Regularity and Other Properties of Hausdorff Measures

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Abstract

In the first part of the thesis the centred Hausdorff measures are studied. These measures are an often used tool in multifractal geometry and are increasingly becoming an object of study themselves. The main result of this part is that the centred Hausdorff measures are Borel regular under certain natural conditions. This question has been open for many years and has various interesting applications in the field of multifractal geometry. The significant step in proving this relation is achieved by showing the equivalence of the centred Hausdorff measure and the spherical measure. A counterexample is also given, for showing that this equivalence does not necessarily hold, if certain conditions are not fulfilled.

The other part of the thesis concerns the Besicovitch 1/2-problem. This problem has been open since 1928 and it is arguably the most famous questions in classical geometric measure theory. The general version of this conjecture states that a subset of a separable metric space with lower n-density strictly greater than 1/2 almost everywhere is n-rectifiable. There have been partial results since, but the question remained open. A non-rectifiable metric space on the real line is constructed in this thesis, so that the lower 2-density is strictly greater than 1/2. This proves that the generalized Besicovitch conjecture cannot be true. An important tool for the study of the densities is the isodiametric inequality. For various metrics and in particular for the Heisenberg group several results concerning this inequality and the lower densities are proved.

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Notation

We list here some notation which we will use throughout the thesis. Some of these concepts will be described to a greater extent later on in the text.

$\mathbf{N}$, Set of natural numbers: $1, 2, 3, \ldots$.

$\mathbf{Z}$, Set of integers.

$\mathbf{Q}$, Set of rational numbers.

$\mathbf{R}$, Set of real numbers.

$\mathbf{C}$, Set of complex numbers.

$\mathbf{R}^n$, The n-dimensional Euclidean space.

$[a, b]$, (a, b), $[a, b)$, $(a, b]$, Closed, open and half-open intervals with $a, b \in \mathbf{R} \cup \{-\infty, \infty\}$.

$\operatorname{Im} w$, The imaginary part of the complex number w.

$\subset$, $\supset$, Set inclusions which can mean equality as well.

$\mathcal{P}(\,.\,)$, Power set of a set.

$|\,.\,|$, Cardinality of a set or the usual absolute value on $\mathbf{C}$.

diam, Diameter of a set (with respect to a specific metric).

$B(x, r)$, Closed ball (with respect to a specific metric) with centre x and radius $r > 0$.

$U(x, r)$, Open ball (with respect to a specific metric) with centre x and radius $r > 0$.

$\operatorname{dist}(x, A)$, Distance between the point x and the set A with respect to a specific metric.

$\mathrm{dist}(A,B)$, Distance between two sets A and B with respect to a specific metric.

$\overline{A}$, Topological closure of a set A.

∂, Boundary of a set.

χ_A, Characteristic function of a set A.

$\mathrm{Lip}(\varphi)$, Lipschitz constant of a Lipschitz function φ.

$C_0(\mathbf{R}^n)$, Set of compactly supported continuous real-valued functions on $\mathbf{R}^n$.

$\limsup_{S\to x} T(S)$, $= \lim_{\tau\to 0} \sup\{T(S) \mid \mathrm{diam}\, S \leq \tau,\ x \in S \subset M\}$, where M is a metric space and T is a set function on $\mathcal{P}(M)$.

$\mu|_A$, Restriction of a measure μ to the set A, with the same domain of definition as μ.

$\mathrm{spt}\,\mu$, Support of a measure μ.

δ_x, The Dirac measure of a point x.

$\mathcal{H}^s$, The s-dimensional Hausdorff measure.

$\mathcal{S}^h$, Spherical measure defined by the gauge function h.

μ^h, Centred Hausdorff measure defined by the gauge function h.

$\mathcal{L}^n$, The n-dimensional Lesbegue measure.

$\alpha(n)$, Volume of the unit ball in $\mathbf{R}^n$.

$\overline{D}_s(E,x)$, Upper s-density of E in x.

$\underline{D}_s(E,x)$, Lower s-density of E in x.

$D_s(E,x)$, The s-density of E in x.

$\overline{d}_s(E,x)$, Centred upper s-density of E in x.

$\underline{d}_s(E,x)$, Centred lower s-density of E in x.

$\overline{D}_s(\mu,x)$, Upper s-density of μ in x.

$\underline{D}_s(\mu,x)$, Lower s-density of μ in x.

Chapter 1

Geometric Measure Theory

1.1 An Introduction to the Thesis

The main theme of this thesis is the study of various properties related to Hausdorff measures. The main results are the proof that the centred Hausdorff measure is Borel regular and the (negative) answer to the generalized Besicovitch 1/2-problem.

In this section we will outline our most interesting results. The formal definitions of the concepts used here will be given in the next section.
In Chapter 2 we focus on the Borel regularity of the centred Hausdorff measure. The centred Hausdorff measure is defined similarly to the Hausdorff measure but one is just allowed to use centred coverings by balls. We will in particular be interested in centred Hausdorff measures defined with the help of the gauge functions $h(x,r) = h_{\nu,t,q}(x,r) = r^t(\nu B(x,r))^q$, where ν is a Borel measure and t and q are real numbers. First we will show that if ν is a doubling measure, then the centred Hausdorff measure (defined by the gauge

function as above), is equivalent to the usual spherical measure (with the same gauge function), and thus they define the same dimension. Moreover, we will show that this statement is true even without the doubling condition, if $q \geq 1$ and $t \geq 0$ or if $q \leq 0$. For the simpler case, where h is a doubling gauge function depending only on the radius of the balls, this equivalence has already been shown in [34]. In the last section of the chapter we will give an example in $\mathbf{R}^2$ to show the surprising fact that the above equivalence is in general not true for all $0 < q < 1$ and $t \geq 0$.

This equivalence will be used to show the Borel regularity of the centred Hausdorff measure for the cases described before. This is an essential basic property of measures, which was unknown for this measure for many years. The interesting question whether this regularity property is at all true for the centred Hausdorff measure has been asked at various conferences and for example in [8] and in Section 1.8 of [9]. The theory that will be developed in Chapter 2 to give an affirmative answer to this problem is based on the author's paper [36].

The main goal of Chapter 3 and Chapter 4 will be to give estimates on the lower densities defined by the Hausdorff measure and the relation of this problem to a form of generalized isodiametric inequality. In the rest of this section we will now explain this further and put it in relation to some major concepts of geometric measure theory.

Now let us explain how the last two chapters of this thesis connect to some fundamental concepts in geometric measure theory.

One of the important theorems in geometric measure theory is that in Euclidean spaces a $\mathcal{H}^n$ measurable set E of finite $\mathcal{H}^n$ measure is n-rectifiable if and only if the n-density of E exists and equals 1 in almost all of its points. This has been proved over many decades. Besicovitch proved the first part in 1938 (see [2]) and it was finally proved in this generality by Mattila in 1975 (see [20]). Preiss even proved in 1987 (see [31]) that n-rectifiability already follows from the existence of the n-density. It is quite easy to show that this cannot be true in all metric spaces (see Theorem 55). However, Kirchheim proved in 1994:

Theorem 1 *Let M be a separable metric space. If $E \subset M$ is of finite n-dimensional Hausdorff measure and n-rectifiable, then the density of E exists and equals* 1 *in almost all of its points.*

Proof. See Theorem 9 of [15].

The other implication (i.e., if from the lower n-density of E equal one in almost all of its points it follows that E is n-rectifiable) is still an open question in the non-Euclidean case. But let us now return for a moment to the $\mathbf{R}^k$ where this is known to be true. Can we there improve this statement, by replacing 1 with some smaller, 'optimal' number? In other words, we are looking for the smallest number σ_n, such that every set with a larger lower n-density than σ_n in almost all of its points is n-rectifiable (see Definition 12 for a precise definition of σ_n). Is it perhaps true that $\sigma_n(\mathbf{R}^k)$ is smaller than, say 1/2? Or is it even true that $\sigma_n(M) \leq 1/2$, for any separable metric space M (it is not difficult to prove $\sigma_n(M) \leq 1$ - see Corollary 19)? We will call this last question the *generalized Besicovitch* 1/2*-problem.* Let us present now what is known in connection to this question.

Besicovitch conjectured already in in his famous publication [1] from 1928 that $\sigma_1(\mathbf{R}^2) = 1/2$:

'In fact I can construct a set at almost all of points of which the lower [1-] density is equal to 1/2, and also I have some reasons (though nothing like a proof) to expect that it cannot be greater than 1/2.'

This conjecture is now commonly known as *Besicovitch's* 1/2*-problem.* This is probably the most famous and oldest open problems in classical geometric measure theory. It is surely an extremely fascinating assertion and its fame has also increased by the fact that every other interesting problem regarding 1-densities in Euclidean spaces has been solved a long time ago.

There have been various publication in order to estimate σ_n for different values of n in Euclidean or other metric spaces (see in particular Preiss and Tišer's publication [32] from 1987, which gave a new best bound on σ_1 in $\mathbf{R}^2$, that was also valid in any separable metric space). Figure 1.1 shows the most important progress connected to the generalized Besicovitch 1/2-problem since its (partial) birth in 1928[1].

We show in Chapter 4 that the answer to the generalized Besicovitch 1/2-problem is 'no'. We will do that by constructing a purely 2-unrectifiable, translation-invariant metric ρ on the real line, that metrizes the Euclidean topology and that fulfills $\sigma_2(\mathbf{R}, \rho) > 1/2$.

[1]As can be seen from the citation above, Besicovitch claimed already in 1928 to be able to construct a set with the lower 1-density equal to 1/2 in almost all of its points (the author does not in any way intend to challenge this claim), but the first published construction of such a set is due to Dickinson in 1939.

Year	Author	Result
1928	A. S. Besicovitch in [1]	$\sigma_1(\mathbf{R}^2) \leq 1 - 10^{-2576}$
1938	A. S. Besicovitch in [2]	$\sigma_1(\mathbf{R}^2) \leq 3/4$
1939	D. R. Dickinson in [7]	$\sigma_1(\mathbf{R}^2) \geq 1/2$
1950	E. F. Moore in [22]	$\sigma_1(\mathbf{R}^k) \leq 3/4$
1961	J. M. Marstrand in [18]	$\sigma_2(\mathbf{R}^3) < 1$
1975	P. Mattila in [20]	$\sigma_n(\mathbf{R}^k) < 1$
1984	M. Chlebík in [6]	$\sup_k \sigma_n(\mathbf{R}^k) < 1$
1987	D. Preiss and J. Tišer in [32]	$\sigma_1(M) \leq (2+\sqrt{46})/12 \approx .7319$
1998	A. Schechter in [35]	$\sigma_1(M) < .7266$

Figure 1.1: The progress connected to the generalized Besicovitch 1/2-problem so far

In Chapter 3 we study the relation of 'generalized isodiametric inequalities' (recall, that in Euclidean spaces the usual isodiametric inequality states that the ball is the set with maximal Lebesgue measure for a given diameter) and densities in general metric spaces. This will in particular help us to give the answer to the generalized Besicovitch 1/2-problem in the last chapter.

1.2 Basic Definitions and Standard Theorems

In this section we will present some of the elementary concepts of geometric measure theory.

Definition 2 *Let X be a set. We call a set function $\mu : \mathcal{P}(X) \to \mathbf{R}_0^+ \cup \{\infty\}$ a* measure *if* *(i)* $\mu\emptyset = 0,$

(ii) $\mu A \leq \mu B$ *if* $A \subset B \subset X,$

(iii) $\mu(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu A_i$ *if* $A_1, A_2, \ldots \subset X.$

In other words a measure will be for us a non-negative, monotonic, subadditive set function vanishing for the empty set. This differs from the usual definition in the classical measure theory where a measure means a non-negative, countably additive set function defined on a σ-algebra of X, which is not necessarily the whole power set. However, our definition is not only more convenient for our purpose, but it is also consistent with most of the modern work in geometric measure theory.

For the rest of the section we take M to be a separable metric space. We will now provide some basic definitions in relation to measures on metric spaces.

Definition 3 *Let* μ *and* ν *be measures on* M.

(i) $A \subset M$ *is called* μ-measurable *if for all* $E \subset M$

$$\mu E = \mu(E \cap A) + \mu(E \backslash A).$$

(ii) We say that μ *is a* Borel measure *if all Borel sets are* μ*- measurable.*

(iii) We say that μ *is* Borel regular *if it is a Borel measure and for every* $E \subset M$ *we find a Borel set* $B \supset E$ *so that* $\mu E = \mu B$.

(iv) We say that μ *is a* Radon measure *if it is a Borel measure and*

- $\mu K < \infty$ *for compact sets* $K \subset M$,
- $\mu U = \sup\{\mu K \mid K \subset U$ *is compact* $\}$ *for open sets* $U \subset M$,

– $\mu A = \inf\{\mu U \mid U \supset A \text{ is open }\}$ *for* $A \subset M$.

(v) *We say that* μ *and* ν *are* equivalent *if we find two positive constants* c_0 *and* c_1 *such that*

$$\mu A \leq c_0 \nu A \leq c_1 \mu A \quad \textit{for all} \quad A \subset M.$$

(vi) *We say that* μ *is a* metric measure *if for all* $A, B \subset M$ *with* $dist(A, B) > 0$ *we have*

$$\mu(A \cup B) = \mu A + \mu B.$$

(vii) *We say that* μ *is* locally-finite *if for all* $x \in M$ *we can find* $r > 0$ *so that* $\mu B(x, r) < \infty$.

The next theorem about the properties of measurable sets follows easily from the definition of measurability.

Theorem 4 *Let* μ *be a measure on* M *and let* $\{A_i\}_i$ *be a sequence of* μ*-measurable sets in* M.

(i) *If the* A_i, $i = 1, 2, ...$ *are pairwise disjoint, then*

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu A_i.$$

(ii) *If* $A_1 \subset A_2 \subset \ldots$, *then*

$$\lim_{i \to \infty} \mu A_i = \mu(\bigcup_{i=1}^{\infty} A_i).$$

(iii) *If* $A_1 \supset A_2 \supset \ldots$ *and* $\mu A_1 < \infty$, *then*

$$\lim_{i \to \infty} \mu A_i = \mu(\bigcap_{i=1}^{\infty} A_i).$$

Proof. See for example Section 1.1, Theorem 1 in [10].

Now to a rather technical, but nevertheless quite useful theorem.

Theorem 5 *Let μ be a Borel regular measure on M, $A \subset M$ a μ-measurable set and $\epsilon > 0$. If there are open sets $U_1, U_2, \ldots \subset M$ such that $A \subset \cup_{i=1}^{\infty} U_i$ and $\mu U_i < \infty$ for all $i \in \mathbf{N}$, then there is an open set $U \supset A$ with $\mu(U) - \mu(A) < \epsilon$.*

Proof. See for example Theorem 1.10 in [21].

The following theorem will be useful in the proof that our Hausdorff measures are Borel measures.

Theorem 6 *Let μ be a measure on M. Then μ is a Borel measure if and only if μ is a metric measure.*

Proof. See for example Theorem 1.5 in [11].

Definition 7 *We say that a measure ν on M fulfills the* doubling condition *if there are $R, c > 0$ such that for all $x \in M$ and $0 < r < R$ we have*

$$\nu B(x, 2r) \leq c \nu B(x, r).$$

Let us now define the measures that are the central object of study in this thesis (we set $0^0 = 1$ and $0^q = \infty$ for $q < 0$).

Definition 8 *Let $A \subset M$ and let ν be a Borel measure on M, that is finite on balls. Define for $q, t \in \mathbf{R}$, $r > 0$ and $x \in M$ the gauge function $h(x, r) = h_{\nu,t,q}(x, r) = r^t (\nu B(x, r))^q$.*

(i) *For* $\delta > 0$ *we call* $\{E_i\}_i$ *a* δ-covering of A *if* $E_i \subset M$, $\operatorname{diam} E_i \leq \delta$ *for* $i \in \mathbf{N}$ *and* $A \subset \bigcup_i E_i$. *If all* E_i *are closed balls centred in* A *we call it a* centred δ-covering of A.

(ii) *For* $s \geq 0$, $\delta > 0$ *we define*

$$\mathcal{H}^s_\delta A = \inf\{\sum_{i=1}^{\infty}(\operatorname{diam} E_i)^s \mid \{E_i\}_i \text{ is a } \delta-\text{covering of } A\}$$

and

$$\mathcal{H}^s A = \sup_{\delta>0} \mathcal{H}^s_\delta A = \lim_{\delta\downarrow 0} \mathcal{H}^s_\delta A.$$

We call $\mathcal{H}^s$ *the* s-dimensional Hausdorff measure.

(iii) *For* $\delta > 0$ *we define*

$$\mathcal{S}^h_\delta A = \inf\{\sum_{i=1}^{\infty} h(x_i, r_i) \mid \{B(x_i, r_i)\}_i \text{ is a } \delta-\text{covering of } A\}, A \neq \emptyset,$$

$$\mathcal{S}^h_\delta \emptyset = 0$$

and

$$\mathcal{S}^h A = \sup_{\delta>0} \mathcal{S}^h_\delta A = \lim_{\delta\downarrow 0} \mathcal{S}^h_\delta A.$$

We call $\mathcal{S}^h$ *the* spherical measure.

(iv) *For* $\delta > 0$ *we define*

$$\mu^h_\delta A = \inf\{\sum_{i=1}^{\infty} h(x_i, r_i) \mid \{B(x_i, r_i)\}_i \text{ is a centred } \delta-\text{covering of } A\},$$

$$A \neq \emptyset,$$

$$\mu^h_\delta \emptyset = 0\,,$$

$$\mu^h_0 A = \sup_{\delta>0} \mu^h_\delta A = \lim_{\delta\downarrow 0} \mu^h_\delta A$$

and

$$\mu^h A = \sup_{B \subset A} \mu_0^h B.$$

We call μ^h the centred Hausdorff measure. *If the gauge function in the above definition equals $(2r)^s$ we will write μ^s for this measure.*

We don't really need the assumption that ν is finite on balls but without some similar condition, the measures $\mathcal{S}^h$ and μ^h could be infinite on all non-empty sets. This would cause unnecessary technical difficulties later on, so we excluded this extreme case in the above definition.

Note that the definition of these measures depends very much on the metric on the underlying set, even though for clarity reasons we have decided not to add this extra parameter in the symbols. However, as two different metrics on the same set usually define two different Hausdorff (spherical, centred Hausdorff) measures, the reader should keep this fact in mind throughout this thesis and especially in the last chapter. The reader should also keep in mind (especially in Chapter 2) that we will simply write h for gauge functions $h_{\nu,t,q}$ depending on ν, q and t. We believe that putting too many different variables in formulas is rather unpleasant for the reader and that omitting these extra parameters will create no confusion.

The set function μ_0^h is not necessarily monotone, since the smaller set may not have the centre points for the ideal covering (e.g., Section 1.4 of [9] or [34]; the example in Section 2.3 also shows that this may indeed happen). However, as we will soon see, the additional step used in the definition of the centred Hausdorff measure provides us with the monotonicity as well.

Some authors also call μ^h the multifractal (Hausdorff) measure or the covering measure.

The proof that the Hausdorff, spherical and centred Hausdorff measures are Borel measures is a standard result. However as these measures are our main object of study we have decided to include the not too difficult proof.

Theorem 9 *Let $s \geq 0$ and h be as in Definition 8. Then $\mathcal{H}^s$, $\mathcal{S}^h$ and μ^h are Borel measures and for μ_0^h we have*

$$\mu_0^h(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu_0^h A_i \quad \textit{for all} \quad A_1, A_2, \ldots \subset M.$$

Proof. i) We show that μ^h is monotone. We have for $A, B \subset M$ with $A \subset B$

$$\mu^h A = \sup_{E \subset A} \mu_0^h E \leq \sup_{E \subset B} \mu_0^h E = \mu^h B.$$

ii) We show that μ_0^h is subadditive. Suppose that $A = \cup_{i=1}^{\infty} A_i \subset M$ and $\delta > 0$. For each A_j, $j = 1, 2, \ldots$ let $\{B(x_i^j, r_i^j)\}_i$ be a centred δ-covering of A_j with

$$\sum_{i=1}^{\infty} h(x_i^j, r_i^j) \leq \mu_\delta^h A_j + \frac{\delta}{2^j}.$$

Then $\{B(x_i^j, r_i^j)\}_{i,j}$ is a centred δ-covering of A with

$$\mu_\delta^h A \leq \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} h(x_i^j, r_i^j) \leq \sum_{j=1}^{\infty} \mu_\delta^h A_j + \delta.$$

By letting $\delta \downarrow 0$ we have $\mu_0^h A \leq \sum_{j=1}^{\infty} \mu_0^h A_j$, as required.

iii) We show that μ^h is subadditive. Suppose $A = \cup_{n=1}^{\infty} A_i \subset M$ and $E \subset A$. Then

$$\mu_0^h E \leq \sum_{i=1}^{\infty} \mu_0^h (A_i \cap E) \leq \sum_{i=1}^{\infty} \mu^h A_i.$$

By taking the supremum over all $E \subset A$ we get $\mu^h A \leq \sum_{i=1}^{\infty} \mu^h A_i$, as required.

iv) We show that μ^h is a Borel measure. Suppose that $A, B \subset M$ with $\mathrm{dist}(A, B) > 0$. Then for $E \subset A$ and $F \subset B$ we also have $\mathrm{dist}(E, F) > 0$.

Hence any centred δ-covering of E is disjoint from any δ-covering of F, provided $\delta <$dist$(E,F)/2$. So

$$\mu_\delta^h(E \cup F) = \mu_\delta^h E + \mu_\delta^h F$$

and thus for $\delta \downarrow 0$

$$\mu_0^h(E \cup F) = \mu_0^h E + \mu_0^h F.$$

By taking the supremum over all $E \subset A$ and all $F \subset B$ we obtain

$$\mu^h(A \cup B) = \mu^h A + \mu^h B.$$

Therefore μ^h is a metric measure and the claim follows immediately from Theorem 6.

v) The proofs that $\mathcal{H}^s$ and $\mathcal{S}^h$ are Borel measures follow with a similar, simpler argument.

The Hausdorff measure was introduced in an even more general form at the beginning of the last century by Carathéodory [5] and Hausdorff [14]. The spherical measure is also just a specific case of the initial definition. It is very hard to overestimate the importance of the Hausdorff measure in geometric measure theory. But beside playing a central role in this area, it is also of great importance to a wide range of other mathematical areas. For further applications and a more general definition one should consult the classical reference book on Hausdorff measures [33].

The centred Hausdorff measure was introduced by Raymond and Tricot in [34] for general continuous and doubling gauge functions depending only on the radius of the ball and by Olsen in [24] as defined above. The dimension defined by these measures was used for example in [24], [25], [26], [27] and

[29] to give estimates for the multifractal spectrum of a measure. They have been used for other purposes as well (e.g. [8] ,[28] and [34]) and have recently become an object of study themselves (e.g. [9] and [17]).

Definition 10 *Let E be a subset of M, ν a finite Borel measure on M, $s > 0$ and $x \in M$.*

(i) The lower *and* upper s-densities of E at x *are defined by*

$$\underline{D}_s(E,x) = \liminf_{r \downarrow 0} \frac{\mathcal{H}^s(E \cap B(x,r))}{(2r)^s}$$

and

$$\overline{D}_s(E,x) = \limsup_{r \downarrow 0} \frac{\mathcal{H}^s(E \cap B(x,r))}{(2r)^s}.$$

If the upper and lower densities of E in x agree, then the common value is called the s-dimensional density of E at x. *We will denote it by $D_s(E,x)$.*

(ii) The centred lower *and* upper s-densities of E at x *are defined by*

$$\underline{d}_s(E,x) = \liminf_{r \downarrow 0} \frac{\mu^s(E \cap B(x,r))}{(2r)^s}$$

and

$$\overline{d}_s(E,x) = \limsup_{r \downarrow 0} \frac{\mu^s(E \cap B(x,r))}{(2r)^s}.$$

(iii) The lower *and* upper s-densities of ν at x *are defined by*

$$\underline{D}_s(\nu,x) = \liminf_{r \downarrow 0} \frac{\nu B(x,r)}{(2r)^s}$$

and

$$\overline{D}_s(\nu,x) = \limsup_{r \downarrow 0} \frac{\nu B(x,r)}{(2r)^s}.$$

Immediately from the definition we have that $\underline{D}_s(E,x) \geq \underline{D}_s(F,x)$ if $E \supset F$. The same is of course true for the other densities of (i) and (ii) in the above definition.

Definition 11 *Let E be a subset of M and $n \in \mathbf{N}$. We say that E is* n-rectifiable *if there are $A_i \subset \mathbf{R}^n$ and Lipschitz mappings $\phi_i : A_i \to M$, $i = 1, 2, \dots$ such that*

$$\mathcal{H}^n(E \setminus \bigcup_{i=1}^{\infty} \phi_i(\mathbf{A}_i)) = 0.$$

We say that E is purely n-unrectifiable *if E contains no n-rectifiable set of positive $\mathcal{H}^n$-measure.*

Note that if a set is n-rectifiable, then every subset is n-rectifiable as well. Likewise, if a set is purely n-unrectifiable, then every subset is also purely n-unrectifiable.

Definition 12 *Let $n \in \mathbf{N}$. We denote by $\sigma_n(M)$ the smallest number, such that every subset E of M with $\mathcal{H}^n(E) < \infty$ and with*

$$\underline{D}_n(E,x) > \sigma_n(M) \quad \textit{for} \;\; \mathcal{H}^n\textit{- almost every } x \in E$$

is n-rectifiable.

If we allow σ_n to be infinite, its existence is clear, but in the next section we will show that it cannot even exceed 1. For M n-rectifiable we have of course $\sigma_n(M) = 0$. If M is purely n-unrectifiable and $\mathcal{H}^n$ is locally-finite on M, then $\sigma_n(M) = \text{ess sup}_{x \in M} \underline{D}_n(M,x)$.

The following version of the Vitali Covering Theorem will be needed in the next section.

Theorem 13 *Let $E \subset M$ and $\mathcal{B}$ be a collection of closed balls in M such that for every $x \in E$ and $\epsilon > 0$ we find $r \in (0, \epsilon)$ with $B(x, r) \in \mathcal{B}$. Then there exist either*

(i) an infinite disjoint sequence $(B(x_i, r_i))_i \subset \mathcal{B}$ with $\inf_i r_i > 0$

or

(ii) a countable (possibly finite) disjoint sequence $(B(x_i, r_i))_i \subset \mathcal{B}$ so that for any $j \in \mathbf{N}$

$$E \setminus \bigcup_{i=1}^{j} B(x_i, r_i) \subset \bigcup_{i=j+1}^{\infty} B(x_i, 3r_i).$$

Proof. See for example Theorem 1.3.1 in [9].

1.3 First Density Theorems

We will provide in this section some basic facts about the densities defined in the previous section. Our main goal here is to show that σ_n cannot exceed 1 on any separable metric space. On the way to this goal, we will however be able to prove without a lot of extra work some interesting facts about other densities. The results in this section were basically proved by Besicovitch. Our presentation is based on Section 1.5 of [9].

For the rest of this section we take M to be a separable metric space. The next theorem also follows immediately from Theorem 25. But as the proof is quite simple and this result is needed later in this section, before Theorem 25 is shown, we have included the proof below.

Theorem 14 *Let $E \subset M$ and $s \geq 0$. Then*

$$2^{-s} \mu^s E \leq \mathcal{H}^s E \leq \mu^s E.$$

Proof. As every centred δ-covering is also a δ-covering, the right inequality is trivial. We will now prove the left inequality. Let $A \subset E$, fix $\delta > 0$ and let $\{E_i\}_i$ be a δ-covering of A with

$$\sum_{i=1}^{\infty}(\operatorname{diam} E_i)^s - \delta \leq \mathcal{H}^s A.$$

We may assume that we can find $x_j \in E_j \cap A$, $j = 1, 2, \ldots$. Then $\{B(x_i, \operatorname{diam} E_i)\}_i$ is a centred 2δ-covering of A with

$$\mu^s_{2\delta} A \leq \sum_{i=1}^{\infty}(2\operatorname{diam} E_i)^s \leq 2^s(\mathcal{H}^s A + \delta) \leq 2^s(\mathcal{H}^s E + \delta).$$

As $\delta > 0$ is arbitrary we have $\mu^s_0 A \leq 2^s \mathcal{H}^s E$ and by taking the supremum over all $A \subset E$ we obtain $\mu^s E \leq 2^s \mathcal{H}^s E$.

We now come to the first of our density theorems.

Theorem 15 *Let E be a subset of M, $s \geq 0$ and ν a finite Borel measure on M. Then*

(i)

$$\mu^s E \inf_{x \in E} \overline{D}_s(\nu, x) \leq \nu E \leq \mu^s E \sup_{x \in E} \overline{D}_s(\nu, x),$$

(ii)

$$\mathcal{H}^s E \inf_{x \in E} \overline{D}_s(\nu, x) \leq \nu E \leq 2^s \mathcal{H}^s E \sup_{x \in E} \overline{D}_s(\nu, x).$$

Proof. Note that (ii) follows immediately from (i) with Theorem 14. We will therefore just prove (i).

(1) Here we will prove the left inequality of (i). We may assume that $\inf_{x \in E} \overline{D}_s(\nu, x) > 0$. Let $h \in (0, \inf_{x \in E} \overline{D}_s(\nu, x))$ and $F \subset E$. We need to show that $h\mu^s_0 F \leq \nu E$. Let $V \supset E$ be an open set and choose $\delta > 0$. We define

$$\mathcal{F} = \{B(x, r) \mid x \in F,\ r \in (0, \min\{\delta, \operatorname{dist}(x, M \backslash V)\}),\ h \leq \frac{\nu B(x, r)}{(2r)^s}\}.$$

Let $\{B(y_i, s_i)\}_i \subset \mathcal{F}$ be any disjoint sub-collection. As $\cup_{i=1}^{\infty} B(y_i, s_i) \subset V$ we have

$$\sum_{i=1}^{\infty}(2s_i)^s \leq \frac{1}{h}\sum_{i=1}^{\infty} \nu B(y_i, s_i) = \frac{1}{h}\nu(\bigcup_{i=1}^{\infty} B(y_i, s_i)) \leq \frac{1}{h}\nu V < \infty. \qquad (1.1)$$

Hence $\inf_i s_i = 0$ and by applying Theorem 13 we find therefore a sub-collection $\{B(x_i, r_i)\}_i \subset \mathcal{F}$ with

$$F\backslash(\bigcup_{i=1}^{j} B(x_i, r_i)) \subset \bigcup_{i=j+1}^{\infty} B(x_i, 3r_i)$$

for all $j \in \mathbf{N}$. Thus by (1.1) for all $j \in \mathbf{N}$

$$\mu_\delta^s F \leq \sum_{i=1}^{j}(2r_i)^s + \sum_{i=j+1}^{\infty} (6r_i)^s < \infty$$

and so

$$\mu_\delta^s F \leq \sum_{i=1}^{\infty}(2r_i)^s \leq \frac{1}{h}\nu V.$$

As $V \supset E$ was an arbitrary open set we have $\mu_\delta^s F \leq \frac{1}{h}\nu E$ and for $\delta \downarrow 0$ we obtain $h\mu_0^s F \leq \nu E$, as required.

(2) We will now prove the right inequality of (i). We may assume that $\sup_{x\in E} \overline{D}_s(\nu, x) < \infty$. Let $h \in (\sup_{x\in E} \overline{D}_s(\nu, x), \infty)$. We need to show that $\nu E \leq h\mu^s E$. Define for $n \in \mathbf{N}$

$$E_n = \{x \in E \mid \frac{\nu B(x,r)}{(2r)^s} \leq h \text{ for all } r < \frac{1}{n}\}$$

and let $\delta \in (0, 1/n)$. For any centred δ-covering $\{B(x_i, r_i)\}_i$ of E_n we then have

$$\frac{1}{h}\nu E_n \leq \frac{1}{h}\nu(\bigcup_{i=1}^{\infty} B(x_i, r_i)) \leq \frac{1}{h}\sum_{i=1}^{\infty} \nu B(x_i, r_i) \leq \sum_{i=1}^{\infty}(2r_i)^s.$$

Thus

$$\frac{1}{h}\nu E_n \leq \mu_\delta^s E_n \leq \mu_0^s E_n \leq \mu^s E_n \leq \mu^s E.$$

By letting $n \to \infty$ we obtain $\nu E \leq h\mu^s E$, as required.

We will now present without proof, a theorem, that is in some way a generalization of the above theorem. We will need it in Chapter 3.

Theorem 16 *Let ν be a measure on M and $A \subset M$. Then*

$$\nu A \leq \mathcal{H}^s A \sup_{x \in A} \limsup_{S \to x} \frac{\nu(A \cap S)}{(\operatorname{diam} S)^s},$$

and if A is open

$$\nu A \geq \mathcal{H}^s A \inf_{x \in A} \limsup_{S \to x} \frac{\nu S}{(\operatorname{diam} S)^s}.$$

Proof. See Theorems 2.10.17 and 2.10.18 in [12].

In order to obtain estimates for the other densities with the help of Theorem 15, we need the following technical lemma.

Lemma 17 *Let $s > 0$, E be a subset of M with finite s-dimensional Hausdorff measure and ν a finite Borel measure on M. Then all the density functions from Definition 10 are Borel measurable functions from M to $\mathbf{R}_0^+ \cup \{\infty\}$.*

Proof. Fix $r > 0$. We show first that $x \mapsto \nu B(x,r)$ is a Borel function. To prove this, it is enough to show that for $t > 0$ the set $V = \{x \in M \mid \nu B(x,r) < t\}$ is open. For $y \in V$ we find $n \in \mathbf{N}$ so that $\nu B(y, r + 1/n) < t$. Suppose now $z \in B(y, 1/n)$. Then $B(z,r) \subset B(y, r+1/n)$ and so $z \in V$. This shows that $x \mapsto \nu B(x,r)$ is Borel measurable. The Borel regularity of $\underline{D}_s(\nu, \,.\,)$ and $\overline{D}_s(\nu, \,.\,)$ follows from this by elementary properties of measurable functions. The Borel regularity of the other densities from Definition 10 follows from the above and the fact that $\mathcal{H}^s|_E$ and $\mu^s|_E$ are both finite Borel measures by Theorem 14.

Theorem 18 *Let $s \geq 0$ and let E be a subset of M with $0 < \mathcal{H}^s E < \infty$. Then*

(i) $\overline{d}_s(E,x) = 1$ for μ^s-almost every $x \in E$,

(ii) $2^{-s} \leq \overline{D}_s(E,x) \leq 1$ for $\mathcal{H}^s$-almost every $x \in E$,

(iii) $\overline{d}_s(E,x) = \overline{D}_s(E,x) = 0$ for $\mathcal{H}^s$-almost every $x \in M \backslash E$.

Proof. By Theorem 14 it is enough to prove the statements for μ^s. Note that $\nu = \mu^s|_E$ is a finite Borel measure by the same theorem.
(i) Let $c > 1$. By Lemma 17 we have that

$$E_c = \{x \in E \mid \overline{D}_s(\nu, x) \geq c\}$$

is a Borel set. Hence by Theorem 15 we have

$$c\mu^s E_c \leq \inf_{x \in E_c} \overline{D}_s(\nu, x)\mu^s E_c \leq \nu E_c = \mu^s E_c$$

and we obtain $\mu^s E_c = 0$. As $c > 1$ was arbitrary, we have just proven $\overline{d}_s(E,x) \leq 1$ for μ^s-almost every x in E.
Now let $k < 1$. Then we apply again Theorem 15 to the Borel set (see again Lemma 17)

$$F_k = \{x \in E \mid \overline{D}_s(\nu, x) \leq k\}$$

to obtain $\nu F_k \leq k\mu^s F_k = k\nu F_k$. Hence $\mu^s F_k = 0$, and because $k < 1$ was arbitrary we infer $\overline{d}_s(E,x) \geq 1$ for μ^s-almost every x in E, as required.
(iii) Let $c > 0$ and define the Borel set

$$E_c = \{x \in M \backslash E \mid \overline{D}_s(\nu, x) \geq c\}.$$

Then by Theorem 15 $c\mu^s E_c \leq \nu E_c = 0$ and so $\mu^s E_c = 0$. As $c > 0$ was arbitrary we obtain $\overline{d}_s(E,x) = 0$ for μ^s- almost every x in $M \backslash E$, as required.

The statements of the previous theorem might appear quite surprising at first, as it is very easy to construct a set of positive and finite 1-dimensional Hausdorff measure in $\mathbf{R}^2$, that has infinite density at, say 0 (take for example a countable union of line segments that 'quickly decrease' in length and meet in the origin).

The promised estimate for σ_n follows now as an easy corollary to the last theorem.

Corollary 19 *Let* $n \in \mathbf{N}$. *Then* $\sigma_n(M) \leq 1$.

Proof. Note that any set of zero $\mathcal{H}^n$-measure is n-rectifiable. The claim therefore follows immediately from Theorem 18 (ii).

Chapter 2

On the Centred Hausdorff Measure

2.1 Introduction

In Section 1.1 we have already presented the main topics of this chapter. In this section we will therefore just state some well-known theorems and make a short comment on the proof of the main result of this chapter - the affirmative result regarding the Borel regularity of the centred Hausdorff measure. The core of the proof is to some extent based on the proof of the Borel regularity of the spherical measure (which is the same as for the Hausdorff measure). Let us therefore state this result as a theorem together with a proof.

The reader is reminded (see comment after Definition 8) that throughout this chapter the notation h stands for a whole family of gauge functions $h_{\nu,q,t}$ depending on a Borel measure ν that is finite on balls, and on two real numbers q and t.

Theorem 20 *Let M be a separable metric space. Then $\mathcal{S}^h$ is Borel regular for any gauge function h as in Definition 8.*

Proof. Let $A \subset M$, $j \in \mathbf{N}$ and $\{B(x_i^j, r_i^j)\}_i$ a $1/j$-covering of A such that

$$\sum_i h(x_i^j, r_i^j) \leq \mathcal{S}^h A + \frac{1}{j}.$$

Therefore $B = \cap_j \cup_i B(x_i^j, r_i^j)$ is a Borel set with $B \supset A$ and $\mathcal{S}^h A = \mathcal{S}^h B$.

Anyone trying to prove the Borel regularity of the centred Hausdorff measure would first attempt to use the same construction of the Borel set B as in the above proof of Theorem 20. This is basically what we are going to do as well, even though the proof is not as straightforward as for the spherical measure. For now observe that at first this type of construction used above would only yield $\mu_0^h B = \mu_0^h A$. That this equality holds, is shown in Lemma 26 where a bit more work is needed than in Theorem 20. But of course the really interesting problems are just starting now: the centred Hausdorff measure of B might be strictly greater than that of A as we have the additional step in the construction of this measure. Or to be more precise, there could be a set C situated 'between' A and B with $\mu_0^h C > \mu^h A$. To show that this cannot happen, we need to 'switch over' to the spherical measure (where such strange things do of course not happen). That we are indeed allowed to do this 'switch' is showed in Theorem 25 where we prove the equivalence of the centred Hausdorff measure to the spherical measure (which is an interesting fact in its own right) for the most interesting classes of gauge functions.

In the Section 2.3 we will give a counter-example in order to show that this equivalence does not necessarily hold for gauge functions not satisfying the assumption of Theorem 25. We will not give here the idea of this construction, as it will be fully explained at the beginning of that section. We felt that the idea would rather fit there, as it is a bit technical and very related to the actual formal construction.

We will now introduce the Besicovitch covering theorem in $\mathbf{R}^n$, which was developed by Besicovitch in [3] and [4] and by Morse in [23]. This classical theorem has many applications in geometric measure theory and it will be also useful for us in the next section.

Theorem 21 *Let $n \in \mathbf{N}$, $A \subset \mathbf{R}^n$ and $\mathcal{B}$ a family of closed and nondegenarate balls in $\mathbf{R}^n$such that each point of A is the centre of some ball of $\mathcal{B}$ and*

$$\sup\{\operatorname{diam} B \mid B \in \mathcal{B}\} < \infty.$$

Then we can find an integer $\xi(n)$ depending solely on n, so that there is a countable sub-collection $\{B_i\}_i \subset \mathcal{B}$ with

$$\chi_A \leq \sum_{i=1}^{\infty} \chi_{B_i} \leq \xi(n).$$

Proof. See for example Theorem 2.2 of [21] .

We shall now remind the reader of the definition of weak convergence of measures.

Definition 22 *We say that a sequence $(\mu_i)_i$ of Radon measures on $\mathbf{R}^n$* converges weakly *to a measure μ if*

$$\lim_{i\to\infty} \int \varphi d\mu_i = \int \varphi d\mu \quad \textit{for all} \quad \varphi \in C_0(\mathbf{R}^n).$$

The following well-known theorem is a very useful consequence of the above definition.

Theorem 23 *Let $(\mu_i)_i$ be a sequence of Radon measures on $\mathbf{R}^n$ converging weakly to a measure μ. If $K \subset \mathbf{R}^n$ is compact and $U \subset \mathbf{R}^n$ is open we have*

$$\mu K \geq \limsup_{i\to\infty} \mu_i K$$

and

$$\mu U \leq \liminf_{i\to\infty} \mu_i U.$$

Proof. See for example Theorem 1.24 of [21].

2.2 Affirmative Results

For the rest of this section we take M to be a separable metric space where the Besicovitch covering theorem holds. The reason behind this generality is that our definition of the doubling condition (see Definition 7) is rather restrictive. A more usual definition would require the validity of the equation $\nu B(x, 2r) \leq c\nu B(x, r)$ for x from the support of ν only. However, if a measure ν on $\mathbf{R}^n$ satisfies this weaker notion of doubling, we may take for M the support of ν and use our results to deduce the Borel regularity of μ^h on M (recall that by Theorem 21 the Besicovitch covering theorem holds if we define M in this way). Since the behaviour of μ^h is trivial outside the support of ν, the Borel regularity results immediately transfer to this more general setting (i.e., ν satisfying only the weaker doubling condition mentioned above).

The following lemma also follows from the results of [24] and [30]. The proof presented here is based on a suggestion by the referee of [36].

Lemma 24 *If $q \geq 1$ and $t > 0$, then*

$$\mu^h A = \mathcal{S}^h A = 0$$

for all $A \subset M$.

Proof. Let $R > 0$, $z \in M$, $A \subset B(z, R)$ and $\delta \in (0, R)$. Define now

$$\mathcal{F} = \{B(x, r) \mid x \in A,\ r \in (0, \delta/2)\}.$$

By the Besicovitch Covering Theorem (see Theorem 21) we can find $\xi = \xi(n)$ and $\{B(x_i, r_i)\}_i \subset \mathcal{F}$ of A with

$$\chi_A \leq \sum_{i=1}^{\infty} \chi_{B(x_i, r_i)} \leq \xi.$$

Note that by the definition of $\mathcal{F}$ the collection $\{B(x_i, r_i)\}_i$ is a centred δ-covering of A. Hence

$$\begin{aligned} \mu_\delta^h(A) &\leq \sum_{i=1}^{\infty} r_i^t (\nu B(x_i, r_i))^q \leq \delta^t (\sum_{i=1}^{\infty} \nu B(x_i, r_i))^q \\ &\leq \delta^t \xi^q (\nu B(z, 2R))^q. \end{aligned}$$

Letting $\delta \downarrow 0$ shows that $\mu_0^h A = 0$ for all $A \subset B(z, R)$, whence $\mu^h B(z, R) = 0$ for all $R > 0$. Thus $\mu^h M = 0$. As every centred covering of a set is also a covering with balls of this set we get the same result for the spherical measure.

Next we will present a useful theorem about the relation of the spherical measure to μ_0^h and μ^h.

Theorem 25 *If ν satisfies the doubling condition, then there exists a constant η depending only on ν, q, t and n such that*

$$\mathcal{S}^h A \leq \mu_0^h A \leq \eta \mathcal{S}^h A$$

and

$$\mathcal{S}^h A \leq \mu^h A \leq \eta \mathcal{S}^h A.$$

for all $A \subset M$. Moreover, if $q \geq 1$ and $t > 0$ or if $q \leq 0$ the above inequalities hold even without the doubling condition.

Proof. Let $A \subset M$. Since every centred covering of A is also a covering of A by balls the inequality $\mathcal{S}^h A \leq \mu_0^h A$ is obvious. In i) - iii) we will now prove $\mu_0^h A \leq \eta \mathcal{S}^h A$ for a constant $\eta \geq 0$.

i) Let ν be a measure fulfilling the doubling condition, $R > 0$ as in Definition 7 and $q > 0$. Furthermore let $0 < \delta < R/2$ and let $\{B(x_i, r_i)\}_i$ be a δ-covering of A. Choose an arbitrary $j \in \mathbf{N}$. We may assume that there exists $y_j \in B(x_j, r_j) \cap A$. We can then find by Definition 7 a constant $C > 0$ (not depending on x_j, r_j or y_j), such that

$$h(y_j, 2r_j) \leq C h(x_j, r_j).$$

By applying the above to our covering $\{B(x_i, r_i)\}_i$ we obtain a centred 2δ-covering $\{B(y_i, 2r_i)\}_i$ of A with

$$\sum_{i=1}^{\infty} h(y_i, 2r_i) \leq C \sum_{i=1}^{\infty} h(x_i, r_i).$$

Hence $\mu_{2\delta}^h A \leq C \mathcal{S}_\delta^h A$ and by letting $\delta \downarrow 0$ we obtain $\mu_0^h A \leq C \mathcal{S}^h A$, as required.

ii) If $q \leq 0$ we observe that $(\nu B(x, 2r))^q \leq (\nu B(x, r))^q$ for all $x \in M$, so that the above proof works even without the doubling condition.

iii) If $q \geq 1$ and $t \geq 0$ the statement follows directly from Lemma 24 without the use of the doubling condition.

iv) To prove the inequalities for μ^h under the conditions in the assumption we apply i) - iii) to obtain

$$\mathcal{S}^h A \leq \mu_0^h A \leq \mu^h A = \sup_{B \subset A} \mu_0^h B \leq C \sup_{B \subset A} \mathcal{S}^h B \leq C\mathcal{S}^h A.$$

The next lemma is the analogue of Theorem 20 for μ_0^h. Even though it is not as straightforward as the proof for the spherical measure, we can still prove that a set defined similarly to B in that proof, is as required. This is due to the fact that our initial set is dense in the set defined in this way.

Lemma 26 *Let $A \subset M$ and $B_0 \supset A$ Borel. Then we can find a Borel set B with $A \subset B \subset B_0$ such that*

$$\mu_0^h B = \mu_0^h A.$$

Proof. For each $n \in \mathbf{N}$ we find a centred $1/n$-covering $\{B(z_i^n, \rho_i^n)\}_i$ of A such that $\sum_i h(z_i^n, \rho_i^n) \leq \mu_{1/n}^h A + 1/n$ and $B(z_i^n, \rho_i^n) \cap A \neq \emptyset$ for all $n, i \in \mathbf{N}$. Define

$$B = \bigcap_n \bigcup_i B(z_i^n, \rho_i^n) \cap B_0.$$

Then B is Borel and $A \subset B \subset B_0$. We will now show that $\mu_0^h B = \mu_0^h A$. As all $B(z_i^n, \rho_i^n)$ are centred in A, they are also centred in B and we obtain $\mu_0^h B \leq \mu_0^h A$. To get the reversed inequality let $\delta > 0$ and $\{B(x_i, s_i)\}_i$ be an arbitrary centred δ-covering of B. Let further $\epsilon > 0$ and $B(x, s) \in \{B(x_i, s_i)\}_i$. By Theorem 4 (iii) (if $q \leq 0$ we don't need Theorem 4) we find $s < s_0 < 2s$ such that

$$(\nu B(x, s_0))^q \leq (\nu B(x, s))^q + \frac{\epsilon}{2} \tag{2.1}$$

and by decreasing s_0, if necessary, we get

$$(\nu B(x, s_0))^q s_0^t \leq (\nu B(x, s))^q s^t + \epsilon. \tag{2.2}$$

As A is dense in B, we can find $y \in A$ and $r \in (s, s_0)$ with $B(x, s) \subset B(y, r) \subset B(x, s_0)$. Hence, depending on whether t and q are positive or negative, we get from (2.1), (2.2) or directly that

$$(\nu B(y, r))^q r^t \leq (\nu B(x, s))^q s^t + \epsilon.$$

Applying the above to each $B(x_i, s_i)$, $i \in \mathbf{N}$, of our centred covering with $\epsilon = (\frac{\delta}{1+\delta})^i$ we obtain a centred 2δ-covering $\{B(y_i, r_i)\}_i$ of A with

$$\sum_{i=1}^{\infty} h(y_i, r_i) \leq \sum_{i=1}^{\infty} (h(x_i, s_i) + (\frac{\delta}{1+\delta})^i) = \sum_{i=1}^{\infty} h(x_i, s_i) + \delta.$$

Finally, we have $\mu^h_{2\delta} A \leq \mu^h_\delta B + \delta$ and letting $\delta \downarrow 0$ we get the required reversed inequality.

We will now show that under certain conditions on the gauge function the Borel set constructed in Lemma 26 is as required in the definition of Borel regularity.

Lemma 27 *Suppose that there exists a constant η such that for every $A \subset M$ we have*

$$\mu^h_0 A \leq \eta \mathcal{S}^h A \leq \eta \mu^h A.$$

Then μ^h is Borel regular.

Proof. Let $A \subset M$. We may assume that $\mu^h A < \infty$. This implies $\mathcal{S}^h A < \infty$. By Theorem 20 we can find a Borel set $B \supset A$ with $\mathcal{S}^h B = \mathcal{S}^h A$. We will show that this B is the required Borel set. Suppose on the contrary, that $\mu^h B > \mu^h A$. Then, by definition, there exists $C \subset B$ with $\mu^h_0 C > \mu^h A$. By Lemma 26 we may assume that C is Borel. In particular we have

$$\mu^h_0 C > \mu^h_0 (C \cap A). \tag{2.3}$$

Applying Lemma 26 again we obtain a Borel set D with $C \cap A \subset D \subset C$ and

$$\mu_0^h(C \cap A) = \mu_0^h D. \tag{2.4}$$

From $C \backslash D \subset C \backslash A \subset B \backslash A$, it follows that $A \subset B \backslash (C \backslash D)$. As $C \backslash D$ is Borel we can therefore infer

$$\mathcal{S}^h B = \mathcal{S}^h(B \backslash (C \backslash D)) + \mathcal{S}^h(C \backslash D) \geq \mathcal{S}^h A + \mathcal{S}^h(C \backslash D) \geq \mathcal{S}^h A = \mathcal{S}^h B.$$

Hence $\mathcal{S}^h(C \backslash D) = 0$, and we obtain $\mu_0^h(C \backslash D) = 0$. By (2.3), (2.4) and Theorem 9 we finally have

$$\mu_0^h C \leq \mu_0^h D + \mu_0^h(C \backslash D) = \mu_0^h D = \mu_0^h(C \cap A) < \mu_0^h C,$$

which provides us with the required contradiction.

We have now done all the work required to prove the main theorem. Note that the underlying measure does not need to be Borel regular in order to have Borel regularity for the resulting centred Hausdorff measure.

Theorem 28 *If ν fulfills the doubling condition, then μ^h is Borel regular. Moreover, if $q \geq 1$ and $t > 0$ or if $q \leq 0$, then μ^h is Borel regular even without the doubling condition.*

Proof. This theorem follows immediately if we combine Theorem 25 and Lemma 27.

2.3 A Counter-Example

Note. Unless otherwise stated, all the balls referred to in this section will be closed and nondegenerate. Also, for each ball $B \subset \mathbf{R}^2$ and each $c > 0$ we will write cB for the ball with the same centre and c times the radius of B.

In this section we will construct a counter-example to Theorem 25. As John Howroyd pointed out, one can easily see that there are counter-examples to Theorem 25 for $q > 0$ and $t < 0$. In fact in his example the measure satisfies the weaker doubling condition mentioned in the introduction of Section 2.2. Here we produce a more difficult example in which $0 < q < 1$ and $t \geq 0$. For clarity we will limit our construction to $q = 1/2$ and $t = 0$ (i.e., $h(x,r) = \sqrt{\nu B(x,r)}$), though the same type of construction can be also done for other q and t.

In addition to the required measure ν we will also construct a subsidiary set $A \subset \mathbf{R}^2$. Before we formally define A and ν, we shall first explain the idea behind the construction. We begin constructing our set by taking an arbitrary ball $B_{1,1} \subset \mathbf{R}^2$. We then place in this ball two very small disjoint balls $B_{2,1}, B_{2,2}$ of the same radius. In addition we require that they have the same distance to the centre of the initial ball. This means that none of the smaller balls includes the centre of the initial ball. In the next step we place in each of those balls three very small disjoint balls $B_{3,j}$ of the same radius. Like before we require that they have the same distance to the centre of the previous construction ball and in addition we require that there is a fixed distance between the neighbouring balls. We continue this construction in the same way and define eventually the set A as the intersection of the unions of the construction balls at step i. We observe that we have $i!$ construction balls at step i.

To construct our measure ν we first choose for each ball $B_{i,j} \subset B_{i-1,k}$ a very close point $x_{i,j} \in B_{i-1,k} \backslash B_{i,j}$. We then define the measure by $\nu B_{1,1} = 1$, $\nu B_{i,j} = 1/i!^2$ and $\nu\{x_{i,j}\} = (i-1)/i!^2$. By simple calculation we observe that this measure is a well-defined, non-doubling probability measure (i.e., $\nu \mathbf{R}^2 = 1$).

As mentioned before one could generalize this type of construction for other $q \in (0,1)$ by setting $\nu B_{i,j} = 1/i!^{1/q}$ and $\nu\{x_{i,j}\} = (i-1)/i!^{1/q}$. However, note that for $q \geq 1$

$$\sum_{j=1}^{i!} \nu B_{i,j} = i!(1/i!)^{1/q} \geq 1 = \nu B_{1,1} \ .$$

Hence this type of construction does not work for $q \geq 1$ as we do not have any 'spare measure' to distribute on our extra points.

We denote by $\tilde{B}_{i-1,k}$ a ball concentric to $B_{i-1,k}$, which does not include the $x_{i,j}$, but still contains the construction balls of the next level. To calculate the spherical measure we use the $\tilde{B}_{i,k}$, which yields, as we will show later, $\mathcal{S}^h A = 0$. However, as this is not a centred covering, we are not allowed to use it to calculate the centred Hausdorff measure. By shifting and increasing these balls (to get a centred covering) we cover most of the special points $\{x_{i+1,j}\}$ and because of $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ we get no advantage by covering with smaller balls. We will calculate later that this leads to $\mu^h A \geq \mu_0^h A \geq 1/3$, so that Theorem 25 cannot hold in this situation.

We will now formally define our set A.

Definition 29 *(Construction of A)*

- *Let* $B_{1,1}$ *be an arbitrary ball in* $\mathbf{R}^2$.

- *Let* $\{B_{m,j}\}_{j=1}^{m!}$ *be already constructed for each* $m \leq i-1$. *We place for* $1 \leq k \leq (i-1)!$ *in each* $B_{i-1,k}$ *exactly* i *balls* $\{B_{i,j}\}_{j=i(k-1)+1}^{ik}$ *such that:*

  (i) $\operatorname{diam} B_{i,j} = \operatorname{diam} B_{i,j_0}$ *for all* $1 \leq j, j_0 \leq i!$; *denote* $\rho_i = \operatorname{diam} B_{i,1}$,

  (ii) for all $1 \leq j \leq i!$ *we have* $3B_{i,j} \subset B_{i-1,k}$ *for the corresponding* $1 \leq k \leq (i-1)!$,

  (iii) let $y_{i-1,k}$ *be the centre point of* $B_{i-1,k}$. *For some* $0 < s_{i-1} < \rho_{i-1}/2$ *all the centres of the* $\{B_{i,j}\}_{j=1}^{i!}$ *are on the corresponding* $\partial B(y_{i-1,k}, s_{i-1})$,

  (iv) there is a fixed distance ζ_i *between every two neighbouring balls, such that* $\zeta_i > 3\rho_i$,

  (v) for each $1 \leq j \leq i!$, *each* $x \in B_{i,j}$ *and each* $0 < r < \rho_{i-1}$ *we have* $\partial B(x,r) \cap 2B_{i,m} \neq \emptyset$ *for at most two different* $B_{i,m}$.

- *Set* $A = \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{i!} B_{i,j}$.

- $\tilde{B}_{i,k} = B(y_{i,k}, s_i + \rho_{i+1}/2)$.

To see that this construction is possible, we first observe that at each scale (i)-(iv) can be fulfilled for sufficiently small radii of the construction balls. To justify (v) we note that all the centre points of the relevant construction balls which could be cut, are on the boundary of a circle (this is because of (iii) and the fact that from (iv) it follows that all these balls are included in the same construction ball of the previous stage). As two balls either coincide, or their boundaries intersect in at most two points we have (v) for sufficiently small radii. By decreasing the radii of the construction balls, when necessary, (i)-(iv) are still fulfilled for these smaller balls. This shows that the above construction is indeed possible.

We will now define our non-doubling measure ν.

Definition 30 *(Construction of ν)*

- *Let $x_{1,1}$ be an arbitrary point in $\mathbf{R}^2 \backslash B_{1,1}$. For $i \geq 2$, $1 \leq j \leq i!$ and $1 \leq k \leq (i-1)!$ such that $B_{i,j} \subset B_{i-1,k}$ we choose $x_{i,j} \in \partial 2B_{i,j} \backslash \tilde{B}_{i-1,k}$.*

- *Define*
$$\nu_i = \sum_{k=1}^{i!} \frac{4}{\pi i!^2 \rho_i^2} \mathcal{L}^2|_{B_{i,k}} + \sum_{j=1}^{i} \sum_{k=1}^{j!} \frac{j-1}{j!^2} \delta_{x_{j,k}}.$$

- *Finally define the measure ν by taking the weak limit of the ν_i.*

- *For $i \geq 1$ and $1 \leq k \leq i!$ denote by $C_{i,k}$ a ball centred in A with $B_{i,k} \subset C_{i,k}$ and $x_{i,k} \notin C_{i,k}$.*

Note that all the ν_i are defined by a finite sum of Radon-measures and thus are Radon-measures themselves. Hence taking the weak limit over the ν_i makes sense in terms of Definition 22.

Let us now prove some basic facts about the measure ν.

Lemma 31 *Let $l \in \mathbf{N}$ and $1 \leq k \leq l!$. Then we have*

(i)
$$\nu B_{l,k} = \frac{1}{l!^2} \quad \text{and} \quad \nu\{x_{l,k}\} = \frac{l-1}{l!^2},$$

(ii)
$$\nu \mathbf{R}^2 = \nu\Big(\bigcup_{i=1}^{\infty} \bigcup_{j=1}^{i!} \{x_{i,j}\}\Big) = 1,$$

(iii)
$$spt\nu = \overline{\bigcup_{i=1}^{\infty} \bigcup_{j=1}^{i!} \{x_{i,j}\}},$$

(iv)

$$\nu\tilde{B}_{l,k} = \frac{l+1}{(l+1)!^2} \text{ and } \nu C_{l,k} = \frac{1}{l!^2}\,.$$

Proof. (i) We will first show by induction that

$$\nu_m B_{l,k} = \frac{1}{l!^2} \quad \text{for every} \quad m \geq l. \tag{2.5}$$

We have $\nu_l B_{l,k} = \frac{1}{l!^2}$ and for $m > l$

$$\begin{aligned} \nu_m B_{l,k} - \nu_{m-1} B_{l,k} &= \frac{m!}{l!m!^2} + \sum_{j=l+1}^{m} \frac{j!}{l!}\frac{j-1}{j!^2} - \frac{(m-1)!}{l!(m-1)!^2} - \sum_{j=l+1}^{m-1} \frac{j!}{l!}\frac{j-1}{j!^2} \\ &= \frac{1-m}{l!m!} + \frac{m-1}{l!m!} = 0\,. \end{aligned}$$

Hence (2.5) holds. Let $U_{l,k} \supset B_{l,k}$ be a (concentric) open ball such that $x_{l,k} \notin U_{l,k}$. By Definition 29 (iv) and Definition 30 we get by the same argument as in (2.5) that $\nu_m U_{l,k} = 1/l!^2 = \nu_m B_{l,k}$ for $m \geq l$. Thus by Theorem 23

$$\begin{aligned} \nu B_{l,k} &\leq \nu U_{l,k} \leq \liminf_{n\to\infty} \nu_n U_{l,k} \leq \limsup_{n\to\infty} \nu_n U_{l,k} \\ &= \limsup_{n\to\infty} \nu_n B_{l,k} \leq \nu B_{l,k}. \end{aligned}$$

Hence $\nu B_{l,k} = (1/l!)^2$.

To prove $\nu\{x_{l,k}\} = (l-1)/l!^2$ we observe that we may find $r > 0$ so that

$$\nu_m U(x_{l,k}, r) = \nu_m\{x_{l,k}\} = (l-1)/l!^2 \quad \text{for every} \quad m \geq l.$$

Therefore by Theorem 23

$$\begin{aligned} \nu\{x_{l,k}\} &\leq \nu U(x_{l,k}, r) \leq \liminf_{n\to\infty} \nu_n U(x_{l,k}, r) \leq \limsup_{n\to\infty} \nu_n U(x_{l,k}, r) \\ &= \limsup_{n\to\infty} \nu_n\{x_{l,k}\} \leq \nu\{x_{l,k}\}. \end{aligned}$$

Hence $\nu\{x_{l,k}\} = (l-1)/l!^2$.

(ii) We have by (i) and Theorem 23

$$\begin{aligned} 1 &= \liminf_{i\to\infty} \nu_i \mathbf{R}^2 \geq \nu\mathbf{R}^2 \geq \nu(\bigcup_{i=1}^{\infty}\bigcup_{j=1}^{i!}\{x_{i,j}\}) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{i!}\nu\{x_{i,j}\} = \sum_{i=1}^{\infty}\frac{i-1}{i!} \\ &= \sum_{i=1}^{\infty}\frac{1}{(i-1)!} - \sum_{i=1}^{\infty}\frac{1}{i!} = e-(e-1) = 1. \end{aligned}$$

This proves (ii).

(iii) follows immediately from (ii).

(iv) By (i) and (iii) we get

$$\nu\tilde{B}_{l,j} = \sum_{B_{l+1,k}\subset B_{l,j}} \nu B_{l+1,k} = (l+1)\nu B_{l+1,1} = \frac{l+1}{(l+1)!^2}$$

and

$$\nu C_{l,k} = \nu B_{l,k} = \frac{1}{l!^2}.$$

This proves the last statement of the lemma.

The next lemma will help us later to give a lower bound for the centred Hausdorff measure. It basically states that for this purpose we are allowed to replace an arbitrary centred covering by a covering with the $C_{i,k}$.

Lemma 32 *Let $B(x,r)$ be a ball centred in A and $i \in \mathbf{N}$ minimal with the property, that for some $1 \leq j \leq i!$ we have $x \in B_{i,j}$ and $2B_{i,j} \subset B(x,r)$. Further let $I_i, J_i \subset \{1,2,\ldots,i!\}$ be such that*

$$2B_{i,k} \cap B(x,r) \neq \emptyset \text{ if and only if } k \in I_i,$$

and

$$2B_{i,k} \cap \partial B(x,r) \neq \emptyset \text{ if and only if } k \in J_i\,.$$

Then

(i) $1 \leq |I_i| \leq i$ *and* $0 \leq |J_i| \leq 2$,

(ii) $\sqrt{\nu B(x,r)} \geq \frac{1}{3} \sum_{k \in I_i} \sqrt{\nu C_{i,k}}$,

(iii) $A \cap B(x,r) \subset \bigcup_{k \in I_i} C_{i,k}$.

Proof. (i) This follows directly from Definition 29 (iv) and (v).

(ii) First note that $x_{i,j} \in B(x,r)$. Applying (i) and Lemma 31 (iii) and (iv) we get for $|I_i| \geq 3$:

$$\begin{aligned}
\sqrt{\nu B(x,r)} &\geq \sqrt{(|I_i|-2)\nu(B_{i,1} \cup \{x_{i,1}\})} \\
&= \sqrt{(|I_i|-2)(\frac{1}{i!^2} + \frac{i-1}{i!^2})} \\
&= \frac{\sqrt{i(|I_i|-2)}}{i!} > \frac{|I_i|-2}{i!} \\
&\geq \frac{1}{3}|I_i|\sqrt{\nu C_{i,1}} = \frac{1}{3}\sum_{k \in I_i} \sqrt{\nu C_{i,k}}\,.
\end{aligned}$$

For $|I_i| = 1$ the statement is trivial. For $|I_i| = 2$ we have even $|J_i| \leq 1$. Thus using (i) and Lemma 31 (iii) and (iv) again, we obtain similarly to the above that

$$\begin{aligned}
\sqrt{\nu B(x,r)} &\geq \sqrt{(|I_i|-1)\nu(B_{i,1} \cup \{x_{i,1}\})} \\
&= \sqrt{\frac{1}{i!^2} + \frac{i-1}{i!^2}} = \frac{\sqrt{i}}{i!} \\
&> \frac{2}{3i!} = \frac{1}{3}|I_i|\sqrt{\nu C_{i,1}} \\
&= \frac{1}{3}\sum_{k \in I_i} \sqrt{\nu C_{i,k}}\,.
\end{aligned}$$

(iii) This follows immediately from the definition of the $C_{i,k}$.

We can now formulate the main result of this section.

Theorem 33 *For* $h(x,r) = \sqrt{\nu B(x,r)}$ *we have*

(i) $\mathcal{S}^h A = 0$,

(ii) $\mu_0^h A \geq 1/3$,

(iii) $\mu^h A \geq 1/3$.

Proof. (i) Let $\delta > 0$ and $i \in \mathbf{N}$ such that $\rho_{i-1} \leq \delta$. By Lemma 31 (iv) we infer that $\{\tilde{B}_{i-1,k}\}_{k=1}^{(i-1)!}$ is a δ-covering of A with

$$\mathcal{S}_\delta^h A \leq \sum_{k=1}^{(i-1)!} \sqrt{\nu \tilde{B}_{i-1,k}} = (i-1)! \sqrt{\frac{i}{i!^2}} = \frac{1}{\sqrt{i}}\,.$$

On taking the limit $i \to \infty$, we obtain then $\mathcal{S}^h A = \mathcal{S}_\delta^h A = 0$.

(ii) Let $\{\tilde{B}_n\}_{n=1}^\infty$ be an arbitrary centred δ-covering of A. As A is compact we can find a finite centred 2δ-covering $\{B_n\}_{n=1}^N$ of A with

$$\sum_{n=1}^N \sqrt{\nu B_n} \leq \sum_{n=1}^\infty \sqrt{\nu \tilde{B}_n} + \delta.$$

Applying Lemma 32 (ii) to $\{B_n\}_{n=1}^\infty$ we obtain a centred covering $\{C_{i_n,j}\}_{(n,j)\in I}$ of A with a finite $I = \{(n,j) \mid 1 \leq n \leq N,\ \ j \in I_{i_n}\}$ (with I_{i_n} defined as in Lemma 32) and

$$\sum_{n=1}^N \sqrt{\nu B_n} \geq \frac{1}{3} \sum_{(n,j)\in I} \sqrt{\nu C_{i_n,j}}\,.$$

By Lemma 31 (iv) we get for every $i \in \mathbf{N}$, $1 \leq j \leq i!$ and $1 \leq k \leq (i+1)!$ that

$$\sqrt{\nu C_{i,j}} = \frac{1}{i!} = (i+1)\sqrt{\nu C_{i+1,k}}\,.$$

Thus, writing $i_0 = \max\limits_{1\leq n\leq N} i_n < \infty$ we finally conclude by Lemma 32 (iii)

$$\begin{aligned}\sum_{n=1}^\infty \sqrt{\nu \tilde{B}_n} + \delta &\geq \sum_{n=1}^N \sqrt{\nu B_n} \geq \frac{1}{3} \sum_{(n,j)\in I} \sqrt{\nu C_{i_n,j}} \\ &\geq \frac{1}{3} \sum_{j=1}^{i_0!} \sqrt{\nu C_{i_0,j}} = \frac{1}{3}\,.\end{aligned}$$

Hence $\mu_\delta^h A \geq 1/3 - \delta$ and for $\delta \downarrow 0$ we get

$$\mu^h A \geq \mu_0^h A \geq \frac{1}{3},$$

as required.

The above theorem shows in particular that we cannot use our method to prove Lemma 27 and Theorem 28 for the centred Hausdorff measure defined by this gauge function. Another immediate consequence is that as mentioned in Section 1.2 we have provided an example which shows that μ_0^h is not necessarily monotone. Let us formulate this fact in a more precise manner:

Corollary 34 *There exists a set $B \supset A$ such that*

$$\mu_0^h B < \mu_0^h A.$$

Proof. Let

$$B = A \cup \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{i!} \{y_{i,j}\}.$$

For each $\delta > 0$ we have that $\{\tilde{B}_{i-1,k}\}_{k=1}^{(i-1)!}$ is a centred δ-covering of B. Therefore, using the same calculations as in the proof of Theorem 33 (i) we obtain that $\mu_0^h B = 0$. Hence by Theorem 33 (ii)

$$\mu_0^h B = 0 < \frac{1}{3} = \mu_0^h A,$$

and also trivially $A \subset B$.

Chapter 3

Isodiametric Inequalities

3.1 The General Case

We start with a standard definition.

Definition 35 *Let $(G, \cdot)$ be a group.*

(i) We say that a metric d on G is left invariant *if $d(g, h) = d(a \cdot g, a \cdot h)$ for all $a, g, h \in G$.* Right invariance *is defined similarly.*

(ii) A left invariant measure *μ on G is a measure on G that satisfies $\mu H = \mu(g \cdot H)$ for all $g \in G$ and all $H \subset G$.* Right invariance *is defined similarly.*

For the rest of this section let $(G, \cdot)$ be a group with the neutral element 1, d a left (or right) invariant, separable metric on G and μ a left (right) invariant Borel regular measure on G. Even though we work with this general setting here, later on we will be just interested in the special case where μ is locally-finite and positive on open balls and G is a locally compact topological group that is metrized by d. Such a measure is called a *left (right) Haar measure.* One can prove the existence and uniqueness (up to a positive constant multiple) of the left (right) Haar measure in that setting (see for example Chapter 11 of [13]).

Theorem 36 *Suppose that for some $\alpha > 0$*

$$0 < \limsup_{S\to 1} \frac{\mu S}{(\operatorname{diam} S)^\alpha} < \infty.$$

Then

(i) the α-dimensional Hausdorff measure is a positive multiple of μ,

(ii)

$$\underline{D}_\alpha(G,g) = \frac{\liminf_{r\to 0}\ (2r)^{-\alpha}\mu B(1,r)}{\limsup_{S\to 1}\ (\operatorname{diam} S)^{-\alpha}\mu S}$$

and

$$\overline{D}_\alpha(G,g) = \frac{\limsup_{r\to 0}\ (2r)^{-\alpha}\mu B(1,r)}{\limsup_{S\to 1}\ (\operatorname{diam} S)^{-\alpha}\mu S}\ .$$

Proof. (i) Set $c = \limsup_{S\to 1}(\operatorname{diam} S)^{-\alpha}\mu S$. By the left (right) invariance of d and μ we obtain by Theorem 16

$$\mathcal{H}^\alpha U = \frac{1}{c}\mu U \quad \text{for all open sets } \ U \subset G.$$

As

$$\limsup_{r\downarrow 0} \frac{\mu B(1,r)}{(2r)^\alpha} \le \limsup_{r\downarrow 0} \frac{\mu B(1,r)}{(\operatorname{diam} B(1,r))^\alpha} \le \limsup_{S\to 1} \frac{\mu S}{(\operatorname{diam} S)^\alpha} < \infty,$$

μ is in particular finite on sufficiently small balls around every point (here we use again the invariance of μ). By the separability of G we can therefore find $\{g_i\}_i \subset G$ and $R > 0$ with

$$G = \bigcup_{i=1}^{\infty} U(g_i, R)$$

and

$$\frac{1}{c}\mu U(g_j, R) = \mathcal{H}^\alpha U(g_j, R) < \infty$$

for $j \in \mathbf{N}$. As $\mathcal{H}^\alpha$ and μ are both Borel regular measures (the Borel regularity of $\mathcal{H}^\alpha$ can be proven in absolutely the same way as for the spherical measure - see Theorem 20) we infer by Theorem 5

$$\mathcal{H}^\alpha B = \frac{1}{c}\mu B \quad \text{for all Borel sets} \quad B \subset G.$$

We use again the Borel regularity of $\mathcal{H}^\alpha$ and μ to obtain this identity for arbitrary subsets of G. This proves (i).
(ii) This follows from the proof of (i) and the left (right) invariance of d and μ.

If the ratio between the diameter and the radius of the balls around 1 has a positive lower bound for 'small' radii (this is for example the case if G is locally connected) we may replace the condition on μ in the above theorem by the more natural condition $0 < \overline{D}_\alpha(\mu, 1) < \infty$.
Even though Theorem 36 is an immediate consequence of the well-known Theorem 16 it is still very useful in calculating the densities and σ_n of a metric space in the above setting. There is no need to compute the exact value of the Hausdorff measure but one can instead choose another appropriate (Haar) measure. This is in particular useful if μ is the Lebesgue measure, which is much easier to calculate. In fact, this is what we are going to do in the next section and in the whole last chapter. Even though we will not explicitly use the above theorem in the next chapter (since the setting is much simpler there, we will give a direct proof), the basic idea is still the same.
Let us now briefly explain what this has to do with the isodiametric inequality. Recall, that the essence of the isodiametric inequality is that the

(Euclidean) ball has maximal (Lebesgue) measure for a given diameter. If we know which set has maximal μ-measure for a given diameter (we could say that this set fulfills the isodiametric inequality for a given diameter and for μ), we can calculate the lower and upper densities and σ_n with the help of Theorem 36 (ii). Of course, we also need to know the measure of balls but this is usually much the easier part to prove.

3.2 The Heisenberg Group

The Heisenberg group is of interest in many areas ranging from physics to analysis on metric spaces. For more information on this group, the reader could in particular see [16], [37] or [38] (out of the vast amount of related literature) because of the connection to the area of this thesis.

Definition 37 The Heisenberg group $(H,\,\cdot\,)$ *with the neutral element* θ *is defined on the set* $\mathbf{C}\times\mathbf{R}$ *with the group operation defined in the following way. If* $(w,s),(z,t)\in H$*, then*

$$(w,s)\cdot(z,t)=(w+z,s+t+2\mathrm{Im}\,(w\overline{z})).$$

We define the Heisenberg metric $h : H\times H\to\mathbf{R}_0^+$ *by*

$$h((w,s),(z,t))=(|z-w|^4+(t-s-2\mathrm{Im}\,(w\overline{z}))^2)^{1/4},$$

for $(w,s),(z,t)\in H$*.*

Let us summarize some of the basic properties of the Heisenberg group and its metric.

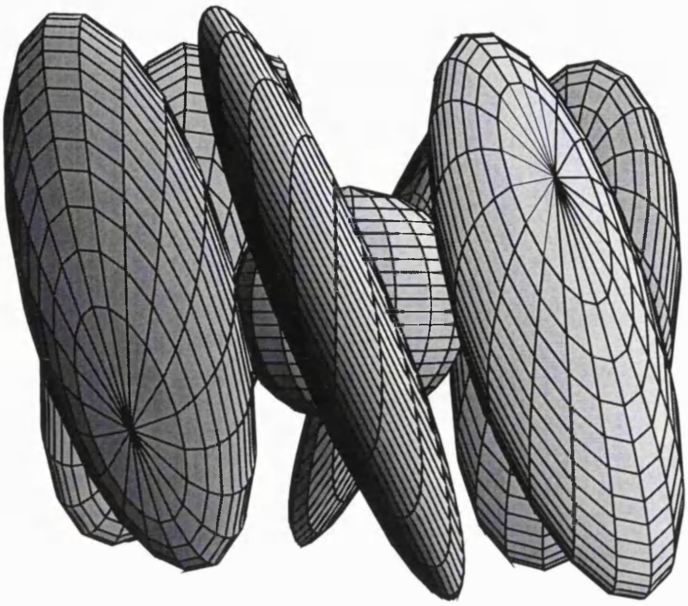

Figure 3.1: The left translations of the Heisenberg group applied to the Euclidean ball centred at the origin

Lemma 38 *(i)* $(H,\cdot)$ *is a non-Abelian topological group with the neutral element* $\theta = (0,0) \in \mathbf{C} \times \mathbf{R}$. *The inverse element of* $(w,s) \in H$ *is* $(-w,-s)$.

(ii) h *is a left invariant metric on* H *that induces the same topology on* $\mathbf{R}^3$ *as the Euclidean topology.*

(iii) The 3*-dimensional Lebesgue measure is a left invariant measure on* H.

(iv) For $(w,s) \in H$ *and* $r \geq 0$

$$\mathcal{L}^3 B((w,s),r) = \frac{1}{2}\pi^2 r^4.$$

(v) We have $\operatorname{diam} B((w,s),r) = 2r$ *for* $(w,s) \in H$ *and* $r \geq 0$.

Proof. (i) Follows directly from the definition of H.

(ii) The only non-trivial part is the triangle inequality for h, which we will now show. Let $(w,s),(z,t) \in H$. Then

$$\begin{aligned}(h((w,s),(z,t)))^2 &= \sqrt{|z-w|^4 + (s-t+2\mathrm{Im}\,(w\overline{z}))^2} \\ &= ||z-w|^2 + i(s-t+2\mathrm{Im}\,(w\overline{z}))| \\ &= ||z|^2 + |w|^2 - w\overline{z} - \overline{w}z + is - it + w\overline{z} - \overline{w}z| \\ &\leq ||z|^2 - it| + ||w|^2 + is| + 2|\overline{w}z| \\ &\leq (h((z,t),\theta) + h(\theta,(w,s)))^2.\end{aligned}$$

The triangle inequality follows from this and the left invariance of h.

(iii) To prove this, we observe that for each $(w,s) \in H$ the Jacobian of the left translation $(z,t) \mapsto (w,s)(z,t)$ is lower triangular with 1 on the diagonal. Thus the values of the Jacobian determinants are 1 and the Lebesgue measure is preserved.

(iv) By (ii) we may assume that $(w,s) = \theta$. We have

$$\int \sqrt{r^4 - t^2}dt = \frac{1}{2}(t\sqrt{r^4-t^2} + r^4 \arcsin\frac{t}{r^2}).$$

Therefore

$$\begin{aligned}\mathcal{L}^3 B(\theta,r) &= \mathcal{L}^3\{(x,y,s) \in \mathbf{R}^3 \mid (x^2+y^2)^2 + s^2 \leq r^4\} \\ &= \mathcal{L}^3\{(x,y,s) \in \mathbf{R}^3 \mid x^2+y^2 \leq \sqrt{r^4-s^2},\ s^2 \leq r^4\} \\ &= \pi\int_{-r^2}^{r^2} \sqrt{r^4-s^2}ds = \frac{1}{2}\pi^2 r^4,\end{aligned}$$

as required.

(v) We may assume by (ii) that $(w,s) = \theta$. Note that by identifying $\mathbf{C} \times \mathbf{R} =$

$\mathbf{R}^3$ we have

$$h((r,0,0),\theta) = h((-r,0,0),\theta) = r$$

Therefore

$$2r \geq \operatorname{diam} B(\theta, r) \geq h((r,0,0),(-r,0,0)) = 2r,$$

and (v) follows immediately.

According to the above lemma, one might interpret the Heisenberg group as a subgroup of (Lebesgue) measure preserving functions $\mathbf{R}^3 \to \mathbf{R}^3$. We will now apply Theorem 36 to obtain an estimate of $\sigma_4(H)$.

Theorem 39

$$\sigma_4(H) < 1.$$

Proof. Let $r > 0$. For every $(z,t) \in B((0,r^2),(2-\sqrt{2})r)$ and $(w,s) \in B(\theta,r)$

$$\begin{aligned} h((w,s),(z,t)) &\leq h((w,s),(0,r^2)) + h((0,r^2),(z,t)) \\ &\leq (|w|^4 + (r^2 - s)^2)^{\frac{1}{4}} + (2-\sqrt{2})r \\ &\leq (|w|^4 + s^2 + r^4)^{\frac{1}{4}} + (2-\sqrt{2})r \\ &\leq \sqrt{2}r + (2-\sqrt{2})r = 2r. \end{aligned}$$

Hence setting

$$A_r = B(\theta,r) \cup B((0,r^2),(2-\sqrt{2})r)$$

we obtain by Lemma 38 (v), that $\operatorname{diam} A_r = 2r$. Set

$$\kappa = 1 + \frac{1}{2}(2-\sqrt{2})^4 > 1.$$

We then have by Lemma 38 (iv)

$$\mathcal{L}^3 A_r \;=\; \mathcal{L}^3 B(\theta,r) + \mathcal{L}^3(B((0,r^2),(2-\sqrt{2})r)\backslash B(\theta,r))$$

$$
\begin{aligned}
&> \frac{1}{2}\pi^2 r^4 + \frac{1}{2}(\frac{1}{2}\pi^2(2-\sqrt{2})^4 r^4) \\
&= \frac{1}{2}\kappa\pi^2 r^4.
\end{aligned}
$$

Note that by Lemma 38

$$
\begin{aligned}
0 &< \frac{\pi^2}{32} = \limsup_{t\downarrow 0} \frac{\mathcal{L}^3 B(\theta,t)}{(\operatorname{diam} B(\theta,t))^4} \leq \limsup_{S\to\theta} \frac{\mathcal{L}^3 S}{(\operatorname{diam} S)^4} \\
&\leq \limsup_{t\downarrow 0} \frac{\mathcal{L}^3 B(\theta,2t)}{(\operatorname{diam} B(\theta,t))^4} \leq \frac{\pi^2}{2} < \infty.
\end{aligned}
$$

We can therefore apply Theorem 36 and Lemma 38 to obtain for all $g \in H$ (see comment after Definition 12)

$$
\begin{aligned}
\sigma_4(H) &\leq \underline{D}_4(H,g) \leq \frac{\liminf_{t\downarrow 0} (2t)^{-4}\mathcal{L}^3 B(\theta,t)}{\limsup_{t\downarrow 0} (\operatorname{diam} A_t)^{-4}\mathcal{L}^3 A_t} \\
&\leq \frac{\frac{\pi^2}{32}}{\frac{\kappa\pi^2}{32}} = \frac{1}{\kappa} < 1,
\end{aligned}
$$

as required.

3.3 The Real Line

In this section we will look at a very special case of the setting in Section 3.1. Recall, that the Heisenberg group in the last section may be interpreted as a subgroup of the volume (Lebesgue measure) preserving functions on the 3-dimensional Euclidean space. On the real line the group of length (Lebesgue measure) preserving functions is of course much simpler, as it consists only of functions of the form $x \mapsto x + y$ for $y \in \mathbf{R}$ and we can identify this group with the real line equipped with the usual addition. The translation invariant metrics on $\mathbf{R}$ can also be characterized quite easily by functions. This will help us to show that if the metric is not too 'unusual', then the interval has maximal Lebesgue measure of all the other sets with the same diameter (as in the Euclidean case). This will be used in the last chapter to give a surprising lower bound for σ_n on a metric constructed on the real line. The functions we will define now characterize all the translation invariant metrics on the real line.

Definition 40 *We say that $f : \mathbf{R}_0^+ \longrightarrow \mathbf{R}_0^+$ is a* metric defining function *if*

(1) $f(x) = 0$ if and only if $x = 0$,

(2) $f(|x+y|) \leq f(|x|) + f(|y|)$ for all $x, y \in \mathbf{R}$.

Remark 41 *If we have a metric defining function f, then we define a unique translation invariant metric d by setting $d(x,y) = f(|x-y|)$ for all $x, y \in \mathbf{R}$. If we have a translation invariant metric d then we define a unique metric defining function by setting $f(x) = d(x,0)$ for $x \in \mathbf{R}_0^+$.*

From now on we will identify translation invariant metrics on the real line with their corresponding metric defining functions. Next we will for the first time give a precise definition of what we mean by saying that a set fulfills the isodiametric inequality. Note that the definition below differs slightly from the intuition given in Section 3.1, but it is more convenient for our needs here.

Definition 42 *Let $\eta > 0$ and d be a translation invariant metric on the real line.*

(i) We say that d is monotone *if its corresponding metric defining function is monotone. Similarily, we say that d is* continous *if the corresponding metric defining function is continous with respect to the Euclidean metric.*

(ii) We say that $A \subset \mathbf{R}$ is η-maximal (with respect to d) *if*

(1) $\operatorname{diam}_d A \leq \eta$,

(2) $\mathcal{L}A \geq \mathcal{L}\tilde{A}$ *for all* $\tilde{A} \subset \mathbf{R}$ *with* $\operatorname{diam}_d \tilde{A} \leq \eta$,

(3) If $\hat{A} \subset \mathbf{R}$ with $A \subset \hat{A}$ and such that (1) and (2) hold, then this implies $\hat{A} = A$.

(iii) We say that d fulfills the isodiametric inequality for η, *if any η-maximal subset of the real line is an interval.*

We will now give some interesting examples of metric defining functions.

Example 43 *For each $p \geq 1$ we will call $x^{1/p}$ the* p-snowflake metric *and denote the corresponding metric space by S_p. The 1-snowflake metric is obviously the usual Euclidean metric.*

Next we will give an example of a whole class of discontinuous metrics.

Example 44 *Let f be a metric defining function and $c > 0$. We can always define a new metric defining function φ by*

$$\varphi_{f,c}(x) = \begin{cases} f(x) + c & \text{for } x \in \mathbf{R}^+ \backslash \mathbf{Q} \\ \\ f(x) & \text{for } x \in \mathbf{Q}_0^+ \end{cases}$$

Remark 45 *$\varphi_{f,c}$ as defined above is indeed a metric defining function.*

Proof. Let $x, y \in \mathbf{R}$ and write $\varphi = \varphi_{f,c}$. We need to show that

$$\varphi(|x+y|) \leq \varphi(|x|) + \varphi(|y|).$$

If $x, y \in \mathbf{Q}$, then $x + y \in \mathbf{Q}$ and the claim follows. Otherwise

$$\varphi(|x+y|) \leq f(|x+y|) + c \leq f(|x|) + f(|y|) + c \leq \varphi(|x|) + \varphi(|y|).$$

The following example will be important later in this chapter in connection with conditions for the isodiametric inequality.

Example 46

$$\psi(x) = \begin{cases} 0 & \text{if } x = 0 \\ -\left(\frac{25}{7}\right)^{z-1} x + 2(\frac{4}{5})^{z+2} & \text{if } x \in [(\frac{28}{125})^z, \frac{16}{7}(\frac{28}{125})^z) \text{ for some } z \in \mathbf{Z} \\ \left(\frac{25}{7}\right)^{z-1} x & \text{if } x \in [\frac{16}{7}(\frac{28}{125})^z, (\frac{28}{125})^{z-1}) \text{ for some } z \in \mathbf{Z}, \end{cases}$$

Definition 47 *For any $z \in \mathbf{Z}$ and $x \in \mathbf{R}_0^+$ we define the functions $h_z(x) = \left(\frac{25}{7}\right)^z x$ and $g_z(x) = -\left(\frac{25}{7}\right)^z x + 2(\frac{4}{5})^{z+3}$.*

To prove that ψ is a metric defining function we need the following observation (see Figure 3.2):

Remark 48 *(i) $\psi(x) \geq h_z(x)$ for all $x \in [0, (\frac{28}{125})^z]$, $z \in \mathbf{Z}$*

(ii) $\psi(x) \leq h_z(x)$ for all $x \in [\frac{16}{7}(\frac{28}{125})^{z+1}, \infty)$, $z \in \mathbf{Z}$

(iii) $\psi(x) \leq g_z(x)$ for all $x \in [0, \frac{16}{7}(\frac{28}{125})^{z+1}]$, $z \in \mathbf{Z}$

(iv) $\psi(x) \geq g_z(x)$ for all $x \in [(\frac{28}{125})^{z+1}, \infty)$, $z \in \mathbf{Z}$.

Remark 49 *ψ is a metric defining function.*

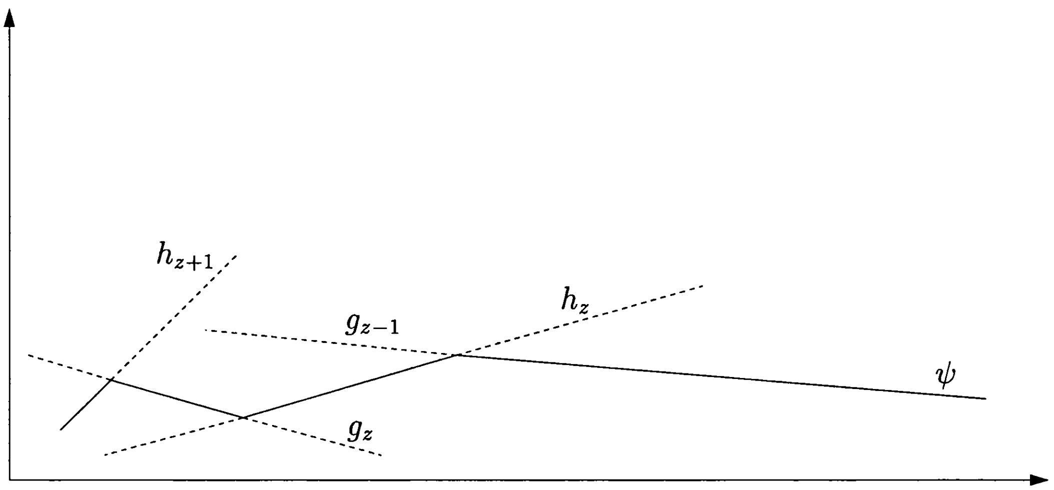

Figure 3.2: The functions ψ, g_{z-1}, g_z, h_z and h_{z-1}

Proof. Let $x, y \in \mathbf{R}$. To show (2) of Definition 40 we may assume that $y \geq |x| > 0$. Let $z \in Z$ be so that $y \in [(\frac{28}{125})^z, (\frac{28}{125})^{z-1})$.

First case: $x > 0$

We have $x + y \in ((\frac{28}{125})^z, 2(\frac{28}{125})^{z-1})$. If $x + y \in ((\frac{28}{125})^z, \frac{16}{7}(\frac{28}{125})^z)$, then

$$\psi(x+y) \leq \psi(y) \leq \psi(x) + \psi(y).$$

Otherwise by Remark 48 (ii) and (i)

$$\begin{aligned} \psi(x+y) &\leq h_{z-1}(x+y) = \left(\frac{25}{7}\right)^{z-1} x + \left(\frac{25}{7}\right)^{z-1} y \\ &= h_{z-1}(x) + h_{z-1}(y) \leq \psi(x) + \psi(y) \end{aligned}$$

Second case: $x < 0$

We have $x + y \in [0, (\frac{28}{125})^{z-1})$. If $x + y \in [\frac{16}{7}(\frac{28}{125})^z, (\frac{28}{125})^{z-1}]$, then

$$\psi(x+y) \leq \psi(y) \leq \psi(x) + \psi(y).$$

Otherwise by Remark 48 (iii), (i) and (iv)

$$\psi(x+y) \leq g_{z-1}(x+y) = \left(\frac{25}{7}\right)^{z-1} |x| - \left(\frac{25}{7}\right)^{z-1} y + 2\left(\frac{4}{5}\right)^{z+2}$$

$$= \quad h_{z-1}(|x|) + g_{z-1}(y) \le \psi(|x|) + \psi(y),$$

and we are done.

One can define similar continuous metric defining functions consisting of line segments. To give a more intuitive approach we have decided not to give a very technical general class of functions but to include only this specific example, which will be needed later in this section. As we will see in the next chapter (see Definition 60) it is even possible to construct similar continuous metric defining functions using 'square-root segments'.

The next theorem will prove very useful in the next chapter and will also yield a natural condition on when the isodiametric inequality is fulfilled.

Theorem 50 *Let $r > 0$, d a continuous metric and f the corresponding metric defining function such that for*

$$x = \sup\{y > 0 \mid f(z) \le r \textit{ for all } z \in (0, y]\}$$

we have

$$x < \infty$$

and

$$r < f(y) \quad \textit{for all} \quad y \ge 2x.$$

Then

(i) $\mathcal{L}D \le x = \mathcal{L}[0, x]$ *for all* $D \subset \mathbf{R}$ *with* $\operatorname{diam} D \le r$.

(ii) d *fulfills the isodiametric inequality for* r.

Proof. Let $D \subset \mathbf{R}$ with $\operatorname{diam} D = r$. We may assume that $0 = \inf D$. Let $B = D \cap (x, \infty)$. By the assumption $B \subset (x, 2x)$. Let $\epsilon \in (0, x)$. We will show that $\mathcal{L}D \leq x + \epsilon$. By the definition of x we may find $z \in (x, x+\epsilon)$ with $f(z) > r$. Note that for $y \in B \cap [x+\epsilon, 2x]$ we have $0 < y - z < x$ and $r < f(z) = d(y, y-z)$, which implies $y - z \in [0, x] \backslash D$. Hence

$$D \subset ([0,x] \backslash ((B \backslash (x, x+\epsilon)) - z)) \cup B$$

and therefore

$$\mathcal{L}D \leq \mathcal{L}[0,x] - (\mathcal{L}B - \mathcal{L}(x, x+\epsilon)) + \mathcal{L}B = x + \epsilon.$$

As $\epsilon \in (0, x)$ is arbitrary we have $\mathcal{L}D \leq x$, which proves (i).

Let $A \supset [0,x]$, $A \neq [0,x]$ and $u \in A \backslash [0,x]$. In order to prove (ii) we need to show that $\operatorname{dist}(u, [0,x]) > r$. By the assumption we may assume $u \in (-2x, 0) \cup (x, 2x)$. If $u \in (x, 2x)$, then we find by the definition of x a $\zeta \in (x, u)$ with $f(\zeta) > r$. We have $0 < u - \zeta < x$ and $d(u, u-\zeta) = f(\zeta) > r$. If $u \in (-x, 0)$, then we find again by the definition of x a $\xi \in (x, x-u)$ with $f(\xi) > r$. We have $0 < \xi - x < \xi + u < x$ and $d(u, u+\xi) = f(\xi) > r$. Finally, if $u \in (-2x, -x]$ we have $x - u \geq 2x$, and we therefore have by the assumption that $d(x, u) = f(x-u) > r$, as required.

Corollary 51 *Let d be a continuous and monotone translation invariant metric. Then d fulfills the isodiametric inequality for any $r > 0$.*

Proof. Let $r > 0$ and write f for the metric defining function corresponding to d. If

$$\sup\{y > 0 \mid f(z) \leq r \text{ for all } z \in (0, y]\} = \infty$$

then the interval $[0, \infty)$ is r-maximal and we are done. Otherwise the assumptions of Theorem 50 are fulfilled and we may apply it to finish the proof.

We show in the next chapter that there exists indeed a non-monotone continuous metric defining function satisfying the assumptions of Theorem 50 (see Definition 60, Lemma 61 (iii) and the proof of Lemma 64).

Next we will show that Theorem 50 does not necessarily hold without the assumptions stated there.

Theorem 52 *There is a continuous translation invariant metric so that the isodiametric inequality does not hold for any $\eta > 0$.*

Proof. We will show that the metric defined by ψ from Example 46 is as required. Note that in Remark 49 we have already shown that ψ is a metric defining function. Let $\eta > 0$. We need to show that the interval with diameter η and maximal length is not η-maximal. Choose $z \in \mathbf{Z}$ such that $\eta = a(\frac{4}{5})^z$ for some $a \in [\frac{4}{5}, 1)$. Then the interval of maximal length is $[0, a(\frac{28}{125})^z]$. Set

$$\tau = \max\{\frac{18}{7}a\Big(\frac{28}{125}\Big)^z, \frac{2}{7}(16-9a)\Big(\frac{28}{125}\Big)^z\}.$$

Note that $\tau < \frac{25}{7}a(\frac{28}{125})^z$. Set

$$A = [0, a\Big(\frac{28}{125}\Big)^z] \cup [\tau, \frac{25}{7}a\Big(\frac{28}{125}\Big)^z].$$

We then have

$$\mathcal{L}[0, a\Big(\frac{28}{125}\Big)^z] < \mathcal{L}A.$$

It remains to show that $\operatorname{diam} A \leq \eta$. We have

$$\frac{25}{7}a\Big(\frac{28}{125}\Big)^z - \tau \leq \frac{25}{7}a\Big(\frac{28}{125}\Big)^z - \frac{18}{7}a\Big(\frac{28}{125}\Big)^z = a\Big(\frac{28}{125}\Big)^z$$

and

$$\tau - a\Big(\frac{28}{125}\Big)^z \geq \frac{2}{7}(16-9a)\Big(\frac{28}{125}\Big)^z - a\Big(\frac{28}{125}\Big)^z = \frac{1}{7}(32-25a)\Big(\frac{28}{125}\Big)^z.$$

Thus for $x, y \in A$ with $y \geq x$ we have

$$y - x \in [0, a\Big(\frac{28}{125}\Big)^z] \cup [\frac{1}{7}(32 - 25a)\Big(\frac{28}{125}\Big)^z, \frac{25}{7}a\Big(\frac{28}{125}\Big)^z]$$

and therefore

$$\begin{aligned} \frac{16}{7}\Big(\frac{28}{125}\Big)^{z+1} &= \frac{64}{125}\Big(\frac{28}{125}\Big)^z < a\Big(\frac{28}{125}\Big)^z < \Big(\frac{28}{125}\Big)^z < \frac{1}{7}(32 - 25a)\Big(\frac{28}{125}\Big)^z \\ &< \frac{25}{7}a\Big(\frac{28}{125}\Big)^z = \frac{4}{5}a\Big(\frac{28}{125}\Big)^{z-1} < \Big(\frac{28}{125}\Big)^{z-1}. \end{aligned}$$

Hence (recall Definition 47)

$$\psi(y - x) \leq \max\{h_z(a\Big(\frac{28}{125}\Big)^z), g_{z-1}(\frac{1}{7}(32 - 25a)\Big(\frac{28}{125}\Big)^z), h_{z-1}(\frac{25}{7}a\Big(\frac{28}{125}\Big)^z)\}$$

and this implies

$$\operatorname{diam} A \leq a\Big(\frac{4}{5}\Big)^z = \eta$$

and we are done.

Chapter 4

Metrics and Hausdorff Measures

4.1 Introduction

In this chapter we show that the answer to the generalized Besicovitch 1/2-problem (i.e., if $\sigma_n(M) \leq 1/2$ for all integers n and all separable metric spaces M) is 'no'. Before we give a brief idea on this proof, let us start with a definition, a related basic theorem and by proving some facts about a specific metric space.

Definition 53 *Let $n \in \mathbf{N}$ and $E \subset \mathbf{R}^n$. We call $x \in \mathbf{R}^n$ a* density point *of E if*

$$\lim_{r \downarrow 0} \frac{\mathcal{L}^n(B(x,r) \cap E)}{\mathcal{L}^n B(x,r)} = 1.$$

Theorem 54 *Let $n \in \mathbf{N}$ and $E \subset \mathbf{R}^n$. Then $\mathcal{L}^n$-almost every $x \in E$ is a density point of E.*

Proof. See for example Corollary 2.14 and Remark 2.15 in [21].

Throughout this chapter we will use the notation of Section 3.3. In particular, recall (see Definition 43) that for $p \geq 1$ the p-snowflake metric is the Euclidean metric on the real line to the power of $1/p$ and that we denote the corresponding metric spaces by S_p. Although the underlying set of S_p is the real line, we will still write S_p for this set in order to emphasize that (for example) the diameter and the Hausdorff measure are calculated here using the snowflake metric and not the Euclidean metric.

Theorem 55 *Let $t \geq 1$ and $p \in \mathbf{N}$.*

(i) $\mathcal{L}D \leq (\operatorname{diam} D)^t$ *for all* $D \subset S_t$,

(ii) $\mathcal{L}B(x,r) = \mathcal{H}^t B(x,r) = 2r^t$ *for all* $x \in S_t$ *and all* $r > 0$,

(iii) $D_t(S_t, x) = 2^{1-t}$ *for all* $x \in S_t$,

(iv) S_p *is purely p-unrectifiable if and only if* $p \geq 2$,

(v) $\sigma_p(S_p) = 2^{1-p}$ *if* $p \geq 2$.

Proof. (i) Let $D \subset S_t$. We may assume that $r = \operatorname{diam} D < \infty$. Observe that (up to translation) $[0, r^t]$ is the only closed interval of diameter r. As $x^{1/t}$ is monotone, we can apply Corollary 51 to obtain

$$\mathcal{L}D \leq \mathcal{L}[0, r^t] = r^t = (\operatorname{diam} D)^t.$$

(ii) Let $x \in S_t$ and $r > 0$. Take an arbitrary $\delta > 0$ and let $\{E_i\}_i$ be a δ-covering of $B(x,r)$ with

$$\sum_{i=1}^{\infty} (\operatorname{diam} E_i)^t - \delta \leq \mathcal{H}^t_\delta B(x,r).$$

By (i) we get

$$\mathcal{L}B(x,r) - \delta \le \sum_{i=1}^{\infty} \mathcal{L}E_i - \delta \le \sum_{i=1}^{\infty} (\operatorname{diam} E_i)^t - \delta \le \mathcal{H}^t_\delta B(x,r).$$

By letting $\delta \downarrow 0$ we obtain

$$\mathcal{L}B(x,r) \le \mathcal{H}^t B(x,r).$$

To prove the reversed inequality we may assume that $x = 0$. Take $\rho > 0$ and choose $\xi \in (0, \rho^t)$ such that r^t/ξ is an integer. As $B(0,r) = [-r^t, r^t]$ we have that $\{[(i-1)\xi, i\xi]\}_{i=1-r^t/\xi}^{r^t/\xi}$ is a ρ-covering of $B(0,r)$. Hence

$$\begin{aligned} \mathcal{H}^t_\delta B(0,r) &\le \sum_{i=1-r^t/\xi}^{r^t/\xi} (\operatorname{diam}[(i-1)\xi, i\xi])^t \\ &= 2r^t = \mathcal{L}[-r^t, r^t] = \mathcal{L}B(0,r) \end{aligned}$$

and we are finished by letting $\delta \downarrow 0$.

(iii) This follows directly from (ii).

(iv) If $p = 1$, then S_1 is the real line equipped with the usual Euclidean metric, which is 1-rectifiable. Thus it is enough to look at the case where $p \ge 2$ and show that S_p is purely p-unrectifiable. Let $F \subset \mathbf{R}^p$ and $\varphi : F \to S_p$ be Lipschitz. Fixing $\rho > 0$ and defining $E = B(0,\rho) \cap F$ it is enough to show that $\mathcal{H}^p(\varphi E) = 0$ (recall that sets of measure 0 are always measurable and we can therefore apply Theorem 4).

Let $x \in E$ be a density point of E. We will now prove that

$$\lim_{E \ni y \to x} \frac{|\varphi(y) - \varphi(x)|}{|y-x|^p} = 0. \tag{4.1}$$

Let $n \ge 2$ be an integer. We then find $R > 0$ so that for all $0 < r \le R$

$$\frac{\mathcal{L}^p(B(x,r) \cap E)}{\mathcal{L}^p B(x,r)} > 1 - \frac{1}{n^p}. \tag{4.2}$$

Choose $y \in B(x,r) \cap E$. We then have for $j = 0, 1, \ldots, n$

$$\begin{aligned} \mathcal{L}^p(B(x,|y-x|)\backslash B(x+\frac{j}{n}(y-x),\frac{|y-x|}{n})) &= \alpha(p)|y-x|^p(1-\frac{1}{n^p}) \\ &= (1-\frac{1}{n^p})\mathcal{L}^p B(x,|y-x|) \end{aligned}$$

and therefore by (4.2)

$$B(x+\frac{j}{n}(y-x),\frac{|y-x|}{n}) \cap E \neq \emptyset.$$

In conclusion we find points $(y_j)_{j=0}^n \subset E$ with $y_0 = x$, $y_n = y$ and for $j = 0, 1, \ldots, n$

$$y_j \in B(x+\frac{j}{n}(j-x),\frac{|y-x|}{n}) \cap E.$$

Therefore

$$\begin{aligned} |\varphi(y)-\varphi(x)| &\leq \sum_{j=1}^n |\varphi(y_i)-\varphi(y_{i-1})| \\ &\leq (\mathrm{Lip}(\varphi))^p \sum_{j=1}^n |y_j - y_{j-1}|^p \\ &\leq (\mathrm{Lip}(\varphi))^p \sum_{j=1}^n (|y_j-(x+\frac{j}{n}(y-x))| + |x+\frac{j}{n}(y-x) \\ &\quad -(x+\frac{j-1}{n}(y-x))| + |x+\frac{j-1}{n}(y-x)-y_{j-1}|)^p \\ &\leq (\mathrm{Lip}(\varphi))^p n(\frac{3|y-x|}{n})^p \\ &= 3^p(\mathrm{Lip}(\varphi))^p n^{1-p}|y-x|^p. \end{aligned}$$

As $n \geq 2$ was an arbitrary integer (4.1) follows from the above.

For $i, j \in \mathbf{N}$ define

$$A_{i,j} = \{z \in E \mid \frac{|\varphi(z)-\varphi(y)|}{|z-y|^p} \leq \frac{1}{i} \text{ for all } y \in E \text{ with } |z-y| \in (0,\frac{1}{j})\}.$$

Fix now $k, m \in \mathbf{N}$ and $\delta \in (0, 1/m)$. To avoid confusion we will write for the rest of this proof diam_e for the Euclidean diameter and diam_s for the diameter with respect to the p-snowflake metric. Let $\{E_i\}_i$ be a $\delta/\mathrm{Lip}(\varphi)$-covering of $A_{k,m}$ with $E_i \subset A_{k,m}$ for $i = 1, 2, \dots$ and

$$\sum_{i=1}^{\infty}(\mathrm{diam}_e E_i)^p \le \mathcal{H}^p A_{k,m} + \delta.$$

Thus $\{\varphi E_i\}_i$ is a δ-covering of $\varphi A_{k,m}$ with

$$\begin{aligned}\mathcal{H}^p_\delta(\varphi A_{k,m}) &\le \sum_{i=1}^{\infty}(\mathrm{diam}_s \varphi E_i)^p = \sum_{i=1}^{\infty}\mathrm{diam}_e \varphi E_i \\ &\le \frac{1}{k}\sum_{i=1}^{\infty}(\mathrm{diam}_e E_i)^p \le \frac{1}{k}(\mathcal{H}^p A_{k,m} + \delta) \le \frac{1}{k}(\mathcal{H}^p E + \delta).\end{aligned}$$

Hence $\mathcal{H}^p(\varphi A_{k,m}) \le \frac{1}{k}\mathcal{H}^p E$. From (4.1) and Theorem 54 we also have that

$$\mathcal{H}^p(\bigcup_{j=1}^{\infty}\varphi A_{k,j}) = \mathcal{H}^p \varphi E.$$

As $m \in \mathbf{N}$ was arbitrary we thus have

$$\mathcal{H}^p(\varphi E) \le \frac{1}{k}\mathcal{H}^p E < \infty,$$

and for $k \to \infty$ we obtain $\mathcal{H}^p(\varphi E) = 0$, as required.

(v) Observe that for $E \subset S_p$ and $x \in S_p$ we have $\underline{D}_p(E, x) \le D_p(S_p, x)$. Therefore the claim follows directly from (iii) and (iv).

One should note that (ii) of the last theorem can be improved to $\mathcal{L} = \mathcal{H}^t$. This is because from (ii) it follows that both measures are finite and positive on all closed balls (such measures are called *uniformly distributed*). By a classic result (see for example Theorem 3.4 of [21]) this implies that $\mathcal{L} = c\mathcal{H}^t$ for a constant $c > 0$. That $c = 1$ again follows from (ii) of the preceding theorem.

Now let us briefly explain how we plan to construct a metric space M with

$\sigma_2(M) > 1/2$. If we choose an $\epsilon > 0$ it is easy to see that by multiplying the 2-snowflake metric by $1+\epsilon$ (i.e., the new metric defining function is $(1+\epsilon)\sqrt{x}$)) we still have $\sigma_2 = 1/2$. Hence, we have two metric defining functions, that both yield $\sigma_2 = 1/2$ and are therefore 'very close' to giving a negative answer to the generalized Besicovitch 1/2-problem. Thus, the basic idea is to construct a metric defining function 'between' $\sqrt{x}$ and $(1+\epsilon)\sqrt{x}$ and hope for a $\tau > 1$ with $\tau\mathcal{L}D \leq (\operatorname{diam} D)^2$. That $\sigma_2 > 1/2$ for this metric would then follow in a similar way as in the proof of Theorem 55 (and in the spirit of Section 3.1). In the next two sections we will show that we can indeed find such a metric and in Section 4.4 we will explore the 'optimal' σ_2 using this type of construction.

4.2 An Interesting Metric

Throughout this chapter we will have a fixed $\epsilon \in (0, 1/4)$ and most of the definitions will depend on this ϵ even if it is not explicitly written. This might sometimes look a bit strange, but it is done for two good reasons. The first one is that by taking a concrete numerical value for ϵ the calculations would look more confusing. The second reason is that in Section 4.4 we will explore what the 'best' ϵ would be in order to get a maximal value for the lower density with the technique used. By doing the work here for a general $\epsilon \in (0, 1/4)$, the task of referring to this section will be much easier. Readers unsympathetic to this idea are invited to go through this section with, say, $\epsilon = 1/20$.

Definition 56 *We define $a, b > 1/2$ as the solutions of*

$$\sqrt{1-a} + \sqrt{a} = 1 + \epsilon = \sqrt{b-1} + \sqrt{b}.$$

Note that as $\sqrt{1-x} + \sqrt{x}$ is decreasing on $(1/2, 1)$ and $\sqrt{x-1} + \sqrt{x}$ is increasing on $(1, \infty)$, a and b are well-defined.

Lemma 57

$$1 - 2\epsilon^2 < a < 1 < b < 1 + \epsilon^2$$

and

$$b < 2a.$$

Proof. i) We have

$$2\epsilon + (\sqrt{2}-1)^2\epsilon < \frac{1}{2} + (\frac{3}{2} - 1)^2\frac{1}{4} < \frac{4}{5} < 2(\sqrt{2}-1).$$

Hence

$$(1 - (\sqrt{2}-1)\epsilon)^2 = 1 - 2(\sqrt{2}-1)\epsilon + (\sqrt{2}-1)^2\epsilon^2 < 1 - 2\epsilon^2$$

and therefore

$$1 + \epsilon < \sqrt{1 - (1 - 2\epsilon^2)} + \sqrt{1 - 2\epsilon^2}.$$

As $\sqrt{1-x} + \sqrt{x}$ is strictly decreasing on $(1/2, 1)$ we infer $1 - 2\epsilon^2 < a$.

ii) As $\sqrt{x-1} + \sqrt{x}$ is strictly increasing on $(1, \infty)$ and $\sqrt{(1+\epsilon^2) - 1} + \sqrt{1+\epsilon^2} > 1 + \epsilon$ we obtain $b < 1 + \epsilon^2$.

iii)

$$b < 1 + \epsilon^2 < \frac{7}{4} < 2(1 - 2\epsilon^2) < 2a.$$

Definition 58 *Let $z > 0$ and a, b as in Definition 56. We set*

$$f_z(x) = \begin{cases} \sqrt{x} & \text{for } x \in [0, az) \cup [bz, \infty) \\ \sqrt{x - az} + \sqrt{az} & \text{for } x \in [az, z) \\ \sqrt{bz - x} + \sqrt{bz} & \text{for } x \in [z, bz). \end{cases}$$

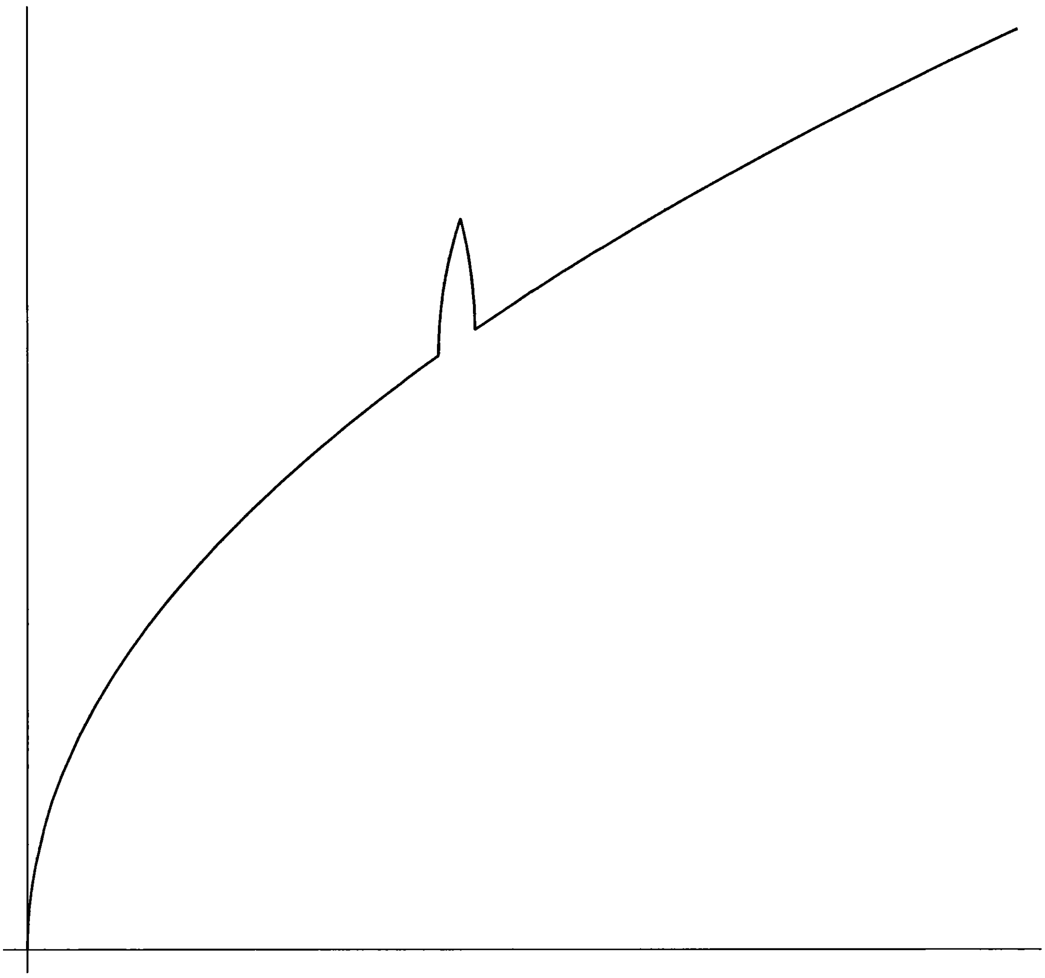

Figure 4.1: The function $f_{1/2}$ for $\epsilon = 1/5$

Lemma 59 *f_z is a metric defining function for all $z > 0$.*

Proof. Fix $z > 0$ and write $f = f_z$. Let $x, y \in \mathbf{R}$. We may assume that $y \geq |x| > 0$. If $x + y \in (0, az] \cup [bz, \infty)$ we have

$$f(x+y) = \sqrt{x+y} \leq \sqrt{|x|} + \sqrt{y} \leq f(|x|) + f(y).$$

Thus we may assume that $x + y \in (az, bz)$ and we therefore have

$$f(x+y) \leq \sqrt{x+y-az} + \sqrt{az} \quad \text{and} \quad f(x+y) \leq \sqrt{bz-(x+y)} + \sqrt{bz}.$$

First case: $|x| \in [bz, \infty)$

If $x > 0$, then $x + y \in [bz, \infty)$. Hence we may assume that $x < 0$. We have

$$\begin{aligned} f(x+y) &\leq \sqrt{bz-(x+y)} + \sqrt{bz} \leq \sqrt{-2x-y} + \sqrt{-x} \\ &\leq \sqrt{y} + \sqrt{|x|} \leq f(|x|) + f(y) \end{aligned}$$

Second case: $|x| \in [az, bz)$

For $x > 0$ we have by Lemma 57 that $x + y \geq 2az > bz$.

If $x < 0$ and $y \in [az, bz)$ we have again by Lemma 57 that $x + y \leq bz - az < az$.

For $x < 0$ and $y \in [bz, \infty)$ we get

$$f(x+y) \leq \sqrt{bz-(x+y)} + \sqrt{bz} \leq \sqrt{y-(x+y)} + \sqrt{y} \leq f(|x|) + f(y).$$

Third case: $|x| \in [0, az)$

If $y \in [bz, \infty)$ we may assume that $x < 0$ (otherwise $x + y \in [bz, \infty)$). Hence

$$f(x+y) \leq \sqrt{bz-(x+y)} + \sqrt{bz} \leq \sqrt{-x} + \sqrt{y} = f(|x|) + f(y).$$

For $y \in [z, bz)$ we have

$$f(x+y) \leq \sqrt{bz-(x+y)} + \sqrt{bz} \leq \sqrt{|x|} + \sqrt{bz-y} + \sqrt{bz} = f(|x|) + f(y).$$

If $y \in [az, z)$ we have

$$f(x+y) \leq \sqrt{x+y-az} + \sqrt{az} \leq \sqrt{|x|} + \sqrt{y-az} + \sqrt{az} = f(|x|) + f(y).$$

For $y \in (0, az)$ and $x < 0$ we have $x + y \in [0, az)$. We may therefore assume that $x > 0$. Introduce a new variable $\xi = x/y$. Note that $az/y - 1 < x/y = \xi \leq 1$. Define

$$F(\xi) = \sqrt{(1+\xi)y - az} + \sqrt{az} - \sqrt{y}(1 + \sqrt{\xi}).$$

Note that $f(x+y) - f(x) - f(y) \leq F(\xi)$. To finish the proof we therefore only need to show that $F(\xi) \leq 0$. As

$$\xi y = x > y + x - az = (1+\xi)y - az,$$

we have

$$F'(\xi) = \frac{y}{2\sqrt{(1+\xi)y - az}} - \frac{\sqrt{y}}{2\sqrt{\xi}} = \frac{y\sqrt{\xi} - \sqrt{y}\sqrt{(1+\xi)y - az}}{2\sqrt{\xi}\sqrt{(1+\xi)y - az}} > 0.$$

It is therefore enough to show (recall that $x + y > az$)

$$G(y) = F(1) = \sqrt{2y - az} + \sqrt{az} - 2\sqrt{y} \leq 0$$

for $y \in (az/2, az)$. As G is increasing on $[az/2, az]$ we get for $y \in (az/2, az)$, that $G(y) \leq G(az) = 0$, as required.

We are now ready to define the metric which will yield $\sigma_2 > 1/2$.

Definition 60 *Set for* $x \in [0, \infty)$

$$f(x) = \sup_{i \in \mathbf{Z}} f_{(1+\epsilon)^{-i}}(x).$$

Lemma 61 *(i)*

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \sqrt{x} & \text{if } x \in [\frac{b}{(1+\epsilon)^{i+1}}, \frac{a}{(1+\epsilon)^i}) \text{ for some } i \in \mathbf{Z} \\ \sqrt{x - \frac{a}{(1+\epsilon)^i}} + \sqrt{\frac{a}{(1+\epsilon)^i}} & \text{if } x \in [\frac{a}{(1+\epsilon)^i}, \frac{1}{(1+\epsilon)^i}) \text{ for some } i \in \mathbf{Z} \\ \sqrt{\frac{b}{(1+\epsilon)^i} - x} + \sqrt{\frac{b}{(1+\epsilon)^i}} & \text{if } x \in [\frac{1}{(1+\epsilon)^i}, \frac{b}{(1+\epsilon)^i}) \text{ for some } i \in \mathbf{Z}, \end{cases}$$

(ii) $f(\frac{1}{(1+\epsilon)^{i+2}}) = \sqrt{\frac{1}{(1+\epsilon)^i}}$,

(iii) f is a continuous metric defining function.

Proof. (i) We need to show that $b/(1+\epsilon)^{i+1} < a/(1+\epsilon)^i$. As $\epsilon < 1/4$ we have $1 - 3\epsilon - 2\epsilon^2 > 0$. Hence by Lemma 57

$$b < 1 + \epsilon^2 < 1 + \epsilon - 2\epsilon^2 - 2\epsilon^3 = (1 - 2\epsilon^2)(1+\epsilon) < a(1+\epsilon).$$

(ii) This follows directly from the definition of b.

(iii) By Lemma 59 we get for all $x, y \geq 0$

$$\begin{aligned} f(|x+y|) &= \sup_{i \in \mathbf{Z}} f_{(1+\epsilon)^{-i}}(|x+y|) \\ &\leq \sup_{i \in \mathbf{Z}} f_{(1+\epsilon)^{-i}}(|x|) + \sup_{i \in \mathbf{Z}} f_{(1+\epsilon)^{-i}}(|y|) = f(|x|) + f(|y|). \end{aligned}$$

Definition 62 *Define ρ_ϵ as the unique translation-invariant metric, corresponding to f in the usual way (see Remark 41). We will write M_ϵ for the metric space $(\mathbf{R}, \rho_\epsilon)$.*

Next we will give lower and upper estimates for the 2-dimensional Hausdorff measure of sets in M_ϵ. We start with the simpler upper estimate for balls.

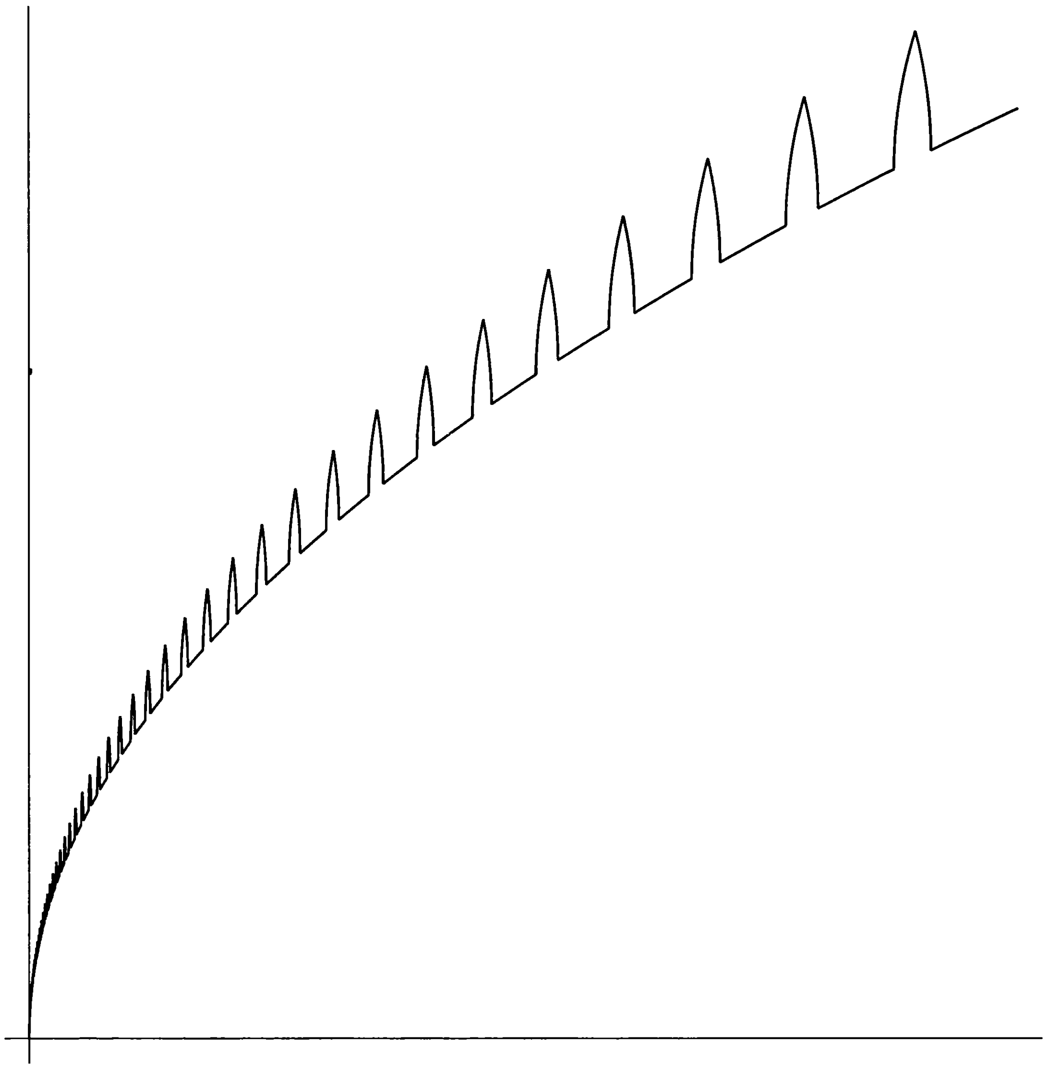

Figure 4.2: The function f for $\epsilon = 1/7$

Lemma 63 *For* $x \in \mathbf{R}$ *and* $r > 0$

$$\mathcal{H}^2 B(x,r) \leq 2(1+\epsilon)^2 r^2.$$

Proof. We may assume that $x = 0$. Let $\xi = \sup B(0,r)$. We have $\sqrt{\xi} \leq f(\xi) \leq r$ and thus $B(0,r) \subset [-r^2, r^2]$. Let $\delta > 0$ and choose $j \in \mathbf{Z}$ so that $\sqrt{1/(1+\epsilon)^{j-2}} < \delta$. Define $k = \min\{i \in \mathbf{N} \mid i \geq (1+\epsilon)^j r^2\}$. We have that

$\{[\frac{i}{(1+\epsilon)^j}, \frac{i+1}{(1+\epsilon)^j}]\}_{i=-k}^{k-1}$ is a δ-covering of $B(0,r)$. Hence

$$\begin{aligned}
\mathcal{H}^2_\delta B(0,r) &\leq \sum_{i=-k}^{k-1} (\operatorname{diam}[\frac{i}{(1+\epsilon)^j}, \frac{i+1}{(1+\epsilon)^j}])^2 \\
&= 2k \sup_{y\in[0,(1+\epsilon)^{-j}]} f^2(y) = 2kf^2(\frac{1}{(1+\epsilon)^j}) \\
&\leq 2(r^2(1+\epsilon)^j + 1)\frac{1}{(1+\epsilon)^{j-2}} \\
&< 2(1+\epsilon)^2 r^2 + 2\delta^2.
\end{aligned}$$

We finish the proof by letting $\delta \downarrow 0$.

Now we will give a lower estimate of the 2-dimensional Hausdorff measure for arbitrary subsets of M_ϵ.

Lemma 64 *For every $D \subset \mathbf{R}$, we have*

(i) $(1+\epsilon)\mathcal{L}D < (\operatorname{diam} D)^2$

and

(ii) $(1+\epsilon)\mathcal{L}D \leq \mathcal{H}^2 D$.

Proof. (i) Let $D \subset \mathbf{R}$. We may assume that $r = \operatorname{diam} D < \infty$. Set

$$x = \sup\{y > 0 \mid f(z) \leq r \text{ for all } z \in (0,y]\}.$$

We have

$$r = f(x) \leq (1+\epsilon)\sqrt{x}$$

and thus for $y > 2x$

$$r < \frac{\sqrt{2}}{1+\epsilon} r \leq \sqrt{2x} < \sqrt{y} \leq f(y).$$

Hence we may apply Theorem 50 to obtain $\mathcal{L}D \leq x$. Now let $i \in \mathbf{Z}$ be such that $r^2 \in [(1+\epsilon)^{-i}, (1+\epsilon)^{-i+1})$. We have

$$f((1+\epsilon)^{-i-1}) = \sqrt{(1+\epsilon)^{-i+1}} > r.$$

Thus we obtain from the definition of x that $x < (1+\epsilon)^{-i-1}$. Hence

$$(1+\epsilon)\mathcal{L}D \leq (1+\epsilon)x \leq (1+\epsilon)^{-i} \leq r^2 = (\operatorname{diam} D)^2.$$

(ii) Let $\delta > 0$ and let $\{E_i\}_i$ be a δ-covering of D with

$$\mathcal{H}^2_\delta D \geq \sum_{i=1}^{\infty} (\operatorname{diam} E_i)^2 - \delta.$$

By (i) we get

$$\mathcal{H}^2_\delta D \geq \sum_{i=1}^{\infty} (\operatorname{diam} E_i)^2 - \delta \geq (1+\epsilon) \sum_{i=1}^{\infty} \mathcal{L}E_i - \delta \geq (1+\epsilon)\mathcal{L}D - \delta.$$

We finish the proof by letting $\delta \downarrow 0$.

Let us now estimate the length of a ball in M_ϵ.

Lemma 65 *For all $r > 0$ and $x \in \mathbf{R}$*

$$\mathcal{L}B(x,r) > 2(1-10\epsilon^2)r^2.$$

Proof. We may assume $x = 0$. Let $r > 0$ and $j \in \mathbf{Z}$ be such that we have $r \in (\sqrt{a/(1+\epsilon)^j}, \sqrt{a/(1+\epsilon)^{j-1}}]$. Observe that $f(y) \leq r$ if $y \leq r^2$ and $f(y) = \sqrt{y}$. Note also that $f = \sqrt{\cdot}$ on $[b/(1+\epsilon)^{i+1}, a/(1+\epsilon)^i]$ for all $i \in \mathbf{Z}$. Thus by repeatedly using Lemma 57

$$\begin{aligned}
\frac{1}{2}\mathcal{L}B(0,r) &\geq r^2 - \frac{b-a}{(1+\epsilon)^j} - \frac{b-a}{(1+\epsilon)^{j+1}} - \frac{b-a}{(1+\epsilon)^{j+2}} \\
&> r^2 - \frac{b-a}{(1+\epsilon)^j} - \frac{2}{(1+\epsilon)^{j+1}}(b-a) \\
&\geq r^2 - \frac{b-a}{a}r^2 - \frac{2r^2}{a(1+\epsilon)}(b-a) \\
&> (2 - \frac{b}{a} - 2\frac{(1-2\epsilon^2)}{a}(b-a))r^2 \\
&> (2 - \frac{b}{a} - 2(b-a))r^2
\end{aligned}$$

$$\begin{aligned}
&\geq (2 - \frac{1+\epsilon^2}{1-2\epsilon^2} - 6\epsilon^2)r^2 \\
&> (2 - (1 + \frac{5}{2}\epsilon^2)(1+\epsilon^2) - 6\epsilon^2)r^2 \\
&= (1 - \frac{19}{2}\epsilon^2 - \frac{5}{2}\epsilon^4)r^2 \\
&> (1 - 10\epsilon^2)r^2,
\end{aligned}$$

as required.

We can now give an estimate on the lower density of M_ϵ. Note the similarity between what we have used in this section for the proof below and the theory of Section 3.3.

Lemma 66 *For all $x \in \mathbf{R}$*

$$\frac{1}{2}(1+\epsilon)(1-10\epsilon^2) \leq \underline{D}_2(M_\epsilon, x) \leq \frac{1}{2}(1+\epsilon)^2.$$

Proof. By Lemma 63, Lemma 64 and Lemma 65 we have for all $x \in \mathbf{R}$

$$\begin{aligned}
\frac{1}{2}(1+\epsilon)(1-10\epsilon^2) &\leq \liminf_{r \downarrow 0} \frac{(1+\epsilon)\mathcal{L}B(x,r)}{(2r)^2} \\
&\leq \liminf_{r \downarrow 0} \frac{\mathcal{H}^2 B(x,r)}{(2r)^2} \leq \frac{1}{2}(1+\epsilon)^2.
\end{aligned}$$

4.3 Main Results

Now we have everything together to solve the generalized Besicovitch 1/2-problem.

Theorem 67 *There exists a purely 2 - unrectifiable metric space M such that $\sigma_2(M) > 1/2$.*

Proof. Let $\rho_{1/20}$ be the metric defined in Definition 62 . We will show that the metric space $M_{1/20} = (\mathbf{R}, \rho_{1/20})$ is as required. By Lemma 66 we have for all $x \in M_{1/20}$

$$\underline{D}_2(M_{1/20}, x) \geq \frac{1}{2}(1 + \frac{1}{20})(1 - 10(\frac{1}{20})^2) = \frac{819}{1600} > \frac{1}{2}.$$

To show that $M_{1/20}$ is purely n-unrectifiable we observe that $M_{1/20}$ is bi-Lipschitz equivalent to the 2-snowflake metric (see Example 43) by the identity map. The statement then follows immediately from Theorem 55.

We have actually proven even more:

Theorem 68 *One can construct a metric ρ on the real line, such that*

(i) ρ is translation - invariant,

(ii) ρ metrizes the Euclidean topology,

(iii) $1/2 < \sigma_2(\mathbf{R}, \rho) < 1$,

(iv) the real line equipped with this metric is purely 2-unrectifiable.

Proof. If we set $\rho = \rho_{1/20}$, then the statement follows immediately from Lemma 66 and the proof of Theorem 67.

We can also prove Theorem 67 and Theorem 68 (iv) without using Theorem 55 . One can use the fact that by Lemma 66 we have $\underline{D}_2(M_\epsilon, x) < 1$ for all $\epsilon \in (0, 1/4)$ and all $x \in M_\epsilon$ and then apply Theorem 1. However in order to have the proof as complete as possible we have decided to take the above approach as the 'official' proof.

4.4 Comments on Theorem 67

In this section we will do an exact calculation of σ_2 for the metric spaces constructed in the last section. We will also give an idea as to why it is very unlikely that this method could yield a higher value for σ_2 than the known upper bound for the σ_1 (i.e. approximately .7266, see Figure 1.1). In doing so we will put away all the mathematical rigour and we will not attempt to prove anything - everything that is written here in the direction of a proof is just an attempt to illustrate why our statements seem plausible. Most results in this section have been obtained using the Mathematica computer programme and recalculating them by hand should prove nearly impossible.

In this first part of the section we explore the precise value of $\sigma_2(M_\epsilon)$. The main reason for doing so is because some of our estimates in Section 4.2 might appear as quite rough at first. We will show here that even 'exact' calculations lead only to a slightly improved result which is 'far away' from the know upper bound for σ_1. We will do so by first giving the exact equations for the 'optimal' bound and then using Mathematica to calculate the results. To simplify the explanations we will refer to the section of f between $a/(1+\epsilon)^i$ and $b/(1+\epsilon)^i$ as a 'hill'. First we will study how we can improve the estimate in Lemma 64. Note that every improvement in the estimate in (i) automatically leads to the same improvement in (ii). Now look at the first part of the proof of (i). By using Theorem 50 we have shown that the interval of diameter r has maximal $\mathcal{L}$-measure for all r. To give the exact bound one therefore only needs to study for which intervals I the ratio $\operatorname{diam}^2 I/\mathcal{L}I$ is minimal and use the value for those intervals to give the 'optimal' estimate (again, observe the similarity between this and the theory of Section 3.1). Intuitively it is clear that these intervals are of the form $[0, d/(1+\epsilon)^i]$, where $d \in (a, 1)$ is such that

$$\sqrt{\frac{d}{(1+\epsilon)^i} - \frac{a}{(1+\epsilon)^i}} + \sqrt{\frac{a}{(1+\epsilon)^i}} = f(\frac{d}{(1+\epsilon)^i}) = f(\frac{1}{(1+\epsilon)^{i+1}}) = \frac{1}{(1+\epsilon)^{\frac{i-1}{2}}}.$$

This intuition could be gained by observing that for infinitesimally smaller diameter one 'looses a lot of measure'.

To get a precise estimate for Lemma 65 one needs to find out for which radii the ratio $\mathcal{L}B(0,r)/r^2$ is minimal. The intuition is that this is the case when $r = \sqrt{b/(1+\epsilon)^i}$. This can also be checked by calculation using a computer. By using inverse functions it is then quite straightforward to calculate $\mathcal{L}B(0, \sqrt{b/(1+\epsilon)^i})$.

The figures 4.3 and 4.4 are drawn with Mathematica using the 'precise' calculations mentioned above.

From the more detailed Figure 4.4 below one can see that the 'optimal' bound in this metric space is approximately 0.545.

Of course one might think that a slightly modified metric defining function f could yield a much higher value for σ_2. The basic idea behind our construction was to 'put' some hills between $\sqrt{x}$ and $(1+\epsilon)\sqrt{x}$. It is therefore natural to ask if we cannot improve the bound by using for instance 'more' hills (e.g. 'put' one or more hills between the existing hills of our construction). This is what we are going to explore now.

Observe first of all that in Lemma 63 we have only used the fact that f is between $\sqrt{\cdot}$ and $(1+\epsilon)\sqrt{\cdot}$. Thus, however we construct the function between these function that lemma will hold. Therefore we can never get a

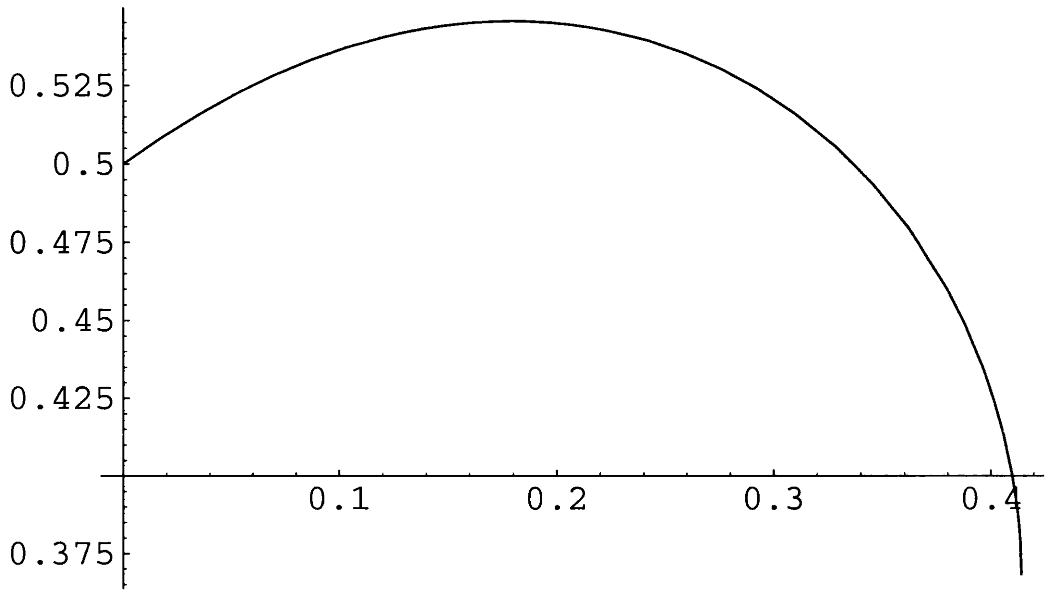

Figure 4.3: $\sigma_2(M_\epsilon)$ as a function of ϵ

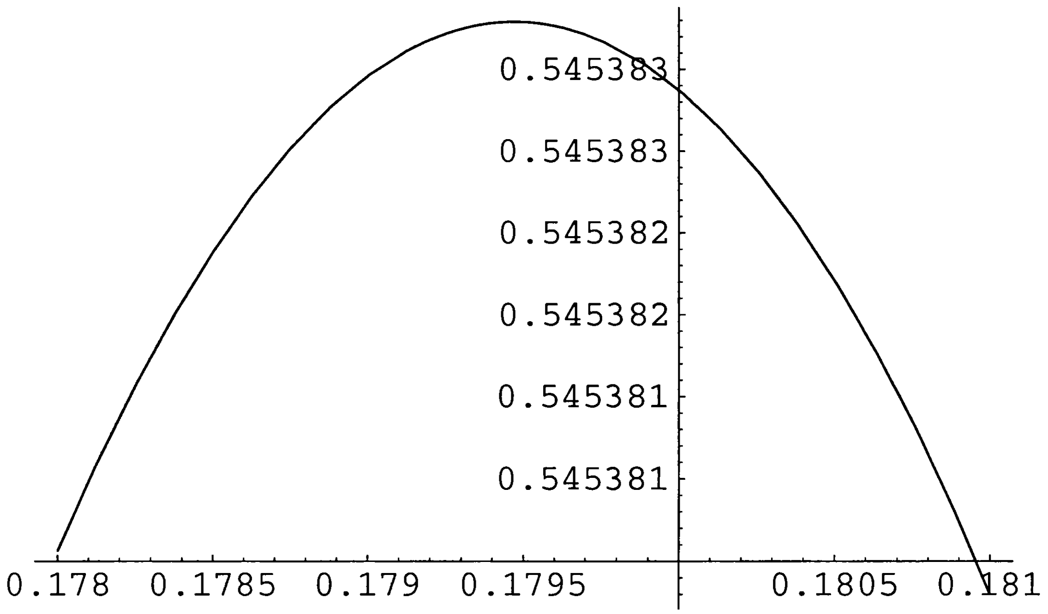

Figure 4.4: A more detailed figure of $\sigma_2(M_\epsilon)$ as a function of ϵ

better bound for σ_2 than $(1+\epsilon)^2/2$. But even this bound is Utopian for many reasons. We will explain two of them, which are easy to understand and easy to write down. The first one is that the inequality in Lemma 63 can never be turned into an equality as one can easily see from the proof. Indeed, we are in some way very far from equality as we completely neglected the gaps in the ball, which are the main point in our construction. The second reason is that for ϵ too large the hills will overlap resulting in the argument on which Theorem 67 is based collapsing. In our construction of f, for example, this would happen for ϵ bigger than approximately $1/3$. If one adds more hills between the existing ones, our ϵ has of course to be taken even smaller.

We should emphasize that the aim of this section was by no means to argue that the value found in this thesis is (close to) the best one could obtain in any metric space (or even in continuous translation-invariant metric spaces on the real line). On the contrary, the writer has the strong belief that this bound can be improved with a different construction. The aim of the section was just to illustrate the main ideas of our construction and to give an intuition why there are no straightforward ways to improve this method significantly.

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