Packing measures, packing dimensions, and the existence of sets of positive finite measure

PhD Thesis

Helen Janeith Joyce

Department of Mathematics

University College London

University of London

September 1995

Abstract

A number of definitions of packing measures have been proposed at one time or another, differing from each other both in the notion of packing they employ, and in whether the radii or the diameters of the balls of the packing are used. In Chapter 1 various definitions of packing measures are considered and relationships between these definitions established.

Chapter 2 presents work which was done jointly with Professor D. Preiss, and which has been published as such. It is shown here that, with one of the possible radius-based definitions of packing measure, every analytic metric space of infinite packing measure contains a compact subset of positive finite measure. It is also indicated how this result carries over to other radius-based packing measures in the case of Hausdorff functions satisfying a doubling condition.

In Chapter 3 a construction is described which provides, for every Hausdorff function h, a compact metric space of infinite diameter-based h-packing measure, with no subsets of positive finite measure. It is then indicated how such a construction may be modified to deal with certain Hausdorff functions which do not satisfy a doubling condition, and a radius-based definition of packing measure.

In Chapter 4 we consider topological and packing dimensions, and show that if X is a separable metric space, then

$$\dim_{\mathcal{T}}(X) = \min\left\{\dim_{\mathcal{Q}}(X') : X' \text{ is homeomorphic to } X\right\},$$

where $\dim_{\mathcal{Q}}$ denotes the packing dimension associated with any one of the packing measures considered in this work, and $\dim_{\mathcal{T}}$ denotes topological dimension.

Chapter 5 answers the question, for which Hausdorff functions h may the Hausdorff and packing measures, $\mathcal{H}^h|_A$ and $\mathcal{P}^h|_A$, agree and be positive and finite for some $A \subseteq \mathbf{R}^n$. We show that the assumption that the two measures agree and are positive and finite on some subset of $\mathbf{R}^n$ implies that the function h is a regular density function (in the sense of Preiss). The converse result is also provided, that for each regular density function h, there is a subset A of $\mathbf{R}^n$ such that $\mathcal{H}^h|_A = \mathcal{P}^h|_A$ and this common measure is positive and finite.

Contents

Acknowledgements

I would like to thank Professor D. Preiss for first suggesting that I should study packing measures, and in particular the problem of subsets of positive finite measure. I deeply appreciated his invaluable advice, constant helpfulness and generosity with his time.

I would also like to thank Professors P. Mattila and R. D. Mauldin for suggesting the problem considered in Chapter 3 of this work; and Professor C. A. Rogers and others at University College London for various helpful comments.

I acknowledge the generous support of The British Council and of SERC (now EPSRC) during my studies.

Basic notation

There follows a list of the notation used in the text without definition, which, although basic, is not completely standard.

$\mathbf{R}^+$	the strictly positive real numbers
$B(x,r)$	the closed ball centred at x with radius r
$U(x,r)$	the open ball centred at x with radius r
$B(A,r)$	$A + B(0,r) = \{a + y : a \in A,\ y \in B(0,r)\}$
$U(A,r)$	$A + U(0,r) = \{a + y : a \in A,\ y \in U(0,r)\}$
$\mathrm{Clos}(A)$	the closure of the set A
∂A	the boundary of the set A
$\mathrm{diam}(A)$	the diameter of the set A, that is, $\sup\{\mathrm{dist}(x,y) : x, y \in A\}$
$h(\cdot -)$	the left continuous modification of a function $h : \mathbf{R} \to \mathbf{R}$
$h(\cdot +)$	the right continuous modification of a function $h : \mathbf{R} \to \mathbf{R}$
$\mu\vert_A$	the restriction of the measure μ to the set A, that is, the measure defined by $\mu\vert_A(S) = \mu(A \cap S)$
$\mathrm{spt}\ \mu$	the support of the measure μ, that is, the smallest closed set C such that the complement of C has μ measure 0
$\mathrm{dist}(A,x)$	the distance between the set A and the point x, that is, $\inf\{\mathrm{dist}(a,x) : a \in A\}$

$\text{dist}(f, g)$	the distance between functions f and g, that is, $\sup\{\text{dist}(f(x), g(x))\}$
$\text{conv}(A)$	the convex hull of the set A, that is, $\{\sum_{i=1}^{n} w_i a_i : 1 \leq n < \infty,\ a_i \in A,\ w_i > 0,\ \sum w_i = 1\}$
$\text{Lip}(f)$	if there is c such that $\text{dist}(f(x), f(y)) \leq c\,\text{dist}(x, y)$ for all x, y, we say that f is *Lipschitz*, and write $\text{Lip}(f) = \inf\{c : \text{dist}(f(x), f(y)) \leq c\,\text{dist}(x, y) \text{ for all } x, y\}$
polytope	we define a polytope in $\mathbb{R}^m$ as follows: A vertex or 0-cell is a point; a 1-cell is a line segment without its end points; a 2-cell a triangle without its sides; a 3-cell a tetrahedron without its faces, and so on. the k-cells, $k = 0, 1, 2 \cdots$ determined by the vertices, sides, faces, ... of a p-cell are called the k-faces of the p-cell; we also include the p-cell itself among its faces. An n-polytope is a point-set contained in $\mathbb{R}^m$ and decomposed in a definite manner into a finite collection of disjoint p-cells, $0 \leq p \leq n$, at least one of which is an n-cell, and such that every face of each cell of the collection belongs to the collection.

Introduction

This work largely concerns itself with packing measures, which were introduced to complement the theory of Hausdorff measures in [TaTr1, TaTr2, TrC]. (For a new treatment see also [MaP, Chapter 5].) While Hausdorff measures are intimately connected to upper density estimates (see, e.g., [FeH, 2.10.18]), much of the importance of packing measures stems from their connection to lower density estimates.

Given a *Hausdorff function*, that is, a non-decreasing function $h : \mathbf{R}^+ \to \mathbf{R}^+$ with $h(0+) = 0$, we may define a number of packing measures, which differ from one another both in the notion of packing they employ, and in whether we consider the radii or the diameters of the balls of the packing. To deal with the inevitable notational difficulties this presents, we shall use the symbol $\mathcal{P}$ to indicate a diameter-based definition, and the symbol $\mathcal{R}$ to indicate a radius-based definition. We shall also distinguish between packing measures which rely on different notions of packing, that is, packings which consist of balls which satisfy different types of disjointness conditions.

In Chapters 2 and 3 we consider the question, when do sets of infinite packing measure have subsets of positive finite packing measure. In Chapter 2, which is the joint work of the author and D. Preiss, a positive result is provided, that is, a result which tells us that for analytic metric spaces, such subsets exist in the case of the Besicovitch-type radius-based packing measure denoted below by $\tilde{\mathcal{R}}^h$. It is also shown that this result carries over to the radius-based packing measure denoted by $\mathcal{R}^h$ in the case of Hausdorff functions satisfying a *doubling condition*, that is, Hausdorff functions h for which we may find a constant c such that for every $r > 0$,

$$h(2r) \leq c\, h(\ r).$$

A similar property for Hausdorff measures, proved by Besicovitch [BeA5], (see also [BeA2],) for compact sets in Euclidean spaces, is an important tool in the study of Hausdorff dimension, see, e.g., [FaK]. For Hausdorff measures, the problem has been studied also in general metric spaces: Davies [DaR] generalized Besicovitch's theorem to analytic sets in Euclidean spaces, Davies and Rogers [DaRo] gave an example of a function h for which there is a compact metric space without this property, Larman [LaD] found a class of "finite dimensional" spaces for which Besicovitch's theorem holds, and Howroyd [HoJ] generalized Besicovitch's result to arbitrary analytic metric spaces provided that the Hausdorff function in question satisfied a doubling condition. (See [RoC] for more information about the theory of Hausdorff measures on metric spaces.) For packing measures the only previous result was that of Haase [HaH2] proving the statement in ultrametric spaces.

The work of this chapter has been used in [IkTa] to prove for packing measures, an analogue of the Frostman Lemma for Hausdorff measures. It has also been used by Mattila and Mauldin in [MaMa], where the packing measure function, which maps a non-empty compact subset K of a complete separable metric space X to its radius-based packing measure $\mathcal{R}^h(K)$, is shown to be measurable with respect to the σ-algebra generated by the analytic subsets of X, provided that h satisfies a doubling condition. It is then remarked that since their proof depended on the existence of compact subsets of positive finite measure, which cannot be guaranteed for the diameter-based packing measure $\mathcal{P}^h$, or for $\mathcal{R}^h$ unless h satisfies a doubling condition, the question of measurability of these packing dimension functions remains open.

In Chapter 3 we provide a negative result for the diameter-based packing measure $\mathcal{P}^h$, that is, for each Hausdorff function we provide a construction of a (compact) metric space of infinite diameter-based packing measure which has no subsets of positive finite measure. This work appears in [JoH], and answers a question that was asked by P. Mattila and R. D. Mauldin at the Conference on Fractal Geometry and Stochastics held at Finsterbergen in June 1994. We also show that this result carries over to the measure $\mathcal{R}^h$ in the case of certain Hausdorff functions which satisfy no doubling condition. The results of these two chapters are constructive, and show clearly the way in which the existence or non-existence of such a subset depends crucially on both the definition of packing measure which is adopted, and the properties of the function h.

In Chapter 4 we consider the relationship between the topological dimension of a separable metric space and the packing dimensions of its homeomorphic images. Given any family of packing measures defined for each non-negative real number, we may define the related dimension; below we provide a number of different families of packing measures, each of which will have its own related packing dimension. Therefore the question of when these dimensions may differ is important for Chapter 4, and is considered in Chapter 1. The relationship between Hausdorff dimension and topological dimension is considered in [SzE] and [HuWa].

In Chapter 5 we consider the situation where there is equality of Hausdorff and packing measures on a subset of $\mathbf{R}^n$, and show that the functions h for which there may exist such a set are precisely those named *regular functions* by D. Preiss, (see [PrD].) This extends the theorem of St. Raymond and Tricot, see [SaTr] (a proof of which may be found in [MaP]), that if $\mathcal{H}^s|_A = \mathcal{P}^s|_A$ is a positive finite measure, then s is an integer and A is s-rectifiable. For this chapter, we use theorems which adapt and extend the standard density-type theorems, and we rely heavily on the concepts and results of [PrD] and [MaPr].

Chapter 1

Some definitions and preliminary results

1.1 Introduction

In this chapter we define the concepts of *packing measure* and *packing dimension* which shall be used throughout this work. We derive some simple consequences of these definitions, and consider the inequalities satisfied by the measures and dimensions thus defined. Since in the following two chapters we concern ourselves with the question of existence of subsets of positive finite measure, it is of interest to establish the inequalities which hold for the various packing measures defined below, and to establish if and when these measures may differ to the extent of one being zero or infinite, and another positive and finite. For the work presented in Chapter 4, we shall also wish to know the inequalities which hold for the dimensions defined by the different packing measures, and on which spaces these dimensions may differ. For use in Chapter 5, we consider an inequality relating packing and

Hausdorff measures.

1.2 Packing measures defined

For the first type of disjointness condition we consider in this work, we will take a *packing* of a subset S of a metric space M to be a finite collection of closed balls $\{B(x_i, r_i) : x_i \in S\}$ such that, for each $i \neq j$,

$$B(x_i, r_i) \cap B(x_j, r_j) = \emptyset.$$

For $\delta > 0$, a *(radius-type) δ-packing* is a packing such that $r_i \leq \delta$ for each i.

We then define $\mathcal{R}^h(S)$, the *radius-based packing measure of S*, thus:

$$\begin{aligned} R^h_\delta(S) &= \sup\left\{\sum h(r_i) : \{B(x_i, r_i)\} \text{ a } \delta\text{-packing of } S\right\}, \\ R^h_0(S) &= \lim_{\delta \to 0} R^h_\delta(S), \\ \mathcal{R}^h(S) &= \inf\left\{\sum_1^\infty R^h_0(S_i) : S \subset \bigcup_1^\infty S_i\right\}. \end{aligned}$$

If we take a *(diameter-type) δ-packing* to be a packing such that for each i, diam $B(x_i, r_i) \leq \delta$, and utilize a similar notion of disjointness to that above, we may define the *diameter-based packing measure* $\mathcal{P}^h$ thus,

$$\begin{aligned} P^h_\delta(S) &= \sup\left\{\sum h(\text{diam } B(x_i, r_i)) : \{B(x_i, r_i)\} \text{ a } \delta\text{-packing of } S\right\}, \\ P^h_0(S) &= \lim_{\delta \to 0} P^h_\delta(S), \\ \mathcal{P}^h(S) &= \inf\left\{\sum_1^\infty P^h_0(S_i) : S \subset \bigcup_1^\infty S_i\right\}. \end{aligned}$$

The second type of disjointness condition we will consider is a *Besicovitch-type* condition, where a *Besicovitch packing* of a subset S of a metric space

M is a finite collection of closed balls $\{B(x_i, r_i) : x_i \in S\}$ such that, for each $i \neq j$,

$$x_j \notin B(x_i, r_i).$$

A *δ-Besicovitch packing* is a Besicovitch packing such that $r_i \leq \delta$ for each i.

We will distinguish packing measures defined using such a notion of packing from those using the stricter notion given above, by using a tilde to indicate a Besicovitch-type definition. Then, in a similar manner to the above, we may define $\tilde{\mathcal{R}}^h(S)$, the *Besicovitch-type radius-based packing measure of* S, thus:

$$\begin{aligned} \tilde{R}^h_\delta(S) &= \sup\left\{\sum h(r_i) : \{B(x_i, r_i)\} \text{ a } \delta\text{-Besicovitch packing of } S\right\}, \\ \tilde{R}^h_0(S) &= \lim_{\delta \to 0} \tilde{R}^h_\delta(S), \\ \tilde{\mathcal{R}}^h(S) &= \inf\left\{\sum_1^\infty \tilde{R}^h_0(S_i) : S \subset \bigcup_1^\infty S_i\right\}. \end{aligned}$$

If $s > 0$ and $h(r) = r^s$, we shall write Q^s_δ, Q^s_0, $\mathcal{Q}^s$, for Q^h_δ, Q^h_0, $\mathcal{Q}^h$, where Q stands for any of $R, \tilde{R}$ or P, and $\mathcal{Q}$ for any of $\mathcal{R}, \tilde{\mathcal{R}}$ or $\mathcal{P}$. We may then define the *packing dimension associated with the family of measures* $\{\mathcal{Q}^s : s \geq 0\}$ thus:

$$\dim_{\mathcal{Q}}(S) = \sup\{s : \mathcal{Q}^s(S) = \infty\} = \inf\{s : \mathcal{Q}^s(S) = 0\}.$$

1.3 A comparison of packing and Hausdorff measures

For the sake of comparison, and for use in Chapters 4 and 5, we now provide a definition of *Hausdorff measure*;

We say a (finite or countable) collection of open sets $\{U_i\}$ is a *covering* of a subset S of a metric space M if $S \subseteq \bigcup_i U_i$.

For $\delta > 0$, a *δ-covering* is a covering such that $\operatorname{diam}(U_i) \le \delta$ for each i.

Given a Hausdorff function h, we define $\mathcal{H}^h(S)$, the *Hausdorff measure of* S, thus:

$$\begin{aligned} H^h_\delta(S) &= \inf\left\{\sum h(\operatorname{diam} U_i) : \{U_i\} \text{ a } \delta\text{-covering of } S\right\}, \\ \mathcal{H}^h(S) &= \lim_{\delta\to 0} H^h_\delta(S). \end{aligned}$$

As for packing measures, we write $\mathcal{H}^s$ for $\mathcal{H}^h$, where $h(r) = r^s$, and we define *Hausdorff dimension* by

$$\dim_{\mathcal{H}}(S) = \sup\{s : \mathcal{H}^s(S) = \infty\} = \inf\{s : \mathcal{H}^s(S) = 0\}.$$

The next two lemmas are proved elsewhere, see [FeH, 2.8.4] and [MaP, 5.12], but for convenience and completeness we here provide the proofs of the precise forms we need.

Lemma 1.3.1 *If $\mathcal{F}$ is a family of closed subsets of a metric space X, with* $\sup\{\operatorname{diam}(F) : F \in \mathcal{F}\} < \infty$, *then $\mathcal{F}$ has a disjoint subfamily $\mathcal{G}$ such that for each $F \in \mathcal{F}$ there exists $G \in \mathcal{G}$ with*

$$F \cap G \neq \emptyset \text{ and } \operatorname{diam}(F) \le 2\operatorname{diam}(G).$$

Proof. Consider the class Ω of all disjoint subfamilies $\mathcal{H}$ of $\mathcal{F}$ with the following property: Whenever $F \in \mathcal{F}$,

$$\begin{aligned} &\text{either } F \cap H = \emptyset \text{ for all } H \in \mathcal{H}, \\ &\text{or } F \cap H \neq \emptyset \text{ and } \operatorname{diam}(F) \le 2\operatorname{diam}(H) \text{ for some } H \in \mathcal{H}. \end{aligned}$$

Since this family is partially ordered by set inclusion and nonempty (if $G \in \mathcal{F}$ is such that $2\operatorname{diam}(G) \ge \sup\{\operatorname{diam}(F) : F \in \mathcal{F}\}$, then $\{G\} \in \Omega$), we may use Hausdorff's maximal principle to choose a maximal totally ordered subset of

Ω, that is, we may find $\mathcal{G} \in \Omega$ such that $\mathcal{G}$ is not a proper subset of any member of Ω.

Let $\mathcal{K} = \{F \in \mathcal{F} : F \cap \bigcup\{G \in \mathcal{G}\} = \emptyset\}$. If $\mathcal{K} \neq \emptyset$ we could select $K \in \mathcal{K}$ so that

$$2\,\mathrm{diam}(K) \geq \sup\{\mathrm{diam}(L) : L \in \mathcal{K}\}.$$

Then we would have that $\mathcal{G} \cup \{K\} \in \Omega$, contrary to the maximal choice of $\mathcal{G}$.

Lemma 1.3.2 *Let X be a separable metric space, and let h be a Hausdorff function satisfying a doubling condition. Then $\mathcal{H}^h(A) \leq \mathcal{P}^h(A)$ for all subsets A of X. It follows immediately that $\dim_{\mathcal{H}}(A) \leq \dim_{\mathcal{P}}(A)$ for each $A \subseteq X$.*

Proof. Write $c = \limsup_{r \searrow 0} h(5r)/h(r)$. (The doubling condition ensures that c is finite.)

It suffices to show that $\mathcal{H}^h(A) \leq P_0^h(A)$ for each subset A of X, and for this we need only consider those sets A for which $P_0^h(A) < \infty$. Let $\varepsilon > 0$, and choose $\delta > 0$ such that $P_\delta^h(A) \leq P_0^h(A) + \varepsilon$, and such that if $r \leq \delta$ then $h(5r) \leq 2c\,h(r)$. Let $\{B_i\}_1^k$ be disjoint closed balls with centres in A such that $\mathrm{diam}(B_i) \leq \delta$ for each i, and

$$\sum_{i=1}^{k} h(\mathrm{diam}\ B_i) \leq P_\delta^h(A) \leq \sum_{i=1}^{k} h(\mathrm{diam}\ B_i) + \varepsilon.$$

Choose a countable dense subset S of A, and apply Lemma 1.3.1 to the family of closed balls $B(x,r)$ such that x belongs to S, $10r \leq \delta$, and

$$B(x,r) \subset X \setminus \bigcup_{i=1}^{k} B_i,$$

to find disjoint closed balls $B_1', B_2', \ldots$, of diameter at most $\delta/5$ with centres in A such that

$$A \setminus \bigcup_{i=1}^{k} B_i \subset \bigcup_j 5B_j'$$

and that the combined collection $\{B_i : 1 \le i \le k\} \bigcup \{B'_j : j = 1, 2, \ldots\}$ is also disjoint. Then

$$\begin{aligned}
\sum_{i=1}^{k} h(\text{diam } B_i) + \sum_{j=1}^{l} h(\text{diam } B'_j) &\le P_\delta^h(A) \text{ for each } l \ge 1; \\
\sum_{i=1}^{k} h(\text{diam } B_i) + \sum_{j=1}^{\infty} h(\text{diam } B'_j) &\le P_\delta^h(A) \\
&\le \sum_{i=1}^{k} h(\text{diam } B_i) + \varepsilon,
\end{aligned}$$

and so

$$\sum_{j=1}^{\infty} h(\text{diam } B'_j) \le \varepsilon.$$

Consequently,

$$\begin{aligned}
H_\delta^h(A) &\le \sum_{i=1}^{k} h(\text{diam } B_i) + \sum_{j=1}^{\infty} h(5 \,\text{diam } B'_j) \\
&\le \sum_{i=1}^{k} h(\text{diam } B_i) + 2c \sum_{j=1}^{\infty} h(\text{diam } B'_j) \\
&\le P_\delta^h(A) + 2c\varepsilon \\
&\le P_0^h(A) + (1 + 2c)\varepsilon.
\end{aligned}$$

Letting $\delta \searrow 0$ and $\varepsilon \searrow 0$ we see that $\mathcal{H}^h(A) \le P_0^h(A)$, as required.

1.4 Some easy properties of packing measures

Lemma 1.4.1 *Writing Q for any of $R, \tilde{R}$ or P, we have*

(i) $Q_\sigma^h(S) \le Q_\tau^h(S)$ *if* $0 < \sigma \le \tau$.

(ii) $Q_\delta^h(S) \le Q_\delta^h(T)$ *if* $S \subset T$, $\delta \geqslant 0$.

(iii) $Q^h_\delta(S \cup T) \le Q^h_\delta(S) + Q^h_\delta(T)$, $\delta \geq 0$.

(iv) $Q^h_\delta(S \cup T) = Q^h_\delta(S) + Q^h_\delta(T)$ *if* $\operatorname{dist}(S,T) > \delta$.

(v) $Q^h_0(S \cup T) = Q^h_0(S) + Q^h_0(T)$ *if* $\operatorname{dist}(S,T) > 0$.

Proof. The statements (i) and (ii) are obvious from the definitions, and (iii) and (iv) follow by decomposing an arbitrary packing of $S \cup T$ into packings of S and T, respectively. The statement (v) follows from (iv).

Lemma 1.4.2 *For every Hausdorff function h and every subset S of a metric space X*

$$\tilde{R}^h_0(S) = \tilde{R}^h_0(\text{Clos } S).$$

As a direct result,

$$\tilde{\mathcal{R}}^h(S) = \inf\left\{\sum_1^\infty \tilde{R}^h_0(C_i) : S \subseteq \bigcup_1^\infty C_i, \text{ and } C_i \text{ are closed subsets of } X\right\}.$$

Proof. Suppose $\eta > 0$, and the balls $B(x_1,r_1),\dots,B(x_n,r_n)$ form an η-Besicovitch packing of Clos(S). Choose $\beta > 0$ such that the balls $B(x_1,r_1+\beta),\dots,B(x_n,r_n+\beta)$ form an $(\eta+\beta)$-Besicovitch packing of Clos(S). If we then choose points $y_1,\dots,y_n$ of S with $\operatorname{dist}(x_i,y_i) < \beta$ for each i, the balls $B(y_1,r_1),\dots,B(y_n,r_n)$ form an η-Besicovitch packing of S, so

$$\tilde{R}^h_\eta(S) \ge \sum h(r_i).$$

So, for each $\eta > 0$,

$$\tilde{R}^h_\eta(S) \ge \tilde{R}^h_\eta(\text{Clos } S).$$

The result follows.

Lemma 1.4.3 *If h is a left-continuous Hausdorff function, then for every subset S of a metric space X*

$$R_0^h(S) = R_0^h(\text{Clos } S).$$

As a direct result, for such h

$$\mathcal{R}^h(S) = \inf\left\{\sum_1^\infty R_0^h(C_i) : S \subseteq \bigcup_1^\infty C_i, \text{ and } C_i \text{ are closed subsets of } X\right\}.$$

Proof. Let $\eta > 0$ and suppose the balls $B(x_1, r_1)$, ..., $B(x_n, r_n)$ form an η-packing of Clos(S). If $0 < \varepsilon < \min\{r_i : 1 \leq i \leq n\}$ we may choose points $y_1, \ldots, y_n$ of S with $\text{dist}(x_i, y_i) \leq \varepsilon$ for each i. Then $B(y_1, r_1 - \varepsilon)$, $\ldots, B(y_n, r_n - \varepsilon)$ is an η-packing of S. So

$$R_\delta^h(S) \geq \sum h(r_i-) = \sum h(r_i),$$

and the result follows.

A proof like those above could not be used for diameter-based packing measure, or for radius-based packing measure in the case where h is not left-continuous. The problem is that replacing a ball of a packing of Clos(S) by a smaller ball centred in S itself may greatly reduce the diameter of this ball, or the value that h takes at its radius.

Lemma 1.4.4 *There is a metric space M which has a subset A such that*

$$P_0^1(A) < \infty; \qquad P_0^1(\text{Clos } A) = \infty.$$

Proof. We choose an increasing sequence of integers $(n_k)_k$, and decreasing sequences of real numbers $(\zeta_k)_k$ and $(\xi_k)_k$ by taking $n_1 = 2$, $\zeta_1 = 1$, and $\xi_1 = 1/8$; then, for each $k \geq 1$, we choose $\zeta_{k+1} < \xi_k/8$ so small that

$$n_k \zeta_{k+1} \leq \zeta_k/2;$$

n_{k+1} so large that

$$n_{k+1}\zeta_{k+1} \geq 2\zeta_k;$$

and $\xi_{k+1} < \zeta_k/8$ so small that

$$n_{k+1}\xi_{k+1} \leq \xi_k/2.$$

For each $k \geq 1$, we construct a metric space (M_k, dist_k), thus:

Let $\{a_{k,j} : 1 \leq j \leq n_k\}$, $\{b_{k,j} : 1 \leq j \leq n_k\}$, $\{y_{k,i} : 1 \leq i < \infty\}$, and $\{z_{k,j,i} : 1 \leq j \leq n_k,\ 1 < i < \infty\}$, satisfy

$$\text{dist}_k(a_{k,i}, a_{k,j}) = \zeta_k \text{ if } i \neq j,$$

$$\text{dist}_k(b_{k,i}, b_{k,j}) = 2\zeta_k \text{ if } i \neq j,$$

$$\text{dist}_k(a_{k,i}, b_{k,i}) = \zeta_k/2,$$

$$\text{dist}_k(a_{k,i}, b_{k,j}) = 3\zeta_k/2 \text{ if } i \neq j,$$

$$\text{dist}_k(y_{k,i}, y_{k,j}) = \sum_{l=i}^{j-1} 2^{-l}\xi_k \text{ if } i < j,$$

$$\text{dist}_k(y_{k,i}, a_{k,j}) = 2^{-i+1}\xi_k + \zeta_k/2,$$

$$\text{dist}_k(y_{k,i}, b_{k,j}) = 2^{-i+1}\xi_k + \zeta_k,$$

$$\text{dist}_k(z_{k,j,l}, z_{k,j,p}) = \sum_{q=l}^{p-1} 2^{-q}\xi_k \text{ if } l < p,$$

$$\text{dist}_k(z_{k,j,l}, a_{k,j}) = 2^{-l+1}\xi_k,$$

$$\text{dist}_k(z_{k,j,l}, b_{k,j}) = \zeta_k/2 + 2^{-l+1}\xi_k,$$

$$\text{dist}_k(z_{k,j,l}, a_{k,p}) = \zeta_k - 2^{-l+1}\xi_k \text{ if } j \neq p,$$

$$\text{dist}_k(z_{k,j,l}, b_{k,p}) = 3\zeta_k/2 - 2^{-i+1}\xi_k \text{ if } j \neq p,$$

$$\text{dist}_k(z_{k,j,l}, y_{k,p}) = \zeta_k/2 + (2^{-p+1} - 2^{-l+1})\xi_k.$$

For each k, write

$$S_k = \{z_{k,j,1} : 1 \leq j \leq n_k\},$$

$$T_k = M_k \setminus S_k.$$

Let

$$M = T_1 \cup \bigcup_{l=1}^{\infty} \left(\prod_{k=1}^{l} S_k \times T_{k+1} \right).$$

Note that no sequence of M may be a truncation of another. So if two sequences (i) and (j) in M differ from each other, then there must be some $l \geq 1$ such that $i_l \neq j_l$, and we may define a metric on M thus; if $(i), (j) \in M$, $(i) \neq (j)$, and l is the least such that $i_l \neq j_l$, then

$$\operatorname{dist}((i), (j)) = \operatorname{dist}_l(i_l, j_l).$$

Let A be the set of those finite sequences in M which have $z_{k,j,i}$ as their final term, for some $k \geq 1,\ 1 \leq j \leq n_k,\ i > 1$. Then Clos$(A)$ also contains all those finite sequences in M which have $a_{k,j}$ as their final term, for some $k \geq 1,\ 1 \leq j \leq n_k$. Now

$$P^1_{\frac{\zeta_k}{2}}(\text{Clos } A) \geq \left(\prod_{l=1}^{k} n_l \right) \zeta_k/2,$$

since the balls of radius $\zeta_k/2$ centred at the points of Clos(A) which terminate in the kth place with the term $a_{k,j}$ $(1 \leq j \leq n_k)$ are disjoint, and there are $\prod_{l=1}^{k} n_l$ such balls, each of diameter greater than $\zeta_k/2$.

We now introduce some notation.

Let $(i) \in M$. We write $|(i)|$ for the length of (i), and if $|(i)| > k$, we write $(i)|_k = (i_1, \ldots, i_k)$.

Fixing k, and regarding two elements (i) and (j) of M as equivalent if $(i)|_k = (j)|_k$, we write

$$\begin{aligned} M|_k &= \{(i)|_k : (i) \in M\}, \\ A|_k &= \{(i)|_k : (i) \in A\}. \end{aligned}$$

We shall also use the notation

$$A_k = A|_k \cup \{(i) \in A : |(i)| \leq k\}.$$

The usefulness of this notation lies in the following fact;

> Packings of A by balls of radius greater than or equal to $2\zeta_{k+1}$ are in one-one correspondence with packings of A_k by balls of radius greater than or equal to $2\zeta_{k+1}$.

This is not hard to see since if (i) and (j) are elements of A, both of length greater than k, such that $(i)|_k = (j)|_k$, then $B((i),r) = B((j),r)$ for each $r \geq 2\zeta_{k+1}$. On the other hand, suppose $(i) \in A_k \setminus A|_k$, and $r \geq 2\zeta_{k+1}$, then either $B((i),r)$ contains only points in $A_k \setminus A|_k$, in which case, $B((i),r)$ is also a ball in A, or it contains some point $(j) \in A|_k$, in which case it necessarily contains every point (l) such that $(j)|_k = (l)|_k$.

Therefore, since packings are finite collections of balls,

$$P^1_{2\zeta_1}(A) \geq \sup_{k \geq 1} P^1_{2\zeta_1}(A_k).$$

Fix $(i_1, \ldots i_{k-1}) \in A|_{k-1}$. Let

$$A_{(i_1,\ldots i_{k-1})} = \left\{(i_1, \ldots i_{k-1}, j) : j = z_{k,l,i}, \text{ some } 1 \leq l \leq n_k,\ i \geq 1\right\}.$$

Then any two balls centred in $A_{(i_1,\ldots i_{k-1})}$ with diameter greater than ξ_k must intersect each other, as each must contain $(i_1, \ldots i_{k-1}, y_{k,q})$ for all $q \geq p$, for some $p \geq 1$. The maximum diameter of such a ball is clearly $2\zeta_k$, that is, the diameter of $A_{(i_1,\ldots i_{k-1})}$. Summing over the diameters of any balls of radius less than or equal to ξ_k in a packing of $A_{(i_1,\ldots i_{k-1})}$, gives a total of at most $n_k\xi_k$, so the contribution to any packing of $A_{(i_1,\ldots i_{k-1})}$ by balls of radius greater than or equal to $2\zeta_{k+1}$ is at most $2\zeta_k + n_k\xi_k$.

Now

$$A|_k \subseteq \bigcup_{(i_1,\ldots i_{k-1})} A_{(i_1,\ldots i_{k-1})},$$

so the contribution to any packing of $A|_k$ by balls of radius greater than or equal to $2\zeta_{k+1}$ is at most

$$2\left(\prod_{i=1}^{k-1} n_i\right)\zeta_k + \left(\prod_{i=1}^{k} n_i\right)\xi_k.$$

For each $k \geq 1$, we have

$$A_k = A|_k \cup (A_{k-1} \setminus \{(i_1, \ldots, i_{k-2}, z_{k-1,j,1}) : 1 \leq j \leq n_{l-1}\}).$$

So

$$\begin{aligned} P^1_{2\zeta_1}(A_k) &\leq P^1_{2\zeta_1}(A|_k) + P^1_{2\zeta_1}\left(A_{k-1} \setminus \{(i_1, \ldots, i_{k-2}, z_{k-1,j,1}) : 1 \leq j \leq n_{l-1}\}\right) \\ &\leq 2\left(\prod_{i=1}^{k-1} n_i\right)\zeta_k + \left(\prod_{i=1}^{k} n_i\right)\xi_k + P^1_{2\zeta_1}(A_{k-1}). \end{aligned}$$

It is easy to see that

$$P^1_{2\zeta_1}(A_1) \leq 2\zeta_1 + n_1\xi_1,$$

so

$$P^1_{2\zeta_1}(A_k) \leq 2\zeta_1 + 2\sum_{j=2}^{k}\left(\prod_{i=1}^{j-1} n_i\right)\zeta_j + \sum_{j=1}^{k}\left(\prod_{i=1}^{j} n_i\right)\xi_j,$$

and

$$P^1_{2\zeta_1}(A) \leq 2\zeta_1 2\sum_{j=2}^{\infty}\left(\prod_{i=1}^{j-1} n_i\right)\zeta_j + \sum_{j=1}^{\infty}\left(\prod_{i=1}^{j} n_i\right)\xi_j.$$

Now the choices of the sequences $(n_j)_j$, $(\zeta_j)_j$ and $(\xi_j)_j$ ensure that

$$\left(\prod_{i=1}^{j} n_i\right)\zeta_j/2 \geq 2\left(\prod_{i=1}^{j-1} n_i\right)\zeta_{j-1}/2,$$

so $\left(\prod_{i=1}^{j} n_i\right)\zeta_j/2 \to \infty$ as $j \to \infty$, and

$$P^1_0(\text{Clos } A) = \infty.$$

But

$$2\left(\prod_{i=1}^{j-1} n_i\right)\zeta_j \leq \left(\prod_{i=1}^{j-2} n_i\right)\zeta_{j-1},$$

so $\sum_{j=1}^{\infty}\left(\prod_{i=1}^{j-1} n_i\right) 2\zeta_j$ is bounded. Also

$$2\left(\prod_{i=1}^{j} n_i\right)\xi_j \leq \left(\prod_{i=1}^{j-1} n_i\right)\xi_{j-1},$$

so $\sum_{j=1}^{\infty}\left(\prod_{i=1}^{j} n_i\right)\xi_j$ is bounded. Therefore $P_0^1(A) \leq P_{2\zeta_1}^1(A) < \infty$, as claimed.

However, with a certain (rather restrictive) condition on the metric space X, we may prove that $P_0^h(S) = P_0^h(\text{Clos } S)$, and that $R_0^h(S) = R_0^h(\text{Clos } S)$ even when h is not left-continuous. This less general result will be useful in Chapter 3.

Lemma 1.4.5 *Let h be a Hausdorff function. Let X be a metric space satisfying the following condition:*

> *For each ball $B(x,r)$ in X, there is $\varepsilon > 0$ such that, if $\operatorname{dist}(x,y) < \varepsilon$, then $B(x,r) = B(y,r)$.*

Then

$$P_0^h(S) = P_0^h(\text{Clos } S), \text{ for each } S \subseteq X,$$

$$R_0^h(S) = R_0^h(\text{Clos } S), \text{ for each } S \subseteq X.$$

It follows immediately that

$$\mathcal{P}^h(S) = \inf\left\{\sum_1^{\infty} P_0^h(C_i) : S \subseteq \bigcup_1^{\infty} C_i, \text{ and } C_i \text{ are closed subsets of } X\right\},$$

$$\mathcal{R}^h(S) = \inf\left\{\sum_1^{\infty} R_0^h(C_i) : S \subseteq \bigcup_1^{\infty} C_i, \text{ and } C_i \text{ are closed subsets of } X\right\}.$$

Proof. Suppose the balls $B(x_1,r_1),\ldots,B(x_n,r_n)$ form a packing of $\text{Clos}(S)$, and pick ε small enough that for each $1 \leq i \leq n$ we have $B(x_i,r_i) = B(y_i,r_i)$ for each $y_i \in B(x_i,\varepsilon)$. For each i, choose $y_i \in S \cap B(x_i,\varepsilon)$. Then $B(x_i,r_i) =$

$B(y_i, r_i)$, and so each packing of $\mathrm{Clos}(S)$ may be viewed as a packing of S. The result follows.

We note that, even though Lemma 1.4.1 (iii) implies that $\tilde{R}_0^h$, R_0^h and P_0^h are (finitely) subadditive, they are not, in general, countably subadditive, which is why the last step in the definitions of $\tilde{\mathcal{R}}^h$, $\mathcal{R}^h$ and $\mathcal{P}^h$ is needed. In Chapter 2 we are mostly concerned with Besicovitch-type radius-based packing measure, and we will use there the proposition below, which provides a simple criterion for the equality of $\tilde{\mathcal{R}}^h$ and $\tilde{R}_0^h$.

Lemma 1.4.6 *If M is a compact metric space and if for every $\varepsilon > 0$, every $\delta > 0$ and every subset S of M one can find an open set $G \supset S$ such that $\tilde{R}_0^h(G) \le \tilde{R}_\delta^h(S) + \varepsilon$, then $\tilde{\mathcal{R}}^h(M) = \tilde{R}_0^h(M)$.*

Proof. Let $M \subset \bigcup_{i=1}^\infty S_i$ and let $\varepsilon > 0$. For each $i = 1, 2, \ldots$ we choose $\delta_i > 0$ such that

$$\tilde{R}_{\delta_i}^h(S_i) \le \tilde{R}_0^h(S_i) + 2^{-i-1}\varepsilon.$$

Let $G_i \supset S_i$ be open sets such that

$$\tilde{R}_0^h(G_i) \le \tilde{R}_{\delta_i}^h(S_i) + 2^{-i-1}\varepsilon.$$

Since M is compact, the cover $\{G_i\}$ of M has a finite subcover. So we may use Lemma 1.4.1(iii) to infer that

$$\tilde{R}_0^h(M) \le \sum_{i=1}^\infty \tilde{R}_0^h(G_i) \le \sum_{i=1}^\infty \left(\tilde{R}_0^h(S_i) + 2^{-i}\varepsilon\right) = \sum_{i=1}^\infty \tilde{R}_0^h(S_i) + \varepsilon.$$

Hence, given any $\{S_i\}$ with $M \subset \bigcup_{i=1}^\infty S_i$, we have $\tilde{R}_0^h(M) \le \sum_{i=1}^\infty \tilde{R}_0^h(S_i)$, which shows that $\tilde{R}_0^h(M) \le \tilde{\mathcal{R}}^h(M)$. The opposite inequality is obvious.

1.5 A comparison of different packing measures and dimensions

Lemma 1.5.1 *For every Hausdorff function h, $\mathcal{R}^h \leq \tilde{\mathcal{R}}^h$.*

Proof. This follows since a δ-packing of a set S is clearly also a δ-Besicovitch packing of S.

Lemma 1.5.2 *If h satisfies a doubling condition, then the two measures $\tilde{\mathcal{R}}^h$ and $\mathcal{R}^h$ are zero, positive and finite, and infinite, respectively, on precisely the same subsets of a metric space X. As a direct consequence, $\dim_{\tilde{\mathcal{R}}}(S) = \dim_{\mathcal{R}}(S)$ for each subset S of X.*

Proof. If $\{B(x_i, r_i)\}$ is a Besicovitch packing of S then $\{B(x_i, r_i/2)\}$ is a packing of S. Since h satisfies a doubling condition, there is $c > 0$ such that, for each i, $h(r_i) \leq c\,h(r_i/2)$. So $\tilde{R}^h_\delta(S) \leq cR^h_\delta(S)$ and $\tilde{\mathcal{R}}^h(S) \leq c\mathcal{R}^h(S)$, which, together with Lemma 1.5.1, proves the result.

Lemma 1.5.3 *If h satisfies a doubling condition, then $\mathcal{P}^h(S) \leq c\mathcal{R}^h(S)$ for each subset S of a metric space X, where c is the doubling constant. This immediately implies that $\dim_{\mathcal{P}}(S) \leq \dim_{\mathcal{R}}(S)$.*

Proof. The proof is obvious since the diameter of a ball is less than or equal to twice its radius.

We now introduce some terminology due to Federer [FeH, 2.8.9] (in a simplified form which will be sufficient for our purposes). One of the consequences of the results presented in Chapters 2 and 3 is that, for each Hausdorff function h, there is a compact metric space of infinite $\tilde{\mathcal{R}}^h$ and $\mathcal{P}^h$ measure, which has subsets of positive finite $\tilde{\mathcal{R}}^h$ measure but none of positive finite $\mathcal{P}^h$ measure. For certain functions h which do not satisfy a doubling condition, we may also find a compact metric space of infinite $\tilde{\mathcal{R}}^h$ and $\mathcal{R}^h$ measure, which has subsets of positive finite $\tilde{\mathcal{R}}^h$ measure but none of positive finite $\mathcal{R}^h$ measure. It is natural to ask if such examples could be constructed inside $\mathbf{R}^n$. The next two lemmas imply that they could not.

A metric space (X, d) is said to be *ζ-directionally limited*, for some positive integer ζ, if the following condition holds:

> For each $a \in X$ and $B \subseteq X \setminus \{a\}$ such that $d(x,c)/d(a,c) \geq 1/3$ whenever $b, c \in B$, $b \neq c$, $d(a,b) \geq d(a,c)$, and the point x is chosen thus,
>
> $$d(a,x) = d(a,c),\ d(x,b) = d(a,b) - d(a,c);$$
>
> we have
>
> $$|B| \leq \zeta.$$

A metric space (X, d) is *directionally limited* if it is ζ-directionally limited for some positive integer ζ. Note that Euclidean spaces are directionally limited.

A collection of closed balls $\{B(x_i, r_i)\}$ in X is said to be *τ-controlled* if $1 < \tau < \infty$, and for each $B(x_i, r_i) \neq B(x_j, r_j)$, one of the following holds;

$$d(x_i, x_j) > r_i > r_j/\tau \text{ or } d(x_i, x_j) > r_j > r_i/\tau.$$

A proof of Lemma 1.5.4 may be found in [FeH, 2.8.12].

Lemma 1.5.4 *If $1 < \tau < 1.04$, a τ-controlled collection of closed balls in a ζ-directionally limited metric space is a union of $2\zeta + 1$ subcollections of disjoint sets.*

Lemma 1.5.5 *If (X,d) is a directionally limited metric space then there is a constant c such that $\tilde{\mathcal{R}}^h(S) \le c\mathcal{P}^h(S)$ for each subset S of X. If, in addition, h is a Hausdorff function which satisfies a doubling condition, then the measures $\tilde{\mathcal{R}}^h$ and $\mathcal{P}^h$ are zero, positive and finite, and infinite, respectively, on the same subsets of X. This directly implies that for each subset S of X,*

$$\dim_{\tilde{\mathcal{R}}}(S) = \dim_{\mathcal{P}}(S).$$

Proof. We first note that Lemmas 1.5.1 and 1.5.3 together imply that it is sufficient to prove the first statement. Let ζ be such that X is ζ-directionally limited. Let $S \subseteq X$ and $\varepsilon > 0$. Choose $\delta > 0$ so small that

$$\tilde{R}^h_\delta(S) \le (1+\varepsilon)\tilde{R}^h_0(S).$$

Choose $0 < \delta_1 < \delta$ so small that $2h(\delta_1) \le h(\delta)$. Let $\{B(x_i, r_i)\}$ be a δ_1-Besicovitch packing of S such that

$$\sum h(r_i) \ge (1-\varepsilon)\tilde{R}^h_0(S).$$

Write

$$\begin{aligned} S_1 &= \{x_i : \operatorname{dist}(B(x_i,r_i), S \setminus B(x_i,r_i)) < \delta\}, \\ S_2 &= \{x_i : x_i \notin S_1\}. \end{aligned}$$

Then $\{B(x_i,r_i) : x_i \in S_1\} \cup \{B(x_i,\delta) : x_i \in S_2\}$ is a δ-Besicovitch packing of S, so

$$\begin{aligned} \sum\{h(r_i) : x_i \in S_1\} + \sum\{h(\delta) : x_i \in S_2\} &\le \tilde{R}^h_\delta(S), \\ \sum\{h(r_i) : x_i \in S_1\} + 2\sum\{h(r_i) : x_i \in S_2\} &\le (1+\varepsilon)\tilde{R}^h_0(S), \\ (1-\varepsilon)\tilde{R}^h_0(S) + \sum\{h(r_i) : x_i \in S_2\} &\le (1+\varepsilon)\tilde{R}^h_0(S), \\ \sum\{h(r_i) : x_i \in S_2\} &\le 2\varepsilon\tilde{R}^h_0(S). \end{aligned}$$

So

$$\sum\{h(r_i) : x_i \in S_1\} \geq (1-\varepsilon)\tilde{R}_0^h(S) - 2\varepsilon\tilde{R}_0^h(S) = (1-3\varepsilon)\tilde{R}_0^h(S).$$

For each i such that $x_i \in S_1$, let $s_i = \min\{\text{dist}(x_i, x_j) : x_i \neq x_j\}$; then it is easy to see that there is $1 < \tau < \infty$ such that $\{B(x_i, s_i) : x_i \in S_1\}$ is τ-controlled. So $\{B(x_i, s_i) : x_i \in S_1\}$ is a union of $2\zeta+1$ subcollections of disjoint balls. Therefore, there is a disjoint subcollection $\mathcal{B}$ of $\{B(x_i, s_i) : x_i \in S_1\}$ such that

$$\sum_{x_i \in S_1} h(r_i) \leq \sum_{x_i \in S_1} h(s_i) \leq (2\zeta+1) \sum_{x_i \in \mathcal{B}} h(\text{diam}(B(x_i, s_i))).$$

Therefore

$$(1-3\varepsilon)\tilde{R}_0^h(S) \leq (2\zeta+1)P_\delta^h(S).$$

Since this is true for each $\varepsilon > 0$, each $\delta > 0$, and each subset S of X,

$$\begin{aligned} \tilde{R}_0^h(S) &\leq (2\zeta+1)P_0^h(S), \\ \tilde{\mathcal{R}}^h(S) &\leq (2\zeta+1)\mathcal{P}^h(S), \end{aligned}$$

which proves the first statement of the lemma.

Lemma 1.5.6 *If (X, d) is a directionally limited metric space, then the measures $\tilde{\mathcal{R}}^h$ and $\mathcal{R}^h$ are zero, positive and finite, and infinite, respectively, on the same subsets of X. This directly implies that for each subset S of X,*

$$\dim_{\tilde{\mathcal{R}}}(S) = \dim_{\mathcal{R}}(S).$$

Proof. Let ζ be such that X is ζ-directionally limited. Let $S \subseteq X$, and $\mathcal{B}$ be a collection of points of X such that $\{B(x_i, r_i) : x_i \in \mathcal{B}\}$ is a Besicovitch packing of S. It is easy to see that there is $1 < \tau < \infty$ such that

$\{B(x_i, r_i) : x_i \in \mathcal{B}\}$ is τ-controlled. So $\{B(x_i, r_i) : x_i \in \mathcal{B}\}$ is a union of $2\zeta+1$ subcollections of disjoint balls. Therefore there is a subcollection $\mathcal{B}'$ of $\mathcal{B}$ such that

$$\sum_{x_i \in \mathcal{B}} h(r_i) \leq (2\zeta + 1) \sum_{x_i \in \mathcal{B}'} h(\text{diam } B(x_i, s_i)).$$

Therefore

$$\begin{aligned} \tilde{R}^h_\delta(S) &\leq (2\zeta + 1) R^h_\delta(S), \\ \tilde{R}^h_0(S) &\leq (2\zeta + 1) R^h_0(S). \end{aligned}$$

Since this is true for each subset S of X,

$$\tilde{\mathcal{R}}^h(S) \leq (2\zeta + 1)\mathcal{R}^h(S).$$

Together with Lemma 1.5.1, this proves the result.

Lemma 1.5.7 *For any Hausdorff function h there is a metric space X with*

$$\mathcal{R}^h(X) = \tilde{\mathcal{R}}^h(X) = \infty, \ \mathcal{P}^h(X) = 0.$$

Proof. Given a Hausdorff function h, we choose an increasing sequence of integers $(n_i)_i$ and a sequence $(d_i)_i$ of real numbers decreasing to 0, such that the points d_k are points of left continuity of h, and

$$\begin{aligned} \lim_{k \to \infty} n_1 \dots n_k h(d_k) &= \infty, \\ \sum_{k=2}^{\infty} n_1 \dots n_{k-1} h(d_k) &< \infty. \end{aligned}$$

Take X to be the space

$$X = \{(i_1, i_2, \dots) : 1 \leq i_j \leq n_j \text{ for each } j \geq 1\},$$

with a metric on X defined thus:

If $(i) = (i_1, i_2, \ldots)$, and $(j) = (j_1, j_2, \ldots)$ are distinct points of X, and l is the least index for which $i_l \neq j_l$, then $\operatorname{dist}((i),(j)) = d_l$.

Write

$$X_{i_1,\ldots i_k} = \{(i_1, \ldots i_k, j_{k+1}, j_{k+2}, \ldots) : 1 \leq j_{k+l} \leq n_{k+l} \text{ for each } l \geq 1\}.$$

Then these sets are the nonempty balls of the metric space X.

We now show that $\tilde{\mathcal{R}}^h(X) = \mathcal{R}^h(X) = \infty$.

Fix $p \geq 1$ and let $k \geq p+1$. Then if $d_{k+1} \leq d < d_k$, there are precisely $n_{p+1} \ldots n_k$ disjoint balls of radius d in $X_{i_1,\ldots i_p}$. These balls constitute both a d_k-packing and a d_k-Besicovitch packing of $X_{i_1,\ldots i_p}$, and since d_k is a point of left continuity of h,

$$R^h_{d_k}\left(X_{i_1,\ldots i_p}\right) = \tilde{R}^h_{d_k}\left(X_{i_1,\ldots i_p}\right) \geq n_{p+1} \ldots n_k h(d_k-) = n_{p+1} \ldots n_k h(d_k),$$

which increases without bound as k increases, by choice of $(n_i)_i$ and $(\sigma_i)_i$.

By Lemmas 1.4.2 and 1.4.5, for each $S \subseteq X$,

$$\begin{aligned}
\mathcal{R}^h(S) &= \inf\left\{\sum_1^\infty R^h_0(C_i) : S \subseteq \bigcup_1^\infty C_i, \text{ and } C_i \text{ are closed subsets of } S\right\}, \\
\tilde{\mathcal{R}}^h(S) &= \inf\left\{\sum_1^\infty \tilde{R}^h_0(C_i) : S \subseteq \bigcup_1^\infty C_i, \text{ and } C_i \text{ are closed subsets of } S\right\}.
\end{aligned}$$

We may use the Baire Category Theorem to see that if $\{C_i\}$ are such that $X \subseteq \bigcup_1^\infty C_i$, then there is i such that C_i contains an open set, and hence a set of the form $X_{i_1,\ldots i_p}$ for some $p \geq 1$, $1 \leq i_j \leq n_j$. This C_i then satisfies $R^h_0(C_i) = \tilde{R}^h_0(C_i) = \infty$.

It remains to show that $\mathcal{P}^h(X) = 0$. For each $j \geq 1$, there are precisely $n_1 \ldots n_{j-1}$ disjoint balls of diameter d_j in X, so clearly, $P^h_{d_k}(X) \leq \sum_{j=k}^\infty n_1 \ldots n_{j-1} h(d_j)$. Therefore, $P^h_0(X) = \mathcal{P}^h(X) = 0$ as claimed.

Chapter 2

Subsets of positive finite measure - a positive result

2.1 Introduction

The work presented in this chapter appears in [JoPr]. We show that analytic sets of infinite Besicovitch-type radius-based packing measure contain subsets of finite positive packing measure. We also indicate how this result carries over to the measure $\mathcal{R}^h$ in the case of Hausdorff functions h satisfying a doubling condition. We recall that a metric space is said to be analytic if it is a continuous image of the set $\mathcal{N}$ of infinite sequences of natural numbers (with its product topology).

2.2 Properties of the measure $\tilde{\mathcal{R}}^h$

Lemma 2.2.1 *If $\delta > 0$, $\eta > 0$ and Q is a finite subset of a metric space M, then there is $\sigma > 0$ such that $\tilde{R}^h_\delta(R) > \tilde{R}^h_\delta(Q) - \eta$ whenever $R \subset M$ meets* *each of the balls $B(x, \sigma)$, $x \in Q$.*

Proof. Let $\{B(v, r(v)) : v \in V\}$ be a δ-Besicovitch packing of Q such that $\sum_{v \in V} h(r(v)) > \tilde{R}^h_\delta(Q) - \eta$. Since the balls $B(v, r(v))$ are closed, there is $\sigma > 0$ such that, if $v, w \in V$, and $x(v) \in B(v, \sigma)$, $x(w) \in B(w, \sigma)$, then $x(w) \notin B(x(v), r(v))$. So if $R \subset M$ meets each of the balls $B(x, \sigma)$, $x \in Q$, there is a packing $\{B(w, r(w)) : w \in W\}$ of R such that $\sum_{w \in W} h(r(w)) > \tilde{R}^h_\delta(Q) - \eta$. So this σ has the desired property.

Lemma 2.2.2 *If $\delta > 0$ and if M is a finite metric space containing at least two points, with* $\mathrm{diam}(M) < \delta$ *and $\tilde{R}^h_\delta(N) < \tilde{R}^h_\delta(M)$ for every proper subset N of M,* and $r(x, M) = \min\{\mathrm{dist}(x, y) : y \in M, y \neq x\}$, then

$$\tilde{R}^h_\delta(M) = \sum_{x \in M} h(r(x, M)-).$$

Proof. For any $0 < r_x < r(x, M)$ the family $\{B(x, r_x) : x \in M\}$ is a δ-Besicovitch packing of M. Hence $\tilde{R}^h_\delta(M) \geq \sum_{x \in M} h(r_x)$, which shows that

$$\tilde{R}^h_\delta(M) \geq \sum_{x \in M} h(r(x, M)-).$$

Let α be the maximum of $\tilde{R}^h_\delta(N)$, where N is a proper subset of M.

Every δ-Besicovitch packing $\{B(x_i, r_i)\}$ of M which does not have balls centred at each $x \in M$ is a δ-Besicovitch packing of some proper subset of M and therefore verifies

$$\sum_i h(r_i) \leq \alpha.$$

Since $\alpha < \tilde{R}^h_\delta(M)$, the supremum defining $\tilde{R}^h_\delta(M)$ is realized by δ-Besicovitch packings $\{B(x_i, r_i)\}$ of M having balls centred at each point of M. Since for such Besicovitch packings we clearly have

$$\sum_i h(r_i) \leq \sum_{x \in M} h(r(x, M)-),$$

we conclude that

$$\tilde{R}^h_\delta(M) = \sum_{x\in M} h(r(x,M)-).$$

2.3 The inductive construction

Lemma 2.3.1 *Suppose that M is a metric space, $\varepsilon > 0$, $0 < \alpha < \infty$, and that $\delta > 0$ satisfies $h(\delta) < \min\{\varepsilon, \alpha\}$. Suppose further that $x_0 \in M$ is such that $\tilde{R}^h_\delta(B(x_0, \delta/5)) > \alpha$.*

Then there exist a finite subset K of $B(x_0, \delta/4)$ containing at least two points and a positive number $\sigma < \frac{1}{4}\min\{r(z,K) : z \in K\}$ such that

(i) $\alpha < \tilde{R}^h_\delta(K) = \sum_{z\in K} h(r(z,K)-) < \alpha + \varepsilon$, *and*

(ii) *whenever $S \subset B(K,\sigma)$ and $T \subset B(K,\sigma) \cap B(S, 2\sigma)$ have the property that, for each $s \in S$, the set $T \cap B(s, 2\sigma)$ is either empty or a singleton, then $\tilde{R}^h_\delta(S) > \tilde{R}^h_\delta(T) - \varepsilon$.*

Proof. Write $G = U(x_0, \delta/4)$. Noting that $\tilde{R}^h_\delta(G) \geq \tilde{R}^h_\delta(B(x_0, \delta/5)) > \alpha$, we use the definition of $\tilde{R}^h_\delta(G)$ to infer that there are finite subsets C of G with $\tilde{R}^h_\delta(C) > \alpha$. So we may choose a finite subset L of G with the least possible number of elements, say m, such that $\tilde{R}^h_\delta(L) > \alpha$. Then $m \geq 2$, since otherwise $\tilde{R}^h_\delta(L) \leq h(\delta) < \alpha$.

Let $\mathcal{C}$ denote the family of all m-element subsets of G. We first show that, whenever $C \in \mathcal{C}$ satisfies $\tilde{R}^h_\delta(C) > \alpha$, then

$$\alpha < \tilde{R}^h_\delta(C) = \sum_{z\in C} h(r(z,C)-) < \alpha + \varepsilon.$$

Indeed, each proper subset N of C satisfies $\tilde{R}^h_\delta(N) \le \alpha < \tilde{R}^h_\delta(C)$ since it has fewer than m elements. Hence Lemma 2.2.2 implies that

$$\tilde{R}^h_\delta(C) = \sum_{z \in C} h(r(z, C)-).$$

Picking any $c \in C$ and observing that $\tilde{R}^h_\delta(C \setminus \{c\}) < \alpha$, we estimate

$$\tilde{R}^h_\delta(C) \le \tilde{R}^h_\delta(C \setminus \{c\}) + \tilde{R}^h_\delta(\{c\}) \lneq \alpha + h(\delta) < \alpha + \varepsilon,$$

which proves that

$$\alpha < \tilde{R}^h_\delta(C) < \alpha + \varepsilon.$$

Write

$$\gamma = 2^{-m} \left(\tilde{R}^h_\delta(L) - \alpha\right) \alpha^{-1}$$

and define, for every $C \in \mathcal{C}$,

$$F(C) = \tilde{R}^h_\delta(C) + \gamma \sum_{R \subset C, R \neq C} \tilde{R}^h_\delta(R).$$

Observing that $F(C) \le m\,h(\delta) + (2^m - 1)\gamma\alpha$, we see that

$$\tau = \sup\{F(C) : C \in \mathcal{C}\}$$

is finite. Let $\omega = \min\{\alpha\gamma, \varepsilon\gamma/4\}$ and let $K \in \mathcal{C}$ be such that $F(K) > \tau - \omega$. Then

$$\begin{aligned} \tilde{R}^h_\delta(K) &= F(K) - \gamma \sum_{R \subset K, R \neq K} \tilde{R}^h_\delta(R) > \tau - \omega - (2^m - 1)\gamma\alpha \\ &\ge F(L) + \gamma\alpha - \omega - \left(\tilde{R}^h_\delta(L) - \alpha\right) \ge \alpha. \end{aligned}$$

So (i) holds for K.

Let $\eta = \frac{1}{4}\varepsilon\gamma(1 + 2^m\gamma)^{-1}$. Using Lemma 2.2.1, we find

$$0 < \sigma < \tfrac{1}{4} \min\{r(z, K) : z \in K\}$$

such that $B(K,\sigma) \subset G$, and

$$\tilde{R}^h_\delta(R) \geq \tilde{R}^h_\delta(Q) - \eta$$

whenever $Q \subset K$ and $R \subset M$ meets each of the balls $B(x,\sigma)$, $(x \in Q)$. Writing, for each $R \subset M$,

$$K(R) = \{x \in K : R \cap B(x,\sigma) \neq \emptyset\},$$

we observe that this implies that

$$\tilde{R}^h_\delta(R) \geq \tilde{R}^h_\delta(K(R)) - \eta$$

for every $R \subset B(K,\sigma)$.

To prove (ii), let $S \subset B(K,\sigma)$ and $T \subset B(K,\sigma) \cap B(S,2\sigma)$ have the property that, for each $s \in S$, the set $T \cap B(s,2\sigma)$ is either empty or a singleton. Let $C = T \cup (K \setminus K(T))$. Then C belongs to $\mathcal{C}$ and the correspondence $R \subset C \mapsto K(R)$ is a bijection between subsets of C and subsets of K. Thus

$$\begin{aligned}
\tau &\geq F(C) \geq \tilde{R}^h_\delta(C) + \gamma \tilde{R}^h_\delta(T) + \gamma \sum_{R \subset C, R \neq C, R \neq T} \tilde{R}^h_\delta(R) \\
&> \tilde{R}^h_\delta(K(C)) - \eta + \gamma \tilde{R}^h_\delta(T) + \gamma \sum_{R \subset C, R \neq C, R \neq T} \left(\tilde{R}^h_\delta(K(R)) - \eta\right) \\
&\geq \tilde{R}^h_\delta(K) + \gamma \tilde{R}^h_\delta(T) + \gamma \sum_{R \subset K, R \neq K, R \neq K(T)} \tilde{R}^h_\delta(R) - (1 + 2^m\gamma)\eta \\
&= F(K) + \gamma\left(\tilde{R}^h_\delta(T) - \tilde{R}^h_\delta(K(T))\right) - \varepsilon\gamma/4 \\
&> \tau + \gamma\left(\tilde{R}^h_\delta(T) - \tilde{R}^h_\delta(K(T))\right) - \varepsilon\gamma/4 - \omega \\
&\geq \tau + \gamma\left(\tilde{R}^h_\delta(T) - \tilde{R}^h_\delta(K(T))\right) - \varepsilon\gamma/2.
\end{aligned}$$

Hence $\tilde{R}^h_\delta(T) - \tilde{R}^h_\delta(K(T)) < \varepsilon/2$, which, since $K(S) \supset K(T)$, implies that

$$\tilde{R}^h_\delta(S) \geq \tilde{R}^h_\delta(K(S)) - \eta > \tilde{R}^h_\delta(K(T)) - \eta > \tilde{R}^h_\delta(T) - \varepsilon/2 - \eta > \tilde{R}^h_\delta(T) - \varepsilon.$$

Lemma 2.3.2 *Let M be a metric space such that $\tilde{R}_0^h(G) = \infty$ for every non-empty open subset G of M, and let $\varepsilon > 0$. Suppose further that L is a finite subset of M having at least two elements.*

Then there is $\Delta > 0$ such that for every $0 < \delta < \Delta$ one can find a finite subset K of $B(L, \delta/4)$, and a positive number $\sigma < \frac{1}{4}\min\{r(z,K) : z \in K\}$ such that

(i) *for each $x \in L$ the set $K \cap B(x, \delta/4)$ has at least two elements,*

(ii) $\sum_{x\in L} h(r(x,L)-) < \tilde{R}_\delta^h(K) = \sum_{z\in K} h(r(z,K)-) < \sum_{x\in L} h(r(x,L)-) + \varepsilon,$

(iii) *whenever $S \subset B(K,\sigma)$ and $T \subset B(K,\sigma) \cap B(S, 2\sigma)$ have the property that, for each $s \in S$, the set $T \cap B(s, 2\sigma)$ is either empty or a singleton, then $\tilde{R}_\delta^h(S) > \tilde{R}_\delta^h(T) - \varepsilon$,*

(iv) *if $S \subset B(K,\sigma)$ meets each ball $B(z,\sigma)$ ($z \in K$) in at most one point, then for every $\eta \geq \max\{r(x,L) : x \in L\}$ there is $T \subset S$ meeting each ball $B(x,\delta)$, ($x \in L$) in at most one point and such that $\tilde{R}_\eta^h(S) \leq \tilde{R}_\eta^h(T) + \varepsilon$.*

Proof. Let $\omega = \frac{\varepsilon}{6|L|}$, where $|L|$ denotes the number of elements of the set L. Let $0 < \Delta < \frac{1}{2}\min\{r(x,L) : x \in L\}$ be such that

$$h(\Delta) < \min\{\omega, \min\{h(r(x,L)-) : x \in L\}\},$$

and such that, for every $x \in L$,

$$h(r(x,L) - \Delta) > h(r(x,L)-) - \omega.$$

Assuming that $0 < \delta < \Delta$, we use, for each $x \in L$, Lemma 2.3.1 with ε replaced by ω, $\alpha = h(r(x,L)-)$, and $x_0 = x$, to find a finite subset K_x of $B(x, \delta/4)$ having at least two elements, and a positive number $\sigma_x < \frac{1}{4}\min\{r(z,K_x) : z \in K_x\}$ such that

(a) $h(r(x,L)-) < \tilde{R}^h_\delta(K_x) = \sum_{z\in K_x} h(r(z,K_x)-) < h(r(x,L)-) + \omega$, and

(b) whenever $S \subset B(K,\sigma_x)$ and $T \subset B(K,\sigma_x) \cap B(S, 2\sigma_x)$ have the property that, for each $s \in S$, the set $T \cap B(s, 2\sigma_x)$ is either empty or a singleton, then $\tilde{R}^h_\delta(S) > \tilde{R}^h_\delta(T) - \omega$.

We prove that the statement of the lemma holds with

$$K = \bigcup_{x\in L} K_x.$$

The statements that $K \subset B(L, \delta/4)$ and (i) are clear. Moreover, (ii) follows immediately from (a), since $r(z,K) = r(z,K_x)$ whenever $z \in K_x$ and since Lemma 1.4.1(iv) implies that $\tilde{R}^h_\delta(K) = \sum_{x\in L} \tilde{R}^h_\delta(K_x)$.

Let $0 < \sigma < \frac{1}{3}\min\{\sigma_x : x \in L\}$ be so small that $B(K,\sigma) \subset B(L,\delta/3)$ and $\sigma < \frac{1}{6}\min\{r(z,K) : z \in K\}$. Then Lemma 1.4.1(iv) gives that, for each $R \subset B(K,\sigma)$,

$$\tilde{R}^h_\delta(R) = \sum_{x\in L} \tilde{R}^h_\delta(R \cap B(x,\delta)),$$

which shows that (iii) follows immediately from (b).

Finally, to prove (iv), let $\eta \geq \max\{r(x,L) : x \in L\}$ and let $S \subset B(K,\sigma)$ meet each ball $B(z,\sigma)$ $(z \in K)$ in at most one point. We find an η-Besicovitch packing $\{B(v,r(v)) : v \in V\}$ of S such that $\sum_{v\in V} h(r(v)) > \tilde{R}^h_\delta(S) - \varepsilon/2$. Since $V \subset S \subset B(K,\sigma) \subset B(L,\delta/3)$ and since $\delta < \frac{1}{2}\min\{r(x,L) : x \in L\}$, there is $T \subset V$ such that, for every $v \in V$, the set $T \cap B(v,\delta)$ has precisely one point. Let T_0 be the set of those $t \in T$ for which $V \cap B(t,\delta)$ contains only one point. We define $s(t) = r(t)$ if $t \in T_0$ and $s(t) = r(x,L) - \delta$ if $t \in T_1 = T \setminus T_0$, $x \in L$ and $t \in B(x,\delta)$.

Let $t \in T_1$ and $x \in L$ be such that $t \in B(x,\delta)$. Using the fact that

$$V \cap B(t,\delta) \subset B(L,\delta/3) \cap B(x,\delta/3) = B(x,\delta/3),$$

and that $V \cap B(t,\delta)$ has at least two points, we infer that $r(v) < \delta$ for $v \in V \cap B(t,\delta)$. So $\{B(v, r(v)) : v \in V \cap B(t,\delta)\}$ is a δ-Besicovitch packing of $V \cap B(t,\delta)$. Using (b) with S replaced by K_x and T replaced by $V \cap B(t,\delta)$, we infer that

$$\tilde{R}^h_\delta(V \cap B(t,\delta)) < \tilde{R}^h_\delta(K_x) + \omega < h(r(x,L)-) + 2\omega.$$

Hence

$$\sum_{v \in V \cap B(t,\delta)} h(r(v)) < h(r(x,L)-) + 2\omega < h(r(x,L) - \delta) + 3\omega = h(s(t)) + 3\omega.$$

Since this inequality obviously holds also if $t \in T_0$, and since $\{B(t, s(t)) : t \in T\}$ is an η-Besicovitch packing of T, we obtain

$$\begin{aligned} \tilde{R}^h_\eta(S) &< \sum_{v \in V} h(r(v)) + \varepsilon/2 = \sum_{t \in T} \sum_{v \in V \cap B(t,\delta)} h(r(v)) + \varepsilon/2 \\ &< \sum_{t \in T} (h(s(t)) + 3\omega) + \varepsilon/2 \le \sum_{t \in T} h(s(t)) + \varepsilon \\ &\le \tilde{R}^h_\eta(T) + \varepsilon. \end{aligned}$$

Lemma 2.3.3 *Let X be a non-empty metric space and let φ be a continuous mapping of a closed subset Z of $\mathcal{N}$ onto X. Suppose that $\tilde{R}^h_0(\varphi(G)) = \infty$ for each non-empty relatively open subset G of Z. Then there exists a compact subset K of X with $0 < \tilde{\mathcal{R}}^h(K) < \infty$.*

Proof. Let $Z_{n_0,\ldots,n_p}$ be the set of those points $(m_0, m_1, \ldots) \in Z$ such that $m_0 \le n_0, \ldots, m_p \le n_p$, and let $F_{n_0,\ldots,n_p}$ be the image of $Z_{n_0,\ldots,n_p}$ under φ.

By induction we will construct a sequence $K_0, K_1, \ldots$ of finite subsets of X having at least two elements, sequences $\delta_0, \delta_1, \ldots$ and $\sigma_0, \sigma_1, \ldots$ of positive numbers and a sequence $n_0, n_1, \ldots$ of natural numbers such that, for each $j = 0, 1, \ldots,$

(i) $K_j \subset F_{n_0,\dots,n_j}$,

(ii) $\delta_{j+1} < \sigma_j < \frac{1}{4}\min\{r(x, K_j) : x \in K_j\} < \frac{1}{4}\delta_j$,

(iii) $1 < \tilde{R}^h_{\delta_j}(K_j) = \sum_{x\in K_j} h(r(x, K_j)-)$,

(iv) $B(K_{j+1}, \sigma_{j+1}) \subset B(K_j, \sigma_j)$,

(v) $K_{j+1} \cap B(x, \sigma_j) \neq \emptyset$ for each $x \in K_j$,

(vi) whenever $S \subset B(K_j, \sigma_j) \cap F_{n_0,\dots,n_j}$ and $T \subset B(K_j, \sigma_j) \cap B(S, 2\sigma_j) \cap F_{n_0,\dots,n_j}$ have the property that, for each $s \in S$, the set $T \cap B(s, 2\sigma_j)$ is either empty or a singleton, then $\tilde{R}^h_{\delta_j}(S) > \tilde{R}^h_{\delta_j}(T) - 2^{-j}$, and

(vii) if $S \subset B(K_{j+1}, \sigma_{j+1}) \cap F_{n_0,\dots,n_{j+1}}$ meets each ball $B(z, \sigma_{j+1})$ ($z \in K_{j+1}$) in at most one point, then for every $\eta \geq \max \delta_j$ there is $T \subset S$ meeting each ball $B(x, \sigma_j)$ ($x \in K_j$) in at most one point and such that $\tilde{R}^h_\eta(S) \leq \tilde{R}^h_\eta(T) + 2^{-j}$.

The inductive construction starts by picking an arbitrary $x_0 \in X$, choosing δ_0 such that $h(\delta_0) < 1$, using Lemma 2.3.1 to find K_0 and σ_0 such that all the statements pertinent for $j = 0$ hold, and then choosing n_0 such that $K_0 \subset F_{n_0}$. Assuming that, for some j, the sets K_j and numbers δ_j, σ_j, and n_j have already been defined, we choose $0 < \delta_{j+1} < \sigma_j$ so small that Lemma 2.3.2 can be used with $M = F_{n_0,\dots,n_j}$ and $L = K_j$. This Lemma then provides us with K_{j+1} and σ_{j+1} for which all the requirements hold; n_{j+1} is chosen so that $K_{j+1} \subset F_{n_0,\dots,n_{j+1}}$.

Let $K = \bigcap_{k=0}^{\infty} \mathrm{Clos}\left(\bigcup_{p\geq k} K_p\right)$. Using (i), continuity of φ and compactness of the set $\bigcap_{p=0}^{\infty} Z_{n_0,\dots,n_p}$, we infer that K is a compact subset of $\bigcap_{p=0}^{\infty} F_{n_0,\dots,n_p}$.

Now $\sigma_j < \frac{1}{4}\min\{r(x, K_j) : x \in K_j\}$, so we may infer from (v) and (iv) that $K_i \cap B(x, \sigma_j) \neq \emptyset$ for each $i \geq j$, each $x \in K_j$. So, for every k,

$\mathrm{Clos}\left(\bigcup_{p\geq k} K_p\right) \cap B(x,\sigma_j) \neq \emptyset$, which, because of (i), continuity of φ and compactness of $\bigcap_{p=0}^{\infty} Z_{n_0,\dots,n_p}$, shows that $K \cap B(x,\sigma_j) \neq \emptyset$ for each $x \in K_j$.

Since clearly $K \subset B(K_k,\sigma_k)$ for each $k = 0,1,\dots$, we infer from (vi) that for every $S \subset K$

$$\tilde{R}^h_{\delta_k}(K_k \cap B(S,\sigma_k)) \leq \tilde{R}^h_{\delta_k}(S) + 2^{-k}. \tag{2.1}$$

Let $k = 0,1,\dots$ and let $\{B(x_q,r_q) : q \in Q\}$ be an arbitrary δ_k-Besicovitch packing of a subset S of K. Choose $l > k$ such that $r_q > 2\sigma_l$ for each $q \in Q$. Let $T_l = \{x_q : q \in Q\}$. Then T_l meets each ball $B(x,\sigma_l)$ $(x \in K_l)$ in at most one point. We use (vii) to define, by backward induction, sets $T_l \supset T_{l-1} \supset \cdots \supset T_k$ such that, for each $j = k, k+1, \dots, l$, the set T_j meets each ball $B(x,\sigma_j)$ $(x \in K_j)$ in at most one point and such that $\tilde{R}^h_{\delta_k}(T_{j+1}) \leq \tilde{R}^h_{\delta_k}(T_j) + 2^{-j}$ for each $j = k,\dots,l-1$. Hence

$$\tilde{R}^h_{\delta_k}(T_l) \leq \tilde{R}^h_{\delta_k}(T_k) + 2^{-k+1}.$$

Defining $T = \{x \in K_k : T_k \cap B(x,\sigma_k) \neq \emptyset\}$, we conclude from (vi) that

$$\sum_{q\in Q} h(r_q) \leq \tilde{R}^h_{\delta_k}(T_l) \leq \tilde{R}^h_{\delta_k}(T_k) + 2^{-k+1} \leq \tilde{R}^h_{\delta_k}(T) + 2^{-k+2}.$$

Observing that $T = K_k \cap B(S,\sigma_k)$, we therefore have

$$\tilde{R}^h_{\delta_k}(S) \leq \tilde{R}^h_{\delta_k}(K_k \cap B(S,\sigma_k)) + 2^{-k+2} \tag{2.2}$$

for every $S \subset K$.

Using (2.1) with $S = K$ and (iii), we get that $\tilde{R}^h_{\delta_k}(K) \geq \tilde{R}^h_{\delta_k}(K_k) - 2^{-k} > 1 - 2^{-k}$. Since $\delta_k \to 0$ as $k \to \infty$, this shows that $\tilde{R}^h_0(K) \geq 1$. On the other hand, (2.2) with $S = K$ and $k = 0$ gives $\tilde{R}^h_0(K) \leq \tilde{R}^h_{\delta_0}(K) \leq \tilde{R}^h_{\delta_0}(K_0) + 4 < \infty$. Consequently,

$$0 < \tilde{R}^h_0(K) < \infty.$$

For arbitrary $S \subset K$ we use (2.2) with S replaced by $K \cap B(S, 2\sigma_k)$ to infer that

$$\tilde{R}_0^h(K \cap B(S, 2\sigma_k)) \le \tilde{R}_{\delta_k}^h(K \cap B(S, 2\sigma_k)) \le \tilde{R}_{\delta_k}^h(K_k \cap B(S, \sigma_k)) + 2^{-k+2}.$$

Hence (2.1) gives

$$\tilde{R}_0^h(K \cap B(S, 2\sigma_k)) \le \tilde{R}_{\delta_k}^h(S) + 2^{-k+3}$$

for any $k = 0, 1, \ldots$ and any $S \subset K$. Since $K \cap B(S, 2\sigma_k)$ ~~is~~ contains an open subset of K containing S, and since $\delta_k \to 0$, this shows that the assumptions of Lemma 1.4.6 are verified. Hence we may use it to conclude that

$$\tilde{\mathcal{R}}^h(K) = \tilde{R}_0^h(K).$$

2.4 The main results

Theorem 2.4.1 *Let X be an analytic metric space such that $\tilde{\mathcal{R}}^h(X) = \infty$. Then X has a compact subset K with $0 < \tilde{\mathcal{R}}^h(K) < \infty$.*

Proof. Suppose not. Let $\varphi : \mathcal{N} \to X$ be continuous and onto. Let

$$H = \bigcup \left\{ G \subset \mathcal{N} : G \text{ is open, } \tilde{\mathcal{R}}^h(\varphi(G)) = 0 \right\}.$$

Then $\tilde{\mathcal{R}}^h(\varphi(H)) = 0$, by the separability and metrizability of $\mathcal{N}$.

Let $Z = \mathcal{N} \backslash H$ and $Y = \varphi(Z)$. Then, for each non-empty open subset G of $\mathcal{N}$, either $\tilde{\mathcal{R}}^h(\varphi(Z \cap G)) = \infty$ or $Z \cap G = \emptyset$, because if $\tilde{\mathcal{R}}^h(\varphi(Z \cap G)) \ne \infty$, then

$$\tilde{\mathcal{R}}^h(\varphi(G)) \le \tilde{\mathcal{R}}^h(\varphi(G \cap Z)) + \tilde{\mathcal{R}}^h(\varphi(G \setminus Z)) = 0,$$

implying that $G \subseteq H$, and so $G \cap Z = \emptyset$.

Apply the previous lemma to the sets Z and Y, and the restriction of the function φ to Z, to find a compact subset K of Y with $\tilde{\mathcal{R}}^h(K)$ positive and finite, and hence a contradiction.

Theorem 2.4.2 *Let X be an analytic metric space such that $\mathcal{R}^h(X) = \infty$, and let h be a Hausdorff function satisfying a doubling condition. Then X has a compact subset K with $0 < \mathcal{R}^h(K) < \infty$.*

Proof. This follows from Theorem 2.4.1 and Lemma 1.5.2.

Theorem 2.4.3 *If X is a directionally limited analytic metric space, the function h satisfies a doubling condition, and $\mathcal{P}^h(X) = \infty$, then X has a compact subset K with $0 < \mathcal{P}^h(K) < \infty$.*

Proof. This follows from Theorem 2.4.1 and Lemma 1.5.5.

Chapter 3

Subsets of positive finite measure - a negative result

3.1 Introduction

This chapter deals with the work which appears in [JoH]. In Section 3.2 we fix a Hausdorff function h, and construct a compact metric space K of infinite diameter-based packing measure which has no subsets of positive finite measure. In Section 3.3 this construction is modified to deal with the case of certain Hausdorff functions h which do not satisfy a doubling condition, and the radius-based packing measure $\mathcal{R}^h$.

We now consider some of the properties of the measure $\mathcal{P}^h$ which mean we cannot guarantee the existence of a subset of positive finite measure of an analytic (or even a compact) metric space of infinite measure. In taking the diameters of the closed balls of a metric space rather than the radii one is restricted to values which are attained by the metric, whereas at every point there are, of course, balls of any positive radius. So if the metric on a space is extremely sparse, the possibilities for sums of the form $\sum h(\text{diam } B)$ are more restricted than those for sums of the form $\sum h(\text{radius } B)$.

If a set S has $\tilde{R}_0^h(S)$ positive and finite, then we may find a Besicovitch packing $\{B(x,r)\}$ of S such that $\sum h(r)$ closely approximates $\tilde{R}_0^h(S)$. The construction below is of a metric space which has different balls of the same diameter, even centred at the same point, which contain in some sense widely differing amounts of the space. The hope is that if $0 < \mathcal{P}^h(S) < \infty$, this will make it difficult to approximate $P_0^h(S)$ using some sum of the form $\sum h(\text{diam } B(x,r))$.

3.2 A counter-example for diameter-based packing measure

Fix $\sigma_1 > 0$, and choose an integer $n_1 \geq 16$ sufficiently large that h takes a value in the range

$$\left[5/2\ h(\sigma_1)(n_1 - 1)^{-2}, 3h(\sigma_1)n_1^{-2}\right]$$

at more than one point of $(0, \sigma_1/2)$. This is possible, since $h(0+) = 0$ and since the intervals

$$\left\{\left[5/2\ h(\sigma_1)(n - 1)^{-2}, 3h(\sigma_1)n^{-2}\right] : n \geq 16\right\}$$

overlap, and together cover the interval $(0, d)$, for some $d > 0$. So for each $N \geq 1$, there is $n > N$ such that the interval $[5/2\ h(\sigma_1)(n - 1)^{-2}, 3h(\sigma_1)n^{-2}]$ contains the image under h of more than one point.

Set $n_2 = n_1$, and choose $0 < \sigma_2 < \varepsilon_1 < \sigma_1/2$ so that h takes values in this range at σ_2 and ε_1. Let $\delta_1 = \sigma_1 - \varepsilon_1$.

In general, if σ_{k+1}, ε_k, δ_k and $n_{2k-1} = n_{2k}$ are chosen, we choose integers $n_{2k+1} = n_{2k+2}$ sufficiently large that

$$n_{2k+1} \geq 2^{k+4},$$

$$\left(\prod_{i=1}^{2k} n_i\right) h(\sigma_{k+1}) n_{2k+1}^{-1} < 2^{-k},$$

and h takes a value in the range

$$\left[5/2\ h(\sigma_{k+1})(n_{2k+1}-1)^{-2}, 3h(\sigma_{k+1})n_{2k+1}^{-2}\right]$$

at more than one point of $(0, \sigma_{k+1}/2)$. Then $0 < \sigma_{k+2} < \varepsilon_{k+1} < \sigma_{k+1}/2$ may be chosen so that h takes a value in this range at σ_{k+2} and ε_{k+1}. Let $\delta_{k+1} = \sigma_{k+1} - \varepsilon_{k+1}$.

These choices ensure that

(i) $\sigma_{k+1} = \delta_{k+1} + \varepsilon_{k+1}$,

(ii) $\varepsilon_{k+1} < \delta_{k+1}$,

(iii) $\sigma_{k+2} < \varepsilon_{k+1}$,

(iv) $h(\sigma_{k+2})h(\sigma_{k+1})^{-1}n_{2k+1}^2 \leq h(\varepsilon_{k+1})h(\sigma_{k+1})^{-1}n_{2k+1}^2 \leq 3$,

(v) $h(\sigma_{k+2})h(\sigma_{k+1})^{-1}(n_{2k+1}-1)^2 \geq 5/2$,

(vi) $\left(\prod_{i=1}^{2k+1} n_i\right)\left(h\left(\varepsilon_{k+1}\right) - h\left(\sigma_{k+2}\right)\right) < 2^{-k+1}$

The sequences thus inductively chosen clearly satisfy

(vii) $\sum_{k=1}^{\infty} n_k^{-1} < \infty$, (since $n_{2k+1} = n_{2k+2} \geq 2^{k+4}$),

(viii) $\sum_{k=1}^{\infty} \frac{h(\sigma_{k+1})n_{2k}}{h(\sigma_k)} < \infty$, (using (iv) and (vii)),

(ix) $\lim_{k\to\infty}\left(\prod_{i=1}^{2k} n_i\right) h\left(\sigma_{k+1}\right) = \infty$, (using (v)).

We take K to be the space

$$K = \{(i_1, i_2, \ldots) : \text{ for each } j \geq 1, 1 \leq i_j \leq n_j\},$$

and provide K with a metric thus; if $(i) = (i_1, i_2, \ldots)$, and $(j) = (j_1, j_2, \ldots)$ are distinct points of K, and l is the least index for which $i_l \neq j_l$, then

$$\begin{aligned} \operatorname{dist}((i),(j)) &= \varepsilon_{\frac{l}{2}} \text{ if } l \text{ is even,} \\ \operatorname{dist}((i),(j)) &= \delta_{\frac{l+1}{2}} \text{ if } l \text{ is odd, and } i_{l+1} = j_{l+1}, \\ \operatorname{dist}((i),(j)) &= \sigma_{\frac{l+1}{2}} \text{ if } l \text{ is odd, and } i_{l+1} \neq j_{l+1}. \end{aligned}$$

It is not hard to see that this is indeed a metric on K, and that K is a compact metric space, since it is complete and totally bounded; only the triangle inequality requires verification.

Suppose $(i), (j)$ and (k) are three distinct points of K. First, suppose $\operatorname{dist}((i),(k)) = \varepsilon_{\frac{l}{2}}$ for some even l. Then $(i_1, \ldots i_{l-1}) = (k_1, \ldots k_{l-1})$, and $i_l \neq k_l$. If one of $\operatorname{dist}((i),(j))$ or $\operatorname{dist}((j),(k)) \geq \varepsilon_{\frac{l}{2}}$, then the triangle inequality certainly holds. If not, then $(j_1, \ldots j_l) = (i_1, \ldots i_l)$, and $(j_1, \ldots j_l) = (k_1, \ldots k_l)$, implying that $i_l = k_l$, which is a contradiction.

The case $\operatorname{dist}((i),(k)) = \delta_{\frac{l+1}{2}}$ for odd l is similar.

If $\operatorname{dist}((i),(k)) = \sigma_{\frac{l+1}{2}}$ for odd l, then $(i_1, \ldots i_{l-1}) = (k_1, \ldots k_{l-1}), i_l \neq k_l$, and $i_{l+1} \neq k_{l+1}$. If one of $\operatorname{dist}((i),(j))$ or $\operatorname{dist}((j),(k)) \geq \sigma_{\frac{l+1}{2}}$, then the triangle inequality holds. Otherwise, $\operatorname{dist}((i),(j)) < \sigma_{\frac{l+1}{2}}$ so either $\operatorname{dist}((i),(j)) = \delta_{\frac{l+1}{2}}$, that is, $(j_1, \ldots j_{l-1}) = (i_1, \ldots i_{l-1}), i_l \neq j_l$, and $i_{l+1} = j_{l+1}$, or $(j_1, \ldots j_l) = (i_1, \ldots i_l)$; and similarly for $\operatorname{dist}((j),(k))$.

We cannot have $i_l = k_l$, so one of $\operatorname{dist}((i),(j)) = \delta_{\frac{l+1}{2}}$ or $\operatorname{dist}((j),(k)) = \delta_{\frac{l+1}{2}}$. Without loss of generality, suppose the former. Then $i_{l+1} = j_{l+1}$. Since $i_{l+1} \neq k_{l+1}$, we must have $j_{l+1} \neq k_{l+1}$, and so $\operatorname{dist}((j),(k)) = \varepsilon_{\frac{l+1}{2}}$. But, by choice, $\sigma_{\frac{l+1}{2}} = \delta_{\frac{l+1}{2}} + \varepsilon_{\frac{l+1}{2}}$, and so the triangle inequality holds.

If we write

$$K_{i_1,\dots i_k} = \{(i_1, \dots i_k, j_{k+1}, j_{k+2}, \dots) : 1 \leq j_{k+l} \leq n_{k+l} \text{ for each } l \geq 1\},$$

then, for (i) in K,

$$\begin{aligned}
&B((i), \varepsilon_k) = K_{i_1,\dots i_{2k-1}},\\
&B((i), \sigma_k) = K_{i_1,\dots i_{2k-2}},\\
&B((i), \delta_k) = K_{i_1,\dots i_{2k-1}} \cup \bigcup \left\{K_{i_1,\dots i_{2k-2},j,i_{2k}} : 1 \leq j \leq n_{2k-1}, j \neq i_{2k-1}\right\},\\
&\operatorname{diam}(B((i), \varepsilon_k)) = \varepsilon_k,\\
&\operatorname{diam}(B((i), \sigma_k)) = \operatorname{diam}(B((i), \delta_k)) = \sigma_k.
\end{aligned}$$

These balls are the only nonempty balls in K.

The construction described above is a variation on a familiar theme. Suppose we have sequences $(n_j)_j$ and $(d_j)_j$, with $d_j \searrow 0$. Consider the following metric space;

$$M = \{(i_1, i_2, \dots) : 1 \leq i_j \leq n_j \text{ for each } j \geq 1\},$$

with a metric on M defined thus:

If $(i) = (i_1, i_2, \dots)$, and $(j) = (j_1, j_2, \dots)$ are distinct points of M, and l is the least index for which $i_l \neq j_l$, then $\operatorname{dist}((i), (j)) = d_l$.

We can think of this space as constructed by dividing the space into *islands*, that is, into some finite number of disjoint parts which satisfy the condition that everything on one island is at some fixed distance from everything on any other island. Then each of these islands is subdivided into smaller *sub-islands*, which are at some new and smaller distance from each other sub-island of the same island, and so on. Clearly the balls of this metric space are precisely the sets $M_{i_1,\dots i_k}$, that is, the islands.

To get the metric space K described above, we now alter this construction by taking two steps at a time, that is, we divide K up into n_1 islands and then each of these into n_2 sub-islands. The metric this time is such that each sub-island has a "partner" sub-island on each other island, which is slightly nearer to it than are the other sub-islands of that island, and the distance between sub-islands of the same island is chosen to make the triangle inequality exact. And so on.

Lemma 3.2.1 $\mathcal{P}^h(K) = \infty$.

Proof. Fix $p \geq 1$ and let $1 \leq i_1 \leq n_1, \ldots, 1 \leq i_p \leq n_p$. Then, for $k \geq p+1$, $P^h_{\sigma_k}\left(K_{i_1,\ldots i_p}\right) \geq n_{p+1}\ldots n_{2k-2}h(\sigma_k)$. To see this, note that $B\left((i), \sigma_k\right) = K_{i_1,\ldots i_{2k-2}}$, and so the number of distinct balls of radius (and diameter) σ_k centred in $K_{i_1,\ldots i_p}$ is precisely $n_{p+1}\ldots n_{2k-2}$. So $P^h_{\sigma_k}\left(K_{i_1,\ldots i_p}\right) \geq n_{p+1}\ldots n_{2k-2}h(\sigma_k)$, which increases without bound as k increases, by choice of $(n_i)_i$ and $(\sigma_i)_i$, so $P^h_0\left(K_{i_1,\ldots i_p}\right) = \infty$.

The metric on K satisfies the conditions of Lemma 1.4.5, so we see that

$$\mathcal{P}^h(K) = \inf\left\{\sum_1^\infty P^h_0(C_i) : K \subseteq \bigcup_1^\infty C_i, \text{ and } C_i \text{ are closed subsets of } K\right\}.$$

If $K \subseteq \bigcup_1^\infty C_i$ for some collection $\{C_i\}$ of closed sets, then by the Baire Category Theorem there is i such that C_i contains a nonempty open set, and therefore some set $K_{i_1,\ldots i_p}$, and so satisfies $P^h_0(C_i)$ infinite. The result follows.

Lemma 3.2.2 *If $S \subseteq K$ satisfies $0 < P^h_0(S) < \infty$, then, for each $\xi > 0$ and $\alpha > 0$, there is $\eta > 0$ such that, if $\{B_i\}$ is an η-packing of S, then*

$$P^h_0\left(S \cap \bigcup\left\{B_i : h(\text{diam } B_i) \geq (1+\xi)P^h_0(S \cap B_i)\right\}\right) < \alpha.$$

Proof. Suppose not. Then there exist $\xi > 0$ and $\alpha > 0$ such that for every $\eta > 0$ we may find an η-packing $\{B_i\}$ of S such that

$$P_0^h\left(S \cap \bigcup\left\{B_i : h(\text{diam } B_i) \geq (1+\xi)P_0^h(S \cap B_i)\right\}\right) \geq \alpha.$$

Fix $\eta > 0$, and choose such a packing $\{B_i\}$. Write

$$S_1 = S \cap \bigcup\left\{B_j : h(\text{diam } B_j) \geq (1+\xi)P_0^h(S \cap B_j)\right\},$$

and

$$S_2 = S \setminus S_1.$$

Note that, due to the structure of K, the sets S_1 and S_2 are at a positive distance from each other and so by Lemma 1.4.1(v)

$$P_0^h(S) = P_0^h(S_1) + P_0^h(S_2).$$

Choose $\eta' < \eta$ sufficiently small that no ball of radius η' centred in S_2 intersects those balls B_j of the packing centred in S_1. Then, for any $\beta > 0$, we may choose $\{D_i\}$ an η'-packing of S_2 satisfying

$$\sum h(\text{diam } D_i) \geq (1-\beta)P_0^h(S_2).$$

Those B_j centred in S_1 together with $\{D_i\}$ form an η-packing of S, and β may be taken arbitrarily small, so

$$\begin{aligned} P_\eta^h(S) &\geq (1+\xi)P_0^h(S_1) + P_0^h(S_2) \\ &\geq P_0^h(S) + \xi\alpha. \end{aligned}$$

$\eta > 0$ was arbitrary, so $\xi\alpha = 0$, which is a contradiction.

Although Lemma 3.2.3 will not actually be used again, we provide it for the sake of completeness.

Lemma 3.2.3 *If $S \subseteq K$ satisfies $0 < P_0^h(S) < \infty$, then for each $\xi > 0$, $\alpha > 0$, and $\eta > 0$ there is an η-packing $\{B_i\}$ such that*

$$\sum \left\{h(\text{diam } B_i) : h(\text{diam } B_i) \leq (1-\xi)P_0^h(S \cap B_i)\right\} < \alpha.$$

Proof. Suppose not. Then, for some $\xi > 0$, $\alpha > 0$ and $\eta > 0$, every η-packing $\{B_i\}$ of S satisfies

$$\sum \left\{h(\text{diam } B_i) : h(\text{diam } B_i) \leq (1-\xi)P_0^h(S \cap B_i)\right\} \geq \alpha.$$

Write

$$\gamma = \min\left\{\frac{\xi}{2}, \frac{\alpha\xi}{4(1-\xi)P_0^h(S)}\right\}.$$

Then for each $\eta > 0$ we may find an η-packing $\{B_i\}$ of S such that

$$\sum h(\text{diam } B_i) \geq (1-\gamma)P_0^h(S).$$

Fix $\eta > 0$, let $\{B_i\}$ be such an η-packing, and write

$$S_1 = S \cap \bigcup\left\{B_j : h(\text{diam } B_j) \leq (1-\xi)P_0^h(S \cap B_j)\right\},$$

and

$$S_2 = S \cap \bigcup\left\{B_j : h(\text{diam } B_j) > (1-\xi)P_0^h(S \cap B_j)\right\}.$$

Then $\sum\{h(\text{diam } B_i) : B_i \cap S_1 \neq \emptyset\} \geq \alpha$. Due to the structure of K,

$$P_0^h(S_1 + S_2) = P_0^h(S_1) + P_0^h(S_2) = \sum_i P_0^h(S \cap B_i).$$

So

$$\begin{aligned}
&\sum h(\text{diam } B_j) \\
&\quad\geq (1-\gamma)P_0^h(S_1 \cup S_2) \\
&\quad= (1-\gamma)(P_0^h(S_1) + P_0^h(S_2)) \\
&\quad\geq (1-\gamma)\left((1-\xi)^{-1}\sum_{B_i \cap S_1 \neq \emptyset} h(\text{diam } B_i) + P_0^h(S_2)\right) \\
&\quad\geq \frac{1-\xi/2}{1-\xi}\sum_{B_i \cap S_1 \neq \emptyset} h(\text{diam } B_i) + \left(1 - \frac{\alpha\xi}{4(1-\xi)P_0^h(S)}\right)P_0^h(S_2).
\end{aligned}$$

$$\begin{aligned}
&\sum_{B_i \cap S_2 \neq \emptyset} h(\text{diam } B_i) \\
&\quad \geq \frac{\xi}{2(1-\xi)} \sum_{B_i \cap S_1 \neq \emptyset} h(\text{diam } B_i) + \left(1 - \frac{\alpha\xi}{4(1-\xi)P_0^h(S)}\right) P_0^h(S_2) \\
&\quad \geq P_0^h(S_2) + \frac{\alpha\xi}{2(1-\xi)}\left(1 - \frac{P_0^h(S_2)}{2P_0^h(S)}\right) \\
&\quad \geq P_0^h(S_2) + \frac{\alpha\xi}{4(1-\xi)}.
\end{aligned}$$

Choose $\eta' < \eta$ sufficiently small that no ball of radius η' centred in $S \setminus S_2$ intersects those balls B_j of the packing centred in S_2. Then, for any $\beta > 0$, we may choose $\{D_i\}$ an η'-packing of $S \setminus S_2$ satisfying

$$\sum h(\text{diam } D_i) \geq (1-\beta) P_0^h(S \setminus S_2).$$

Therefore, for every $\eta > 0$,

$$P_\eta^h(S) \geq P_0^h(S \setminus S_2) + P_0^h(S_2) + \frac{\alpha\xi}{4(1-\xi)}.$$

Since $P_0^h(S) = P_0^h(S \setminus S_2) + P_0^h(S_2)$, and since $\eta > 0$ was arbitrary, we get $\varepsilon\alpha = 0$, which is a contradiction.

Lemma 3.2.4 *No subset S of K satisfies* $0 < P_0^h(S) < \infty$.

Proof. Suppose some subset S of K satisfies $0 < P_0^h(S) < \infty$. Writing $\xi = \sqrt{68/67} - 1$, we may use Lemma 3.2.2 to choose $\eta > 0$ sufficiently small that, for every η-packing $\{B_i\}$ of S,

$$P_0^h\left(S \cap \bigcup \left\{B_i : h(\text{diam } B_i) \geq (1+\xi) P_0^h(S \cap B_i)\right\}\right) < P_0^h(S)/798.$$

Choose k so large that $\sigma_k \leq \eta$, and that each of the following holds,

$$\left(\prod_{i=1}^{2j-1} n_i\right)(h(\varepsilon_j) - h(\sigma_{j+1})) < P_0^h(S)/798 \text{ for each } j \geq k,$$

$$\sum_{p=k}^{\infty} n_{2p} h(\sigma_p)^{-1} h(\sigma_{p+1}) < 1/3990.$$

Let $\{B_i\}$ be a σ_k-packing of S satisfying

$$\sum h(\text{diam } B_i) \geq \frac{398}{399} P_0^h(S).$$

If we then remove from $\{B_i\}$ all of those balls B_j such that $h(\text{diam } B_j) \geq (1+\xi)P_0^h(S \cap B_j)$, and replace all of the at most $\prod_{i=1}^{2j-1} n_i$ balls of diameter ε_j by the (smaller) concentric balls of radius and diameter σ_{j+1}, the resulting collection, which we will refer to as $\{C_i\}$, is again a packing of S, and satisfies both

$$h(\text{diam } C_j) < (1+\xi)P_0^h(S \cap C_j) \text{ for each } j,$$

and

$$\sum h(\text{diam } C_i) \geq \sum h(\text{diam } B_i) - \frac{P_0^h(S)}{399} \geq \frac{397}{399} P_0^h(S).$$

Suppose that the largest ball appearing in the packing has diameter σ_{l+1}. For each $p \geq l$, label the sets $K_{i_1,\dots i_{2p-2}}$ (that is, the distinct balls of diameter equal to radius σ_p within K) which intersect S by $\mathcal{I}_{p,1}, \dots \mathcal{I}_{p,N(p)}$. For each $1 \leq j \leq N(p)$, write $M(p,j)$ for the number of balls of the packing of diameter σ_{p+1} contained in $\mathcal{I}_{p,j}$. (Note that $M(p,j)$ may be zero for some p, j.)

Then, for each p and j, one of the following holds,

(i) $M(p,j) > 0$ and $h(\sigma_p)/5 \leq 2n_{2p} M(p,j)^{-1} P_0^h(\mathcal{I}_{p,j} \cap S)$,

(ii) Either $M(p,j) > 0$ and $h(\sigma_p)/5 > 2n_{2p} M(p,j)^{-1} P_0^h(\mathcal{I}_{p,j} \cap S)$, or $M(p,j) = 0$.

For every p and j for which case (i) holds,

$$M(p,j) \leq 10 \frac{n_{2p}}{h(\sigma_p)} P_0^h(\mathcal{I}_{p,j} \cap S),$$

$$M(p,j)h(\sigma_{p+1}) \leq 10\frac{n_{2p}h(\sigma_{p+1})}{h(\sigma_p)}P_0^h(\mathcal{I}_{p,j}\cap S).$$

Fixing p, and summing over all such j, we see that the contribution to the packing by balls of diameter σ_{p+1} contained in those balls $\mathcal{I}_{p,j}$ for which case (i) holds is at most

$$10\frac{n_{2p}h(\sigma_{p+1})}{h(\sigma_p)}P_0^h(S),$$

and that the contribution by all such balls is at most

$$10P_0^h(S)\sum_{p=k}^{\infty}\frac{n_{2p}h(\sigma_{p+1})}{h(\sigma_p)} \leq 10P_0^h(S)\frac{1}{3990} = \frac{P_0^h(S)}{399}.$$

For each p and j for which case (i) holds, remove all balls of diameter σ_{p+1} contained in $\mathcal{I}_{p,j}$ from the packing $\{C_i\}$. Write $\mathcal{D}$ for the packing of S consisting of the balls which remain, and partition $\mathcal{D}$ into two subcollections $\mathcal{D}_1$ and $\mathcal{D}_2$, the first consisting of the balls of $\mathcal{D}$ of diameters σ_{2j-1}, the second, those of diameters σ_{2j}, for $j \geq 1$. Then, since $\mathcal{D} \subseteq \{C_i\}$,

$$h(\text{diam } D) < (1+\xi)P_0^h(S \cap D) \text{ for each } D \in \mathcal{D}.$$

Also

$$\sum\{h(\text{diam } D) : D \in \mathcal{D}\} \geq \frac{132}{133}P_0^h(S),$$

and at least one of the following holds,

$$\sum_{D\in\mathcal{D}_1} h(\text{diam } D) \geq 1/2 \sum_{D\in\mathcal{D}} h(\text{diam } D),$$

or

$$\sum_{D\in\mathcal{D}_2} h(\text{diam } D) \geq 1/2 \sum_{D\in\mathcal{D}} h(\text{diam } D).$$

Without loss of generality, suppose the former holds. Let p be even, and p and j satisfy case (ii). If $M(p,j) \neq 0$, write $D_1, \ldots D_{M(p,j)}$ for the balls of diameter σ_{p+1} contained in $\mathcal{I}_{p,j}$, and $D'_1, \ldots D'_{M(p,j)}$ for the concentric balls of radius δ_p and diameter σ_p. If we write $D_{i,1}(\sigma_{p+1}), \ldots D_{i,(2n_{2p}-1)}(\sigma_{p+1})$ for the distinct balls in K of radius σ_{p+1} contained in D_i', then

$$\begin{aligned} P_0^h(D_i' \cap S) &= \sum_{k=1}^{2n_{2p}-1} P_0^h(D_{i,k}(\sigma_{p+1}) \cap S), \\ \sum_{i=1}^{M(p,j)} P_0^h(D_i' \cap S) &= \sum_{i=1}^{M(p,j)} \sum_{k=1}^{2n_{2p}-1} P_0^h(D_{i,k}(\sigma_{p+1}) \cap S) \\ &\leq (2n_{2p}-1) P_0^h(\mathcal{I}_{p,j} \cap S), \end{aligned}$$

since no ball of diameter σ_{p+1} is contained in more than $(2n_{2p}-1)$ distinct balls of radius δ_p.

So we may choose $1 \leq i \leq M(p,j)$ with

$$P_0^h(D_i' \cap S) \leq 2n_{2p} M(p,j)^{-1} P_0^h(\mathcal{I}_{p,j} \cap S).$$

We shall write $\mathcal{D}_{p,j}$ for the chosen ball D_i'.

In this manner we may choose such a ball $D_{p,j}$ for each even $p \geq l$, and each $1 \leq j \leq N(p)$ for which $M(p,j) \neq 0$.

Choose i so large that $\sigma_i < \text{diam}(D)$ for every ball D of $\mathcal{D}$, and use Lemma 3.2.2 to find a σ_i-packing of $S \cap \bigcup \{D : D \in \mathcal{D}_2\}$, such that each ball Q of this new packing satisfies

$$h(\text{diam } Q) < (1+\xi) P_0^h(S \cap Q),$$

and, summing over this packing,

$$\sum h(\text{diam } Q) \geq (1+\xi)^{-1} P_0^h\left(S \cap \bigcup \{D : D \in \mathcal{D}_2\}\right).$$

We replace all the balls of $\mathcal{D}_2$ by the balls of this new packing. Note that

$$\sum_{D \in \mathcal{D}_2} h(\text{diam } D) < (1+\xi)^2 \sum h(\text{diam } Q).$$

For ease of reference we will write $\{Q_i\}$ for the packing of S consisting of $\mathcal{D}_1$ together with this new packing of $S \cap \bigcup\{D : D \in \mathcal{D}_2\}$. Then we may estimate

$$\sum_{D \in \mathcal{D}} h(\text{diam } D) < (1+\xi)^2 \sum_i h(\text{diam } Q_i).$$

Note that the balls $D_{p,j}$ chosen above either contain, or are disjoint from, any ball $\{Q_i\}$ of this packing. This is because a ball of diameter σ_p and radius δ_p may only intersect but not contain another ball of the same diameter, and radius less than diameter, or a ball of diameter ε_p, neither of which may appear in $\{Q_i\}$, for p even.

Now, for every ball $D_{p,j}$ we have

$$\begin{aligned}\sum \{h(\text{diam } Q_i) : Q_i \subset D_{p,j}\} &\leq (1+\xi)P_0^h(S \cap \bigcup\{Q_i : Q_i \subset D_{p,j}\}) \\ &\leq (1+\xi)P_0^h(S \cap D_{p,j}) \\ &\leq (1+\xi)\frac{h(\sigma_p)}{5}.\end{aligned}$$

Let p be the smallest such that there is at least one ball $D_{p,j}$. We modify the packing $\{Q_i\}$ by replacing all those balls contained in $D_{p,j}$ by $D_{p,j}$ itself, for each such j. Write $M'_{p,j}$ for the number of balls of diameter σ_{p+1} contained in $\mathcal{I}_{p,j} \setminus D_{p,j}$. Then $M'_{p,j} \leq (n_{2p}-1)^2$. Taking sums over all j such that there is a ball $D_{p,j}$,

$$\begin{aligned}&\frac{\sum_j h(\sigma_p) + \sum_j M'_{p,j} h(\sigma_{p+1})}{\sum_j \sum_i \{h(\text{diam } Q_i : Q_i \subset D_{p,j}\} + \sum_j M'_{p,j} h(\sigma_{p+1})} \\ &\geq \frac{\sum_j [h(\sigma_p) + (n_{2p}-1)^2 h(\sigma_{p+1})]}{\sum_j [(1+\xi)h(\sigma_p)/5 + (n_{2p}-1)^2 h(\sigma_{p+1})]} \\ &\geq \frac{1 + (n_{2p}-1)^2 h(\sigma_{p+1}) h(\sigma_p)^{-1}}{(1+\xi)/5 + n_{2p}^2 h(\sigma_{p+1}) h(\sigma_p)^{-1}} \\ &\geq \frac{1+5/2}{(1+\xi)/5+3} \\ &\geq \frac{35}{33}\end{aligned}$$

using $\xi < 1/2$. Letting p increase, and modifying the packing in this manner at each stage for which there is a ball $D_{p,j}$ such that those Q_i contained in $D_{p,j}$ have not already been replaced, we obtain a new packing $\mathcal{P}$ for the set $S \cap (\{D : D \in \mathcal{D}_1\} \cup \{Q_i : Q_i \subset D_{p,j} \text{ for some } p, j\})$ such that

$$\begin{aligned}\sum_{P \in \mathcal{P}} h(\text{diam } P) &\geq \frac{35}{33} \sum \{h(\text{diam } D) : D \in \mathcal{D}_1\} \\ &+ \frac{35}{33} \sum \{h(\text{diam } Q_i) : Q_i \notin \mathcal{D}_1, Q_i \subset D_{p,j} \text{ for some } p, j\}.\end{aligned}$$

Write $\mathcal{P}'$ for the packing of S consisting of the balls of $\mathcal{P}$, together with $\{Q_i : Q_i \subset \mathcal{D}_2,\ Q_i \cap D_{p,j} = \emptyset \text{ for each } p, j\}$. By assumption,

$$\sum \{h(\text{diam } D) : D \in \mathcal{D}_1\} \geq 1/2 \sum \{h(\text{diam } D) : D \in \mathcal{D}\},$$

and so

$$\sum_{P \in \mathcal{P}'} h(\text{diam } P) \geq \frac{34}{33} \sum_i h(\text{diam } Q_i) \geq \frac{67}{66} \sum_{\mathcal{D}} h(\text{diam } D) \geq \frac{134}{133} P_0^h(S),$$

by the choice of the balls $\{Q_i\}$ to satisfy

$$(1+\xi)^2 \sum_i h(\text{diam } Q_i) > \sum_{\mathcal{D}} h(\text{diam } D),$$

and the choice of $\mathcal{D}$ to satisfy

$$\sum_{\mathcal{D}} h(\text{diam } D) \geq \frac{132}{133} P_0^h(S).$$

So $P_{\sigma_l}^h(S) \geq \frac{134}{133} P_0^h(S)$ for every l, and hence $P_0^h(S)$ could not have been positive and finite.

Theorem 3.2.5 *Every subset S of K satisfies one of $\mathcal{P}^h(S) = 0$ or $\mathcal{P}^h(S) = \infty$.*

Proof. If $S \subseteq K$ satisfies $\mathcal{P}^h(S) < \infty$, then for each $\varepsilon > 0$ we can find sets $\{S_i\}$, with $S \subseteq \bigcup S_i$, $\mathcal{P}^h(S) \lessapprox (1+\varepsilon) \sum_i P_0^h(S_i)$ and $P_0^h(S_i) < \infty$, for each i. So by Lemma 3.2.4, $P_0^h(S_i) = 0$ for each i, and $\mathcal{P}^h(S) = 0$.

3.3 A counter-example for radius-based packing measure

We now turn our attention to the radius-based packing measure $\mathcal{R}^h$, and prove that if a Hausdorff function h satisfies the following conditions, (a stronger assumption than that h does not satisfy a doubling condition), we may construct a compact metric space $\tilde{K}$, such that $\mathcal{R}^h(\tilde{K}) = \infty$, and each subset S of $\tilde{K}$ satisfies one of $R_0^h(S) = 0$, or $R_0^h(S) = \infty$.

We require of h that we may find sequences $(n_i)_i$ and $(\sigma_i)_i$ satisfying

(i) $n_{2k-1} = n_{2k}$,

(ii) $\sigma_{k+1} < \sigma_k/2$,

such that there are constants c_1 and c_2 with $1 < c_1 < c_2$ and $c_2 - c_1 < 1$, with

(iii) $\sum_{k=1}^{\infty} n_{2k}^{-1} < \infty$,

(iv) $h\left(\frac{\sigma_k}{2}-\right) h(\sigma_k-)^{-1} n_{2k}^2 \leq c_2$,

(v) $h\left(\frac{\sigma_k}{2}-\right) h(\sigma_k-)^{-1} n_{2k}^2 \geq c_1$,

(vi) $\lim_{k\to\infty} \left(\prod_{i=1}^{2k} n_i\right) h\left(\frac{\sigma_k}{2}-\right) = \infty$.

Using (iii) and (iv) we see that

(vii) $\sum_{k=1}^{\infty} \frac{h(\frac{\sigma_k}{2}-)n_{2k}}{h(\sigma_k-)} < \infty$.

To see that such functions exist, choose a sequence of integers $(n_i)_i$ and a positive sequence $(\sigma_i)_i$ such that (i), (ii), and (iii) are satisfied, and use the requirements (iv), (v) and (vi) to prescribe the values $h(\sigma_i)$ and $h(\sigma_i/2)$. Then let h be constant on the intervals $[\sigma_i/2, \sigma_i)$ and $[\sigma_{i+1}, \sigma_i/2)$.

We then take $\tilde{K}$ to be the space

$$\tilde{K} = \{(i_1, i_2, \ldots) : \text{ for each } j \geq 1, 1 \leq i_j \leq n_j\},$$

and provide $\tilde{K}$ with a metric thus; if $(i) = (i_1, i_2, \ldots)$, and $(j) = (j_1, j_2, \ldots)$ are distinct points of $\tilde{K}$, and l is the least index for which $i_l \neq j_l$, then

$$\begin{aligned} \operatorname{dist}((i),(j)) &= \sigma_p/2 \text{ if } l = 2p, \\ \operatorname{dist}((i),(j)) &= \sigma_p/2 \text{ if } l = 2p-1, \text{ and } i_{l+1} = j_{l+1}, \\ \operatorname{dist}((i),(j)) &= \sigma_p \text{ if } l = 2p-1, \text{ and } i_{l+1} \neq j_{l+1}. \end{aligned}$$

The proof that $\tilde{K}$ is a compact metric space is identical to the proof in Section 3.2 that the space K constructed there is.

Note that, if $(i) \in \tilde{K}$, and $\sigma_k/2 \leq r < \sigma_k$, then

$$\begin{aligned} B((i), r) &= B((i), \sigma_k/2) \\ &= \tilde{K}_{i_1,\ldots i_{2k-1}} \cup \bigcup \left\{ \tilde{K}_{i_1,\ldots i_{2k-2},j,i_{2k}} : 1 \leq j \leq n_{2k-1}, j \neq i_{2k-1} \right\}, \end{aligned}$$

and

$$\sup \{h(r) : \sigma_k/2 \leq r < \sigma_k\} = h(\sigma_k-).$$

If $\sigma_k \leq r < \sigma_{k-1}/2$, then

$$B((i), r) = B((i), \sigma_k) = \tilde{K}_{i_1,\ldots i_{2k-2}},$$

$$\sup \{h(r) : \sigma_k \leq r < \sigma_{k-1}/2\} = h\left(\frac{\sigma_{k-1}}{2}-\right).$$

These are the only nonempty balls in $\tilde{K}$.

We now define a new function g by taking g constant, with values $h(\sigma_k-)$ and $h(\sigma_k/2-)$, respectively, on the intervals $[\sigma_k/2, \sigma_k)$, and $[\sigma_{k+1}, \sigma_k/2)$. It is not hard to see that $\mathcal{R}^g$ and $\mathcal{R}^h$ are identical on $\tilde{K}$. However, the function g has the useful property that, if $B((i),r) = B((i),s)$, then $g(r) = g(s)$. So when we pack $\tilde{K}$, we may assume that every ball has radius σ_k or $\sigma_k/2$, for some $k \geq 1$.

Note that, by Lemma 1.4.5

$$\mathcal{R}^g(S) = \inf\left\{\sum_1^\infty R_0^g(C_i) : S \subset \bigcup_1^\infty C_i, \text{ and } C_i \text{ are closed in } M\right\}.$$

With proofs similar to those of Lemmas 3.2.1 and 3.2.2, we have

Lemma 3.3.1 $\mathcal{R}^g(\tilde{K}) = \infty$.

Lemma 3.3.2 *If $S \subseteq K$ satisfies $0 < R_0^g(S) < \infty$, then, for each $\xi > 0$ and $\alpha > 0$, there is $\eta > 0$ such that, if $\{B(x_i, r_i)\}$ is an η-packing of S, then*

$$R_0^g\left(S \cap \bigcup\{B_i : g(r_i) \geq (1+\xi)R_0^g(S \cap B_i)\}\right) < \alpha.$$

Lemma 3.3.3 *No subset S of $\tilde{K}$ satisfies $0 < R_0^g(S) < \infty$.*

Proof. The following is very similar to the proof of Lemma 3.2.4, however the differences are sufficiently significant for this proof to be given in full.

Suppose some subset S of $\tilde{K}$ satisfies $0 < R_0^g(S) < \infty$. Write $\xi = [(6+6c_1+2c_2)/(5+5c_1+3c_2)]^{\frac{1}{2}}-1$, and $\zeta = 1-(9+9c_1+7c_2)/(10+10c_1+6c_2)$. We choose $\eta > 0$ sufficiently small that, for every η-packing $\{B(x_i, r_i)\}$ of S,

$$R_0^g\left(S \cap \bigcup\{B(x_i, r_i) : g(r_i) \geq (1+\xi)R_0^g(S \cap B(x_i, r_i))\}\right) < \frac{\zeta}{3}R_0^g(S).$$

Choose k so large that $\sigma_k \leq \eta$, and that

$$\sum_{p=k}^{\infty} n_{2p}\, g(\sigma_{p+1})\, g(\sigma_p/2)^{-1} < \frac{\zeta}{18}(1+c_1-c_2).$$

Let $\{B(x_i, r_i)\}$ be a σ_k-packing of S satisfying

$$\sum g(r_i) \geq (1-\zeta/3)\, R_0^g(S).$$

If we then remove from $\{B(x_i, r_i)\}$ all of those balls $B(x_j, r_j)$ such that $g(r_j) \geq (1+\xi)R_0^g(S \cap B(x_j, r_j))$, the resulting collection, which we will refer to as $\{C_i\}$, is again a packing of S, and satisfies both

$$g(\text{radius } C_j) < (1+\xi)R_0^g(S \cap C_j) \text{ for each } j,$$

and

$$\sum g(\text{radius } C_i) \geq \sum g(\text{radius } B_i) - \frac{\zeta}{3}R_0^g(S) \geq \left(1 - \frac{2\zeta}{3}\right) R_0^g(S).$$

Suppose that the largest ball appearing in the packing has diameter σ_{l+1}. For each $p \geq l$, label the sets $\tilde{K}_{i_1,\dots i_{2p-2}}$ which intersect S by $\mathcal{I}_{p,1}, \dots \mathcal{I}_{p,N(p)}$. For each $1 \leq j \leq N(p)$, write $M(p,j)$ for the number of balls of the packing of diameter σ_{p+1}, that is, of the form $\tilde{K}_{i_1,\dots i_{2p}}$ or $\tilde{K}_{i_1,\dots i_{2p+1}} \cup \bigcup \left\{\tilde{K}_{i_1,\dots i_{2p},j,i_{2p+2}} : 1 \leq j \leq n_{2p+1}, j \neq i_{2p+1}\right\}$, contained in $\mathcal{I}_{p,j}$. (Note that $M(p,j)$ may be zero for some p, j.)

Then, for each p and j, one of the following holds,

(i) $M(p,j) > 0$ and $(1 + c_1 - c_2)g(\sigma_p/2)/3 \leq 2n_{2p}M(p,j)^{-1}R_0^g\left(\mathcal{I}_{p,j} \cap S\right)$,

(ii) $M(p,j) > 0$ and $(1 + c_1 - c_2)g(\sigma_p/2)/3 > 2n_{2p}R_0^g\left(\mathcal{I}_{p,j} \cap S\right)/M(p,j)$; or else $M(p,j) = 0$.

For every p and j for which case (i) holds,

$$\begin{aligned} M(p,j) &\leq \frac{6n_{2p}}{(1 + c_1 - c_2)g(\sigma_p/2)} R_0^g\left(\mathcal{I}_{p,j} \cap S\right), \\ M(p,j)g(\sigma_{p+1}) &\leq \frac{6n_{2p}g(\sigma_{p+1})}{(1 + c_1 - c_2)g(\sigma_p/2)} R_0^g\left(\mathcal{I}_{p,j} \cap S\right). \end{aligned}$$

Fixing p, and summing over all such j, we see that the contribution to the packing by balls of diameter σ_{p+1} contained in those balls $\mathcal{I}_{p,j}$ for which case (i) holds is at most

$$\frac{6n_{2p}g(\sigma_{p+1})}{(1 + c_1 - c_2)g(\sigma_p/2)} R_0^g\left(S\right),$$

and that the contribution by all such balls is at most

$$\frac{6R_0^g(S)}{(1+c_1-c_2)}\sum_{p=k}^{\infty}\frac{n_{2p}g(\sigma_{p+1})}{g(\sigma_p/2)} \le \frac{6R_0^g(S)}{(1+c_1-c_2)}\frac{\zeta(1+c_1-c_2)}{18} = \frac{\zeta}{3}R_0^g(S).$$

For each p and j for which case (i) holds, remove all balls of diameter σ_{p+1} contained in $\mathcal{I}_{p,j}$ from the packing $\{C_i\}$. Write $\mathcal{D}$ for the packing of S consisting of the balls which remain, and partition $\mathcal{D}$ into two subcollections $\mathcal{D}_1$ and $\mathcal{D}_2$, the first consisting of the balls of $\mathcal{D}$ of diameters σ_{2j-1}, the second, those of diameters σ_{2j}, for $j \ge 1$. Then, since $\mathcal{D} \subseteq \{C_i\}$,

$$g(\text{radius } D) < (1+\xi)R_0^g(S \cap D) \text{ for each } D \in \mathcal{D}.$$

Also

$$\sum \{g(\text{radius } D) : D \in \mathcal{D}\} \ge (1-\zeta)\, R_0^g(S),$$

and at least one of the following holds,

$$\sum_{D\in\mathcal{D}_1} g(\text{radius } D) \ge 1/2 \sum_{D\in\mathcal{D}} g(\text{radius } D),$$

or

$$\sum_{D\in\mathcal{D}_2} g(\text{radius } D) \ge 1/2 \sum_{D\in\mathcal{D}} g(\text{radius } D).$$

Without loss of generality, suppose the former holds. Let p be even, and p and j satisfy case (ii). If $M(p,j) \neq 0$, write $D_1, \ldots D_{M(p,j)}$ for the balls of diameter σ_{p+1} contained in $\mathcal{I}_{p,j}$, and $D'_1, \ldots D'_{M(p,j)}$ for the concentric balls of radius $\sigma_p/2$. If we write $D_{i,1}(\sigma_{p+1}), \ldots D_{i,(2n_{2p}-1)}(\sigma_{p+1})$ for the distinct balls in $\tilde{K}$ of radius σ_{p+1} contained in D'_i, then

$$\begin{aligned}
R_0^g(D'_i \cap S) &= \sum_{k=1}^{2n_{2p}-1} R_0^g(D_{i,k}(\sigma_{p+1}) \cap S), \\
\sum_{i=1}^{M(p,j)} R_0^g(D'_i \cap S) &= \sum_{i=1}^{M(p,j)} \sum_{k=1}^{2n_{2p}-1} R_0^g(D_{i,k}(\sigma_{p+1}) \cap S) \\
&\le (2n_{2p}-1)R_0^g(\mathcal{I}_{p,j} \cap S),
\end{aligned}$$

since no ball of radius σ_{p+1} is contained in more than $(2n_{2p}-1)$ distinct balls D'_i.

So we may choose $1 \leq i \leq M(p,j)$ with

$$R_0^g(D'_i \cap S) \leq 2n_{2p}M(p,j)^{-1}R_0^g(\mathcal{I}_{p,j} \cap S).$$

Rename this ball $D_{p,j}$.

In this manner we may choose such a ball $D_{p,j}$ for each even $p \geq l$, and each $1 \leq j \leq N(p)$ for which $M(p,j) \neq 0$.

Choose i so large that $\sigma_i <$ radius(D) for every ball D of $\mathcal{D}$, and use Lemma 3.3.2 to find a σ_i-packing of $S \cap \bigcup\{D : D \in \mathcal{D}_2\}$, such that each ball Q of this new packing satisfies

$$g(\text{radius } Q) < (1+\xi)R_0^g(S \cap Q),$$

and, summing over this packing,

$$\sum g(\text{radius } Q) \geq (1+\xi)^{-1}R_0^g\left(S \cap \bigcup\{D : D \in \mathcal{D}_2\}\right).$$

We replace all the balls of $\mathcal{D}_2$ by the balls of this new packing. Note that

$$\sum_{\mathcal{D}_2} g(\text{radius } D) \leq (1+\xi)^2 \sum g(\text{radius } Q).$$

For ease of reference we will write $\{Q_i\}$ for the packing of S consisting of $\mathcal{D}_1$ together with this new packing of $S \cap \bigcup\{D : D \in \mathcal{D}_2\}$. Then we may estimate

$$\sum_{\mathcal{D}} g(\text{radius } D) \leq (1+\xi)^2 \sum_i g(\text{radius } Q_i).$$

Note that the balls $D_{p,j}$ either contain, or are disjoint from, any ball $\{Q_i\}$ of this packing. This is because a ball of radius $\sigma_p/2$ may only intersect but not contain another ball of the same diameter σ_p and radius at least than $\sigma_p/2$, which may not appear in $\{Q_i\}$, for p even.

Now, for every ball $D_{p,j}$ we have

$$\begin{aligned}\sum\{g(\text{radius } Q_i) : Q_i \subset D_{p,j}\} &\le (1+\xi)R_0^g(S \cap \bigcup\{Q_i : Q_i \subset D_{p,j}\}) \\ &\le (1+\xi)R_0^g(S \cap D_{p,j}) \\ &\le (1+\xi)(1+c_1-c_2)g(\sigma_p/2)/3.\end{aligned}$$

Let p be the least such that there is at least one ball $D_{p,j}$. We modify the packing $\{Q_i\}$ by replacing all those balls contained in $D_{p,j}$ by $D_{p,j}$ itself, for each such j. Write $M'_{p,j}$ for the number of balls of diameter σ_{p+1} (and radius less than $\sigma_p/2$) contained in $\mathcal{I}_{p,j} \setminus D_{p,j}$. Taking sums over all j such that there is a ball $D_{p,j}$, recalling that $g(\text{radius } D_{p,j}) = g(\sigma_p/2)$, and using the fact that $M'_{p,j} \le (n_{2p}-1)^2$, we see that

$$\begin{aligned}&\frac{\sum_j g(\sigma_p/2) + \sum_j M'_{p,j} g(\sigma_{p+1})}{\sum_j \sum_i \{g(\text{radius } Q_i : Q_i \in D_{p,j}\} + \sum_j M'_{p,j} g(\sigma_{p+1})} \\ &\ge \frac{\sum_j [g(\sigma_p/2) + (n_{2p}-1)^2 g(\sigma_{p+1})]}{\sum_j [(1+\xi)(1+c_1-c_2)g(\sigma_p/2)/3 + (n_{2p}-1)^2 g(\sigma_{p+1})]} \\ &\ge \frac{1+(n_{2p}-1)^2 g(\sigma_{p+1}) g(\sigma_p/2)^{-1}}{(1+\xi)(1+c_1-c_2)/3 + (n_{2p})^2 g(\sigma_{p+1}) g(\sigma_p/2)^{-1}} \\ &= \frac{1+(n_{2p}-1)^2 h(\frac{\sigma_p}{2}-) h(\sigma_p-)^{-1}}{(1+\xi)(1+c_1-c_2)/3 + (n_{2p})^2 h(\frac{\sigma_p}{2}-) h(\sigma_p-)^{-1}} \\ &\ge \frac{1+c_1}{(1+\xi)(1+c_1-c_2)/3 + c_2} \\ &\ge \frac{2(1+c_1)}{(1+c_1+c_2)}\end{aligned}$$

using $\xi < 1/2$. Letting p increase, and modifying the packing in this manner at each stage for which there is a ball $D_{p,j}$ such that those Q_i contained in $D_{p,j}$ have not already been replaced, we obtain a new packing $\mathcal{P}$ for the set $S \cap (\{D : D \in \mathcal{D}_1\} \cup \{Q_i : Q_i \subset D_{p,j} \text{ for some } p, j\})$ such that

$$\begin{aligned}\sum_{P\in\mathcal{P}} g(\text{radius } P) &\ge \frac{2(1+c_1)}{(1+c_1+c_2)} \Big(\sum\{g(\text{radius } D) : D \in \mathcal{D}_1\} \\ &+ \sum\{g(\text{radius } Q_i) : Q_i \notin \mathcal{D}_1, Q_i \subset D_{p,j} \text{ for some } p, j\}\Big).\end{aligned}$$

Write $\mathcal{P}'$ for the packing of S consisting of the balls of $\mathcal{P}$, together with $\{Q_i : Q_i \subset \mathcal{D}_2,\ Q_i \cap D_{p,j} = \emptyset \text{ for each } p, j\}$. By assumption,

$$\sum \{g(\text{radius } D) : D \in \mathcal{D}_1\} \geq 1/2 \sum \{g(\text{radius } D) : D \in \mathcal{D}\},$$

and so

$$\begin{aligned}\sum_{P \in \mathcal{P}'} g(\text{radius } P) &\geq \frac{3 + 3c_1 + c_2}{2 + 2c_1 + 2c_2} \sum_i g(\text{radius } Q_i) \\ &\geq \frac{5 + 5c_1 + 3c_2}{4 + 4c_1 + 4c_2} \sum_{\mathcal{D}} g(\text{radius } D) \\ &\geq \frac{9 + 9c_1 + 7c_2}{8 + 8c_1 + 8c_2} R_0^g(S),\end{aligned}$$

by the choice of the balls $\{Q_i\}$ to satisfy

$$(1 + \xi)^2 \sum_i g(\text{radius } Q_i) \geq \sum_{\mathcal{D}} g(\text{radius } D),$$

and the choice of $\mathcal{D}$ to satisfy

$$\sum_{\mathcal{D}} g(\text{radius } D) \geq (1 - \zeta) R_0^g(S).$$

So $R_{\sigma_l}^g(S) \geq \frac{9+9c_1+7c_2}{8+8c_1+8c_2} R_0^g(S)$ for every l, and hence $R_0^g(S)$ could not have been positive and finite.

This leads us directly to the analogue of Theorem 3.2.5;

Theorem 3.3.4 *Every subset S of $\tilde{K}$ satisfies one of $\mathcal{R}^h(S) = 0$ or $\mathcal{R}^h(S) = \infty$.*

Remark As noted earlier, the conditions imposed above on h are somewhat stronger than the condition that h does not satisfy a doubling condition. To see this, consider the following inequalities, implied by these conditions;

$$c_2 h(\sigma_k -) \geq n_{2k}^2 h\left(\frac{\sigma_k}{2} -\right) \geq h\left(\frac{\sigma_{k-1}}{2} -\right).$$

The inequality $c_2 h(\sigma_k-) \geq h(\frac{\sigma_{k-1}}{2}-)$ imposes a lower bound on σ_k, and hence an upper bound on the ratio $h(\sigma_k-)/h(\frac{\sigma_k}{2}-)$, and on n_{2k}. We cannot therefore be certain that $\sum n_{2k}^{-1}$ converges, and so, on its own, the fact that h does not satisfy a doubling condition is not sufficient for the proof above. If h satisfies neither a doubling condition nor these conditions, it is not clear whether we can necessarily find a subset of finite measure, given a metric space of infinite measure.

Chapter 4

A relationship between packing and topological dimensions

4.1 Introduction

In Chapter 1 we provided a number of definitions of packing dimension and considered the inequalities those dimensions satisfied. In this chapter we consider topological and packing dimensions, and show that if X is a separable metric space, then

$$\dim_{\mathcal{T}}(X) = \min\left\{\dim_{\mathcal{Q}}(X') : X' \text{ is homeomorphic to } X\right\},$$

where $\mathcal{Q}$ denotes any of $\mathcal{R}$, $\tilde{\mathcal{R}}$, or $\mathcal{P}$, and $\dim_{\mathcal{T}}(X)$ denotes the topological dimension of X, (defined below).

The result which relates the topological dimension of a separable metric space to the Hausdorff dimensions of its homeomorphic images has been known for some time. The Szpilrajn inequality (see [SzE]) tells us that if X

is a separable metric space, then $\dim_{\mathcal{T}}(X) \leq \dim_{\mathcal{H}}(X)$. In fact,

$$\dim_{\mathcal{T}}(X) = \min\{\dim_{\mathcal{H}}(X') : X' \text{ is homeomorphic to } X\}.$$

For a proof, see, for example, [HuWa].

This proof uses the idea of approximation of a compact metric space by the images of polytope mappings, and relies on the fact that a finite open cover of a set S will also cover $B(S,\varepsilon)$, for some $\varepsilon > 0$. Since we define Hausdorff pre-measure by taking an infimum over covers, this means that, given $\delta > 0$, we may be sure there is $\varepsilon > 0$ such that the δ pre-measure H^s_δ is not much greater on $B(S,\varepsilon)$ than on S. The same is not true for the packing pre-measure of S. Firstly, a packing of S does not necessarily pack a set S', no matter how close all the points of S' lie to S. Secondly, we cannot be sure that there are no substantially better packings of a set S' which is close to S than there are of S itself. This problem arises because we use the supremum in the definition of packing pre-measure Q^s_δ, rather than the infimum. So in choosing successive approximating subsets of polytopes which admit only packings which are in some sense similar to each other, we must take more care than would be needed in the case of coverings.

4.2 Definitions and notation

We now provide the definitions we shall need for this chapter. In the main, we follow the notation of [HuWa].

(i) By a *covering* of a subset S of a space X we mean a finite collection $U_1, \ldots, U_r$ of non-empty open subsets of X whose union contains S. In this chapter we shall reserve the symbols $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ for coverings.

(ii) The *order* of a covering of S is the largest integer n such that there are $n+1$ members of the covering with nonempty intersection in S.

(iii) If X is bounded the *mesh* of a covering $\{U_1, \ldots, U_r\}$ of a subset of X is the largest of $\{\text{diam}(U_i),\ 1 \leq i \leq r\}$.

(iv) A covering $\mathcal{V}$ is a *refinement* of a covering $\mathcal{U}$ if each member of $\mathcal{V}$ is contained in some member of $\mathcal{U}$.

(v) If $\mathcal{U}$ and $\mathcal{V}$ are coverings of S, then we write $\mathcal{U} \wedge \mathcal{V}$ for the covering $\{U \cap V : U \in \mathcal{U},\ V \in \mathcal{V},\ U \cap V \neq \emptyset\}$ of S. Clearly, $\mathcal{U} \wedge \mathcal{V}$ is a refinement of each of $\mathcal{U}$ and $\mathcal{V}$.

(vi) If $\mathcal{U}$ is a covering of X and $g : X \to Y$, we say g is a *$\mathcal{U}$-mapping* if every point of Y has a neighbourhood in Y whose inverse image is entirely contained in some member of $\mathcal{U}$. If X is compact, and $\varepsilon > 0$, we say g is an *ε-mapping* if the inverse image of each point of Y is of diameter less than ε. We write $g^{-1}(\mathcal{U})$ for the collection $\{g^{-1}(U) : U \in \mathcal{U}\}$ of subsets of X.

(vii) If $\mathcal{U}$ is a covering of X, we write $St(\mathcal{U}, x)$ for the open set which is the union of those members of $\mathcal{U}$ which contain x.

(viii) A countable sequence $\mathcal{U}_1,\ \mathcal{U}_2, \ldots$ of coverings of a space X is called a *basic sequence of coverings* if, given a point x in X, and a neighbourhood U of x, at least one of $St(\mathcal{U}_1, x), St(\mathcal{U}_2, x), \ldots$ is contained in U.

We shall take as our definition of topological dimension the following:

A separable metric space X is of topological dimension $\dim_{\mathcal{T}}(X)$ less than or equal to n if and only if every covering $\mathcal{U}$ of X has a refinement $\mathcal{V}$ of order less than or equal to n.

With these definitions, we have the following results;

Theorem 4.2.1 *If $\mathcal{U}_1, \mathcal{U}_2, \ldots$ is a basic sequence of coverings for a metric space X, and a continuous mapping $g : X \to Y$ is a $\mathcal{U}_i$-mapping for each i, then g is a homeomorphism* onto its range.

Theorem 4.2.2 *Any separable metric space can be embedded in a compact metric space of the same topological dimension.*

We omit proofs; see, for example [HuWa].

Our goal is to construct a homeomorphism from a separable metric space X of topological dimension less than or equal to n to an image space of packing dimension less than or equal to n. If X is compact, we shall do this by constructing successive polytope ε_i-mappings, where $\varepsilon_i \searrow 0$, in such a way that no mapping differs too much from the previous mapping in the packings its image admits. To do this, we shall utilize certain ε_i-coverings of X, whose existence we shall ascertain in the following two lemmas. Theorem 4.4.1 contains most of the work of this chapter. In Theorem 4.4.3 we provide our main result.

4.3 Two covering lemmas

Lemma 4.3.1 *Any covering $\mathcal{U}$ of a compact metric space X has a refinement $\mathcal{V}$ of the same order as $\mathcal{U}$ which satisfies the following condition:*

$$\text{For any } V_1, \ldots, V_p \in \mathcal{V} \text{ either } \cap_1^p \operatorname{Clos}(V_i) = \emptyset \text{ or } \cap_1^p V_i \neq \emptyset. \qquad (4.1)$$

Proof. Choose $\delta > 0$ sufficiently small that whenever $U_1, \ldots, U_p \in \mathcal{U}$ satisfy $\cap_1^p U_i \neq \emptyset$, there is a ball of radius δ contained in $\cap_1^p U_i$. Choose $0 < \varepsilon < \delta$ sufficiently small that $\mathcal{V} = \{U \setminus B(X \setminus U, \varepsilon) : U \in \mathcal{U}\}$ is still a covering for X. (To see that this is possible, choose a sequence $\varepsilon_i \searrow 0$, then $\{U \setminus B(X \setminus U, \varepsilon_i) : U \in \mathcal{U},\ 1 \leq i < \infty\}$ is an open cover for X. Choose a finite subcover, then if j is the greatest such that, for some $U \in \mathcal{U}$, $U \setminus B(X \setminus U, \varepsilon_j)$ appears in the subcover, we may take $\varepsilon = \varepsilon_j$.)

Then $\cap_1^p (U_i \setminus B(X \setminus U_i, \varepsilon)) \neq \emptyset$ whenever $\cap_1^p U_i \neq \emptyset$, since $\varepsilon < \delta$, and $\cap_1^p \mathrm{Clos}((U_i \setminus B(X \setminus U_i, \varepsilon))) = \emptyset$ whenever $\cap_1^p U_i = \emptyset$. Also, since $\varepsilon < \delta$, the covering $\{U \setminus B(X \setminus U, \varepsilon) : U \in \mathcal{U}\}$ has the same order as $\mathcal{U}$.

Lemma 4.3.2 *Let a covering $\mathcal{U}$ of a compact metric space X satisfy condition 4.1, and $\mathcal{V}$ be a refinement of $\mathcal{U}$. Then there exists a refinement $\mathcal{W}$ of $\mathcal{V}$ satisfying* $\mathrm{order}(\mathcal{W}) \leq \dim_{\mathcal{T}}(X)$, *and*

$$\cap \{U \in \mathcal{U} : U \cap St(\mathcal{W}, x) \neq \emptyset\} \neq \emptyset, \text{ for each } x \in X. \tag{4.2}$$

Proof. Suppose not. Then we may find a sequence $(\mathcal{W}_k)_1^\infty$ of coverings of X, such that $\mathrm{mesh}(\mathcal{W}_k) \searrow 0$, and such that for each $k \geq 1$, $\mathrm{order}(\mathcal{W}_k) \leq \dim_{\mathcal{T}}(X)$ and $\mathcal{W}_{k+1}$ is a refinement of $\mathcal{W}_k$; and a sequence $(x_k)_1^\infty$ of points in X such that $\cap \{U \in \mathcal{U} : U \cap St(\mathcal{W}_k, x_k) \neq \emptyset\} = \emptyset$ for each k. If necessary passing to a subsequence, we may assume that $(x_k)_1^\infty$ tends to some point x in X.

Consider $\cap \{U \in \mathcal{U} : U \cap St(\mathcal{W}_k, x) \neq \emptyset\}$. This intersection is taken over a decreasing collection of elements of $\mathcal{U}$ as $k \to \infty$, so for K large enough, the set $\cap \{U \in \mathcal{U} : U \cap St(\mathcal{W}_k, x) \neq \emptyset\}$ is constant for $k \geq K$. Therefore $x \in \mathrm{Clos}(U)$ whenever $U \cap St(\mathcal{W}_K, x) \neq \emptyset$. This, together with the fact that $\mathcal{U}$ satisfies condition 4.1, implies that $\cap \{U \in \mathcal{U} : U \cap St(\mathcal{W}_k, x) \neq \emptyset\} \neq \emptyset$ for each $k \geq K$. But for each $k \geq K$, $\cap\{U \in \mathcal{U} : U \cap St(\mathcal{W}_k, x_k) \neq \emptyset\} = \emptyset$, so for each $k \geq K$ there is $U_k \in \mathcal{U}$ such that

$$U_k \cap St(\mathcal{W}_k, x_k) \neq \emptyset, \text{ and } U_k \cap St(\mathcal{W}_K, x) = \emptyset.$$

$\mathcal{U}$ is a finite collection, so for some $U \in \mathcal{U}$,

$$U \cap St(\mathcal{W}_k, x_k) \neq \emptyset \text{ infinitely often, and } U \cap St(\mathcal{W}_K, x) = \emptyset.$$

The former implies that $x \in \mathrm{Clos}(U)$, contradicting the latter, and proving the lemma.

4.4 The inductive construction and main result

Recall that a set $\{p_i\}_1^r$ in $\mathbf{R}^n$ is said to be in *general position* if for each $1 \leq m \leq n-1$, no $m+2$ of these points lie in an m-dimensional affine subspace of $\mathbf{R}^n$.

Theorem 4.4.1 *If X is a compact metric space with $\dim_{\mathcal{T}}(X) \leq n$, and I^{2n+1} is the set of points in $\mathbf{R}^{2n+1}$ each of whose coordinates $x_1, \ldots, x_{2n+1}$ satisfies $|x_i| \leq 1$, then there is a homeomorphism $f : X \to f(X) \subseteq I^{2n+1}$ with $\dim_{\mathcal{R}} f(X) \leq n$.*

Proof. We will use Lemmas 4.3.1 and 4.3.2 to construct by induction a sequence of mappings from X to I^{2n+1} which have a homeomorphism as their uniform limit, and such that for each $\varepsilon > 0$, $R_0^{n+\varepsilon}$ is finite on the image of X under this homeomorphism.

Choose a sequence $(\mathcal{U}_i)_1^\infty$ of coverings for X, with $\varepsilon_i = \text{mesh}(\mathcal{U}_i) \searrow 0$ and $\mathcal{U}_{i+1}$ a refinement of $\mathcal{U}_i$, for each $i \geq 1$. (This is not hard to do, since X is compact.) Let $c_i = \sum_{j=1}^{i} 2^{-j}$.

To start the induction we choose the simplest possible covering for I^{2n+1}, namely $\mathcal{W}_1 = \{I^{2n+1}\}$, and choose a covering $\mathcal{V}_1$ for X which refines $\mathcal{U}_1$, is of order less than or equal to n, satisfies condition 4.1, that is, satisfies

$$\text{For any } V_1, \ldots, V_p \in \mathcal{V}_1 \text{ either } \cap_1^p \text{Clos}(V_i) = \emptyset \text{ or } \cap_1^p V_i \neq \emptyset,$$

and is such that for no $V \in \mathcal{V}_1$ does $\mathcal{V}_1 \setminus \{V\}$ cover X. Choose a collection $Y_1 = \{y_1(V) : V \in \mathcal{V}_1\} \subseteq I^{2n+1}$ of $|\mathcal{V}_1|$ points in general position.

Define a function $f_1 : X \to I^{2n+1}$ thus;

$$f_1(x) = \frac{\sum_{V \in \mathcal{V}_1} \text{dist}\,(x, X \setminus V)\, y_1(V)}{\sum_{V \in \mathcal{V}_1} \text{dist}\,(x, X \setminus V)}.$$

We now show that f_1 is an ε_1-mapping.

Let $x \in X$, and suppose $V_{i_1}, \ldots, V_{i_s}$ are all the members of $\mathcal{V}_1$ containing x. Consider the affine $(s-1)$-space $L(x)$ spanned by the vertices $y_1(V_{i_1}), \ldots, y_1(V_{i_s})$. It is clear that $f_1(x)$ is in $L(x)$. Let x' be another point of X. Suppose $L(x')$ is spanned by the vertices $y_1(V_{j_1}), \ldots, y_1(V_{j_t})$. Then, since order$(\mathcal{V}_1) \leq n$, we have $s, t \leq n+1$ and $L(x')$ is a $(t-1)$-space. If $L(x)$ and $L(x')$ meet, the affine space spanned by all these vertices has dimension $\leq s+t-2 \leq 2n$. Since Y_1 is in general position in I^{2n+1}, we see that if $L(x)$ and $L(x')$ meet they contain a common vertex.

Therefore if $f_1(x) = f_1(x')$, x and x' must be contained in a common member of $\mathcal{V}_1$, and by the fact that mesh $(\mathcal{V}_1) \leq \varepsilon_1$, we see that $\text{dist}(x, x') \leq \varepsilon_1$, as required.

We also see that $f_1(X)$ is a subset of the polytope Q_1 in I^{2n+1}, where Q_1 has vertex set Y_1, and conv $(\{y_1(V_1), \ldots, y_1(V_s)\})$ is a face of Q_1 if and only if $V_1 \cap \ldots \cap V_s \neq \emptyset$.

Since at most $n+1$ elements of $\mathcal{V}_1$ may intersect in a common point, the polytope Q_1 is at most n-dimensional, and so we may choose $\delta_1 > 0$ sufficiently small that $R_{\delta_1}^{n+\varepsilon_1}(Q_1) \leq c_1$.

Write $\xi_1 = \inf\{\text{dist}(f_1(x_1), f_1(x_2)) : x_1, x_2 \in X,\ \text{dist}(x_1, x_2) \geq \varepsilon_1\}$. ($X$ is a compact space, so this infimum is attained at some points x_1 and x_2 of X, and is greater than zero, otherwise f would not be an ε_1-mapping.)

Now suppose we can also find functions $f_2, \ldots, f_k$ mapping X into I^{2n+1}, coverings $\mathcal{W}_2, \ldots, \mathcal{W}_k$ of I^{2n+1}, coverings $\mathcal{V}_2, \ldots, \mathcal{V}_k$ of X, and positive numbers $\delta_2, \ldots, \delta_k$ with $\delta_i \leq \delta_{i-1}/2$, such that for each $2 \leq i \leq k$,

(i) $\mathcal{W}_i$ is a refinement of $\mathcal{W}_{i-1}$, with

$$\text{mesh}\,(\mathcal{W}_i) \leq \frac{1}{2}\min\left(\delta_{i-1}, \min_{y_1 \neq y_2 \in Y_{i-1}} \text{dist}(y_1, y_2)\right).$$

(ii) $\mathcal{V}_i$ is a refinement of $\mathcal{U}_i \wedge f_{i-1}^{-1}(\mathcal{W}_i) \wedge \mathcal{V}_{i-1}$ of order less than or equal to n, satisfying condition 4.1, such that for no $V \in \mathcal{V}_i$ does $\mathcal{V}_i \setminus \{V\}$ cover X, and such that condition 4.2 is satisfied with $\mathcal{V}_{i-1}$ in place of $\mathcal{U}$, and $\mathcal{V}_i$ in place of $\mathcal{W}$. That is,

For any $V_1, \ldots, V_p \in \mathcal{V}_i$ either $\cap_1^p \text{Clos}(V_i) = \emptyset$ or $\cap_1^p V_i \neq \emptyset$;

and for each $x \in X$, $\cap\{V \in \mathcal{V}_{i-1} : V \cap St(\mathcal{V}_i, x) \neq \emptyset\} \neq \emptyset$.

(iii) There are sets $Y_2 = \{y_2(V) : V \in \mathcal{V}_2\}, \ldots, Y_k = \{y_k(V) : V \in \mathcal{V}_k\}$ in I^{2n+1} with $Y_{i-1} \subseteq Y_i$ and $|Y_i| = |\mathcal{V}_i|$, with the elements of Y_i in general position, and satisfying

(a) For each $V \in \mathcal{V}_i$, $\text{dist}\,(y_i(V), f_{i-1}(V)) \leq \delta_{i-1}/4$.

(b) $f_i : X \to I^{2n+1}$ is the function

$$f_i(X) = \frac{\sum_{V \in \mathcal{V}_i} \text{dist}\,(x, X \setminus V)\, y_i(V)}{\sum_{V \in \mathcal{V}_i} \text{dist}\,(x, X \setminus V)}.$$

(We then see that f_i is an ε_i-mapping just as we saw that f_1 was an ε_1-mapping.)

(c) If we write Q_i for the polytope in I_{2n+1} with vertex set Y_i, and satisfying

$$\operatorname{conv}(\{y_i(V_1), \ldots, y_i(V_s)\}) \text{ is a face of } Q_i \text{ if and only if } V_1 \cap \ldots \cap V_s \neq \emptyset,$$

then $f_i(X) \subset Q_i$, and for each $1 \leq j < i$, $R_{\delta_j}^{n+\varepsilon_j}(Q_i) \leq c_{i-j+1}$.

(iv) If $\xi_i = \inf \{\operatorname{dist}(f_i(x_1), f_i(x_2)) : x_1, x_2 \in X,\ \operatorname{dist}(x_1, x_2) \geq \varepsilon_i\}$, then for each $1 \leq i \leq k$, $\xi_i > 0$ and for each $1 \leq j < i$, $\operatorname{dist}(f_j, f_i) < \xi_j/2$.

We now show that we may choose $\mathcal{W}_{k+1}$, $\mathcal{V}_{k+1}$, f_{k+1}, Y_{k+1}, Q_{k+1}, and δ_{k+1} so that the same conditions are satisfied with $i = k+1$.

Let

$$\zeta = \min \{\xi_j/2 - \operatorname{dist}(f_j, f_k) : 1 \leq j < k\}.$$

Since f_k is continuous and $\zeta > 0$, we may choose $\eta > 0$ such that if $x_i, x_2 \in X$ satisfy $\operatorname{dist}(x_1, x_2) < \eta$, then $\operatorname{dist}(f_k(x_1), f_k(x_2)) < \zeta/3$.

To make the inductive step, choose σ such that

$$0 < \sigma \leq \min \left(\eta,\ \delta_k,\ \min_{y_1 \neq y_2 \in Y_k} \operatorname{dist}(y_1, y_2)\right).$$

Choose a refinement $\mathcal{W}_{k+1}$ of $\mathcal{W}_k$, with $\operatorname{mesh}(\mathcal{W}_{k+1}) \leq \sigma/4$. Use Lemmas 4.3.1 and 4.3.2 to choose a refinement $\mathcal{V}_{k+1}$ of $\mathcal{U}_{k+1} \wedge f_k^{-1}(\mathcal{W}_{k+1}) \wedge \mathcal{V}_k$, of order less than or equal to n, such that $\mathcal{V}_{k+1}$ satisfies condition 4.1, such that for no $V \in \mathcal{V}_{k+1}$ is $\mathcal{V}_{k+1} \setminus \{V\}$ a covering for X, and such that condition 4.2 is satisfied with $\mathcal{V}_k$ in place of $\mathcal{V}$, and $\mathcal{V}_{k+1}$ in place of $\mathcal{W}$. (Then conditions (i) and (ii) are satisfied with $i = k+1$). By choice of $\mathcal{W}_{k+1}$ to satisfy $\operatorname{mesh}(\mathcal{W}_{k+1}) \leq \sigma/4 \leq \min_{y_1 \neq y_2 \in Y_k} \operatorname{dist}(y_1, y_2)/4$, for any $V \in \mathcal{V}_{k+1}$, no two elements of Y_k may both belong to $f_k(V)$. So we may choose sets $Z = \{z(V) : V \in \mathcal{V}_{k+1}\} \supset Y_k$, and, for each $l \geq 1$, $Z_l = \{z_l(V) : V \in \mathcal{V}_{k+1}\}$, thus:

> For each $V \in \mathcal{V}_{k+1}$ choose $z(V) \in f_k(V)$ so that $Y_k \subseteq \{z(V) : V \in \mathcal{V}_{k+1}\}$; and points $z_l(V)$, for each $l \geq 1$, such that the elements of Z_l are in general position, $\operatorname{dist}(z_l(V), f_k(V)) < \sigma$, and $z_l(V) \to z(V)$ as $l \to \infty$. Note that $z(v) = y_k$ for each $V \in \mathcal{V}_{k+1}$ such that $f_k(V) = y_k$.

Then, for each $l \geq 1$, we may define the maps $g, g_l : X \to I_{2n+1}$ by

$$g(x) = \frac{\sum_{V \in \mathcal{V}_{k+1}} \operatorname{dist}(x, X \setminus V)\, z(V)}{\sum_{V \in \mathcal{V}_{k+1}} \operatorname{dist}(x, X \setminus V)},$$
$$g_l(x) = \frac{\sum_{V \in \mathcal{V}_{k+1}} \operatorname{dist}(x, X \setminus V)\, z_l(V)}{\sum_{V \in \mathcal{V}_{k+1}} \operatorname{dist}(x, X \setminus V)}.$$

(Then g, g_l are ε_{k+1}-mappings.) The function f_{k+1} will be chosen from amongst these functions $(g_l)_1^\infty$; note that conditions (iii:a) and (iii:b) will then be satisfied for $i = k+1$. It remains to choose l large enough that conditions (iii:c) and (iv) are also satisfied.

Write S_l for the polytope in I_{2n+1} with vertex set Z_l, and such that $\operatorname{conv}(\{z_l(V_1), \ldots, z_l(V_s)\})$ is a face of S_l f and only if $V_1, \ldots, V_s \in \mathcal{V}_{k+1}$ and $V_1 \cap \ldots \cap V_s \neq \emptyset$. (Then $g_l(X) \subseteq S_l$.)

We claim that if $\{w_V : V \in \mathcal{V}_{k+1}\}$ is such that $\sum_{V \in \mathcal{V}_{k+1}} w_V z_1(V) \in S_1$, then

$$\lim_{l \to \infty} \sum_{V \in \mathcal{V}_{k+1}} w_V z_l(V) \in Q_k.$$

Clearly, $\lim_{l \to \infty} \sum_{V \in \mathcal{V}_{k+1}} w_V z_l(V) = \sum_{V \in \mathcal{V}_{k+1}} w_V z(V)$. Also, the definition of S_l implies that $\cap \{V \in \mathcal{V}_{k+1} : w_V \neq 0\} \neq \emptyset$. Then since $\mathcal{V}_k$ and $\mathcal{V}_{k+1}$ together satisfy condition 4.2, we have

$$\cap \{U \in \mathcal{V}_k : U \cap V \neq \emptyset \text{ for some } V \in \mathcal{V}_{k+1} \text{ with } w_V \neq 0\} \neq \emptyset.$$

So the face

$$\mathrm{conv}\left(\{y_k(U) : U \in \mathcal{V}_k,\ U \cap V \neq \emptyset \text{ for some } V \in \mathcal{V}_{k+1} \text{ with } w_V \neq 0\}\right),$$

which contains $\sum_{V \in \mathcal{V}_{k+1}} w_V z(V)$, is itself in Q_k.

For each $l_1, l_2 \geq 1$, the mapping between S_{l_1} and S_{l_2} which associates the points $\sum_{V \in \mathcal{V}_{k+1}} w_V z_{l_1}(V)$ and $\sum_{V \in \mathcal{V}_{k+1}} w_V z_{l_2}(V)$ is a Lipschitz map, and for l_1, l_2 large enough its Lipschitz constant is close to 1. To see this, let $\beta > 0$, and choose $m(\beta)$ large enough that whenever $l \geq m(\beta)$, we have $z_l(V) \in B(z(V), \beta)$. Then if $l_1, l_2 \geq m(\beta)$,

$$\begin{aligned} \frac{|\sum u_V z_{l_1}(V) - \sum w_V z_{l_1}(V)|}{|\sum u_V z_{l_2}(V) - \sum w_V z_{l_2}(V)|} &\leq \frac{|\sum u_V z_{l_2}(V) - \sum w_V z_{l_2}(V)|}{|\sum u_V z_{l_2}(V) - \sum w_V z_{l_2}(V)|} + 4\beta \sum w_V \\ &= 1 + 4\beta. \end{aligned}$$

So we may fix m_1 large enough that this mapping between S_{l_1} and S_{l_2} has Lipschitz constant smaller than 2, say, whenever $l_1, l_2 \geq m_1$.

Now choose $\tau < \delta_k$ sufficiently small that

$$R_{\tau}^{n+\varepsilon_1}(S_{m_1}) \leq 2^{-(k+2+n+\varepsilon_1)}.$$

Then, for each $l \geq m_1$,

$$R_{\frac{\tau}{2}}^{n+\varepsilon_1}(S_l) \leq 2^{-(k+2)} = \frac{1}{2}(c_{k+1} - c_k),$$

and so

$$R_{\frac{\tau}{2}}^{n+\varepsilon_i}(S_l) \leq R_{\frac{\tau}{2}}^{n+\varepsilon_1}(S_l) \leq \frac{1}{2}(c_{k+1} - c_k) < \frac{1}{2}(c_{k-i+2} - c_{k-i+1}). \tag{4.3}$$

For $\delta > \tau$, call a packing of a set S by balls with radii in the range $[\tau, \delta]$ a $[\tau, \delta]$*-packing*, and write

$$R_{[\tau,\delta]}^{n+\varepsilon_i}(S) = \sup\left\{\sum r_j^{n+\varepsilon_i} : \{B(x_j, r_j)\} \text{ is a } [\tau,\delta]\text{-packing of } S\right\}.$$

As $l \to \infty$, $\sup_{x \in X} \mathrm{dist}(g_l(x), g(x)) = \eta(l) \to 0$. Choose $m_2 \geq m_1$ large enough that $\eta(l) < \tau$ for each $l \geq m_2$.

If for some $1 \leq i \leq k$, $\{B(x_j, r_j)\}$ is a $[\tau, \delta_i]$-packing of S_l, then for some collection $\{w_j\}$, where $w_j \in B(x_j, \eta(l)) \cap Q_k$ for each j, we see that $\{B(w_j, r_j - \eta(l))\}$ is a δ_i-packing of Q_k, and so

$$\sum (r_j - \eta(l))^{n+\varepsilon_i} \leq c_{k-i+1}.$$

Since the number of balls in a $[\tau, \delta_1]$-packing of any subset of I^{2n+1} is bounded, this means we may choose $m_3 \geq m_2$ so large that for each $1 \leq i \leq k$ and each $l \geq m_3$, if $\{B(x_j, r_j)\}$ is a $[\tau, \delta_i]$-packing of S_l, then

$$\sum r_j^{n+\varepsilon_i} \leq \frac{1}{2} \left(c_{k-i+2} + c_{k-i+1} \right). \tag{4.4}$$

Since any δ_i-packing of S_l may be split into a τ-packing and a $[\tau, \delta_i]$-packing, we may combine (4.3) and (4.4) to see that for each $1 \leq i \leq k$, each $l \geq m_3$,

$$R_{\delta_i}^{n+\varepsilon_i}(S_l) \leq c_{k-i+2}.$$

Now if x and x' are both contained in $V \in \mathcal{V}_{k+1}$, then by the choice of $\mathcal{V}_{k+1}$ to have sufficiently small mesh, we have $\mathrm{dist}(f_k(x), f_k(x')) < \zeta/3$. In particular, since $z(V) \in f_k(V)$ for each $V \in \mathcal{V}_{k+1}$, we have $\mathrm{dist}(f_k(x), z(V)) < \zeta/3$ for each $x \in V$. Now $g(x)$ is a convex combination of those points $z(V)$ such that $x \in V \in \mathcal{V}_{k+1}$, so

$$\mathrm{dist}(f_k(x), g(x)) < \zeta/3.$$

Since g_l tends uniformly to g as l tends to infinity, we may therefore choose $m_4 \geq m_3$ sufficiently large that $\mathrm{dist}(f_k(x), g_l(x)) < 2\zeta/3$ whenever $l \geq m_4$. So for each $1 \leq j < k+1$,

$$\mathrm{dist}(f_j(x), g_l(x)) \leq \mathrm{dist}(f_j(x), f_k(x)) + \mathrm{dist}(f_k(x), g_l(x))$$

$$\begin{aligned} &< \frac{2}{3}\mathrm{dist}(f_j(x), f_k(x)) + \frac{\xi_j}{6} + \frac{2}{3}(\xi_j/2 - \mathrm{dist}(f_j(x), f_k(x))) \\ &< \xi_j/2. \end{aligned}$$

Setting $f_{k+1} = g_{m_4}$ and $Q_{k+1} = S_{m_4}$, choosing $\delta_{k+1} < \sigma$ sufficiently small that $R^{n+\varepsilon_{k+1}}_{\delta_{k+1}}(Q_{k+1}) \leq c_1$, and setting

$$\xi_{k+1} = \inf \{\mathrm{dist}(f_{k+1}(x_1), f_{k+1}(x_2)) : x_1, x_2 \in X,\ \mathrm{dist}(x_1, x_2) \geq \varepsilon_{k+1}\},$$

we see that $\xi_{k+1} > 0$ since f_{k+1} is an ε_{k+1}-mapping, and that conditions (iii:c) and (iv) are satisfied with $i = k+1$. So the induction is complete.

Let $f : X \to I^{2n+1}$ be the pointwise limit of the sequence $(f_i)_1^\infty$. Condition (iv) ensures that, for each $l \geq 1$ and $1 \leq i < l$, $\mathrm{dist}(f_i, f_l) < \xi_i/2$. So f is the uniform limit of $(f_i)_1^\infty$ and so continuous, and $\mathrm{dist}(f_i, f) \leq \xi_i/2$. Suppose now that $x_1, x_2 \in X$ are such that $f(x_1) = f(x_2)$. Then $\mathrm{dist}(f_i(x_1), f_i(x_2)) < \xi_i$, for each i, and so $\mathrm{dist}(x_1, x_2) \leq \varepsilon_i$. So f is an ε_i-mapping for each $i \geq 1$, and hence, by Theorem 4.2.1, is a homeomorphism.

It is also not hard to see that $R^n_0(f(X)) < \infty$. Suppose $i \geq 1$ and $\{B(x_j, r_j)\}$ is a δ_i-packing of $f(X)$. Choose $l \geq i$ so large that $\xi_l \leq \min\{r_j/2\}$. Then we may find points $w_j \in B(x_j, r_j) \cap Q_l$ such that $\{B(w_j, r_j - \xi_l)\}$ is a packing of Q_l.

$R^{n+\varepsilon_i}_{\delta_i}(Q_l) \leq c_{l-i+1}$ for each $1 \leq i \leq l$, so

$$\begin{aligned} \sum (r_j/2)^{n+\varepsilon_i} &\leq \sum (r_j - \xi_l)^{n+\varepsilon_i} \leq c_{l-i+1}, \\ \sum r_j^{n+\varepsilon_i} &\leq c_{l-i+1} 2^{-(n+\varepsilon_i)} \leq 2^{1-(n+\varepsilon_i)}. \end{aligned}$$

So for each ε_i we have that $R^{n+\varepsilon_i}_{\delta_i}(f(X))$, (and hence $R^{n+\varepsilon_i}_0(f(X))$,) are bounded, and so, since $\delta_k \searrow 0$, we have $\dim_{\mathcal{R}} f(X) \leq n$, as required.

Lemmas 1.5.2, 1.5.3, 1.3.2, and the Szpilrajn inequality, together show that

Lemma 4.4.2 *If X is a separable metric space, then for each $S \subseteq X$,*

$$\dim_{\mathcal{T}}(S) \leq \dim_{\mathcal{H}}(S) \leq \dim_{\mathcal{P}}(S) \leq \dim_{\mathcal{R}}(S) = \dim_{\tilde{\mathcal{R}}}(S).$$

Theorem 4.4.3 *For any separable metric space X,*

$$\dim_{\mathcal{T}}(X) = \min\left\{\dim_{\mathcal{Q}}(X') : X' \textit{ is homeomorphic to } X\right\},$$

where $\mathcal{Q}$ stands for any of $\mathcal{P}$, $\mathcal{R}$, or $\tilde{\mathcal{R}}$.

Proof. If $\dim_{\mathcal{T}}(X) = \infty$, then by the above lemma there is nothing to prove. By Lemma 4.4.2 and Theorem 4.2.2, it is sufficient to show that if X is compact and $\dim_{\mathcal{T}}(X) < \infty$, then

$$\dim_{\mathcal{T}}(X) = \min\left\{\dim_{\mathcal{R}}(X') : X' \text{ is homeomorphic to } X\right\},$$

and this follows from Theorem 4.4.1.

Chapter 5

Conditions for equality of Hausdorff and packing measures on $\mathbf{R}^n$

5.1 Introduction

This chapter answers the question, for which Hausdorff functions h may the measures $\mathcal{H}^h|_A$ and $\mathcal{P}^h|_A$ agree for some subset A of $\mathbf{R}^n$, and be positive and finite. We show that these conditions imply that h is a regular density function, in the sense of Preiss, (see [PrD]), using the fact that this common measure necessarily has h-density equal to 1 almost everywhere.

In [PrD] it was shown that regular density functions are exactly those functions $h : \mathbf{R}^+ \to \mathbf{R}^+$ which satisfy particular limiting conditions near 0. We also show here that for each function h satisfying these limiting conditions, there is a subset A of $\mathbf{R}^n$ such that $\mu = \mathcal{H}^h|_A = \mathcal{P}^h|_A$ is a positive finite measure. The situation for functions $h(r) = r^s$ has been dealt with previously, and the sets A for which it is possible that measures $\mathcal{H}^s|_A$ and $\mathcal{P}^s|_A$ agree have been fully characterised. The existence of such a set implies that s is an integer and that A is s-rectifiable.

In this chapter we rely heavily on the concepts and results of [PrD], (which in part grew out of the seminal work of Besicovitch, [BeA1, BeA3, BeA4]), and also quote a result from [MaPr]. As a consequence, the preliminaries below are rather lengthy, in order to provide all the prerequisites for the work that follows.

5.2 Definitions

In what follows, a 'measure' is a Borel regular outer measure on $\mathbf{R}^n$, (that is, every subset of $\mathbf{R}^n$ is contained in a Borel set of the same μ measure,) such that the Borel sets are measurable. If μ is also *locally finite*, that is, if for every x in $\mathbf{R}^n$ there is $r > 0$ such that $\mu B(x,r) < \infty$, we call μ a *Radon measure.* We note that μ is locally finite if and only if every compact subset of $\mathbf{R}^n$ has finite μ-measure.

(i) If $h : \mathbf{R}^+ \to \mathbf{R}^+$, μ measures $\mathbf{R}^n$, and $x \in \mathbf{R}^n$, we define $\overline{D}^h(\mu,x)$ and $\underline{D}^h(\mu,x)$, the *upper and lower h-densities of μ at x*, by the formulae

$$\overline{D}^h(\mu,x) = \limsup_{r\searrow 0} \mu B(x,r)/h(2r)$$

and

$$\underline{D}^h(\mu,x) = \liminf_{r\searrow 0} \mu B(x,r)/h(2r).$$

If the upper and lower h-densities of μ at x coincide and are positive and finite, we denote their common value by $D^h(\mu,x)$, and say that x is an *h-density point of μ.*

(ii) A positive function h defined on $\mathbf{R}^+$ is said to be a *density function in* $\mathbf{R}^n$ if there is a non-zero measure μ over $\mathbf{R}^n$ such that μ almost every $x \in \mathbf{R}^n$ is an h-density point of μ.

(iii) If $\{\mu_k\}$ is a sequence of measures over $\mathbf{R}^n$ and μ measures $\mathbf{R}^n$ then we say that $\mu_k \to \mu$ if

(a) μ is locally finite.

(b) $\limsup_{k\to\infty} \mu_k(D) < \infty$ for every compact set $D \subset \mathbf{R}^n$.

(c) $\lim_{k\to\infty} \int f\, d\mu_k = \int f\, d\mu$ for every continuous function f with compact support.

If $\mu_k \to \mu$ then for each compact set $D \subset \mathbf{R}^n$ and each open set $G \subset \mathbf{R}^n$,

$$\begin{aligned} \mu(D) &\geq \limsup_{k\to\infty} \mu_k(D), \\ \mu(G) &\leq \liminf_{k\to\infty} \mu_k(G). \end{aligned}$$

(For a proof, see [PrD, 1.11(4)].)

(iv) Let $x \in \mathbf{R}^n$ and $r \in \mathbf{R} \setminus \{0\}$. We define the map $T_{x,r} : \mathbf{R}^n \to \mathbf{R}^n$ by $T_{x,r}(z) = (z - x)/r$.

(v) If $T : \mathbf{R}^n \to \mathbf{R}^m$ is Borel measurable and μ measures $\mathbf{R}^n$, we define $T[\mu]$, the image of μ under T, by

$$T[\mu](E) = \mu(T^{-1}(E)) \text{ for every Borel set } E \subseteq \mathbf{R}^m.$$

(vi) Let μ measure $\mathbf{R}^n$ and $x \in \mathbf{R}^n$. A nonzero locally finite measure ψ is said to be a *tangent measure of* μ *at* x if there are sequences $r_k \searrow 0$ and $c_k > 0$ such that $\psi = \lim_{k\to\infty} c_k T_{x,r_k}[\mu]$. We write $\psi \in \mathrm{Tan}(\mu, x)$.

(vii) A density function h will be called *regular* (in the sense of Preiss) if $\lim_{r \searrow 0} h(tr)/h(r)$ exists for each $t > 0$. In [PrD, 6.5] it is shown that h is a regular density function if and only if there is $m \in \{1, \dots, n\}$ such that either

$$0 < \lim_{r \searrow 0} \frac{h(r)}{r^m} < \infty,$$

or $m \neq n$ and there is a positive non-decreasing function $\tilde{h} : \mathbf{R}^+ \to \mathbf{R}^+$ such that

$$\begin{aligned} &\lim_{r \searrow 0} \tilde{h}(r) = 0, \\ &\lim_{r \searrow 0} \frac{\tilde{h}(tr)}{\tilde{h}(r)} = 1, \text{ each } t > 0, \text{ and} \\ &\lim_{r \searrow 0} \frac{h(r)}{\tilde{h}(r) r^m} = 1. \end{aligned}$$

(viii) A measure μ on $\mathbf{R}^n$ is said to be *uniformly distributed* if $\mu B(x, r) = \mu B(y, r) < \infty$ whenever $x, y \in \operatorname{spt} \mu$ and $0 < r < \infty$.

(ix) Let μ be a Radon measure on $\mathbf{R}^n$. Then $x \in \operatorname{spt} \mu$ is called a *symmetric point of* μ if for every $\rho > 0$

$$\int_{B(x,\rho)} z \, d\mu(z) = x \mu B(x, \rho).$$

(x) A Radon measure μ on $\mathbf{R}^n$ is called *flat* if $\mu = c\mathcal{H}^m|_V$ for some constant c and some m-dimensional linear subspace V of $\mathbf{R}^n$, $(1 \leq m \leq n)$.

5.3 Some prerequisites

We now have all the concepts required to state both the theorem of Saint Raymond and Tricot, [SaTr], for functions $h(r) = r^s$, and the results from [PrD] and [MaPr] that we will need. A proof of Theorem 5.3.1 may be found in [MaP].

Theorem 5.3.1 *If $A \subseteq \mathbf{R}^n$ satisfies $\mathcal{P}^s(A) < \infty$, then $\mathcal{H}^s(A) = \mathcal{P}^s(A)$ if and only if the density $D^s(\mathcal{H}^s|_A, x)$ exists and equals 1 for $\mathcal{P}^s$ almost all $x \in A$. This in turn implies that s is an integer, and that A is s-rectifiable.*

The following results may be found in [PrD, 2.12, 4.7, 4.11, 6.1, 6.5] and [MaPr].

Theorem 5.3.2 *Let μ measure $\mathbf{R}^n$. Then μ almost every $x \in \mathbf{R}^n$ is a* point of translational invariance of $\mathrm{Tan}(\mu, x)$, *that is, μ almost every $x \in \mathbf{R}^n$ has the following property: Whenever $\psi \in \mathrm{Tan}(\mu, x)$ and $u \in \mathrm{spt}\ \psi$ then*

$$T_{u,1}[\psi] \in \mathrm{Tan}(\mu, x).$$

Theorem 5.3.3 *If μ is a locally finite measure on $\mathbf{R}^n$, then every tangent measure to μ at x is flat at μ almost every point x, if and only if*

$$\lim_{r \to 0} \frac{\mu B(x, tr)}{\mu B(x, r)} \text{ exists for some (equivalently for all) } t > 0, t \neq 1.$$

Corollary 5.3.4 *If μ measures $\mathbf{R}^n$, h is a regular density function, and μ almost every point of $\mathbf{R}^n$ is an h-density point of μ, then at μ almost every point x of $\mathbf{R}^n$, every tangent measure to μ at x is flat.*

Theorem 5.3.5 *If μ is a locally finite measure on $\mathbf{R}^n$ and almost every point of $\mathbf{R}^n$ is an h-density point of μ, then at almost every point of $\mathbf{R}^n$, every tangent measure ψ to μ at x is uniformly distributed, with $0 \in \mathrm{spt}\ \psi$.*

Theorem 5.3.6 *Let μ be a Radon measure on $\mathbf{R}^n$. If for μ almost every point x in $\mathbf{R}^n$, every tangent measure to μ at x has 0 as a symmetric point, then at μ almost every point x in $\mathbf{R}^n$, every tangent measure to μ at x is flat.*

5.4 Two density lemmas

As a first step towards proving a result for more general functions h, we now prove two simple density lemmas for the measures $\mathcal{H}^h$ and $\mathcal{P}^h$, which replace the standard density lemmas for $\mathcal{H}^s$ and $\mathcal{P}^s$. The proofs scarcely differ from those of the standard lemmas, and use the following covering theorem, which is due to Morse (see [MoA]).

Theorem 5.4.1 *Let μ be a Radon measure in $\mathbf{R}^n$, $A \subseteq \mathbf{R}^n$, $0 \leq \alpha < 1$, and let $\mathcal{B}$ be a family of closed balls such that for each point y of A and each $r > 0$ we may find a ball $B(x, s) \in \mathcal{B}$ with $s \leq r$ and $y \in B(x, \alpha s)$. Then there is a countable collection of disjoint balls $\{B_i\} \subseteq \mathcal{B}$ such that $\mu(A \setminus \bigcup_i B_i) = 0$.*

Lemma 5.4.2 *If $A \subseteq \mathbf{R}^n$ satisfies $0 < \mathcal{P}^h(A) < \infty$, then for $\mathcal{P}^h|_A$ almost every $x \in \mathbf{R}^n$,*

$$\underline{D}^h\left(\mathcal{P}^h|_A, x\right) \geq 1.$$

Proof. Since $\mathcal{P}^h$ is Borel regular, we may assume that A is a Borel set. Then $\mathcal{P}^h|_A$ is a Radon measure. It clearly suffices to show that if $0 < t < 1$ and

$$A_t = \left\{x \in A : \liminf_{r \searrow 0} \mathcal{P}^h(A \cap B(x, r))/h(2r) < t\right\},$$

then $\mathcal{P}^h(A_t) = 0$.

Let $E \subseteq A_t$, and let $\varepsilon > 0$. Choose $\delta > 0$ sufficiently small that $P^h_\delta(E) \leq (1 + \varepsilon)P^h_0(E)$. Since $\mathcal{P}^h|_A$ is a Radon measure we may use Theorem 5.4.1 with $\alpha = 0$ to choose disjoint balls $B_i = B(x_i, r_i)$ such that for each i,

(i) $x_i \in E$,

(ii) $r_i < \delta/2$,

(iii) $\mathcal{P}^h|_A(B_i) < t\,h(2r_i)$,

(iv) $\mathcal{P}^h(E \setminus \bigcup B_i) = 0$.

Then

$$\begin{aligned}\mathcal{P}^h(E) &\leq \sum_i \mathcal{P}^h(E \cap B_i) \leq \sum_i \mathcal{P}^h(A \cap B_i) \\ &< t\sum_i h(2r_i) \leq tP^h_\delta(E) \leq t(1+\varepsilon)P^h_0(E).\end{aligned}$$

Therefore $\mathcal{P}^h(E) \leq tP^h_0(E)$, for any $E \subset A_t$. So if $A_t \subset \cup_{i=1}^\infty E_i$, then $\mathcal{P}^h(A_t) \leq t\mathcal{P}^h(A_t)$, and $\mathcal{P}^h(A_t) = 0$.

Lemma 5.4.3 *Let $A \subseteq \mathbf{R}^n$ satisfy $\mathcal{H}^h(A) < \infty$. Then, for $\mathcal{H}^h|_A$ almost every x, for any $0 < \alpha < 1$ and $t > 1$ there is $r > 0$ such that, for every $s \leq r$ and every $y \in B(x, \alpha s)$,*

$$\frac{\mathcal{H}^h\left(B\left(y,s\right) \cap A\right)}{h(2s)} \leq t.$$

In particular,

$$\overline{D}^h(\mathcal{H}^h|_A, x) \leq 1.$$

Proof. Choose a measurable set C which contains A, such that $\mathcal{H}^h(C) = \mathcal{H}^h(A)$. For $0 < \alpha < 1$ and $t > 1$ write

$$\begin{aligned}C_{\alpha,t} = \{x \in C : &\text{ for each } r > 0, \text{ there are } s \leq r \text{ and } y \in U(x, \alpha s), \\ &\text{such that } \mathcal{H}^h\left(C \cap B\left(y,s\right)\right) > t\,h(2s)\}.\end{aligned}$$

It is sufficient to show that, for any $0 < \alpha < 1$ and $t > 1$, $\mathcal{H}^h\left(C_{\alpha,t}\right) = 0$.

Fix $0 < \alpha < 1$ and $t > 1$, choose $\varepsilon > 0$, and let K be a compact subset of $C_{\alpha,t}$ satisfying $\mathcal{H}^h(K) \geq (1-\varepsilon)\mathcal{H}^h\left(C_{\alpha,t}\right)$. (This is possible since C is $\mathcal{H}^h$ measurable, $\mathcal{H}^h|_C$ is Radon, and $C_{\alpha,t}$ is a G_δ subset of C). We may now choose $\delta_i \searrow 0$, and use Theorem 5.4.1 to choose disjoint balls $\{B_{i,j}\}_{j=1}^\infty = \{B\left(y_{i,j}, r_{i,j}\right)\}_{j=1}^\infty$ for each i, such that

(i) $K \cap B(y_{i,j}, \alpha r_{i,j}) \neq \emptyset$,

(ii) $r_{i,j} \leq \delta_i/2$,

(iii) $\mathcal{H}^h(C \cap B_{i,j}) > t\,h(2r_{i,j})$,

(iv) $\mathcal{H}^h(K \setminus \cup_j B_{i,j}) = 0$.

If $y \in \bigcup_{i\geq k}\bigcup_j B_{i,j}$, then $\operatorname{dist}(K,y) < \delta_k$. So if $y \in \bigcap_{k\geq 1}\bigcup_{i\geq k}\bigcup_j B_{i,j}$, then $\operatorname{dist}(K,y) = 0$, so $y \in K$. Therefore

$$\begin{aligned}
\mathcal{H}^h(C_{\alpha,t}) &\geq \mathcal{H}^h(K) \geq \mathcal{H}^h\left(\bigcap_{k\geq 1}\bigcup_{i\geq k}\bigcup_j B_{i,j}\right) \\
&= \lim_{k\to\infty}\mathcal{H}^h\left(C \cap \bigcup_{i\geq k}\bigcup_j B_{i,j}\right) \\
&\geq \limsup_{k\to\infty}\mathcal{H}^h\left(C \cap \bigcup_j B_{k,j}\right) \\
&= \limsup_{k\to\infty}\sum_{j=1}^{\infty}\mathcal{H}^h(C \cap B_{k,j}) \\
&> \limsup_{k\to\infty}\; t\sum_{j=1}^{\infty} h(2r_{k,j}) \\
&\geq t\limsup_{k\to\infty} H^h_{\delta_k}\left(K \cap \bigcup_j B_{k,j}\right) \\
&= t\mathcal{H}^h(K) \geq t(1-\varepsilon)\mathcal{H}^h(C_{\alpha,t}).
\end{aligned}$$

Letting $\varepsilon \searrow 0$, we see that $\mathcal{H}^h(C_{\alpha,t}) = 0$. The second statement of the theorem follows immediately.

5.5 The main result

Theorem 5.5.1 *Let $A \subseteq \mathbf{R}^n$, and $\mu = \mathcal{P}^h|_A = \mathcal{H}^h|_A$ be a positive finite measure. Then, for μ almost every $x \in A$, every tangent measure to μ at x is flat and h is a regular density function.*

Proof. Lemmas 5.4.2 and 5.4.3 together show that the h-density of μ exists and equals 1 μ almost everywhere. Therefore, by Theorem 5.3.5, at μ almost every $x \in \mathbf{R}^n$ every tangent measure ψ to μ at x has 0 in its support, and is uniformly distributed, that is, $\psi B(x,\rho) = \psi B(y,\rho) < \infty$ whenever $x, y \in \text{spt } \psi$ and $0 < \rho < \infty$.

Write A^* for those points of A which are exceptional points of neither Lemma 5.4.2 nor Lemma 5.4.3, at which all tangent measures are uniformly distributed, and which are points of translational invariance of $\text{Tan}(\mu, x)$ (see Theorem 5.3.2). Then $\mu(A^*) = \mu(A)$.

Fix $x \in A^*$. We now show that for each $z \in \mathbf{R}^n$, each $\rho > 0$, and each tangent measure ψ to μ at x,

$$\psi B(z,\rho) \leq \psi B(0,\rho).$$

Fix $\psi \in \text{Tan}(\mu, x)$. Since ψ is uniformly distributed and $0 \in \text{spt } \psi$, it only remains to show the required inequality for $z \notin \text{spt } \psi$. Let $\rho > 0$.

We first suppose that $z \in U(0,\rho)$. Since

$$\psi = \lim_{k\to\infty} c_k T_{x,r_k}[\mu],$$

we have that

$$\psi U(z,\rho) \leq \liminf_{k\to\infty} c_k \mu U(x + r_k z, r_k\rho).$$

Since $z \in B(0, \alpha\rho)$ for some $\alpha < 1$, since x is an exceptional point of neither Lemma 5.4.2 nor Lemma 5.4.3, and since $x + r_k z \in U(x, r_k\rho)$, we see that for each $t > 1$ there is a number κ such that if $k > \kappa$ then

$$\mu U(x + r_k z, r_k\rho) \leq \mu B(x + r_k z, r_k\rho) \leq t\mu B(x, r_k\rho).$$

Therefore

$$\psi U(z,\rho) \leq \liminf_{k\to\infty} c_k \mu U(x + r_k z, r_k\rho) \leq \limsup_{k\to\infty} c_k \mu B(x, r_k\rho) \leq \psi B(0,\rho).$$

The measure ψ is Radon, so for each $\varepsilon > 0$ we may find $\delta > 0$ such that

$$\psi U(0, \rho + \delta) \leq \psi B(0, \rho) + \varepsilon.$$

Replacing ρ by $\rho + \delta/2$ in the above calculations, we see that

$$\psi U(z, \rho + \delta/2) \leq \psi B(0, \rho + \delta/2),$$

so

$$\psi B(z, \rho) \leq \psi U(z, \rho + \delta/2) \leq \psi B(0, \rho + \delta/2) \leq \psi U(0, \rho + \delta) \leq \psi B(0, \rho) + \varepsilon.$$

The choice of $\varepsilon > 0$ was arbitrary, so

$$\psi B(z, \rho) \leq \psi B(0, \rho).$$

Now suppose that $z \in \partial B(0, \rho)$; then for each $\rho_1 > \rho$ we have $z \in U(0, \rho_1)$, and $\psi B(z, \rho_1) \leq \psi B(0, \rho_1)$. Therefore

$$\psi B(z, \rho) \leq \psi B(0, \rho_1) \text{ for each } \rho_1 > \rho,$$

and

$$\psi B(z, \rho) \leq \lim_{\rho_1 \to \rho} \psi B(0, \rho_1) = \psi \left(\bigcap_{\rho_1 > \rho} B(0, \rho_1) \right) = \psi B(0, \rho).$$

The third case we must consider is that where $B(z, \rho) \cap B(0, \rho) = \emptyset$. If $B(z, \rho) \cap \operatorname{spt} \psi = \emptyset$, the inequality $B(z, \rho) \leq B(0, \rho)$ is obvious. If $B(z, \rho) \cap \operatorname{spt} \psi \neq \emptyset$, we may choose $w \in B(z, \rho) \cap \operatorname{spt} \psi$ and use the fact that x is a point of translational invariance of $\operatorname{Tan}(\mu, x)$ to see that

$$T_{w,1}[\psi] B(z + w, \rho) \leq T_{w,1}[\psi] B(0, \rho),$$

and so

$$\psi B(z, \rho) \leq \psi B(w, \rho) \leq \psi B(0, \rho).$$

Therefore, the required inequality holds for each $z \in \mathbb{R}^n$.

It is now not hard to show that 0 is a symmetric point of each measure $\psi \in \mathrm{Tan}(\mu, x)$.

Let $\rho > 0$. For $y \in \mathbf{R}^n$, define

$$F(y) = \int \left(\rho^2 - (z-y)^2\right) \chi_{B(y,\rho)}(z) d\psi(z).$$

Since $0 \leq \rho^2 - (z-y)^2 \leq \rho^2$ for $z \in B(y,\rho)$ and $\psi B(y,\rho)$ is finite; and since

$$\nabla_y \left(\rho^2 - (z-y)^2\right) = 2(z-y)$$

and $\rho^2 - (z-y)^2$ vanishes on $\partial B(y,\rho)$,
we see that F is differentiable.

Then, using Fubini's Theorem,

$$\begin{aligned} F(y) &= \int_0^\infty \psi\left\{z : \left(\rho^2 - (z-y)^2\right)\chi_{B(y,\rho)}(z) > t\right\} dt \\ &= \int_0^{\rho^2} \psi B\left(y, \sqrt{\rho^2 - t}\right) dt \\ &\leq \int_0^{\rho^2} \psi B\left(0, \sqrt{\rho^2 - t}\right) dt \\ &= F(0). \end{aligned}$$

Therefore 0 is a maximum for F, so $F'(0) = 0$, and

$$\int_{B(0,\rho)} z\, d\psi(z) = 0,$$

that is, 0 is a symmetric point of ψ. So we may use Theorem 5.3.6 to see that for almost every $x \in A$ every tangent measure to μ at x is flat.

Lemma 5.5.2 *If μ is a Radon measure on $\mathbf{R}^n$, A is a compact subset of $\mathbf{R}^n$, and $\underline{D}^h(\mu, x) \geq 1$ for all $x \in A$, then $\mu(A) \geq \mathcal{P}^h(A)$.*

Proof. Since μ is Radon, $\mu(A) < \infty$. For $t < 1$ and $\delta > 0$ write

$$A_{t,\delta} = \{x \in A : \mu B(x,r) \geq t\, h(2r) \text{ whenever } r \leq \delta/2\}.$$

Then, for every $0 < \eta \le \delta$,

$$\begin{aligned} \mu B(A_{t,\delta},\eta) &\ge tP^h_\eta(A_{t,\delta}), \\ \mu(\operatorname{Clos}(A_{t,\delta})) &\ge tP^h_0(A_{t,\delta}), \end{aligned}$$

since if $\{B(x_i,r_i)\}$ is an η-packing of $A_{t,\delta}$, then $\mu B(x_i,r_i) \ge t\,h(2r_i)$, and the compact set $B(A_{t,\delta},\eta)$ contains the disjoint union $\bigcup B(x_i,r_i)$.

By assumption, $A = \bigcup_{\delta>0}\operatorname{Clos}(A_{t,\delta})$ for each $t<1$. The measures μ and $\mathcal{P}^h$ are Borel regular and $A_{t,1/n} \subseteq A_{t,1/(n+1)}$ for each n, so for each $t<1$,

$$\begin{aligned} \mathcal{P}^h(A) &= \mathcal{P}^h\Big(\bigcup_{n=1}^\infty \operatorname{Clos}(A_{t,1/n})\Big) = \lim_{n\to\infty}\mathcal{P}^h(A_{t,1/n}) \\ &\le \lim_{n\to\infty} P^h_0(A_{t,1/n}) \le t^{-1}\lim_{n\to\infty}\mu\left(\operatorname{Clos}\left(A_{t,1/n}\right)\right) \\ &= t^{-1}\mu\Big(\bigcup_{n=1}^\infty \operatorname{Clos}(A_{t,1/n})\Big) = t^{-1}\mu(A). \end{aligned}$$

Since $t<1$ was arbitrary, we have $\mu(A) \ge \mathcal{P}^h(A)$, as required.

Lemma 5.5.3 *If μ is a Radon measure on $\mathbf{R}^n$, $\overline{D}^h(\mu,x) \le 1$ for all $x \in A$, and all tangent measures to μ are flat at each $x \in A$, then for each $x \in A$,*

$$\lim_{\varepsilon\searrow 0}\left(\sup\left\{\frac{\mu(D)}{h(\operatorname{diam} D)} : x \in D,\ \operatorname{diam}(D) < \varepsilon,\ D \text{ compact, convex}\right\}\right) \le 1.$$

Proof. Suppose not, then for some $x \in A$, without loss of generality $x = 0$, we may find numbers $c_k > 0$, $t > 1$, $r_k \searrow 0$, compact convex sets D_k of diameter 1 and containing 0, $m \in \{1,\dots,n\}$ and $V \in G(n,m)$, such that

(i) $D_k \to D$ (a nonempty compact convex set with $\operatorname{diam}(D) \le 1$) in the Hausdorff metric,

(ii) $c_k T_{x,r_k}[\mu] \to \mathcal{H}^m|_V \in \operatorname{Tan}(\mu,x)$,

(iii) $\mu(r_k D_k)/h(r_k) \ge t$ for each k.

Write $E_k = \text{Clos}\left(\bigcup_{n\geq k} D_k\right)$, then $E_{k+1} \subseteq E_k$, and $E_k \to D$ in the Hausdorff metric. So

$$\mathcal{H}^m|_V(E_k) \to \mathcal{H}^m|_V(D).$$

Also,

$$\mu(r_l E_l) \geq \mu(r_l D_l) \geq t\, h(r_l), \text{ for each } l.$$

Since

$$\mathcal{H}^m|_V = \lim_{l\to\infty} c_l T_{x,r_l}[\mu],$$

we have

$$\mathcal{H}^m|_V(E_k) \geq \limsup_{l\to\infty} c_l \mu(r_l E_k) \text{ for each } k.$$

Choose k_1 so large that whenever $k \geq k_1$,

$$\mathcal{H}^m|_V(E_k) \leq \frac{(3+t)}{4}\mathcal{H}^m|_V(D).$$

Then, using the isodiametric inequality, for each $k \geq k_1$ we have

$$\frac{(3+t)}{4}\mathcal{H}^m|_V B(0,1/2) \geq \frac{(3+t)}{4}\mathcal{H}^m|_V(D) \geq \limsup_{l\to\infty} c_l \mu(r_l E_k).$$

Since $E_{k+1} \subseteq E_k$ for each k,

$$\limsup_{l\to\infty} c_l \mu(r_l E_k) \geq \limsup_{l\to\infty} c_l \mu(r_l E_l).$$

So

$$\frac{(3+t)}{4}\mathcal{H}^m|_V B(0,1/2) \geq \limsup_{l\to\infty} c_l \mu(r_l E_l) \geq \limsup_{l\to\infty} c_l t\, h(r_l).$$

By assumption $\overline{D}^h(\mu,x) \leq 1$ for each $x \in A$, so

$$\limsup_{l\to\infty} \mu\left(B(x,r_l/2)\right) h(r_l)^{-1} \leq 1.$$

Therefore

$$\begin{aligned}
\frac{(3+t)}{4}\mathcal{H}^m|_V B(0,1/2) &\geq t \limsup_{l\to\infty} c_l \mu B(x,r_l/2) \\
&\geq t \liminf_{l\to\infty} c_l \mu U(x,r_l/2) \\
&\geq t\mathcal{H}^m|_V\left(U\left(0,1/2\right)\right).
\end{aligned}$$

So $t \leq (1+t)/2$, and $t \leq 1$.

Lemma 5.5.4 *If μ is a Radon measure on $\mathbf{R}^n$ and $A \subseteq \mathbf{R}^n$ such that for every point of A*

$$\lim_{\varepsilon \searrow 0}\left(\sup\left\{\frac{\mu(D)}{h(\operatorname{diam} D)} : x \in D,\ \operatorname{diam}(D) < \varepsilon,\ D \textit{ compact, convex}\right\}\right) \leq 1,$$

then

$$\mu(A) \leq \mathcal{H}^h(A).$$

Proof. For $t > 1$ and $\delta > 0$, let

$$A_{t,\delta} = \{x \in A : \mu(D) \leq t\, h(\operatorname{diam} D) \text{ whenever } x \in D,$$
$$\operatorname{diam}(D) < \delta, \text{ and } D \text{ is compact and convex}\}.$$

Suppose $A_{t,\delta} \subseteq \bigcup_{i=1}^{\infty} D_i$, where the sets D_i are compact and convex, with $\operatorname{diam}(D) < \delta$. Then

$$\mu(A_{t,\delta}) \leq \sum_{i=1}^{\infty} \mu(D_i) \leq t \sum_{i=1}^{\infty} h(\operatorname{diam} D_i),$$

so $\mu(A_{t,\delta}) \leq t\, H^h_\delta(A_{t,\delta})$. By assumption, $A = \bigcup_{n=1}^{\infty} A_{t,1/n}$, so, since $A_{t,1/n} \subseteq A_{t,1/(n+1)}$, we have

$$\begin{aligned}\mu(A) &= \mu\left(\bigcup_{n=1}^{\infty} A_{t,1/n}\right) = \lim_{n\to\infty} \mu(A_{t,1/n}) \\ &\leq t \lim_{n\to\infty} H^h_{1/n}(A_{t,1/n}) \leq t \lim_{n\to\infty} H^h_{1/n}(A) = t\mathcal{H}^h(A).\end{aligned}$$

Letting $t \searrow 1$ gives the result.

Theorem 5.5.5 *If $A \subseteq \mathbf{R}^n$ and $\mu = \mathcal{H}^h|_A = \mathcal{P}^h|_A$ is a positive finite measure, then h is a regular density function and μ has h-density* 1 *almost everywhere. Conversely, for each function h which satisfies the limiting conditions listed in Section 5.2* (vii)*, there is a positive finite measure μ on $\mathbf{R}^n$ with h-density* 1 *almost everywhere, such that $\mu = \mathcal{H}^h|_A = \mathcal{P}^h|_A$ for some $A \subseteq \mathbf{R}^n$.*

Proof. Lemmas 5.4.2 and 5.4.3 together imply that if $\mu = \mathcal{H}^h|_A = \mathcal{P}^h|_A$, and μ is positive and finite, then μ has density 1 almost everywhere. Theorem 5.5.1 implies that h is regular.

In [PrD, 6.5], for each regular density function h there is given a construction of a non-zero Radon measure μ in $\mathbf{R}^n$ which has positive finite constant h-density μ almost everywhere in $\mathbf{R}^n$. We normalize μ to have h-density 1 almost everywhere and write D for the set where the h-density of μ is 1. Now D is a G_δ set with $\mathcal{P}^h(D) > 0$, so we may find a compact subset C of D with $\mu(C) > 0$. Then Lemma 5.5.2 tells us that $\mu|_C(S) \geq \mathcal{P}_h|_C(S)$ for all closed subsets S of $\mathbf{R}^n$.

Corollary 5.3.4 ensures that, for μ almost every x, every tangent measure to μ at x is flat, and so we may use Lemmas 5.5.3 and 5.5.4 to show that $\mu|_C(S) \leq \mathcal{H}^h|_C(S)$ for all measurable subsets S of $\mathbf{R}^n$. Lemma 1.3.2 implies that if h is regular, then $\mathcal{H}^h(A) \leq \mathcal{P}^h(A)$ for all $A \subseteq \mathbf{R}^n$. Therefore $\mathcal{P}_h|_C$, $\mu|_C$ and $\mathcal{H}^h|_C$ agree on closed, and therefore on all, subsets of $\mathbf{R}^n$.

Bibliography

[BeA1] A. S. Besicovitch, On the fundamental geometrical properties of linearly measurable plane sets of points. *Math. Ann.* **98** (1928), 422–464.

[BeA2] A. S. Besicovitch, Concentrated and rarefied sets of points. *Acta. Math.* **62** (1934), 289–300.

[BeA3] A. S. Besicovitch, On the fundamental geometrical properties of linearly measurable plane sets of points II. *Math. Ann.* **115** (1938), 296–329.

[BeA4] A. S. Besicovitch, On the fundamental geometrical properties of linearly measurable plane sets of points III. *Math. Ann.* **116** (1939), 349–357.

[BeA5] A. S. Besicovitch, On existence of subsets of finite measure of sets of infinite measure. *Indag. Math.* **14** (1952), 339–344.

[DaR] R. O. Davies, Subsets of finite measure in analytic sets. *Indag. Math.* **14** (1952), 488–489.

[DaRo] R. O. Davies and C. A. Rogers, The problem of subsets of finite positive measure. *Bull. London Math. Soc.* **1** (1969), 47–54.

[FaK] K. J. Falconer, Fractal Geometry: Mathematical Foundations and Applications. John Wiley, 1990.

[FeH] H. Federer, Geometric Measure Theory. Springer-Verlag, 1969.

[HaH1] H. Haase, Non-σ-finite sets for packing measure. *Mathematika* **33** (1986), 129–136.

[HaH2] H. Haase, Packing measures in ultra-metric spaces. *Studia Math.* **91** (1988), 189–203.

[HoJ] J. D. Howroyd, On dimension and on the existence of sets of finite positive Hausdorff measure. *Proc. London Math. Soc. (3)* **70** (1995), 581–604.

[HuWa] Witold Hurewicz and Henry Wallman, Dimension Theory. Princeton University Press, 1941.

[IkTa] S. Ikeda and M. Tamashiro, Frostman theorem for the packing measure. *C. R. Acad. Sci. Paris T.p. Série I* **320** (1995), 1445–1448.

[JoH] H. Joyce, Concerning the problem of subsets of finite positive packing measure. Preprint.

[JoPr] H. Joyce and D. Preiss, On the existence of subsets of finite positive packing measure. *Mathematika* **42** (1995), 15–24.

[LaD] D. G. Larman, On Hausdorff measure in finite-dimensional compact metric spaces. *Proc. London Math. Soc. (3)* **17** (1967), 193–206.

[MaJ1] J. Marstrand, Some fundamental geometrical properties of plane sets of fractional dimensions. *Proc. London Math. Soc. (3)* **4** (1954), 257–302.

[MaJ2] J. Marstrand, Circular density of plane sets. *J. London Math. Soc.* **30** (1955), 238–246.

[MaJ3] J. Marstrand, The (φ, s)-regular subsets of n-space. *Trans. Amer. Math. Soc.* **113** (1964), 369–392.

[MaP] P. Mattila, Geometry of Sets and Measures in Euclidean Spaces. Cambridge University Press, 1995.

[MaMa] P. Mattila and R. D. Mauldin, Measure and dimension functions: measurability and densities. Preprint.

[MaPr] P. Mattila and D. Preiss, Rectifiable measures in $\mathbf{R}^n$ and existence of principal values for singular integrals. To appear.

[MoA] A. P. Morse, Perfect Blankets. *Trans. Amer. Math. Soc.* **61** (1947), 418–442.

[PrD] D. Preiss, Geometry of measures in $\mathbf{R}^n$: Distributions, rectifiability and densities. *Ann. Math.* **125** (1987), 537–643.

[RoC] C. A. Rogers, Hausdorff Measure. Cambridge University Press, 1970.

[SaTr] X. Saint Raymond and C. Tricot, Packing regularity of sets in n-space. *Math. Proc. Camb. Phil. Soc.* **103** (1988), 133–145.

[SzE] E. Szpilrajn, La dimension et la mesure. *Fund. Math.* **28** (1937), 81–89.

[TaTr1] S. J. Taylor and C. Tricot, Packing measure and its evaluation for a Brownian path. *Trans. Amer. Math. Soc.* **288** (1985), 679–699.

[TaTr2] S. J. Taylor and C. Tricot, The packing measure of rectifiable subsets of the plane. *Math. Proc. Camb. Phil. Soc.* **99** (1986), 285–296.

[TrC] C. Tricot, Two definitions of fractional dimension. *Math. Proc. Camb. Phil. Soc.* **91** (1982), 57–74.