determine the constants a_1 and a_2, we firstly observe that $\phi(\cdot,0) = 0$. This then implies that $a_1 = -a_2 := a$. We now deduce that the function ψ is given by the following expression:

$$\psi(x) = a(x^{c_+} - x^{c_-}), \qquad \forall x \in D_1 \cap D_2. \tag{2.80}$$

In order to characterise the function over the entire state space and find the value a, using conditions (i) - (v) of Theorem 2.7, we study the behaviour of the function ϕ given each player's control. Firstly, we consider the player I impulse control problem. In particular, we seek conditions on the impulse intervention applied when $\mathscr{M}\phi = \phi$. To this end, let us firstly conjecture that the player I continuation region D_1 takes the following form:

$$D_1 = \{x \in \mathbb{R}; 0 < x < \tilde{x}\}, \tag{2.81}$$

for some constant $\tilde{x}$ which we shall later determine.

Our first task is to determine the optimal value of the impulse intervention. We now define the following two functions which will be of immediate relevance:

$$\psi_0(x) := a(x^{c_+} - x^{c_-}), \tag{2.82}$$

$$h(\xi) := \psi(x - \kappa_1 - (1+\lambda)\xi) + \xi, \tag{2.83}$$

$$\forall x \in \mathbb{R}, \forall \xi \in \mathscr{Z}.$$

In order to determine the value $\hat{\xi}$ that maximises $\Gamma(x(\tau^-), \xi)$ at the point of intervention, we investigate the first order condition on h i.e. $0 = h'(\xi)$. This implies the following:

$$\psi'(\tilde{x} - \kappa_1 - (1+\lambda)\xi) = \frac{1}{1+\lambda}. \tag{2.84}$$

Using the expression for ψ (2.79) we also observe the following:

$$\psi_0'(x) = c_+ x^{c_+ - 1} - c_- x^{c_- - 1} > 0, \qquad \forall x \in \mathbb{R}, \tag{2.85}$$

$$\psi_0''(x) = c_+(c+-1)x^{c_+ - 2} - c_-(c_- - 1)x^{c_- - 1} < 0, \qquad \forall x < x^\# := \left|\frac{c_-(c_- - 1)}{c_+(c_+ - 1)}\right|^{\frac{1}{c_+ - c_-}}, \tag{2.86}$$

from which we deduce the existence of two points $x^\star, x_\star$ for which the condition $\psi_0'(\cdot) = (1+\lambda)^{-1}$ holds. W.l.o.g. we assume $x^\star > x_\star$. Now by (i) of Theorem 2.7 we require that $e^{-\delta s}\psi(x) = \mathscr{M}e^{-\delta s}\psi_0(x)$ for any $s \in [0,T]$ whenever $x \geq \tilde{x}$ (c.f. D_1 in equation (2.81)), hence we find that:

$$\psi(x) = \psi_0(x_\star) + \hat{\xi}(x), \qquad \forall x \geq \tilde{x}, \tag{2.87}$$

where $x - \kappa_1 - (1+\lambda)\hat{\xi}(x) = x_\star$ from which we readily find that the optimal impulse intervention value is given by:

$$\hat{\xi}(x) = \frac{x - x_\star - \kappa_1}{1+\lambda}, \qquad \forall x \geq \tilde{x}. \tag{2.88}$$

Having determined the optimal impulse intervention and constructed the form of the continuation region for Player I, we now turn to the optimal stopping criterion for Player II. We conjecture that the continuation region for player II, D_2, takes the following form:

$$D_2 = \{x \in \mathbb{R};\ x > \hat{x}\}. \tag{2.89}$$

Now using condition (v) of Theorem 2.7 we observe that $\psi(x) = (x - \kappa_2), \quad \forall x \notin D_2$. Additionally, we recall that by (2.79) we have that $\psi(x) = a(x^{c_+} - x^{c_-})$ for any $x \in D_1 \cup D_2$ where the constant a is to be determined. Putting the above facts together we can give a characterisation for the function ψ:

$$\psi(x) = \begin{cases} a(x^{c_+} - x^{c_-}), & \forall x \in D_1 \cap D_2, \\ (x - \kappa_2), & \forall x \notin D_2, \\ a(x_\star^{c_+} - x_\star^{c_-}) + \dfrac{x - x_\star - \kappa_1}{1+\lambda}, & \forall x \notin D_1, \end{cases} \tag{2.90}$$

where the constants c_+ and c_- are specified in equation (2.77).

Using the facts above, we are now in a position to determine the value of the constants $a, \hat{x}$ and $\tilde{x}$. To do this, we assume the *high contact principle* — a condition that asserts the continuity of the value function at the boundary of the continuation region (for exhaustive discussions on the condition, see [ØS07; Øks90]).

For player II, using (v) it then follows that the following conditions must hold:

$$\phi(\cdot,\hat{x}) = G(\cdot,\hat{x}) \quad \Longrightarrow \quad a(\hat{x}^{c_+} - \hat{x}^{c_-}) = \hat{x} - \kappa_2, \tag{2.91}$$

$$\phi'(\cdot,\hat{x}) = G'(\cdot,\hat{x}) \quad \Longrightarrow \quad a(c_+\hat{x}^{c_+-1} - c_-\hat{x}^{c_--1}) = 1, \tag{2.92}$$

Using (2.91) - (2.92) we deduce that the value a is given by:

$$a = \kappa_2[(1-c_-)\hat{x}^{c_-} - (1-c_+)\hat{x}^{c_+})]^{-1}. \tag{2.93}$$

Additionally, by (2.91) - (2.92) we find that the value of $\hat{x}$ is the solution to the equation:

$$p(\hat{x}) = 0, \tag{2.94}$$

where $p(x) = \kappa_2{}^{-1}[(1-c_-)x^{c_- -1} - (1-c_+)x^{c_+ -1})](c_+ x^{c_+ -1} - c_- x^{c_- -1})^{-1}$. Lastly, we apply the high contact principle to find the boundary of the continuation region D_1. Indeed, continuity at $\tilde{x}$ leads to the following:

$$\psi(\tilde{x}) = \psi_0(x_\star) + \hat{\xi}(\tilde{x}), \implies a(\tilde{x}^{c_+} - \tilde{x}^{c_-}) = a(x_\star^{c_+} - x_\star^{c_-}) + \frac{\tilde{x} - x_\star - \kappa_1}{1+\lambda}, \tag{2.95}$$

from which we find that $\tilde{x}$ is the solution to the following equation:

$$m(\tilde{x}) = 0, \tag{2.96}$$

$$m(x) = x - a(1+\lambda)[x^{c_+} - x^{c_-} + x_\star^{c_-} - x_\star^{c_+}] - x_\star + \kappa_1. \tag{2.97}$$

Equations (2.94) and (2.96) are difficult to solve analytically for the general case but can however, be straightforwardly solved numerically using a root-finding algorithm.

To summarise, the solution is as follows: whenever $X \in D_1 \cap D_2$ neither player intervenes. Player I performs an impulse intervention of size $\hat{\xi}$ given by (2.88) whenever the process reaches the value $\tilde{x}$ and player II terminates the game if the process hits the value $\hat{x}$. The value function for the problem is $\phi(s,x) \equiv e^{-\delta s}\psi(x)$, $\forall (s,x) \in [0,T] \in \mathbb{R}$, where is ψ given by:

$$\psi(x) = \begin{cases} a(x^{c_+} - x^{c_-}), & \forall x \in D_1 \cap D_2, \\ (x - \kappa_2), & \forall x \notin D_2, \\ a(x_\star^{c_+} - x_\star^{c_-}) + \dfrac{x - x_\star - \kappa_1}{1+\lambda}, & \forall x \notin D_1, \end{cases} \tag{2.98}$$

and where the player I and player II continuation regions are given by:

$$D_1 = \{x \in \mathbb{R}; 0 < x < \tilde{x}\}, \tag{2.99}$$

$$D_2 = \{x \in \mathbb{R}; x > \hat{x}\}, \tag{2.100}$$

where the constants $a, \hat{x}$ and $\tilde{x}$ are determined by (2.93), (2.94) and (2.97) respectively and the constants $c_\pm$ are given by (2.77).

We now give an application of the theory developed within the chapter in an investment context. In particular, we are now in a position to apply the results to the optimal liquidity injection investment model. The calculations required to derive the results are lengthy and are therefore delegated to the chapter appendix.

2.7.1 Risk-Minimising Optimal Liquidity Control with Lifetime Ruin (Revisited)

We now revisit the optimal liquidity control and lifetime ruin problem and solve the model. We use the results of the stochastic differential game of impulse control and stopping to solve our model. Before stating results, using (1.2), (2.2) and (2.3), we firstly make the following observation on the stochastic generator $\mathscr{L}^{\theta}$ which is given by the following expression:

$$\begin{aligned}&\mathscr{L}^{\theta}\Phi(s,\cdot)\\ =&erx\frac{\partial\Phi}{\partial x}(s,\cdot)+\Gamma y\frac{\partial\Phi}{\partial y}(s,\cdot)+\frac{1}{2}\sigma_f^2x^2\frac{\partial^2\Phi}{\partial x^2}(s,\cdot)+\frac{1}{2}\pi^2\sigma_I^2y^2\frac{\partial^2\Phi}{\partial y^2}(s,\cdot)+\frac{1}{2}q^2\frac{\partial^2\Phi}{\partial q^2}(s,\cdot)-\sigma_f xq\frac{\partial^2\Phi}{\partial x\partial q}(s,\cdot)\\ &+\int_{\mathbb{R}}\left\{\Phi(s,x+x\gamma_f(z),y,q-q\theta_1(z))-\Phi(s,\cdot)-x\gamma_f(z)\frac{\partial\Phi}{\partial x}(s,\cdot)+q\theta_1(z)\frac{\partial\Phi}{\partial q}(s,\cdot)\right\}\nu(dz).\end{aligned} \tag{2.101}$$

The following result provides a complete characterisation of the investor's value function.

Theorem 2.14

The investor's problem reduces to the following double obstacle variational inequality:

$$\inf\left\{\sup\left[\psi(s,\cdot)-(\kappa_I+\alpha_I(\hat{y}-y)),-\left(\frac{\partial}{\partial_s}+\mathscr{L}^{\hat{\theta}}\right)\psi(s,\cdot)\right],\psi(s,\cdot)-G(s,\cdot)\right\}=0, \tag{2.102}$$

where $G(s,x,y,q)=e^{-\delta s}(g_1xq+\lambda_T+g_2y)$; the constants $g_1\in]0,1]$ and $\lambda_T\geq 0$ represent the fraction of the firm's liquidity process and some fixed amount each received by the investor upon exit respectively, $\hat{y}\in\mathbb{R}$ is an endogenous constant and lastly $q\in\mathbb{R}$ is the value of a stochastic process (later described in Lemma 2.16).

Theorem 2.14 establishes that the complete problem facing the investor can be written as a double obstacle problem from which the value function for the investment can be computed. Explicit solutions for the problem can be derived in cases in which the investor's wealth and firm liquidity processes do not contain jumps (see chapter appendix).

We now state the main theorem of the section.

Theorem 2.15

For the investor's optimal liquidity control and exit problem, the sequence of optimal capital injections $(\hat{\tau},\hat{Z})\equiv[\hat{\tau}_j,\hat{z}_j]_{j\in\mathbb{N}}\equiv\sum_{j\geq 1}\hat{z}_j\cdot 1_{\{\hat{\tau}_j\leq\hat{\rho}\wedge T\}}(s)$ is characterised by the investment times $\{\hat{\tau}_j\}_{j\in\mathbb{N}}$ and magnitudes $\{\hat{z}_j\}_{j\in\mathbb{N}}$ where $[\hat{\tau}_j,\hat{z}_j]_{j\in\mathbb{N}}$ are constructed recursively via the following expressions:

(i) $\hat{\tau}_0\equiv t_0$ and $\hat{\tau}_{j+1}=\inf\{s>\tau_j;Y^{(\hat{\tau},\hat{Z})_{[t_0,s]}}(s)\geq\tilde{y}|s\in\mathscr{T}\}\wedge\hat{\rho}$,

(ii) $\hat{z}_j=\hat{y}-y(\hat{\tau}_j)$, where the duplet $(\tilde{y}, \hat{y}) \in \mathbb{R} \times \mathbb{R}$ consists of endogenous constants.

The investor's non-investment region is given by:

$$D_2 = \{y < \tilde{y} | y \in \mathbb{R}_{>0}\}. \tag{2.103}$$

The optimal exit time $\hat{\rho} \in \mathscr{T}$ for the investor is given by:

$$\hat{\rho} = \inf\{s \geq t_0; (X(s), Q(s)) \notin D_1 | s \in \mathscr{T}\} \wedge \tau_S, \tag{2.104}$$

where Q is a stochastic process (c.f. Lemma 2.16) and the set D_1 (non-stopping region) is defined by:

$$D_1 = \{xq > \omega^\star | x \in \mathbb{R}_{>0}, q \in \mathbb{R}\}, \tag{2.105}$$

where $\omega^\star \in \mathbb{R}$ is an endogenous constant.

Theorem 2.15 says that the investor performs discrete capital injections over a sequence of intervention times $\{\hat{\tau}_k\}_{k \in \mathbb{N}}$ over the time horizon of the problem. The decision to invest is determined by the investor's wealth process — in particular, whenever the investor's wealth process reaches $\tilde{y}$, then the investor performs capital injections of magnitudes $\{\hat{z}_k\}_{k \in \mathbb{N}}$ to increase the firm's liquidity levels in order to provide the firm with maximal liquidity to perform market operations. Therefore, the value $\tilde{y}$ represents an investment threshold. This in turn maximises the liquidity that the investor makes available to the firm whilst the investor remains in the market after which the investor liquidates all investment holdings. However, if the firm's liquidity process exits the region D_1, in order to avoid the prospect of loss of investment, the investor immediately exits the market by liquidating all market holdings in the firm.

The non-stopping region D_1 is defined by (2.105) and the function ψ is the investor's value function. We later provide a full characterisation of the investor's value function and the set of endogenous constants (Proposition 2.17).

From Theorem 2.15 we also arrive at the following result that enables us to state the exact points at which the investor performs an injection, when the investor exits the market and when the investor does nothing.

Corollary 2.15.1

For the optimal liquidity control and lifetime ruin problem, the investor's wealth process X lies within a space that splits into three regions: a region in which the investor performs a capital injection — I_1, a region in which no action is taken — I_2 and lastly a region in which the investor exits the market by selling all firm holdings — I_3. Moreover, the three regions are characterised by the following expressions:

$$
\begin{aligned}
I_1 &= \{y \geq \tilde{y}, qx > \omega^\star | x, y \in \mathbb{R}_{>0}, q \in \mathbb{R}\},\\
I_2 &= \{qx > \omega^\star, y < \tilde{y} | x, y \in \mathbb{R}_{>0}, q \in \mathbb{R}\},\\
I_3 &= \{qx \leq \omega^\star | x \in \mathbb{R}_{>0}, q \in \mathbb{R}\},
\end{aligned}
$$

where $\tilde{y}, \omega^\star$ are fixed endogenous constants.

Lemma 2.16 provides an expression for the process Q:

Lemma 2.16

The process Q is determined by the expression:

$$
Q(s) = Q(0)\exp\left\{\frac{1}{2}\sigma_f^2 s - \sigma_f B_f(s) + \int_0^s \int_{\mathbb{R}} (\ln(1+\hat{\theta}_1(r,z)) - \hat{\theta}_1(r,z))\tilde{N}_f(dr,dz)\right\}, \quad \forall s \in [0,T], \tag{2.106}
$$

where $\hat{\theta}_1$ is a solution to the equation $H(\psi) = 0$ where H is given by:

$$
H(\psi) = \int_{\mathbb{R}} ([\Xi(\psi(z))]^k - 1)\nu(dz), \tag{2.107}
$$

where $\Xi(\psi(z)) := (1 - \hat{\theta}_1(z))(1 + \gamma_f(z)))$ and k is an endogenous constant.

The following result provides a complete characterisation of the investor's value function and the set of endogenous constants within the problem.

Proposition 2.17

The value function $\psi : [0,T] \times \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{R}$ for the investor's joint problem (2.7) - (2.8) is given by:

$$
\psi(s,x,y,q) = \begin{cases} A_1(s,x,y,q), & (\mathbb{R}\backslash\partial D_2) \cap D_1 \\ A_2(s,x,y,q), & \mathbb{R}\backslash\partial D_1 \\ A_3(s,x,y,q), & D_1 \cap D_2 \end{cases} \tag{2.108}
$$

where A_1, A_2, A_3 are given by:

$$
\begin{aligned}
A_1(s,x,y,q) &:= e^{-\delta s} q \left\{ c\left(y^{d_1} - y^{d_2}\right) - q^{-1}\left(\kappa_I + \alpha_I\left(\hat{y} - y\right)\right) + a x^k q^k \right\},\\
A_2(s,x,y,q) &:= e^{-\delta(T \wedge \hat{\rho})}\left(g_1 x q + \lambda_T + g_2 y\right),\\
A_3(s,x,y,q) &:= q e^{-\delta s}\left(c(y^{d_1} - y^{d_2}) + a x^k q^k\right),
\end{aligned}
$$

where the constants $\delta, \kappa_I, \alpha_I$ are the investor's discount factor, the fixed part of the transaction cost and the proportional part of the transaction cost respectively and the constants a, d_1, d_2 and $\omega^\star$ are given by:

$$\omega^\star = \frac{\lambda_T k}{g_1(1-k)} \tag{2.109}$$

$$a = \left(\frac{g_1}{k}\right)^k \left(\frac{\lambda_T k}{1-k}\right)^{1-k} \tag{2.110}$$

$$d_1 = \frac{1}{2} - \frac{1}{\pi^2\sigma_I^2}\left(\sqrt{(\Gamma - \frac{1}{2}\pi^2\sigma_I^2)^2 + 2\pi^2\sigma_I^2\delta} + \Gamma\right) \tag{2.111}$$

$$d_2 = \frac{1}{2} + \frac{1}{\pi^2\sigma_I^2}\left(\sqrt{(\Gamma - \frac{1}{2}\pi^2\sigma_I^2)^2 + 2\pi^2\sigma_I^2\delta} - \Gamma\right). \tag{2.112}$$

The constant k is a solution to the equation $p(k) = 0$ where the function p is given by:

$$p(k) := -\delta + (er - \sigma_f^2)k + k\int_{\mathbb{R}} (\hat{\theta}_1(z) - \gamma_f(z))\nu(dz), \tag{2.113}$$

and lastly the constants $c, \hat{y}_2, \tilde{y}_2$ are determined by the set of equations:

$$\tilde{y}_2^{d_1} - \hat{y}_2^{d_1} + \tilde{y}_2^{d_2} - \hat{y}_2^{d_2} = c^{-1}(\alpha_I(\tilde{y}_2 - \hat{y}_2) - \kappa_I) \tag{2.114}$$

$$d_1\hat{y}_2^{d_1-1} - d_2\hat{y}_2^{d_2-1} = \alpha_I c^{-1} \tag{2.115}$$

$$d_1\tilde{y}_2^{d_1-1} - d_2\tilde{y}_2^{d_2-1} = \alpha_I c^{-1}. \tag{2.116}$$

Proposition 2.17 therefore provides a complete characterisation of the value function for the investor's problem and the endogenous constants appearing in Theorem 2.15 - Lemma 2.16.

Analytic Solvability of the Investment Problem

For the case without jumps, we can compute an exact closed analytic expression for the value function which is presented in Lemma 2.18 (see chapter appendix). For the general case in which jumps are included, an analytic solution is not available. However, numerical approximations to solutions of quasi-variational HJBI equations are accessible through finite difference approximation schemes. In particular, under certain stability conditions, the Howard policy iteration algorithm can be shown to converge to the optimal strategy for the impulse control problem.

Such matters are discussed in [LST03; Kus90] and in particular, the numerical approximation of solutions to the quasi-variational inequality arising from impulse control problems is discussed extensively in [Azi17]. A proof of the convergence the Howard policy iteration algorithm for a general class of problems under which the current impulse control problem falls is discussed is given in [CMS07].

2.8 Chapter Appendix

Proof of Theorem 2.12. Let us fix the player II control $\hat{\rho} \in \mathscr{T}$; we firstly appeal to the Dynkin formula for jump-diffusions, hence:

$$\mathbb{E}[\phi_1(\hat{Y}^{y_0,u}(\tau_{j+1}))] - \mathbb{E}[\phi_1(Y^{y_0,u}(\tau_j))] = \mathbb{E}\left[\int_{\tau_j}^{\tau_{j+1}} \frac{\partial \phi_1}{\partial s} + \mathscr{L}[\phi_1(Y^{y_0,u}(s))]ds\right]. \tag{2.117}$$

Summing (2.117) from $j=0$ to $j=k$ implies that:

$$-\phi_1(y_0) - \sum_{j=1}^{k} \mathbb{E}\left[\phi_1(Y^{y_0,u}(\tau_j)) - \phi_1(\hat{Y}^{y_0,u}(\tau_j^-))\right] + \mathbb{E}\left[\phi_1(\hat{Y}^{y_0,u}(\tau_{k+1}^-))\right] \tag{2.118}$$

$$= \mathbb{E}\left[\int_{t_0}^{\tau_{k+1}} \frac{\partial \phi_1}{\partial s} + \mathscr{L}[\phi_1(Y^{y_0,u_{[t_0,s]}}(s))]ds\right]. \tag{2.119}$$

Now by (ii') we have that:

$$\begin{aligned}&\frac{\partial \phi_1}{\partial s} + \mathscr{L}\phi_1(Y^{y_0,u_{[t_0,s]}}(s))\\ \leq &\frac{\partial \phi_1}{\partial s} + \mathscr{L}\phi_1(Y^{y_0,\hat{u}_{[t_0,s]}}(s)) + (f_1(Y^{y_0,\hat{u}}(s)) - f_1(Y^{y_0,u_{[t_0,s]}}(s))) \leq -f_1(Y^{y_0,u_{[t_0,s]}}(s)).\end{aligned} \tag{2.120}$$

Hence inserting (2.120) into (2.119) yields

$$\begin{aligned}&-\phi_1(y_0) - \sum_{j=1}^{k} \mathbb{E}[\phi_1(Y^{y_0,u}(\tau_j)) - \phi_1(\hat{Y}^{y_0,u}(\tau_j^-))] + \mathbb{E}[\phi_1(\hat{Y}^{y_0,u}(\tau_{k+1}^-))]\\ = &\mathbb{E}\left[\int_{t_0}^{\tau_{k+1}} \frac{\partial \phi_1}{\partial s} + \mathscr{L}[\phi_1(Y^{y_0,u_{[t_0,s]}}(s))]ds\right] \leq -\mathbb{E}\left[\int_{t_0}^{\tau_{k+1}} f_1(Y^{y_0,u_{[t_0,s]}}(s))ds\right].\end{aligned} \tag{2.121}$$

Or equivalently:

$$\phi_1(y_0) + \sum_{j=1}^{k} \mathbb{E}[\phi_1(Y^{y_0,u}(\tau_j)) - \phi_1(\hat{Y}^{y_0,u}(\tau_j^-))] - \mathbb{E}[\phi_1(\hat{Y}^{y_0,u}(\tau_{k+1}^-))] \geq \mathbb{E}\left[\int_{t_0}^{\tau_{k+1}} f_1(Y^{y_0,u_{[t_0,s]}}(s))ds\right]. \tag{2.122}$$

We now use analogous arguments to (2.47) - (2.48). Indeed, by definition of $\mathscr{M}$ we find that:

$$\phi_1(Y^{y_0,u}(\tau_j)) = \phi_1(\hat{\Gamma}(\hat{Y}^{y_0,u}(\tau_j^-),\xi_j)) \leq \mathscr{M}\phi_1(\hat{Y}^{y_0,u}(\tau_j^-)) + c_1(\tau_j,\xi_j)\cdot 1_{\{\tau_j \leq \tau_S \wedge \rho\}}, \tag{2.123}$$

(using the fact that $\inf_{z\in\mathscr{Z}}[\phi_1(\tau',\Gamma(X(\tau'^-),z)) - c_1(\tau',z)\cdot 1_{\{\tau'\leq T\}}] = 0$ whenever $\tau' > \tau_S \wedge \rho$).

After subtracting $\phi_1(\hat{Y}^{y_0,u}(\tau_j^-))$ from both sides of (2.123), summing then negating, we find that:

$$\sum_{j=1}^{k} \mathbb{E}[\phi_1(Y^{y_0,\hat{u}}(\tau_j)) - \phi_1(\hat{Y}^{y_0,\hat{u}}(\tau_j^-))] \tag{2.124}$$

$$\leq \sum_{j=1}^{k} \mathbb{E}[\mathscr{M}\phi_1(\hat{Y}^{y_0,u}(\tau_j^-)) - \phi_1(\hat{Y}^{y_0,\hat{u}}(\tau_j^-))] + \sum_{j=1}^{k} \mathbb{E}[c_1(\tau_j,\xi_j)\cdot 1_{\{\tau_j\leq\tau_S\wedge\rho\}}]. \tag{2.125}$$

After inserting (2.125) into (2.122) we find that:

$$\begin{aligned}\phi_1(y_0) \geq \mathbb{E}\Bigg[&\phi_1(\hat{Y}^{y_0,u}(\tau_{k+1}^-)) - \sum_{j=1}^{k}[\phi_1(Y^{y_0,u}(\tau_j)) - \phi_1(\hat{Y}^{y_0,u}(\tau_j^-))] + \int_{t_0}^{\tau_{k+1}} f_1(Y^{y_0,u_{[t_0,s]}}(s))ds\Bigg]\\ \geq \mathbb{E}\Bigg[&\phi_1(\hat{Y}^{y_0,u}(\tau_{k+1}^-)) - \sum_{j=1}^{k}[\mathscr{M}\phi_1(Y^{y_0,u}(\tau_j^-)) - \phi_1(\hat{Y}^{y_0,u}(\tau_j^-))] - \sum_{j=1}^{k} c_1(\tau_j,\xi_j)\cdot 1_{\{\tau_j\leq\tau_S\wedge\rho\}}\\ &+ \int_{t_0}^{\tau_{k+1}} f_1(Y^{y_0,u_{[t_0,s]}}(s))ds\Bigg].\end{aligned} \tag{2.126}$$

Define $\hat{\rho}_m \equiv \hat{\beta}_m(u) = \hat{\rho}\wedge m; m = 1,2\ldots$. As in the zero-sum case, since the number of interventions in (2.126) is bounded above by $\mu_{[t_0,\hat{\rho}_m\wedge\tau_S]}(u)\wedge m$ for some $m<\infty$ and (2.126) holds for any $k\in\mathbb{N}$, taking the limit as $k\to\infty$ in (2.126) gives:

$$\begin{aligned}\phi_1(y_0) \geq \mathbb{E}\Bigg[&\phi_1(\hat{Y}^{y_0,u}(\tau_{k+1}^-)) - \sum_{j=1}^{\mu_{[t_0,\hat{\rho}_m\wedge\tau_S]}(u)\wedge m}[\mathscr{M}\phi_1(Y^{y_0,u}(\tau_j^-)) - \phi_1(\hat{Y}^{y_0,u}(\tau_j^-))]\\ &+ \int_{t_0}^{\hat{\rho}_m\wedge\tau_s} f_1(Y^{y_0,u_{[t_0,s]}}(s))ds - \sum_{j\geq 1} c_1(\tau_j,\xi_j)\cdot 1_{\{\tau_j\leq\rho\wedge\tau_s\}}\Bigg].\end{aligned} \tag{2.127}$$

Now, $\lim_{m\to\infty}\sum_{j=1}^{\mu_{[t_0,\hat{\rho}_m\wedge\tau_S]}(u)\wedge m}\mathbb{E}[\mathscr{M}\phi_1(Y^{y_0,u}(\tau_j)) - \phi_1(\hat{Y}^{y_0,u}(\tau_j))]$ $= \sum_{j=1}^{\mu_{[\hat{\rho}\wedge\tau_S]}(u)}\mathbb{E}[\mathscr{M}\phi_1(Y^{y_0,u}(\tau_j)) - \phi_1(\hat{Y}^{y_0,u}(\tau_j))] = 0$ and $\lim_{m\to\infty}[\phi_1(\hat{Y}^{y_0,u}(\tau_{\mu_{[t_0,\hat{\rho}_m\wedge\tau_S]}}(u)\wedge m)))] =$ $\phi_1(\hat{Y}^{y_0,u}(\hat{\rho}\wedge\tau_S)) = G_1(\hat{Y}^{y_0,u}(\hat{\rho}\wedge\tau_S))$. Indeed, by (iv') we have that $\lim_{m\to\infty}[\tau_{\mu_{[t_0,\hat{\rho}_m\wedge\tau_S]}(u)}\wedge m)] =$ $\tau_{\mu_{[t_0,\hat{\rho}\wedge\tau_S]}(u)} \equiv \hat{\rho}\wedge\tau_s$. Thus, after taking the limit $k,m\to\infty$ in (2.127) and noting that by definition, $\lim_{m\to\infty}\hat{\rho}_m = \hat{\rho}$, we have that:

$$\phi_1(y_0) \geq \mathbb{E}\left[\int_{t_0}^{\hat{\rho}\wedge\tau_s} f_1(Y^{y_0,u_{[t_0,s]}}(s))ds - \sum_{j\geq 1} c_1(\tau_j,\xi_j)\cdot 1_{\{\tau_j\leq\hat{\rho}\wedge\tau_s\}} + G_1(\hat{Y}^{y_0,u}(\hat{\rho}\wedge\tau_s))\cdot 1_{\{\hat{\rho}\wedge\tau_s<\infty\}}\right]. \tag{2.128}$$

Since this holds for all $u\in\mathscr{U}$ we find:

$$\phi_1(y_0) \geq \sup_{u\in\mathscr{U}}\mathbb{E}\left[G_1(\hat{Y}^{y_0,u}(\hat{\rho}\wedge\tau_s))\cdot 1_{\{\hat{\rho}\wedge\tau_s<\infty\}} + \int_{t_0}^{\hat{\rho}\wedge\tau_s} f_1(Y^{y_0,u_{[t_0,s]}}(s))ds - \sum_{j\geq 1} c_1(\tau_j,\xi_j)\cdot 1_{\{\tau_j\leq\hat{\rho}\wedge\tau_s\}}\right]. \tag{2.129}$$

Hence, we find that

$$\phi_1(y) \geq \sup_{u \in \mathscr{U}} J_1^{(u,\hat{\rho})}[y], \qquad \forall y \in [0,T] \times S. \tag{2.130}$$

Now, applying the above arguments with the controls $(\hat{u},\hat{\rho})$ yields the following equality:

$$\phi_1(y) = \sup_{\rho \in \mathscr{T}} J_1^{(\hat{u},\rho)}[y] = J_1^{(\hat{u},\hat{\rho})}[y], \qquad \forall y \in [0,T] \times S. \tag{2.131}$$

To prove (2.69) - (2.70), we firstly fix $\hat{u} \in \mathscr{U}$ as in (iii'), we again define $\rho_m = \rho \wedge m; m = 1,2\ldots$. Now, by the Dynkin formula for jump-diffusions and by (iii') and (2.117) - (2.119), we have that:

$$\begin{aligned}
&\mathbb{E}[\phi_2(\hat{Y}^{y_0,\hat{u}}(\rho_m^-))] - \phi_2(y_0) - \sum_{j=1}^{\mu_{[t_0,\rho_m]}(\hat{u})} \mathbb{E}[\phi_2(Y^{y_0,\hat{u}}(\hat{\tau}_j)) - \phi_2(\hat{Y}^{y_0,\hat{u}}(\hat{\tau}_j^-))] \\
&= \mathbb{E}\left[\int_{t_0}^{\hat{\tau}_{\mu_{[t_0,\rho_m]}(\hat{u})}} \left(\frac{\partial \phi_2}{\partial s} + \mathscr{L}\phi_2(Y^{y_0,\hat{u}_{[t_0,s]}}(s))ds\right)\right] = -\mathbb{E}\left[\int_{t_0}^{\hat{\tau}_{\mu_{[t_0,\rho_m]}(\hat{u})}} f_2(Y^{y_0,\hat{u}_{[t_0,s]}}(s))ds\right],
\end{aligned}$$

which (as before, similar to (2.39)) and by our choice of $\hat{\xi}_j \in \mathscr{Z}$, implies

$$\begin{aligned}
&\phi_2(y_0) + \sum_{j=1}^{\mu_{[t_0,\rho_m]}(\hat{u})} \mathbb{E}[\mathscr{M}\phi_2(Y^{y_0,\hat{u}}(\hat{\tau}_j^-)) - \phi_2(\hat{Y}^{y_0,\hat{u}}(\hat{\tau}_j^-))] \\
&= \mathbb{E}\left[\phi_2(\hat{Y}^{y_0,\hat{u}}(\hat{\tau}_{\mu_{[t_0,\rho_m]}(\hat{u})})) + \int_{t_0}^{\hat{\tau}_{\mu_{[t_0,\rho_m]}(\hat{u})}} f_2(Y^{y_0,\hat{u}_{[t_0,s]}}(s))ds - \sum_{j=1}^{\mu_{[t_0,\rho_m]}(\hat{u})} c_2(\hat{\tau}_j,\hat{\xi}_j) \cdot 1_{\{\hat{\tau}_j \leq \rho_m\}}\right],
\end{aligned}$$

which we may rewrite as

$$\begin{aligned}
\phi_2(y_0) = \mathbb{E}&\left[\phi_2(\hat{Y}^{y_0,\hat{u}}(\hat{\tau}_{\mu_{[t_0,\rho_m]}(\hat{u})})) + \int_{t_0}^{\hat{\tau}_{\mu_{[t_0,\rho_m]}(\hat{u})}} f_2(Y^{y_0,\hat{u}_{[t_0,s]}}(s))ds - \sum_{j=1}^{\mu_{[t_0,\rho_m]}(\hat{u})} c_2(\hat{\tau}_j,\hat{\xi}_j) \cdot 1_{\{\hat{\tau}_j \leq \tau_{\mu_{[t_0,\rho_m]}(\hat{u})}\}}\right] \\
&- \sum_{j=1}^{\mu_{[t_0,\rho_m]}(\hat{u})} \mathbb{E}[\mathscr{M}\phi_2(Y^{y_0,\hat{u}}(\tau_j^-)) - \phi_2(\hat{Y}^{y_0,\hat{u}}(\tau_j^-))].
\end{aligned} \tag{2.132}$$

Now, since $\mu_{[t_0,\rho_m]}(\hat{u}) \to \mu_{[t_0,\rho\wedge\tau_S]}(\hat{u})$ as $m \to \infty$ and $\lim_{s\to\tau_S} \phi_i(\hat{Y}^{y_0,\cdot}(s)) \to G_i(\hat{Y}^{y_0,\cdot}(\tau_S))$, $i \in \{1,2\}$ using (v) and $\hat{\tau}_{\rho\wedge\tau_S} \equiv \rho \wedge \tau_S$ then using (2.132) and by the Fatôu lemma we find that:

$$\phi_2(y_0) \geq \liminf_{m\to\infty} \mathbb{E}\Bigg[\phi_2(\hat{Y}^{y_0,\hat{u}}(\hat{\tau}_{\mu_{[t_0,\rho_m]}(\hat{u})})) + \int_{t_0}^{\hat{\tau}_{\mu_{[t_0,\rho_m]}(\hat{u})}} f_2(Y^{y_0,\hat{u}_{[t_0,s]}}(s))ds$$
$$- \sum_{j\geq 1} c_2(\hat{\tau}_j,\hat{\xi}_j)\cdot 1_{\{\tau_j\leq\hat{\tau}_{\mu_{[t_0,\rho_m]}(\hat{u})}\}} - \sum_{j=1}^{\mu_{[t_0,\rho_m]}(\hat{u})} \mathbb{E}[\mathscr{M}\phi_2(Y^{y_0,\hat{u}}(\tau_j)) - \phi_2(\hat{Y}^{y_0,\hat{u}}(\tau_j))]]$$
$$= \mathbb{E}\left[G_2(Y^{y_0,u}(\rho\wedge\tau_S))\cdot 1_{\{\rho\wedge\tau_s<\infty\}} + \int_{t_0}^{\rho\wedge\tau_S} f_2(Y^{y_0,u}(s))ds - \sum_{j\geq 1} c_2(\hat{\tau}_j,\hat{\xi}_j)\cdot 1_{\{\hat{\tau}_j\leq\rho\wedge\tau_S\}}\right]. \tag{2.133}$$

Since this holds for all $\rho \in \mathscr{T}$ we find that:

$$\phi_2(y_0) \geq \sup_{\rho\in\mathscr{T}} \mathbb{E}\left[G_2(Y^{y_0,u}(\rho\wedge\tau_S))\cdot 1_{\{\rho\wedge\tau_s<\infty\}} + \int_{t_0}^{\rho\wedge\tau_S} f_2(Y^{y_0,\hat{u}}(s))ds - \sum_{j\geq 1} c_2(\hat{\tau}_j,\hat{\xi}_j)\cdot 1_{\{\hat{\tau}_j\leq\rho\wedge\tau_S\}}\right]. \tag{2.134}$$

Hence, we find that

$$\phi_2(y) \geq \sup_{\rho\in\mathscr{T}} J_2^{(\hat{u},\rho)}[y], \qquad \forall y \in [0,T]\times S. \tag{2.135}$$

Now, applying the above arguments with the controls $(\hat{u},\hat{\rho})$ yields the following equality:

$$\phi_2(y) = \sup_{u\in\mathscr{U}} J_2^{(u,\hat{\rho})}[y] = J_2^{(\hat{u},\hat{\rho})}[y], \qquad \forall y \in [0,T]\times S. \tag{2.136}$$

We therefore observe using (2.136) in conjunction with (2.131) that $(\hat{u},\hat{\rho})$ is a Nash equilibrium and hence the thesis is proven. □

Proof of Theorem 2.15. We prove the theorem by applying Theorem 2.12 to the model.

We separate the proof into components that study the investor's capital injections problem and the investor's optimal stopping individually before combining the calculations to construct the full solution to the problem. The scheme of the proof is as follows:

Step 1: derive the functional form for the value function.

Step 2: characterise the non-investment region and optimal capital injection.

Step 3: characterise the continuation (non-exit) region and exit criterion.

Step 4: compute value function for the complete problem and show that the value function is a solution to a double obstacle problem.

We then finalise with some remarks on the solution and discuss the cases where the underlying processes contain no jumps and the corresponding closed analytic solutions.

We wish to fully characterise the optimal investment strategies for the investor. To put problem (2.5) - (2.6) in terms of the framework of Theorem 2.7, we firstly note that we now seek the triplet $(\hat{\theta},(\tau,Z),\hat{\rho})$ such that

$$J^{\hat{\rho},\hat{u},\hat{\theta}}(t,y_1,y_2,y_3) = \sup_{\rho\in\mathcal{T}}\Big(\inf_{(\tau,Z)\in\Phi}\Big(\inf_{\theta} J^{\rho,u,\theta}(t,y_1,y_2,y_3)\Big)\Big), \tag{2.137}$$

where

$$J^{\rho,u,\theta}(t,y_1,y_2,y_3) = \mathbb{E}\Bigg[-\sum_{m\geq 1} e^{-\delta\tau_m} z_m \cdot 1_{\{\tau_m\leq\tau_S\wedge\rho\}} \\ + e^{-\delta(\tau_S\wedge\rho)}\Big(g_1 Y_1{}^{t,y_1,(\tau,Z)}_{\tau_S\wedge\rho} Y_3^{t,y_3,(\theta_0,\theta_1)} + \lambda_T - g_2 Y_2{}^{t,y_2,(\tau,Z)}_{\tau_S\wedge\rho}\Big)\Bigg], \tag{2.138}$$

and $\theta\equiv(\theta_0,\theta_1):[0,T]\times\Omega\times[0,T]\times\Omega\to\Theta\subset\mathbb{R}^2$ and the dynamics of the state processes $Y:=(Y_0,Y_1,Y_2,Y_3)$ are expressed via the following:

$$dY_0(s)\equiv dt,\ \forall s\in[0,T], \tag{2.139}$$

$$dY_1(s)\equiv dX_s^{t_0,x_0,(\tau,Z)};\ X_{t_0}^{t_0,x_0,\cdot}=y_1,\ \forall s\in[0,T], \tag{2.140}$$

$$dY_2(s)\equiv dY_s^{t_0,y_0,(\tau,Z)};\ Y_{t_0}^{t_0,y_0,\cdot}=y_2,\ \forall s\in[0,T], \tag{2.141}$$

$$dY_3^{t_0,y_3,(\theta_0,\theta_1)}(s) = -Y_3^{t_0,y_3,(\theta_0,\theta_1)}(s)\left[\theta_0(s)dB_f(s)+\int_{\mathbb{R}}\theta_1(s,z)\tilde{N}_f(ds,dz)\right], \tag{2.142}$$

so that Y_1,Y_2 are processes which represent the firm liquidity processes and the investor's wealth process respectively. The processes Y_0 and $Y_3^{t_0,y_3,(\theta_0,\theta_1)}$ represent time and market adjustments to the investor's wealth process respectively.

In this section, we suppress the indices on the process and write $Y_3\equiv Y_3^{t_0,y_3,(\theta_0,\theta_1)}$. We also occasionally employ the following shorthands: $\frac{\partial\phi}{\partial y_i}\equiv\partial_{y_i}\phi$, $\frac{\partial^2\phi}{\partial y_i\partial y_j}\equiv\partial_{y_i,y_j}\phi$ for $i\in\{0,1,2,3\}$.

Lastly, we have the following relations for the state process coefficients:

$$\mu(\cdot,y_2)=\Gamma y_2, \quad \mu(\cdot,y_1)=ery_1. \tag{2.143}$$

We restrict ourselves to the case when:

$$\gamma_I(\cdot,y_2)=0. \tag{2.144}$$

For the case that includes jumps (i.e $\gamma_I\not\equiv 0,\theta_1\not\equiv 0$) we impose a condition on the firm's rate of expenditure relative to the firm's return on capital, in particular, we assume the following condition holds:

$$e > \left(\frac{r}{\sigma_f^2}\right)^{-1}. \tag{2.145}$$

The continuation regions D_2 and D_1 for the controller and the stopper respectively now take the form:

$$D_2 = \{y \in [0,T] \times \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R}; \psi(y) < \mathscr{M}\psi(y)\}, \tag{2.146}$$

$$D_1 = \{(y_0,y_1,y_2) \in [0,T] \times \mathbb{R}_{>0} \times \mathbb{R}_{>0}; \psi(y_0,y_1,y_2,\cdot) - G(y_0,y_1,y_2) > 0\}, \tag{2.147}$$

where given some $\phi \in \mathscr{H}$ the intervention operator $\mathscr{M} : \mathscr{H} \to \mathscr{H}$ is given by:

$$\mathscr{M}\phi(y_0,y_1,y_2,y_3) = \inf_{\zeta \in \mathscr{Z}} \{\phi(y_0,y_1,y_2-\zeta,y_3) - (\kappa_I + \alpha_I \zeta), \zeta > 0\}, \tag{2.148}$$

for all $y \in [0,T] \times \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R}$ and the stopping time $\hat{\rho}$ is defined by:

$$\hat{\rho} = \inf\{y_0 > t_0; (y_0,y_1,y_2) \notin D_1 | y_0 \in \mathscr{T}\} \wedge \tau_S. \tag{2.149}$$

Step 1. Our first task is to characterise the value of the game. Now by conditions (ii) - (vi) of Theorem 2.7, we observe that the following expressions must hold:

$$\psi(y_0,y_1,y_2,y_3) = e^{-\delta y_0}(g_1 y_1 y_3 + \lambda_T + g_2 y_2),\ \forall (y_0,y_3) \in [0,T] \times \mathbb{R}; \forall (y_1,y_2) \in \mathbb{R}^2_{>0} \backslash D_1 \text{ (condition (ii))} \tag{2.150}$$

$$\psi(y_0,y_1,y_2,y_3) \geq e^{-\delta y_0}(g_1 y_1 y_3 + \lambda_T + g_2 y_2), \forall y \in [0,T] \times \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R}, \qquad \text{(condition (v))} \tag{2.151}$$

$$\frac{\partial \psi}{\partial y_0} + \mathscr{L}^\theta \psi(y) \geq 0, \qquad \forall y \in [0,T] \times \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R}, \qquad \text{(condition (vi))} \tag{2.152}$$

$$\inf_\theta \left\{\frac{\partial \psi}{\partial y_0} + \mathscr{L}^\theta \psi(y)\right\} = 0, \qquad \forall y \in [0,T] \times D_1 \times D_2 \times \mathbb{R}, \qquad \text{(condition (vi))} \tag{2.153}$$

where the condition labels refer to the conditions of Theorem 2.7. Now using (2.139) -

(2.142) we find that the generator is given by the following expression:

$$\begin{aligned}\mathscr{L}^{\theta}\psi(y) =& ery_1\frac{\partial\psi}{\partial y_1}(y)+\Gamma y_2\frac{\partial\psi}{\partial y_2}(y)+\frac{1}{2}\sigma_f^2y_1^2\frac{\partial^2\psi}{\partial {y_1}^2}(y)+\frac{1}{2}\pi^2\sigma_I^2y_2^2\frac{\partial^2\psi}{\partial {y_2}^2}(y)\\ &+\frac{1}{2}\theta_0^2y_3^2\frac{\partial^2\psi}{\partial y_3^2}(y)-\theta_0y_1y_3\sigma_f\frac{\partial^2\psi}{\partial y_1\partial y_3}(y)+\int_{\mathbb{R}}\Big\{\psi(s,y_1+y_1\gamma_f(z),y_2,y_3-y_3\theta_1(z))\\ &-\psi(y)-y_1\gamma_f(z)\frac{\partial\psi}{\partial y_1}(y)+y_3\theta_1(z)\frac{\partial\psi}{\partial y_3}(y)\Big\}\nu(dz),\end{aligned} \tag{2.154}$$

By (2.153) and (2.154) we readily deduce that the first order condition on $\hat{\theta}_0$ is given by the following expression:

$$\hat{\theta}_0y_3^2\frac{\partial^2\psi}{\partial y_3^2}-y_1y_3\sigma_f\frac{\partial^2\psi}{\partial y_1\partial y_3}=0, \tag{2.155}$$

which after some simple manipulation we find that:

$$\hat{\theta}_0=y_1y_3^{-1}\sigma_f\partial^2_{y_1,y_3}\psi(\partial^2_{y_3}\psi)^{-1}. \tag{2.156}$$

Now by (vi) of Theorem 2.7 we have that on D_1:

$$\frac{\partial\psi}{\partial s}+\mathscr{L}^{\theta}\psi=0, \tag{2.157}$$

(here $f=0$) which implies that:

$$0=\frac{\partial\psi}{\partial y_0}(y)+ery_1\frac{\partial\psi}{\partial y_1}(y)+\Gamma y_2\frac{\partial\psi}{\partial y_2}(y)+\frac{1}{2}\sigma_f^2y_1^2\frac{\partial^2\psi}{\partial {y_1}^2}(y)+\frac{1}{2}\pi^2\sigma_I^2y_2^2\frac{\partial^2\psi}{\partial {y_2}^2}(y) \tag{2.158}$$

$$\begin{aligned}&-\frac{1}{2}\sigma_f^2y_1^2\left[\frac{\partial^2\psi}{\partial y_1\partial y_3}(y)\right]^2\Big(\frac{\partial^2\psi}{\partial y_3^2}(y)\Big)^{-1}+\int_{\mathbb{R}}\Big\{\psi(y_0,y_1+y_1\gamma_f(z),y_2,y_3-y_3\theta_1(z))\\ &-\psi(y)-y_1\gamma_f(z)\frac{\partial\psi}{\partial y_1}(y)+y_3\theta_1(z)\frac{\partial\psi}{\partial y_3}(y)\Big\}\nu(dz).\end{aligned} \tag{2.159}$$

Let us try as our candidate function $\psi(y)=e^{-\delta y_0}y_3(\phi_2(y_2)+\phi_\omega(\omega))$, where $\omega:=y_1y_3$. Then after plugging our expression for ψ into (2.159) we find that:

$$\begin{aligned}0=&-\delta[\phi_2(y_2)+\phi_\omega(\omega)]+er\omega\phi_\omega'(\omega)+\Gamma\phi_2(y_2)+\frac{1}{2}\sigma_f^2\omega^2\phi_\omega''(\omega)+\\ &+\frac{1}{2}\pi^2\sigma_I^2y_2^2\phi_2''(y_2)+\frac{1}{2}\hat{\theta}_0^2\omega(2\phi_\omega'(\omega)+\omega\phi_\omega''(\omega))-\hat{\theta}_0\sigma_f\omega(2\phi_\omega'(\omega)+\omega\phi_\omega''(\omega))\\ &+\int_{\mathbb{R}}\Big\{(1-\theta_1(z))\big[\phi_2(y_2)+\phi_\omega(\omega(1+\gamma_f(z))(1-\theta_1(z)))\big]\\ &-(1-\theta_1(z))[\phi_2(y_2)+\phi_\omega(\omega)]+\omega\phi_\omega(\omega)'(\theta_1(z)-\gamma_I(z))\Big\}\nu(dz),\end{aligned} \tag{2.160}$$

and (2.156) now becomes:

$$\hat{\theta}_0 = \sigma_f \frac{y_1(2y_3\phi'_\omega(\omega) + y_3\omega\phi''_\omega(\omega)}{y_3(2y_1\phi'_\omega(\omega) + y_1\omega\phi''_\omega(\omega))} = \sigma_f. \tag{2.161}$$

Hence, substituting (2.161) into (2.160) we find that:

$$\begin{aligned} 0 = &- \delta[\phi_\omega(\omega) + \phi_2(y_2)] + er + \sigma_f^2)\omega\phi'_\omega(\omega) + \Gamma y_2\phi'_2(y_2) + \frac{1}{2}\pi^2\sigma_I^2 y_2^2\phi''_2(y_2) \\ &+ \int_{\mathbb{R}} \Big\{(1-\theta_1(z))\big[\phi_2(y_2) + \phi_\omega(\omega(1+\gamma_f(z))(1-\theta_1(z)))\big] \\ &- (1-\theta_1(z))[\phi_2(y_2) + \phi_\omega(\omega)] + \omega\phi'_\omega(\omega)(\theta_1(z) - \gamma_I(z))\Big\}\nu(dz). \end{aligned} \tag{2.162}$$

Additionally, our first order condition on $\hat{\theta}_1$ becomes:

$$\int_{\mathbb{R}} \Big\{\phi_\omega(\omega\Xi(\hat{\theta}_1(z))) + \omega\Xi(\hat{\theta}_1(z))\phi'_\omega(\omega\Xi(\hat{\theta}_1(z))) - \phi_\omega - \omega\phi'_\omega\Big\}\nu(dz) = 0, \tag{2.163}$$

where $\Xi(\hat{\theta}_1(z)) := (1-\hat{\theta}_1(z))(1+\gamma_f(z)))$.

We can decouple (2.162) after which we find that when $(y_1, y_2) \in D_1 \times D_2$ we have that:

i)

$$\begin{aligned} &\int_{\mathbb{R}} \Big\{(1-\theta_1(z))\big[\phi_\omega(\omega(1+\gamma_f(z))(1-\theta_1(z)))\big] - (1-\theta_1(z))\phi_\omega(\omega) \\ &\quad + \omega\phi'_\omega(\omega)(\theta_1(z) - \gamma_I(z))\Big\}\nu(dz) - \delta\phi_\omega(\omega) + (er - \sigma_f^2)\omega\phi'_\omega(\omega) = 0, \end{aligned} \tag{2.164}$$

ii) $\quad -\delta\phi_2(y_2) + \Gamma y_2\phi'_2(y_2) + \frac{1}{2}\pi^2\sigma_I^2 y_2^2\phi''_2(y_2) = 0.$

We can solve the Cauchy-Euler equation ii) — after performing some straightforward calculations we find that:

$$\phi_2(y_2) = c_1 y_2^{d_1} + c_2 y_2^{d_2}, \tag{2.165}$$

for some (as yet undetermined) constants c_1 and c_2. The constants d_1 and d_2 are given by:

$$d_1 = \frac{1}{2} - \frac{1}{\pi^2\sigma_I^2}\left(\sqrt{(\Gamma - \frac{1}{2}\pi^2\sigma_I^2)^2 + 2\pi^2\sigma_I^2\delta} + \Gamma\right) \tag{2.166}$$

$$d_2 = \frac{1}{2} + \frac{1}{\pi^2\sigma_I^2}\left(\sqrt{(\Gamma - \frac{1}{2}\pi^2\sigma_I^2)^2 + 2\pi^2\sigma_I^2\delta} - \Gamma\right). \tag{2.167}$$

Since $\psi(0) = Y_2(0) = 0$, we easily deduce that $c_2 = -c_1$, after which we deduce that ϕ_2 is given by the following expression:

$$\phi_2(y_2) = c(y_2^{d_1} - y_2^{d_2}), \tag{2.168}$$

where $c := c_1 = -c_2$ is some as of yet undetermined constant.

To obtain an expression for the function ϕ_ω, in light of (2.164) we conjecture that ϕ_ω takes the form:

$$\phi_\omega = a\omega^k, \tag{2.169}$$

where a and k are some constants. Using (2.169) and (2.164), we find the following:

$$\mathscr{L}^q\phi_\omega(\omega) = a\omega^k p(k), \tag{2.170}$$

where the operator $\mathscr{L}^q$ is defined by the following expression for some function $\phi \in \mathscr{C}^{1,2}$:

$$\begin{aligned}&\mathscr{L}^q\phi(w) := -\delta\phi_w(w) + (er - \sigma_f^2)w\phi_w'(w)\\ &+\int_{\mathbb{R}}\Big\{(1-\hat{\theta}_1(z))\big[\phi(w(1+\gamma_f(z))(1-\hat{\theta}_1(z))) - \phi(w)\big] + w\phi'(w)(\hat{\theta}_1(z)-\gamma_f(z))\Big\}\nu(dz),\end{aligned} \tag{2.171}$$

and $p(k)$ is defined by:

$$p(k) := -\delta + (er-\sigma_f^2)k + \int_{\mathbb{R}}\Big\{(1-\hat{\theta}_1(z))[\Xi(c\hat{\theta}_1(z))^k - 1] + k(\hat{\theta}_1(z)-\gamma_f(z))\Big\}\nu(dz), \tag{2.172}$$

where $\Xi(\hat{\theta}_1(z)) := (1-\hat{\theta}_1(z))(1+\gamma_f(z)))$.

Note that using (2.169), the first order condition on $\hat{\theta}_1$ (c.f. (2.163)) now becomes:

$$\int_{\mathbb{R}}(\Xi^k - 1)\nu(dz) = 0. \tag{2.173}$$

Hence using (2.173), (2.172) becomes:

$$p(k) = -\delta + (er-\sigma_f^2)k + k\int_{\mathbb{R}}(\hat{\theta}_1(z)-\gamma_f(z))\nu(dz). \tag{2.174}$$

We now make the following observations:

$$p(0) = -\delta < 0, \quad p(b) > 1-\delta + b\int_{\mathbb{R}}(\theta_1(z)-\gamma_f(z))\nu(dz) \geq 0, \mathbb{P}-\text{a.s.}, \tag{2.175}$$

for any $b > (er-\sigma_f^2)^{-1}$ using condition (2.145) and the fact that $\delta \in]0,1[$. We therefore deduce the existence of a value $z \in]0,1[$ such that $p(z) = 0$. We now conclude that:

$$\phi_\omega(\omega) = a\omega^k, \tag{2.176}$$

where a is an arbitrary constant and where k is a solution to the equation:

$$p(k) = 0. \tag{2.177}$$

We now split the analysis into two parts in which we study the investor's capital injections (impulse control) problem and the investor's optimal stopping problem separately. We then later recombine the two problems to construct our solution to the problem.

Step 2: The Investor's Capital Injections Problem

We firstly tackle the investor's capital injections problem, in particular we wish to ascertain the form of the function ϕ_2 and describe the intervention region and the optimal size of the investor's capital injections.

Our ansatz for the continuation region D_2 is that it takes the form:

$$D_2 = \{y_2 < \tilde{y}_2, | y_2, \tilde{y}_2 \in \mathbb{R}\}. \tag{2.178}$$

Therefore by (ii) of Theorem 2.7 for $y_2 \notin D_2$ we have that:

$$\begin{aligned} \psi(s,y) = \mathscr{M}\psi(s,y) &= \inf\{\psi(s,y_1,y_2-\zeta,y_3) + (\kappa_I + \alpha_I\zeta), \zeta > 0\} \\ \iff \quad \phi_2(y_2) &= \inf\{\phi_2(y_2-\zeta) + (\kappa_I + \alpha_I\zeta), \zeta > 0\}. \end{aligned} \tag{2.179}$$

Let us define the function h by the following expression:

$$h(\zeta) = \phi_2(y_2-\zeta) - (\kappa_I + \alpha_I\zeta). \tag{2.180}$$

We therefore see that the first order condition for the minimum $\hat{\zeta}(y_2) \in \mathscr{Z}$ of the function h is:

$$h'(\zeta) = \phi_2'(y_2 - \hat{\zeta}) = \alpha_I. \tag{2.181}$$

Let us now consider a unique point $\hat{y}_2 \in]0, \tilde{y}_2[$ such that

$$\phi_2'(\hat{y}_2) = \alpha_I, \tag{2.182}$$

and

$$\hat{y}_2 = y_2 - \hat{\zeta}(y_2) \text{ or } \zeta(\hat{y}_2) = \hat{y}_2 - y_2. \tag{2.183}$$

After imposing a continuity condition at $y_2 = \tilde{y}_2$, by (2.179) we have that:

$$\phi_2(\tilde{y}_2) = \phi_{2,0}(\hat{y}_2) - (\kappa_I + \alpha_I(\hat{y}_2 - \tilde{y}_2)), \tag{2.184}$$

where $\phi_{2,0}(y_2) = \phi_2(y_2)$ on D_2 and where ϕ_2 is given by (2.168). Additionally, by construction of $\tilde{y}_2$ we have that:

$$\phi_2'(\tilde{y}_2) = \alpha_I. \tag{2.185}$$

Hence we deduce that the function ϕ_2 is given by the following expression:

$$\phi_2(y_2) = \begin{cases} c\left(y_2^{d_1} - y_2^{d_2}\right) - \left(\kappa_I + \alpha_I\left(\hat{y}_2 - y_2\right)\right), & y_2 \geq \tilde{y}_2 \\ c\left(y_2^{d_1} - y_2^{d_2}\right), & y_2 < \tilde{y}_2, \end{cases} \tag{2.186}$$

where d_1 and d_2 are given by (2.166) - (2.167).

In order to compute the constants a, $\hat{y}_2$ and $\tilde{y}_2$, we use the system of equations (2.182), (2.184) and (2.185).

Step 3: The Investor's Optimal Stopping Problem

Our ansatz for the continuation region D_1 is that it takes the form:

$$D_1 = \{\omega = y_1 y_3 > y_1^\star y_3^\star = \omega^\star | y_1, y_1^\star \in \mathbb{R}; y_3, y_3^\star \in \mathbb{R}\}. \tag{2.187}$$

If we assume that the *high contact principle*[6] holds, in particular if we have differentiability at $\omega^\star$ then, using (2.169) we obtain the following equations:

(i) $$a\omega^{*k} = g_1\omega^\star + \lambda_T$$

(ii) $$ak\omega^{*k-1} = g_1.$$

Since the system of equations (i) - (ii) completely determine the constants a and $\omega^\star$, we can compute the values of $\omega^\star$ and a in (2.169), after which we find:

$$\omega^\star = \frac{\lambda_T k}{g_1(1-k)}, \quad a = \left(\frac{g_1}{k}\right)^k \left(\frac{\lambda_T k}{1-k}\right)^{1-k}. \tag{2.188}$$

We are now in a position to prove Lemma 2.16; in particular, using (2.142) and (2.161) we now see that the process Y_3 is determined by the expression:

$$dY_3(s) = -\left[\sigma_f Y_3 dB_f(s) + Y_3(s)\int_{\mathbb{R}} \hat{\theta}_1(s,z)\tilde{N}_f(ds,dz)\right], \quad \mathbb{P}-a.s., \tag{2.189}$$

where $\hat{\theta}_1$ is determined by the equation (c.f. (2.173)):

$$\int_{\mathbb{R}} (\Xi^k(\hat{\theta}_1(z)) - 1)\nu(dz) = 0, \tag{2.190}$$

[6]Recall that the high contact principle is a condition that asserts the continuity of the value function at the boundary of the continuation region. In the current case this implies that $\phi_\omega(\omega)|^{\omega=\omega^\star} = G(\omega)|^{\omega=\omega^\star}$, $\frac{\partial}{\partial\omega}\phi_\omega(\omega)|^{\omega=\omega^\star} = \frac{\partial}{\partial\omega}G(\omega)|^{\omega=\omega^\star}$.

where $\Xi(\hat{\theta}_1(z)) := (1-\hat{\theta}_1(z))(1+\gamma_f(z)))$.

Using Itō's formula for Itō-Lévy processes, we can solve (2.190), moreover since $\mathbb{E}_{\mathbb{Q}}\left[X+\lambda_T\right] = \mathbb{E}_{\mathbb{P}}\left[\left(X+\lambda_T\right)Y_3\right]$, (c.f. (2.6)), the process Y_3 represents the Radon-Nikodym derivative of the measure $\mathbb{Q}$ with respect to the measure $\mathbb{P}$ (i.e. $Y_3(s) = \frac{d(\mathbb{Q}|\mathscr{F}_s)}{d(\mathbb{P}|\mathscr{F}_s)}$). Combining these two statements and denoting Y_3 by Q immediately gives the result stated in Lemma 2.16. □

Step 4: The Investor's Value Function and Joint Problem

Our last task is to combine the results together and fully characterise the investor's value function. We firstly note that putting the above results together yields the following double obstacle variational inequality:

$$\sup\left\{\inf\left[\psi(y)-(\kappa_I+\alpha_I(\hat{y}_2-y_2)),-\left(\frac{\partial}{\partial_{y_0}}+\mathscr{L}^{\hat{\theta}}\right)\psi(y)\right],\psi(y)-G(y)\right\}=0, \tag{2.191}$$

where $y=(y_0,y_1,y_2,y_3)$ and $G(y)=e^{-\delta y_0}(g_1y_1y_3+g_2y_2+\lambda_T)$ and the investor's stopping time is given by:

$$\hat{\rho}=\inf\{s\geq t_0; Y_1(s)Y_3(s)\notin D_1|s\in\mathscr{T}\}, \tag{2.192}$$

where the stochastic generator $\mathscr{L}^{\hat{\theta}}$ acting on a test function $\psi\in\mathscr{C}^{1,2}$ is defined via the following expression:

$$\begin{aligned}\mathscr{L}^{\theta}\psi(y)=&\,ery_1\frac{\partial\psi}{\partial y_1}(y)+\Gamma y_2\frac{\partial\psi}{\partial y_2}(y)+\frac{1}{2}\sigma_f^2y_1^2\frac{\partial^2\psi}{\partial {y_1}^2}(y)+\frac{1}{2}\pi^2\sigma_I^2y_2^2\frac{\partial^2\psi}{\partial {y_2}^2}(y)\\&+\frac{1}{2}\theta_0^2y_3^2\frac{\partial^2\psi}{\partial y_3^2}(y)-\theta_0y_1y_3\sigma_f\frac{\partial^2\psi}{\partial y_1\partial y_3}(y)+\int_{\mathbb{R}}\Big\{\psi(y_0,y_1+y_1\gamma_f(z),y_2,y_3-y_3\theta_1(z))\\&-\psi(y)-y_1\gamma_f(z)\frac{\partial\psi}{\partial y_1}(y)+y_3\theta_1(z)\frac{\partial\psi}{\partial y_3}(y)\Big\}\nu(dz),\end{aligned}$$

and where the function $\hat{\theta}_1$ satisfies the first order condition (2.190).

The double obstacle problem in (2.102) characterises the value for the game, this proves Theorem 2.14.

Proposition 2.17 provides a full expression of the value function for the investor's problem. To prove Proposition 2.17, we need to collect the results on the constituent functions of the value function and assemble the complete function. Combining (2.169) and (2.186) and using (2.150) shows that the value function ψ is given by:

$$\psi(y) = \begin{cases} A_1(y), & (\mathbb{R}\backslash\partial D_2)\cap D_1 \\ A_2(y), & \mathbb{R}\backslash\partial D_1 \\ A_3(y), & D_1\cap D_2, \end{cases} \tag{2.193}$$

where

$$\begin{aligned} A_1(y) &:= e^{-\delta y_0} y_3 \left(c\left(y_2^{d_1} - y_2^{d_2} \right) - y_3^{-1}\left[\kappa_I + \alpha_I(\hat{y}_2 - y_2)\right] + a y_1^k y_3^k \right), \\ A_2(y) &:= e^{-\delta(T\wedge\hat{\rho})}\left(g_1 y_1 y_3 + \lambda_T + g_2 y_2 \right), \\ A_3(y) &:= y_3 e^{-\delta y_0}\left(a y_1^k y_3^k + c(y_2^{d_1} - y_2^{d_2}) \right). \end{aligned}$$

The constants $a, \omega^\star$ are given by:

$$\omega^\star = \frac{\lambda_T k}{g_1(1-k)}, \quad a = \left(\frac{g_1}{k}\right)^k \left(\frac{\lambda_T k}{1-k}\right)^{1-k}, \tag{2.194}$$

and the constants d_1 and d_2 are given by:

$$d_1 = \frac{1}{2} - \frac{1}{\pi^2\sigma_I^2}\left(\sqrt{(\Gamma - \frac{1}{2}\pi^2\sigma_I^2)^2 + 2\pi^2\sigma_I^2\delta} + \Gamma \right) \tag{2.195}$$

$$d_2 = \frac{1}{2} + \frac{1}{\pi^2\sigma_I^2}\left(\sqrt{(\Gamma - \frac{1}{2}\pi^2\sigma_I^2)^2 + 2\pi^2\sigma_I^2\delta} - \Gamma \right). \tag{2.196}$$

The constants $c, \hat{y}_2, \tilde{y}_2$ are determined by the set of equations:

$$\tilde{y}_2^{d_1} - \hat{y}_2^{d_1} + \tilde{y}_2^{d_2} - \hat{y}_2^{d_2} = c^{-1}(\alpha_I(\tilde{y}_2 - \hat{y}_2) - \kappa_I) \tag{2.197}$$

$$d_1\hat{y}_2^{d_1-1} - d_2\hat{y}_2^{d_2-1} = \alpha_I c^{-1} \tag{2.198}$$

$$d_1\tilde{y}_2^{d_1-1} - d_2\tilde{y}_2^{d_2-1} = \alpha_I c^{-1}, \tag{2.199}$$

and the constant k is a solution to the equation $p(k) = 0$ where the function p is given by[7]:

$$p(k) := -\delta + (er - \sigma_f^2)k + k\int_{\mathbb{R}} (\hat{\theta}_1(z) - \gamma_f(z))\nu(dz), \tag{2.200}$$

where $\hat{\theta}_1$ is a solution to (2.107). This proves Proposition 2.17. □

[7]For the case that includes jumps in the firm liquidity process, we assume that condition (2.145) holds.

2.8.1 Further remarks on solvability

The Case $\gamma_I = 0, \theta_1 = 0$

If the investor's liquidity process contains no jumps (i.e. $\gamma_f \equiv 0$ and $\theta_1 \equiv 0$ in (2.2) and (2.142) (resp.)) then we can obtain closed analytic solutions for the parameters of the function ϕ_ω. Indeed, when $\gamma_I \equiv 0$ and $\theta_1 \equiv 0$ using (2.200), we see that the expression for $p(k)$ reduces to:

$$p_0(k) := p(k)\big|^{\gamma_I=0,\theta_1=0} = -\delta + (er - \sigma_f^2)k. \tag{2.201}$$

We can therefore solve for k after which we find that the function ϕ_ω is given by:

$$\phi_\omega(\omega) = a\omega^k, \tag{2.202}$$

where $k = \delta(er - \sigma_f^2)^{-1}$. The constants $a, \omega^\star, d_1, d_2$ are determined by (2.194) - (2.196) and the constants $c, \hat{y}, \tilde{y}$ are determined by the set of equations:

$$\tilde{y}^{d_1} - \hat{y}^{d_1} + \tilde{y}^{d_2} - \hat{y}^{d_2} = c^{-1}(\alpha_I(\tilde{y} - \hat{y}) - \kappa_I) \tag{2.203}$$

$$d_1\hat{y}^{d_1-1} - d_2\hat{y}^{d_2-1} = \alpha_I c^{-1} \tag{2.204}$$

$$d_1\tilde{y}^{d_1-1} - d_2\tilde{y}^{d_2-1} = \alpha_I c^{-1}. \tag{2.205}$$

We therefore arrive at the following result which provides a complete characterisation of the value function in terms of a closed analytic solution for the investor's problem when the liquidity process contains no jumps:

Lemma 2.18

For the case in which the investor's liquidity process contains no jumps (i.e. $\gamma_f \equiv 0$ in (2.2)), the function ψ is given by the following:

$$\psi(y) = \begin{cases} A_1'(y), & (\mathbb{R}\backslash\partial D_2) \cap D_1 \\ A_2'(y), & \mathbb{R}\backslash\partial D_1 \\ A_3'(y), & D_1 \cap D_2, \end{cases} \tag{2.206}$$

where

$$A_1'(y) := e^{-\delta y_0} y_3 (c(y_2^{d_1} - y_2^{d_2}) - y_3^{-1}(\kappa_I + \alpha_I(\hat{y} - y_2)) + a y_1^k y_3^k)),$$
$$A_2'(y) := e^{-\delta(T\wedge\hat{\rho})}(g_1 y_1 y_3 + \lambda_T + g_2 y_2),$$
$$A_3'(y) := y_3 e^{-\delta y_0}(c(y_2^{d_1} - y_2^{d_2}) + a y_1^k y_3^k),$$

where $k = \delta(er - \sigma_f^2)^{-1}$, $\omega^\star = \lambda_T k g_1^{-1}(1-k)^{-1}$, $a = g_1^k k^{-k}(\lambda_T k)^{1-k}(1-k)^{-(1-k)}$.

Lastly, the process Q is determined by the expression:

$$Y_3(t) = Y_3(0)\exp\left\{\frac{1}{2}\sigma_f^2 t - \sigma_f B_f(t)\right\}, \qquad \forall t \in [0,T]. \tag{2.207}$$

□

The General Case ($\gamma_1 \neq 0, \theta_1 \neq 0$)

Though obtaining a closed analytic solution to $p(k) = 0$ represents a difficult task, the solution may be approximated using numerical methods.

The following results follow directly from the degeneracy of the game:

Corollary 2.18.1

Consider the above problem when investment set Φ is a singleton, the game collapses to an optimal stopping problem when the investor seeks to minimise risk of ruin.[8] In this case the double-obstacle problem reduces to:

$$\inf\left\{-\left(\frac{\partial}{\partial_s} + \mathscr{L}^{\hat{\theta}}\psi(y)\right), \psi(y) - G(y)\right\} = 0. \tag{2.208}$$

Corollary 2.18.2

Consider the above problem, when $\theta \equiv 0$, the game collapses to a problem of impulse control with discretionary stopping.

[8] This is a specific case of the game considered in for example [NZ+15].

Chapter 3

Viscosity Theory

In this chapter, we perform a formal analysis of the stochastic differential game of control and stopping. Using viscosity theory, we prove that the value of the game exists, is unique and, is a viscosity solution to a double obstacle problem.

The contribution of this chapter is encompassed in the following paper:

David Mguni, ***"A Viscosity Approach to Stochastic Differential Games of Control and Stopping Involving Impulsive Control of a Jump-Diffusion Process"*** **(2018)** [Mgu18a].

Overview

In the Chapter 2 we derived a characterisation of the value function for a stochastic game of impulse and control and stopping by proving a verification theorem (Theorem 2.7). The theorem provides a direct method of computing the value function which can be extracted as a solution to a PDE namely, the Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation. As we then showed, the result provides a means by which explicit solutions to the optimal liquidity and lifetime ruin investment problem described in Chapter 2 can be obtained.

However, the verification theorem in Chapter 2 requires that the solution to the HJBI equation be sufficiently smooth enough everywhere to apply Itō's formula — a condition which is violated in a number of financial investment problems [Rei98; Tou02]. Secondly, the verification theorem is only meaningful if the value of the game exists — a result which itself is not provided by the verification theorem. To resolve these inadequacies in the theory, in this chapter we both prove the existence and uniqueness of the value of the game using a framework known as *viscosity theory*. Viscosity theory allows us to assign meaning to the solution of the HJBI equation in instances when the value function may not be everywhere smooth. With this approach, in addition to proving the existence of a value of the game, we demonstrate that the (everywhere) smoothness assumptions imposed in the verification procedure can be circumvented.

In this chapter, we perform a deep study of the game in Chapter 2 using tools from viscosity theory. The main contribution of the chapter is to derive two key results that prove the existence of a value of the stochastic differential game of control and stopping. Consider a stochastic game of control and stopping in which an impulse controller employs a strategy $\alpha \in \mathscr{A}_{(0,T)}$ and an adversary uses the strategy $\beta \in \mathscr{B}_{(0,T)}$ to decide when to stop the game. The corresponding upper and lower value functions of the game are given by the following expressions respectively:

$$V^-(y_0) = \inf_{\alpha \in \mathscr{A}_{(0,T)}} \sup_{\rho' \in \mathscr{T}} \mathbb{E}\left[\int_{t_0}^{\tau_s \wedge \rho'} f(Y_s^{y_0,\alpha(\rho')})ds + \sum_{m \geq 1} c(\tau_m(\rho'), \xi_m(\rho')) \cdot 1_{\{\tau_m \leq \tau_S \wedge \rho'\}} \right. \\ \left. + G\left(Y_{\tau_S \wedge \rho'}^{y_0,\alpha(\rho')}\right) \cdot 1_{\{\tau_S \wedge \rho' < \infty\}} \right], \tag{3.1}$$

$$V^+(y_0) = \sup_{\beta \in \mathscr{B}_{(0,T)}} \inf_{u' \in \mathscr{U}} \mathbb{E}\left[\int_{t_0}^{\tau_s \wedge \beta(u')} f(Y_s^{y_0,u'})ds + \sum_{m \geq 1} c(\tau_m, \xi_m) \cdot 1_{\{\tau_m \leq \tau_S \wedge \beta(u')\}} \right. \\ \left. + G\left(Y_{\tau_S \wedge \beta(u')}^{y_0,u'}\right) \cdot 1_{\{\tau_S \wedge \beta(u') < \infty\}} \right]. \tag{3.2}$$

In this chapter we show that $V^+ = V^-$ so that the upper and lower value functions of the game coincide. This establishes the existence of an equilibrium for the game and shows that the players are in fact playing the same game (c.f. Definition 1.1.). We therefore establish the existence of a solution to the game presented in Chapter 2.

We then prove that the value function for the game admits a representation as a solution to a *double obstacle quasi-integro-variational inequality* which we show is a unique *viscosity solution* to a HJBI equation. This allows us for the first time to provide a formal demonstration of the existence and uniqueness of an equilibrium solution for stochastic differential games of control and stopping in which the controller uses impulse controls.

A summary of the contributions of this chapter is as follows:

- First, we provide a formal proof of a dynamic programming principle (Theorem 3.10) for the stochastic differential game of impulse control and stopping introduced in Chapter 2.
- Second, we prove that the value is a viscosity solution to a double obstacle quasi-variational inequality (Lemma 3.12).
- Lastly, we prove the uniqueness of the value function ans the existence of a saddle point equilibrium of the stochastic differential game (Theorem 3.13).

In addition to proving that the game has in fact an equilibrium solution, the treatment of the game in this chapter weakens some of the smoothness assumptions required in the verification theorem

(Theorem 2.7) allowing for solutions to be generated in instances in which the value function may not be everywhere smooth enough to apply Dynkin's formula.

The analysis in this chapter is related to the viscosity approach for tackling Dynkin games presented in [BS14] which is a special case of our scenario in which the controller's decision is restricted to a single decision to terminate the game. Similarly, the results of this chapter extend the viscosity arguments and the Hamilton-Jacobi-Bellman-Issacs characterisation of the value function for the single impulse controller case presented in [Sey09; Ish93; Ish95; BL84]. In particular, our results extend the viscosity arguments in [Sey09; Ish93; Ish95; BL84] to a two-player game setting which now includes a player that can choose to stop the game. The results in this chapter are also related to results in [BZ15b] in which a two-player game of *continuous control* and stopping is analysed using viscosity theory. In [BZ15b], the existence of a unique value function which corresponds to a viscosity solution of the associated HJB equation is proved. We tackle a game setting which is similar to that in [BZ15b] however our setting is adapted so as to allow minimally bounded adjustment costs.

Our analysis also differs markedly from that of [BZ15b] as the results in [BZ15b] are derived using a stochastic version of Perron's method using singular controls. Our setup is a game which allows for risk adversity against worst-case scenarios (which the adversary seeks to induce). Additionally, our proofs are derived by way of firstly proving a dynamic programming principle for the game then proving a maximum (comparison) principle then deducing that the value of the game is a viscosity solution to a double obstacle problem.

Organisation

The analysis of the chapter is organised as follows: first, we give a summary of the main results in Sec. 3.2. In Section 3.3 the chapter proceeds to the main analysis beginning with proving regularity results for the upper and lower value functions of the game. Using the regularity results, we then prove a dynamic programming principle for the game from which a HJBI characterisation follows. We then prove that the value of the game is a viscosity solution to a HJBI equation described by a double obstacle problem. Lastly, the analysis is completed with a comparison principle result which establishes the existence and uniqueness of the value of the game — a result that confirms the existence of a saddle point equilibrium of the game. The chapter also has an appendix to which some of the lengthy proofs of the chapter are deferred.

3.1 Introduction to Viscosity Theory

In differential game theory, there are two main approaches to obtaining a solution to the problem. The first approach as we have seen in the previous chapter is a verification method which involves characterising the value function in terms of a set of (in general, non-linear, second order) PDEs or HJBI equations (in the case of single controller problems, HJB equations).

As remarked previously, verification theorems require that the value function must be smooth enough everywhere to apply Itō formula along the diffusion. Such conditions are not likely to hold in classes of control problems beyond those in which continuous controls are applied to diffusion processes without jumps. Indeed, within the current context of a game with both an optimal stopping and impulse control component, the smoothness condition of the value function is likely to fail when the process X approaches the boundary of the continuation region. Therefore, in such cases, it is not possible to invoke Itō's formula to derive the HJBI equations via a classical limiting procedure.

Nonetheless, in deriving a verification theorem, we showed that if a sufficiently smooth function that satisfies the conditions of Theorem 2.7 can be found, then that function is also the value function for the game. However, we have yet to establish whether the converse is true — whether the value function of the game is always a solution to the quasi-integro-variational inequalities of the verification theorem.[1] Thus the question of existence of the value of stochastic differential games and its uniqueness remain thus far unresolved.

The second approach to solving problems within differential game theory uses a framework known as viscosity theory. Viscosity theory provides a means by which value functions of a wide class of stochastic control problems (and consequently, stochastic differential games) can be made to satisfy the PDEs corresponding to HJB(I) equations when the solutions of the PDEs are interpreted in a weaker, *viscosity sense*. Indeed, viscosity solutions generalise the notion of a solution of a PDE to a non-classical definition. The main advantage of the viscosity solution approach is that it does not require strong (everywhere) smoothness of the value function.

Viscosity solutions were introduced by Michael Crandall and Pierre-Louis Lions in 1983 [CL83] and were developed to handle first order HJB equations.[2] The theory was subsequently extended to handle second order PDEs in part due to a comparison principle result introduced by Robert Jensen in 1988 [Jen88]. The name *viscosity theory* is derived from the historical connection of the theory to the *vanishing viscosity method* [GBLX16] — a method by which solutions to a class of first order PDEs can be obtained with an approximation procedure [Pas06] (these PDEs often arise within fluid dynamics in which the notion of physical viscosity appears). However, the vanishing viscosity method is in general irrelevant for second order PDEs [CIL92]. Moreover, since in general, the modern usages of viscosity solutions employ a definition that does not involve notions of physical viscosity nor do they invoke the vanishing viscosity method, the name viscosity theory remains only as a historical artefact.

[1] In fact, in general, there exist an infinite number of Lipschitz continuous functions that satisfy the HJBI equations of the verification theorems (see for example exercise 3.2 in [Car10]).

[2] We recall that Hamiltion-Jacobi-Bellman equations or HJB equations are the dynamic programming equation of single controller optimal stochastic control problems.

3.2 Main Results

We prove two main results for the game of stochastic differential game of control and stopping with impulse controls. First, we formally demonstrate that the game admits a value and in so doing, show that the game has a saddle point equilibrium. We then prove that the value of the game satisfies a *double obstacle quasi-integro-variational equality* and is a unique viscosity solution to a HJBI equation.

In particular, we show that equality (3.1) holds by firstly showing that V^+ (resp., V^-) is a viscosity supersolution (resp., subsolution) to the following non-linear obstacle problem:

$$\max\left\{\min\left[-\frac{\partial V}{\partial s}-\mathscr{L}V-f,V-G\right],V-\mathscr{M}V\right\}=0$$
$$V\left(\tau_s\wedge\rho,y\right)=G\left(\tau_s\wedge\rho,y\right),\quad\forall y\in S,\tag{3.3}$$

where $\mathscr{L}$ is the local stochastic generator operator associated to the process (c.f.(1.2)) and $\mathscr{M}$ is the non-local intervention operator. The result generalises the Hamilton-Jacobi-Bellman quasi-variational inequality results in [Sey09; Azi17; Len89] to a double obstacle problem which now includes the action of an adversary that can terminate the game.

We now state one of the main results of the chapter.

Theorem 3.1

The value of the game exists and is given by: $V(y)=V^-(y)=V^+(y),\quad\forall y\in[0,T]\times S.$

Theorem 3.1 formally establishes the existence of a value of the game. The theorem is proven by generating two inequalities the first of which follows directly from the definitions of the upper and lower value functions. In particular, we have the following remark.

Remark 3.2

By definition of the value functions, we automatically have:

$$V^-(y)\geq V^+(y),\quad\forall y\in[0,T]\times S.\tag{3.4}$$

To prove Theorem 3.1, it therefore remains to establish the reverse inequality of (3.4). To prove that the value function is a solution to an obstacle problem in (3.3), we firstly establish a dynamic programming principle (DPP) for the game which describes the evolution of the value function over a small interval.

Outline

The outline of the scheme of proofs and results of the chapter is as follows:

(i) First, prove a dynamic programming principle (DPP) for the game after establishing regularity properties of the value function.

(ii) Using the DPP for each of the value functions, prove that the upper (resp., lower) value function is a viscosity subsolution (resp., supersolution) to the HJBI equation (3.3).

(iii) Prove a comparison theorem and the reverse inequality of (3.4) therefore proving equality of the value functions.

(iv) Using (ii), deduce the existence of a value of the game and that the value is a unique solution to the HJBI equation.

3.3 Main Analysis

The central idea of viscosity theory is to replace the derivatives of the value function appearing in the HJBI PDE pointwise (as those that arose in Theorem 2.7) with a suitably smooth function $\psi \in \mathscr{C}^{1,2}([0,T],\mathbb{R}^p)$ which is related to the value function V by some local maximisation and minimisation conditions. Our first task is to prove the regularity and boundedness of the value functions associated to the game which are proven in Lemma 3.3 - Proposition 3.5. Using the regularity of the value functions, we then establish an appropriate DPP for the game (Theorem 3.10). The DPP serves as a crucial tool for characterising the value of the game and proving that the value function is a viscosity solution to a HJBI equation which is proven in Lemma 3.12. In order to prove the time-regularity ($\frac{1}{2}$-Hölder-continuity) of the value functions, we require the following result which is a single-controller version of Lemma 3.3 in [Cos13]:

Lemma 3.3

The functions V^- and V^+ can be equivalently expressed by the following:

$$V^-(y) = \inf_{\alpha \in \bar{\mathscr{A}}_{(t,T)}} \sup_{\rho \in \mathscr{T} \backslash \{t\}} J[y;\alpha(\rho),\rho], \tag{3.5}$$

$$V^+(y) = \sup_{\beta \in \hat{\mathscr{B}}_{(t,T)}} \inf_{u \in \bar{\mathscr{U}}_{(t,T)}} J[y;u,\beta(u)], \qquad \forall y \in [0,T]\times S, \tag{3.6}$$

where $\bar{\mathscr{U}}_{(t,T)}$ is the set of player I admissible controls which have no impulses at time t and correspondingly, $\bar{\mathscr{A}}_{(t,T)}$ (resp., $\hat{\mathscr{B}}_{(t,T)}$ is the set of all player I (resp., player II) Elliott-Kalton (non-anticipative) strategies with controls drawn from the set $\bar{\mathscr{U}}_{(t,T)}$ (resp., $\mathscr{T}\backslash\{t\}$).

We defer the proof of the lemma to the chapter appendix.

The following results allow us to establish continuity properties of the value function required to prove the DPP for the game. In particular, the following results demonstrate that small changes in the inputs of the value function produce small changes in the value function itself. Later, we use these results to construct controls that produce values of the payoff function J that are arbitrarily close to the values of the value functions.

Given Lemma 3.3, we can now seek to prove that the upper and lower value functions associated to the game are Lipschitz continuous in the spatial variable and $\frac{1}{2}$-Hölder continuous in time, that is, we prove the following proposition:

Proposition 3.4

We can deduce the existence of constants $c_1, c_2 > 0$ such that the following results hold:

(i) $|V^-(t,x') - V^-(t,x)| + |V^+(t,x') - V^+(t,x)| \leq c_1|x'-x|$,

(ii) $|V^-(t',x) - V^-(t,x)| + |V^+(t',x) - V^+(t,x)| \leq c_2|t'-t|^{\frac{1}{2}}$.

$$\forall (t,x),(t',x') \in [0,T] \times S.$$

Proposition 3.4 establishes an important property of the game — small changes in the input variables of the value functions lead to small changes in the game. This result is crucial for deriving the DPP for the game which describes the behaviour of the value function under infinitesimal variations.

Proof of Proposition 3.4. We separate the proof into two parts, proving the spatial Lipschitzianity (i) first, then the temporal $\frac{1}{2}$-Hölder-continuity (ii) last.

To show that the value functions are Lipschitz continuous in the spatial variable, it suffices to show that the property is satisfied for the function J. The proof follows as an immediate consequence of the Lipschitzianity of the constituent functions. In particular, writing $y \equiv (t,x)$ and $y' \equiv (t,x')$ we have:

$$\begin{aligned}
&\left|J[y;\cdot,\cdot] - J[y';\cdot,\cdot]\right| \\
&\leq \mathbb{E}\left[\int_t^{\tau_s \wedge \rho} \left|f(Y_s^{y,\cdot}) - f(Y_s^{y',\cdot})\right| ds + \left|G(Y_{\tau_S \wedge \rho}^{y,\cdot}) - G(Y_{\tau_S \wedge \rho}^{y',\cdot})\right|\right] \\
&\leq c_f \int_t^T \mathbb{E}\left[\left|X_s^{t,x,\cdot} - X_s^{t,x',\cdot}\right|\right] ds + c_G \mathbb{E}\left[\left|X_{\tau_S \wedge \rho}^{t,x,\cdot} - X_{\tau_S \wedge \rho}^{t,x',\cdot}\right|\right], \forall (t,x'),(t,x) \in [0,T] \times S,
\end{aligned}$$

where $c_f, c_G > 0$ are Lipschitz constants for the function f and G respectively (see assumptions A.2.1 - A.2.2). Therefore, as an immediate consequence of Lemma A.1, we see that we can deduce the existence of a constant $c > 0$ such that

$$|J[t,x;\cdot,\cdot] - J[t,x';\cdot,\cdot]| \leq c|x-x'|. \tag{3.7}$$

We note also that since the constituent functions of J are bounded, J is also bounded; hence by (3.7) and by Lemma 3.6 in [Car10], we therefore conclude that:

$$|V^{\pm}(t,x) - V^{\pm}(t,x')| \leq c|x-x'|, \tag{3.8}$$

for some constant $c > 0$ as required.

To show that the value functions are Lipschitz continuous in the temporal variable, we show that (ii) is satisfied by the function V^+ with the proof for the function V^- being analogous.

We firstly note that:

$$V^+(t',x) - V^+(t,x) = \sup_{\mu \in \mathscr{B}_{(0,T)}} \inf_{u \in \mathscr{U}} J[t',x;u,\mu(u)] - \sup_{\mu \in \mathscr{B}_{(0,T)}} \inf_{u \in \mathscr{U}} J[t,x;u,\mu(u)]$$
$$\forall (t,x'),(t,x) \in [0,T] \times S. \qquad (3.9)$$

By (i), we can deduce the existence of some $\varepsilon-$optimal strategy $\hat{\mu} \in \mathscr{B}_{(0,T)}$ against $V^+(t,x)$ such that $V^+(t,x) - \varepsilon \leq \inf_{u \in \mathscr{U}} J[t,x;u,\hat{\mu}(u)]$, $\forall (t,x) \in [0,T] \times S$ where $\varepsilon > 0$ is arbitrary. Hence, by (3.9) we have that:

$$V^+(t',x) - V^+(t,x) - 2\varepsilon \leq \inf_{u \in \mathscr{U}} J[t',x;u,\hat{\mu}(u)] - \inf_{u \in \mathscr{U}} J[t,x;u,\hat{\mu}(u)]. \qquad (3.10)$$

Let us now construct the control $u_\varepsilon = \sum_{j \geq 1} \xi_j^\varepsilon \cdot 1_{[\tau_j^\varepsilon, T]}$ which is associated with the strategy $\alpha^\varepsilon \in \mathscr{A}_{(0,T)}$. Let us also construct the control $u'_\varepsilon \in \mathscr{U}_{[t',T]}$ using the following expression: $u'_\varepsilon = \sum_{\tau_j^\varepsilon \leq t'} \xi_j^\varepsilon \cdot 1_{t'} + \sum_{\tau_j^\varepsilon > t'} \xi_j^\varepsilon \cdot 1_{[\tau_j^\varepsilon, T]}$ which is associated to the strategy $\hat{\alpha}^\varepsilon$ so that the control u'_ε is simply the control $u_\varepsilon \in \mathscr{U}$ except that the impulse interventions within the interval $[t,t'[$ are now pushed to t'.

Now thanks to Lemma 3.3, we have that $|J[t,x;u,\cdot] - J[t,x;\bar{u},\cdot]| < \varepsilon$ where $\varepsilon > 0$ is arbitrary and where $\bar{u} \in \bar{\mathscr{U}}$ is the set of player I admissible controls that have no impulses at time t. Hence, by Lemma 3.3 and using the $\varepsilon-$optimality of the strategy $\hat{\mu} \in \mathscr{B}_{(0,T)}$, we can therefore deduce the following inequality:

$$V^+(t',x) - V^+(t,x) - 3\varepsilon \leq J[t',x;u'_\varepsilon,\mu(u'_\varepsilon)] - J[t,x;u_\varepsilon,\hat{\mu}(u_\varepsilon)]. \qquad (3.11)$$

Let us denote by $\hat{\rho} := \mu(u'_\varepsilon) \equiv \hat{\mu}(u_\varepsilon)$ and define $\bar{\rho} \in \mathscr{T}$ by:

$$\bar{\rho} := \begin{cases} t', & \{\hat{\rho} < t'\} \\ \hat{\rho}, & \{\hat{\rho} \geq t'\}. \end{cases} \qquad (3.12)$$

Hence, we have that:

$$J[t',x;u'_\varepsilon,\bar{\mu}(u'_\varepsilon)] - J[t,x;u_\varepsilon,\hat{\mu}(u_\varepsilon)] = J[t',x;u'_\varepsilon,\bar{\rho}] - J[t,x;u_\varepsilon,\hat{\rho}].$$

Writing $y' \equiv (t',x)$ and $y \equiv (t,x)$ for any $(t,x),(t',x) \in [0,T]\times S$, we calculate that:

$$
\begin{aligned}
& J[y';u'_\varepsilon,\bar\rho] - J[y;u_\varepsilon,\hat\rho] \\
= & -\mathbb{E}\Bigg[\int_t^{\hat\rho\wedge\tau_S} f(Y_s^{y,u_\varepsilon})ds + \sum_{j\geq 1} c(\tau_j^\varepsilon,\xi_j^\varepsilon)\cdot 1_{\{t<\tau_j^\varepsilon<\hat\rho\wedge\tau_S\}} \\
& \qquad - \left(G\left(Y^{y',u'_\varepsilon}_{\bar\rho\wedge\tau_S}\right) - G\left(Y^{y,u_\varepsilon}_{\hat\rho\wedge\tau_S}\right)\right)\cdot 1_{\{\hat\rho\wedge\tau_s<\infty\}}\Bigg]\cdot 1_{\{\hat\rho\leq t'\}} \\
& +\mathbb{E}\Bigg[\int_{t'}^{\hat\rho\wedge\tau_S)} f(Y_s^{y',u'_\varepsilon})ds - \int_t^{\hat\rho\wedge\tau_S} f(Y_s^{y,u_\varepsilon})ds \\
& \qquad + \sum_{j\geq 1}\left(c\left(\tau_j'^\varepsilon,\xi_j'^\varepsilon\right)\cdot 1_{\{t'<\tau_j'^\varepsilon<k\}} - c(\tau_j^\varepsilon,\xi_j^\varepsilon)\cdot 1_{\{t<\tau_j^\varepsilon<k\}}\right)\cdot\delta^k_{\hat\rho\wedge\tau_S} \\
& \qquad + \left(G\left(Y^{y',u'_\varepsilon}_{\bar\rho\wedge\tau_S}\right) - G\left(Y^{y,u_\varepsilon}_{\hat\rho\wedge\tau_S}\right)\right)\cdot 1_{\{\hat\rho\wedge\tau_s<\infty\}}\Bigg]\cdot 1_{\{\hat\rho>t'\}} \\
= & \,\mathbb{E}\Bigg[\int_{t'}^{\hat\rho\wedge\tau_S} f(Y_s^{y',u'_\varepsilon})ds - \int_t^{\hat\rho\wedge\tau_S} f(Y_s^{y,u_\varepsilon})ds + c\left(t',\sum_{\tau_j^\varepsilon\leq t'}\xi_j^\varepsilon\right) - \sum_{\tau_j^\varepsilon\leq t'} c\left(\tau_j^\varepsilon,\xi_j^\varepsilon\right) \\
& + \left(G\left(Y^{y',u'_\varepsilon}_{\hat\rho\wedge\tau_S}\right) - G\left(Y^{y,u_\varepsilon}_{\hat\rho\wedge\tau_S}\right)\right)\cdot 1_{\{\infty>\hat\rho>t'\}} + \left(G\left(Y^{y,u_\varepsilon}_{\hat\rho\wedge\tau_S}\right) - G\left(Y^{y',u'_\varepsilon}_{\bar\rho\wedge\tau_S}\right)\right)\cdot 1_{\{\hat\rho\leq t'<\infty\}}\Bigg].
\end{aligned}
$$

Now by assumption A.3, we have that:

$$
\sum_{\tau_j^\varepsilon\leq t'} c(\tau_j^\varepsilon,\xi_j^\varepsilon) \geq c(t',\sum_{\tau_j^\varepsilon\leq t'}\xi_j^\varepsilon). \tag{3.13}
$$

Hence, we find that:

$$
\begin{aligned}
& J[y;u'_\varepsilon,\bar\mu(u'_\varepsilon)] - J[y;u_\varepsilon,\hat\mu(u_\varepsilon)] \\
& \leq \mathbb{E}\left[\int_{t'}^{\hat\rho\wedge\tau_S} f(Y_s^{y',u'_\varepsilon})ds - \int_t^{\hat\rho\wedge\tau_S} f(Y_s^{y,u_\varepsilon})ds\right] + \sup_{\hat\rho\in\{t',\hat\rho\}}\mathbb{E}\left[\left|G(Y^{y,u_\varepsilon}_{\hat\rho\wedge\tau_S}) - G(Y^{y',u'_\varepsilon}_{\hat\rho\wedge\tau_S})\right|\right],
\end{aligned} \tag{3.14}
$$

where we have used (3.13) to remove the cost terms. By the Lipschitz continuity of G (c.f. A.1.2.) and Lemma A.1.3, we deduce the existence of a constant $c>0$ such that $\forall s\in[0,T]$:

$$
\mathbb{E}\left[\left|G(Y_s^{y',\cdot}) - G(Y_s^{y,\cdot})\right|\right] \leq c\sup_{s\in[0,T]}\mathbb{E}\left[\left|X_s^{t',x,\cdot} - X_s^{t,x,\cdot}\right|\right] \leq c\left|t-t'\right|^{\frac{1}{2}}, \tag{3.15}
$$

where c denotes some arbitrary constant (which may differ in each step of the proof). Moreover, we can arrive at the result using the Lipschitz property of f and using the properties of X by deducing the existence of a constant $c>0$ for which

$$
\mathbb{E}\left[\int_t^{\hat\rho\wedge\tau_S} f(Y_s^{y,\cdot})ds - \int_{t'}^{\hat\rho\wedge\tau_S} f(Y_s^{y',\cdot})ds\right] \leq c|t-t'|^{\frac{1}{2}}. \tag{3.16}
$$

To see this, for any $0 \leq t' \leq T$, we observe the following estimate:

$$\mathbb{E}\left[\int_t^{\hat{\rho}\wedge\tau_S} f(Y_s^{y,\cdot})ds - \int_{t'}^{\hat{\rho}\wedge\tau_S} f(Y_s^{y',\cdot})ds\right] \tag{3.17}$$

$$\leq \mathbb{E}\left[\int_{t'}^{T} \left|f(Y_s^{y,\cdot}) - f(Y_s^{y',\cdot})\right| ds\right] + \mathbb{E}\left[\int_t^{t'} |f(Y_s^{y,\cdot})|\, ds\right] \tag{3.18}$$

$$\leq c\mathbb{E}\left[\int_{t'}^{T} \left|Y_s^{y,\cdot} - Y_s^{y',\cdot}\right| ds\right] + c\mathbb{E}\left[\int_t^{t'} (1+|Y_s^{y,\cdot}|)ds\right] \tag{3.19}$$

$$\leq c|T-t'|\mathbb{E}\left[\sup_{t'\leq s\leq T} \left|Y_s^{y,\cdot} - Y_s^{y',\cdot}\right| ds\right] + c|t'-t|\left(1+\mathbb{E}\left[\sup_{t\leq s\leq t'} |Y_s^{y,\cdot}|\right]\right) \tag{3.20}$$

$$\leq cT^{1/2}(1+|y|)\left(T^{1/2}+1\right)|t'-t|^{1/2}, \tag{3.21}$$

where we have used the Lipschitz property of f and where we have applied the results of Lemma A.1.3 in the last step (and where the constant c may differ in each step) and hence we arrive at the desired result.

After collecting the above results we deduce that there exists a constant $c > 0$ such that for $\varepsilon > 0$ the following estimate holds:

$$V^+(y) - V^+(y') \leq |J[y';u'_\varepsilon,\hat{\mu}(u'_\varepsilon)] - J[y;u_\varepsilon,\hat{\mu}(u_\varepsilon)]| + 3\varepsilon \leq c|t-t'|^{\frac{1}{2}} + 3\varepsilon. \tag{3.22}$$

Now, since ε in (3.22) is arbitrary, we deduce the existence of a constant $c > 0$ such that

$$|V^+(t,x) - V^+(t',x)| \leq c|t-t'|^{\frac{1}{2}}, \quad \forall (t,x),(t',x) \in [0,T]\times S, \tag{3.23}$$

after which we deduce (ii) holds for the function V^+. To deduce that (ii) holds for the function V^-, we note that analogous to (3.9), we have that:

$$V^-(t',x) - V^-(t,x) = \inf_{\alpha\in\mathscr{A}_{(0,T)}} \sup_{\rho\in\mathscr{T}} J[t',x;\alpha(\rho),\rho] - \inf_{\alpha\in\mathscr{A}_{(0,T)}} \sup_{\rho\in\mathscr{T}} J[t,x;\alpha(\rho),\rho]. \tag{3.24}$$

In a similar way to the proof of (ii) for V^+ we can deduce the existence of a constant $c > 0$ such that $|V^-(t,x) - V^-(t',x)| \leq c|t-t'|^{\frac{1}{2}}$, from which we deduce the thesis. □

The following proposition establishes the boundedness of the value functions:

Proposition 3.5

The value functions $V^\pm$ are both bounded.

Proof

We do the proof for the function V^- with the proof for V^+ being analogous. Recall that:

$$V^-(y_0) = \inf_{\alpha\in\mathscr{A}_{(0,T)}} \sup_{\rho\in\mathscr{T}} \mathbb{E}\left[\int_{t_0}^{\rho\wedge\tau_S} f(Y_s^{y_0,\alpha(\rho)})ds + \sum_{j\geq 1} c(\tau_j(\rho),\xi_j(\rho))\cdot 1_{\{\tau_j(\rho)\leq\rho\wedge\tau_S\}} \right.$$
$$\left. + G(Y_{\rho\wedge\tau_S}^{y_0,\alpha(\rho)})1_{\{\rho\wedge\tau_S<\infty\}}\right] \quad . \tag{3.25}$$

Now let $u_0 \in \mathscr{U}$ be the player I control with which no impulses exercised. Then $X_{\tau_S}^{t_0,x_0,u} \equiv X_{\tau_S}^{t_0,x_0,u_0} + \sum_{j=1}^{\mu_{[t_0,T]}(u)} \xi_j$. If for any $u \in \mathscr{U}$ and for any $s\in[0,T]$, we denote by $Z_s^{z_0,u} := (s, X_s^{t_0,x_0,u_0} + \sum_{j\geq 1}\xi_j \cdot 1_{\{\tau_j<\tau_S\wedge\rho\}})$ and by $Y_s^{y_0,u_0} := (s, X_s^{t_0,x_0,u_0})$ where $z_0 \equiv Z_{t_0}^{t_0,u}$ and $y_0 \equiv Y_{t_0}^{t_0,u_0}$, using Lemma A.1.3 and by Gronwall's lemma we have that $\mathbb{E}\left[G(Z_{\tau_S}^{z_0,\alpha(\rho)}) - G(Y_{\tau_S}^{y_0,u_0})\right] \leq c_2\mathbb{E}[|Z_{\tau_S}^{z_0,\alpha(\rho)} - Y_{\tau_S}^{y_0,u_0}|] \leq c_1\mathbb{E}[((\rho\wedge\tau_S)-t_0)^{\frac{1}{2}})]$ for some $c_1, c_2 > 0$. Moreover, since $u\in\mathscr{U}$ (hence $\mathbb{E}[\mu_{[t_0,T]}(u)]<\infty$) we can find some $\lambda > 0$ such that $\sum_{j\geq 1} c(\tau_j,\xi_j)\cdot 1_{\{\tau_j<\tau_S\wedge\rho\}} \leq \lambda$ and hence:

$$\begin{aligned}
&\mathbb{E}\left[\sum_{j\geq 1} c(\tau_j,\xi_j)\cdot 1_{\{\tau_j<\tau_S\wedge\rho\}} + G(Z_{\rho\wedge\tau_S}^{z_0,\alpha(\rho)})\right] \\
&= \mathbb{E}\left[\sum_{j\geq 1} c(\tau_j,\xi_j)\cdot 1_{\{\tau_j<\tau_S\wedge\rho\}} + \left((G(Z_{\tau_S}^{z_0,\alpha(\rho)}) - G(Y_{\tau_S}^{y_0,u_0})\right) + G(Y_{\tau_S}^{y_0,u_0}))\right] \\
&\leq \mathbb{E}\left[G(Y_{\tau_S}^{y_0,u_0}) + \lambda + c_1((\rho\wedge\tau_S)-t_0)^{\frac{1}{2}}\right] \\
&\leq \mathbb{E}\left[G(Y_{\tau_S}^{y_0,u_0}) + \lambda + c_1(T-t_0)^{\frac{1}{2}}\right].
\end{aligned}$$

Since by similar reasoning we can deduce that $\mathbb{E}\left[\int_{t_0}^{\rho\wedge\tau_S} f(Z_s^{z_0,\alpha(\rho)})ds\right] \leq \mathbb{E}[\int_{t_0}^{\rho\wedge\tau_S} f(Y_s^{y_0,u_0})ds + c(T-t_0)^{\frac{1}{2}}\cdot 1_{\{\mu_{[t_0,\tau_S]}(u)\}}]$ for some $c>0$; using the continuity of the functions f and G we find that:

$$\begin{aligned}
V^-(y_0) &\leq \sup_{\rho\in\mathscr{T}} \mathbb{E}\left[\int_{t_0}^{\rho\wedge\tau_S} f(Y_s^{y_0,u_0})ds + G(Y_{\tau_S}^{y_0,u_0})\cdot 1_{\{\rho\wedge\tau_s<\infty\}} \right. \\
&\qquad \left. + (\lambda + c_1|(\rho\wedge\tau_S)-t_0|^{\frac{1}{2}})\cdot 1_{\{\mu_{[t_0,\tau_S]}(u)\}}\right] \\
&\leq \sup_{\rho\in\mathscr{T}} \mathbb{E}\left[\int_{t_0}^{T} c_2(1+\mathbb{E}[|X_s^{t_0,x_0,u_0}|])ds + c_3(1+\mathbb{E}[|X_{\rho\wedge\tau_S}^{t_0,x_0,u_0}|]) \right. \\
&\qquad \left. + \left(\lambda + c_1|T-t_0|^{\frac{1}{2}}\right)\cdot 1_{\{\mu_{[t_0,\tau_S]}(u)\}}\right] \\
&\leq \sup_{\rho\in\mathscr{T}} \mathbb{E}\left[\alpha + (\lambda + c_1|T-t_0|^{\frac{1}{2}})\cdot 1_{\{\mu_{[t_0,\tau_S]}(u)\}}\right],
\end{aligned}$$

using Lemma A.1.3 and where $\alpha := (T-t_0)\cdot[c_2+c_3](1+|x|)$ and $c_1>0$ and $c_2>0$ are constants. We then deduce the thesis since each of the terms inside the square bracket is bounded. □

Lemma 3.7

Let $V \in \mathscr{H}$ be a bounded function and $(\tau,x) \in [0,T] \times S$ where τ is some $\mathscr{F}-$ measurable stopping time, then the set $\Xi(\tau,x)$ defined by:

$$\Xi(\tau,x) := \left\{\xi \in \mathscr{Z} : \mathscr{M}V(\tau^-,x) = V(\tau,x+\xi) + c(\tau,\xi) \cdot 1_{\{\tau \leq T\}}\right\} \tag{3.26}$$

is non-empty.

The proof of the lemma is straightforward since we need only prove that the infimum is in fact a minimum. This follows directly from the fact that the cost function is minimally bounded (c.f. A.4) and that the value functions are also bounded by Proposition 3.5.

A proof of the following lemma is reported in [DGW10], Lemma 2.6 and similar result may be found in (Lemma 3.10 in [CG13]):

Lemma 3.8

The non-local intervention operator $\mathscr{M}$ is continuous wherein we can deduce the existence of a constants $c_1, c_2 > 0$ such that when $s < s'$:

(i) $|\mathscr{M}V^{\pm}(s,x) - \mathscr{M}V^{\pm}(s,y)| \leq c_1|x-y|$,

(ii) $|\mathscr{M}V^{\pm}(t,x) - \mathscr{M}V^{\pm}(s,x)]| \leq c_2|t-s|^{\frac{1}{2}}, \qquad \forall (t,x),(s,y) \in [0,T] \times S.$

Proof

We prove the results for V^+ since the proof for V^- is analogous.

Statement (i) follows from the uniform continuity of $V^+(s,x) + c(\tau,\xi)$ on compact sets and the compactness of the set $\Xi(s,x)$ for all $(s,x) \in [0,T] \times S$. Indeed, since we are applying the infimum operator over fixed compact sets, we can readily deduce that $\mathscr{M}V^+(s,x)$ is continuous for all $(s,x) \in [0,T] \times S$. For statement (ii), let us firstly assume that $s < t$, we then observe the following for some $c > 0$:

$$\begin{aligned}
&\mathscr{M}V^+(t,x) - \mathscr{M}V^+(s,x) \\
&\leq V^+(t,x+\xi) + c(t,\xi) - (V^+(s,y+\xi) + c(s,\xi)) \\
&\leq |V^+(t,x+\xi) - V^+(s,y+\xi)| \leq c|t-s|^{\frac{1}{2}},
\end{aligned}$$

where we have used assumption A.3 (iv) to remove the cost terms and Proposition 3.4 to deduce the last line, after which we can straightforwardly deduce the result.

□

Having proven the above preliminary results, we now have the necessary grounding to prove a DPP for the game. The following result is of crucial importance for deriving the verification theorem which characterises the value of the game and establishing the existence and uniqueness of the value via a viscosity approach. In particular, the following result enables us to partition the problem into smaller intervals over which we compute the optimal controls individually, letting the size of the intervals tend to 0 then reduces the problem to that of pointwise minimisation.

We now state the DPP for the game:

Theorem 3.10 (Dynamic programming principle for stochastic differential games of control and stopping with Impulse Controls)

Let $u \in \mathscr{U}$ be an admissible player I control and suppose $\rho \in \mathscr{T}$ is an $\mathscr{F}-$measurable stopping time, then for a sufficiently small h the following variants of the DPP hold for the functions V^+ and V^-:

$$V^-(y_0) = \inf_{\alpha \in \mathscr{A}_{(0,T)}} \sup_{\rho \in \mathscr{T}} \mathbb{E}\Bigg[\int_{t_0}^{(t_0+h)\wedge\rho} f(Y_s^{y_0,\alpha(\rho)})ds + \sum_{j\geq 1} c(\tau_j(\rho),\xi_j(\rho)) \cdot 1_{\{\tau_j(\rho)\leq(t_0+h)\wedge\rho\}} \\ +G(Y_{\rho\wedge\tau_S}^{y_0,\alpha(\rho)}) \cdot 1_{\{\rho\wedge\tau_S\leq t_0+h\}} + V^-(Y_{t_0+h}^{y_0,\alpha(\rho)\wedge\rho}) \cdot 1_{\{\rho\wedge\tau_S>t_0+h\}}\Bigg] \tag{3.27}$$

and

$$V^+(y_0) = \sup_{\beta \in \mathscr{B}_{(0,T)}} \inf_{u \in \mathscr{U}} \mathbb{E}\Bigg[\int_{t_0}^{(t_0+h)\wedge\beta(u)} f(Y_s^{y_0,u})ds + \sum_{j\geq 1} c(\tau_j,\xi_j) \cdot 1_{\{\tau_j\leq(t_0+h)\wedge\beta(u)\}} \\ +G(Y_{\beta(u)\wedge\tau_S}^{y_0,u}) \cdot 1_{\{\beta(u)\wedge\tau_S\leq t_0+h\}} + V^-((Y_{t_0+h}^{y_0,u}) \cdot 1_{\{\beta(u)\wedge\tau_S>t_0+h\}}\Bigg]. \tag{3.28}$$

$$\forall y_0 \equiv (t_0,x_0) \in [0,T]\times S.$$

Intuitively, the DPP states that if we compute the optimal controls on the intervals $[t_0,t_0+h]$ and $[t_0+h,t']$ for some $h<(t'\wedge\rho)-t_0$ for any $t'\in[0,T]$, then we would obtain the same result as that which we would obtain if we computed the optimal controls for the interval $[t_0,t']$ as a whole.

A classical consequence of the DPP (3.27) and (3.28) is that we find that the function V^- (resp., V^+) is the subsolution (resp., supersolution) to an associated HJBI equation, namely (3.3). Moreover, if the game admits a value V with $V \in \mathscr{C}^{1,2}([0,T],\mathbb{R}^p)$, then V is a classical solution to an associated HJBI equation. The proof of the DPP is quite technical; we therefore postpone the proof of the theorem to the Chapter appendix.

Viscosity theory is equipped with a set of definitions and results some of which we will now explore. We now introduce a key definition that defines viscosity solutions:

Definition 3.11

A locally bounded lower (resp., upper) semicontinuous function $\psi : [0,T]\times S \to \mathbb{R}$ is a viscosity supersolution (resp., subsolution) to the HJBI equation (3.3) if:

For any $(s,x) \in [0,T]\times S$ and $\psi \in \mathscr{C}^{1,2}([0,T];S)$ such that (s,x) is a local minimum (resp., maximum) of $V-\psi$, we have that:

$$\max\left\{\min\left[-\frac{\partial\psi}{\partial s}(s,x)-\left(\mathscr{L}\psi(s,x)+f(s,x)\right),\psi(s,x)-G(s,x)\right],\psi(s,x)-\mathscr{M}\psi(s,x)\right\}\geq 0$$
$$\text{(resp., }\leq 0\text{).}\tag{3.29}$$

A locally bounded lower (resp., upper) semicontinuous function $\psi:[0,T]\times S\to\mathbb{R}$ is a viscosity solution to the HJBI equation (3.3) if it is both a subsolution and supersolution of (3.29).

The following lemma characterises the conditions in which the value of the game satisfies a HJBI equation:

Lemma 3.12

The function V^- is a viscosity supersolution of (3.3) and the V^+ is a viscosity subsolution of (3.3).

Proof of Lemma 3.12. The lemma is proven by contradiction.

We begin by proving that V^+ is a viscosity subsolution of (3.3). Suppose $\psi:[0,T]\times S\to\mathbb{R}$ is a test function with $\psi\in\mathscr{C}^{1,2}([0,T],S)$ and $(t,x)\in[0,T]\times S$ are such that $V^+-\psi$ attains a local minimum at (t,x) with $V^+(t,x)-\psi(t,x)=0$. We note that it remains only to show that $\forall(s,x)\in[0,T]\times S$, $\frac{\partial\psi}{\partial s}(s,x)+\mathscr{L}\psi(s,x)+f(s,x)\geq 0$ whenever $\psi(s,x)-G(s,x)>0$ which, as we now show, follows as a classical consequence of the DPP. Indeed, by Proposition 3.4 we can deduce the existence of a $\varepsilon-$optimal strategy $\alpha^\varepsilon\in\mathscr{A}_{(0,T)}$ to which the associated control is $\alpha^\varepsilon(v)\equiv u^\varepsilon\in\mathscr{U}$ (against $V^+(y_0)$) such that

$$\begin{aligned}
&\psi(y_0)=V^+(y_0)\\
&\geq\inf_{u\in\mathscr{U}}\mathbb{E}\Bigg[\int_{t_0}^{\tau_S\wedge\rho}f(Y_s^{y_0,u})ds+\sum_{j\geq1}c(\tau_j,\xi_j)\cdot 1_{\{\tau_j\leq\tau_S\wedge\rho\}}+G(Y_{\tau_S}^{y_0,u})\cdot 1_{\{\tau_S\wedge\rho<t_0+h\}}\\
&\qquad\qquad+V^+(Y_{t_0+h}^{y_0,u})\cdot 1_{\{t_0+h=\rho\}}\Bigg]\\
&\geq\mathbb{E}\Bigg[\int_{t_0}^{\tau_S\wedge\rho}f(Y_s^{y_0,u_\varepsilon})ds+\sum_{j\geq1}c(\tau_j^\varepsilon,\xi_j^\varepsilon)\cdot 1_{\{\tau_j\leq\tau_S\wedge\rho\}}+G(Y_{\tau_S\wedge\rho}^{y_0,u_\varepsilon})\cdot 1_{\{\tau_S\wedge\rho<t_0+h\}}\\
&\qquad\qquad+V^+(Y_{t_0+h}^{y_0,u_\varepsilon})\cdot 1_{\{t_0+h=\rho\}}\Bigg]-\varepsilon h.
\end{aligned}\tag{3.30}$$

Let us now define:

$$\phi^{[h]}(y_0):=\mathbb{E}\left[\int_{t_0}^{\tau_s\wedge\rho}f(Y_s^{y_0,u_0})ds+G(Y_{\tau_S\wedge\rho}^{y_0,u_0})\cdot 1_{\{\tau_S\wedge\rho<t_0+h\}}+\psi(Y_{t+h}^{y_0,u_0})\cdot 1_{\{t_0+h=\rho\}}\right],\tag{3.31}$$

where $u_0\in\mathscr{U}$ is the player I control such that no impulses are exercised. We firstly wish to show that given $\varepsilon > 0$ we have that:

$$\psi(y_0) \geq \phi^{[h]}(y_0) - 2\varepsilon h. \tag{3.32}$$

Indeed, we firstly note that: $X^{t_0,x_0,u_\varepsilon}_{t_0+h} \equiv X^{t_0,x_0,u_{0[t_0,t_0+h]}}_{t_0+h} + \sum_{j=1}^{\mu_{[t_0,t_0+h]}(u_\varepsilon)} \xi^\varepsilon_j$. We now exploit the regularity of V and the boundedness of the sequence of intervention costs, indeed we have that:

$$\begin{aligned}
&\mathbb{E}\left[\sum_{j\geq 1} c(\tau^\varepsilon_j, \xi^\varepsilon_j)\cdot 1_{\{\tau^\varepsilon_j \leq (t_0+h)\wedge\rho\}} + V^+(Y^{y_0,u_\varepsilon}_{t_0+h})\cdot 1_{\{t_0+h=\rho\}}\right] \\
&\geq \mathbb{E}\left[(V^+(Y^{y_0}_{t_0+h}) + (\lambda - c(\rho-t_0)^\beta)\cdot 1_{\mu_{[t_0,t_0+h]}(u_\varepsilon)\geq 1}\cdot 1_{\{t_0+h=\rho\}}\right] \\
&\geq \mathbb{E}[(\psi(Y^{y_0}_{t_0+h}) + (\lambda - c(\rho-t_0)^\beta)\cdot 1_{\mu_{[t_0,t_0+h]}(u_\varepsilon)\geq 1}\cdot 1_{\{t_0+h=\rho\}}],
\end{aligned} \tag{3.33}$$

where we have used the fact that $\sum_{j\geq 1}\inf_{z\in\mathscr{Z}} c(\tau^\varepsilon_j, z) \geq \lambda\cdot 1_{\mu_{[t_0,t_0+h]}(u_\varepsilon)}$ for some $\lambda > 0$ and if $Y'_{t_0+h} \equiv (t_0+h, X^{t_0,x_0,u_{0[t_0,t_0+h]}}_{t_0+h} + \sum_{j=1}^{\mu_{[t_0,t_0+h]}(u_\varepsilon)} \xi^\varepsilon_j)$, we have $V^+(Y^{y_0}_{t_0+h}) = V^+(Y'_{t_0+h}) + (V^+(Y^{y_0}_{t_0+h}) - V^+(Y'_{t_0+h})) \leq V^+(Y'_{t_0+h}) + ch^{\frac{1}{2}}$ for some $c > 0$ using Lemma A.3. and Gronwall's lemma. Using the same arguments we can similarly deduce that there exists some constant $c > 0$ such that:

$$G(Y^{y_0,u_\varepsilon}_{t_0+h}) + f(Y^{y_0,u_\varepsilon}_{t_0+h}) \geq f(Y^{y_0}_{t_0+h}) + G(Y^{y_0}_{t_0+h}) - ch^{\frac{1}{2}}. \tag{3.34}$$

Now, since $(\lambda - c(\rho-t_0)^{\frac{1}{2}})\cdot 1_{\mu_{[t_0,t_0+h]}(u_\varepsilon)\geq 1}\cdot 1_{\{t_0+h=\rho\}} = (\lambda - ch^{\frac{1}{2}})\cdot 1_{\mu_{[t_0,t_0+h]}(u_\varepsilon)\geq 1}$ and since there exists $\bar{s} \in]t_0, T[$ such that for $h \in [t_0, \bar{s}]$ for any $\varepsilon > 0$ we have that:

$$(\lambda - ch^{\frac{1}{2}})\cdot 1_{\mu_{[t_0,t_0+h]}(u_\varepsilon)\geq 1} \geq -\varepsilon h, \tag{3.35}$$

we observe that after inserting (3.35) and (3.34) into (3.33) and (3.30), we deduce that (3.32) does indeed hold. Hence, combining (3.31) and (3.32) we find that:

$$\begin{aligned}
\psi(y_0) = V^+(y_0) \geq \inf_{u\in\mathscr{U}} \mathbb{E}\Bigg[&\int_{t_0}^{\tau_S\wedge\rho} f(Y^{y_0,u}_s)ds + \sum_{j\geq 1} c(\tau_j, \xi_j)\cdot 1_{\{\tau_j\leq \tau_S\wedge\rho\}} \\
&+ G(Y^{y_0,u}_{\tau_S\wedge\rho})\cdot 1_{\{\tau_S\wedge\rho < t_0+h\}} + V^+(Y^{y_0,u}_{t_0+h})\cdot 1_{\{t_0+h=\rho\}}\Bigg] \\
\geq \mathbb{E}\left[\int_{t_0}^{\tau_S\wedge\rho} f(Y^{y_0,u_\varepsilon}_s)ds + G(Y^{y_0,u_\varepsilon}_{\tau_S\wedge\rho})\cdot 1_{\{\tau_S\wedge\rho<t_0+h\}} + \psi(Y^{t_0,x_0}_{t_0+h})\cdot 1_{\{t_0+h=\rho\}}\right] &- 2\varepsilon h.
\end{aligned} \tag{3.36}$$

Let us now define as $\Lambda(s,x) := (\frac{\partial}{\partial s} + \mathscr{L})\psi(s,x)$. By Itō's formula for càdlàg[3] semi-martingale (jump-diffusion) processes (see for example Theorem II.33 of [Pro05]), we have that:

$$\psi(y_0) = \psi(Y^{y_0,u_\varepsilon}_{t_0+h}) - \int_{t_0}^{t_0+h\wedge\tau_S\wedge\rho} \langle\nabla_x\psi(Y^{y_0,u_\varepsilon}_s), \mu(Y^{y_0,u_\varepsilon}_s)\rangle\, dB_s - \int_{t_0}^{t_0+h\wedge\tau_S\wedge\rho} \Lambda(Y^{y_0,u_\varepsilon}_s)ds.$$

[3]A function is càdlàg if it is defined on a subset of $\mathbb{R}$, has left limits everywhere and is everywhere right-continuous [Bil13].

In order to generate a contradiction, we assume that $G(s,x)-V^+(s,x)=G(s,x)-\psi(s,x)\geq 0$ and suppose that the supposition of the lemma is false so that $\Lambda(s,x)+f(s,x)>0$. We can therefore consider constants $a,h,\delta>0$ such that $\forall(s,x)\in[t_0,t_0+h]\times B_a(x)$ such that $G(s,x)-\psi(s,x)\geq\delta$ and $\Lambda(s,x)+f(s,x)\geq\delta$. Let us now define the set $E:=\{\inf_{s\in[t_0,t_0+h]}|X_s^{t_0,x_0,\cdot}-x|>a\}$ then using Lemma A.1.3 (i.e. the $\frac{1}{2}$-Hölder continuity of X) and by Tchebyshev's inequality, we can deduce the existence of a constant $c>0$ that depends only on the parameters of $X_s^{t_0,x_0,\cdot}$ such that $\mathbb{P}[E]\leq\inf_{s\in[t_0,t_0+h]}\frac{c(s-t_0)^2}{a^4}\leq\frac{ch^2}{a^4}$.

Then since $\mathbb{E}\left[\psi(Y_{t_0+h}^{y_0,u_\varepsilon})\right]-\psi(y_0)=\mathbb{E}\left[\int_{t_0}^{(t_0+h)\wedge\tau_S}\Lambda(Y_s^{y_0,u_\varepsilon})ds\right]$, we have that:

$$
\begin{aligned}
-\psi(y_0)=&\,\mathbb{E}\left[1_{E^c}\cdot\left(\int_{t_0}^{(t_0+h)\wedge\tau_S}\Lambda(Y_s^{y_0,u_\varepsilon})ds-\psi(Y_{t_0+h}^{y_0,u_\varepsilon})\right)\right]\\
&+\mathbb{E}\left[1_{E}\cdot\left(\int_{t_0}^{(t_0+h)\wedge\tau_S}\Lambda(Y_s^{y_0,u_\varepsilon})ds-\psi(Y_{t_0+h}^{y_0,u_\varepsilon})\right)\right]\\
\geq&\,\mathbb{E}\left[1_{E^c}\cdot\left(\int_{t_0}^{(t_0+h)\wedge\tau_S}\Lambda(Y_s^{y_0,u_\varepsilon})ds-\psi(Y_{t_0+h}^{y_0,u_\varepsilon})\right)\right]-\frac{ch^2}{a^4}.
\end{aligned}
$$

Hence, by the given assumptions, we have that:

$$
\begin{aligned}
-\psi(y_0)\geq&\,\mathbb{E}\Bigg[1_{E^c}\cdot\Bigg(\int_{t_0}^{t_0+h\wedge\tau_S}(\delta-f(Y_s^{y_0,u_\varepsilon}))ds\\
&+\left(\delta-G(Y_{\tau_S\wedge\rho}^{y_0,u_\varepsilon})\right)\cdot 1_{\{\rho<t_0+h\}}-\psi(Y_{t_0+h}^{y_0,u_\varepsilon})\cdot 1_{\{t_0+h=\rho\}}\Bigg)\Bigg]-\frac{ch^2}{a^4}\\
\geq&\,\mathbb{E}\Bigg[\delta\left(h+\mathbb{E}[1_{\{\rho<t_0+h\}}]\right)-\int_{t_0}^{t_0+h\wedge\tau_S}f(Y_s^{y_0,u_\varepsilon})ds\\
&-G(Y_{\tau_S\wedge\rho}^{y_0,u_\varepsilon})\cdot 1_{\{\tau_S\wedge\rho<t_0+h\}}-\psi(Y_{t_0+h}^{y_0})\cdot 1_{\{t_0+h=\rho\}}\Bigg]-2\frac{ch^2}{a^4}-\varepsilon h.
\end{aligned}
\tag{3.37}
$$

Therefore combining (3.37) and (3.36) and after rearranging we find that:

$$
\frac{2ch^2}{a^4}+3\varepsilon h\geq\mathbb{E}\left[\delta\left(h+\mathbb{E}[1_{\{\rho<t_0+h\}})\right)\right]. \tag{3.38}
$$

from which we easily deduce that:

$$
\frac{1}{2}\delta h\leq\frac{ch^2}{a^4}+\frac{3}{2}\varepsilon h. \tag{3.39}
$$

After which after dividing through by h we find that:

$$
\frac{1}{2}\delta-\left(\frac{c}{a^4}h+\frac{3}{2}\varepsilon\right)\leq 0. \tag{3.40}
$$

We then deduce the result since both h and ε can be made arbitrarily small which implies (3.40) yields a contradiction.

Next we prove that V^- is a viscosity supersolution of (3.3). As in part (i), we prove the result by generating a contradiction, hence now suppose $\psi : [0,T] \times S \to \mathbb{R}$ is a test function with $\psi \in \mathscr{C}^{1,2}([0,T],S)$ and suppose $(t,x) \in [0,T] \times S$ is such that $\mathscr{M}V^- - \psi$ attains a local maximum at (t,x). In order to generate a contradiction, we assume that $\mathscr{M}V^-(s,x) - V^-(s,x) = \mathscr{M}V^-(s,x) - \psi(s,x) \leq 0$ and suppose that the supposition of the lemma is false so that $-\Lambda(s,x) - f(s,x) > 0$, and consider constants $h, \delta > 0$ such that $\forall (s,x) \in [t_0, t_0 + h] \times B_h(x)$ and $\mathscr{M}V^-(s,x) - \psi(s,x) \leq -\delta$ and $\Lambda(s,x) + f(s,x) \leq -\delta$. By Proposition 3.4, we can deduce the existence of an $\varepsilon-$optimal strategy $\beta^\varepsilon \in \mathscr{B}_{(0,T)}$ to which the associated stopping time is $\beta^\varepsilon(u) \equiv \rho^\varepsilon \in \mathscr{T}$ for all $u \in \mathscr{U}$ (against V^-) such that

$$
\begin{aligned}
\psi(y_0) \leq \sup_{\rho \in \mathscr{T}} \mathbb{E}\Bigg[&\int_{t_0}^{(t_0+h)\wedge\rho\wedge\tau_S} f(Y_s^{y_0,\alpha(\rho)})ds + \sum_{j\geq 1} c(\tau_j, \xi_j) \cdot 1_{\{\tau_j < (t_0+h)\wedge\rho\wedge\tau_S\}} \\
&+ G(Y_{\rho\wedge\tau_S}^{y_0,\alpha(\rho)}) \cdot 1_{\{\rho\wedge\tau_S \leq t_0+h\}} + V^-(Y_{t_0+h}^{y_0,\alpha(\rho)}) \cdot 1_{\{\rho\wedge\tau_S > t_0+h\}} \Bigg] \\
\leq \mathbb{E}\Bigg[&\int_{t_0}^{(t_0+h)\wedge\rho^\varepsilon\wedge\tau_S} f(Y_s^{y_0,\alpha(\rho^\varepsilon)})ds + \sum_{j\geq 1} c(\tau_j, \xi_j) \cdot 1_{\{\tau_j < (t_0+h)\wedge\rho^\varepsilon\wedge\tau_S\}} \\
&+ G(Y_{\rho^\varepsilon\wedge\tau_S}^{y_0,\alpha(\rho^\varepsilon)}) \cdot 1_{\{\rho^\varepsilon\wedge\tau_S \leq t_0+h\}} + V^-(Y_{t_0+h}^{y_0,\alpha(\rho^\varepsilon)}) \cdot 1_{\{\rho^\varepsilon\wedge\tau_S > t_0+h\}} \Bigg] + \varepsilon h.
\end{aligned}
\tag{3.41}
$$

After re-employing the estimate (3.32), we find that:

$$
\psi(y_0) \leq \mathbb{E}\left[\int_{t_0}^{(t_0+h)\wedge\rho^\varepsilon\wedge\tau_S} f(Y_s^{y_0,u_0})ds + G(Y_{\rho^\varepsilon\wedge\tau_S}^{y_0,u_0}) \cdot 1_{\{\rho^\varepsilon\wedge\tau_S \leq t_0+h\}} + V^-(Y_{t_0+h}^{y_0,u_0}) \cdot 1_{\{\rho^\varepsilon\wedge\tau_S > t_0+h\}} \right] + 2\varepsilon h
$$

Now by Remark 2.3., we have that $-\delta \geq \mathscr{M}V^-(s,x) - \psi(s,x) \geq V^-(s,x) - \psi(s,x)$; that is $\psi(s,x) \geq V^-(s,x) + \delta$, $\forall (s,x) \in [t_0, t_0 + h] \times B_h(x)$. Using the definition of Λ and the set E introduced earlier and, again applying Itō's formula, by similar reasoning as part (i), we find that:

$$
\begin{aligned}
\psi(y_0) &\geq \mathbb{E}\left[\Big(\int_{t_0}^{(t_0+h)\wedge\tau_S} -\Lambda(Y_s^{y_0,\alpha(\rho^\varepsilon)})ds + \psi(Y_{t_0+h}^{t_0,x_0,\alpha(\rho^\varepsilon)}) \Big) \cdot 1_{E^c} \right] - \frac{ch^2}{a^4} \\
&\geq \mathbb{E}\left[\Big(\int_{t_0}^{(t_0+h)\wedge\tau_S} \Big(\delta + f(Y_s^{t_0,x_0,\alpha(\rho^\varepsilon)}) \Big) ds + V^-(Y_{t_0+h}^{y_0,\alpha(\rho^\varepsilon)}) \Big) \cdot 1_{E^c} \right] + \delta - \frac{ch^2}{a^4}.
\end{aligned}
\tag{3.42}
$$

Employing similar reasoning as in part (i), and again re-employing the estimate (3.32) we find that:

$$\begin{aligned}\psi(y_0) \geq \mathbb{E}\Bigg[&\delta\big(h+\mathbb{E}[1_{\{\rho^\varepsilon<t_0+h\}}]\big)+\int_{t_0}^{(t_0+h)\wedge\tau_S} f(Y_s^{y_0,\alpha(\rho^\varepsilon)})ds \\ &+G(Y_{\tau_S\wedge\rho^\varepsilon}^{y_0,\alpha(\rho^\varepsilon)})\cdot 1_{\{\tau_S\wedge\rho^\varepsilon<t_0+h\}}-V^-(Y_{t_0+h}^{t_0,x_0,\alpha(\rho^\varepsilon)})\cdot 1_{\{t_0+h=\rho^\varepsilon\}}\Bigg]-2\frac{ch^2}{a^4}-\varepsilon h \\ \geq \mathbb{E}\Bigg[&\delta\mathbb{E}[1_{(\rho^\varepsilon<t_0+h)}]+\int_{t_0}^{(t_0+h)\wedge\tau_S} f(Y_s^{y_0,u_0})ds+G(Y_{\tau_S\wedge\rho^\varepsilon}^{y_0,u_0})\cdot 1_{\{\tau_S\wedge\rho^\varepsilon<t_0+h\}} \\ &-V^-(Y_{t_0+h}^{t_0,x_0,u_0})\cdot 1_{\{t_0+h=\rho^\varepsilon\}}\Bigg]-2\frac{ch^2}{a^4}-2\varepsilon h,\end{aligned} \tag{3.43}$$

where we have used the fact that $h>0$ which implies that $\delta\big(h+\mathbb{E}[1_{\{(\rho^\varepsilon<t_0+h)\}}]\big)>\delta\mathbb{E}[1_{\{(\rho^\varepsilon<t_0+h)\}}]$.

Hence, combining (3.43) with (3.41) and since:

$$4\varepsilon h \geq \delta h - 2\frac{ch^2}{a^4}, \tag{3.44}$$

for h small enough $h<1$, we therefore find that:

$$\frac{1}{2}\delta-\left(2\varepsilon+\frac{ch}{a^4}\right)\leq 0, \tag{3.45}$$

which is a contradiction since both ε and h can be made arbitrarily small — hence we deduce the thesis. □

Crucially, Lemma 3.12 establishes the viscosity solution property of the game which, in conjunction with the DPP (Theorem 3.10) is derived from first principles. We have therefore succeeded in characterising the value of the game in terms of a viscosity solution which weakens the assumptions of the verification theorem (Theorem 2.7) of Chapter 2.

We now turn our attention to proving the second set of key results of the chapter which are concerned with the existence and uniqueness of the value of the game.

Theorem 3.13

If the value of the game V exists, then V is a viscosity solution to the HJBI equation (3.3).

Remark 3.14

An important observation is that if the value function is in fact smooth enough to obtain a classical solution (to HJBI equations stemming from the verification theories) then that solution is also a viscosity solution — see for example Proposition 5.2, pg.70 in [Tou13].

Proof of Theorem 3.13 Let us firstly recall that by (3.33) and selecting h such that $h<\tau_S-t_0$ we have the following inequality:

$$\psi(y_0) \geq \mathbb{E}\left[\int_{t_0}^{\rho\wedge(t_0+h)} f(Y_s^{y_0,u_\varepsilon})ds + G(Y_\rho^{y_0,u_\varepsilon})\cdot 1_{\{\rho\wedge\tau_S\leq t_0+h\}} + \psi(Y_{t_0+h}^{y_0})\cdot 1_{\{t_0+h<\rho\wedge\tau_S\}}\right] -2\varepsilon h. \quad (3.46)$$

Moreover, since $V^+ - \psi$ attains a local minimum at (t,x), we can deduce the existence of a constant $\delta > 0$ such that for $(t,x) \in [0,T]\times S$:

$$V^+(t,x) - \psi(t,x) \geq 0 \quad \text{for} \quad |(t,x)-(t_0,x_0)| \leq \delta. \quad (3.47)$$

Additionally, by Lemma A.1.3, we can deduce the existence of a constant $c > 0$ such that $\forall t \in [t_0, T[$ we have that:

$$\mathbb{E}|X_t^{t_0,x_0} - x_0| \leq c|t-t_0|^{\frac{1}{2}}. \quad (3.48)$$

We can therefore deduce the existence of a sequence $t_n \downarrow t_0$ for which $X_{t_n}^{t_0,x_0} \to x_0$ as $n\to\infty$. Let us now define the closed balls $\{B_n\}_{n\geq 1}$ by the following:

$$B_n := \{|X_{t_m}^{t_0,x_0} - x_0| \leq \delta \;\forall m \geq n\},$$
$$B_n \downarrow B \equiv \cup_{n\geq 1} B_n.$$

Further, let us now introduce the sequence of stopping times:

$$\tau_m = \sum_{n=1}^{\infty} t_{n+m}\cdot 1_{\{B_n\setminus B_{(n-1)}\}} \wedge \rho - . \quad (3.49)$$

Hence, by (3.33) we have that:

$$\psi(y_0) \geq \mathbb{E}\left[\int_{t_0}^{t_m} f(Y_s^{y_0,u_\varepsilon})ds + \psi(Y_{t_m}^{t_0,x_0})\right] - 2\varepsilon(t_m - t_0). \quad (3.50)$$

After applying Itō's formula for càdlàg semi-martingale (jump-diffusion) processes to (3.50), we find that:

$$0 \geq \mathbb{E}\left[\int_{t_0}^{t_m} \frac{\partial\psi}{\partial t}(Y_s^{y_0,u_\varepsilon}) + \mathscr{L}\psi(Y_s^{y_0,u_\varepsilon}) + f(Y_s^{y_0,u_\varepsilon}))ds\right] - 2\varepsilon(t_m - t_0). \quad (3.51)$$

Then, after dividing both sides of (3.51) by $(t_m - t_0)$ and taking the limit $m \to \infty$, we deduce that:

$$0 \geq \frac{\partial\psi}{\partial t}(y) + \mathscr{L}\psi(y) + f(y), \quad \forall y \in [0,T]\times S, \quad (3.52)$$

which proves the subsolution property. We can prove the supersolution property analogously by firstly using (3.42) and applying similar steps after which the thesis is proved. □

The following result establishes the equality of the two value functions V^- and V^+; we defer the proof of the following result to the chapter appendix:

Theorem 3.15 (Comparison Principle)

Let $V^- : [0,T] \times S \to \mathbb{R}$ be a continuous bounded viscosity subsolution to (3.3) and let $V^+ : [0,T] \times S \to \mathbb{R}$ be a continuous bounded viscosity supersolution to (3.3). Also suppose that for all $t \in [0,T]$ we have that $V^-(\cdot, X_T^{t,\cdot}) \leq V^+(\cdot, X_T^{t,\cdot})$ then we have that $V^-(t,x) \leq V^+(t,x), \qquad \forall (t,x) \in [0,T] \times S$.

Corollary 3.15.1 (The Game Admits a Value)

To prove Theorem 3.1 it remains only to reverse the inequality (3.4) therefore proving that $V^-(\cdot, X_T^{t,\cdot}) \leq V^+(\cdot, X_T^{t,\cdot})$ — a result that follows directly from the comparison principle for the game. Indeed, Theorem 3.1 and Corollary 3.15.1 then follow as direct consequences to the viscosity solutions results of Lemma 3.12 in conjunction with the comparison principle.

The above results formally establish the existence and uniqueness of the value of the game. This proves the existence of a stable equilibrium of the game which goes beyond the results of Chapter 2 which serve only to characterise the equilibrium solution, should it exist. Moreover, the results of this chapter demonstrate that the value function is a unique solution to the HJBI equation in (3.3). Crucially, the results within this chapter can be interpreted in a sense for which the (everywhere) smoothness criteria of the verification theorem are no longer required. This enables meaning to be ascribed to a solution of HJBI equation even when the value function is not everywhere differentiable.

3.4 Chapter Appendix

Proof of Lemma 3.3. The proof of the lemma is similar to that of Lemma 3.3 in [Cos13] with some modifications. The main idea is to prove that for all $(t,x) \in [0,T] \times S$, there exists a control $\bar{u} \in \mathscr{U} \backslash \bar{\mathscr{U}}_{(0,T)}$ and an $\mathscr{F}-$ measurable stopping time $\bar{\rho} \in \mathscr{T} \backslash t'$ where $\bar{\mathscr{U}}_{(0,T)} \subset \mathscr{U}$ is the set of admissible impulse controls $\mathscr{U}$ which excludes impulses at time t', for which the following inequality holds $\forall u \in \mathscr{U}$, given some $\mathscr{F}-$measurable stopping time $\rho \in \mathscr{T}$ and for some $\varepsilon > 0$:

$$\left|J[t',x;u,\beta(u)] - J[t',x;\bar{u},\bar{\beta}(\bar{u})]\right| + \left|J[t',x;\alpha(\rho),\rho] - J[t',x;\bar{\alpha}(\bar{\rho}),\bar{\rho}]\right| \leq \varepsilon. \tag{3.53}$$

We prove the result for the case in which player I exercises only one intervention at the point t since the extension to multiple interventions is straightforward.

W.l.o.g., we can employ the following short-hands $\beta(u) \equiv \rho \in \mathscr{T}, \bar{\beta}(\bar{u}) \equiv \bar{\rho} \in \mathscr{T}_{t'}, \alpha(\rho) \equiv u \in \mathscr{U}$ and $\bar{\alpha}(\bar{\rho}) \equiv \bar{u} \in \bar{\mathscr{U}}_{(0,T)}$. The result is proven by constructing the following control and stopping times:

$$u_n = \xi \cdot 1_{[\tau_n,T]} + u',$$

where $\tau_n = (\tau + \frac{1}{n}) \cdot 1_{\{\tau = t'\}} + \tau \cdot 1_{\{\tau > t'\}}$ and

$$\rho_n = \left(\rho + \frac{1}{n}\right) \cdot 1_{\{\rho = t'\}} + \rho \cdot 1_{\{\rho > t'\}},$$

where $u' = \sum_{j \geq 1} \xi_j \cdot 1_{[\tau_j,T]}$.

Writing $Y_s^{t,x,u} \equiv (s, X_s^{t,x,u})$ and $y' \equiv (t', x_0)$ for any $s \in [0,T]$ and $\forall (t,x) \in [0,T] \times S, \forall u \in \mathscr{U}$, we have:

$$\begin{aligned}
& J[y';u,\rho] - J[y';u_n,\rho_n] \\
=& \mathbb{E}\Bigg[\int_{t'}^{\rho \wedge \tau_S} f(Y_s^{y',u})ds - \int_{t'}^{\rho_n \wedge \tau_S} f(Y_s^{y',u_n})ds - \sum_{j \geq 1} c(\tau_j, \xi_j) \cdot 1_{\{\rho \leq \tau_j < \rho_n\}} \\
& + \left(G(Y_{\rho \wedge \tau_S}^{y',u}) - G(Y_{\rho_n \wedge \tau_S}^{y',u_n})\right) \cdot 1_{\{\tau_S \wedge \rho_n < \infty\}}\Bigg] \\
=& \mathbb{E}\Bigg[\int_{t'}^{\rho \wedge \tau_S} \left(f(Y_s^{y',u}) - f(Y_s^{y',u_n})\right)ds - \int_{\rho \wedge \tau_S}^{\rho_n \wedge \tau_S} f(Y_s^{y',u_n})ds \\
& + \left(G(Y_{\rho \wedge \tau_S}^{y',u}) - G(Y_{\rho_n \wedge \tau_S}^{y',u_n})\right) \cdot 1_{\{\tau_S \wedge \rho_n < \infty\}} - \sum_{j \geq 1} c(\tau_j, \xi_j) \cdot 1_{\{\rho \leq \tau_j < \rho_n\}}\Bigg].
\end{aligned}$$

We now readily observe that $X_s^{t',x_0,u_n} \to X_s^{t',x_0,u}$, $\mathbb{P}-$a.s. Additionally, by construction, $\rho_n \to \rho$ as $n \to \infty, \mathbb{P}$-a.s. hence, after invoking the dominated convergence theorem, we can deduce the existence of an integer $N \geq 1$ such that $\forall \varepsilon > 0$ and $\forall n \geq N$ such that:

$$J[y'; u, \rho] - J[y'; u_n, \rho_n] \leq \varepsilon. \tag{3.54}$$

The proof can be extended to multiple impulse case (at time t') straightforwardly after employing conditions A3 (iii) and A3 (iv) where after the proof easily reduces to the single impulse case. □

We now turn to proving the DPP. Before giving a proof of the DPP (3.27) - (3.28) for the game, we firstly make the following remark.

Remark 3.16

If the value functions are known a priori to be continuous (or the setting is a deterministic or discrete-time case) the derivation of the DPP is straightforward. Otherwise, in general, we must use one of two arguments: a measurable selection argument or establish the regularity of the value functions then construct a measurable selection i.e. partition the state space then construct a measurable selection (this uses the Lindelöf property[4] of the canonical space).

Proof of Theorem 3.10 We begin by proving:

$$\begin{aligned} V^+(y_0) \geq \sup_{\beta \in \mathscr{B}_{(0,T)}} \inf_{u \in \mathscr{U}} \mathbb{E}\Bigg[&\int_{t_0}^{(t_0+h)\wedge\beta(u)\wedge\tau_S} f(Y_s^{y_0,u})ds + \sum_{j\geq 1} c(\tau_j, \xi_j) \cdot 1_{\{\tau_j < t_0+h\wedge\beta(u)\wedge\tau_S\}} \\ &+ G\left(Y^{t_0,x_0,u}_{\beta(u)\wedge\tau_S}\right) \cdot 1_{\{\beta(u)\wedge\tau_S \leq t_0+h\}} \Bigg] + V^+(X^{t_0,x_0,u}_{(t_0+h)\wedge\beta(u)}) \cdot 1_{\{\beta(u)\wedge\tau_S > t_0+h\}}, \end{aligned} \tag{3.55}$$

for some $\infty > h > 0$.

Having established the uniform continuity of the functions V^- and V^+, a countable selection argument is sufficient in order to derive the DPP, therefore avoiding measurable selection arguments directly. Indeed, using Proposition 3.4, we can find a set of controls that produce values of J that are arbitrarily close to the values of V^- and V^+ at some given point.

Hence, let $(A^j)^{j\in\mathbb{N}}$ be a partition of $\mathbb{R}^p$. Let $\hat{\mu} \in \mathscr{B}_{(0,T)}$ be some $\varepsilon-$optimal strategy against $\sup_{\beta(u)\in\mathscr{B}_{(0,T)}} \inf_{u\in\mathscr{U}} V^+(t_0 + h \wedge \beta(u), X^{t_0,x_0,u}_{t_0+h})$. Note by Lemma A.1 we can deduce that since $\hat{\mu}$ is an $\varepsilon-$optimal strategy against $\sup_{\beta(u)\in\mathscr{B}_{(0,T)}} \inf_{u\in\mathscr{U}} V^+(t_0 + h \wedge \beta(u), X^{t_0,x_0,u}_{t_0+h})$ then there exists some $2\varepsilon-$optimal strategy $\hat{\mu}^x \in \mathscr{B}_{(0,T)}$ against $\sup_{\beta(u)\in\mathscr{B}_{(0,T)}} \inf_{u\in\mathscr{U}} V^+(t_0 + h \wedge \beta(u), x)$ such that $\sup_{\beta(u)\in\mathscr{B}_{(0,T)}} \inf_{u\in\mathscr{U}} V^+(t_0 + h \wedge \beta(u), x) - (\varepsilon + \delta) \leq \inf_{u\in\mathscr{U}} J[t_0 + h, x; u_{[t_0+h,T]}, \hat{\mu}^x(u_{[t_0+h,T]})]$ where $\varepsilon > \delta > 0$. Hence, we deduce that the strategy $\hat{\mu}^x$ is a $2\varepsilon-$optimal strategy $\inf_{u\in\mathscr{U}} V^+(t_0 + h \wedge \beta(u), y)$ for all $y \in B_\delta(x)$ within some radius $0 < \delta < \varepsilon$.

[4] A topological space is said to be Lindelöf or have the Lindelöf property if every open cover of X has a countable subcover see for example [BT11] for further details.

Let us therefore construct the strategy μ by:

$$\hat{\mu}(u)(s) = \begin{cases} \mu(u)(s), & s \in [t_0, t_0+h[\\ \hat{\mu}^x(u_{[t_0+h,T]})(s)), & s \in [t_0+h, T], X_{t_0+h}^{t_0,x_0,u} \in B_\delta(x) \end{cases} \tag{3.56}$$

Now for any $(t_0, x_0) \in [0,T] \times S, u \in \mathscr{U}, \mu \in \mathscr{B}_{(0,T)}$ and $\forall u_{[t_0+h,T]} \in \mathscr{U}_{(t_0+h,T)}$ using Lemma 3.3, for some sufficiently small $h > 0$, we have:

$$\begin{aligned}
&\mathbb{E}\left[\int_{t_0}^{\hat{\mu}(u)\wedge\tau_S} f(Y_s^{y_0,u})ds + \sum_{j\geq 1} c(\tau_j', \xi_j') \cdot 1_{\{\tau_j' \leq \hat{\mu}(u)\wedge\tau_S\}} + G\left(Y_{\hat{\mu}(u)\wedge\tau_S}^{y_0,u}\right) \cdot 1_{\{\hat{\mu}(u)\wedge\tau_S<\infty\}}\right] \\
&\geq \mathbb{E}\left[\int_{t_0}^{(t_0+h)\wedge\mu(u)\wedge\tau_S} f(Y_s^{y_0,u})ds + \sum_{j\geq 1} c(\tau_j, \xi_j) \cdot 1_{\{\tau_j \leq (t_0+h)\wedge\mu(u)\wedge\tau_S\}}\right. \\
&\qquad\qquad \left. + G\left(Y_{\mu(u)\wedge\tau_S}^{y_0,u}\right) \cdot 1_{\{\mu(u)\wedge\tau_S < t_0+h\}}\right] \\
&+\mathbb{E}\left[\int_{t_0+h}^{\hat{\mu}(u)\wedge\tau_S} f(Y_s^{y_0,u})ds + \sum_{j\geq 1} c(\tau_j', \xi_j') \cdot 1_{\{\hat{\mu}(u)\wedge\tau_S \geq \tau_j' > t_0+h\}}\right. \\
&\qquad\qquad \left. + G\left(Y_{\hat{\mu}(u)\wedge\tau_S}^{y_0,u}\right) \cdot 1_{\{\hat{\mu}(u)\wedge\tau_S<\infty\}}\right] \cdot 1_{\{\hat{\mu}(u)\wedge\tau_S > t_0+h\}} - \varepsilon,
\end{aligned} \tag{3.57}$$

for some arbitrary $\varepsilon > 0$. Using the properties of X, we can further rewrite (3.57) as:

$$\begin{aligned}
&\mathbb{E}\left[\int_{t_0}^{(t_0+h)\wedge\mu(u)\wedge\tau_S} f(Y_s^{y_0,u})ds + \sum_{j\geq 1} c(\tau_j, \xi_j) \cdot 1_{\{\tau_j \leq (t_0+h)\wedge\mu(u)\wedge\tau_S\}}\right. \\
&\qquad\qquad \left. + G\left(Y_{\mu(u)\wedge\tau_S}^{y_0,u}\right) \cdot 1_{\{\mu(u)\wedge\tau_S \leq t_0+h\}}\right] \\
&+\mathbb{E}\left[\int_{t_0+h}^{\hat{\mu}(u_{[t_0+h,\tau_S]})\wedge\tau_S} f(Y_s^{Y_{t_0+h}^{t_0,x_0,u}, u_{[t_0+h,\tau_S]}})ds + \sum_{j\geq 1} c(\tau_j', \xi_j') \cdot 1_{\{\hat{\mu}(u_{[t_0+h,\tau_S]})\wedge\tau_S \geq \tau_j' > t_0+h\}}\right. \\
&\qquad \left. +G\left(Y_{\hat{\mu}(u_{[t_0+h,\tau_S]})\wedge\tau_S}^{Y_{t_0+h}^{t_0,x_0,u}, u_{[t_0+h,\tau_S]}}\right) \cdot 1_{\{\hat{\mu}(u_{[t_0+h,\tau_S]})\wedge\tau_S<\infty\}}\right] \cdot 1_{\{\hat{\mu}(u_{[t_0+h,\tau_S]})\wedge\tau_S \geq t_0+h\}} - \varepsilon.
\end{aligned} \tag{3.58}$$

Exploiting the regularity of V (Prop. 3.4) and the $\varepsilon-$optimality of μ, we find:

$$\begin{aligned}
&\mathbb{E}\Bigg[\int_{t_0+h}^{\hat{\mu}(u_{[t_0+h,\tau_S]})\wedge\tau_S} f(Y_s^{Y_{t_0+h}^{t_0,x_0,u},u_{[t_0+h,\tau_S]}})ds+\sum_{j\geq 1}c(\tau'_j,\xi'_j)\cdot 1_{\{\hat{\mu}(u_{[t_0+h,\tau_S]})\wedge\tau_S\geq\tau'_j>t_0+h\}} \\
&\quad +G\left(Y_{\hat{\mu}(u_{[t_0+h,\tau_S]})\wedge\tau_S}^{Y_{t_0+h}^{t_0,x_0,u},u_{[t_0+h,\tau_S]}}\right)\cdot 1_{\{\hat{\mu}(u_{[t_0+h,\tau_S]})\wedge\tau_S\geq t_0+h\}}\Bigg]-\varepsilon \\
&\geq \sum_{j\in\mathbb{N}}\mathbb{E}\Bigg[\int_{t_0+h}^{\hat{\mu}(u)\wedge\tau_S} f(Y_s^{t_0+h,y^j,u_{[t_0+h,\tau_S]}})ds+\sum_{j\geq 1}c(\tau'_j,\xi'_j)\cdot 1_{\{\hat{\mu}(u)\wedge\tau_S\geq\tau'_j\geq t_0+h\}} \\
&\qquad +G\left(Y_{\hat{\mu}(u)\wedge\tau_S}^{t_0+h,y^j,u_{[t_0+h,\tau_S]}}\right)\cdot 1_{\{\hat{\mu}(u_{[t_0+h,\tau_S]})\wedge\tau_S\geq t_0+h\}}\Bigg]-c\delta-\varepsilon \\
&\qquad\qquad \geq V^+(t_0+h\wedge\hat{\mu}(u),X_{t_0+h}^{t_0,x_0,u})-2\varepsilon-c\delta, \qquad (3.59)
\end{aligned}$$

using the $\varepsilon-$optimality of the strategy $\hat{\mu}$ against $\sup_{\beta\in\mathscr{B}_{(0,T)}}\inf_{u\in\mathscr{U}}V^+(t_0+h\wedge\beta(u),X_{t_0+h}^{t_0,x_0,u})$. Putting (3.59) together with (3.58) yields:

$$\begin{aligned}
&\mathbb{E}\Bigg[\int_{t_0}^{(t_0+h)\wedge\mu(u)\wedge\tau_S} f(Y_s^{y_0,u})ds+\sum_{j\geq 1}c(\tau_j,\xi_j)\cdot 1_{\{\tau_j\leq(t_0+h)\wedge\mu(u)\}}+G(Y_{\mu(u)\wedge\tau_S}^{y_0,u})\cdot 1_{\{\mu(u)\wedge\tau_S\leq t_0+h\}}\Bigg] \\
&+\mathbb{E}\Bigg[\int_{t_0+h}^{\mu(u)\wedge\tau_S} f(Y_s^{y_0,u})ds+\sum_{j\geq 1}c(\tau_j,\xi_j)\cdot 1_{\{\mu(u)\wedge\tau_S\geq\tau_j>t_0+h\}} \\
&\qquad +G(Y_{\mu(u)\wedge\tau_S}^{y_0,u})\cdot 1_{\{\mu(u)\wedge\tau_S<\infty\}}\Bigg]\cdot 1_{\{\mu(u)\geq t_0+h\}} \\
&\geq \mathbb{E}\Bigg[\int_{t_0}^{(t_0+h)\wedge\mu(u)\wedge\tau_S} f(Y_s^{y_0,u})ds+\sum_{j\geq 1}c(\tau_j,\xi_j)\cdot 1_{\{\tau_j\leq(t_0+h)\wedge\mu(u)\wedge\tau_S\}} \\
&+G(Y_{\mu(u)\wedge\tau_S}^{y_0,u})\cdot 1_{\{\mu(u)\wedge\tau_S\leq t_0+h\}}+V^+(t_0+h\wedge\hat{\mu}(u),X_{t_0+h}^{t_0,x_0,u})\cdot 1_{\{\hat{\mu}(u)\wedge\tau_S>t_0+h\}}\Bigg]-c\delta-2\varepsilon.
\end{aligned}$$

from which after successively applying the $\inf$ and $\sup$ operators we deduce the first result since δ and ε can be chosen arbitrarily.

We prove the reverse inequality in an analogous manner, in particular, we now prove the inequality for the function V^-. Indeed, by Prop. 3.4, we deduce the existence of a strategy $\alpha^{(1,\varepsilon)} \in \mathscr{A}_{(0,T)}$ against $V^-(t,x)$ such that

$$\begin{aligned}
&\inf_{\alpha\in\mathscr{A}_{(0,T)}}\sup_{\rho\in\mathscr{T}}\mathbb{E}\Bigg[\int_{t_0}^{(t_0+h)\wedge\rho\wedge\tau_S} f(Y_s^{y_0,\alpha(\rho)})ds+\sum_{j\geq 1}c(\tau_j,\xi_j)\cdot 1_{\{\tau_j\leq(t_0+h)\wedge\rho\wedge\tau_S\}}\\
&\qquad +G(Y_{\rho\wedge\tau_S}^{y_0,\alpha(\rho)})\cdot 1_{\{\rho\wedge\tau_S\leq t_0+h\}}+V^-(Y_{t_0+h}^{y_0,\alpha(\rho)})\cdot 1_{\{\rho>t_0+h\}}\Bigg]\\
\geq&\mathbb{E}\Bigg[\int_{t_0}^{(t_0+h)\wedge\rho\wedge\tau_S} f(Y_s^{y_0,\alpha^{(1,\varepsilon)}(\rho)})ds+\sum_{j\geq 1}c(\tau_j^{1,\varepsilon},\xi_j^{1,\varepsilon})\cdot 1_{\{\tau_j^{1,\varepsilon}\leq(t_0+h)\wedge\rho\wedge\tau_S\}}\\
&+G(Y_{\rho\wedge\tau_S}^{y_0,\alpha^{(1,\varepsilon)}(\rho))}\cdot 1_{\{\rho\wedge\tau_S\leq t_0+h\}}+V^-(Y_{t_0+h}^{y_0,\alpha^{(1,\varepsilon)}(\rho)})\cdot 1_{\{\rho\wedge\tau_S>t_0+h\}}\Bigg]-\varepsilon,
\end{aligned}\tag{3.60}$$

where we have used the shorthand $\tau_j(\rho)\equiv\tau_j$ and $\xi_j(\rho)\equiv\xi_j$, $\forall j\in\mathbb{N}$.

We now build the strategy α by:

$$\alpha(\rho)(s)=\begin{cases}\alpha^{(1,\varepsilon)}(\rho)(s), & s\in[t_0,t_0+h[\\ \alpha^{A_i}(\rho')(s), & s\in[t_0+h,T], X_{t_0+h}^{t_0,x_0,\alpha^{(1,\varepsilon)}}\in A_i,\end{cases}$$

where we have used ρ' to denote the player II stopping time such that $\rho'\in\mathscr{T}_{(t_0+h,T)}\subset[t_0+h,T]$. Let $\alpha^{(2,\varepsilon)}\in\mathscr{A}_{(0,T)}$ be an $\varepsilon-$optimal strategy against $\sup_{\rho\in\mathscr{T}}V^-((t_0+h)\wedge\rho,x)$ for any $x\in S$. Using Lemma 3.3 and by similar reasoning as in part (i), we can also deduce the existence of a strategy $\alpha^{A_i}\in\mathscr{A}_{(t_0+h,T)}$ such that $\forall q\in A_i,\rho'\in\mathscr{T}_{(t_0+h,T)}$ and some $\varepsilon>0$ such that

$$V^-((t_0+h)\wedge\rho,q)\geq J[(t_0+h)\wedge\rho,q;\alpha^{\mathscr{A}_i}(\rho')]-\varepsilon.\tag{3.61}$$

We therefore observe that:

$$\begin{aligned}
\mathbb{E}\left[V^-\left(Y_{t_0+h}^{y_0,\alpha^{(1,\varepsilon)}(\rho')}\right)\right]&=\mathbb{E}\left[\sum_{i\geq 1}V^-\left(Y_{t_0+h}^{y_0,\alpha^{(1,\varepsilon)}(\rho')}\right)\cdot 1_{\{X_{t_0+h}^{t_0,x_0,\alpha^{(1,\varepsilon)}(\rho'))}\in\mathscr{A}_i\}}\right]\\
&\geq\mathbb{E}\left[\sum_{i\geq 1}J\left[Y_{t_0+h}^{y_0,\alpha^{(1,\varepsilon)}(\rho')};\alpha^{\mathscr{A}_i}(\rho'),\rho'\right]\cdot 1_{\{X_{t_0+h}^{t_0,x_0,\alpha^{(1,\varepsilon)}(\rho')}\in\mathscr{A}_i\}}\right]-\varepsilon\\
&=J\left[Y_{t_0+h}^{y_0,\alpha^{(1,\varepsilon)}(\rho')};\sum_{i\geq 1}\alpha^{A_i}(\rho')\cdot 1_{\{X_{t_0+h}^{t_0,x_0,\alpha^{(1,\varepsilon)}(\rho')}\in\mathscr{A}_i\}},\rho'\right]-\varepsilon.
\end{aligned}\tag{3.62}$$

We construct the strategy $\bar{\alpha}^{(2,\varepsilon)}(\rho)\in\bar{\mathscr{A}}(t_0+h)$ defined by:$\bar{\alpha}^{(2,\varepsilon)}(\rho):=\sum_{i\geq 1}\alpha^{\mathscr{A}_i}(\rho')\cdot$

$1_{\{X^{t_0,x_0,\alpha^{(1,\varepsilon)}(\rho')}_{t_0+h}\in\mathscr{A}_i\}}$. Now, after introducing the strategy $\bar{\alpha}^{(2,\varepsilon)}$ to (3.62), we deduce that:

$$\begin{aligned}
&\mathbb{E}\Bigg[\int_{t_0}^{(t_0+h)\wedge\rho\wedge\tau_S} f(Y_s^{y_0,\alpha^{(1,\varepsilon)}(\rho)})ds+\sum_{j\geq 1}c(\tau_j^{1,\varepsilon},\xi_j^{1,\varepsilon})\cdot 1_{\{\tau_j^{1,\varepsilon}\leq(t_0+h)\wedge\rho\wedge\tau_S\}}\\
&\quad+G\left(Y^{y_0,\alpha^{(1,\varepsilon)}(\rho)}_{\rho\wedge\tau_S}\right)\cdot 1_{\{\rho\wedge\tau_S\leq t_0+h\}}+V^-\left(Y^{y_0,\alpha^{(1,\varepsilon)}(\rho)}_{t_0+h}\right)\cdot 1_{\{\rho\wedge\tau_S>t_0+h\}}\Bigg]-\varepsilon\\
&\geq\mathbb{E}\Bigg[\int_{t_0}^{(t_0+h)\wedge\rho\wedge\tau_S} f(Y_s^{y_0,\alpha^{(1,\varepsilon)}(\rho)})ds+\sum_{j\geq 1}c(\tau_j^{1,\varepsilon},\xi_j^{1,\varepsilon})\cdot 1_{\{\tau_j^{1,\varepsilon}\leq(t_0+h)\wedge\rho\wedge\tau_S\}}\\
&\quad+G\left(Y^{y_0,\alpha^{(1,\varepsilon)}(\rho)}_{\rho\wedge\tau_S}\right)\cdot 1_{\{\rho\wedge\tau_S\leq t_0+h\}}+J\left[Y^{y_0,\alpha^{(1,\varepsilon)}(\rho')}_{t_0+h};\bar{\alpha}^{(2,\varepsilon)}(\rho'),\rho'\right]\cdot 1_{\{\rho\wedge\tau_S>t_0+h\}}\Bigg]-2\varepsilon.
\end{aligned}\tag{3.63}$$

We lastly construct the strategy $\alpha^\varepsilon\in\mathscr{A}(t_0)$ which consists of the strategy $\alpha^{(1,\varepsilon)}$ which is played up to time t_0+h at which point the strategy $\bar{\alpha}^{(2,\varepsilon)}$ is then played.

Hence, after putting (3.63) and (3.60) together we observe that:

$$\begin{aligned}
V^-(y_0)\geq\ &\mathbb{E}\Bigg[\int_{t_0}^{(t_0+h)\wedge\rho\wedge\tau_S} f(Y_s^{y_0,\alpha^{(1,\varepsilon)}(\rho)})ds+\sum_{j\geq 1}c(\tau_j^{1,\varepsilon},\xi_j^{1,\varepsilon})\cdot 1_{\{\tau_j^{1,\varepsilon}\leq(t_0+h)\wedge\rho\wedge\tau_S\}}\\
&\quad+G(Y_\rho^{y_0,\alpha^{(1,\varepsilon)}(\rho)})\cdot 1_{\{\rho\wedge\tau_S\leq t_0+h\}}+V^-(Y^{y_0,\alpha^{(1,\varepsilon)}(\rho)}_{t_0+h})\cdot 1_{\{\rho\wedge\tau_S>t_0+h\}}\Bigg]-\varepsilon\\
\geq\ &\mathbb{E}\Bigg[\int_{t_0}^{(t_0+h)\wedge\rho\wedge\tau_S} f(Y_s^{y_0,\alpha^{\varepsilon}(\rho)})ds+\sum_{j\geq 1}c(\tau_j^{\varepsilon},\xi_j^{\varepsilon})\cdot 1_{\{\tau_j^{\varepsilon}\leq(t_0+h)\wedge\rho\wedge\tau_S\}}\\
&\quad+G(Y^{y_0,\alpha^{\varepsilon}(\rho)}_{\rho\wedge\tau_S})\cdot 1_{\{\rho\wedge\tau_S\leq t_0+h\}}+J\left[Y^{y_0,\alpha^{\varepsilon}(\rho')}_{t_0+h};\alpha^{\varepsilon}(\rho'),\rho'\right]\cdot 1_{\{\rho\wedge\tau_S>t_0+h\}}\Bigg]-2\varepsilon.
\end{aligned}$$

Moreover, since ε is arbitrary, we readily deduce that:

$$\begin{aligned}
V^-(y_0)\geq &\inf_{\alpha\in\mathscr{A}_{(0,T)}}\sup_{\rho\in\mathscr{T}}\mathbb{E}\Bigg[\int_{t_0}^{(t_0+h)\wedge\rho\wedge\tau_S} f(Y_s^{y_0,\alpha(\rho)})ds+\sum_{j\geq 1}c(\tau_j,\xi_j)\cdot 1_{\{\tau_j\leq(t_0+h)\wedge\rho\wedge\tau_S\}}\\
&+G(Y^{y_0,\alpha(\rho)}_{\rho\wedge\tau_S})\cdot 1_{\{\rho\wedge\tau_S\leq t_0+h\}}+V^-(Y^{y_0,\alpha(\rho)}_{t_0+h})\cdot 1_{\{\rho\wedge\tau_S>t_0+h\}}\Bigg],
\end{aligned}$$

from which we readily deduce the required result. We prove the reverse inequality in an analogous manner for which, in conjunction with (3.28) proves the thesis. □

The proof of the comparison principle is an adaptation of the standard comparison theorem result. We prove Theorem 3.13 by making the necessary adjustments to existing comparison theorem results (see for example, [Car10]). We first require the following definitions which allow us to construct a replacement of the notion of a derivative for a non-differentiable function and, thereafter make use of a widely-known equivalence result.

Definition 3.17

We say that a function $\psi \in \mathscr{C}([0,T];\mathbb{R}^p)$ is an upper (resp., lower) semicontinuous function if for any sequence $x_k(s) \in \mathbb{R}^p$ such that $x_k \to x_0 \in \mathbb{R}^p$, we have that $\limsup_{k\to\infty} \psi(s,x_k(s)) \geq \psi(s,x_0(s))$ (resp., $\liminf_{k\to\infty} \psi(s,x_k(s)) \geq \psi(s,x_0(s))$).

To prove Theorem 3.13, we need the following definition and result:

Definition 3.18

Let $\psi \in \mathscr{C}([0,T];\mathbb{R}^p)$ be a lower semicontinuous function, then the parabolic subjet of ψ at the point $(t,x) \in [0,T]\times\mathbb{R}^p$ which we denote by $J^{(2,-)}\psi(t,x)$ is the set of triples $(M,r,q) \in \mathbb{S}(p)\times\mathbb{R}\times\mathbb{R}^p$ such that

$$\psi(s,y) \geq \psi(s,x) + r(s-t) + \langle q, y-x\rangle + \frac{1}{2}\langle M(y-x), y-x\rangle + \mathscr{O}(|s-t| + |y-x|^2)) \qquad (3.64)$$

as $s \to t$ or as $s \downarrow t$ when $t = 0$ and $y \to x$. We can analogously define the parabolic superjet of ψ at the point $(t,x) \in [0,T]\times\mathbb{R}^p$ which we denote by $J^{(2,+)}\psi(t,x)$ by the following:

$$J^{(2,+)}\psi(t,x) \equiv -J^{(2,-)}(-\psi)(t,x). \qquad (3.65)$$

Parabolic (semi-)jets play an important role in second order differential equations (e.g. the HJBI equation) and are motivated from a classic Taylor expansion of the test functions. In particular, (semi-)jets serve as a basis of a generalised derivative which underscores an equivalent definition of a viscosity solution.

Let us also introduce the following notation the convenience of which is immediate: suppose $\Lambda : \mathbb{S}(p)\times\mathbb{R}^p\times\mathscr{C}([0,T];\mathbb{R}^p)\times[0,T]\times\mathbb{R}^p \to \mathbb{R}$ then we define Λ by:

$$\begin{aligned}\Lambda(M,r,\psi,m,q) := m - \sum_{i=1}^{p}\mu_i(m,q)r_i + \frac{1}{2}\sum_{i,j=1}^{p}(\sigma\cdot\sigma^T)_{ij}(q)M_{ij} \\ + \sum_{j=1}^{l}\int_{\mathbb{R}^p}\psi(m,q+\gamma^{(j)})(m,q,z_j)) - \psi(m,q) - r\cdot\gamma^{(j)})(m,q,z_j)\nu_j(dz_j) + f(m,q). \quad (3.66)\end{aligned}$$

We note that using Definition 3.18 we can obtain the following result — the proof of which is standard and therefore omitted:

Lemma 3.19

Let $\psi \in \mathscr{C}([0,T];\mathbb{R}^p)$ be a lower (resp., upper) semicontinuous function, then ψ is a viscosity supersolution (resp., subsolution) to the HJBI equation (3.3) iff:$\forall(t,x) \in [0,T]\times S$ and $\forall(M,m,r) \in \bar{J}^{(2,-)}\psi(t,x)$ (resp., $\bar{J}^{(2,+)}\psi(t,x)$) we have that:

$$\max\{\min[-\Lambda(M,r,\psi,m,x), V-G], V-\mathscr{M}V\} \geq 0 \text{ (resp., } \leq 0), \qquad (3.67)$$

and,

$$\max\{V(\tau_S,x)-G(\tau_S,x),V(\tau_S,x)-\mathscr{M}V(\tau_S,x)\}\geq 0 \text{ (resp., } \leq 0); \qquad \forall x\in S. \tag{3.68}$$

Having stated the above results, we can now prove Theorem 3.13:

Proof of Theorem 3.13. We prove the comparison principle using the standard technique as introduced in [CIL92] — namely we prove the result by contradiction. Suppose that the functions V and U are a viscosity subsolution and supersolution (respectively) to the HJBI equation (3.3), then to prove the theorem we must prove that under assumptions A.1.1 - A.4 we have that $V\leq U$ on $[0,T]\times S$.

Hence, let us firstly assume that $\forall x\in S$:

$$V(T,x)\leq U(T,x). \tag{3.69}$$

Moreover, let us also assume that:

$$M:=\sup_{[0,T]\times S}(V-U)>0. \tag{3.70}$$

Now by Proposition 3.5 we know that V and U are bounded hence for some $\varepsilon,\alpha,\eta>0$ such that $\forall(t,s,x,y)\in[0,T]^2\times S^2$:

$$M_{\varepsilon,\alpha,\eta}:=\max_{(t,s,x,y)\in[0,T]^2\times S^2}V(t,x)-U(s,y)-\frac{(|t-s|^2+|x-y|^2)}{2\varepsilon}-\frac{\alpha}{2}(|x|^2+|y|^2)+\eta t), \tag{3.71}$$

is both a finitely bounded quantity and has some maximum which is achieved by a point (which depends on $(\varepsilon,\alpha,\eta$)) which we shall denote by $(\bar{t},\bar{s},\bar{x},\bar{y})\in[0,T]^2\times S^2$. Now since there exist values $(s,y)\in[0,T]\times S$ such that $M=M_{(\varepsilon,\alpha,\eta)}$, we have that:

$$0<M\leq M_{\varepsilon,\alpha,\eta}=V(\bar{t},\bar{x})-U(\bar{s},\bar{y})-\frac{(|\bar{t}-\bar{s}|^2+|\bar{x}-\bar{y}|^2}{2\varepsilon}-\frac{\alpha}{2}(|\bar{x}|^2+|\bar{y}|^2)+\eta\bar{t}). \tag{3.72}$$

Hence,

$$\lim_{\varepsilon\downarrow 0}\frac{(|\bar{t}-\bar{s}|^2+|\bar{x}-\bar{y}|^2}{2\varepsilon}<V(\bar{t},\bar{x})-U(\bar{s},\bar{y})-\frac{\alpha}{2}(|\bar{x}|^2+|\bar{y}|^2)+\eta\bar{t}). \tag{3.73}$$

Now, since the RHS is composed of finitely bounded terms and the LHS is non-negative, we readily conclude that $\lim_{\varepsilon\downarrow 0}\frac{(|\bar{t}-\bar{s}|^2+|\bar{x}-\bar{y}|^2)}{2\varepsilon}=0$ and hence we observe that $|\bar{t}-\bar{s}|^2+|\bar{x}-\bar{y}|^2\to 0$ as $\varepsilon\downarrow 0$.

Moreover, if we denote by $(s_n,y_n),(t_n,x_n)\in[0,T]\times S$ and $\varepsilon_n>0$ a triple of bounded sequences such that $(s_n,y_n),(t_n,x_n)\to(t,x)$ as $\varepsilon_n\to 0$ we have that:

$$M \le \lim_{\varepsilon,\alpha,\eta\downarrow 0} M_{\varepsilon,\alpha,\eta} \le \lim_{\varepsilon,\alpha,\eta\downarrow 0}\sup_{[0,T]\times S}[V(t_n,x_n)-U(s_n,y_n)-\frac{(|t-s|^2+|x-y|^2)}{2\varepsilon}-\frac{\alpha}{2}(|x|^2+|y|^2)+\eta t]$$
$$=\sup_{[0,T]\times S}\lim_{\varepsilon,\alpha,\eta\downarrow 0}[V(t_n,x_n)-U(s_n,y_n)-\frac{(|t-s|^2+|x-y|^2)}{2\varepsilon}-\frac{\alpha}{2}(|x|^2+|y|^2)+\eta t]$$
$$=\sup_{[0,T]\times S}[V(t,x)-U(t,x)]=M.$$

We therefore readily deduce that:

$$\lim_{(\varepsilon,\alpha,\eta\downarrow 0)} M_{\varepsilon,\alpha,\eta}=M. \tag{3.74}$$

We now invoke Ishii's lemma (e.g. as in [CIL92]) to the sequence $\{(t_n,x_n,y_n)\}_n$ so that we deduce the existence of a pair of triples $(p_V^n,q_V^n,M_n)\in \bar{J}^{(2,+)}V(t_n,x_n)$ and $(p_U^n,q_U^n,N_n)\in \bar{J}^{(2,-)}U(t_n,y_n)$ such that the following statements hold:

$$p_V^n-p_U^n=\partial_s\psi_n(t_n,x_n,y_n)=2(t_n-t_0), \tag{3.75}$$
$$q_V^n=D_x\psi_n(t_n,x_n,y_n), \tag{3.76}$$
$$q_U^n=D_y\psi_n(t_n,x_n,y_n)), \tag{3.77}$$

and

$$\begin{pmatrix} M_n & 0 \\ 0 & -N_n \end{pmatrix} \le A_n+\frac{1}{2}A_n^2,$$

where $A_n:=D^2_{xy}\psi_n(t_n,x_n,y_n)$. Now we note that by the viscosity subsolution property of V we have that:

$$V(t_n,x_n)-p_V^n-\langle\mu(t_n,x_n),q_V^n\rangle-\frac{1}{2}tr(\sigma\cdot\sigma'(t_n,x_n)M_n)-f(t_n,x_n)\le 0, \tag{3.78}$$

And similarly, by the viscosity supersolution property of U we have that:

$$U(t_n,y_n)-p_V^n-\langle\mu(t_n,y_n),q_V^n\rangle-\frac{1}{2}tr(\sigma\cdot\sigma'(t_n,y_n)N_n)-f(t_n,y_n)\ge 0. \tag{3.79}$$

Now subtracting (3.79) from (3.78) yields the following:

$$\begin{gathered} V(t_n,x_n)-U(t_n,y_n) \\ \le p_V^n-p_V^n+\langle\mu(t_n,x_n),q_V^n\rangle-\langle\mu(t_n,y_n),q_V^n\rangle+\frac{1}{2}tr(\sigma\cdot\sigma'(t_n,x_n)M_n) \\ -\frac{1}{2}\mathrm{tr}(\sigma\cdot\sigma'(t_n,y_n)N_n)+f(t_n,x_n)-f(t_n,y_n)\le 0. \end{gathered} \tag{3.80}$$

We now use the fact that $(s_n, y_n), (t_n, x_n) \to (t, x)$ from which we now observe the following limits as $n \to \infty$:

$$\lim_{n\to\infty}[p_V^n - p_V^n] = \lim_{n\to\infty}[t_n - t_0] = 0, \tag{3.81}$$

and for some $c > 0$:

$$\lim_{n\to\infty}\langle\mu(t_n, x_n), q_V^n\rangle - \langle\mu(t_n, y_n), q_V^n\rangle \leq c \lim_{n\to\infty}|x_n - y_n| = 0, \tag{3.82}$$

using the Lipschitzianity of μ. Lastly we observe, using that:

$$\frac{1}{2}\lim_{n\to\infty}[\mathrm{tr}(\sigma\cdot\sigma'(t_n, x_n)M_n) - \mathrm{tr}(\sigma\cdot\sigma'(t_n, y_n)N_n)] = 0. \tag{3.83}$$

Hence, putting (3.81) - (3.83) together with (3.80) yields a contradiction since we observe that:

$$\lim_{n\to\infty}[V(t_n, x_n) - U(t_n, y_n)] \leq 0, \tag{3.84}$$

which is a contradiction to (3.70). □

Chapter 4

Stochastic Differential Games with Impulse Controls

In this chapter, we study a stochastic differential game in which both players modify a jump-diffusion process using impulse controls. We prove that the value of the game can be represented as a double obstacle problem and derive a PDE characterisation of the value.

The contribution of this chapter is encompassed in the following paper:

David Mguni, ***"Duopoly Investment Problems with Minimally Bounded Adjustment Costs"*****, (2018)**[Mgu18b].

Overview

In Chapters 2 and 3 we studied a game in which one of the players modifies the underlying system dynamics using impulse controls and another player selects the stopping time for the game. In this chapter, we extend the analysis to a strategic interaction in which both players use impulse controls to modify the dynamics of a system governed by a jump-diffusion process.

In [Cos13], a zero-sum stochastic differential game in which two controllers modify the values of an Itō diffusion using impulse controls was investigated using a viscosity theoretic approach. Our main contribution in this chapter is to extend the work in [Cos13] to now accommodate both jump-diffusion dynamics and non zero-sum payoff structures. This broader set of features provides an appropriate modelling framework for various strategic interactions that occur within economics. Examples of such interactions include oligopoly which, owing to the possibility of firms expanding the market are non zerosum scenarios and more generally, competitive environments in which each player faces fixed adjustment costs. In order to extend the analysis in [Cos13], the solution concept must be generalised to a Nash equilibrium which accommodates payoff structures beyond zero-sum games. Additionally, studying the strategic interaction with jumps necessitates a new treatment to characterise the equilibrium behaviour.

We motivate the framework within a widely studied duopoly advertising investment problem [Dea79; PS04; Eri95; Jør82] which we augment to include minimally bounded investment costs. As we show, the resulting framework is a non zero-sum stochastic differential game with impulse controls used by both players. Our development leads to a new model of duopoly advertising investments which now takes into account the fixed minimal costs incurred by firms for their advertising investments. This provides a closer modelling description of systems in which competing firms use advertising investments to capture market share.

Our analysis leads to two verification theorems which fully characterise the value for both zero-sum and non zero-sum versions of the game. This yields a characterisation of the value functions of the game in terms of a solution to a PDE for both the zero-sum and non zero-sum cases which we then apply to compute a complete solution to the duopoly investment problem.

Contributions

In this chapter, we introduce a new model of duopoly advertising investments that now accounts for the fact that firms incur minimally bounded costs for their investment adjustments. We perform a detailed analysis of the problem and characterise the optimal strategies of each firm. The model we construct further augments current models such as [PS04; Eri95] by allowing for the occurrence of market expansions following joint investments by the firms. To extract the optimal strategies for this problem, in this chapter we develop the mathematical framework, namely a stochastic differential game of two-sided impulse control to accommodate the necessary features to model the advertising investment market scenario.

A summary of the contributions of this chapter is as follows:

- First, we extend the stochastic differential game of two-sided impulse controls introduced in [Cos13] to the case in which the underlying system dynamics is described by a jump-diffusion process. We perform a detailed analysis of the game beginning with a verification theorem (Theorem 4.9) leading to a full characterisation of the value function and the minimax equilibrium conditions of the game.

- Second, we extend the analysis of the zero-sum game to a stochastic differential game of two-sided impulse control with a non zero-sum payoff structure. In analogy with the zero-sum case, we prove a verification theorem for the non zero-sum setting (Theorem 4.12) and characterise the *Nash equilibrium* controls for the game. In performing this analysis, we generalise the game to describe a broad range of economic and financial settings.

- Last, we apply the theoretical analysis conducted in the chapter to investigate the duopoly advertising investment problem with minimally bounded costs. This results in a new model that extends existing models e.g. [PS04; Eri95] which now accounts for the minimum expenditures incurred by firms when adjusting their positions in addition to capturing the effect of exogenous market shocks and market expansions.

Organisation

The chapter is arranged as follows: first, in Section 4.1, we give a detailed description of the duopoly advertising investment problem. We begin with an in-depth review of models currently employed to tackle the problem then progress to the main model of the chapter which now includes minimally bounded investment costs. We then give a statement of the main results of the chapter and their relevance to the existing literature. Section 4.2 contains the analysis of the zero-sum game and the characterisation of the minimax equilibrium for the game. In Section 4.4, we extend the analysis of Section 4.2 to non zero-sum games. In Section 4.5, we give some examples of solved problems using the theorems developed in the chapter. Lastly, in Section 4.6, we revisit the advertising investment problem.

4.1 A Firm Duopoly Investment Problem: Dynamic Competitive Advertising

The problem with which we are concerned is an advertising investment problem in which two firms compete for market share in a duopoly market. Each firm seeks to maximise its long-term profit by performing investment adjustments to its advertising position. The costs of each advertising investment is bounded from below so that each adjustment incurs some minimal cost.

We introduce our model by firstly studying a classical advertising investment model then progressively developing the model to include economic features which include minimally bounded investment costs and exogenous shocks. To fix ideas, as our first case (case I), we consider a zero-sum setting in which both firms make continuous modifications to their investment positions — this approach reproduces the *Vidale-Wolfe model of advertising*. Such models do not include fixed minimal costs and assume a fixed market size. We refer the reader to [Eri95; Jør82] for exhaustive discussions on duopoly advertising investment models and to [PS04] for a stochastic differential game approach.

Having constructed a description of classical advertising investment models, we then consider scenarios in which each firm's investment costs are minimally bounded which is a more realistic description of advertising investments. We secondly relax the zero-sum payoff feature of classical advertising investment models now allowing for each firm to have its own market share process described by jump-diffusion. In this setting, we also embed into our description *cross-over effects* from exogenous shocks from each firm's market share process.

The resulting description leads to a new model of dynamic competitive advertising which encapsulates some of the key features of the continuous control model but captures market shocks and cross-over effects between firms. Crucially, unlike current models which assume continuous investment adjustments [PS04; Eri95; Jør82] the model we introduce requires that each investment incurs at least some fixed minimal cost; this feature of the setup significantly alters the firms' problems and subsequent investment behaviour.

We now give a detailed analysis of each case. The solution to the model in case II is presented at the end of the chapter.

Case I: Duopoly with Continuous Investments (Review)

Consider a duopoly in which each firm modifies their advertising investment position continuously. At time $t_0 \leq t \leq T$ each Firm $i \in \{1,2\}$ has a revenue stream $S_i(t)$ which is a stochastic process and where T is some possibly infinite time horizon for both firms. Let us suppose that at any point, each Firm i can make a costly investment of size $u_i \in \mathcal{U}_i$ where $\mathcal{U}_i$ denotes the set of admissible investments for Firm i given $i \in \{1,2\}$. The firms act in a market which has a potential size which we denote by $M \in \mathbb{R}_{>0}$ and the response rate to advertising for Firm i is denoted by $b_i \in]0,1]$. The revenue stream for Firm i is therefore given by:

$$dS_i^{t_0,s_i}(t) = b_i u_i(t^-)\left[M - S_i^{t_0,s_i}(t) - S_j^{t_0,s_j}(t)\right]M^{-1}dt - r_i S_i^{t_0,s_i}(t)dt + \sigma_i dB(t),\ \mathbb{P}-\text{a.s}, \tag{4.1}$$

$$\text{where } i,j \in \{1,2\}\ (i \neq j),\ \forall (t_0,s_i) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0},$$

where $s_i \equiv S_i(t_0) \in \mathbb{R}_{>0}$ are the initial sales for Firm i; $r_i, \sigma_i \in \mathbb{R}$ are constants that represent the rate at which firm $j \neq i$ abstracts market share from its rival and the volatility of the sales process for Firm i respectively. The term B is Brownian motion which introduces randomness to the system.

In this setting, each firm seeks to maximise its cumulative profit. For a firm $i \in \{1,2\}$ the cumulative profit, denoted by Π_i is composed of the firm's revenue due to sales $h : \mathbb{R} \to \mathbb{R}$ minus the firm's running advertising costs $c_i : \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{R}$ and lastly, a function of the firm's terminal market share $G : \mathbb{R} \to \mathbb{R}$ leading to the following expression for Π_i:

$$\Pi_i(t_0,s_i;u_i,u_j) = \mathbb{E}\left[\int_{t_0}^{\tau_S}\left(h(S_i^{t_0,s_i,u_i,u_j}(t)) - [c_i(t,u_i) - c_j(t,u_j)]\right)dt + G(S_i^{t_0,s_i,u_i,u_j}(\tau_S))\right], \tag{4.2}$$

where $\tau_S : \Omega \to [0,T]$ is a random exit time for both firms at which point the problem ends.

In an duopoly problem with a zero-sum payoff structure the firms' profit functions satisfy the following condition:

$$\Pi_1 + \Pi_2 = 0. \tag{4.3}$$

In light of of this condition, to ease notation we can denote by $\Pi(t,s;u_1,u_2) := \Pi_1(t,s;u_1,u_2) = -\Pi_2(t,s;u_1,u_2),\ \forall (t,s) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0},\ \forall u_1 \in \mathcal{U}_1, \forall u_2 \in \mathcal{U}_2$.

We can now write the dynamic (zero-sum) duopoly problem as:

Find ϕ and $(\hat{u}_1,\hat{u}_2) \in \mathscr{U}_1 \times \mathscr{U}_2$ such that

$$\phi(t,s) = \sup_{u_1 \in \mathscr{U}_1} \left(\inf_{u_2 \in \mathscr{U}_2} \Pi^{u_1,u_2}(t,s) \right) = \inf_{u_1 \in \mathscr{U}_1} \left(\sup_{u_2 \in \mathscr{U}_2} \Pi^{u_1,u_2}(t,s) \right) = \Pi^{\hat{u}_1,\hat{u}_2}(t,s),\ \forall (t,s) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}, \tag{4.4}$$

where we have used the shorthand $\Pi^{(u_1,u_2)}(t,s) \equiv \Pi(t,s;u_1,u_2)$ for any $(t,s) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ and for any $(u_1,u_2) \in \mathscr{U}_1 \times \mathscr{U}_2$.

We recognise (4.4) as a zero-sum stochastic differential game and is a general version of the Vidale-Wolfe advertising model (see for example the differential game extension of the Vidale-wolfe model in [Dea79]). The stochastic differential game is one in which both players modify the state process using continuous controls. Models of this kind have been used to analyse the strategic interaction within advertising duopoly using a game-theoretic framework [PS04]. Using this framework, the behaviour of the firms in the advertising problem can be characterised by computing the equilibrium policies within a stochastic differential game.

A feature of this model is that firms are permitted to make infinitely fast investments over the horizon of the problem. This follows since the investment adjustments of each firm are described using continuous controls which allows the firms to make arbitrarily small adjustments to their investment positions (which can incur arbitrarily small costs). Additionally, the zero-sum payoff structure produces a notional transfer of wealth from one firm to the other whenever an advertising investment is made.

We now present the main duopoly model of the chapter which addresses the above issues.

Case II: Non Zero-sum Payoff with Impulse Controls with Jumps

We now seek a description of the duopoly setting that does not impose a zero-sum payoff structure and removes the ability of firms to perform investment adjustments with arbitrarily small costs. Additionally, we seek a description that accounts for the effect that both firms have on the market which we assume undergoes exogenous shocks. In order to accurately model duopoly investment settings of this kind, it is necessary to embed into the model, fixed minimal bounds to the investment costs which naturally preclude the execution of continuous investment strategies.

To this end, denote by $c : [0,T] \times \mathscr{Z} \to \mathbb{R}$ and by $\chi : [0,T] \times \mathscr{Z} \to \mathbb{R}$ the cost function associated to the advertising investments of Firm 1 and Firm 2 respectively where $\mathscr{Z} \subset \mathbb{R}$ is a given set. We now consider a setting in which the firms' investments now incur some minimal cost for each investment which means that for any $(\tau,z) \in \mathscr{T} \times \mathscr{Z}$ we have that $c(\tau,z) \geq \lambda_1$ and $\chi(\tau,z) \geq \lambda_2$ for some constants $\lambda_1,\lambda_2 \in \mathbb{R}_{>0}$. In this case, continuous investment would result in immediate bankruptcy, each firm must now modify its advertising investment position by performing a discrete sequence of investments over the horizon of the problem. In particular, denoting by $\mathscr{U}$ the set of admissible investments for Firm 1, the investment strategy for Firm 1 is therefore described by a double sequence

$u = [\tau_j, \xi_j]_{j\in\mathbb{N}} \equiv (\tau_1, \tau_2, \ldots; \xi_1, \xi_2, \ldots) \in \mathscr{U}$ which describe the times and magnitudes of the firm's investments respectively. Analogously, the investment strategy for Firm 2 is a double sequence: $v = [\rho_m, \eta_m]_{m\in\mathbb{N}} \equiv (\rho_1, \rho_2, \ldots; \eta_1, \eta_2, \ldots) \in \mathscr{V}$ which denotes a sequence of investments $\{\eta_m\}_{m\in\mathbb{N}}$ performed over the sequence of points $\{\rho_m\}_{m\in\mathbb{N}}$ where $\mathscr{V}$ is the set of admissible investments for Firm 2 and where $\xi_1, \xi_2, \ldots, ; \eta_1, \eta_2, \ldots, \in \mathscr{Z}$.

In order to increase its revenue stream, each firm may use advertising investments ($\{\xi_j\}_{j\in\mathbb{N}}$ for Firm 1, $\{\eta_m\}_{m\in\mathbb{N}}$ for Firm 2) to abstract market share which reduces the rival firm's revenue stream. However, increases in either firm's market size expands the economy; this also leads to a higher terminal valuation for both firms which is proportional to the square of the terminal cost (this term is often included in models of duopoly with finite horizon — see for example [ØR97]).

The market share processes S_i for each Firm i where $i \in \{1,2\}$ evolve according to the following expressions:

$$\begin{aligned} S_1^{t_0,s_1,u,v}(t) = s_1 &+ \int_{t_0}^t \mu_1 S_1^{t_0,s_1,u,v}(r)dr + \sum_{j\geq 1} \xi_j \cdot 1_{\{\tau_j \leq T\}}(t) + \int_{t_0}^t \sigma_{11}(S_1^{t_0,s_1,u,v}(r))dB_1(r) \\ &+ \int_{\mathbb{R}}\int_{t_0}^t \theta_{11}(S_1^{t_0,s_1,u,v}(r-))z\tilde{N}_{11}(dr,dz) + \int_{t_0}^t \sigma_{12}(S_1^{t_0,s_1,u,v}(r))dB_2(r) \\ &+ \int_{\mathbb{R}}\int_{t_0}^t \theta_{12}(S_1^{t_0,s_1,u,v}(r-))z\tilde{N}_{12}(dr,dz). \\ &S_1^{t_0,s_1,u,v}(t_0) \equiv s_1 \in \mathbb{R}_{>0}, \end{aligned} \tag{4.5}$$

$$\begin{aligned} S_2^{t_0,s_2,u,v}(t) = s_2 &+ \int_{t_0}^t \mu_2 S_2^{t_0,s_2,u,v}(r)dr + \sum_{m\geq 1} \eta_m \cdot 1_{\{\rho_m \leq T\}}(t) + \int_{t_0}^t \sigma_{22}(S_2^{t_0,s_2,u,v}(r))dB_2(r) \\ &+ \int_{\mathbb{R}}\int_{t_0}^t \theta_{22}(S_2^{t_0,s_2,u,v}(r-))z\tilde{N}_{22}(dr,dz) \quad + \int_{t_0}^t \sigma_{21}(S_2^{t_0,s_2,u,v}(r))dB_1(r) \\ &+ \int_{\mathbb{R}}\int_{t_0}^t \theta_{21}(S_2^{t_0,s_2,u,v}(r-))z\tilde{N}_{21}(dr,dz), \\ &S_2^{t_0,s_2,u,v}(t_0) \equiv s_2 \in \mathbb{R}_{>0}, \end{aligned} \tag{4.6}$$

where $\mu_i \in \mathbb{R}$ are given constants, $\tilde{N}_i$ are compensated random measures[1] and B_i are Wiener processes, $i \in \{1,2\}$ and, as before $s_i \equiv S_i^{t_0,s_i,u,v}(t_0) \in \mathbb{R}_{>0}$ are the initial sales for Firm i and $t_0 \in [0,T[$ is the start point of the problem. Moreover, without loss of generality we assume that $S_i^{t_0,s_i,\cdot}(t_0) = s_i$ for any $s \leq t_0$ for $i \in \{1,2\}$. The functions $\sigma_{ii} : \mathbb{R} \to \mathbb{R}$ and $\theta_{ii} : \mathbb{R} \to \mathbb{R}$ represent the *internal* volatility and jump-amplitude for the sales process for Firm i (resp.). The terms $\sigma_{ij} : \mathbb{R} \to \mathbb{R}$ and $\theta_{ij} : \mathbb{R} \to \mathbb{R}$, $i, j \in \{1,2\}, i \neq j$, each represent the volatility and jump-amplitudes from the rival firm's sales activities on the firm's own sales process.

Unlike the competitive advertising models that appeal to differential games in which the players' modifications of their investment positions are modelled using continuous controls, the model

[1] Recall also that $\nu(\cdot) := \mathbb{E}[N(]0,1],V)]$ for $V \subset \mathbb{R}\backslash\{0\}$.

now requires that each advertising investment incurs at least some fixed minimal cost.

Our next modification is to relax the zero-sum payoff structure (4.3); we therefore decouple the payoff criterion (4.2) into two profit functions Π_i for each Firm $i \in \{1,2\}$. Hence, now Firm i seeks to maximise its running profits over the problem horizon plus a valuation of its terminal market sales. The firms objectives are given by the following expressions:

$$\Pi_1(t_0,s_1,s_2;u,v) = \mathbb{E}^{[s_1,s_2]}\Bigg[\int_{t_0}^{\tau_S} e^{-\varepsilon r}\left[\alpha_1 S_1^{t_0,s_1,u,v}(r) - \beta_1 S_2^{t_0,s_2,u,v}(r)\right]dr - \sum_{j\geq 1} c_1(\tau_j,\xi_j)\cdot 1_{\{\tau_j\leq\tau_S\}} + \gamma_1 e^{-\varepsilon\tau_S}\left[S_1^{t_0,s_1,u,v}(\tau_S)\right]^2\left[S_2^{t_0,s_2,u,v}(\tau_S)\right]^2\Bigg], \tag{4.7}$$

$$\Pi_2(t_0,s_1,s_2;u,v) = \mathbb{E}^{[s_1,s_2]}\Bigg[\int_{t_0}^{\tau_S} e^{-\varepsilon r}\left[\alpha_2 S_2^{t_0,s_2,u,v}(r) - \beta_2 S_1^{t_0,s_1,u,v}(r)\right]dr - \sum_{m\geq 1} c_2(\rho_m,\eta_m)\cdot 1_{\{\rho_m\leq\tau_S\}} + \gamma_2 e^{-\varepsilon\tau_S}\left[S_1^{t_0,s_1,u,v}(\tau_S)\right]^2\left[S_2^{t_0,s_2,u,v}(\tau_S)\right]^2\Bigg]. \tag{4.8}$$

where $\alpha_i, \beta_i, \gamma_i \in \mathbb{R}$ ($i \in \{1,2\}$) are constants and $\varepsilon \in \mathbb{R}_{>0}$ is a common discount factor.

The above model has the following interpretation: the market share S_i of Firm i determines the size of the revenue $\alpha_i S_i$ generated from sales; the parameters $\alpha_i \in \mathbb{R}$ and $\beta_i \in \mathbb{R}$ represent the Firm i revenue per unit sale and the sensitivity of Firm i's activities on Firm j's market share respectively. The terminal quadratic terms capture the fact that increases in either firm's market size *heats up the economy* (expands market opportunities) leading to a higher terminal valuation for *both* firms.

The dynamic duopoly problem is now to characterise the optimal investment policies $(\hat{u},\hat{v}) \in \mathcal{U}\times\mathcal{V}$ and to find the functions ϕ_1, ϕ_2 such that

$$\phi_1(t,s_1,s_2) = \sup_{u\in\mathcal{U}} \Pi_1(t,s_1,s_2;u,\hat{v}) = \Pi_1(t,s_1,s_2;\hat{u},\hat{v}), \tag{4.9}$$

$$\phi_2(t,s_1,s_2) = \sup_{v\in\mathcal{V}} \Pi_2(t,s_1,s_2;\hat{u},v) = \Pi_2(t,s_1,s_2;\hat{u},\hat{v}), \qquad \forall (t,s_1,s_2) \in \mathbb{R}_{>0}\times\mathbb{R}_{>0}\times\mathbb{R}. \tag{4.10}$$

The stochastic differential game therefore involves the use of impulse controls exercised by both players who modify system dynamics to maximise some given payoff function.

In this chapter, we provide a complete characterisation of the solution to both the zero-sum and non zero-sum cases in terms of classical solution to a PDE. We then apply the results of the formal analysis of the chapter to characterise the firms' policies for case II.

4.1.1 Main Results

In this chapter, we prove the results for the game that characterise the conditions for a HJBI equation in both zero-sum and non zero-sum games. In particular, we prove a verification theorem (Theorem

4.12) for stochastic differential games with a jump-diffusion process and in which the players use impulse controls. In particular, consider a game whose value is given by the following:

$$V(y) = \inf_{\hat{u}\in\mathscr{U}} \sup_{\hat{v}\in\mathscr{V}} J[y;\hat{u},\hat{v}] = \sup_{\hat{v}\in\mathscr{V}} \inf_{\hat{u}\in\mathscr{U}} J[y;\hat{u},\hat{v}], \qquad \forall y \in [0,T]\times S, \tag{4.11}$$

where for all $\forall y_0 \in [0,T]\times S$, $J[y_0;u,v] = \left[\int_{t_0}^{\tau_S} f(Y_s^{y_0,u,v})ds + \sum_{j\geq 1} c(\tau_j,\xi_j)\cdot 1_{\{\tau_j\leq\tau_S\}} - \sum_{m\geq 1}\chi(\rho_m,\eta_m)\cdot 1_{\{\rho_m\leq\tau_S\}} + G(Y^{y_0,u,v}(\tau_S))\cdot 1_{\{\tau_S<\infty\}}\right]$. We prove the following result:

Theorem 4.1

Suppose that the value of the zero-sum game V exists and that $V\in\mathscr{C}^{1,2}([0,T],S)\cap\mathscr{C}([0,T],\bar{S})$, then V satisfies the following double obstacle quasi-variational inequality:

$$\begin{cases}\max\left\{\min\left[-\left(\partial_s V(y)+\mathscr{L}V(y)+f(y)\right), V(y)-\mathscr{M}_2 V(y)\right], V(y)-\mathscr{M}_1 V(y)\right\}=0\\ V(\cdot,x)=G(\cdot,x),\end{cases} \tag{4.12}$$

$$\forall x\in S,\ \forall y\in[0,T]\times S,$$

where $\mathscr{M}_i$ is the [non-local] player i intervention operator for $i\in\{1,2\}$, $T\in\mathbb{R}_{>0}$ is the time horizon of the game and $S\subset\mathbb{R}^p$ is a given set.

Moreover, denote by $\hat{u}:=[\hat{\tau}_j,\hat{\xi}_j]_{j\in\mathbb{N}}$ and $\hat{v}:=[\hat{\rho}_m,\hat{\eta}_m]_{m\in\mathbb{N}}$ the equilibrium controls, then V satisfies the following expression:

$$\Delta_{\hat{\xi}_j} V(\hat{\tau}_j,\cdot) = -c(\hat{\tau}_j,\hat{\xi}_j), \tag{4.13}$$

where $\Delta_z\phi(\cdot,X(\tau)) := \phi(\cdot,\Gamma(X(\tau^-),z)) - \phi(\cdot,X(\tau^-)) + \Delta_N X(\tau)$ given some $\mathscr{F}_\tau-$measurable intervention $z\in\mathscr{Z}$ and where $\Delta_N X(\tau)$ denotes a jump at some stopping time $\tau\in\mathscr{T}$ due to $\tilde{N}$.

Theorem 4.1 extends the analyses of two-sided impulse controls with stochastic games contained in [Cos13] to now cover games in which the underlying dynamics include jumps. This augments the game in such a way that allows for modelling of various financial systems that face exogenous shocks. The result extends the verification result (Theorem 2.7) of Chapter 2 beyond controller stopper games to the case in which the stopper can now perform a sequence of actions that allow it to alter the dynamics of the game.

For the non zero-sum payoff case with running cost functions f_i and terminal cost functions G_i for $i\in\{1,2\}$, we have the following result:

Theorem 4.2

Denote by ϕ_i the player i value function for the non zero-sum game for $i\in\{1,2\}$ then the value functions ϕ_i satisfy the following quasi-variational inequalities:

$$\begin{cases} \max\{-(\partial_s\phi_i(y)+\mathscr{L}\phi_i(y)+f_i(y)),\phi_i(y)-\mathscr{M}_i\phi_i(y)\}=0, \\ \phi_i(\tau_S,x)=G_i(\tau_S,x), \qquad \forall x\in S,\ y\in[0,T]\times S,\ i\in\{1,2\}. \end{cases} \tag{4.14}$$

Therefore, we extend existing results that provide the conditions for a HJBI equation of a stochastic differential game of impulse control ([Cos13]) to now cover games in which i) the uncontrolled state (diffusion) process now includes jumps and ii) the payoff structure is now non zero-sum.

Having proven these results, we then implement the analysis to characterise the investment behaviour of each firm in the duopoly investment problem.

The Setup

As in Chapter 1, the uncontrolled passive state process evolves according to a jump-diffusion process given by (1.1) and where $S\subset\mathbb{R}^p$. The state process is influenced by a pair of impulse controls $u\in\mathscr{U},v\in\mathscr{V}$ exercised by each player where $u(s)=\sum_{j\geq 1}\xi_j\cdot 1_{\{\tau_j\leq T\}}(s)$ for all $s\in[0,T]$, with impulses $\xi_1,\xi_2,\ldots\in\mathscr{Z}\subset S$ being exercised by player I who intervenes at times $\{\tau_i\}_{i\in\mathbb{N}}$ where $0\leq t_0\leq\tau_1<\tau_2<\cdots<$ and where $v(s)=\sum_{m\geq 1}\eta_m\cdot 1_{\{\rho_m\leq T\}}(s)$ for all $s\in[0,T]$, with impulses $\eta_1,\eta_2,\ldots\in\mathscr{Z}\subset S$ being exercised by player II who intervenes at times $\{\rho_m\}_{m\in\mathbb{N}}$ where $0\leq t_0\leq\rho_1<\rho_2<\ldots<$.

The evolution of the state process with interventions is described by the equation $\forall r\in[0,T]$, $\forall(t_0,x_0)\in[0,T]\times S$, $\forall u\in\mathscr{U},\forall v\in\mathscr{V}$:

$$\begin{aligned} X_r^{t_0,x_0,u,v}=x_0+\int_{t_0}^r\mu(s,X_s^{t_0,x_0,u,v})ds+\int_{t_0}^r\sigma(s,X_s^{t_0,x_0,u,v})dB_s+\sum_{j\geq 1}\xi_j\cdot 1_{\{\tau_j\leq T\}}(r) \\ +\sum_{m\geq 1}\eta_m\cdot 1_{\{\rho_m\leq T\}}(r)+\int_{t_0}^r\int\gamma(X_{s^-}^{t_0,x_0,u,v},z)\tilde{N}(ds,dz),\ \mathbb{P}-\text{a.s.}. \end{aligned} \tag{4.15}$$

Without loss of generality we assume that $X_s^{t_0,x_0,\cdot}=x_0$ for any $s\leq t_0$.

Player I has a gain (or profit) function J which is also a cost function for player II which is given by the following expression $\forall u\in\mathscr{U},\forall v\in\mathscr{V}$, $\forall(t_0,x_0)\in[0,T]\times S$:

$$\begin{aligned} J[t_0,x_0;u,v]=\mathbb{E}\Bigg[\int_{t_0}^{\tau_S}f(s,X_s^{t_0,x_0,u,v})ds+\sum_{m\geq 1}c(\tau_m,\xi_m)\cdot 1_{\{\tau_m\leq\tau_S\}}-\sum_{l\geq 1}\chi(\rho_l,\eta_l)\cdot 1_{\{\rho_l\leq\tau_S\}} \\ +G(\tau_S,X_{\tau_S}^{t_0,x_0,u,v})\cdot 1_{\{\tau_S<\infty\}}\Bigg]. \end{aligned} \tag{4.16}$$

The results contained within this chapter are built under assumptions A.1.1 - A.4 (see Appendix). As in Chapter 2, we assume that the function G satisfies the condition $\lim\limits_{s\to\infty}G(s,x)=0$ for any $x\in S$.

Following Remark 2.1 and in a similar way to Chapter 2, we restrict ourselves to Markov controls and hence the player I control takes the form $u = \tilde{f}_1(s, X_s)$ for any $s \in [0,T]$ where $\tilde{f}_1 : [0,T] \times S \to \mathfrak{U}$ and $\mathfrak{U} \subset \mathbb{R}^p$ and $\tilde{f}_1$ is some measurable map w.r.t. $\mathscr{F}$. Analogously, the player II control can expressed in the form $v = \tilde{f}_2(s, X_s)$ for any $s \in [0,T]$ where $\tilde{f}_2 : [0,T] \times S \to \mathfrak{V}$ and $\tilde{f}_2$ is some measurable map w.r.t. $\mathscr{F}$ and $\mathfrak{V} \subset \mathbb{R}^p$.

Let us now firstly recall the following definition which we shall rely on repeatedly throughout the chapter:

Definition 4.3

Let $\tau \in \mathscr{T}$, we define the [non-local] Player I intervention operator $\mathscr{M}_1 : \mathscr{H} \to \mathscr{H}$ acting at a state $X(\tau^-)$ by the following expression:

$$\mathscr{M}_1 \phi(\tau^-, X(\tau^-)) := \inf_{z \in \mathscr{Z}} \left[\phi(\tau, \Gamma(X(\tau^-), z)) + c(\tau, z) \cdot 1_{\{\tau \leq T\}} \right], \tag{4.17}$$

where $\Gamma : S \times \mathscr{Z} \to S$ is the impulse response function defined earlier. We analogously define the [non-local] Player II intervention operator $\mathscr{M}_2 : \mathscr{H} \to \mathscr{H}$ at $X(\rho^-)$ for some $\rho \in \mathscr{T}$ by:

$$\mathscr{M}_2 \phi(\rho^-, X(\rho^-)) := \sup_{z \in \mathscr{Z}} \left[\phi(\rho, \Gamma(X(\rho -), z)) - \chi(\rho, z) \cdot 1_{\{\rho \leq T\}} \right]. \tag{4.18}$$

We recall also the fact that by Remark 2.3, at any given point an immediate intervention may not be optimal, hence the following inequalities hold:

$$\mathscr{M}_1 V(s,x) \geq V(s,x), \tag{4.19}$$

$$\mathscr{M}_2 V(s,x) \leq V(s,x), \quad \forall (s,x) \in [0,T] \times S. \tag{4.20}$$

The first part of the chapter is devoted to relating the value of the game to some PDE — to do this we firstly establish some preliminary continuity results for the value functions after which we prove a verification theorem for the zero-sum case and the non zero-sum case.

4.2 Stochastic Differential Games Involving Impulse Controls

Preliminaries

We begin by establishing regularity properties of the value functions — we start by stating the following result for which we provide a sketch of the proof.

Lemma 4.4

We can deduce the existence of constants $c_1, c_2 > 0$ such that the following results hold $\forall (s,x'), (s,x), (t',x), (t,x) \in [0,T] \times S$:

(i) $|V^-(s,x') - V^-(s,x)| + |V^+(s,x') - V^+(s,x)| \leq c_1 |x' - x|$,

(ii) $|V^-(t',x) - V^-(t,x)| + |V^+(t',x) - V^+(t,x)| \leq c_2 |t' - t|^{\frac{1}{2}}$.

Lemma 4.4 is proven using the properties (Lipschitzianity and $\frac{1}{2}$-Hölder continuity) of the constituent functions of the players' objective functions. A proof of the lemma without jumps in the underlying state process can be found in (Propositions 2.2 and 2.4, pgs. 7-11 in [Cos13]). Indeed, given Lemma 3.3 (applied to a two-controller game), Lemma 4.4 is proven using the steps as the proof of the regularity of the value function in [Cos13]. In order to include jumps, we note that it is sufficient to note the estimate:

$$\mathbb{E}\left[\left|\int_t^s\int \gamma(r,X_r,z)\tilde{N}(dz,dr)\right|^\beta\right] \leq \mathbb{E}\left(\left[\int_t^s\int |\gamma(r,X_r,z)|^2\nu(dz)dr\right]\right)^{\beta/2}, \quad \forall s,t\in[0,T], \quad (4.21)$$

for some $\beta\in\mathbb{N}$ which follows from an application of Hölder's inequality. Upon inserting the estimate (4.21) into the arguments in Lemma A.1 and Lemma A.1.3 we readily deduce the lemma. We note also that having established the regularity of the value functions, we can invoke Rademacher's Theorem to deduce that the value functions are at least once differentiable almost everywhere in S.

A central component of the proof of the verification theorem is the analysis of the players' non-intervention regions. In the zero-sum case, the opponent's actions produce two changes in the value function: each impulse action performed by the opponent produces an immediate shift in the value of the state process, this in turn causes indirect changes to the value function since the state process enters as one of its inputs.

A second effect is that at each intervention, the player performing the execution incurs an intervention cost which, in the zero-sum case represents a transfer of wealth to the player's opponent — this produces direct instantaneous changes to the value function.

To capture the two effects on the dynamics of the value function, it is necessary to reformulate the impulse control system as a singular control system which has minimally bounded adjustment costs (Lemma 4.6).

Firstly, we provide a description of singular control with the following definition.

Definition 4.5 ([GT08])

Let $I\subset\mathbb{R}$ be an open (and possibly unbounded) interval and denote its closure by $\bar{I}$. Suppose $x\in\bar{I}$, then an admissible singular control is a pair $(\nu_s^+,\nu_s^-)_{s\geq t}$ of $\mathscr{F}-$adapted, non-decreasing càdlàg processes such that $\nu^+(0)=\nu^-(0)=0$, $X_s^{t_0,x_0,\cdot}:=x_0+\nu_s^+-\nu_s^-$ and $d\nu^+,d\nu^-$ are supported on disjoint subsets.

The following result demonstrates that general impulse control problems can be represented as a singular control problem, in particular it shows that the game (4.16) can be represented as a game of singular control.

Lemma 4.6

Let $\nu\equiv(\nu_1(s),\nu_2(s)):[0,T]\times\Omega\times[0,T]\times\Omega\to\mathbb{R}^{2p}$ be a pair of adapted finite variation càdlàg processes with increasing components and let $\Theta_i\in\mathscr{H},i\in\{1,2\}$ be a pair of functions which satisfy conditions A.2.1 - A.2.2. The impulse control problem with cost functions c and χ given by $c(\tau_k, \xi_k) \equiv \lambda_1 \xi_k + \kappa_1$ and $\chi(\rho_m, \eta_m) \equiv \lambda_2 \eta_m + \kappa_2$ for player I and player II respectively where $\xi, \eta \in \mathscr{Z}$; $\lambda_i, \kappa_i \in \mathbb{R}_{>0}$, $i \in \{1,2\}$ is equivalent to the following singular control problem:

Find $\phi \in \mathscr{H}$ and $\hat{v} = (\hat{v}_1, \hat{v}_2)$ such that

$$\phi(t_0, x_0) = \inf_{v_1} \sup_{v_2} J[t_0, x_0; v] = J[t_0, x_0; \hat{v}], \tag{4.22}$$

where $\forall (t_0, x_0) \in [0,T] \times S$

$$J[t_0, x_0; v] = \mathbb{E}\left[\int_{t_0}^{\tau_S} f(s, X_s^{t_0, x_0, v}) ds + \int_{t_0}^{\tau_S} \Theta_1(s) dv_1(s) - \int_{t_0}^{\tau_S} \Theta_2(s) dv_2(s) + G(\tau_S, X_{\tau_S}^{t_0, x_0, v}) \cdot 1_{\{\tau_S < \infty\}}\right], \tag{4.23}$$

and where the state process X evolves according to the following SDE $\forall (t_0, x_0) \in [0,T] \times S$:

$$X_r^{t_0, x_0, v} = x_0 + \int_{t_0}^{r} \mu(s, X_s^{t_0, x_0, v}) ds + \int_{t_0}^{r} \sigma(s, X_s^{t_0, x_0, v}) dB_s + v(r) + \int_{t_0}^{r} \int \gamma(X_{s-}^{t_0, x_0, v}, z) \tilde{N}(ds, dz), \mathbb{P} - \text{a.s.} \tag{4.24}$$

We use Lemma 4.6 to prove a verification theorem for the game with a zero-sum payoff structure. We defer the proof of the lemma to the chapter appendix.

Heuristic Analysis of The Value Function

In order to motivate the origins of the verification theorem, we now give a heuristic set of arguments that elucidate the key features of the verification theorem. Suppose the value of the game exists and that the following variant of the dynamic programming principle holds for some $r \in [t_0, T - t_0]$ (see [Cos13] for a statement and proof of the dynamic programming principle for a diffusion process without jumps) $\forall y \equiv (t_0, x_0) \in [0,T] \times S$:

$$\begin{aligned} V(y_0) = \inf_{u \in \mathscr{U}} \sup_{v \in \mathscr{V}} \mathbb{E}\Bigg[& \int_{t_0}^{r \wedge \tau_S} f(Y_s^{y_0, u, v}) ds + \sum_{j \geq 1} c(\tau_j, \xi_j) \cdot 1_{\{\tau_j \leq r \wedge \tau_S\}} - \sum_{m \geq 1} \chi(\rho_m, \eta_m) \cdot 1_{\{\rho_m \leq r \wedge \tau_S\}} \\ & + V(Y_r^{y_0, u, v}) \cdot 1_{\{r < \tau_S\}} + G(Y_{\tau_S}^{y_0, u, v}) \cdot 1_{\{\infty > r \geq \tau_S\}} \Bigg], \end{aligned} \tag{4.25}$$

Assuming the value of the game V exists then it is invariant under commutation of the infimum and supremum operators. Without loss of generality, we focus on the optimality conditions for player I. Let us therefore fix some player II control $v = [\rho_j, \eta_j]_{j \in \mathbb{N}} \in \mathscr{V}$. Suppose also that we define $\hat{u} \in \mathscr{U}$ by $\inf_{u \in \mathscr{U}} J[y; u, v] = J[y; \hat{u}, v]$, $\forall v \in \mathscr{V}$, $\forall y \in [0,T] \times S$ where $\hat{u} = [\hat{\tau}_j, \hat{\xi}_j]_{j \in \mathbb{N}} \in \mathscr{U}$. We consider the cases for which $\rho_1 > \{t_0 \wedge \tau_1\}$. Let us then consider the case in which neither player makes an immediate intervention (at the initial point of the game), hence we can find a sufficiently small $r \in]t_0, T]$ such that $r < (\hat{\tau}_1 \wedge \rho_1) - t_0$. Therefore, by (4.25) we have that:

$$V(y_0) = \sup_{v \in \mathscr{V}} \mathbb{E}\left[\int_{t_0}^{r \wedge \tau_S} f(Y_s^{y_0, u_{0[t_0,r]}, v}) ds + V(Y_r^{y_0, u_{0[t_0,r]}, v}) \cdot 1_{\{r < \tau_S\}} + G(Y_{\tau_S}^{y_0, u_{0[t_0,r]}, v}) \cdot 1_{\{r \geq \tau_S\}}\right], \tag{4.26}$$

where $u_0 \in \mathscr{U}$ is the player I control with no impulses and where we have used the restriction notation as in Definition 1.5. In the presence of the assumption that $V \in \mathscr{C}^{(1,2)}([0,T], S) \cap \mathscr{C}([0,T], \bar{S})$, V is sufficiently smooth to apply Dynkin's formula for jump-diffusion processes to the last term in (4.26). We note lastly that given the continuity of the term $\partial_{x_i} V$, we can infer the boundedness of the stochastic integral term on $[0,T]$ and by the properties of standard Brownian motion that:

$$\mathbb{E}\left[\sum_{i=1}^{p}\sum_{j=1}^{p}\int_{t_0}^{r \wedge \hat{\rho}} \partial_{x_i} V(Y_s^{y_0, u_{0[t_0, s \wedge \hat{\rho}]}}) \sigma_{ij} dB_s^j\right] = 0. \tag{4.27}$$

Therefore by a classical limiting procedure, that is after taking limit as $s \downarrow t_0$, commuting the expectation operator with the limit then invoking the mean value theorem, we find that:

(i) $0 = \partial_t V(y) + \mathscr{L} V(y) + f(y), \quad \forall y \in [0,T] \times S,$

(ii) $G(\tau_S, x) = V(\tau_S, x), \qquad \forall x \in S,$

where $\mathscr{L}$ is a local operator corresponding to the stochastic generator for the uncontrolled process (1.2).

Since $J[y_0; u, \hat{v}] \geq \inf_{u \in \mathscr{U}} J[y_0; u, \hat{v}] = J[y_0; \hat{u}, \hat{v}] = V(y_0)$ and since $\hat{\tau}_1 > t_0$ (i.e. at the initial point it is optimal for player I to leave the system) we observe that $\mathscr{M}_1 V(y_0) \geq V(y_0)$. Let us now consider the case when $\rho_1 > \hat{\tau}_1 > t_0$, it is a simple matter to see that by the dynamic programming principle we have that:

$$V(y_0) = \mathbb{E}\left[\int_{t_0}^{\hat{\tau}_1 - \wedge \tau_S} f(Y_r^{y_0, u_{0[t_0, \hat{\tau}_1 - \wedge \tau_S]}, v_{[t_0, \hat{\tau}_1 - \wedge \tau_S]}}) dr + V(Y_{\hat{\tau}_1}^{y_0, u_{0[t_0, \hat{\tau}_1 - \wedge \tau_S]}, v_{[t_0, \hat{\tau}_1 - \wedge \tau_S]}}) \cdot 1_{\{\hat{\tau}_1 < \tau_S\}} \right.$$
$$\left. + G(Y_{\tau_S}^{y_0, u_{0[t_0, \hat{\tau}_1 \wedge \tau_S]}, v_{[t_0, \hat{\tau}_1 \wedge \tau_S]}}) \cdot 1_{\{\hat{\tau}_1 \geq \tau_S\}}\right], \tag{4.28}$$

from which we can straightforwardly observe that after invoking steps (4.25) - (4.27) and by the classical limiting procedure, we recover (i) - (ii). Indeed, we can derive (i) - (ii) whenever we have $\{\rho_1 \wedge \hat{\tau}_1\} = \hat{\tau}_1 > t_0$.

Let us lastly consider the case in which an immediate player I intervention (only) is optimal. Define $\hat{v} \in \mathscr{V}$ by $\sup_{v \in \mathscr{V}} J[y; u, v] = J[y; u, \hat{v}]$, $\forall u \in \mathscr{U}$, $\forall y \in [0,T] \times S$ where $\hat{v} = [\hat{\rho}_m, \hat{\eta}_m]_{m \in \mathbb{N}} \in \mathscr{V}$. Since $\hat{\tau}_1 \in \mathscr{T}$, we know in particular that $\{\hat{\tau}_1(\omega, v) = t; \omega \in \Omega, t \in [0,T]\}$ is $\mathscr{F}_t-$measurable. Consider a policy $u = [\tau_j, \xi_j]_{j \in \mathbb{N}} \in \mathscr{U}$ such that $\tau_1 > t_0$. Since an immediate player I intervention is optimal, we can find some sufficiently small $r \in]t_0, T - t_0]$ such that

$$V(y_0) \leq \mathbb{E}\left[\left(\int_{t_0}^{r} f(Y_s^{y_0,u,\hat{v}})ds - \sum_{m\geq 1} \chi(\rho_m, \eta_m) \cdot 1_{\{\rho_m \leq r\}} + V(Y_r^{y_0,u,\hat{v}})\right) \cdot 1_{\{r<\tau_S\}} + G(Y_{\tau_S}^{y_0,u,\hat{v}}) \cdot 1_{\{r\geq\tau_S\}}\right], \tag{4.29}$$

where the inequality arises due to the fact that player I can improve their payoff (decreasing the value of RHS of (4.29)) by performing an intervention in the interval $[t_0, r]$. Since this holds for all $v \in \mathscr{V}$, we have in particular, using the definition of $\hat{v}$, that:

$$\begin{aligned} V(y_0) &\leq \sup_{v\in\mathscr{V}} \mathbb{E}\left[\int_{t_0}^{r} f(Y_s^{y_0,u,v})ds - \sum_{m\geq 1} \chi(\rho_m, \eta_m) \cdot 1_{\{\rho_m \leq r\}} + V(Y_r^{y_0,u,\hat{v}}) \cdot 1_{\{r<\tau_S\}} + G(Y_{\tau_S}^{y_0,u,v}) \cdot 1_{\{r\geq\tau_S\}}\right] \\ &= \mathbb{E}\left[\int_{t_0}^{r} f(Y_s^{y_0,u,\hat{v}})ds - \sum_{m\geq 1} \chi(\hat{\rho}_m, \hat{\eta}_m) \cdot 1_{\{\hat{\rho}_m \leq r\}} + V(Y_r^{y_0,u,\hat{v}}) \cdot 1_{\{r<\tau_S\}} + G(Y_{\tau_S}^{y_0,u,\hat{v}}) \cdot 1_{\{r\geq\tau_S\}}\right]. \end{aligned}$$

Hence, using a classical limiting procedure (and by the fact that $\sum_{j\geq 1} \inf_{z\in\mathscr{Z}} \chi(t_0, z) \geq \lambda$ for some $\lambda > 0$), we find that (i) becomes:

$$\partial_s V(y_0) + \mathscr{L}V(y_0) + f(y_0) \geq 0. \tag{4.30}$$

Since by assumption $\hat{\rho}_1 > t_0$ (an immediate player II intervention is not optimal), from (4.30) we find:

$$\partial_s V(y_0) + \mathscr{L}V(y_0) + f(y_0) \geq 0. \tag{4.31}$$

Moreover, since it is optimal for player I to apply an impulse intervention at the initial point we have that:

$$\mathscr{M}_1 V(y_0) = V(y_0). \tag{4.32}$$

Putting (4.31), (4.32) together with (4.19) yields:

$$\min\left[\partial_s V(y) + \mathscr{L}V(y) + f(y), \mathscr{M}_1 V(y) - V(y)\right] = 0, \quad \forall y \in [0,T] \times S. \tag{4.33}$$

Using identical reasoning as the above steps, we can derive the analogous condition for player II:

$$\max\left[\partial_s V(y) + \mathscr{L}V(y) + f(y), \mathscr{M}_2 V(y) - V(y)\right] = 0, \quad \forall y \in [0,T] \times S. \tag{4.34}$$

Combining the statements (4.33), (4.34) and the terminal condition (ii), leads to the following double obstacle quasi-variational inequality:

$$\begin{cases} \max\left\{\min\left[-\left(\partial_s V(y)+\mathscr{L}V(y)+f(y)\right),\, V(y)-\mathscr{M}_2 V(y)\right],\, V(y)-\mathscr{M}_1 V(y)\right\}=0 \\ \qquad\qquad\qquad\qquad V(\tau_S,x)=G(\tau_S,x) \qquad \forall x\in S,\ \forall y\in[0,T]\times S. \end{cases}$$

The double obstacle problem (4.35) captures the fact that for any given $x\in S$, each player is faced with the decision of whether to intervene or not. For each player $i\in\{1,2\}$, if it is optimal to intervene then the equality $\mathscr{M}_i V(\cdot,x)=V(\cdot,x)$ holds. When the converse is true for player I (player II), intervening even with the best possible action leads to a worse payoff than leaving the system to evolve freely so that $\mathscr{M}_1 V(\cdot,x)>V(\cdot,x)$ ($\mathscr{M}_2 V(\cdot,x)<V(\cdot,x)$) moreover, the above classical arguments imply that in this instance $\mathscr{L}V+f=0$.

In an analogous manner to the verification theorem of Chapter 2 (Theorem 2.7), for each player we can construct continuation regions in which an immediate intervention is *not* optimal — for player I this is $D_1=\{x\in S:\mathscr{M}V(\cdot,x)\geq V(\cdot,x)\}$. The player II continuation region is constructed analogously. Now for $X\notin D_1$, it is optimal to intervene from which we can deduce by induction the definition of the optimal sequence of intervention times: $\hat{\tau}_0\equiv t_0$ and $\hat{\tau}_{j+1}=\inf\{s>\tau_j;X^{\cdot,\hat{u}_{[t_0,s]}}(s)\notin D_1\}\wedge\tau_S$.

Imposing the condition that any candidate solution must be sufficiently smooth to apply Dynkin's formula on the interior of S, then combining the above statements and equation (4.31) (and its player I analogue) generates the required conditions for the verification theorem for stochastic differential games involving impulse controls which we formally state in Theorem 4.9.

Note however, that the assumption that $\rho_1>\{t_0\wedge\tau_1\}$ in the above analysis allowed us to consider intervals with no player II interventions; clearly this does not hold in general. Indeed, to accommodate the influence of impulses exercised by player II on the value function, it is necessary to reformulate the problem as a singular impulse control problem for which we appeal to Lemma 4.6.

The following results follow directly from the results of the single-controller environments in Chapter 3 (c.f. Lemma 3.7 and Lemma 3.8):

Lemma 4.7

Let $(\tau,x)\in\mathscr{T}\times S$ and let $V\in\mathscr{H}$, then the sets Ξ_1 and Ξ_2 defined by:

$$\Xi_1(\tau,x):=\left\{\xi\in\mathscr{Z}:\mathscr{M}_1V(\tau^-,x)=V(\tau^-,x+\xi)+c(\tau,\xi)\cdot 1_{\{\tau\leq\tau_S\}}\right\}, \tag{4.35}$$

$$\Xi_2(\tau,x):=\left\{\xi\in\mathscr{Z}:\mathscr{M}_2V(\tau^-,x)=V(\tau^-,x+\xi)-\chi(\tau,\xi)\cdot 1_{\{\tau\leq\tau_S\}}\right\}, \tag{4.36}$$

are non-empty.

Lemma 4.8

Let $V \in \mathscr{H}$, then the non-local intervention operators $\mathscr{M}_i, i \in \{1,2\}$ are continuous so that we can deduce the existence of constants $c_1 > 0$ and $c_2 > 0$ such that

$$|\mathscr{M}_i V(s,x) - \mathscr{M}_i V(s,y)| \leq c_1 |x-y|, \tag{4.37}$$

$$\left|\mathscr{M}_i V(s,x) - \mathscr{M}_i V(s',x)\right| \leq c_2 \left|s-s'\right|^{\frac{1}{2}}, \qquad \forall x,y \in S, \quad t \leq s < s' \leq T. \tag{4.38}$$

4.3.1 A HJBI Equation for Zero-Sum Stochastic Differential Games with Impulse Controls

In this section, we give a verification theorem for the value of the game therefore giving conditions under which the value of the game is a solution to the HJBI equation. As remarked earlier, to accommodate the influence of impulses exercised by player II on the value function, it is necessary to reformulate the problem as a singular impulse control problem for which we appeal to Lemma 4.6.

The following theorem characterises the conditions in which the value of the game satisfies a HJBI equation:

Theorem 4.9 (Verification theorem for Zero-Sum Games with Impulse Control)

Suppose that the value of the game V exists and that $V \in \mathscr{C}^{1,2}([0,T],S) \cap \mathscr{C}([0,T],\bar{S})$. Suppose also that there exists a function $\phi \in \mathscr{C}^{1,2}([0,T],S) \cap \mathscr{C}([0,T],\bar{S})$ that satisfies technical conditions (T1) - (T4) (see Appendix) and the following conditions:

(i) $\phi \leq \mathscr{M}_1\phi$ in S and $\phi \geq \mathscr{M}_2\phi$ in S where D_1 and D_2 are defined by: $D_1 = \{x \in S; \phi(\cdot,x) < \mathscr{M}_1\phi(\cdot,x)\}$ and $D_2 = \{x \in S; \phi(\cdot,x) > \mathscr{M}_2\phi(\cdot,x)\}$ where we refer to D_1 (resp., D_2) as the player I (resp., player II) continuation region.

(ii) $\frac{\partial \phi}{\partial s} + \mathscr{L}\phi(\cdot, X^{\cdot,\hat{u},v}(\cdot)) + f(\cdot, X^{\cdot,\hat{u},v}(\cdot)) \leq 0$, $\forall v \in \mathscr{V}$ on $S \backslash \partial D_2$.

(iii) $\frac{\partial \phi}{\partial s} + \mathscr{L}\phi(\cdot, X^{\cdot,u,\hat{v}}(\cdot)) + f(\cdot, X^{\cdot,u,\hat{v}}(\cdot)) \geq 0$, $\forall u \in \mathscr{U}$ on $S \backslash \partial D_1$.

(iv) $\frac{\partial \phi}{\partial s} + \mathscr{L}\phi(\cdot, X^{\cdot,\hat{u},\hat{v}}(\cdot)) + f(\cdot, X^{\cdot,\hat{u},\hat{v}}(\cdot)) = 0$ in $D \equiv D_1 \cap D_2$ in S.

(v) $X^{\cdot,u,v}(\tau_S) \in \partial S$, $\mathbb{P}-$a.s. on $\{\tau_S < \infty\}$ and $\phi(s, X^{\cdot,u,v}(\cdot)) \to G(\tau_S, X^{\cdot,u,v}(\tau_S)) \cdot 1_{\{\tau_S < \infty\}}$ as $s \to \tau_S^-$ $\mathbb{P}-$a.s.,$\forall x \in S, \forall u \in \mathscr{U}$, $\forall v \in \mathscr{V}$.

(vi) $\hat{\xi}_k \in \text{arginf}_{z \in \mathscr{Z}}\{\phi(\tau_k^-, \Gamma(x,z)) + c(\tau_k, z)\}$ is a Borel Measurable selection and similarly, $\hat{\eta}_j \in \text{argsup}_{z \in \mathscr{Z}}\{\phi(\rho_j^-, \Gamma(x,z)) - \chi(\rho_j, z)\}$ is a Borel Measurable selection $\forall x \in S$; and for any $\tau_1, \tau_2, \ldots; \rho_1, \rho_2, \ldots \in \mathscr{T}$.

Put $\hat{\tau}_0 \equiv t_0$ and define $\hat{u} := [\hat{\tau}_j, \hat{\xi}_j]_{j \in \mathbb{N}}$ inductively by: $\hat{\tau}_{j+1} = \inf\{s > \tau_j;\ X^{\cdot,\hat{u},v}(\cdot) \notin D_1\} \wedge \tau_S$, $\forall v \in \mathscr{V}$. Similarly, put $\hat{\rho}_0 \equiv t_0$ and define $\hat{v} := [\hat{\rho}_m, \hat{\eta}_m]_{m \in \mathbb{N}}$ inductively by $\hat{\rho}_{m+1} = \inf\{s > \rho_m;\ X^{\cdot,u,\hat{v}}(\cdot) \notin D_2\} \wedge \tau_S$, $\forall u \in \mathscr{U}$.

(vii) $\Delta_z\phi(\hat{\rho}_m),X(\hat{\rho}_m)) = \chi(\hat{\rho}_m,z)$ and $\Delta_{\hat{\xi}_j}\phi(\hat{\tau}_j,X(\hat{\tau}_j)) = -c(\hat{\tau}_j,z)$, $\forall z\in\mathscr{Z}$.

Then

$$\phi(t,x) = J[t,x;\hat{u},\hat{v}] = \inf_{u\in\mathscr{U}}\sup_{v\in\mathscr{V}} J[t,x;u,v] = \sup_{v\in\mathscr{V}}\inf_{u\in\mathscr{U}} J[t,x;u,v], \qquad \forall(t,x)\in[0,T]\times S. \tag{4.39}$$

Theorem 4.12 provides a full characterisation of the value for the game. In particular, the theorem states that in equilibrium, both players play QVI controls and, provided that the value of the game exists and is sufficiently smooth to apply the Dynkin formula and, should a solution to the HJBI equation (iv) exist, then the value of the function coincides with the HJBI equation solution.

In a similar way to the verification theorems of Chapter 2, conditions (i) - (iv) of Theorem 4.9 follow directly from the QVI conditions motivated in the heuristic analysis. Additionally, the function ϕ is required to be smooth on the interior of S, that is $\phi\in\mathscr{C}^{1,2}([0,T],S)\cap\mathscr{C}([0,T],\bar{S})$ to allow application of the integro-differential operator $\mathscr{L}$ in (ii) - (iv) for the use of Dynkin's formula which is central to the proof of the theorem.

From Theorem 4.9, we also see that the sample space splits into three regions that consist of a continuation region, in which neither player performs an intervention and intervention regions for each player within which a player performs an impulse execution. In particular, we have the following corollary.

Corollary 4.9.1

The sample space splits into three regions which, when playing their equilibrium strategies represent a region in which player I executes interventions I_1, a region in which player II executes interventions I_2, and a region I_3 in which no action is taken by either player; moreover the three regions are characterised by the following expressions:

$$\begin{aligned}
I_1 &= \{y\in[0,T]\times S : V(y) = \mathscr{M}_1V(y),\ \mathscr{L}V(y)+f(y)\geq 0\},\\
I_2 &= \{y\in[0,T]\times S : V(y) = \mathscr{M}_2V(y), \mathscr{L}V(y)+f(y)\geq 0\},\\
I_3 =&\{y\in[0,T]\times S : V(y) < \mathscr{M}_1(y), V(y) > \mathscr{M}_2V(y); \mathscr{L}V(y)+f(y) = 0\}.
\end{aligned}$$

We now give a proof of the verification theorem:

Proof of Theorem 4.9. In the following, we make the distinction between the jumps due to the players' impulse controls and the jumps due to $\tilde{N}$. Indeed, for any $\tau\in\mathscr{T}$, we denote by $\hat{X}(\tau) := X(\tau^-)+\Delta_N X(\tau)$ where $\Delta_N X(\tau)$ is the jump at τ due to $\tilde{N}$ where $\Delta_N X(s) = \int\gamma(X(s-),z)\tilde{N}(ds,dz)$ and $\tilde{N}(ds,dz) = \tilde{N}(s,dz)-\tilde{N}(s,dz)$.

Similarly, given an impulse $\xi\in\mathscr{Z}$ (resp., $\eta\in\mathscr{Z}$) exercised by player I (resp., player II), we denote the jump induced by the player I (resp., player II) impulse by Δ_ξ (resp., Δ_η). That is, for any $\tau\in\mathscr{T}$, we define $\Delta_\xi\phi(\tau,X^{t_0,x_0,u,v}(\tau)) := \phi(\tau^-,\Gamma(X^{t_0,x_0,u,v}(\tau^-),\xi)) - \phi(\tau^-,X^{t_0,x_0,u,v}(\tau^-)) +$

$\Delta_N\phi(\tau, X^{t_0,x_0,u,v}(\tau))$ to be the change in ϕ due to the player I impulse $\xi \in \mathscr{Z}$ where $\Gamma : S\times Z \to S$ is the impulse response function.

We define Δ_η analogously so that $\Delta_\eta\phi(\rho, X^{t_0,x_0,u,v}(\rho)) := \phi(\rho^-, \Gamma(X^{t_0,x_0,u,v}(\rho^-),\eta)) - \phi(\rho^-, X^{t_0,x_0,u,v}(\rho^-)) + \Delta_N\phi(\rho, X^{t_0,x_0,u,v}(\rho))$ is the change in ϕ due to the player II impulse $\eta \in \mathscr{Z}$ at some intervention time $\rho \in \mathscr{T}$.

To prove the theorem, we use a singular control representation of the combined impulse controls for each player. For our first case, we define ν by $\nu(s) \equiv \eta(s) + \xi(s)$ so that ν is a process consisting of the combined player I and player II controls. Note that by Lemma 4.6, we have the following equivalences $\forall r \in [0,T]$:

(a) $\xi(r) = \sum_{m=1}^{\mu_{[t_0,r]}} \xi_j \cdot 1_{\{\tau_j \leq T\}}$

(b) $\eta(r) = \sum_{m=1}^{\mu_{[t_0,r]}} \eta_m \cdot 1_{\{\rho_m \leq T\}}$

(c) $\int_r^{r'} d\xi(s) = \sum_{j=1}^{\mu_{[r,r']}(\nu)} c(\tau_j, \xi_j)\cdot 1_{\{\tau_j \leq T\}}$

(d) $\int_r^{r'} d\eta(s) = \sum_{m=1}^{\mu_{[r,r']}(\nu)} \chi(\rho_m, \eta_m)\cdot 1_{\{\rho_m \leq T\}}$

We now fix the player II impulse control as $\hat{v} = [\hat{\rho}_m, \hat{\eta}_m]_{m\geq 1} \in \mathscr{V}$ and hence using (a) and (b) we find that $\nu(s)$ is now given by $\nu(s) = \sum_{m=1}^{\mu_{[t_0,s]}} \hat{\eta}_m \cdot 1_{\{\rho_m \leq T\}} + \sum_{m=1}^{\mu_{[t_0,r]}} \xi_j \cdot 1_{\{\tau_j \leq T\}}$.

As before, we employ the shorthand:

$$Y^{y_0,\cdot}(s) \equiv (s+t_0, X^{t_0,x_0,\cdot}(t_0+s)), \quad y_0 \equiv (t_0,x_0),\ \forall s \in [0,T-t_0], \tag{4.40}$$

$$\hat{Y}^{y_0,\cdot}(\tau) = Y^{y_0,\cdot}(\tau^-) + \Delta_N Y^{y_0,\cdot}(\tau), \quad \tau \in \mathscr{T}, \tag{4.41}$$

where $\Delta_N Y(\tau)$ denotes a jump at time τ due to $\tilde{N}$.

Correspondingly, we adopt the following impulse response function $\hat{\Gamma} : \mathscr{T}\times S \times \mathscr{Z} \to \mathscr{T}\times S$ acting on $y' \equiv (\tau, x') \in \mathscr{T}\times S$ where $x' \equiv X^{t_0,x_0,\cdot}(t_0+\tau^-)$ is given by:

$$\hat{\Gamma}(y',\zeta) \equiv (\tau, \Gamma(x',\zeta)) = (\tau, X^{t_0,x_0,\cdot}(\tau)), \quad \forall \xi \in \mathscr{Z},\ \forall \tau \in \mathscr{T}. \tag{4.42}$$

By Itō's formula for càdlàg semi-martingale (jump-diffusion) processes (see for example theorem II.33 of [Pro05] in conjunction with Theorem 1.24 of [ØS05]), we have that:

$$\mathbb{E}[\phi(Y^{y_0,u,\hat{v}}(\tau_j))] - \mathbb{E}[\phi(\hat{Y}^{y_0,u,\hat{v}}(\tau_{j+1}^-))] \tag{4.43}$$

$$= -\mathbb{E}\left[\int_{\tau_j}^{\tau_{j+1}} \frac{\partial\phi}{\partial s} + \mathscr{L}\phi(Y^{y_0,u,\hat{v}}(s))ds + \sum_{\nu_{t_0,\tau_j}(\hat{v}) < m \leq \nu_{t_0,\tau_{j+1}}(\hat{v})} \Delta_\nu\phi(Y^{y_0,u,\hat{v}}(\hat{\rho}_m))\right]. \tag{4.44}$$

We note firstly that by definition of the intervention times $\{\tau_j\}_{j\in\mathbb{N}}$ we have that $\mu_{[\tau_j,\tau_{j+1}[}(u) = 0$ since no player I interventions occur in the interval $[\tau_j, \tau_{j+1}[$. Hence, on the interval $[\tau_j, \tau_{j+1}[$ we have that $\Delta_\nu = \Delta_{\hat{\eta}}$ in particular, $\Delta_\nu\phi = \Delta_{\hat{\eta}}\phi$ so that $\Delta_\nu\phi(Y^{y_0,u,\hat{v}}(\hat{\rho}_m)) = \Delta_{\hat{\eta}}\phi(Y^{y_0,u,\hat{v}}(\hat{\rho}_m))$.

Hence, by (vii) we have that:

$$\mathbb{E}[\phi(Y^{y_0,u,\hat{v}}(\tau_j))] - \mathbb{E}[\phi(\hat{Y}^{y_0,u,\hat{v}}(\tau_{j+1}^-))] \tag{4.45}$$

$$= -\mathbb{E}\left[\int_{\tau_j}^{\tau_{j+1}} \frac{\partial \phi}{\partial s} + \mathscr{L}\phi(Y^{y_0,u,\hat{v}}(s))ds + \sum_{\nu_{t_0,\tau_j}(\hat{v})<m\leq \nu_{t_0,\tau_{j+1}}(\hat{v})} \chi(\hat{\rho}_m,\hat{\eta}_m)\right]. \tag{4.46}$$

Summing both sides from $j=0$ to $j=k<\infty$, we obtain the following:

$$\phi(y_0) + \sum_{j=1}^{k} \mathbb{E}[\phi(\hat{Y}^{y_0,u,\hat{v}}(\tau_j)) - \phi(\hat{Y}^{y_0,u,\hat{v}}(\tau_j^-))] - \mathbb{E}[\phi(\hat{Y}^{y_0,u,\hat{v}}(\tau_{k+1}^-))] \tag{4.47}$$

$$\leq \mathbb{E}\left[\int_{t_0}^{\tau_{k+1}} f(Y^{y_0,u,\hat{v}}(s))ds - \sum_{m\leq \nu_{t_0,\tau_{k+1}}(\hat{v})} \chi(\hat{\rho}_m,\hat{\eta}_m)\right]. \tag{4.48}$$

Now by definition of the non-local intervention operator $\mathscr{M}_1$, we have that:

$$\phi(Y^{y_0,u,\hat{v}}(\tau_j)) = \phi(\Gamma(\hat{Y}^{y_0,u,\hat{v}}(\tau_j-),\xi_j)) \geq \mathscr{M}_1\phi(\hat{Y}^{y_0,u,\hat{v}}(\tau_j-)) - c(\tau_j,\xi_j)\cdot 1_{\{\tau_j\leq\tau_S\}}.$$

(using the fact that $\inf_{z\in\mathscr{Z}}[\phi(\tau',\Gamma(X(\tau'^-),z)) + c(\tau',z)\cdot 1_{\{\tau'\leq T\}}] = 0$ whenever $\tau' > \tau_S$).

Hence,

$$\phi(Y^{y_0,u,\hat{v}}(\tau_j)) - \phi(\hat{Y}^{y_0,u,\hat{v}}(\tau_j-)) \geq \mathscr{M}_1\phi(\hat{Y}^{y_0,u,\hat{v}}(\tau_j-)) - \phi(\hat{Y}^{y_0,u,\hat{v}}(\tau_j-)) - c(\tau_j,\xi_j)\cdot 1_{\{\tau_j\leq\tau_S\}}, \tag{4.49}$$

and by (vi) we readily observe $\phi(Y^{y_0,u,\hat{v}}(\tau_S)) - \phi(\hat{Y}^{y_0,u,\hat{v}}(\tau_S)) = 0$. After plugging (4.49) into (4.48) we obtain the following:

$$\phi(y_0) + \sum_{j=1}^{k} \mathbb{E}[\mathscr{M}_1\phi(\hat{Y}^{y_0,u,\hat{v}}(\tau_j^-)) - \phi(\hat{Y}^{y_0,u,\hat{v}}(\tau_j^-)) - c(\tau_j,\xi_j)\cdot 1_{\{\tau_j\leq\tau_S\}}] - \mathbb{E}[\phi(\hat{Y}^{y_0,u,\hat{v}}(\tau_{k+1}^-))]$$
$$\leq \mathbb{E}\left[\int_{t_0}^{\tau_{k+1}} f(Y^{y_0,u,\hat{v}}(s))ds - \sum_{m\leq \nu_{t_0,\tau_{k+1}}(\hat{v})} \chi(\hat{\rho}_m,\hat{\eta}_m)\right].$$

Hence,

$$\phi(y_0) + \sum_{j=1}^{k} \mathbb{E}[\mathscr{M}_1\phi(\hat{Y}^{y_0,u,\hat{v}}(\tau_j^-)) - \phi(\hat{Y}^{y_0,u,\hat{v}}(\tau_j^-))] \tag{4.50}$$

$$\leq \mathbb{E}\left[\int_{t_0}^{\tau_{k+1}} f(Y^{y_0,u,\hat{v}}(s))ds + \phi(\hat{Y}^{y_0,u,\hat{v}}(\tau_{k+1}^-)) + \sum_{j\geq 1} c(\tau_j,\xi_j)\cdot 1_{\{\tau_j\leq\tau_S\}} - \sum_{m\leq \nu_{t_0,\tau_{k+1}}} \chi(\hat{\rho}_m,\hat{\eta}_m)\right], \tag{4.51}$$

Now $\lim_{k\to\infty}\sum_{j=1}^{k}\mathbb{E}[\mathscr{M}_1\phi(\hat{Y}^{y_0,u,\hat{v}}(\tau_j-)) - \phi(\hat{Y}^{y_0,u,\hat{v}}(\tau_j^-))] = 0$ since by (v) we have that $\phi(\hat{Y}^{y_0,\cdot}(\tau_j)) - \phi(\hat{Y}^{y_0,\cdot}(\tau_j^-)) = 0, \mathbb{P}-$a.s. when $\tau_j = \tau_S$; we can then deduce the statement by Lemma

3.10 in [CG13] i.e. using the $\frac{1}{2}$-Hölder continuity of the non-local operator $\mathscr{M}_1$. Similarly we have by (v) that $\phi(Y^{\cdot}(s)) \to G(Y^{\cdot}(\tau_S))$ as $s \to \tau_S^-$, $\mathbb{P}-$a.s. Hence, letting $k \to \infty$ in (4.51) gives:

$$\phi(y_0) \leq \mathbb{E}\Bigg[\int_{t_0}^{\tau_S} f(Y^{y_0,u,\hat{v}}(s))ds + \sum_{j\geq 1} c(\tau_j,\xi_j)\cdot 1_{\{\tau_j\leq\tau_S\}} - \sum_{n\geq 1}\chi(\hat{\rho}_n,\hat{\eta}_n)\cdot 1_{\{\hat{\rho}_n\leq\tau_S\}} \\ + G(Y^{y_0,u,\hat{v}}(\tau_S))\cdot 1_{\{\tau_S<\infty\}}\Bigg].$$

Since this holds for all $u \in \mathscr{U}$, we have that:

$$\phi(y_0) \leq \inf_{u\in\mathscr{U}} \mathbb{E}\Bigg[\int_{t_0}^{\tau_S} f(Y^{y_0,u,\hat{v}}(s))ds + \sum_{j\geq 1} c(\tau_j,\xi_j)\cdot 1_{\{\tau_j\leq\tau_S\}} - \sum_{n\geq 1}\chi(\hat{\rho}_n,\hat{\eta}_n)\cdot 1_{\{\hat{\rho}_n\leq\tau_S\}} \\ + G(Y^{y_0,u,\hat{v}}(\tau_S))\cdot 1_{\{\tau_S<\infty\}}\Bigg].$$

In particular, we have that:

$$\phi(y_0) \leq \sup_{v\in\mathscr{V}}\inf_{u\in\mathscr{U}} \mathbb{E}\Bigg[\int_{t_0}^{\tau_S} f(Y^{y_0,u,v}(s))ds + \sum_{j\geq 1} c(\tau_j,\xi_j)\cdot 1_{\{\tau_j\leq\tau_S\}} - \sum_{n\geq 1}\chi(\rho_n,\eta_n)\cdot 1_{\{\rho_n\leq\tau_S\}} \\ + G(Y^{y_0,x_0,u,v}(\tau_S))\cdot 1_{\{\tau_S<\infty\}}\Bigg] = V(y). \tag{4.52}$$

Using an analogous argument, namely replacing $\hat{v}$ with $\hat{u}$ in (4.44), then performing similar steps (using condition (iii)), we can similarly prove that:

$$\phi(y_0) \geq \inf_{u\in\mathscr{U}}\sup_{v\in\mathscr{V}} \mathbb{E}\Bigg[\int_{t_0}^{\tau_S} f(Y^{y_0,u,v}(s))ds + \sum_{j\geq 1} c(\tau_j,\xi_j)\cdot 1_{\{\tau_j\leq\tau_S\}} - \sum_{m\geq 1}\chi(\rho_m,\eta_m)\cdot 1_{\{\rho_m\leq\tau_S\}} \\ + G(Y^{y_0,u,v}(\tau_S))\cdot 1_{\{\tau_S<\infty\}}\Bigg] = V(y). \tag{4.53}$$

Let us now fix the pair of controls $(\hat{u},\hat{v}) \in \mathscr{U}\times\mathscr{V}$, using the definition of Δ_z and by (vii) we have that:

$$\begin{aligned} 0 &= \Delta_z\phi(Y(\hat{\rho}_m) - \chi(\hat{\rho}_m,z) = \phi(\hat{\Gamma}(Y(\hat{\rho}_m^-,z))) - \phi(Y(\hat{\rho}_m^-)) + \Delta_N Y(\hat{\rho}_m) - \chi(\hat{\rho}_m,z) \\ &= \phi(\hat{\Gamma}(\hat{Y}(\hat{\rho}_m^-,z))) - \phi(Y(\hat{\rho}_m^-)) - \chi(\hat{\rho}_m,z). \end{aligned} \tag{4.54}$$

Now since (4.54) holds for all $z \in \mathscr{Z}$, after applying the sup operator to both sides of (4.54) we find that:

$$0 = \sup_{z \in \mathscr{Z}} [\phi(\hat{\Gamma}(\hat{Y}(\hat{\rho}_m^-), z)) - \chi(\hat{\rho}_m, z)] - \phi(\hat{Y}(\hat{\rho}_m^-)) = \mathscr{M}_2 \phi(\hat{Y}(\hat{\rho}_m^-)) - \phi(\hat{Y}(\hat{\rho}_m^-)),$$

from which we immediately deduce the statement:

$$\mathscr{M}_2 \phi(\hat{Y}(\hat{\rho}_m^-)) = \phi(\hat{Y}(\hat{\rho}_m^-)). \tag{4.55}$$

We now see that an immediate impulse intervention at $\hat{\rho}_m$ is indeed optimal for player II. Using analogous arguments we can deduce that:

$$\mathscr{M}_1 \phi(\hat{Y}(\hat{\tau}_j^-)) = \phi(Y(\hat{\tau}_j^-)). \tag{4.56}$$

We hereafter straightforwardly observe using (iv) and (T4) we find the following *equality*:

$$\phi(y_0) = \mathbb{E}\left[\int_{t_0}^{\tau_S} f(Y^{y_0,\hat{u},\hat{v}}(s))ds + \sum_{j \geq 1} c(\hat{\tau}_j, \hat{\xi}_j) \cdot 1_{\{\hat{\tau}_j \leq \tau_S\}} - \sum_{m \geq 1} \chi(\hat{\rho}_m, \hat{\eta}_m) \cdot 1_{\{\hat{\rho}_m \leq \tau_S\}} \right.$$
$$\left. + G(Y^{y_0,\hat{u},\hat{v}}(\tau_S)) \cdot 1_{\{\tau_S < \infty\}} \right]. \tag{4.57}$$

Hence, we can deduce the following statement:

$$\sup_{v \in \mathscr{V}} \inf_{u \in \mathscr{U}} J^{u,v}[y] \geq \phi(y) = J^{\hat{u},\hat{v}}[y] \geq \inf_{u \in \mathscr{U}} \sup_{v \in \mathscr{V}} J^{u,v}[y], \quad \forall y \in [0,T] \times S. \tag{4.58}$$

Now, since $\inf_{u \in \mathscr{U}} \sup_{v \in \mathscr{V}} J^{u,v}[y] \geq \sup_{v \in \mathscr{V}} \inf_{u \in \mathscr{U}} J^{u,v}[y] (= V^+(y))$ it then follows that:

$$V(y) = \phi(y) = J^{\hat{u},\hat{v}}[y], \qquad \forall y \in [0,T] \times S, \tag{4.59}$$

after which we deduce the thesis. □

4.4 A HJBI Equation for Non Zero-sum Stochastic Differential Games with Impulse Controls

In this section, we now extend the results to non zero-sum stochastic differential games. Expectedly, proving existence results of Nash equilibria for stochastic differential games that involve impulse controls relies on a similar set of arguments as those constructed in the continuous control case. Indeed, Nash equilibria existence and characterisation results have been established for differential games in which continuous controls were used see e.g. [Kon76; Kle93] and in the stochastic case in [BCR04] in which a method of generalising Folk Theorems[2] in classical game theory (i.e.

[2]Folk Theorems are a set of fundamental results within repeated games that state that any feasible payoff for which the player is weakly better off than their minmax payoff (i.e. individual rational payoff) can in fact be supported in subgame

deterministic repeated games) was used.

We firstly prove a non zero-sum verification theorem for the game in which both players use impulse controls to modify the state process.

In order to describe a non zero-sum game, we now consider a game in which each player has their own individual payoff function. The payoff functions for player I and player II, J_1 and J_2 respectively, are given by the following:

$$J_1^{(\tilde{u},\tilde{v})}[y_0] = \mathbb{E}\left[\int_{t_0}^{\tau_S} f_1(Y_s^{y_0,\tilde{u},\tilde{v}})ds - \sum_{j\geq 1} c_1(\tilde{\tau}_j,\tilde{\xi}_j)\cdot 1_{\{\tau_j\leq\tau_S\}} + G_1(Y_{\tau_S}^{y_0,\tilde{u},\tilde{v}})\cdot 1_{\{\tau_S<\infty\}}\right], \tag{4.60}$$

$$J_2^{(\hat{u},\tilde{v})}[y_0] = \mathbb{E}\left[\int_{t_0}^{\tau_S} f_2(Y_s^{y_0,\tilde{u},\tilde{v}})ds - \sum_{m\geq 1} c_2(\tilde{\rho}_m,\tilde{\eta}_m)\cdot 1_{\{\rho_m\leq\tau_S\}} + G_2(Y_{\tau_S}^{y_0,\tilde{u},\tilde{v}})\cdot 1_{\{\tau_S<\infty\}}\right], \tag{4.61}$$

$$\forall y_0 \equiv (t_0,x_0) \in [0,T]\times S,$$

where $\tilde{u} = [\tilde{\tau}_j,\tilde{\xi}_j]_{j\geq 1}$ and $\tilde{v} = [\tilde{\rho}_m,\tilde{\eta}_m]_{m\geq 1}$ are admissible controls for player I and player II respectively.

We note that the function $J_1^{(\tilde{u},\hat{v})}[y]$ (resp., $J_2^{(\hat{u},\tilde{v})}[y]$) defines the payoff received by the player I (resp., player II) when it uses the control $\tilde{u}\in\mathscr{U}$ (resp., $\tilde{v}\in\mathscr{V}$) and player II (resp., player I) uses the control $\hat{v}\in\mathscr{V}$ (resp. $\hat{u}\in\mathscr{U}$) given some initial point $y\in[0,T]\times S$.

Since we are now handling a game with a non zero-sum payoff structure, we must adapt the definitions of the non-local intervention operators (Definition 2.2) to the following:

Definition 4.10

Let $\tau\in\mathscr{T}$. For $i\in\{1,2\}$, we define the [non-local] Player i-intervention operator $\mathscr{M}_i:\mathscr{H}\to\mathscr{H}$ acting at a state $X(\tau)$ by the following expression:

$$\mathscr{M}_i\phi(\tau,X(\tau)) := \sup_{z\in\mathscr{Z}}[\phi(\tau,\Gamma(X(\tau^-),z)) - c_i(\tau,z)\cdot 1_{\{\tau\leq T\}}], \tag{4.62}$$

where $\Gamma: S\times\mathbb{R}\to S$ is the impulse response function.

Definition 4.11 (Nash Equilibrium for Non Zero-sum Games with Impulse Control)

We say that a pair $(\hat{u},\hat{v})\in\mathscr{U}\times\mathscr{V}$ is a Nash equilibrium of the stochastic differential game with impulse controls $\hat{u} = [\hat{\tau}_j,\hat{\xi}_j]_{j\in\mathbb{N}}\in\mathscr{U}, \hat{v} = [\hat{\rho}_m,\hat{\eta}_m]_{m\in\mathbb{N}}\in\mathscr{V}$ if the following statements hold:

(i) $J_1^{(u,\hat{v})}[y] \geq J_1^{(\hat{u},\hat{v})}[y] \quad \forall u\in\mathscr{U},\forall y\in[0,T]\times S,$

(ii) $J_2^{(\hat{u},v)}[y] \geq J_2^{(\hat{u},\hat{v})}[y] \quad \forall v\in\mathscr{V},\forall y\in[0,T]\times S.$

Condition (i) states that given some fixed player II control policy $\hat{v}\in\mathscr{V}$, player I cannot profitably deviate from playing the control policy $\hat{u}$. Analogously, condition (ii) is the equivalent statement

perfect Nash equilibrium when the players are sufficiently patient that is, whenever the players of the game are sufficiently patient then the repeated game can allow any outcome in the average payoff sense.

given the player I's control policy is fixed as $\hat{u}$, player II cannot profitably deviate from $\hat{v}$. We therefore see that $(\hat{u},\hat{v})$ is an equilibrium in the sense of a Nash equilibrium since neither player has an incentive to deviate whenever their opponent plays their equilibrium policy.

We now generalise our the zero-sum result of Theorem 4.9 to cover non zero-sum payoff structures with the use of a Nash Equilibrium solution concept.

Theorem 4.12 (Verification theorem for Non Zero-sum Games with Impulse Control)

Let us suppose that the value of the game exists and that there exists functions $\phi_i \in \mathscr{C}^{1,2}([0,T],S) \cap \mathscr{C}([0,T],\bar{S}), i \in \{1,2\}$ such that ϕ_i satisfy technical conditions (T1) - (T4) (see Appendix) and the following conditions:

(i') $\phi_i \geq \mathscr{M}_i\phi_i$ on S and the regions D_i are defined by:
$D_i = \{x \in S; \phi_i(\cdot,x) > \mathscr{M}_i\phi_i(\cdot,x)\}, i \in \{1,2\}$ where we refer to D_1 (resp., D_2) as the player I (resp., player II) continuation region.

(ii') $\frac{\partial \phi_1}{\partial s} + \mathscr{L}\phi_1(\cdot,X^{\cdot,u,\hat{v}}(\cdot)) + f_1(\cdot X^{\cdot,u,\hat{v}}(\cdot))$
$\geq \frac{\partial \phi_1}{\partial s} + \mathscr{L}\phi_1(\cdot,X^{\cdot,\hat{u},\hat{v}}(\cdot)) + f_1(\cdot,X^{\cdot,\hat{u},\hat{v}}(\cdot)) \geq 0, \quad \forall u \in \mathscr{U}$ on $S\backslash D_1$.

(iii') $\frac{\partial \phi_2}{\partial s} + \mathscr{L}\phi_2(\cdot,X^{\cdot,\hat{u},v}(\cdot)) + f_2(\cdot,X^{\cdot,\hat{u},v}(\cdot))$
$\geq \frac{\partial \phi_2}{\partial s} + \mathscr{L}\phi_2(\cdot,X^{\cdot,\hat{u},\hat{v}}(\cdot)) + f_2(\cdot,X^{\cdot,\hat{u},\hat{v}}(\cdot)) \geq 0, \quad \forall v \in \mathscr{V}$ on $S\backslash D_2$.

(iv') $\frac{\partial \phi_i}{\partial s} + \mathscr{L}\phi_i(\cdot,X^{\cdot,\hat{u},\hat{v}}(\cdot)) + f_i(\cdot,X^{\cdot,\hat{u}_{[t_0,s]},\hat{v}_{[t_0,s]}}(\cdot)) = 0$ on $D_1 \cap D_2$.

(v') $\hat{\xi}_k \in \operatorname{argsup}_{z\in\mathscr{Z}}\{\phi_i(\tau_k,\Gamma(x,z)) - c(\tau_k,z)\}$ is a Borel Measurable selection $\forall x \in S$, $\tau_k \in \mathscr{T}$. Similarly, $\hat{\eta}_j \in \operatorname{argsup}_{z\in\mathscr{Z}}\{\phi_i(\rho_j,\Gamma(x,z)) - \chi(\rho_j,z)\}$ is a Borel Measurable selection $\forall x \in S$, $\forall \rho_j \in \mathscr{T}$.

Put $\hat{\tau}_0 \equiv t_0$ and define $\hat{u} := [\hat{\tau}_j,\hat{\xi}_j]_{j\in\mathbb{N}}$ inductively by $\hat{\tau}_{j+1} = \inf\{s > \tau_j;\ X^{\hat{u}_{[t_0,s]},v}(\cdot) \notin D_1\} \wedge \tau_S$, $\forall v \in \mathscr{V}$. Similarly, put $\hat{\rho}_0 \equiv t_0$ and define $\hat{v} := [\hat{\rho}_m,\hat{\eta}_m]_{m\in\mathbb{N}}$ inductively by $\hat{\rho}_{m+1} = \inf\{s > \rho_m; X^{u,\hat{v}_{[t_0,s]}}(\cdot) \notin D_2\} \wedge \tau_S$, $\forall u \in \mathscr{U}$.

Then $(\hat{u},\hat{v})$ is a Nash equilibrium for the game, that is to say the following statements hold:

$$\phi_1(y) = \sup_{u\in\mathscr{U}} J_1^{(u,\hat{v})}[y] = J_1^{(\hat{u},\hat{v})}[y], \tag{4.63}$$

and

$$\phi_2(y) = \sup_{v\in\mathscr{V}} J_2^{(\hat{u},v)}[y] = J_2^{\hat{u},\hat{v}}[y]. \tag{4.64}$$

$$\forall y \in [0,T] \times S.$$

As in Theorem 4.9, we note that conditions (i') - (iii') of Theorem 4.12 follow directly from QVI conditions which can be motivated by a heuristic analysis similar to that of the zero-sum case. Condition (i') is used to allow for the integro-differential operator $\mathscr{L}$ in (i') - (iii') to be applied in addition to permitting an application of Dynkin's formula which is central to the proof of the theorem.

The proof of Theorem 4.12 follows a similar path to that of Theorem 4.9, we therefore defer the proof of the theorem to the chapter appendix.

In the following, we apply Theorem 4.12 to solve the duopoly investment problem and in doing so, provide an example of a function that satisfies the properties of the theorem.

Before doing so, in analogy to Corollary 4.9.1, we give the following result which follows directly from Theorem 4.12:

Corollary 4.12.1

The sample space splits into three regions that represent a region in which player I intervenes in I_1, a region in which player II intervenes I_2, and a region in which no action is taken by either player I or player II; moreover the three regions are characterised by the following expressions for $j \in \{1,2\}$:

$$I_j = \{y \in [0,T] \times S : V_j(y) = \mathscr{M}_j V_j(y),\ \mathscr{L}V_j(y) + f_j(y) \geq 0\},$$
$$I_3 = \{y \in [0,T] \times S : V_j(y) \geq \mathscr{M}_j V_j(y);\ \mathscr{L}V_j(y) + f_j(y) = 0\}.$$

4.5 Examples

In order to demonstrate the workings of the theorem, we give an example calculation.

The first example solves a zero-sum stochastic differential game of two-sided impulse control. The example exemplifies the method by which solutions to the game can be calculated using the verification theorem (Theorem 4.9).

Example 4.13

Consider a system with passive dynamics that are described by the following stochastic process:

$$dX(r) = \alpha dr + \beta dB(r), \qquad \forall r \in [0,T], \tag{4.65}$$

where $\alpha, \beta \in \mathbb{R}_{>0}$ are fixed constants, $B(r)$ is a 1-dimensional Brownian motion and $T \in \mathbb{R}_{>0}$ is some finite time horizon. The process X in (4.65) is known as *Brownian motion with drift* and models and number of processes in finance such as insurance claim processes and risk-neutral price-processes in options pricing, for example [DPY05; Pec99].

The state process X is modified by two controllers, player I, that exercises an impulse control policy $u = [\tau_j, \xi_j] \in \mathscr{U}$ and player II that exercises an impulse control policy $v = [\rho_m, \eta_m] \in \mathscr{V}$. The controlled state process evolves according to the following expression $\forall t \in [0,T]$:

$$X(t) = x_0 + \alpha \int_0^{t\wedge\tau_S} ds + \beta \int_0^{t\wedge\tau_S} dB(s) - \sum_{j\geq 1} (\kappa_1 + (1+\lambda_1)\xi_j) \cdot 1_{\{\tau_j \leq t\wedge\tau_S\}}$$
$$- \sum_{m\geq 1} (\kappa_2 + (1+\lambda_2)\eta_m) \cdot 1_{\{\rho_m \leq t\wedge\tau_S\}},\ X(0) \equiv x_0,\ \mathbb{P} - \text{a.s.} \tag{4.66}$$

where $\tau_S := \inf\{s > 0 : X(s) \leq 0\}$ and the constants $\kappa_i > 0$ and $\lambda_i > 0$ are the fixed part and the proportional part of the transaction cost incurred by player $i \in \{1,2\}$ for each intervention (resp.).

Player I seeks to choose an admissible impulse control $u = [\tau_j, \xi_j]$ that maximises its reward J where $\{\tau_j\}_{\{j\geq 1\}}$ are player I intervention times and each $\xi_{j\geq 1} \in \mathscr{Z}$ is a player I impulse intervention. Player II seeks to choose an admissible impulse control $v = [\rho_m, \xi_m]$ that minimises the same quantity J where $\{\rho_m\}_{\{m\geq 1\}}$ are player II intervention times and each $\eta_{m\geq 1} \in \mathscr{Z}$ is a player II impulse intervention. The function J is given by the following expression:

$$J^{u,v}[t,x] = \mathbb{E}\left[\sum_{j\geq 1} e^{-\delta\tau_j}\xi_j \cdot 1_{\{t<\tau_j\leq\tau_S\}} - \sum_{m\geq 1} e^{-\delta\rho_m}\eta_m \cdot 1_{\{t<\rho_m\leq\tau_S\}}\right], \qquad \forall (t,x) \in [0,T]\times\mathbb{R}, \tag{4.67}$$

where $\delta \in]0,1]$ is common discount factor.

An example of a setting for this game is an interaction between two players that consume a common rivalrous and exhaustible good (e.g. public funds, extractable resources, labour supply etc.) which is vulnerable to stochastic extinction. For each act of consumption, each player incurs both a fixed cost and a proportional cost.

The problem is to find a function $\phi \in \mathscr{C}^{1,2}([0,T],\mathbb{R})$ such that

$$\sup_u \inf_v J^{u,v}(s,x) = \inf_v \sup_u J^{u,v}(s,x) = \phi(s,x), \quad \forall (s,x) \in [0,T]\times\mathbb{R}. \tag{4.68}$$

By (4.65) and using (1.2), the generator $\mathscr{L}$ for the process X is given by:

$$\mathscr{L}\psi(s,x) = \frac{\partial\psi}{\partial s}(s,x) + \alpha\frac{\partial\psi}{\partial x}(s,x) + \frac{1}{2}\beta^2\frac{\partial^2\psi}{\partial x^2}(s,x), \tag{4.69}$$

for some test function $\psi \in \mathscr{C}^{1,2}([0,T],\mathbb{R})$.

We wish to firstly derive the functional form of the function ϕ. Applying (iv) of Theorem 4.9 leads to the *heat equation* $\mathscr{L}\phi = 0$ (here, $f \equiv 0$ in Theorem 4.9). Following this, we make the following ansatz for the function ϕ, $\phi(s,x) = e^{-\delta s}\psi(x), \psi(x) := ae^{bx}$ for some as yet, undetermined constants $a,b \in \mathbb{R}$. Plugging the ansatz for the function ϕ and using (iv) of Theorem 4.9 into (4.69) immediately gives:

$$-\delta+\alpha b+\frac{1}{2}\beta^2 b^2=0. \tag{4.70}$$

After some manipulation, we deduce that there exist two solutions for b which we denote by b_1 and b_2 such that $b_1 > b_2$ with $b_1 > 0$ and $|b_2| > 0$ which are given by the following:

$$b_1=\frac{1}{\beta^2}\sqrt{\alpha^2+2\beta^2\delta}-\frac{\alpha}{\beta^2}, \qquad b_2=-\frac{\alpha}{\beta^2}-\frac{1}{\beta^2}\sqrt{\alpha^2+2\beta^2\delta}. \tag{4.71}$$

We now apply the HJBI equation (iv) of Theorem 4.9 to characterise the function ϕ on the region $D_1\cap D_2$. Following our ansatz, we observe, using (iv), the following expression for the function ϕ:

$$\phi(s,x)=e^{-\delta s}\psi(x), \qquad \forall(s,x)\in[0,T]\times D_1\cap D_2, \tag{4.72}$$

$$\psi(x)=(a_1e^{b_1x}+a_2e^{b_2x}), \qquad \forall x\in D_1\cap D_2, \tag{4.73}$$

where a_1 and a_2 are constants that are yet to be determined and D_1 and D_2 are the continuation regions for player I and player II respectively. In order to determine the constants a_1 and a_2, we firstly observe that $\phi(\cdot,0)=0$. This implies that $a_1=-a_2:=a>0$. We now deduce that the function ψ is given by the following expression:

$$\psi(x)=a(e^{b_1x}-e^{b_2x}), \qquad \forall x\in D_1\cap D_2. \tag{4.74}$$

In order to characterise the function over the entire state space and find the value a, using conditions (i) - (vii) of Theorem 4.9, we study the behaviour of the function ϕ given each of the players' equilibrium controls. Firstly, we consider the player I impulse control problem. In particular, we seek to obtain conditions on the impulse intervention applied when $\mathscr{M}_1\phi=\phi$. To this end, let us firstly conjecture that the player I continuation region D_1 takes the following form:

$$D_1=\{x\in\mathbb{R};0<x<\tilde{x}\}, \tag{4.75}$$

for some constant $\tilde{x}$ which we shall later determine.

Our first task is to determine the optimal value of the impulse intervention. We now define the following two functions which will be of immediate relevance:

$$\psi_0(x):=a(e^{b_1x}-e^{b_2x}), \tag{4.76}$$

$$h(\xi):=\psi(x-\kappa_1-(1+\lambda_1)\xi)+\xi, \tag{4.77}$$

$$\forall x\in\mathbb{R},\forall\xi\in\mathscr{Z}.$$

In order to determine the value $\hat{\xi}$ that maximises $\Gamma(x(\tau^-),\xi)$ at the point of intervention, we investigate the first order condition on h i.e. $0 = h'(\xi)$. This implies the following:

$$\psi'(\tilde{x} - \kappa_1 - (1+\lambda_1)\xi) = \frac{1}{1+\lambda_1}. \tag{4.78}$$

Using the expression for ψ (4.73), we also observe the following:

$$\psi_0'(x) = b_1 e^{b_1 x} - b_2 e^{b_2 x} > 0, \qquad \forall x \in \mathbb{R}, \tag{4.79}$$

$$\psi_0''(x) = b_1^2 e^{b_1 x} - b_2^2 e^{b_2 x} < 0, \qquad \forall x < x^\# := \frac{2}{b_1 - b_2} \ln\left[\frac{|b_2|}{b_1}\right], \tag{4.80}$$

from which we deduce the existence of two points $x^\star, x_\star$ for which the condition $\psi_0'(x) = (1+\lambda_1)^{-1}$ holds. W.l.o.g. we assume $x^\star > x_\star$. Now by (i) of Theorem 4.9 we require that $\phi(\cdot,x) = \mathscr{M}_1\phi(\cdot,x)$ whenever $x \geq \tilde{x}$ (c.f. D_1 in equation (4.75)) for which we have $e^{-\delta t}\psi_0(x) = \mathscr{M}_1\phi(t,x)$ whenever $x \geq \tilde{x}$, hence we find that:

$$\psi(x) = \psi_0(x_\star) + \hat{\xi}(x), \qquad \forall x \geq \tilde{x}, \tag{4.81}$$

where $x - \kappa_1 - (1+\lambda_1)\hat{\xi}(x) = x_\star$ from which we readily find that the optimal player I impulse intervention value is given by:

$$\hat{\xi}(x) = \frac{x - x_\star - \kappa_1}{1+\lambda_1}, \qquad \forall x \geq \tilde{x}. \tag{4.82}$$

Having determined the optimal impulse intervention and constructed the form of the continuation region for Player I, we can derive the optimal impulse intervention for player II straightforwardly by analogous arguments from which we find that the continuation region for player II takes the form:

$$D_2 = \{x \in \mathbb{R}; 0 < x < \bar{x}\}, \tag{4.83}$$

and the optimal player II impulse intervention value is given by

$$\hat{\eta}(x) = \frac{x_\# - x - \kappa_2}{1+\lambda_2}, \qquad \forall x \geq \bar{x}. \tag{4.84}$$

Putting the above facts together yields the following characterisation of the function ψ:

$$\psi(x) = \begin{cases} a(e^{b_1 x} - e^{b_2 x}), & \forall x \in D_1 \cap D_2, \\ a(e^{b_1 x_\#} - e^{b_2 x_\#}) + \dfrac{x_\# - x - \kappa_2}{1+\lambda_2}, & \forall x \notin D_2, \\ a(e^{b_1 x_\star} - e^{b_2 x_\star}) + \dfrac{x - x_\star - \kappa_1}{1+\lambda_1}, & \forall x \notin D_1, \end{cases} \tag{4.85}$$

where the constants b_1 and b_2 are specified in equation (4.71).

We are now in a position to determine the value of the constants $a, \hat{x}$ and $\tilde{x}$. In particular, we apply the high contact principle[3] to find the boundary of the continuation region D_2. Indeed, continuity at $\bar{x}$ leads to the following:

$$\psi(\bar{x}) = \psi_0(x_\#) + \hat{\eta}(\bar{x}) \implies a(e^{b_1\bar{x}} - e^{b_2\bar{x}}) = a(e^{b_1 x_\#} - e^{b_2 x_\#}) + \frac{x_\# - \bar{x} - \kappa_2}{1+\lambda_2}, \tag{4.86}$$

from which we find that $\bar{x}$ is the solution to the following equation:

$$m_2(\bar{x}) = 0, \tag{4.87}$$

where the function m_2 is given by:

$$m_2(x) = x - a(1+\lambda_2)(e^{b_1 x} - e^{b_2 x} - e^{b_1 x_\#} + e^{b_2 x_\#}) + x_\# - \kappa_2. \tag{4.88}$$

Lastly, we reapply the high contact principle to find the boundary of the continuation region D_1. Indeed, continuity at $\tilde{x}$ leads to the following relationship:

$$\psi(\tilde{x}) = \psi_0(x_\star) + \hat{\xi}(\tilde{x}) \implies a(e^{b_1\tilde{x}} - e^{b_2\tilde{x}}) = a(e^{b_1 x_\star} - e^{b_2 x_\star}) + \frac{\tilde{x} - x_\star - \kappa_1}{1+\lambda_1}, \tag{4.89}$$

from which we find that $\tilde{x}$ is the solution to the following equation:

$$m_1(\tilde{x}) = 0, \tag{4.90}$$

where the function m is given by:

$$m_1(x) = x - a(1+\lambda_1)(e^{b_1 x} - e^{b_2 x} - e^{b_1 x_\star} + e^{b_2 x_\star}) - x_\star - \kappa_1. \tag{4.91}$$

Equations (4.88) and (4.90) are difficult to solve analytically for the general case but can however, be straightforwardly solved numerically using a root-finding algorithm.

To summarise, the solution is as follows: whenever $X \in D_1 \cap D_2$ neither player intervenes. Player I performs an impulse intervention of size $\hat{\xi}$ given by (4.82) whenever the process reaches the value $\tilde{x}$ and player II performs an impulse intervention of size $\hat{\eta}$ given by (4.84) whenever the process reaches the value $\bar{x}$. The value function for the problem is $\phi(s,x) = e^{-\delta s}\psi(x)$, $\forall (s,x) \in [0,T] \in \mathbb{R}$,

[3] Recall that the high contact principle is a condition that asserts the continuity of the value function at the boundary of the continuation region.

where is ψ given by:

$$\psi(x) = \begin{cases} a(e^{b_1 x} - e^{b_2 x}), & \forall x \in D_1 \cap D_2, \\ a(e^{b_1 x_\#} - e^{b_2 x_\#}) + \dfrac{x_\# - x - \kappa_2}{1+\lambda_2}, & \forall x \notin D_2, \\ a(e^{b_1 x_\star} - e^{b_2 x_\star}) + \dfrac{x - x_\star - \kappa_1}{1+\lambda_1}, & \forall x \notin D_1, \end{cases} \tag{4.92}$$

and where the player I and player II continuation regions are given by:

$$D_1 = \{x \in \mathbb{R}\,; 0 < x < \tilde{x}\}, \tag{4.93}$$

$$D_2 = \{x \in \mathbb{R}\,; 0 < x < \bar{x}\}, \tag{4.94}$$

where the constants $\bar{x}$ and $\tilde{x}$ are determined by (4.88) and (4.91) respectively and the constants b_1, b_2 are given by (4.71).

We now tackle the case of scenario in which the payoff structure is not zero-sum. We therefore give an application of the Theorem 4.12. In particular, we are now in a position to apply the results to the duopoly investment model.

4.6 A Firm Duopoly Investment Problem: Dynamic Competitive Advertising (Revisited)

Theorem 4.14

Suppose that the market share X_i of Firm i, ($i \in \{1,2\}$) evolves according to (4.5) - (4.6) and let the firm payoff functions be given by (4.7) - (4.8), then the sequence of optimal investments $\hat{u} = [\hat{\tau}_j, \hat{\xi}_j]_{j\in\mathbb{N}} \equiv \sum_{j\geq 1} \hat{\xi}_j \cdot 1_{\{\hat{\tau}_j \leq \tau_S\}}(s)$ for Firm 1 is characterised by the investment times $\{\hat{\tau}_j\}_{j\in\mathbb{N}}$ and investment magnitudes $\{\hat{\xi}_j\}_{j\in\mathbb{N}}$ where $[\hat{\tau}_j, \hat{\xi}_j]_{j\in\mathbb{N}}$ are constructed via the following expressions:

(i) $\hat{\tau}_0 \equiv t_0$ and $\hat{\tau}_{j+1} = \inf\{s > \tau_j; X_1(s) \leq x_1^\star\} \wedge \tau_S,\ \forall v \in \mathscr{V}$.

(ii) $\hat{\xi}_j = \hat{x}_1 - X_1(\hat{\tau}_j)$.

Similarly for Firm 2, the optimal sequence of investments $\hat{v} := [\hat{\rho}_m, \hat{\eta}_m]_{m\in\mathbb{N}} = \sum_{m\geq 1} \hat{\eta}_m \cdot 1_{\{\hat{\rho}_m \leq \tau_S\}}(s)$ is given by:

(i) $\hat{\rho}_0 \equiv t_0$ and $\hat{\rho}_{m+1} = \inf\{s > \rho_m; X_2(s) \leq x_2^\star\} \wedge \tau_S,\ \forall u \in \mathscr{U}$,

(ii) $\hat{\eta}_m = \hat{x}_2 - X_2(\hat{\rho}_m)$, where the quadruplet $(x_1^\star, x_2^\star, \hat{x}_1, \hat{x}_2)$ is determined by the following equations ($i \in \{1,2\}$):

$$C_1 r_{1,i} e^{r_{1,i} x_i^\star} + C_2 r_{2,i} e^{r_{2,i} x_i^\star} + \frac{\alpha_i}{\varepsilon} = \lambda_i \tag{4.95}$$

$$C_1 r_{1,i} e^{r_{1,i} \hat{x}_i} + C_2 r_{2,i} e^{r_{2,i} \hat{x}_i} + \frac{\alpha_i}{\varepsilon} = \lambda_i \tag{4.96}$$

$$C_1 (e^{r_{1,i} x_i^\star} - e^{r_{1,i} \hat{x}_i}) + C_2 (e^{r_{2,i} x_i^\star} - e^{r_{2,i} \hat{x}_i}) = -\kappa_i + \left(\lambda_i - \frac{\alpha_i}{\varepsilon}\right)(x_i^\star - \hat{x}_i), \tag{4.97}$$

where C_1 and C_2 are endogenous constants whose values are determined by (4.95) - (4.97), λ_i and κ_i are the Firm i proportional and fixed intervention costs (respectively), α_i is the Firm i margin parameter, ε is the discount factor and the values $r_{1,i}$ and $r_{2,i}$ are roots of the equation:

$$q(r_{k,i}) := \frac{1}{2}\sigma_{ii}^2 r_{k,i}^2 + \mu_i r_{k,i} - \varepsilon + \int_{\mathbb{R}} \{e^{r_{k,i}\theta_{ij} z} - 1 - \theta_{ij} r_{k,i} z\} \nu_j(dz), \tag{4.98}$$

for $i, j, k \in \{1,2\}$.

Theorem 4.14 says that each firm performs a sequence of investments over the time horizon of the problem. The decision to invest is determined by the firm's market share process — in particular, at the point at which Firm i's share of the market falls below the level $x_i^\star$, then the firm performs an investment in order to raise its market share to $\hat{x}_i$, where the fixed values $\hat{x}_i$ and $x_i^\star$ are determined by the given parameters $\lambda_i, \lambda_j, \kappa_i, \kappa_j, \alpha_i, \varepsilon$ via (4.95) - (4.97). In particular, each Firm i seeks to retain a market share of at least $x_i^\star$, where $x_i^\star$ is a quantity determined by the size and influence of both firms. Therefore, if S is the total size of the market the value $S - x_i^\star$ represents the maximum level of market share that Firm i is prepared to cede to the rival firm.

In summary, each firm observes its own market share and only intervenes at the points at which the firm's market share has fallen below some fixed level. Each firm's intervention policy is reactant to the investment and subsequent market acquisition of the other firm, each firm therefore reacts by performing the best sequence of response investments to the other firm's investment strategy.

The following corollary follows directly from Theorem 4.14 and establishes when each firm performs investments under the optimal Nash equilibrium strategy:

Corollary 4.14.1

The sample space splits into three regions: a region in which Firm i performs an advertising investment — I_1, a region in which Firm 2 performs an advertising investment — I_2 and a region in which no action is taken by either firm I_3. Moreover, the three regions are characterised by the following expressions:

$$I_j = \{x < x_j^\star | x, x_j^\star \in \mathbb{R}\}, \qquad i, j \in \{1,2\},$$
$$I_3 = \{x \geq x_i^\star \wedge x_j^\star | x, x_i^\star, x_j^\star \in \mathbb{R}\},$$

where the $x_i^\star, x_j^\star, i,j \in \{1,2\}$ are determined by (4.95) - (4.97).

The following result characterises the value function for each firm:

Proposition 4.15

The value function $\phi_i(t,x_1,x_2) : \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ for each Firm i is given by the following expression:

$$\phi_i(t,x_1,x_2) = e^{-\varepsilon t}\Big\{C_1(e^{r_1x_1}+e^{r_1x_2})+C_2(e^{r_2x_1}+e^{r_2x_2})+\frac{\alpha_i}{\varepsilon}x_1-\frac{\beta_i}{\varepsilon}x_2+\frac{1}{2\varepsilon^2}(\mu_i\alpha_i-\mu_j\beta_i)\Big\},\ i\neq j,\ i,j\in\{1,2\}, \tag{4.99}$$

where C_1, C_2, r_1, r_2 are endogenous constants.

Proposition 4.15 provides a full characterisation of each firm's value function which in turn, quantifies each firm's future expected payoff. As we show in the chapter appendix, the endogenous constants can be recovered by approximating the solutions to a system of simultaneous equations.

In the following analysis, we use the results of the stochastic differential game involving impulse controls and a non zero-sum payoff to solve our model of investment duopoly (case II) presented in the chapter.

The Duopoly Investment Problem Revisited

In this section, we apply the results of the chapter to prove Theorem 4.14. Let us denote by Y the process $Y(s) = (s+t_0, X_1(s), X_2(s))$, where $X_1 : \mathbb{R}_{>0} \times \Omega \to \mathbb{R}$, $X_2 : \mathbb{R}_{>0} \times \Omega \to \mathbb{R}$ are processes which represent the market share processes for Firm 1 and Firm 2 respectively and whose evolution is described by (4.5) - (4.6). We wish to fully characterise the optimal investment strategies for each firm, in order to do this we apply Theorem 4.12. We restrict ourselves to the case when $\theta_{ij}(s) \equiv \bar{\theta}_{ij} \in \mathbb{R}\backslash\{0\}$ and $\sigma_{ij}(s) \equiv \bar{\sigma}_{ij} \in \mathbb{R}\backslash\{0\}$.

Proof of Theorem 4.14 Given an admissible Firm 1 (resp., Firm 2) investment policy $u = [\tau_j, \xi_j]_{j\in\mathbb{N}} \in \mathscr{U}$ (resp., $v = [\rho_m, \eta_m]_{m\in\mathbb{N}} \in \mathscr{V}$) we note that the following identities hold:

$$X_1(\tau_j) = \Gamma(X_1(\tau_j-)+\Delta_N X_1(\tau_j), \xi_j) = \hat{X}_1(\tau_j)+\xi_j, \tag{4.100}$$

$$X_2(\rho_m) = \Gamma(X_2(\rho_m-)+\Delta_N X_2(\rho_m), \eta_m) = \hat{X}_2(\rho_m)+\eta_m. \tag{4.101}$$

The Firm 1 and Firm 2 investment intervention operator (acting on some function $\psi : [0,T] \times$

$S \to \mathbb{R}$) are given by the following expressions:

$$\mathscr{M}_1\psi(\tau^-,x) = \sup_{\xi\in\mathscr{Z}}\{\psi(\tau,x+\xi)-(\lambda_1\xi+\kappa_1)\}, \tag{4.102}$$

$$\mathscr{M}_2\psi(\tau^-,x) = \sup_{\eta\in\mathscr{Z}}\{\psi(\tau,x+\eta)-(\lambda_2\eta+\kappa_2)\}, \quad \forall(\tau,x)\in\mathscr{T}\times\mathbb{R}. \tag{4.103}$$

Recall that the Firm 1 and the Firm 2 profit functions are given by:

$$\begin{aligned}\Pi_1(y;u,v) = \mathbb{E}^{[y]}\Bigg[\int_{t_0}^{\tau_S} e^{-\varepsilon r}[\alpha_1 X_1(r)dr-\beta_1 X_2(r)] - \sum_{j\geq 1} c_1(\tau_j,\xi_j)\cdot 1_{\{\tau_j\leq\tau_S\}} \\ +\gamma_1 e^{-\varepsilon\tau_S}[X_1(\tau_S)]^2[X_2(\tau_S)]^2\Bigg],\end{aligned} \tag{4.104}$$

$$\begin{aligned}\Pi_2(y;u,v) = \mathbb{E}^{[y]}\Bigg[\int_{t_0}^{\tau_S} e^{-\varepsilon r}[\alpha_2 X_2(r)-\beta_2 X_1(r)]dr - \sum_{m\geq 1} c_2(\rho_m,\eta_m)\cdot 1_{\{\rho_m\leq\tau_S\}} \\ +\gamma_2 e^{-\varepsilon\tau_S}[X_1(\tau_S)]^2[X_2(\tau_S)]^2\Bigg],\end{aligned} \tag{4.105}$$

where $x_i := X_i(t_0)\in\mathbb{R}_{>0}$.

Given the setup of Theorem 4.12, at time $s\in\mathbb{R}_{>0}$ the Firm i running cost f_i is now given by: $f_i(Y(s)) = e^{-\varepsilon s}(\alpha_i X_i(s)-\beta_i X_j(s))$; $i,j\in\{1,2\}$, the Firm i intervention costs are given by: $c_i(\tau,\xi) = \lambda_i\xi+\kappa_i$ for some intervention time $\tau\in\mathscr{T}$ and intervention $\xi\in\mathscr{Z}$ and the Firm i terminal reward is given by: $G_i(Y(\tau_S)) = \gamma_i e^{-\varepsilon\tau_S}[X_i(\tau_S)]^2[X_j(\tau_S)]^2$.

W.l.o.g. we shall focus on the case for Firm 1, the arguments for Firm 2 being analogous. We can now apply the conditions of Theorem 4.12 to show that the value function is a solution to the following Stefan problem:

$$\mathscr{L}\phi_i(y)+f_i = 0, \qquad \forall(x_1,x_2)\in D\equiv D_1\cap D_2,\ i\in\{1,2\}, \tag{4.106}$$

$$\frac{\partial}{\partial z}\phi_1(x_1+z,x_2) = e^{-\varepsilon t}\lambda_1, \qquad \forall(x_1,x_2)\notin D\equiv D_1\cap D_2. \tag{4.107}$$

Indeed (4.106) is immediately observed using (iii') of Theorem 4.12. This implies that:

$$\begin{aligned}0 =&\alpha_1 e^{-\varepsilon t}x_1-\beta_1 e^{-\varepsilon t}x_2+\frac{\partial\phi_1}{\partial t}+\sum_{j=1}^{2}\mu_j\frac{\partial\phi_1}{\partial x_j}+\frac{1}{2}\sum_{i,j=1}^{2}\sigma_{ij}^2\frac{\partial^2\psi_1(x_1,x_2)}{\partial x_i\partial x_j} \\ &+\int_{\mathbb{R}}\{\phi_1(s,x_1+\theta_{1j}z,x_2+\theta_{2j}z)-\phi_1(s,x_1,x_2)-\theta_{1j}z\frac{\partial\phi_1}{\partial x_1}-\theta_{2j}z\frac{\partial\phi_1}{\partial x_2}\}\nu_j(dz).\end{aligned} \tag{4.108}$$

We now try a candidate for the function, i.e. we specify the form:

$$\phi_1(y) = e^{-\varepsilon t}\psi_1(x_1,x_2), \tag{4.109}$$

for some $\varepsilon > 0$ and, as of yet, undetermined function $\psi \in \mathscr{C}^{1,2}$.

After plugging (4.109) into (4.108), we find that:

$$\alpha_1 x_1 - \beta_1 x_2 - \varepsilon \psi_1(x_1,x_2) + \sum_{j=1}^{2} \mu_j \frac{\partial \psi_1(x_1,x_2)}{\partial x_j} + \frac{1}{2}\sum_{i,j=1}^{2} \sigma_{ij}^2 \frac{\partial^2 \psi_1(x_1,x_2)}{\partial x_i \partial x_j}$$
$$+ \sum_{j=1}^{2} \int_{\mathbb{R}} \{\psi_1(x_1 + \theta_{1j}z, x_2 + \theta_{2j}z) - \psi_1(x_1,x_2) - z\theta_{1j}\frac{\partial \psi_1(x_1,x_2)}{\partial x_1} - z\theta_{2j}\frac{\partial \psi_1(x_1,x_2)}{\partial x_2}\}\nu_j(dz) = 0. \tag{4.110}$$

Let us now suppose that:

$$\psi_1(x_1,x_2) \equiv \hat{\psi}_1(x_1) + \hat{\psi}_1(x_2). \tag{4.111}$$

Hence using (4.110) we deduce that:

$$\alpha_1 x_1 - \beta_1 x_2 + \sum_{i=1}^{2} \Big\{ -\varepsilon \hat{\psi}_1(x_i) + \mu_i \frac{\partial \hat{\psi}_1(x_i)}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{2} \sigma_{ii}^2 \frac{\partial^2 \hat{\psi}_1(x_i)}{\partial x_i^2}\Big\}$$
$$+ \sum_{i,j=1}^{2} \int_{\mathbb{R}} \{\hat{\psi}_1(x_i + \theta_{ij}z) - \hat{\psi}_1(x_i) - z\theta_{ij}\frac{\partial \hat{\psi}_1(x_i)}{\partial x_i}\}\nu_j(dz) = 0. \tag{4.112}$$

After which we find that $\hat{\psi}_1$ is a solution to

$$h(y) = A_1 e^{r_1 x_1} + A_2 e^{r_2 x_1} + \frac{\alpha_1}{\varepsilon} x_1 + B_1 e^{r_1 x_2} + B_2 e^{r_2 x_2} - \frac{\beta_1}{\varepsilon} x_2 + \frac{1}{2\varepsilon^2}(\mu_1 \alpha_1 - \mu_2 \beta_1), \tag{4.113}$$

where $A_1, A_2, B_1, B_2 \in \mathbb{R}$ are unknown constants and r_1 and r_2 are roots of the equation:

$$q(r_k) := \frac{1}{2}\sigma_{ii}^2 r_k^2 + \mu_i r_k - \varepsilon + \int_{\mathbb{R}} \{e^{r_k \theta_{ij} z} - 1 - \theta_{ij} r_k z\}\nu_j(dz), \quad i,j,k \in \{1,2\}. \tag{4.114}$$

W.l.o.g. let us set $r_1 < r_2$. Now since $\lim_{|r|\to\infty} q(r) = \infty$ $\mathbb{P}-$a.s., and $q(0) = -\varepsilon < 0$ and since $\forall r, z$ we have that: $\{e^{r_k \theta_{ij} z} - 1 - \theta_{ij} r_k z\}\nu_j(dz) > 0$ we find that:

$$|r_1| > r_1, \tag{4.115}$$

and

$$r_1 < 0 < r_2. \tag{4.116}$$

Our ansatz for the continuation region D_1 is that it takes the form:

$$D_1 = \{x_1 > x_1^\star | x_1, x_1^\star \in \mathbb{R}\}. \tag{4.117}$$

We now derive (4.107) and in doing so we shall determine $x_1^\star$. Now for all $x_1 \leq x_1^\star$ we have that:

$$\psi_1(x_1,x_2) = \mathscr{M}_1\psi_1(x_1,x_2) = \sup_{z\in\mathscr{Z}}\{\psi_1(x_1+z,x_2) - (\kappa_1+\lambda_1 z)\}. \tag{4.118}$$

We wish to determine the value z that maximises (4.118), hence let us now define the function G by the following expression:

$$G(\xi) = \psi_1(x_1+\xi,x_2) - (\kappa_1+\lambda_1\xi), \tag{4.119}$$

$\forall \xi \in \mathscr{Z}, x_1, x_2 \in \mathbb{R}$.

We now seek to evaluate the maxima of (4.119), i.e. when $G'(\xi) = 0$. We therefore see that the following expression holds :

$$\psi_1'(x_1+\xi,x_2) = \lambda_1, \qquad \forall x_1,x_2 \in \mathbb{R}, \xi \in \mathscr{Z}. \tag{4.120}$$

Let us now consider a unique point $\hat{x}_1 \in]0,x_1^\star[$ then:

$$\psi_1'(\hat{x}_1,x_2) = \lambda_1. \tag{4.121}$$

Hence, we have that $\hat{x}_1 = x_1 + \hat{\xi}(x_1)$ or $\hat{\xi}(x_1) = \hat{x}_1 - x_1$ from which we deduce that for $x \in]0,x_1^\star[$, we have that:

$$\psi_1(x_1,x_2) = \psi_1(\hat{x}_1,x_2) - \kappa_1 + \lambda_1(x_1 - \hat{x}_1), \tag{4.122}$$

or which by (4.111) may be equivalently expressed as:

$$\psi_1(x_1,x_2) = \hat{\psi}_1(\hat{x}_1) + \hat{\psi}_1(x_2) - \kappa_1 + \lambda_1(x_1 - \hat{x}_1). \tag{4.123}$$

Using (4.120) - (4.121) and (4.123) and inserting (4.113), we can construct the following system of equations:

$$A_1 r_1 e^{r_1 x_1^\star} + A_2 r_2 e^{r_2 x_1^\star} + \frac{\alpha_1}{\varepsilon} = \lambda_1, \tag{4.124}$$

$$A_1 r_1 e^{r_1 \hat{x}_1} + A_2 r_2 e^{r_2 \hat{x}_1} + \frac{\alpha_1}{\varepsilon} = \lambda_1, \tag{4.125}$$

$$A_1(e^{r_1 x_1^\star} - e^{r_1\hat{x}_1}) + A_2(e^{r_2 x_1^\star} - e^{r_2\hat{x}_1}) = -\kappa_1 + \left(\lambda_1 - \frac{\alpha_1}{\varepsilon}\right)(x_1^\star - \hat{x}_1). \tag{4.126}$$

Repeating the above steps for ϕ_2 leads to an analogous set of equations as (4.124)-(4.126) with $(A_1, A_2, x_1^\star, \hat{x}_1, \alpha_1, \lambda_1)$ replaced by $(B_1, B_2, x_2^\star, \hat{x}_2, \alpha_2, \lambda_2)$.

Now, since the system (4.104) - (4.105) is invariant under the transformations $\{1 \leftrightarrow 2\}$ then we must have $A_1 = B_1 (:= C_1)$ and $A_2 = B_2 (:= C_2)$ (since (4.113) must still be a solution to (iv) after the transformation $\{1 \leftrightarrow 2\}$). Hence, we are left with a system of 6 unknowns $(C_1, C_2, x_1^\star, \hat{x}_1, x_2^\star, \hat{x}_2)$ and 6 equations. We can therefore uniquely determine the values $(C_1, C_2, x_1^\star, \hat{x}_1, x_2^\star, \hat{x}_2)$ — this proves

Theorem 4.14. □

Proof of Proposition 4.15 The proof follows immediately after combining equations (4.109) (the value function ansatz) and (4.113) (the general solution of the value function), together with the system of equations (4.124) - (4.126) (and exploiting symmetry in accordance with the above remarks). We therefore find that the value function for Firm i is given by the following:

$$\phi_i(y) = e^{-\varepsilon t}\Big\{C_1(e^{r_1 x_1}+e^{r_1 x_2})+C_2(e^{r_2 x_1}+e^{r_2 x_2})+\frac{\alpha_i}{\varepsilon}x_1-\frac{\beta_i}{\varepsilon}x_2+\frac{1}{2\varepsilon^2}(\mu_i\alpha_i-\mu_j\beta_i)\Big\}, i\neq j, i,j\in\{1,2\},$$

where the constants $(C_1,C_2,x_1^\star,\hat{x}_1,x_2^\star,\hat{x}_2)$ are determined by the solutions to the following system of equations:

$$C_1 r_1 e^{r_1 x_i^\star}+C_2 r_2 e^{r_2 x_i^\star}+\frac{\alpha_i}{\varepsilon} = \lambda_i, \tag{4.127}$$

$$C_1 r_1 e^{r_1 \hat{x}_i}+C_2 r_2 e^{r_2 \hat{x}_i}+\frac{\alpha_i}{\varepsilon} = \lambda_i, \tag{4.128}$$

$$C_1(e^{r_1 x_i^\star}-e^{r_1 \hat{x}_i})+C_2(e^{r_2 x_i^\star}-e^{r_2 \hat{x}_i}) = -\kappa_i+\Big(\lambda_i-\frac{\alpha_i}{\varepsilon}\Big)(x_i^\star-\hat{x}_i) \tag{4.129}$$

$$i\in\{1,2\},$$

and r_i are the roots of the equation

$$q(r_k) := \frac{1}{2}\sigma_{ii}^2 r_k^2+\mu_i r_k-\varepsilon+\int_{\mathbb{R}}\{e^{r_k\theta_{ij}z}-1-\theta_{ij}r_k z\}\nu_j(dz), \tag{4.130}$$

for $i,j,k\in\{1,2\}$. □

The transcendental nature of the system of equations (4.127) - (4.129) means that the solutions to the constants $(C_1,C_2,x_1^\star,\hat{x}_1,x_2^\star,\hat{x}_2)$ cannot be expressed in closed form. Similarly, we cannot find closed solutions for the constants r_1,r_2 fir the integral equation (4.130). Nonetheless, as the following result shows, if we restrict our attention to the case in which the market does not contain exogenous shocks, we can recover closed solutions to the constants r_1,r_2.

4.6.1 The case without jumps ($\theta_{\mathbf{ij}}=\mathbf{0};\mathbf{i},\mathbf{j}\in\{\mathbf{1},\mathbf{2}\}$)

For the case in which the market contains no exogenous shocks using (4.114), we readily observe that the constants r_1,r_2 can be solved analytically. In this case, each value function ϕ_i is given by

$$\phi_i(y) = e^{-\varepsilon t}\left\{\tilde{C}_1(e^{r_1 x_1}+e^{r_1 x_2})+\tilde{C}_2(e^{r_2 x_1}+e^{r_2 x_2})+\frac{\alpha_i}{\varepsilon}x_1-\frac{\beta_i}{\varepsilon}x_2+\frac{1}{2\varepsilon^2}(\mu_i\alpha_i-\mu_j\beta_i)\right\},$$

$$i\neq j, i,j\in\{1,2\},$$

where the constants $(\tilde{C}_1, \tilde{C}_2, x_1^\star, \hat{x}_1, x_2^\star, \hat{x}_2)$ are determined by the solutions to the following system of equations:

$$\tilde{C}_1 r_1 e^{r_1 x_i^\star} + \tilde{C}_2 r_2 e^{r_2 x_i^\star} + \frac{\alpha_i}{\varepsilon} = \lambda_i, \tag{4.131}$$

$$\tilde{C}_1 r_1 e^{r_1 \hat{x}_i} + \tilde{C}_2 r_2 e^{r_2 \hat{x}_i} + \frac{\alpha_i}{\varepsilon} = \lambda_i, \tag{4.132}$$

$$\tilde{C}_1 (e^{r_1 x_i^\star} - e^{r_1 \hat{x}_i}) + \tilde{C}_2 (e^{r_2 x_i^\star} - e^{r_2 \hat{x}_i}) = -\kappa_i + \left(\lambda_i - \frac{\alpha_i}{\varepsilon}\right)(x_i^\star - \hat{x}_i), \tag{4.133}$$

$$i \in \{1,2\},$$

where the constants r_1 and r_2 are given by:

$$r_1 = -\frac{1}{\sigma_{ii}^2}\left(\mu_i + \sqrt{\mu_i^2 + 2\sigma_{ii}^2 \varepsilon}\right), \qquad r_2 = \frac{1}{\sigma_{ii}^2}\left(\sqrt{\mu_i^2 + 2\sigma_{ii}^2 \varepsilon} - \mu_i\right). \tag{4.134}$$

4.7 Chapter Appendix

The results of the chapter are built under the following assumptions: A.3.

Let $\rho, \rho', \tau, \tau' : \Omega \to [0,T]$ be $\mathscr{F}-$measurable stopping times and let $\eta, \eta', \xi, \xi' \in \mathscr{Z}$ be $\mathscr{F}-$measurable impulse interventions such that $t_0 \leq \tau < \tau' \leq T$ and $t_0 \leq \rho < \rho' \leq T$. Then we assume that for some strictly positive function $\Theta(\tau) \in \mathscr{C}([0,T];\mathbb{R}^p)$, the following statements hold:

(i) $\chi(\rho, \eta + \eta') \leq \chi(\rho, \eta) + \chi(\rho, \eta') - \Theta(\tau)$,

(ii) $\chi(\rho, \eta) \geq \chi(\rho', \eta)$.

(iii) $c(\tau, \xi + \xi') \leq c(\tau, \xi) + c(\tau, \xi') - \Theta(\tau)$,

(iv) $c(\tau, \xi) \geq c(\tau', \xi)$.

A.4.

We assume there exist constants $\lambda_c, \lambda_\chi \in \mathbb{R}_{>0}$ such that $\inf_{\xi \in \mathscr{Z}} c(\cdot, \xi) \geq \lambda_c$ and $\inf_{\eta \in \mathscr{Z}} \chi(\cdot, \eta) \geq \lambda_\chi$ where $\xi, \eta \in \mathscr{Z}$ are $\mathscr{F}-$measurable impulse interventions.

Proof of Lemma 4.6 Firstly, let us set $\Theta_i(s) \equiv \lambda_i, i \in \{1,2\}$ and suppose that $\xi(s) \equiv v_1^+(s) - v_1^-(s)$ and $\eta(s) \equiv v_2^+(s) - v_2^-(s)$ for player I and player II controls (resp.) where $s \in [0,T]$ and v_i^+ and v_i^-, $i \in \{1,2\}$ are given by the using expressions:

$$v_1^+(s) = \frac{1}{2}\left[\sum_{j\geq 1}(\xi_j \cdot 1_{\{\xi_j>0\}} + \lambda_1^{-1}\kappa_1 \cdot 1_{\{\tau_j \leq s\}} - \lambda_1^{-1}\kappa_1) \cdot 1_{\{\tau_j \leq s\}}\right], \tag{4.135}$$

$$v_1^-(s) = -\frac{1}{2}\left[\sum_{j\geq 1}(\xi_j \cdot 1_{\{\xi_j<0\}} + \lambda_1^{-1}\kappa_1 \cdot 1_{\{\tau_j \leq s\}} - \lambda_1^{-1}\kappa_1) \cdot 1_{\{\tau_j \leq s\}}\right] \tag{4.136}$$

and similarly for the player II control:

$$v_2^+(s) = \frac{1}{2}\left[\sum_{m\geq 1}(\eta_m \cdot 1_{\{\eta_m>0\}} + \lambda_2^{-1}\kappa_2 \cdot 1_{\{\rho_m \leq s\}} - \lambda_2^{-1}\kappa_2) \cdot 1_{\{\rho_m \leq s\}}\right] \tag{4.137}$$

$$v_2^-(s) = -\frac{1}{2}\left[\sum_{m\geq 1}(\eta_m \cdot 1_{\{\eta_m<0\}} + \lambda_2^{-1}\kappa_2 \cdot 1_{\{\rho_m \leq s\}} - \lambda_2^{-1}\kappa_2) \cdot 1_{\{\rho_m \leq s\}}\right] \tag{4.138}$$

We do the proof for the player II impulse controls, the proof for the player I part is analogous.

Using (4.135) - (4.136) we readily deduce that:

$$\begin{aligned}
d\eta(s) &= dv_2^+(s) - dv_2^-(s) \\
&= \frac{1}{2}\sum_{m\geq 1}(\eta_m \cdot 1_{\{\eta_m>0\}} + 2\lambda_2^{-1}\kappa_2 \cdot 1_{\{\rho_m\leq s\}} - \lambda_2^{-1}\kappa_2)\cdot\delta_{\rho_m}(s) \\
&\quad + \frac{1}{2}\sum_{m\geq 1}(\eta_m \cdot 1_{\{\eta_m<0\}} + 2\lambda_2^{-1}\kappa_2 \cdot 1_{\{\rho_m\leq s\}} - \lambda_2^{-1}\kappa_2)\cdot\delta_{\rho_m}(s) \\
&= \sum_{m\geq 1}(\eta_m + (2\cdot 1_{\{\rho_m\leq s\}} - 1)\lambda_2^{-1}\kappa_2)\cdot\delta_{\rho_m}(s)
\end{aligned}$$

Using the properties of the Dirac-delta function and by Fubini's theorem we find that:

$$\begin{aligned}
\int_{t_0}^{\tau_S}\Theta_2(s)d\eta(s) &= \sum_{m\geq 1}\int_{t_0}^{\tau_S}(\lambda_2\eta_m + (2\cdot 1_{\{\rho_m\leq s\}} - 1)\kappa_2)\cdot\delta_{\rho_m}(s) \\
&= \sum_{m\geq 1}\int_{t_0}^{\tau_S}(\lambda_2\eta_m + (2\cdot 1_{s\in[\rho_m,\infty[} - 1)\kappa_2)\cdot\delta_{\rho_m}(s) \\
&= \sum_{m=1}^{\mu_{[t_0,\tau_S]}(v)}(\lambda_2\eta_m + \kappa_2) \\
&= \sum_{m\geq 1}\chi(\rho_m,\eta_m)\cdot 1_{\{\rho_m\leq\tau_S\}}.
\end{aligned}$$

Lastly, we compute $\eta(s)$, indeed we observe that:

$$\begin{aligned}
\eta(s) &= v_2^+(s) + v_2^-(s) \\
&= \sum_{m\geq 1}((\eta_m + \lambda_2^{-1}\kappa_2\cdot 1_{\{\rho_m\leq s\}}) - \lambda_2^{-1}\kappa_2)\cdot 1_{\{\rho_m\leq s\}}\cdot 1_{\{\eta_m>0\}} \\
&\quad + \sum_{m\geq 1}((\eta_m + \lambda_2^{-1}\kappa_2\cdot 1_{\{\rho_m\leq s\}}) - \lambda_2^{-1}\kappa_2)\cdot 1_{\{\rho_m\leq s\}}\cdot 1_{\{\eta_m<0\}}.
\end{aligned}$$

Now, since $1_{\{\rho_m\leq s\}}\cdot 1_{\{\rho_m\leq s\}} = 1_{\{\rho_m\leq s\}}$ we find that:

$$\eta(s) = \sum_{m\geq 1}(\eta_m + \lambda_2^{-1}\kappa_2 - \lambda_2^{-1}\kappa_2)\cdot 1_{\{\rho_m\leq s\}} = \sum_{m\geq 1}\eta_m\cdot 1_{\{\rho_m\leq s\}} = \sum_{m=1}^{\mu_{[t_0,s]}(v)}\eta_m.$$

Hence, after repeating the exercise for the player I controls (using that $\xi(s) := v_1^+(s) - v_1^-(s)$) and setting $v(s) = v^+(s) - v^-(s)$ where $v^+(s) \equiv v_1^+(s) + v_2^+(s)$, $v^- \equiv v_1^- + v_2^-$ we recover the impulse control game. □

Proof of Theorem 4.12. We prove the theorem for player I with the proof for player II being analogous.

As in the proof of Theorem 4.9, we begin by adopting the following notation:

$$Y^{y_0,\cdot}(s) \equiv (s+t_0, X^{t_0,x_0,\cdot}(t_0+s)), \quad y_0 \equiv (t_0,x_0),\, \forall s \in [0,T-t_0], \tag{4.139}$$

$$\hat{Y}^{y_0,\cdot}(\tau) = Y^{y_0,\cdot}(\tau^-) + \Delta_N Y^{y_0,\cdot}(\tau), \quad \tau \in \mathscr{T}, \tag{4.140}$$

where $\Delta_N Y(\tau)$ denotes a jump at time τ due to $\tilde{N}$.

Correspondingly, we adopt the following impulse response function $\hat{\Gamma} : \mathscr{T} \times S \times \mathscr{Z} \to \mathscr{T} \times S$ acting on $y' \equiv (\tau, x') \in \mathscr{T} \times S$ where $x' \equiv X^{t_0,x_0,\cdot}(t_0+\tau^-)$ is given by:

$$\hat{\Gamma}(y',\zeta) \equiv (\tau, \Gamma(x',\zeta)) = (\tau, X^{t_0,x_0,\cdot}(\tau)), \quad \forall \xi \in \mathscr{Z},\, \forall \tau \in \mathscr{T}. \tag{4.141}$$

Let us also now fix the player II control $\hat{v} \in \mathscr{V}$; we firstly appeal to Dynkin's formula for jump-diffusion processes hence, we have the following:

$$\mathbb{E}[\phi_1(Y^{y_0,u,\hat{v}}(\tau_j))] - \mathbb{E}[\phi_1(\hat{Y}^{y_0,u,\hat{v}}(\tau_{j+1}-))] = -\mathbb{E}\left[\int_{\tau_j}^{\tau_{j+1}} \frac{\partial \phi_1}{\partial s} + \mathscr{L}\phi(Y^{y_0,u,\hat{v}}(s))ds\right].$$

Summing from $j=0$ to $j=k$ implies that:

$$\phi_1(y_0) + \sum_{j=1}^{k} \mathbb{E}\left[\phi_1(\hat{Y}^{y_0,u,\hat{v}}(\tau_j)) - \phi_1(\hat{Y}^{y_0,u,\hat{v}}(\tau_j^-))\right] - \mathbb{E}\left[\phi_1(\hat{Y}^{y_0,u,\hat{v}}(\tau_{k+1}^-))\right] \tag{4.142}$$

$$= -\mathbb{E}\left[\int_{t_0}^{\tau_{k+1}} \frac{\partial \phi_1}{\partial s} + \mathscr{L}\phi_1(Y^{y_0,u,\hat{v}}(s))ds\right]. \tag{4.143}$$

Now by similar reasoning as in the zero-sum case (c.f. (4.49)), we have that:

$$\mathscr{M}_1\phi_1(\hat{Y}^{y_0,u,\hat{v}}(\tau_j^-)) - \phi_1(\hat{Y}^{y_0,u,\hat{v}}(\tau_j^-)) + c_1(\tau_j,\xi_j)\cdot 1_{\{\tau_j \leq \tau_S\}} \geq \phi_1(Y^{y_0,u,\hat{v}}(\tau_j)) - \phi_1(\hat{Y}^{y_0,u,\hat{v}}(\tau_j^-)). \tag{4.144}$$

Inserting (4.144) into (4.143) implies that:

$$\phi_1(y_0) + \sum_{j=1}^{k} \mathbb{E}[\mathscr{M}_1\phi_1(\hat{Y}^{y_0,u,\hat{v}}(\tau_j^-)) - \phi_1(\hat{Y}^{y_0,u,\hat{v}}(\tau_j^-)) + c_1(\tau_j,\xi_j)\cdot 1_{\{\tau_j \leq \tau_S\}} - \mathbb{E}[\phi_1(\hat{Y}^{y_0,u,\hat{v}}(\tau_{k+1}^-))]$$
$$\geq -\mathbb{E}\left[\int_{t_0}^{\tau_{k+1}} \frac{\partial \phi_1}{\partial s} + \mathscr{L}\phi_1(Y^{y_0,u,\hat{v}}(s))ds\right]. \tag{4.145}$$

Additionally, by (i') we have that:

$$\begin{aligned}
&\frac{\partial \phi}{\partial s} + \mathscr{L}\phi_1(Y^{y_0,u_{[t_0,s]},\hat{v}_{[t_0,s]}}(s)) \\
&\geq \frac{\partial \phi}{\partial s} + \mathscr{L}\phi_1(Y^{\hat{u}_{[t_0,s]},\hat{v}_{[t_0,s]}}(s)) + f_1(Y^{y_0,\hat{u}_{[t_0,s]},\hat{v}_{[t_0,s]}}(s)) - f_1(Y^{y_0,u_{[t_0,s]},\hat{v}_{[t_0,s]}}(s))) \\
&\geq -f_1(Y^{y_0,u_{[t_0,s]},\hat{v}_{[t_0,s]}}(s)).
\end{aligned}$$

Or

$$-\left(\frac{\partial \phi}{\partial s}+\mathscr{L}\phi_1(Y^{y_0,u_{[t_0,s]},\hat{v}_{[t_0,s]}}(s))\right)\leq f_1(Y^{y_0,u_{[t_0,s]},\hat{v}_{[t_0,s]}}(s)). \tag{4.146}$$

Hence, inserting (4.146) into (4.145) yields:

$$\begin{aligned}&\phi_1(y_0)+\sum_{j=1}^{k}\mathbb{E}[\mathscr{M}_1\phi_1(\hat{Y}^{y_0,u,\hat{v}}(\tau_j^-))-\phi_1(\hat{Y}^{y_0,u,\hat{v}}(\tau_j^-))+c_1(\tau_j,\xi_j)\cdot 1_{\{\tau_j\leq\tau_S\}}]-\mathbb{E}[\phi_1(\hat{Y}^{y_0,u,\hat{v}}(\tau_{k+1}^-))]\\ &\geq\mathbb{E}\left[\int_{t_0}^{\tau_{k+1}}f_1(Y^{y_0,u_{[t_0,s]},\hat{v}_{[t_0,s]}}(s))ds\right].\end{aligned}$$

Now, as in the proof for the zero-sum case, we have, using (ii) that $\lim_{s\to\tau_S^-}[\mathscr{M}_i\phi_i(Y^{y_0,u,v}(s))-\phi_i(\hat{Y}^{y_0,u,v}(s))]=0, i\in\{1,2\}$ and $\lim_{s\to\tau_S^-}\phi(Y^{\cdot,u,v}(s))=G(Y^{\cdot,u,v}(\tau_S))]$ $\forall u\in\mathscr{U}$, $\forall v\in\mathscr{V}$, $\mathbb{P}-$a.s.. Hence, after taking the limit $k\to\infty$, we recognise:

$$\phi_1(y_0)\geq\mathbb{E}\left[\int_{t_0}^{\tau_S}f_1(Y^{y_0,u_{[t_0,s]},\hat{v}_{[t_0,s]}}(s))ds-\sum_{j\geq 1}c_1(\tau_j,\xi_j)\cdot 1_{\{\tau_j\leq\tau_S\}}+G_1(Y^{y_0,u,v}(\tau_S))\cdot 1_{\{\tau_S<\infty\}}\right], \tag{4.147}$$

or

$$\phi_1(y)\geq J_1^{(u,\hat{v})}[y],\quad \forall y\in[0,T]\times S. \tag{4.148}$$

Since this holds for all $u\in\mathscr{U}$ we have

$$\phi_1(y)\geq\sup_{u\in\mathscr{U}}J_1^{(u,\hat{v})}[y],\quad \forall y\in[0,T]\times S. \tag{4.149}$$

Now, applying the above arguments and fixing the pair of controls $(\hat{u},\hat{v})\in\mathscr{U}\times\mathscr{V}$ yields the following equality:

$$\phi_1(y)=\sup_{u\in\mathscr{U}}J_1^{(u,\hat{v})}[y]=J_1^{\hat{u},\hat{v}}[y],\ \forall y\in[0,T]\times S. \tag{4.150}$$

After using an analogous argument for the player II policy $v\in\mathscr{V}$ (fixing the player I control as $\hat{u}$), we deduce that:

$$\phi_2(y)\geq\sup_{u\in\mathscr{U}}J_2^{(u,\hat{v})}[y],\quad \forall y\in[0,T]\times S, \tag{4.151}$$

after which we observe the following statements:

$$\phi_2(y)=\sup_{v\in\mathscr{V}}J_2^{(\hat{u},v)}[y]=J_2^{\hat{u},\hat{v}}[y], \tag{4.152}$$

$$\phi_1(y)=\sup_{u\in\mathscr{U}}J_1^{(u,\hat{v})}[y]=J_1^{\hat{u},\hat{v}}[y], \tag{4.153}$$

from which we deduce that $(\hat{u},\hat{v})$ is a Nash equilibrium and hence the thesis is proven. □

Part II

Strategic Interactions with Impulse Control: Principal-Agent Problems

Chapter 5

Optimal Selection of Transaction Costs in a Dynamic Principal-Agent Problem

In this chapter, we analyse of the effect of modifying transaction costs on agent behaviour and in doing so, for the first time study incentive-distortion theory within an optimal stochastic impulse control model. We consider an agent that maximises their utility criterion by performing a sequence of purchases for which they face transaction costs and, a principal that chooses the transaction cost faced by the agent. This results in a principal-agent model in which the agent uses impulse controls.

The contribution of this chapter is encompassed in the following paper:

David Mguni, *"Optimal Selection of Transaction Costs in a Dynamic principal-agent Problem"*, (2018) [Mgu18d].

Introduction

So far we have tackled problems in which, given a set of self-interested and *rational* players[1] with fixed reward functions, the task is to characterise the outcomes following a strategic interaction.

In this chapter, we reverse the question and ask — under which conditions would a self-interested agent choose to execute a given policy and subsequently produce a desired outcome? In particular, we wish to ascertain a set of *distortionary incentives* that adequately incentivise a self-interested controller to maximise some other external objective.

[1] The term *rationality* describes both the player's self-interested motives and their ability to compute solutions to their problems in a way that is logically consistent with their beliefs and observations and, without cognitive limitations. This in turn produces behaviour in which players execute best-response strategies against the actions of other players.

The problem involves an agent that seeks to maximise its purchases of a divisible good that its cash-flow process can tolerate. For each purchase, the agent incurs at least some fixed minimal cost which is deducted from the agent's cash-flow. Consequently, the agent must make discrete purchases of the good (of possibly of varying size) over the horizon of the problem. The setting involves a principal that is not able to perform any direct purchases but derives its rewards from the agent's sequence of purchases. Although the principal cannot make direct purchases, the principal can influence the agent's behaviour by choosing the value of the fixed minimal costs incurred by the agent for each purchase.

The central question we seek to address is the following: what is the fixed minimal cost the principal should choose in order to induce the principal's desired purchasing pattern?

The aim of this analysis is twofold: the first goal is to perform an analysis of the effect that the magnitude of the transaction costs has on the agent's consumption policy. The second goal is to fully determine the value of the transaction cost that induces an agent policy that is desirable for the principal. In the latter case, the choice of transaction cost serves to condition the agent's preferences so that the timing, magnitude (and total number) of the agent's investment adjustments coincide with the principal's objectives.

We show that with an appropriate choice of transaction cost, the agent's preferences can be sufficiently distorted so that the agent finds it optimal to execute an impulse control policy that maximises the principal's payoff. We perform the analysis with sufficient generality to allow for the principal to be uninformed about the agent's preferences and cash-flow process.

The results of this chapter generalise [KS15] and [DZ00] beyond optimal stopping and entry and exit decisions (respectively) to a scenario that involves distorting the incentives of an agent that performs *many* investments over some time horizon. A component of the analysis performed in this chapter is concerned with studying the effect of varying the transaction cost incurred by an agent on the agent's impulse control policy. This part of the study extends the analysis in [Øks99; ØUZ02; Fra04] which study the limiting behaviour as the intervention costs approach zero to a general analysis of the relationship between marginal changes in the cost function components and the executed policy.

Background

The problem tackled in this chapter lies within a widely-studied class of problems known as principal-agent models [MCWG+95; GH92]. In this setting, a self-interested agent makes decisions or acts in an environment in which another agent namely, the principal cannot affect directly and may not even observe. The goal of the principal is to choose an incentive scheme that affects the agent's decisions and leads to the agent taking actions that maximise the principal's payoff. Principal-agent models therefore undertake a systematic analysis of the incentive schemes that lead to an external objective being maximised by self-interested agents.

Consider firstly the example of a single irreversible investment for a firm that privately observes the demand process. In order to maximise its overall profit, the firm strategically selects a profit-maximising time to enter the market. Secondly, consider the case of a firm that wishes to adjust its production capacity according its observations of market (demand) fluctuations in order to maximise its cumulative profits. For the firm, increasing production capacity involves paying investment costs which include fixed costs with which the increases in production yield additional firm revenue. In this case to maximise overall profits, the firm selects an optimal *sequence* of capital adjustments implemented over the firm's time horizon [BHM93].

In the case of the single irreversible firm investment, it is widely known that the optimal firm strategy is to delay investment beyond the point at which the expected returns of investment becomes positive — from a system perspective, the late entry of investment results in a socially inefficient outcome [DP94]. Similarly, in the multiple production capacity case, the firm's decision process relating to profit maximising production capital levels may also produce socially inefficient outcomes.

In both cases, it is therefore natural to ask whether it is possible for an (uninformed) central planner to sufficiently modify the firm's preferences so that the firm's investment decisions produce socially efficient outcomes. The case of a single irreversible firm investment was analysed in [KS15] in which it was shown that a regulator can induce socially efficient entry decisions through the use of a *posted-price mechanism*.[2]

In particular, in [KS15] it is shown that by performing a transfer of wealth at the point of an agent's entry or exit decision, a central authority or principal who does not observe the state of the world can sufficiently distort an informed agent's preferences in a way that induces a decision by the agent which is desirable for the principal. The framework for this problem is an optimal stopping problem in which the agent's decision to stop the process is altered by way of incentives to coincide with the principal's optimal stopping time.

Presently however, the literature concerning multiple sequential investment analysis has been primarily limited to (single) entrance and (single) exit problems within environments of complete information (see for example [DZ00]). More generally, there is a vast literature on dynamic principal-agent models in which the agent performs actions continuously over the horizon of their problem [AW08; Zhu13] which has been extended to included jump-diffusion dynamics [DLT13]. However, despite the breadth of the literature in both the static and continuous-time domains, the important case of principal-agent models with discrete multiple sequential investments has thus far, not been studied. Additionally, there has been an increasing focus on understanding the effect of transaction costs on the behaviour of self-interested agents. To this end, various experimental investigations have been performed that study the effect of transaction costs [HHKS10; ACY03].

The analysis of the chapter is selected with appeal to conduct a theoretical investigation of

[2] A posted price mechanism presents each agent with a (possibly different) price, thereafter each agent can choose to either accept or reject the mechanism offer [BKS12].

financial environments with transaction costs and in which the optimal choice of transaction cost is unknown. The study of public-private partnerships (e.g. employment initiatives or capital investments within), trading with transaction costs and central authorities that seek to condition the behaviour of players in a given financial environment are some examples [ACBRS07; GJP12].

Contribution

The analysis of this chapter addresses the absence of dynamic principal-agent models with fixed minimal costs. Our main result is to determine the value of the transaction cost that induces the principal's desired consumption policy to be executed by the agent. We also conduct an analysis of the transaction cost parameter and the solution to the agent's optimal control policy.

Integral to the study of the incentive-distortion problem is an analysis of the changes in the optimal behaviour with varying transaction costs. In particular, as part of our study of the principal-agent problem, in this chapter we analyse the effect of changes to the parameter λ on the quantities $(\hat{x}, x^\star)$. We determine the values of the fixed cost parameters λ and κ such that given some desired fixed pair of values $(\hat{x}, x^\star)$ where $\mathscr{D} \equiv \{x \in S : x < x^\star\}$ and $\hat{z} = \hat{x} - x^\star$. That is, we address the question of how to induce a particular impulse control policy through a choice of the transaction costs. These results augment the studies presented in [Øks99; ØUZ02; Fra04] which analyse the behaviour of impulse controls systems in the limiting cases when the fixed minimal cost goes to 0.

The results presented in this chapter generalise those in [DZ00] beyond single entrance and single exit problems to the case of multiple sequential investments. Similarly, our results are related to the analysis in [KS15] in which a transfer rule for a principal-agent problem that involves incentive-distortion in an optimal stopping problem is derived. Impulse controls generalise optimal stopping problems to instances in which the controller now affects the state process at a sequence of intervention times rather than affecting the process only once. The results of this chapter therefore generalise the results in [KS15] to now cover incentive-distortions in settings with multiple agent interventions.

We note also that the method in [KS15] requires a construction of a *reflected state process* — a version of the underlying system process which is constrained to remain below some fixed value. Our approach works by appealing to arguments similar to those presented in Chapter 2 and Chapter 4 thus avoiding the need to construct a reflected process. In particular, the methodology inverts the verification theorem (specifically, Corollary 2.8.1 in Chapter 2) to characterise the conditions under which a given control policy will be executed by a rational agent.

The analysis in this chapter leads to a solution of the following *inverse impulse control problem*: let X be a one-dimensional diffusion and denote by $x_0 \in S$ and $t_0 \in [0, T]$ parameters that represent the initial point and start time of the process respectively and fix $X_s^{t_0, x_0, \cdot} \equiv x_0$ for all $s \leq t_0$.

The agent's impulse control problem is specified by the following objective which the agent seeks to maximise by a choice of the control $u \in \mathscr{U}$:

$$J[t_0, x_0; u] = \mathbb{E}\left[\int_{t_0}^{\tau_S} h(s, X_s^{t_0,x_0,u}) + \sum_{j\geq 1} c(\tau_j, z_j)\cdot 1_{\{\tau_j \leq \tau_S\}} + \phi(X_{\tau_S}^{t_0,x_0,u})\cdot 1_{\{\tau_S<\infty\}}\right], \tag{5.1}$$

where $\tau_S : \Omega \to [0,T]$ is some random exit time and where the control policy takes the form $u(s) = \sum_{j\geq 1} z_j \cdot 1_{\{\tau_j \leq T\}}(s) \in \mathscr{U}$ for any $s \in [0,T]$. The quantities $z_1, z_2, \ldots \in \mathscr{Z}$ and $\tau_1, \tau_2, \ldots \in \mathscr{T}$ are $\mathscr{F}-$ measurable intervention times and $\mathscr{F}-$ measurable stopping times where $\mathscr{Z}$ is some admissible set of interventions and $\mathscr{U}$ is an admissible control set. The functions $h : [0,T] \times S \to \mathbb{R}$ and $\phi : S \to \mathbb{R}$ are the running cost and the terminal payoff functions (resp.) where $S \subset \mathbb{R}^q$ is a given fixed domain (solvency region) for some $q \in \mathbb{N}$ and $c : \mathscr{T} \times \mathscr{Z} \to \mathbb{R}$ is an intervention cost function.

Let $\mathscr{D} := \{x \in S : x < x^\star\}$ be a given continuation region, that is, a region in which the agent finds it suboptimal to execute an intervention of any size and suppose there exists an optimal intervention magnitude $\hat{z}$ that is given by $\hat{z} = \hat{x} - x^\star$ for some real-valued constant $\hat{x}$. Lastly, denote by $\lambda \in \mathbb{R}_{>0}$ and $\kappa \in \mathbb{R}_{>0}$ the parameters that represent the *proportional cost* and *fixed cost* parts respectively so that an impulse execution of magnitude $z \in \mathscr{Z}$ incurs a cost $(1+\lambda)z + \kappa$. The inverse impulse control problem is to determine the value of κ and λ that induces a given fixed pair $(\hat{x}, x^\star)$ given the objective function J in (5.1).

The analysis of the inverse impulse control problem extends to the multiple intervention cases the study performed in [KS18] in which an inverse optimal stopping problem is investigated. Our last result shows the solutions to two distinct optimal impulse control problems can be made to be identical after a transformation that acts purely on the intervention cost function.

A summary of the contributions of this chapter is as follows:

- We perform an in-depth investigation of a dynamic principal-agent problem in which the agent performs a sequence of purchases each of which incur costs that are minimally bounded. This framework and the corresponding results generalise [KS15; DZ00] beyond optimal stopping and entry and exit decisions (respectively) to a problem in which the agent performs a sequence of purchases.

- Firstly, we analyse the dynamic-principal agent problem and determine the precise characterisation of the cost function that the principal is required to impose on the agent for the agent to find it optimal to execute the principal's desired policy (Theorem 5.3).

- Second, we develop the study of the principal-agent problem to tackle an inverse optimal impulse control problem. Here we show how the optimal policies of two distinct optimal impulse control problems can be made identical by an alteration of the control cost (Lemma 5.6).

- Last, we perform an investigation into the relationship between the policy of an rational agent with minimally bounded adjustment costs and the size of the cost function components. The analysis establishes an analytic relationship between marginal changes in the agent's cost function components and both the agent's policy and the values of the state process the agent chooses to perform an impulse intervention (Proposition 5.4 and Proposition 5.5). This extends the limiting behaviour analyses in [Øks99; ØUZ02; Fra04].

Organisation

First, we give a description of the problem and outline some of the underlying concepts required for the incentive-distortion analyses. Next, in Section 5.3, we give a statement of the main results of the chapter and give a descriptive overview of the mathematical statements. In Section 5.4 we perform the main analysis of the chapter and derive the key results.

5.1 Leading Example: Consumption with Transaction Costs

Consider an agent that observes its liquidity process (cash-flow) which is subject to exogenous shocks and a principal that does not observe the process. The agent makes costly purchases and seeks to maximise their consumption over some given time horizon before the point at which the liquidity process hits 0 (bankruptcy). Each purchase incurs at least some fixed minimal cost or *transaction cost* which is drawn from the agent's cash-flow.

We assume that the market consists of one infinitely divisible good that the agent is able to purchase and consume. The principal and agent have misaligned payoffs, the principal however is given the choice of the transaction costs paid by the agent. The principal therefore aims to choose a fixed value of the transaction cost so as to modify the agent's consumption pattern to satisfy some given objective.

A formal description of the problem is as follows: let X be a stochastic process which represents the agent's cash-flow process and denote by $t_0 \in [0,T]$ and $x_0 \in \mathbb{R}$ the start time of the problem and the initial amount of cash held by the agent respectively and denote by T the horizon of the problem which may be infinite. When there are no purchases, the agent's cash-flow process evolves according to the following expression:

$$X_s^{t_0,x_0} = x_0 + \int_{t_0}^{s\wedge\tau_S} \Gamma X_r^{t_0,x_0} dr + \int_{t_0}^{s\wedge\tau_S} \sigma X_r^{t_0,x_0} dB_r + \int_{t_0}^{s\wedge\tau_S} \int X_{r-}^{t_0,x_0} \gamma(r,z)\tilde{N}(dr,dz), \ \mathbb{P}-\text{a.s} \qquad (5.2)$$

$$X_{t_0}^{t_0,x_0} := x_0,$$

where without loss of generality we assume that $X_s^{t_0,x_0,\cdot} = x_0$ for any $s \le t_0$. The random variable $\tau_S : \Omega \to [0,T]$ is a random exit time or *bankruptcy time* which is defined by $\tau_S(\omega) := \inf\{s \in \mathbb{R}_{>0} | X_s^{t_0,x_0,\cdot} \le 0\}$ so that τ_S is the time at which the agent's cash-flow process first hits 0. The parameter $\Gamma := r_0 + \alpha$ consists of $r_0 \in \mathbb{R}_{>0}$ which is the interest rate and $\alpha \in \mathbb{R}$ which is some constant. The constant $\sigma \in \mathbb{R}$ is the diffusion coefficient and $S \subset \mathbb{R}$ is the state space. The term

$B(r)$ is a $1-$dimensional standard Brownian motion and $\tilde{N}(ds,dz)$ is a compensated Poisson random measure. We assume that N and B are independent.

At any time, the agent may make a purchase which incurs some fixed minimal cost. The inclusion of a transaction cost precludes agent control policies for which the agent makes purchases continuously, hence the agent makes purchases over a sequence of times over the horizon of the problem. The sizes of the purchases are $\{z_k\}_{k\in\mathbb{Z}}$ and the sequence of times of the agent's purchases is given by $\{\tau_k\}_{k\in\mathbb{N}}$ — an increasing sequence of ($\mathscr{F}_{\tau_k}-$measurable) discretionary stopping times so that the agent's control policy is given by the double sequence $(\tau,Z)\equiv\sum_{j\in\mathbb{N}} z_j\cdot 1_{\{\tau_j\leq T\}}\in\mathscr{U}$ where $\mathscr{Z}\subset\mathbb{R}$ is the set of feasible agent purchases. The agent's cash-flow process is affected sequentially at the points of purchases performed by the agent and is described by a stochastic process that obeys the following expression $\forall(t_0,x_0)\in[0,T]\times S$, $\forall s\in[0,T]$, $\forall(\tau,Z)\in\mathscr{U}$:

$$\begin{aligned}X_s^{t_0,x_0,(\tau,Z)}=x_0+\int_{t_0}^{s\wedge\tau_S}\Gamma X_r^{t_0,x_0,(\tau,Z)}dr-\sum_{j\geq1}((1+\lambda)z_j+\kappa)\cdot 1_{\{\tau_j\leq T\}}+\int_{t_0}^{s\wedge\tau_S}\sigma X_r^{t_0,x_0,(\tau,Z)}dB_r\\+\int_{t_0}^{s\wedge\tau_S}\int X_{r-}^{t_0,x_0,(\tau,Z)}\gamma(r,z)\tilde{N}(dr,dz),\ X_{t_0}^{t_0,x_0,\cdot}:=x_0,\ \mathbb{P}-\text{a.s.}\end{aligned}\tag{5.3}$$

where $\kappa,\lambda\in\mathbb{R}_{>0}$ are fixed constants which we shall refer to as the *fixed part* of the transaction cost and *proportional part* of the transaction cost respectively whose pair we denote by $\mathfrak{c}:=(\kappa,\lambda)$.

Agent Payoff Function

The agent's goal is to maximise its purchases which its wealth process can tolerate. Given a cash-flow process given by (5.3), the agent's payoff function Π is given by the following expression $\forall(t_0,x_0)\in[0,T]\times S,\forall(\tau,Z)\in\mathscr{U}$:

$$\Pi^{(\mathfrak{c},(\tau,Z))}[t_0,x_0]=\mathbb{E}\left[\int_{t_0}^{\tau_S}e^{-\delta r}R(X_r^{t_0,x_0,(\tau,Z)})dr+\sum_{j\geq1}e^{-\delta\tau_j}c(\tau_j^-,z_j)\cdot 1_{\{\tau_j\leq\tau_S\}}\right],\tag{5.4}$$

where $R:S\to\mathbb{R}$ is some utility function (we shall later specialise to the case in which R is a power utility function) and $\delta\in]0,1]$ is the agent's discount factor. The function c is given by $c(\cdot,z_j)=z_j$ which quantifies the reward endowed to the agent after each purchase. The agent's problem is to find a sequence of selected purchases i.e. an impulse control policy that alters its cash-flow process in such a way that maximises its payoff.

In this setting, a principal chooses a transaction cost which consists of a fixed cost $\kappa\in\mathbb{R}_{>0}$ and a marginal cost parameter $\lambda\in\mathbb{R}_{>0}$ which is proportional to the size of the agent's purchase both of which are incurred by the agent at the point of each purchase. The principal has a payoff function $Q^{(\tau,Z)}$ which is composed of a running gain function $W:[0,T]\times S\to\mathbb{R}$ and a purchase gain function $c_P:[0,T]\times\mathscr{Z}\to\mathbb{R}$.

Principal Payoff Function

The principal's payoff depends on the actions of the agent. Moreover, since the agent decides when to make a purchase, the principal's payoff is dependent on the agent's decisions. Let $(\tau, Z) \equiv [\tau_j, z_j]_{j\in\mathbb{N}} \in \mathscr{U}$ be the agent's policy, we suppose that the principal's payoff function is given by the following expression $\forall (t_0, x_0) \in [0,T] \times S$, $\forall (\tau, Z) \in \mathscr{U}$:

$$Q^{(\tau,Z)}[t_0,x_0] = \mathbb{E}\left[\int_{t_0}^{\tau_S} W(r, X_r^{t_0,x_0,(\tau,Z)})dr + \sum_{j\geq 1} e^{-\delta_P \tau_j} c_P(\tau_j^-, z_j) \cdot 1_{\{\tau_j \leq \tau_S\}}\right], \tag{5.5}$$

where $W : [0,T] \times S \to \mathbb{R}$ is the principal's running reward function, the function $c_P : [0,T] \times \mathscr{Z} \to \mathbb{R}$ quantifies the reward endowed to the principal after each agent purchase and lastly the constant $\delta_p \in]0,1]$ is the principal's discount factor. We assume that the principal purchase gain function c_P is given by $c_P(\tau_j, z_j) = \lambda_P z_j + \bar{c}_P \tau_j + \alpha_P$ where $\lambda_P, \bar{c}_P, \alpha_P \in \mathbb{R}_{>0}$ are constants.

The problem faced by the principal is to determine the parameters $(\lambda, \kappa) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ that induce agent purchases at the times and by the magnitudes that the principal would like (i.e. that coincide with the policy that maximises (5.5)), given that the agent seeks to maximise its own objective function (5.4).

In this chapter, we study the effect of the fixed cost parameters (λ, κ) associated to the agent's control costs on the agent's consumption pattern. A central aim of this analysis is determine a pair $\mathfrak{c}^\star := (\kappa^\star, \lambda^\star)$ that maximises the principal's objective and the conditions under which a desirable agent control policy is induced — that is, determining the transaction cost that leads to the agent finding it optimal to exercise a control that maximises the principal's payoff (5.5). The result of this chapter are built under assumptions A.1.1 - A.4 (see Appendix).

5.2 Preliminaries

Definition 5.1

The agent and principal have value functions v_A and v_P that are respectively given by the following expressions:

$$v_A(y_0) = \sup_{u\in\mathscr{U}} \Pi^{(\mathfrak{c},u)}[y_0], \qquad v_P(y_0) = \sup_{u\in\mathscr{U}} Q^{(u)}[y_0],\ \forall y_0 \in [0,T] \times S. \tag{5.6}$$

Where it will not cause confusion, we write $v_A(y_0) \equiv v(y_0)$ for any $y_0 \in [0,T] \times S$. With reference to the principal's problem (5.6), we can express the principal's problem as the following: find $\mathfrak{c}^\star \in \mathbb{R}^2_{>0}$ such that

$$\Pi^{(\mathfrak{c}^\star,u^\star)}[y_0] = v_A(y_0), \qquad Q^{(u^\star)}[y_0] = v_P(y_0), \tag{5.7}$$

We now give a definition which is central to the problem:

Definition 5.2 (Implementability)

We say that $\mathfrak{c}$ implements an impulse control policy $u^\star = [\tau_j^\star, z_j^\star]_{j\geq 1} \in \mathscr{U}$ if the following condition is satisfied:

$$\Pi^{(\mathfrak{c},u^\star)}[y_0] \geq \Pi^{(\mathfrak{c},u')}[y_0], \qquad \forall y_0 \in [0,T]\times S,\ \forall u' \in \mathscr{U}, \tag{5.8}$$

The implementability condition asserts the optimality of the policy $u^\star \in \mathscr{U}$ for the agent, given the transaction cost parameters $\mathfrak{c}$. Therefore, to analyse the principal's problem it suffices to characterise $\mathfrak{c}^\star$ and the conditions on the principal's policy for which the agent always finds it optimal to enact the prefixed impulse control policy $u^\star \in \mathscr{U}$ (so that the inequality in (5.8) is satisfied).

5.3 Main Results

We now present the main results of the chapter; we postpone the proofs until the following section.

The following results characterise the implementability conditions that ensure that the agent behaves in such a way that maximises the principal's objective. In particular, the following result characterises the transaction cost parameters that ensure that the agent's optimal impulse control policy and the policy that maximises the principal's objective coincide.

Theorem 5.3

Let $x^\star \in S$ be the principal's target for the agent's consumption threshold so that whenever the agent's cash flow is less than $x^\star$ no purchases are made by the agent. Define $\hat{x} = \hat{z} + x^\star$ where $\hat{z} \in \mathscr{Z}$ is the fixed optimal purchase magnitude. Then the agent adopts the principal's target for the pair $(\hat{x}, x^\star)$ whenever the transaction cost parameter pair $\mathfrak{c}$ is set to the following:

$$\begin{aligned}
\lambda^\star(\hat{x},x^\star) &= \left(\frac{z}{b}\right)\frac{l_2^{-1}z^{-l_2} - l_1^{-1}z^{-l_1}}{l_1^{-1}z^{-l_1} + l_2^{-1}z^{-l_2}} - 1 \\
\kappa^\star(\hat{x},x^\star) &= z\left[l_1^{-1} + l_2^{-1} - 1\right] - z\frac{l_1^{-1}z^{-l_1} - l_2^{-1}z^{-l_2}}{l_1^{-1}z^{-l_1} + l_2^{-1}z^{-l_2}}\left[l_1^{-1} - l_2^{-1} + \ln\hat{x} - \ln x^\star\right],
\end{aligned} \tag{5.9}$$

where $b := \varepsilon\delta^{-1}$, $z^m := \hat{x}^m - x^{\star m}$ and where $\varepsilon \in \mathbb{R}/\{0\}$ is a constant that parameterises the agent's risk aversion for the CRRA utility function (c.f. (5.25)) and δ is the agent's discount factor.

When the agent's liquidity process contains no jumps ($\gamma(z) \equiv 0$ in (5.2)), the parameters l_1 and l_2 in (5.12) can be expressed exactly in closed form by:

$$l_1 = \frac{-1}{\sigma^2}\left(\sqrt{c^2\varepsilon^{-2} + 2\sigma^2\delta} + c\delta b^{-1}\right), \quad l_2 = \frac{1}{\sigma^2}\left(\sqrt{c^2\varepsilon^{-2} + 2\sigma^2\delta} - c\delta b^{-1}\right), \tag{5.10}$$

where $c := \varepsilon\left(\Gamma - \frac{1}{2}\sigma^2\right)$.

For the general case ($\gamma(z) \not\equiv 0$ in (5.2)), the constants l_1 and l_2 are solutions to the equation:

$$h(l) = 0 \tag{5.11}$$

where the function h is defined by $h(l) := \frac{1}{2}\sigma^2 l(l-1) + l\Gamma - \delta + \int_{\mathbb{R}}\left\{(1+\gamma(z))^l - 1 - l\gamma(z)\right\}\nu(dz)$.

If the proportional part λ is exogenously fixed, then the value of the fixed part κ for which the agent finds it optimal to adopt the principal's target is given by the following:

$$\kappa^{\star}(\hat{x},x^{\star},\lambda)=z\left[l_1^{-1}+l_2^{-1}-1\right]+b\,(1+\lambda)\left[l_1^{-1}-l_2^{-1}+\ln\hat{x}-\ln x^{\star}\right]. \tag{5.12}$$

Theorem 5.3 says that if the principal imposes a transaction cost with proportional part and fixed part given by (5.9), then the agent's continuation region is given by $D=\{x<x^{\star}|x,x^{\star}\in S\}$ i.e. the agent makes a purchase whenever the agent's cash-flow attains the value $x^{\star}$. Moreover, the agent's purchase times are $\hat{\tau}_{j+1}=\inf\{s>\tau_j;\ x\geq x^{\star}\}\wedge\tau_S$ and the agent's purchases have a size given by $\hat{z}=\hat{x}-x^{\star}$ which are exactly the intervention times and magnitudes that are optimal for the principal.

The first result of the theorem relates to the case when the principal is free to choose the value of the proportional part of the transaction cost parameter λ *and* the fixed part of the transaction cost parameter κ. The second result relates to the case when the principal is free to choose the value of the fixed part of the transaction cost parameter κ but the proportional cost parameter λ is exogenous and fixed. Theorem 5.3 characterises implementability conditions under which the principal can sufficiently distort the agent's incentives so that the agent plays actions that maximise the principal's objective.

We now turn to describing the change to the agent's behaviour following changes to the transaction cost parameters. In particular, the following results characterise the changes in the agent's policy following a change in the agent's transaction costs. The first result follows from Theorem 5.3:

Proposition 5.4

Let the values l_1 and l_2 be as in Theorem 5.3 and suppose the initial fixed and proportional costs are given by $\kappa_0\in\mathbb{R}_{>0}$ and $\lambda_0\in\mathbb{R}_{>0}$ respectively. Suppose now that the fixed and proportional costs undergo the transformations $\kappa_0\to\kappa_1$ and $\lambda_0\to\lambda_1$, then the agent's intervention threshold and consumption magnitude attain the values $x_1^{\star}=x_0^{\star}+h^{\star}$ and $\hat{x}_1=\hat{x}_0+\hat{h}$ (respectively) whenever the values λ_1 and κ_1 are given by the following expressions:

$$\lambda_1^{\star}(\kappa_0,\lambda_0,\hat{h},h^{\star})=\left(\frac{\tilde{z}}{b}\right)\frac{l_2^{-1}\tilde{z}^{-l_2}-l_1^{-1}\tilde{z}^{-l_1}}{l_1^{-1}\tilde{z}^{-l_1}+l_2^{-1}\tilde{z}^{-l_2}}-1 \tag{5.13}$$

$$\begin{aligned}\kappa_1^{\star}(\kappa_0,\lambda_0,\hat{h},h^{\star})&=\tilde{z}\left[l_1^{-1}+l_2^{-1}-1\right]\\&\quad-\tilde{z}\frac{l_1^{-1}\tilde{z}^{-l_1}-l_2^{-1}\tilde{z}^{-l_2}}{l_1^{-1}\tilde{z}^{-l_1}+l_2^{-1}\tilde{z}^{-l_2}}\left[l_1^{-1}-l_2^{-1}+\ln(\hat{m}+\hat{h})-\ln(m^{\star}+h^{\star})\right],\end{aligned} \tag{5.14}$$

where $\tilde{z}^k:=(\hat{m}+\hat{h})^k-(m^{\star}+h^{\star})^k$ where $\hat{m}(\kappa_0,\lambda_0)$ and $m^{\star}(\kappa_0,\lambda_0)$ are the solutions to the equations:

$$\mathbf{Q}(\hat{m},m^{\star},\kappa_0,\lambda_0)=\begin{bmatrix}Q_1(\hat{m},m^{\star},\kappa_0,\lambda_0)\\Q_2(\hat{m},m^{\star},\kappa_0,\lambda_0)\end{bmatrix}=0 \tag{5.15}$$

where Q_1 and Q_2 are given by:

$$Q_1(x,y,q,k) := \left(l_1 x^{l_1} + l_2 x^{l_2}\right)(y - x - q + b(1+k)[\ln x - \ln y]) - (x - b(1+k))(y^{l_1} - x^{l_1} + y^{l_2} - x^{l_2}),$$
$$Q_2(x,y,q,k) := \left(l_1 y^{l_1} + l_2 y^{l_2}\right)(y - x - q + b(1+k)[\ln x - \ln y]) - (y - b(1+k))(y^{l_1} - x^{l_1} + y^{l_2} - x^{l_2}).$$

and where $b := \varepsilon\delta^{-1}$.

Proposition 5.4 says that given an initial fixed and proportional cost for the agent, κ_0 and λ_0 respectively, a shift of size $h^\star$ and $\hat{h}$ in the agent intervention threshold and consumption magnitudes (respectively) can be induced whenever the fixed and proportional costs are set to $\kappa_1^\star$ and $\lambda_1^\star$. The result extends existing analyses that study the induced changes on a controller's behaviour in limiting cases as the fixed part of the transaction cost approaches 0 [Øks99; Fra04]. The usefulness of the result stems from considering circumstances in which a policy-maker or central authority seeks to examine the effect of changes to its transaction costs on self-interested actors. Indeed, Proposition 5.4 provides an exact description of the change in behaviour by a self-interested agent following changes to the transaction costs.

Here, interestingly the initial agent intervention threshold $x_0^\star$ and initial consumption magnitude $\hat{x}_0$ do not feature in any of the equations that determine the values κ_1 and λ_1, hence the only required data are the shift targets $(h^\star, \hat{h})$ and the initial cost parameters (κ_0, λ_0). This is useful for the case in which the principal does not observe the agent's current consumption threshold and magnitude but seeks to induce a change in those quantities by some given magnitudes.

Proposition 5.4 tackles instances in which the transaction cost undergoes a transformation. This allows us to compare the agent's behaviour following a switch in transaction cost. The following result analyses the change in the agent's behaviour following (continuous) changes in the transaction costs:

Proposition 5.5

The marginal rates of change in $\hat{x}(\kappa,\lambda)$ and $x^\star(\kappa,\lambda)$ w.r.t. λ and κ are given by the following expressions:

$$\begin{aligned} \frac{\partial \hat{x}}{\partial \lambda} &= [f_1(\hat{x}, x^\star)]^{-1}, \\ \frac{\partial x^\star}{\partial \lambda} &= [f_2(\hat{x}, x^\star)]^{-1}, \\ \frac{\partial \hat{x}}{\partial \kappa} &= [f_3(\hat{x}, x^\star)]^{-1}, \\ \frac{\partial x^\star}{\partial \kappa} &= [f_4(\hat{x}, x^\star)]^{-1}. \end{aligned} \tag{5.16}$$

where the parameters l_1 and l_2 are solutions to the equation (5.11) and the functions f_1, f_2, f_3 and f_4 are given by (5.59) - (5.62).

Proposition 5.5 therefore evaluates the change in the intervention threshold and consumption magnitudes due to a marginal change in the cost parameters λ and κ. The result extends the results in [Øks99; ØUZ02; Fra04] which describe the asymptotic behaviour of the controller's value function in the limit as $\kappa \downarrow 0$.

The following result follows from Theorem 5.3 and relates two general stochastic impulse control problems:

Lemma 5.6

Let X be a stochastic process that evolves according to (5.2).

Consider the following pair of impulse control problems:

(i) Find $u_1^\star = [\tau_{1_j}^\star, z_{1_j}^\star]_{j\in\mathbb{N}} \in \mathscr{U}$ and $\phi_1 \in \mathscr{H}$ such that

$$\phi_1(t_0,x_0) = J_1^{(u_1^\star)}[t_0,x_0] = \sup_{u_1\in\mathscr{U}} J_1^{(u_1)}[t_0,x_0], \quad \forall (t_0,x_0) \in [0,T]\times S.$$

(ii) Find $u_2^\star = [\tau_{2_j}^\star, z_{2_j}^\star]_{j\in\mathbb{N}} \in \mathscr{U}$ and $\phi_2 \in \mathscr{H}$ such that

$$\phi_2(t_0,x_0) = J_2^{(u_2^\star)}[t_0,x_0] = \sup_{u_2\in\mathscr{U}} J_2^{(u_2)}[t_0,x_0], \quad \forall (t_0,x_0) \in [0,T]\times S,$$

where the objective functions for problem (i) and (ii) are given by the following expressions:

$$J_1^{(u_1)}[t_0,x_0] = \mathbb{E}^{[x]}\left[\int_{t_0}^{\tau_S} \alpha e^{-\delta s}\ln(X_s^{t_0,x_0,u_1})ds + \sum_{j\geq 1}(\lambda_1 z_j + \kappa_1)\cdot 1_{\{\tau_{1_j}\leq \tau_S\}} + \Psi_1(X_{\tau_S}^{t_0,x_0,u_1})\cdot 1_{\{\tau_S<\infty\}}\right], \tag{5.17}$$

$$J_2^{(u_2)}[t_0,x_0] = \mathbb{E}^{[x]}\left[\int_{t_0}^{\tau_S} F(s,X_s^{t_0,x_0,u_2})ds + \sum_{j\geq 1} l_2(X_{\tau_{2_j}-}^{t_0,x_0,u_2}, z_j)\cdot 1_{\{\tau_{2_j}\leq \tau_S\}} + \Psi_2(X_{\tau_S}^{t_0,x_0,u_2})\cdot 1_{\{\tau_S<\infty\}}]\right], \tag{5.18}$$

where $\alpha \in \mathbb{R}$ and F, l_2, Ψ_1, Ψ_2 are bounded Lipschitz continuous functions. Suppose also that the controlled process (with interventions) evolves according to (5.3). Then if $u_2^\star \in \operatorname{argsup}_{u_2\in\mathscr{U}} J_2^{(u_2)}(t_0,x_0)$, then $u_1^\star = u_2^\star$ whenever:

$$\lambda_1^\star = \left(\frac{z_2}{b}\right)\frac{l_2^{-1}z_2^{-l_2} - l_1^{-1}z_2^{-l_1}}{l_1^{-1}z_2^{-l_1} + l_2^{-1}z_2^{-l_2}} - 1 \tag{5.19}$$

$$\kappa_1^\star = z\left[l_1^{-1} + l_2^{-1} - 1\right] - z_2\frac{l_1^{-1}z_2^{-l_1} - l_2^{-1}z_2^{-l_2}}{l_1^{-1}z_2^{-l_1} + l_2^{-1}z_2^{-l_2}}\left[l_1^{-1} - l_2^{-1} + \ln\hat{x}_2 - \ln x_2^\star\right], \tag{5.20}$$

where $\hat{x}_2 = x_2^\star - z_2^\star$ and $z_2^m := \hat{x}_2^m - x_2^{\star m}$ and the constants l_1 and l_2 are solutions to the equation

$m(l) = 0$ where m is defined by:

$$m(l) := \frac{1}{2}\sigma^2 l(l-1) + l\Gamma - \delta + \int_{\mathbb{R}} \Big\{(1+\gamma(z))^l - 1 - l\gamma(z)\Big\}\nu(dz). \tag{5.21}$$

The parameter $x_2^\star \in S$ is the parabolic boundary of the continuation region for problem (ii), that is to say, given some continuation region for the problem (ii), D_2, each $x_2^\star$ is of the form $x_2^\star = \{x \in S : x \in \partial D_2\}$ and $z^\star := \text{argsup}_{z\in\mathscr{Z}}\{\phi_2(\tau_k, \Gamma(X(\tau_k-), z)) + l_2(X(\tau_k), z)\}$ quantifies the optimal intervention magnitude for the problem with payoff function J_2.

Lemma 5.6 says that impulse control problem (i) has the same optimal control policy solution as that of problem (ii) whenever the intervention cost function in (5.17) has a proportional cost and fixed cost given by $\lambda_1^\star$ and $\kappa_1^\star$ respectively. The result generalises the inverse optimal stopping problems in [KS18] to the case involving multiple agent interventions.

5.4 Main Analysis

We begin by proving Theorem 5.3 which is demonstrated by showing that given $\mathfrak{c}^\star := (\kappa^\star, \lambda^\star)$ defined in (5.9), it is optimal for the agent to execute the sequence of interventions that maximises the principal's payoff Q. Before deriving the main results, we require some background results. In particular, we require a verification theorem for the single controller optimal stochastic control problem which was reported in Corollary 2.8.1. We recall Corollary 2.8.1 for the reader's convenience:

Corollary 2.8.1 Consider the impulse control problem in which the dynamics under the influence of impulse controls $u = [\tau_j, \xi_j]_{j\geq 1} \in \mathscr{U}$ evolves according to the jump-diffusion process $\forall r \in [0,T]$; $\forall (t_0, x_0) \in [0,T]\times S$:

$$X_r^{t_0,x_0,u} = x_0 + \int_{t_0}^r \mu(s, X_s^{t_0,x_0,u})ds + \int_{t_0}^r \sigma(s, X_s^{t_0,x_0,u})dB_s + \sum_{j\geq 1}\xi_j \cdot 1_{\{\tau_j \leq r\}}(r) \\ + \int_{t_0}^r \int \gamma(X_{s-}^{t_0,x_0,u}, z)\tilde{N}(ds,dz), \ \ \mathbb{P}-\text{a.s.},$$

where $X_s^{t_0,x_0,\cdot} = x_0$ for any $s \leq t_0$ and for which the agent seeks to maximise the following objective function $\forall y_0 \in [0,T]\times S$:

$$J[y_0;u] = \mathbb{E}\left[\int_{t_0}^{\tau_S} f(Y_s^{y_0,u})ds + \sum_{m\geq 1} c(\tau_m, \xi_m)\cdot 1_{\{\tau_m \leq \tau_S\}} + G(Y_{\tau_S}^{y_0,u})1_{\{\tau_S<\infty\}}\right]. \tag{5.22}$$

Suppose that there exists a function $\phi \in \mathscr{C}^{1,2}([0,T],S)\cap\mathscr{C}([0,T],\bar{S})$ that satisfies technical conditions (T1) - (T4) (see Appendix) and the following conditions:

(I) $\phi \leq \mathscr{M}\phi$ on S and define the region D by:
$D = \{x \in S; \phi(\cdot,x) < \mathscr{M}\phi(\cdot,x)\}$ where D is the controller continuation region and where $\mathscr{M}$ is the (non-local) intervention operator defined in (2.14).

(II) $\frac{\partial \phi}{\partial s} + \mathscr{L}\phi(\cdot, X^{\cdot,u}(\cdot)) + f(\cdot, X^{\cdot,u}(\cdot)) \geq 0, \ \forall u \in \mathscr{U}$ on $S \backslash \partial D$.

(III) $\frac{\partial \phi}{\partial s} + \mathscr{L}\phi(\cdot, X^{\cdot,\hat{u}}(\cdot)) + f(\cdot, X^{\cdot,\hat{u}}(\cdot)) = 0$ in D.

(IV) $X^{\cdot,u}(\tau_S) \in \partial S, \mathbb{P}-$a.s. on $\tau_S < \infty$ and $\phi(s, X^{\cdot,u}(s)) \to G(\tau_S, X^{\cdot,u}(\tau_S))$ as $s \to \tau_S^-$ $\mathbb{P}-$a.s.,$\forall u \in \mathscr{U}$.

Put $\hat{\tau}_0 \equiv t_0$ and define $\hat{u} := [\hat{\tau}_j, \hat{\xi}_j]_{j \in \mathbb{N}}$ inductively by:
$\hat{\tau}_{j+1} = \inf\{s > \tau_j; X^{\cdot,\hat{u}_{[t_0,s]}}(s) \notin D\} \wedge \tau_S$, then $\hat{u} \in \mathscr{U}$ is an optimal control for the agent's impulse control problem, that is to say we have:

$$\phi(t_0, x_0) = \inf_{u \in \mathscr{U}} J[t_0, x_0; u] = J[t_0, x_0; \hat{u}]; \qquad \forall (t_0, x_0) \in [0, T] \times S. \tag{5.23}$$

Remark 5.7

Let us denote by $\mathscr{D}$ the region $\mathscr{D} = \{x \in S : v(\cdot, x) < \mathscr{M}v(\cdot, x)\}$ so that $\mathscr{D}$ represents the region in which the agent finds an immediate intervention suboptimal. We can infer the existence of a value $x^\star \in S$ for which $\partial \mathscr{D} = \{X_s^{\cdot} = x^\star | x^\star \in S, s \in [0, T]\}$, that is to say the agent performs an intervention as soon as the cash-flow process X attains a value $x^\star$, hence we shall hereon refer to the value $x^\star$ as the agent's intervention threshold.

Proof of Theorem 5.3 We now seek to characterise the cost function parameters λ and κ which implement the principal's control policy. Suppose that the agent makes purchases according to the policy $[\tau_k, z_k]_{k \geq 1} \equiv (\tau, Z) \in \mathscr{U}$, hence the agent's payoff function is given by the expression:

$$\Pi^{(\mathfrak{c},(\tau,Z))}[t_0, x_0] = \mathbb{E}\left[\int_{t_0}^{\tau_S} e^{-\delta r} R(X_r^{t_0,x_0,(\tau,Z)}) dr + \sum_{j \geq 1} e^{-\delta \tau_j} z_j \cdot 1_{\{\tau_j \leq T\}}\right].$$

Let us define the control $(\tau^\star, Z^\star) \in \mathscr{U}$ by the following construction:

$$\Pi^{(\mathfrak{c},(\tau^\star,Z^\star))}[s, x] = \sup_{(\tau,Z) \in \mathscr{U}} \Pi^{(\mathfrak{c},(\tau,Z))}[s, x], \ \forall (s, x) \in [0, T] \times S,$$

so that given some $\mathfrak{c} \in \mathbb{R}^2$, the agent's optimal purchase strategy is given by $(\tau^\star, Z^\star) \in \mathscr{U}$.

Recall that the state process obeys the following $\forall (s, x_0), (t_0, x_0) \in [0, T] \times S, \forall (\tau, Z) \in \mathscr{U}$:

$$\begin{aligned} X_s^{t_0,x_0,(\tau,Z)} = x_0 + \int_{t_0}^{s \wedge \tau_S} \Gamma X_r^{t_0,x_0,(\tau,Z)} dr - \sum_{j \geq 1} ((1+\lambda) z_j + \kappa) \cdot 1_{\{\tau_j \leq \tau_S\}} + \int_{t_0}^{s \wedge \rho} \sigma X_r^{t_0,x_0,(\tau,Z)} dB_r \\ + \int_{t_0}^{s} \int X_{r-}^{t_0,x_0,(\tau,Z)} \gamma(r,z) \tilde{N}(dr, dz), \ \mathbb{P}-\text{a.s. } X_{t_0}^{t_0,x_0} := x_0, \end{aligned} \tag{5.24}$$

We now specialise to the case in which the agent's utility function R is given by:

$$R(x) = \varepsilon \ln x, \quad \forall x \in S, \tag{5.25}$$

for some constant $\varepsilon \in \mathbb{R}\backslash\{0\}$ so that R can be viewed as a limiting case of the CRRA utility function that is $R(x) = \lim_{\eta\to 1} \varepsilon \frac{x^{1-\eta}-1}{1-\eta}$. We note also that given some test function $\phi \in \mathscr{C}^{\{1,2\}}([0,T],\mathbb{R})$, the generator $\mathscr{L}$ for (5.24) is given by the following expression (c.f. (1.2)) $\forall (s,x) \in \mathbb{R}_{>0} \times \mathbb{R}$:

$$\mathscr{L}\phi(s,x) = \Gamma x \frac{\partial \phi}{\partial x}(s,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 \phi}{\partial x^2}(s,x) + \int_{\mathbb{R}} \left\{ \phi(s,x(1+\gamma(z)) - \phi(s,x) - x\gamma(z)\frac{\partial \phi}{\partial x} \right\} \nu(dz). \tag{5.26}$$

By (III) of Corollary 2.8.1, we have that on D the following expression holds:

$$R + \frac{\partial \phi}{\partial s} + \mathscr{L}\phi = 0. \tag{5.27}$$

Hence, using (5.26) and by (5.27) we have that:

$$\begin{aligned} 0 = e^{-\delta s}\varepsilon \ln x + \frac{\partial \phi}{\partial s}(s,x) + \Gamma x \frac{\partial \phi}{\partial x}(s,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 \phi}{\partial x^2}(s,x) \\ + \int_{\mathbb{R}} \left\{ \phi(s,x(1+\gamma(z)) - \phi(s,x) - x\gamma(z)\frac{\partial \phi}{\partial x} \right\} \nu(dz). \end{aligned} \tag{5.28}$$

Let us try the following ansatz for the candidate function for ϕ:

$$\phi \equiv \phi_a + \phi_b, \tag{5.29}$$

where $\phi_a(s,x) = e^{-\delta s} a x^l$, $\quad \phi_b(s,x) = e^{-\delta s}(b \ln x + c)$ for some constants $a,b,c \in \mathbb{R}$. We firstly seek to ascertain the values of the constants a,b and c hence, inserting the expression for ϕ into (5.28) we find that $h_a(l) + h_b(x) = 0$ where the functions h_a and h_b are given by:

$$h_a(l) = \frac{1}{2}\sigma^2 l(l-1) + l\Gamma - \delta + \int_{\mathbb{R}} \left\{ (1+\gamma(z))^l - 1 - l\gamma(z) \right\} \nu(dz) \tag{5.30}$$

$$h_b(x) = \varepsilon \ln x - \delta(b \ln x + c) + b\Gamma - \frac{1}{2}\sigma^2 b + \int_{\mathbb{R}} \left\{ b \ln(1+\gamma(z)) - b\gamma(z) \right\} \nu(dz), \tag{5.31}$$

from which we find that the equation $h_b(x) = 0$ is solved by the following values for b and c:

$$b = \varepsilon \delta^{-1}, \tag{5.32}$$

$$c = \varepsilon \delta^{-2}\left(\Gamma - \frac{1}{2}\sigma^2\right) + \varepsilon \delta^{-2} \int_{\mathbb{R}} \left\{ \ln(1+\gamma(z)) - \gamma(z) \right\} \nu(dz). \tag{5.33}$$

Let us make a brief excursion to discuss the case when the process (5.2) contains no jumps i.e. when $\gamma \equiv 0$. In this case, we readily observe that the constants b and c are given by:

$$b = \varepsilon \delta^{-1}, \tag{5.34}$$

$$c = \varepsilon \delta^{-2}\left(\Gamma - \frac{1}{2}\sigma^2\right). \tag{5.35}$$

Additionally, (5.30) now reduces to the following expression:

$$h_{a,0}(l) := h_a(l)\big|^{\gamma\equiv 0} = \frac{1}{2}\sigma^2 l^2 + c\delta b^{-1} l - \delta. \tag{5.36}$$

After some simple algebra, we then deduce that in this case there exist two solutions to the equation $h_{a,0}(l) = 0$, namely $l_{1,0}$ and $l_{2,0}$ given by:

$$l_{1,0} = \frac{-1}{b\sigma^2}\left(\sqrt{c^2\delta^2 + 2b^2\sigma^2\delta} + c\delta\right), \quad l_{2,0} = \frac{1}{b\sigma^2}\left(\sqrt{c^2\delta^2 + 2b^2\sigma^2\delta} - c\delta\right). \tag{5.37}$$

Let us now return to the case when the process (5.2) contains jumps. Using (5.30), we now observe that $\lim_{m\to\infty} h_a(m) = +\infty$ and $h_a(m)\big|^{m=0} = -\delta$. Hence, we deduce the existence of values l_1, l_2 such that $h_a(l_1) = h_a(l_2) = 0$. W.l.o.g. let us assume that $l_1 < l_2$, since $\forall\, l, z$ we have that $(1+\gamma(z))^l - 1 - l\gamma(z)\}\nu(dz) > 0$ so that $|l_1| > l_1$ and $l_1 < 0 < l_2$. Therefore, the function ϕ is given by the following (c.f. (5.29)):

$$\phi(s,x) = e^{-\delta s}[a_1 x^{l_1} + a_2 x^{l_2} + b\ln x + c], \tag{5.38}$$

where a_1 and a_2 are a pair of as of yet, undetermined constants and b and c are given by (5.32) - (5.33) and l_1 and l_2 are solutions to (5.30). Our ansatz for the continuation region D is that it takes the form $D = \{x < x^\star | x, x^\star \in S\}$. We now seek to determine the value of $x^\star$ and characterise the optimal intervention magnitude $\hat{z}$.

Now by Corollary 2.8.1, we find that for all $x_1 \geq x^\star$ we have:

$$\phi(\cdot,x) = \mathscr{M}\phi(\cdot,x) = \sup_{z\in\mathscr{Z}}\{\phi(\cdot, x - \kappa - (1+\lambda)z) + z)\}. \tag{5.39}$$

We wish to determine the value z that maximises (5.39), hence let us now define the function G by the following expression:

$$G(t,z) = \phi(t, x - \kappa - (1+\lambda)z) + z, \quad \forall\, z \in \mathscr{Z}, \forall (t,x) \in [0,T]\times S. \tag{5.40}$$

Our task now is to evaluate the maxima of (5.40) from which we readily observe that the first order condition for the maximum of G is given by $\phi'(\cdot, x - \kappa - (1+\lambda)\hat{z}) = \frac{1}{1+\lambda}$. Let us now consider a unique point $\hat{x} \in]0, x^\star[$ then using (5.39) we find that$\phi'(\cdot,\hat{x}) = \frac{1}{1+\lambda}$. We now observe that the following expression holds $x^\star - \kappa - (1+\lambda)\hat{z} = \hat{x}$ and hence we deduce that $\hat{z}(x) = \frac{x - \hat{x} - \kappa}{(1+\lambda)}$. From this we deduce that ϕ is given by the following expression $\forall x \in S$:

$$\phi(\cdot,x) = \phi(\cdot,\hat{x}) + \hat{z}. \tag{5.41}$$

Using (5.39) - (5.41) we readily obtain the following equations:

$$\phi'(\cdot,\hat{x}) = \frac{1}{1+\lambda} \tag{5.42}$$

$$\phi'(\cdot,x^\star) = \frac{1}{1+\lambda} \tag{5.43}$$

$$\phi(\cdot,x^\star) - \phi(\cdot,\hat{x}) = \frac{x^\star - \hat{x} - \kappa}{1+\lambda}. \tag{5.44}$$

We now separate the analysis into two cases; case I in which the proportional part of the transaction cost λ is fixed and, case II in which the principal is free to choose both values κ, λ.

Case I

Inserting (5.38) into (5.42) - (5.44) and by the high contact principle[3], we arrive at the following system of equations:

(i) $$a_1 l_1 \hat{x}^{l_1-1} + a_2 l_2 \hat{x}^{l_2-1} + \tfrac{b}{\hat{x}} = \tfrac{1}{1+\lambda}$$

(ii) $$a_1 l_1 x^{*l_1-1} + a_2 l_2 x^{*l_2-1} + \tfrac{b}{x^\star} = \tfrac{1}{1+\lambda}$$

(iii) $$a_1(x^{*l_1} - \hat{x}^{l_1}) + a_2(x^{*l_2} - \hat{x}^{l_2}) = \tfrac{x^\star - \hat{x} - \kappa}{1+\lambda} + b\ln\left(\tfrac{\hat{x}}{x^\star}\right)$$

where $b := \varepsilon\delta^{-1}$.

The system of 3 equations (i) - (iii) contains 3 unknowns (κ, a_1, a_2), hence we can solve it, in particular using (i) - (ii) to solve for a_1 and a_2 we find that:

$$a_1 = \left[\frac{z}{1+\lambda} + b\right] l_1^{-1} z^{-l_1}, \tag{5.45}$$

$$a_2 = \left[\frac{z}{1+\lambda} - b\right] l_2^{-1} z^{-l_2}, \tag{5.46}$$

where $b := \varepsilon\delta^{-1}$ and $z^m := \hat{x}^m - x^{\star m}$.

After substituting (5.45) - (5.46) into (iii) we readily obtain the expression for the fixed cost parameter κ:

$$\kappa(\hat{x}, x^\star, \lambda) = z\left[l_1^{-1} + l_2^{-1} - 1\right] + b(1+\lambda)\left[l_1^{-1} - l_2^{-1} + \ln\hat{x} - \ln x^\star\right], \tag{5.47}$$

Case II

We now seek to identify the parameters κ and λ, after setting $a_1 = a_2 := a$ in (5.38), then substituting into (5.42) - (5.44) we arrive at the following system of equations:

(i) $$a(l_1 \hat{x}^{l_1-1} + l_2 \hat{x}^{l_2-1}) + \tfrac{b}{\hat{x}} = \tfrac{1}{1+\lambda}$$

(ii) $$a(l_1 x^{*l_1-1} + l_2 x^{*l_2-1}) + \tfrac{b}{x^\star} = \tfrac{1}{1+\lambda}$$

[3]Recall that the high contact principle is a condition that asserts the continuity of the value function at the boundary of the continuation region.

(iii) $$a(x^{*l_1}-\hat{x}^{l_1}+x^{*l_2}-\hat{x}^{l_2})=\frac{x^\star-\hat{x}-\kappa}{1+\lambda}+b\ln\left(\frac{\hat{x}}{x^\star}\right)$$

where the constant b is given by Equation (5.32).

The system (i) - (iii) which involves 3 equations now consists of 3 unknowns (κ,λ,a), hence we can solve for the three unknown parameters. Eliminating the constant a from the system (i) - (iii) yields the following expressions for the cost parameters:

$$\lambda(\hat{x},x^\star)=\left(\frac{z}{b}\right)\frac{l_2^{-1}z^{-l_2}-l_1^{-1}z^{-l_1}}{l_1^{-1}z^{-l_1}+l_2^{-1}z^{-l_2}}-1 \tag{5.48}$$

$$\kappa(\hat{x},x^\star)=z\left[l_1^{-1}+l_2^{-1}-1\right]-z\frac{l_1^{-1}z^{-l_1}-l_2^{-1}z^{-l_2}}{l_1^{-1}z^{-l_1}+l_2^{-1}z^{-l_2}}\left[l_1^{-1}-l_2^{-1}+\ln\hat{x}-\ln x^\star\right], \tag{5.49}$$

where $z^m:=\hat{x}^m-x^{\star m}$.

By inverting the procedure, we can further deduce that using equations (i) - (iii), we can derive the values $m^\star$, $\hat{m}$ such that

$$x^\star=m^\star(\kappa,\lambda) \tag{5.50}$$

$$\hat{x}=\hat{m}(\kappa,\lambda), \tag{5.51}$$

where $\hat{m}$ and $m^\star$ are solutions to the system of equations:

$$\mathbf{Q}(\hat{m},m^\star,\kappa,\lambda)=\begin{bmatrix} Q_1(\hat{m},m^\star,\kappa,\lambda) \\ Q_2(\hat{m},m^\star,\kappa,\lambda)\end{bmatrix}=0 \tag{5.52}$$

where Q_1 and Q_2 are given by:

$$Q_1(x,y,q,k):=\left(l_1x^{l_1}+l_2x^{l_2}\right)(y-x-q+b(1+k)[\ln x-\ln y])-(x-b(1+k))(y^{l_1}-x^{l_1}+y^{l_2}-x^{l_2}), \tag{5.53}$$

$$Q_2(x,y,q,k):=\left(l_1y^{l_1}+l_2y^{l_2}\right)(y-x-q+b(1+k)[\ln x-\ln y])-(y-b(1+k))(y^{l_1}-x^{l_1}+y^{l_2}-x^{l_2}). \tag{5.54}$$

where l_1 and l_2 are solutions to (5.30) and $b:=\varepsilon\delta^{-1}$. □

Though it is not possible to obtain a closed analytic solution to (5.53) - (5.54), the values $\hat{m}$ and $m^\star$ can be approximated using numerical methods.

Having proven Theorem 5.3, we can straightforwardly prove Proposition 5.4:

Proof of Proposition 5.4 To prove Proposition 5.4, we firstly note that using (5.50) - (5.51), we can express the unobservable parameter pair $(x_0^\star,\hat{x}_0)$ in terms of the observable parameter pair $(\tilde{\lambda}_0,\kappa_0)$, that is $x_0^\star=m^\star(\kappa_0,\lambda_0)$ and $\hat{x}_0=\hat{m}(\kappa_0,\lambda_0)$. Hence, we have that $x_1^\star=m^\star(\kappa_0,\lambda_0)+h^\star$ and $\hat{x}_1=\hat{m}(\kappa_0,\lambda_0)+\hat{h}$, where $x_1^\star$ and $\hat{x}$ are the target consumption level and target consumption threshold respectively. Inserting these expressions for $x_1^\star$ and $\hat{x}_1$ into (5.48) and (5.48) yields the result. □

We have therefore succeeded in providing a full characterisation of the parameters of transaction costs that sufficiently distort the incentives of a rational agent so that the agent finds it optimal to maximise the principal's payoff. In particular, if the above values for the transaction cost are adopted by the principal, the rational agent finds it optimal to adopt a consumption pattern that is optimal for the principal.

We now give a sketch of the remaining proofs, the first of which follows from direct calculation:

Proof of Proposition 5.5 To prove Proposition 5.5, we differentiate (i) and (ii) w.r.t. $\hat{x}$ and $x^\star$ respectively and plugging in (5.50) and (5.51). We now observe that $\frac{\partial \hat{x}}{\partial \lambda}, \frac{\partial x^\star}{\partial \lambda}, \frac{\partial \hat{x}}{\partial \kappa}, \frac{\partial x^\star}{\partial \kappa}$ are given by the following expressions:

$$\frac{\partial \hat{x}}{\partial \lambda} = [f_1(\hat{x},x^\star)]^{-1}, \tag{5.55}$$
$$\frac{\partial x^\star}{\partial \lambda} = [f_2(\hat{x},x^\star)]^{-1}, \tag{5.56}$$
$$\frac{\partial \hat{x}}{\partial \kappa} = [f_3(\hat{x},x^\star)]^{-1} \tag{5.57}$$
$$\frac{\partial x^\star}{\partial \kappa} = [f_4(\hat{x},x^\star)]^{-1}. \tag{5.58}$$

where the functions f_1, f_2, f_3, f_4 are given by:

$$f_1(\hat{x},x^\star) = (\lambda+1)\left(\frac{1}{z}+\frac{1}{\hat{x}}\left[\frac{\hat{x}^{-l_1}+\hat{x}^{-l_2}}{l_1^{-1}z^{-l_1}+l_2^{-1}z^{-l_2}}-\frac{\hat{x}^{-l_1}-\hat{x}^{-l_2}}{l_1^{-1}z^{-l_1}-l_2^{-1}z^{-l_2}}\right]\right) \tag{5.59}$$
$$f_2(\hat{x},x^\star) = (\lambda+1)\left(-\frac{1}{z}+\frac{1}{x^\star}\left[\frac{x^{\star-l_1}-x^{\star-l_2}}{l_1^{-1}z^{-l_1}-l_2^{-1}z^{-l_2}}+\frac{x^{\star-l_1}+x^{\star-l_2}}{l_1^{-1}z^{-l_1}+l_2^{-1}z^{-l_2}}\right]\right) \tag{5.60}$$
$$\begin{aligned} f_3(\hat{x},x^\star) = {} & \frac{\kappa}{z}-\frac{1}{\hat{x}}\left(\kappa-z[l_1^{-1}+l_2^{-1}-1]\right)\left(\frac{\hat{x}^{-l_1}-\hat{x}^{-l_2}}{l_1^{-1}z^{-l_1}-l_2^{-1}z^{-l_2}}-\frac{\hat{x}^{-l_1}+\hat{x}^{-l_2}}{l_1^{-1}z^{-l_1}+l_2^{-1}z^{-l_2}}\right) \\ & -\frac{z}{\hat{x}}\frac{l_1z^{-l_1}-l_2z^{-l_2}}{l_1^{-1}z^{-l_1}+l_2^{-1}z^{-l_2}} \end{aligned} \tag{5.61}$$
$$\begin{aligned} f_4(\hat{x},x^\star) = {} & -\frac{\kappa}{z}-\frac{1}{x^\star}\left(\kappa-z[l_1^{-1}+l_2^{-1}-1]\right)\left(\frac{x^{\star-l_1}-x^{\star-l_2}}{l_1^{-1}z^{-l_1}-l_2^{-1}z^{-l_2}}-\frac{x^{\star-l_1}+x^{\star-l_2}}{l_1^{-1}z^{-l_1}+l_2^{-1}z^{-l_2}}\right) \\ & +\frac{z}{x^\star}\frac{l_1z^{-l_1}-l_2z^{-l_2}}{l_1^{-1}z^{-l_1}+l_2^{-1}z^{-l_2}} \end{aligned} \tag{5.62}$$

where $z^m := \hat{x}^m - x^{\star m}$ and the parameters l_1 and l_2 are solutions to the equation (5.11).

Proof of Lemma 5.6 To prove Lemma 5.6, we firstly consider a control solution to the problem (5.18) (which can by obtained using Corollary 2.8.1). Denote the optimal policy $u_2^\star \in \arg\sup_{u\in\mathscr{U}} J_2^u(t,x)$ where $u_2^\star = [\tau_{2_j}^\star, z_{2_j}^\star]_{j\geq 1} \in \mathscr{U}$ and the sets $\{\tau_{2_j}^\star\}_{j\in\mathbb{N}}$ and $\{z_{2_j}^\star\}_{j\in\mathbb{N}}$ are sequences of $\mathscr{F}_{\tau_j}-$measurable intervention times and intervention magnitudes respectively. Then by Corollary 2.10.1, there exist constants $\hat{x}_2 \in S$ and $x_2^\star \in S$ such that $\hat{\tau}_{j+1} = \inf\{s > \tau_j; X^{\cdot,\hat{u}}(s) \geq x_p^\star\} \wedge \tau_S$ and $\hat{z} = \hat{x}_p - x_p^\star$. Hence, by setting $x^\star = x_2^\star$ and $\hat{x} = \hat{x}_2$ in Theorem 5.3 we immediately deduce the result after applying the theorem.

Lemma 5.6 demonstrates that the results of the chapter can be applied to any pair of impulse control problems so that the cost parameters can be fixed so as to change the optimal control to match that of some other external objective function.

Conclusion

This thesis performs a detailed study of dynamic competitions in which players incur fixed minimal costs for each adjustment of their position. In doing so, the thesis addresses the absence of fixed adjustment costs as a modelling feature of multiplayer financial environments. Underpinning this analysis is the formal development of the mathematical structures for studying strategic interactions namely, stochastic (differential) games to now accommodate fixed adjustment costs. These theoretical results are developed within context of competitive financial scenarios in which in order to maximise their payoff, players must determine an optimal (investment) strategy in response to the actions of other players. The results of the thesis therefore establish a framework that prescribes investment strategies appropriate for settings with transaction costs, the presence of which rules out continuous investment strategies (which result from classical investment models).

In the introduction to the thesis, we provided a compelling motivation for the topic of the thesis and its relevance for tackling problems in finance and economics. In particular, it has been observed that the presence of transaction costs produces large deviations in the behaviour of financial agents since transaction costs induce rigidities and adjustment stickiness [LMW04]. Moreover, even slight changes to the payoff structures of a multiplayer system can induce significant changes owing to the interdependence of the agents' actions. Therefore, the presence of features such as transaction costs in such systems can cause a profound change in outcomes. However, despite their relevance, the task of incorporating transaction costs in multiplayer models within finance had been largely unaddressed. From this a clear motivation for addressing the task of incorporating transaction costs or more generally, minimally bounded adjustment costs into multiplayer models of finance follows. Next, we argued that in order to perform the task of incorpating minimally bounded costs in multiplayer financial models, it is necessary to develop the underlying mathematical frameworks that model non-cooperative strategic interactions in market scenarios, namely stochastic (differential) games to now incorporate minimally bounded control costs.

To this end, we performed a detailed investigation of the following environments:

- Stochastic differential games of impulse control and stopping
- Stochastic differential games of two-sided impulse control
- Dynamic principal-agent problems with minimally bounded adjustment costs

Each of the above frameworks underpin important applications within finance in which transaction costs are present. In the thesis, we demonstrated that the results generated in the formal analyses of the above frameworks can be applied to extract optimal investment strategies within financial models of optimal investment.

In Chapter 2, we commenced our investigation into multiplayer financial settings. We began by introducing the optimal liquidity control and lifetime ruin problem — a problem that admits a representation as a stochastic differential game of control and stopping. We then motivated the need to incorporate transaction costs within the model the inclusion of which leads to a departure from existing models which assume continuous investment. Due to the absence of a stochastic differential game of control and stopping with minimally bounded costs, tackling this problem required a new mathematical framework namely, a new stochastic differential game of impulse control and stopping. In contrast to existing controller and stopper games such as [BY11; BHY11; BHØT13; BH13; BZ15a], the necessary underlying framework involves a controller that now faces control costs that are bounded from beneath

Accordingly, in Chapter 2 we introduced a new stochastic differential game which consists of an impulse controller and an adversary that chooses when to termination the game. We progressively developed a set of arguments that describe the features of each player's equilibrium strategy which led to a set of *variational inequalities*. We then formally proved a *verification theorem* which yields a full characterisation of the value of the game in terms of a Hamilton-Jacobi-Bellman-Isaacs equation. This enables the equilibrium payoffs and controls to be extracted from candidate solutions to a PDE. These results extend existing analyses of stochastic differential games of control and stopping in [BY11; BHY11; BHØT13; BH13; BZ15a] to now accommodate minimally bounded adjustment costs. Moreover, the analysis of the chapter extends to jump-diffusion processes the stochastic differential games of control and stopping in [KS01; BY11; NZ+15; KZ+08; BH13] in which the game dynamics are described by Itō diffusions with continuous sample paths.

The stochastic differential game introduced in Chapter 2 gives rise to a general mathematical framework for analysing financial investment problems in which we seek both an optimal market exit criterion and an optimal investment strategy in the presence of minimally bounded adjustment costs. The general results of the chapter are accompanied by worked examples to elucidate the workings of the theory in context of investment problems within finance.

After performing an analytic treatment of the game, we returned to the investment problem and computed the solution to the optimal liquidity and lifetime ruin problem. In doing so, we demonstrated that the optimal investment strategy and optimal exit criterion can be recovered from the equilibrium controls of the stochastic differential game of control and stopping.

Within the analysis of the verification theorems performed in Chapter 2, we identified several deficiencies of the theory. The first such deficiency is the smoothness requirement of the value function in order to apply Dynkin's formula along the diffusion. This presented a shortcoming since in practice, the value function may not be differentiable at all points of the solvency region. In such cases, the non-differentiability of the value function leads to the arguments for proving the verification procedure to break down. A second deficiency we identified is that the question of existence and uniqueness of the value function had been left unaddressed. Indeed, in order to execute the verification approach, it was necessary to assume the existence of a value of the game and that the value function satisfied a dynamic programming principle which underpinned the verification theorem.

To address these issues, in Chapter 3 we performed a formal treatment of the game using viscosity theory. Viscosity-theoretic approaches serve as a tool within optimal control theory that enable the strong smoothness assumptions required in verification theorems to be relaxed. In particular, we showed that the value function of the game is a solution to a HJBI equation even when the value function is not everywhere smooth when the solution is interpreted in a weak, viscosity sense. Next, we then formally proved that the value of the game exists, is unique and is solution to the HJBI equation presented in Chapter 3. Crucially however, we show that the value function need not be everywhere smooth when interpreted in a weaker, viscosity sense.

The analysis in Chapter 3 is related to many existing works that employ viscosity approaches. These include the viscosity-theoretic analyses for tackling Dynkin games presented in [BS14] (which is a degenerate scenario in which the controller's decision is restricted to a single decision to terminate the game) and the single impulse controller case presented in [Sey09; Ish93; Ish95; BL84] and the viscosity-theoretic analysis of a two-player game of *continuous control* and stopping in [BZ15b]. Nonetheless, each game setting requires its own separate treatment using viscosity theory to establish the existence and uniqueness of the value function.

In Chapter 4, we performed a detailed investigation of a stochastic differential game of two-sided impulse control. The investigation was initiated by a study of a well-known investment problem namely a duopoly advertising problem. Here, we argued that the absence of fixed minimal investment costs leads to a deficiency in current models to describe the duopoly investment environment. Accordingly, we introduced a new model of duopoly advertising investments that now accounts for the fact that firms incur minimally bounded costs for their investment adjustments in addition to allowing for market expansion following firm (advertising) investments.

Solving this model necessitated a treatment of a non zero-sum stochastic differential game of two-sided impulse control the analysis of which was absent within the literature. To this end, we extended the game in [Cos13] to a setting that firstly accommodates jump-diffusion process and secondly admits a non zero-sum payoff structure. We derived a verification theorem that characterised the value function and best-response strategies for each player in both the zero-sum and non zero-sum scenarios leading to a characterisation of the saddle point equilibrium and Nash equilibrium of the game. The resulting framework is one that can describe a broad range of economic and financial settings. Last, we applied the theoretical analysis conducted in the chapter to investigate the duopoly advertising investment problem with minimally bounded costs. The outcome is a new model that extends existing models e.g. [PS04; Eri95] which now accounts for the minimum expenditures incurred by firms when adjusting their positions in addition to capturing the effect of exogenous market shocks and market expansions.

In part II of the thesis we performed a detailed investigation of a dynamic principal-agent problem in which an agent incurs a fixed minimal costs for each investment. In the setting we investigated, an agent that is able to modify the system dynamics but is subject to transaction costs the magnitude of which is chosen by a principal in advance of the agent performing their interventions.

The goal of this analysis is to extend the current work on incentive-distortion which currently tackles entry and exit problems [Zer03] and optimal stopping scenarios [KS15], to a sequential decision-making setting. We demonstrated that in this setting, the principal can find a transaction cost for which the agent finds it optimal to maximise the principal's external objective. Within the chapter, we additionally performed some computational analyses on the influence of the transaction cost on the agent's behaviour. The study performed in Chapter 5 is contextualised within a consumer model in which an external agent or principal seeks to induce a given consumption policy for the agent or maximise some external objective which depends on the agent's consumption pattern.

The analysis also provides the solutions of two inverse optimal control problems — for a given control policy, the results of Chapter 5 determine the cost functions that lead to a policy being executed by a payoff-maximising impulse controller.

Summary in Relation to The General Objectives

The aim of this thesis is to investigate modelling strategic interactions and financial investment models in which players face fixed minimal costs. The thesis aims are contextualised within three prominent problems within finance namely an optimal liquidity control and lifetime ruin problem, a dynamic duopoly investment advertising problem and lastly a dynamic principal-agent problem involving minimally bounded adjustment costs.

The investment models associated to these problems are widely applied and serve as important tools within theoretical finance. Moreover, each of these scenarios admits a stochastic differential game representation. The increasing reliance on models of this kind to describe and solve investment problems results in a deep need for such models to accurately model market conditions. Accordingly, addressing the absence of transaction costs provides a crucial modelling benefit in addition to allowing feasible investment strategies to be computed when investors face transaction costs.

To address this, we proposed a unified framework, namely a non zero-sum stochastic differential game of impulse controls presented in Chapter 4. As we showed, the various instantitions of this framework allow for computing optimal investment strategies in the problems we addressed within this thesis. In particular, in Chapter 2 we applied a specific case of the game in which one of the players exercised impulse controls while the other player's decision determined only whether the game continues or terminates. Similarly, in Chapter 5, we applied a case of the game in which one of the players exercised impulse control whilst another passive player decides in advance what transaction costs should be paid. Despite being instantiations of the game presented in Chapter 4, each setting required its own detailed treatment to obtain the best-response strategy of both players.

Accordingly, this thesis set out to perform an in-depth analysis of the games in each case leading to a contribution to the theory of stochastic differential games.

Lastly, a central goal of the thesis was to demonstrate the practical value of the theory derived in the thesis by applying the results to well-known investment problems. By applying the theoretical frameworks to tackle the investment problems, the method of obtaining precise characterisations of optimal investment strategies was demonstrated within the three investment problems tackled in the thesis. This methodology is intended to outline a general template for applying the theoretical results of this research in investment problems beyond those tackled in this thesis.

Further work

- The appearance of stochastic differential games in which the players use impulse control has revealed a vast fertile ground for future research for both symmetric and asymmetric environments. Indeed, for the complete information component, after the introduction classical stochastic differential game theory in which the players use continuous controls, the results of Fleming [FS89] have been extended to cases in which the underlying process is non-Markovian. A viscosity theoretic approach was also used to establish the existence of Nash equilibria in non zero-sum stochastic differential games [BCR04]. A single player impulse controller version of a framework in which players have partial state information was studied in [ØS08], here the controller's actions are subject to some execution delay so that there is some non-zero lag between the decision to execute an intervention and the execution being carried out. All of these results have yet to realised in differential games in which either one or more players use impulse controls.

- An interesting avenue for future research is to investigate the above stochastic differential game of control and stopping framework (in which the controller uses impulse controls to modify the state process) when either or both of the players only have access to state partial information — a system in which the state process is adapted to some subset of the canonical filtration.

  These questions remain unanswered in both a mechanism design framework in which many players modify the process within a system with a passive principal (a degenerate case of which is that discussed in Chapter 5) and games of impulse control with asymmetric information.

- Lastly, advances in machine learning (ML) methods have led to vast improvements in modelling dynamic systems [WR06]. In addition to being valuable tools for modelling physical settings, ML methods have been deployed to model the underlying dynamics within various financial environments. Moreover, with the availability of large (financial) data sets, ML modelling methods can be trained to achieve highly accurate predictions [HPW16]. Additionally, reinforcement learning (RL) — a subfield of machine learning concerned with determining optimal sequences of actions enables some decision problems to be solved purely through repeated interaction with the system [SB18]. RL methods achieve high (sample) efficiency when combined with model estimation methods leading to what is known as *model-based reinforcement learning* [PN17]. Therefore, notwithstanding the challenges of deploying such methodologies in non-stationary environments, exploring model-based RL extensions to multiplayer settings offers a lucrative potential to learn optimal investment strategies in competitive settings. Crucially this approach avoids the need to solve the PDEs that appear within the stochastic differential games framework.

Appendix

The results contained within the thesis are built under the following assumptions (unless otherwise stated):

A.1.1 Lipschitz continuity of the state coefficients

There exist real-valued constants $c_\mu, c_\sigma > 0$ and $c_\gamma(\cdot) \in \mathbb{L} \cap \mathbb{L}^2(\mathbb{R}^l, \nu)$ such that $\forall s \in [0,T], \forall x, y \in \mathbb{R}^p$ and $\forall z \in \mathbb{R}^l$ we have:

$$\begin{aligned}
|\mu(s,x) - \mu(s,y)| &\leq c_\mu |x-y| \\
|\sigma(s,x) - \sigma(s,y)| &\leq c_\sigma |x-y| \\
\int_{|z|\geq 1} |\gamma(x,z) - \gamma(y,z)| &\leq c_\gamma(z)|x-y|.
\end{aligned}$$

A.1.2. The state process functions are deterministic and measurable functions.

A.1.3. Growth Conditions

There exist real-valued constants $d_\mu, d_\sigma > 0$ and $d_\gamma(\cdot) \in \mathbb{L} \cap \mathbb{L}^2(\mathbb{R}^l, \nu), \rho \in [0,1[$ such that

$$\begin{aligned}
|\mu(s,x)| &\leq d_\mu(|1+|x|^\rho|) \\
|\sigma(s,x)| &\leq d_\sigma(|1+|x|^\rho|) \\
\int_{|z|\geq 1} |\gamma(x,z)| &\leq d_\gamma(|1+|x|^\rho|), \qquad \forall (s,x) \in [0,T] \times \mathbb{R}^p; \forall z \in \mathbb{R}^l.
\end{aligned}$$

A.2.1. Lipschitz continuity of the running cost and bequest functions

The running cost functions and terminal cost functions are Lipschitz that is, given a running cost function f and a terminal cost function G, for $K \in \{f, G\}$ there exists real-valued constants $c_K > 0$ such that

$$|K(s,x) + K(s,y)| \leq c_K|x-y|, \quad \forall s \in [0,T], \forall (x,y) \in \mathbb{R}^p.$$

A.2.2. The running cost and terminal cost functions are deterministic and measurable functions.

Assumptions A.1.1 - A.1.3 ensure the existence and uniqueness of a solution to (1.1) (c.f. [IW14]). Assumptions A.2.1 - A.2.2 are required to prove the regularity of the value function (and also appears in the single-player case, see for example [DGW10]).

A.3.

Let $\rho,\rho',\tau,\tau':\Omega\to[0,T]$ be $\mathscr{F}-$measurable stopping times and let $\eta,\eta',\xi,\xi'\in\mathscr{Z}$ be measurable impulse interventions s.th $t\leq\tau<\tau'\leq T$ and $t\leq\rho<\rho'\leq T$. Then for some strictly positive function $\Theta:\mathscr{T}\to\mathbb{R}$ such that $\Theta\in\mathscr{C}([0,T];\mathbb{R}^p)$, the following statements hold:

(i) $\chi(\rho,\eta+\eta')\leq\chi(\rho,\eta)+\chi(\rho,\eta')-\Theta(\tau)$,

(ii) $\chi(\rho,\eta)\geq\chi(\rho',\eta)$.

(iii) $c(\tau,\xi+\xi')\leq c(\tau,\xi)+c(\tau,\xi')-\Theta(\tau)$,

(iv) $c(\tau,\xi)\geq c(\tau',\xi)$.

A.4.

There exist constants $\lambda_c>0,\lambda_\chi>0$ such that $\inf_{\xi\in\mathscr{Z})}c(s,\xi)\geq\lambda_c$ and $\inf_{\eta\in\mathscr{Z})}\chi(s,\eta)\geq\lambda_\chi,\forall s\in[0,T]$ where $\xi,\eta\in\mathscr{Z}$ are $\mathscr{F}-$measurable impulse interventions.

Assumptions A.3 (i), (iii) (subadditivity) are required in the proof of the uniqueness of the value function. Assumptions A.3 (ii), (iv) (the player cost function is a decreasing function in time) may be interpreted as a discounting effect on the cost of interventions. Assumptions A.3 (ii) and (iv) were introduced (for the two-player case) in [Yon94] though are common in the treatment of single-player case problems (e.g. [DGW10; CG13]) and is needed to prove regularity properties of the value function. Assumption A.4 is integral to the definition of the impulse control problem.

Technical Conditions: (T1) - (T4)

(T1) Assume that $\mathbb{E}\left[\int_0^T 1_{\partial\mathscr{D}}(X^{\cdot,u}(s))ds\right]=0$ for all $x\in S$, $\forall u\in\mathscr{U}$ where $\mathscr{D}\equiv D_1\cup D_2$.

(T2) ∂D is a Lispchitz surface, that is to say that ∂D is locally the graph of a Lipschitz continuous function: $\phi\in\mathscr{C}^2(S\backslash\partial D)$ with locally bounded derivatives.

(T3) The sets $\{\phi^-(X^{\cdot,u}(\tau_m));\ \tau_m\in\mathscr{T},\forall m\in\mathbb{N}\}$ and $\{\phi^-(X^{\cdot,u}(\rho));\rho\in\mathscr{T}\}$ are uniformly integrable $\forall x\in S,u\in\mathscr{U}$.

(T4) $\mathbb{E}[|\phi(X^{\cdot,u}(\tau_m))|+|\phi(X^{\cdot,u}(\rho))|+\int_0^T|\mathscr{L}\phi(X^{\cdot,u}(s))|ds]<\infty$,
for all intervention times $\tau_m\in\mathscr{T},\rho\in\mathscr{T}$ and $\forall u\in\mathscr{U}$.

Assumptions T3 and T4 hold if for example ϕ satisfies a polynomial growth condition and guarantee that $\phi(X(\tau)$ is both well-defined and finite — in particular the uniform integrability assumption (T3) implies the existence of a finite constant $c > 0$ such that $\mathbb{E}[|\phi(X(\tau))|1_{\{\tau<\infty\}}] \leq c$ for all $\tau \in \mathscr{T}$.

Sobolev Spaces

In order to define Sobolev spaces, it is firstly necessary to introduce the notion of a weak derivative which weakens the notion of a classical partial derivative.

Definition (Weak derivative)

Fix an open set $U \subset \mathbb{R}^n$. Let $u, v \in L^1_{\text{loc}}(U)$ where $L^1_{\text{loc}} := (U)\{f : U \to \mathbb{R}, \text{measurable} \mid f|_H \in L_1(H),\ \forall H \subset U, H \text{ compact}\}$ is the space of locally integrable functions (or locally summable functions). Let α be a multiindex (that is an $n-$tuple $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ for some $n \in \mathbb{N}$) then v is the $\alpha^{th}-$ weak partial derivative of u if:

$$\int_U uD^\alpha \phi ds = (-1)^{|\alpha|} \int_U v\phi dx \tag{5.63}$$

for all infinitely differentiable test functions $\phi : U \to \mathbb{R}$ with compact support in U.

We write $D^\alpha u = v$ to note that $\alpha^{th}-$weak derivative of u, in particular given some function u and if there exists a function v such that (5.63) holds then it is said that $D^\alpha u = v$ in the *weak sense*.

Having defined the notion of a weak derivative, we are now in position to define the Sobolev space:

Definition (Sobolev space)

Fix $1 \leq p < \infty$ and let $k \in \mathbb{N}$, then a Sobolev space is a function space which has elements that have weak derivatives in L^p spaces, formally, a Sobolev space $W^{k,p}(U)$ is a space that consists of all locally summable functions $u : U \to \mathbb{R}$ such that for each multindex α such that $|\alpha| \leq k$, $D^\alpha u$ exists in the weak sense and $D^\alpha u \in L^p(U)$.

Risk Measures

In this section, we give some important details concerning risk measures. These definitions and results relate to the investment problem studied in Chapter 2.

Definition

Let $\mathbb{F}$ be a family of all lower bounded $\mathscr{F}-$measurable random variables. A risk measure on $\mathbb{F}$ is a map $\rho : \mathbb{F} \to \mathbb{R}$ with the following properties:

(i) [Monotonicity] If $X \leq Y$ then $\rho(X) \geq \rho(Y), \qquad \forall X, Y \in \mathbb{F}$,

(ii) [Translation invariance] $\rho(X+m) = \rho(X) + m, \quad \forall X \in \mathbb{F}, \forall m \in \mathbb{R}$

The intuition behind the monotonicity condition is the following — given two investment positions X and Y, if for all possible states of the world the position X outperforms Y then the risk associated to X is *less than* Y.

Translation invariance captures the idea that adding (deducting) a risk-free asset to (from) a portfolio and investing it leads to a reduction (increase) in risk position by that exact amount.

Definition (Coherent risk measure [ADEH99])
A risk measure $\rho : \mathbb{F} \to \mathbb{R}$ is said to be *coherent* if it satisfies the monotonicity condition and translation invariance in addition to the following conditions:

(iii) [Sub-additivity] $\rho(X+Y) \leq \rho(X)+\rho(Y), \quad \forall X,Y \in \mathbb{F}$

(iv) [Positive Homogeneity] $\rho(\lambda X) = \lambda\rho(X), \quad \forall\lambda \in \mathbb{R}_{>0}, \forall X \in \mathbb{F}$

Subadditivity captures the notion of diversification within portfolio finance, that is, a portfolio composed by several assets is strictly less risky than a portfolio which consists of a single asset (or instrument) provided that the correlation among the assets is not unity.

Positive homogeneity states that if the investor's holdings increases by some factor then the investor's risk exposure is multiplied by that same factor.

To handle (the many) situations in which the investor's risk profile evolves nonlinearly with the magnitude of the investor's position the notion of a convex risk measure was introduced. This risk measure relaxes the positive homogeneity and sub-additivity axioms which are now replaced with a weaker property which is known as convexity of risk measures.

Definition (Convex risk measure [FS02; FG02])
A risk measure is said to be *convex* if it satisfies the monotonicity condition and translation invariance in addition to the following condition:

(iii') [Convexity] $\rho(\lambda X+(1-\lambda)Y) \leq \lambda\rho(X)+(1-\lambda)\rho(Y), \qquad \forall X,Y \in \mathbb{F}, \forall\lambda \in]0,1[.$

The following results are built under assumptions A.1.1 - A.1.3 and establish the Lipschitz continuity, a growth condition and $\frac{1}{2}$-Hölder continuity of the state process:

Lemma (A.1.)
There exists a constant $c>0$ such that

$$\mathbb{E}|X_s^{t,x,\cdot} - X_s^{t,x',\cdot}| \leq c|x'-x|, \qquad \forall(t,x'),(t,x) \in [0,T]\times\mathbb{R}^p,\ \forall s \in [0,T].$$

Proof
In the following, $c>0$ denotes a constant which may vary from line to line.

Now, using Itō's lemma we readily observe that for all $(t,x),(t,x') \in [0,T]\times\mathbb{R}^p$:

$$|X_s^{t,x,\cdot}-X_s^{t,x',\cdot}|^2 \le |x-x'|^2+2\left|\int_t^s \left\langle X_r^{t,x,\cdot}-X_r^{t,x',\cdot}, \mu\left(r,X_r^{t,x,\cdot}\right)-\mu\left(r,X_r^{t,x',\cdot}\right)\right\rangle dr\right|$$
$$+2\left|\int_t^s \left\langle X_r^{t,x,\cdot}-X_r^{t,x',\cdot}, \sigma(r,X_r^{t,x,\cdot})-\sigma(r,X_r^{t,x',\cdot})\right\rangle dW_r\right|+\left|\int_t^s \left(\sigma(r,X_r^{t,x,\cdot})-\sigma(r,X_r^{t,x',\cdot})\right)dW_r\right|^2$$
$$+2\left|\int_t^s \left\langle X_r^{t,x,\cdot}-X_r^{t,x',\cdot}, \int\left[\gamma(X_{r-}^{t,x,\cdot},z)-\gamma(X_{r-}^{t,x',\cdot},z)\right]\tilde{N}(dr,dz)\right\rangle\right|+\left|\int_t^s\int\left[\gamma(X_{r-}^{t,x,\cdot},z)-\gamma(X_{r-}^{t,x',\cdot},z)\right]\tilde{N}(dr,dz)\right|^2. \tag{5.64}$$

Now as in [CG13] we apply Hölder's inequality to deduce that:

$$\mathbb{E}\left[\left|\int_t^s\int\gamma(r,X_r,z)\tilde{N}(dz,dr)\right|^\beta\right] \le \mathbb{E}\left[\int_t^s\left(\int|\gamma(r,X_r,z)|^2\nu(dz)dr\right)^{\beta/2}\right], \tag{5.65}$$

then, after taking expectations in (5.64) and using the properties of standard Brownian Motion and the jump measure $\tilde{N}$ and Itō isometry, we observe that:

$$\mathbb{E}\left|X_s^{t,x,\cdot}-X_s^{t,x',\cdot}\right|^2 \le |x-x'|^2+\mathbb{E}\left[\int_t^s\left|\left\langle X_r^{t,x,\cdot}-X_r^{t,x',\cdot},\mu(r,X_r^{t,x,\cdot})-\mu(r,X_r^{t,x',\cdot})\right\rangle\right|dr\right.$$
$$\left.+\int_t^s\left|\sigma(r,X_r^{t,x,\cdot})-\sigma(r,X_r^{t,x',\cdot})\right|^2dr+\int_t^s\int\left|\gamma(X_{r-}^{t,x,\cdot},z)-\gamma(X_{r-}^{t,x',\cdot},z)\right|^2\nu(dz)dr\right]$$
$$=|x-x'|^2+\mathbb{E}\left[\int_t^s\left|\left\langle X_r^{t,x,\cdot}-X_r^{t,x',\cdot},\mu(r,X_r^{t,x,\cdot})-\mu(r,X_r^{t,x',\cdot})\right\rangle\right|dr\right.$$
$$\left.+\int_t^s\left|\sigma(r,X_r^{t,x,\cdot})-\sigma(r,X_r^{t,x',\cdot})\right|^2dr\right]+\mathbb{E}\left[\int_t^s\int\left|\gamma(X_{r-}^{t,x,\cdot},z)-\gamma(X_{r-}^{t,x',\cdot},z)\right|^2dr\right]$$
$$\le|x-x'|^2+c\int_t^s\mathbb{E}|X_r^{t,x,\cdot}-X_r^{t,x',\cdot}|^2dr\le c|x-x'|^2,$$

thanks to assumption A.1.1 and where the last line follows from Gronwall's lemma for some $c>0$, after which we can readily deduce the result. □

Lemma (A.1.3.)

There exist a constant $c>0$ such that for any $p\le 2$ and for any $h<\infty$ and for any $(t,x),(t',x')\in[0,T]\times S$

(i) $\mathbb{E}\left[\sup_{s\in[0,T]}|X_s^{t,x,\cdot}|\right]^p\le c(1+|x|^p)$.

(ii) $\mathbb{E}\left[\sup_{s\in[t,h]}|X_s^{t,x,\cdot}-x|\right]^p\le ch^{p/2}(1+|x|^p)$.

(iii) $\mathbb{E}\left[\sup_{s\in[t,t']}|X_s^{t',x,\cdot}-X_s^{t,x,\cdot}|\right]^p\le c|t'-t|^{p/2}(1+|x|^p)$,

where $x\equiv X_t^{t,x}$ and where the constant c may vary in each line.

Proof

Using Itō's lemma and by similar reasoning as above, we readily observe that for all $(t,x),(s,x)\in[0,T]\times\mathbb{R}^p$ there exist constants (which may vary from line to line) $c',c>0$ such that

$$\mathbb{E}|X_s^{t,x,\cdot}|^2 \leq |x|^2 + \mathbb{E}\left[\int_t^s |\mu(r, X_r^{t,x,\cdot})|^2 dr\right] + \mathbb{E}\left[\int_t^s |\sigma(r, X_r^{t,x,\cdot})|^2 dr\right] + \mathbb{E}\left[\int_t^s \int |\gamma(X_{r-}^{t,x,\cdot}, z)|^2 \nu(dz) dr\right]$$

$$\leq |x|^2 + c\mathbb{E}\left[\left(1 + \int_t^s |X_r^{t,x,\cdot}|^2\right) dr + \mathbb{E}\left[\int_t^s \int \left(|\gamma(X_{r-}^{t,x,\cdot}, z)|^2 + \mathbb{E}\left[\left|\gamma(X_{r-}^{t,x,\cdot}, z) - \gamma(X_t^{t,x,\cdot}, z)\right|^2\right]\right) \nu(dz) dr\right]\right] \tag{5.66}$$

$$\leq |x|^2 + c\mathbb{E}\left[\int_t^s \left(1 + |X_r^{t,x,\cdot}|^2 + \mathbb{E}[|X_r^{t,x,\cdot} - X_t^{t,x,\cdot}|^2]\right) dr\right]$$

$$\leq c(1 + |x|^2) + c'\mathbb{E}\left[\int_t^s (1 + \sup_{r\in[t,s]} \mathbb{E}[|X_r^{t,x,\cdot}|^2]) dr\right], \tag{5.67}$$

where we have used (5.65) and assumptions A.1.3, A.1.1 and Fubini's Theorem (and the smoothing Theorem). Hence, after applying Gronwall's lemma to (5.67), we immediately deduce the existence of some real-valued constant $c > 0$ such that

$$\mathbb{E}[|X_s^{t,x,\cdot}|^2] \leq c(1 + |x|^2). \tag{5.68}$$

From here it is straightforward to deduce (i).

The proof of (ii) proceeds in a very much a similar way to the proof of Lemma A.1 and is omitted.

To prove statement (iii), given Lemma A.1; it suffices to prove that:

$$\mathbb{E}\left[\sup_{s\in[t,t']} \left|X_s^{t',x,\cdot} - x\right|\right]^2 \leq ch(1 + |x|^2), \quad x := X_t^{t,x,\cdot}. \tag{5.69}$$

The proof follows from Doob's inequality for martingales. Indeed, we observe that

$$\mathbb{E}\left[\sup_{s\in[t,t']} \left|X_s^{t',x,\cdot} - x\right|\right]^2$$
$$\leq c\mathbb{E}\left[\int_t^{t'} \left|\mu(r + t', X_r^{t',x,\cdot})\right|^2 dr + \int_t^{t'} \left|\sigma(r + t', X_r^{t',x,\cdot})\right|^2 dr + \int_t^{t'} \int \left|\gamma(X_{r-}^{t',x,\cdot}, z)\nu(dz) dr\right|^2\right]$$
$$\leq c\left(1 + |x|^2\right)(t' - t),$$

which is the desired result. □

Lemma (A.3.)

Let $\hat{\mu} \in \mathscr{B}_{(0,T)}$ be some $\varepsilon-$optimal strategy against $V^{\pm}(t,x)$ for any $(t,x) \in [0,T] \times \mathbb{R}^p$ then there exists some $\eta > 0$ such that the strategy $\hat{\mu}$ remains $2\varepsilon-$optimal against $V^{\pm}(t,y^j)$ for any $y^j \in B(x,\eta)$.

Proof

We do the proof for the function V^- with the proof for V^+ being analogous. Denote by $\hat{\rho} \equiv \hat{\mu}(u)$ where $\hat{\rho} \in \mathscr{T}$. Since the strategy $\hat{\mu}$ is ε-optimal against $V^-(t,x)$ we have that for some $\varepsilon > 0$:

$$V^-(t,x) \geq \inf_{u \in \mathscr{U}} J^{(u,\hat{\rho})}(t,x) + \varepsilon, \qquad \forall (t,x) \in [0,T] \times S. \tag{5.70}$$

Now by Proposition 3.4, we can deduce the existence of some constants $c_1, c_2 > 0$ such that for any $y^j \in B(x,\eta)$ and $\forall u \in \mathscr{U}, \forall \rho \in \mathscr{T}$:

$$|J^{(u,\rho)}(t,y^j) - J^{(u,\rho)}(t,x)| \leq c_1 |x - y^j|. \tag{5.71}$$

Hence,

$$\begin{aligned}\inf_{u \in \mathscr{U}} J^{(u,\hat{\rho})}(t,y^j) &\geq \inf_{u \in \mathscr{U}} J^{(u,\hat{\rho})}(t,x) - c_1|x-y^j| \geq V^-(t,x) - c_1|x-y^j| - \varepsilon \\ &\geq V^-(t,y^j) - (c_1+c_2)|x-y^j| - \varepsilon \geq V^-(t,y^j) - 2\varepsilon.\end{aligned}$$

where the last line follows provided that $|x - y^j| \geq \varepsilon(c_1+c_2)^{-1}$, from which we deduce the thesis since ε is arbitrary. $\square$

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