LAGRANGIANS OF HYPERGRAPHS AND OTHER COMBINATORIAL RESULTS

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Ph.D. Thesis

2001

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Abstract

We consider two types of extremal problems for hypergraphs. In chapters two, three and four these are problems related to Lagrangians of hypergraphs. In the final chapter we examine a problem on intersecting families of sets.

After giving an introduction to extremal problems for hypergraphs and Lagrangians in chapter one, we consider a question due to Frankl and Füredi. They asked how large the Lagrangian of an r-graph with m edges can be. We prove the first "interesting" case of their conjecture on this problem, namely that the 3-graph with $\binom{k}{3}$ edges and largest Lagrangian is $[k]^{(3)}$. We also prove a result for general r-graphs: for k sufficiently large, the r-graph supported on $k+1$ vertices with $\binom{k}{r}$ edges and largest Lagrangian is $[k]^{(r)}$.

In the third chapter we consider Erdős' jumping constant conjecture and give a new result on values which are not jumps for hypergraphs. We also discuss an unresolved case of this conjecture which is of particular interest.

Our main result in chapter four is a bound for a Turán-type problem related to Erdős' jumping constant conjecture. We also review Turán's original problem and the use of Lagrangians in this context.

In the final chapter we consider a problem due to Holroyd and Johnson on

intersecting families of separated sets. Our main result here is a new version of the Erdős-Ko-Rado theorem for weighted separated sets.

Acknowledgements

Firstly, I would like to thank my supervisor Imre Leader for his encouragement, advice and guidance. The problems considered in this thesis were suggested by him and the progress I have made owes much to his comments and suggestions.

I would also like to thank my mother and father for the support and encouragement they have always given me. Without their help I could never have achieved this.

For their generous financial support I would like to thank EPSRC, GCHQ and UCL.

Finally, I would like to thank all the other friends, family and fellow students who have put up with the trail of paper I have left in my wake over the past few years.

Sötaste, jag vill att du ska veta hur viktig du är för mig. Det var med energin och lyckan du gav mig som jag skrev den här avhandlingen. Tack för allt...

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Chapter 1

Introduction

Extremal problems for hypergraphs are questions of the form *how many (or few) edges may a particular type of hypergraph contain given that it has a certain property?* Such questions often place restrictions on which sets may occur as edges in the hypergraph, perhaps by insisting that their size is fixed. Examples of properties we may consider include the condition that all the edges of the hypergraph are disjoint or that a particular forbidden subhypergraph does not occur.

Many extremal problems for hypergraphs arise as natural generalizations of problems from extremal graph theory. Extremal graph theory is a broad and rich area of combinatorics providing answers to questions such as: *how many edges may a triangle-free graph of order n contain?* In sharp contrast to the wealth of results for graphs there are many seemingly simple extremal problems for hypergraphs which remain unanswered.

We will find the following definitions useful. For a set V and a positive integer r let $V^{(r)}$ be the collection of all subsets of V of size r. An *r-uniform hypergraph* or *r-graph* G consists of a set V of vertices and a set $E \subseteq V^{(r)}$ of edges. An edge $e = \{a_1, a_2, \ldots, a_r\}$ will often be denoted by $a_1a_2 \ldots a_r$. So for example if $r = 3$ then 137 will represent the edge $\{1, 3, 7\}$. In the special case of $r = 2$ we will often refer to G simply as a *graph.* The *order* of a hypergraph is the cardinality of its vertex set, while its *size* is the cardinality of its edge set.

For an r-graph $G = (V, E)$ and a subset of vertices $W \subseteq V$ we define the subhypergraph induced on W by G to be $G[W] = (W, E \cap W^{(r)})$. We may also denote the set of edges of such an induced subhypergraph by $E[W]$.

When considering extremal problems for r-graphs it is useful to have a measure of the number of edges present in an r-graph as a proportion of the collection of all possible edges, $V^{(r)}$. We define the *density* of an r-graph

$G = (V, E)$ by

$$d(G) = \frac{|E|}{\binom{|V|}{r}}.$$

A fundamental extremal problem for hypergraphs is to ask: how dense may an r-graph be without containing a copy of a forbidden hypergraph from a family $\mathcal{G}$?

Given a family of r-graphs, $\mathcal{G}$, and an r-graph, H, we say that H is *$\mathcal{G}$-free* if no member of $\mathcal{G}$ is a subhypergraph of H. The maximal size of a $\mathcal{G}$-free r-graph of order n is denoted by $ex(n, \mathcal{G})$. A simple averaging result due to Katona, Nemetz and Simonovits [17] tells us that the sequence $\left\{ \frac{ex(n,\mathcal{G})}{\binom{n}{r}} \right\}_{n=r}^{\infty}$ is decreasing. Hence the limit of this sequence exists and we may define the *extremal density* of a family of r-graphs $\mathcal{G}$ by

$$\gamma(\mathcal{G}) = \lim_{n \to \infty} \frac{ex(n, \mathcal{G})}{\binom{n}{r}}.$$

There are some special hypergraphs which we will encounter frequently. We will denote the *complete r-graph of order t* by $K_t^{(r)}$. This is the r-graph of order t containing all possible edges. For any positive integer n we will denote the set $\{1, 2, \ldots, n\}$ by $[n]$, and so in this notation the complete r-graph of order n may also be denoted by $[n]^{(r)}$. Finally, for integers l, r and t we will denote the *complete l-partite r-graph with vertex class size t* by $K_l^{(r)}(t)$. This is the r-graph with vertex set $\bigcup_{i=1}^{l} V_i$, where $V_1, V_2, \ldots, V_l$ are disjoint sets each of order t. The edges of $K_l^{(r)}(t)$ are all r-sets from $\bigcup_{i=1}^{l} V_i$ meeting each V_i in at most one point.

A very natural and simple function to consider for any r-graph is its Lagrangian. For an r-graph G of order n we define the *weight polynomial* of G to be

$$w(G, \mathbf{x}) = \sum_{e \in E} \prod_{i \in e} x_i.$$

We will call $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{R}^n$ a *legal weighting* (for G) if

(i) $\forall i \in [n] \quad x_i \geq 0$,

(ii) $\sum_{i=1}^{n} x_i = 1$.

The *Lagrangian* of G is then defined by

$$\lambda(G) = \max w(G, \mathbf{x}),$$

where the maximum is taken over all legal weightings for G. (Note that this maximum is clearly always attained.) We will call a legal weighting $\mathbf{x}$ *optimal* if in addition to the above the following condition holds

(iii) $w(G, \mathbf{x}) = \lambda(G)$.

Why are Lagrangians interesting? Firstly, the Lagrangian of an r-graph is related to its density. In particular we have the following trivial but useful bound

$$\frac{|E|}{n^r} = w\left(G, \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)\right) \leq \lambda(G).$$

Another important property of $\lambda(G)$ is that if H is a subhypergraph of G then $\lambda(H) \leq \lambda(G)$.

An interesting probabilistic interpretation of $\lambda(G)$ is the following: select r elements of $\mathbf{N}$ independently with the probability that i is selected being x_i. Denoting the resulting set by F we have $\mathbf{P}(F \text{ is an edge in } G) = r!w(G, \mathbf{x})$. Hence $r!\lambda(G)$ is the maximum probability, over all distributions, of obtaining an edge of G in this way.

Lagrangians were introduced for 2-graphs by Motzkin and Straus in 1965 [23]. They determined a simple expression for the Lagrangian of a 2-graph, namely that it is given by taking a clique of maximal order and giving each vertex within this clique equal weight. (A *clique* is a subset of the vertices that induces a complete graph.) We omit the proof since it will follow immediately from Lemma 2.3(b).

Theorem 1.1 (Motzkin and Straus [23])

Let G be a 2-graph in which the maximal order of a clique is t. Then

$$\lambda(G) = \lambda(K_t) = \frac{1}{2}\left(1 - \frac{1}{t}\right).$$

This result allowed Motzkin and Straus to give a new simple proof of the first major result in extremal graph theory, Turán's theorem. More precisely Motzkin and Straus gave a new proof of the extremal density version of Turán's theorem.

Recall that $K_t^{(r)}$ denotes the complete r-graph of order t,

$$\mathrm{ex}(n, K_t^{(r)}) = \max\{|E| : G = (V, E) \text{ is a } K_t^{(r)}\text{-free } r\text{-graph}, |V| = n\},$$

and

$$\gamma(K_t^{(r)}) = \lim_{n\to\infty} \frac{\mathrm{ex}(n, K_t^{(r)})}{\binom{n}{r}}.$$

Theorem 1.2 (Turán's Theorem [28], [23])

The extremal density of the complete 2-graph of order t is

$$\gamma(K_t^{(2)}) = 1 - \frac{1}{t-1}.$$

Proof: If G is $K_t^{(2)}$-free then Theorem 1.1 provides an upper bound for the Lagrangian of G

$$\lambda(G) \leq \lambda(K_{t-1}^{(2)}) = \frac{1}{2}\left(1 - \frac{1}{t-1}\right).$$

Also, as noted earlier, $\lambda(G)$ is bounded below by the value of $w(G, \mathbf{x})$ given by placing weights equal to $\frac{1}{n}$ at each vertex. This yields the upper bound

$$\gamma(K_t^{(2)}) \leq 1 - \frac{1}{t-1}.$$

For the other direction of the inequality note that the complete $(t-1)$-partite 2-graph of order n formed by taking the vertex classes to be as equal as possible in size is $K_t^{(2)}$-free. □

The problem of determining $\mathrm{ex}(n, K_t^{(r)})$ and $\gamma(K_t^{(r)})$, for $t \geq r \geq 2$, is known as Turán's problem. In contrast with the case of $r = 2$ very little is known concerning Turán's problem for $r \geq 3$ (see for example Sidorenko [27]). The new proof of the 2-graph case using Lagrangians aroused interest in the study of Lagrangians for general r-graphs. However, as may be expected given the difficulty of Turán's problem for $r \geq 3$, determining the Lagrangian of a general r-graph seems to be non-trivial.

In the next three chapters we will examine three related problems. We will first consider a question due to Frankl and Füredi [12] as to how large the

Lagrangian of an r-graph with m edges may be. We prove the first "interesting" case of their conjecture. In particular we show that the 3-graph with $\binom{k}{3}$ edges and largest Lagrangian is $[k]^{(3)}$. Our other main result in this chapter is that if k is sufficiently large then the r-graph of size $\binom{k}{r}$ supported on $k+1$ vertices and with largest Lagrangian is $[k]^{(r)}$.

In chapter three we will consider a conjecture due to Erdős [7] as to which values may occur as the extremal densities of r-graphs. We adapt an idea due to Frankl and Rödl [13], using a new construction, to show that certain limiting densities are not jumps.

Finally in chapter four we will consider some Turán-type problems. Our main result here is a non-trivial upper bound for a particular forbidden subhypergraph problem, related to Erdős' jumping constant conjecture.

In the final chapter we will examine a different type of extremal hypergraph problem. In 1961 Erdős, Ko and Rado [8] answered the following question: how many edges may an r-graph of order n contain given that no two edges are disjoint? We consider a problem due to Holroyd and Johnson [15] that asks the same question of a restricted subhypergraph of $[n]^{(r)}$, the collection of separated r-sets.

Chapter 2

The Frankl-Füredi Conjecture

2.1 Introduction

In this chapter we will consider a very natural question due to Frankl and Füredi [12]. For integers $m \geq r \geq 3$ they asked how large the Lagrangian of an r-graph with m edges can be. In order to state their conjecture on this problem we require the following definition. For $A, B \in \mathbf{N}^{(r)}$ with $A \neq B$ we say that A is less than B in the *colex ordering*, and write $A < B$, if $\max(A \triangle B) \in B$. So for example as 3-sets we have $246 < 156$. Note that for integers $r \leq k$ the r-graph consisting of the first $\binom{k}{r}$ sets in the colex ordering of $\mathbf{N}^{(r)}$ is simply the complete r-graph of order k, $[k]^{(r)}$.

Conjecture 2.1 (Frankl and Füredi [12])

The $r-$graph with m edges formed by taking the first m sets in the colex ordering of $\mathbb{N}^{(r)}$ has the largest Lagrangian of all r-graphs with m edges.

In particular the r-graph with $\binom{k}{r}$ edges and largest Lagrangian is $[k]^{(r)}$.

The validity of this conjecture for $r = 2$ follows directly from Theorem 1.1. However, for $r \geq 3$ very little was previously known (see for example [24]). In the next section of this chapter we will prove our main result for 3-graphs on this problem (Theorem 2.2). In particular we will show that Conjecture 2.1 is true for $m = \binom{k}{3}$.

Following a brief discussion of the remaining cases for 3-graphs we give a result for general r-graphs. We show that for k sufficiently large the r-graph supported on $k+1$ vertices with $\binom{k}{r}$ edges and largest Lagrangian is $[k]^{(r)}$.

2.2 An exact result for $r = 3$

Our aim now is to prove the following result.

Theorem 2.2 *Let m and k be integers satisfying*

$$\binom{k-1}{3} \leq m \leq \binom{k-1}{3} + \binom{k-2}{2} - (k-1).$$

Then Conjecture 2.1 is true for $r = 3$ and this value of m.

In particular Conjecture 2.1 is true for 3-graphs with $\binom{k}{3}$ edges.

We first need to establish the following three easy lemmas concerning simple properties of Lagrangians. They provide assumptions that we may make about any r-graph G of size m satisfying $\lambda(G) = \max\{\lambda(H) : H \text{ is an } r\text{-graph of size } m\}$.

The first lemma tells us that we may assume that any such r-graph is *covering*, in the sense that any two vertices lie in at least one common edge. (Note that when $r = 2$ this single lemma is enough to establish the truth of Conjecture 2.1 since a covering 2-graph is simply a complete 2-graph.) This lemma also provides a way of comparing the weights of distinct vertices.

The second lemma tells us that we may assume that G is left compressed. This is useful since it allows us to infer the existence of various edges in E simply by knowing that a certain special edge belongs to E.

The third lemma implies that we need not compare $\lambda(G)$ directly with the Lagrangian of the r-graph formed by taking the first m edges in the colex ordering of $\mathbf{N}^{(r)}$. For the values of m which we are interested in it is sufficient to check that $\lambda(G) \leq \lambda([k-1]^{(r)})$. The proofs of all three lemmas are immediate.

For an r-graph $G = (V, E)$ we will denote the $(r-1)$-neighbourhood of a vertex $i \in V$ by $E_i = \{A \in V^{(r-1)} : A \cup \{i\} \in E\}$. Similarly we will denote the $(r-2)$-neighbourhood of a pair of vertices $i, j \in V$ by $E_{ij} = \{B \in V^{(r-2)} : B \cup \{i, j\} \in E\}$. We will denote the complement of E_i by $E_i^c = \{A \in V^{(r-1)} : A \cup \{i\} \in V^{(r)} \backslash E\}$. Similarly we define $E_{ij}^c = \{A \in V^{(r-2)} : A \cup \{i, j\} \in V^{(r)} \backslash E\}$. Note that the vertex i does not belong to any set in E_i or E_i^c. Similarly neither i nor j belong to any set in either E_{ij} or E_{ij}^c.

We will assume throughout the remainder of this chapter that any optimal legal weighting $\mathbf{x} = (x_1, \ldots, x_n)$ satisfies $x_1 \geq x_2 \geq \cdots \geq x_n$.

Lemma 2.3 (Frankl and Rödl [13])

Let $G = (V, E)$ be an r-graph and $\mathbf{x} = (x_1, \ldots, x_n)$ be an optimal legal weighting for G. Let $k \in [n]$ be the number of non-zero weights, so $x_i > 0$ for $i = 1, \ldots, k$ and $x_i = 0$ for $i > k$. Further, let us suppose that k is minimal, in the sense that any other optimal legal weighting for G uses at least k non-zero weights.

Then for every pair $\{i, j\} \in [k]^{(2)}$

(a) $w(E_i, \mathbf{x}) = w(E_j, \mathbf{x})$,

(b) there is an edge in E *containing both* i *and* j.

Proof: Suppose, for a contradiction, that there exist vertices $\{i, j\} \in [k]^{(2)}$ with $w(E_i, \mathbf{x}) > w(E_j, \mathbf{x})$. We define a new legal weighting $\mathbf{y}$ for G as follows. Let $0 < \delta \leq x_j$ and define $y_l = x_l$ for $l \neq i, j$, $y_i = x_i + \delta$ and $y_j = x_j - \delta$. Then $\mathbf{y}$ is clearly a legal weighting for G and

$$\begin{aligned} w(G, \mathbf{y}) - w(G, \mathbf{x}) &= \delta \left(w(E_i, \mathbf{x}) - w(E_j, \mathbf{x}) \right) - \delta^2 w(E_{ij}, \mathbf{x}) \qquad (2.1) \\ &> 0, \end{aligned}$$

for δ sufficiently small and positive, contradicting $w(G, \mathbf{x}) = \lambda(G)$. Hence part (a) holds.

For part (b) suppose that there exist $\{i, j\} \in [k]^{(2)}$ such that no edge in E contains both i and j. Let $\mathbf{y}$ be defined as above with $\delta = x_j$. Since $E_{ij} = \emptyset$, part (a) and (2.1) imply that $w(G, \mathbf{y}) = w(G, \mathbf{x}) = \lambda(G)$. However $|\{i : y_i > 0\}| = k - 1$, contradicting the minimality of k. Hence part (b) also holds. $\square$

For the next lemma we require the following definition. Let $E \subset \mathbf{N}^{(r)}$, $e \in E$ and $i, j \in \mathbf{N}$ with $i < j$. Then define

$$L_{ij}(e) = \begin{cases} (e \backslash \{j\}) \cup \{i\} & \text{if } i \notin e \text{ and } j \in e \\ e & \text{otherwise} \end{cases}$$

and

$$\mathcal{C}_{ij}(E) = \{L_{ij}(e) : e \in E\} \cup \{e : e, L_{ij}(e) \in E\}.$$

We say that E is *left compressed* if $\mathcal{C}_{ij}(E) = E$ for every $1 \leq i < j$.

Lemma 2.4 *Let $G = (V, E)$ be an r-graph of order n, $i, j \in [n]$ with $i < j$ and $\mathbf{x} = (x_1, \ldots, x_n)$ be an optimal legal weighting for G. Then*

$$w(G, \mathbf{x}) \leq w(G_{ij}, \mathbf{x}),$$

where $G_{ij} = (V, \mathcal{C}_{ij}(E))$.

Proof: Consider the difference

$$w(G_{ij}, \mathbf{x}) - w(G, \mathbf{x}) = \sum_{\substack{e \in E,\ L_{ij}(e) \notin E \\ i \notin e,\ j \in e}} w(e \backslash \{j\}, \mathbf{x})(x_i - x_j).$$

This is non-negative since $i < j$ implies that $x_i \geq x_j$. □

We will denote the r-graph with m edges formed by taking the first m elements in the colex ordering of $\mathbf{N}^{(r)}$ by $C_{r,m}$.

Lemma 2.5 *For any integers m, k and r satisfying*

$$\binom{k-1}{r} \leq m \leq \binom{k-1}{r} + \binom{k-2}{r-1}$$

we have

$$\lambda(C_{r,m}) = \lambda([k-1]^{(r)}).$$

Proof: First we note that $\lambda(C_{r,m}) \geq \lambda([k-1]^{(r)})$ since $[k-1]^{(r)} \subseteq C_{r,m}$.

Let $\mathbf{x} = (x_1, \ldots, x_k)$ be an optimal legal weighting for $C_{r,m}$ using $l \leq k$ non-zero weights. So $x_i > 0$ for $1 \leq i \leq l$ and $x_i = 0$ for $i = l+1, \ldots k$. Further, let us suppose that l is minimal, in the sense that any other optimal legal weighting for $C_{r,m}$ uses at least l non-zero weights.

As the pair of vertices $k-1$ and k do not appear in a common edge of $C_{r,m}$, Lemma 2.3(b) implies that $x_k = 0$. Hence $l \leq k-1$ and

$$\lambda(C_{r,m}) = w(C_{r,m}, \mathbf{x}) = w([k-1]^{(r)}, \mathbf{x}) \leq \lambda([k-1]^{(r)}).$$

So $\lambda(C_{r,m}) = \lambda([k-1]^{(r)})$. □

We require one final definition. Let

$$\lambda_m^r = \max\{\lambda(G) : G \text{ is an } r\text{-graph with } m \text{ edges}\}.$$

We will now give an outline of the proof of Theorem 2.2. Let $G = (V, E)$ be a 3-graph satisfying $\lambda(G) = \lambda_m^3$ for $m = |E|$. The basic argument involves a type of "compression" on the edges of G. We remove certain edges from E and insert others. We then need to check two things. Firstly, that the total weight of the 3-graph (with a slightly modified weighting) has not decreased and secondly that the number of edges we have added does not exceed the number previously removed.

Going into a little more detail, let us suppose that an optimal legal weighting for G uses k strictly positive weights, $x_1, \ldots, x_k$ and that k is minimal. Our aim is to show that most of the edges in $[k]^{(3)}$ are contained in E. We show that if too many edges in $[k-1]^{(3)}$ are missing from E then we can remove the weight from the lightest vertex, k, and place it at vertex $k-1$. This reduces the weight of G but also reduces the number of edges in E (since any edge containing the vertex k now has zero weight and so may be discarded). We may then insert new edges into E (using some of the edges in $[k-1]^{(3)} \backslash E$) and hence produce a new 3-graph G' with the same number of edges as G but with a larger Lagrangian, clearly contradicting the maximality of $\lambda(G)$. The same type of argument is then repeated but this time the weight from vertex $k-1$ is removed and added to vertex k. We can again construct a new 3-graph with a larger Lagrangian than G if the number of edges in $([k-2]^{(2)} \times \{k\}) \backslash E$ is too large. Combining these two results tells us that

$|[k]^{(3)} \backslash E|$ must be small. Hence, if m is in the range given in the statement of Theorem 2.2 then any optimal legal weighting for G can only use at most $k-1$ strictly positive weights. So $\lambda(G) \leq \lambda([k-1]^{(3)})$, which as Lemma 2.5 tells us is enough to prove the result.

Proof of Theorem 2.2:

Let G be a 3-graph with m edges satisfying $\lambda(G) = \lambda^3_m$. Suppose that $\mathbf{x} = (x_1, \ldots, x_n)$ is an optimal legal weighting for G satisfying $x_1 \geq x_2 \geq \cdots \geq x_k > x_{k+1} = \cdots = x_n = 0$. We may suppose that k is minimal in the sense that any other optimal legal weighting for G uses at least k non-zero weights.

We will show that the number of edges in G must satisfy

$$|E| \geq \binom{k-1}{3} + \binom{k-2}{2} - (k-2). \tag{2.2}$$

So if

$$\binom{k-1}{3} \leq m \leq \binom{k-1}{3} + \binom{k-2}{2} - (k-1)$$

then the Lagrangian of G is achieved on $k-1$ vertices. Hence we have $\lambda(G) \leq \lambda([k-1]^{(3)})$. So, by Lemma 2.5, Conjecture 2.1 is true for such values of m.

We aim to show that if $|E|$ is small compared to k, i.e. if (2.2) does not hold, then we can find another 3-graph, G', with the same number of edges as G satisfying $\lambda(G') > \lambda(G)$, contradicting the maximality of $\lambda(G)$.

We know, by Lemma 2.3(b), that the vertices $k-1$ and k appear in some common edge $e \in E$. Also, by Lemma 2.4, we may suppose that E is left compressed and hence $1k-1k \in E$. (Recall that $1k-1k$ denotes the edge $\{1, k-1, k\}$.) Define $b = \max\{i : ik-1k \in E\}$. Then, since E is left compressed, we have $E_i = \{uv : iuv \in E\} = \{1, \ldots, i-1, i+1, \ldots, k\}^{(2)}$, for $1 \leq i \leq b$. Hence, by Lemma 2.3(a), we have $x_1 = x_2 = \cdots = x_b$. Note that $b \leq k-2$.

The following three lemmas will provide the lower bound on $|E|$, proving (2.2). In particular Lemma 2.6 implies that E contains most of the the first $\binom{k-1}{3}$ edges in the colex ordering of $\mathbf{N}^{(3)}$, while Lemma 2.8 implies that E also contains most of the next $\binom{k-2}{2}$ edges.

Lemma 2.6

$$|[k-1]^{(3)} \backslash E| \leq \left\lceil b\left(1 + \frac{k-(b+2)}{k-3}\right)\right\rceil .$$

Lemma 2.7

$$|[k-2]^{(2)} \backslash E_{k-1}| \leq b.$$

Lemma 2.8

$$|[k-2]^{(2)} \backslash E_k| \leq b.$$

Once these lemmas are verified we obtain, using Lemma 2.6 and Lemma 2.8,

$$\begin{aligned} |E| &= |[k-1]^{(3)} \cap E| + |[k-2]^{(2)} \cap E_k| + |E_{k-1k}| \\ &\geq \binom{k-1}{3} - \left\lceil b\left(1 + \frac{k-(b+2)}{k-3}\right)\right\rceil + \binom{k-2}{2} - b + b. \end{aligned}$$

So

$$|E| \geq \binom{k-1}{3} + \binom{k-2}{2} - \left\lceil b\left(1 + \frac{k-(b+2)}{k-3}\right)\right\rceil.$$

It is then easy to check that $\left\lceil b\left(1 + \frac{k-(b+2)}{k-3}\right)\right\rceil \leq k-2$ and so (2.2) holds and the theorem is proved. □

We must now prove the three lemmas.

Proof of Lemma 2.6: We define a new legal weighting for G, $\mathbf{y}$, as follows. Let $y_i = x_i$ for $i \neq k-1, k$, $y_{k-1} = x_{k-1} + x_k$ and $y_k = 0$. (We think of this new weighting as being given by moving the weight from vertex k and placing it at vertex $k-1$.) Clearly $\mathbf{y} = (y_1, \ldots, y_k)$ is a legal weighting.

By Lemma 2.3(a) $w(E_{k-1}, \mathbf{x}) = w(E_k, \mathbf{x})$, so

$$w(G, \mathbf{y}) - w(G, \mathbf{x}) = x_k(w(E_{k-1}, \mathbf{x}) - w(E_k, \mathbf{x})) - x_k^2 \sum_{i=1}^{b} x_i$$

$$= \quad -bx_1x_k^2. \tag{2.3}$$

Since $y_k = 0$ we can remove all edges containing k from E to give a new 3-graph $\overline{G} = (V, \overline{E})$ with $w(\overline{G}, \mathbf{y}) = w(G, \mathbf{y})$ and $|\overline{E}| = |E| - |E_k|$. We will show that if Lemma 2.6 fails, then there exists a set of edges $F \subseteq [k-1]^{(3)} \backslash E$ satisfying

$$w(F, \mathbf{y}) > bx_1x_k^2 \tag{2.4}$$

and

$$|F| \leq |E_k|. \tag{2.5}$$

Then, using (2.3), (2.4) and (2.5), the 3-graph $G' = (V, E')$, where $E' = \overline{E} \cup F$, satisfies $|E'| \leq |E|$ and

$$\begin{aligned} w(G', \mathbf{y}) &= w(\overline{G}, \mathbf{y}) + w(F, \mathbf{y}) \\ &> w(G, \mathbf{y}) + bx_1x_k^2 \\ &= w(G, \mathbf{x}) \\ &= \lambda(G). \end{aligned}$$

Hence $\lambda(G') > \lambda(G)$, contradicting the maximality of $\lambda(G)$. Our next task is to construct the set of edges F.

Consider the equality given by Lemma 2.3(a), $w(E_1,\mathbf{x}) = w(E_{k-1},\mathbf{x})$. Since E is left compressed we know that if $ab \in E_{k-1}$ and $a,b \neq 1$ then $ab \in E_1$. Cancelling such pairs we obtain

$$x_1 w(E_{1k-1},\mathbf{x}) = x_{k-1} w(E_{1k-1},\mathbf{x}) + w(E_1 \cap E^c_{k-1},\mathbf{x}).$$

Hence we have

$$x_1 = x_{k-1} + \frac{w(E_1 \cap E^c_{k-1},\mathbf{x})}{w(E_{1k-1},\mathbf{x})}.$$

Multiplying by bx_k^2 and considering those pairs of the form $ak \in E_1 \cap E^c_{k-1}$ separately we obtain

$$bx_1x_k^2 = bx_{k-1}x_k^2 + \frac{bx_k^3 \sum\limits_{i=b+1}^{k-2} x_i}{\sum\limits_{i=2,\, i\neq k-1}^{k} x_i} + \frac{bx_k^2 w(C,\mathbf{x})}{\sum\limits_{i=2,\, i\neq k-1}^{k} x_i},$$

where $C = [k-2]^{(2)} \backslash E_{k-1}$. Then, since $x_1 \geq x_2 \geq \cdots \geq x_k$,

$$bx_1x_k^2 \leq bx_{k-1}x_k^2\left(1 + \frac{k-(b+2)}{k-3}\right) + \frac{bx_k w(C,\mathbf{x})}{k-2}. \tag{2.6}$$

Define $\alpha = \left\lceil \frac{b|C|}{k-2} \right\rceil$ and $\beta = \left\lceil b\left(1 + \frac{k-(b+2)}{k-3}\right)\right\rceil$. Let the set $F_1 \subseteq [k-1]^{(3)} \backslash E$ consist of the α heaviest edges in $[k-1]^{(3)} \backslash E$ containing the vertex $k-1$. Recalling that $y_{k-1} = x_{k-1} + x_k$ we have

$$w(F_1,\mathbf{y}) \geq \frac{bx_k w(C,\mathbf{x})}{k-2} + \alpha x_{k-1} x_k^2.$$

So using (2.6)

$$w(F_1, \mathbf{y}) - bx_1x_k^2 \geq -x_{k-1}x_k^2(\beta - \alpha). \tag{2.7}$$

We now distinguish two cases.

Case 1 $\alpha > \beta$

In this case $w(F_1, \mathbf{y}) - bx_1x_k^2 > 0$ so defining $F = F_1$ satisfies (2.4). We need to check that (2.5) holds, i.e. that $|F| \leq |E_k|$. We have $|F| = \alpha = \left\lceil \frac{b|C|}{k-2} \right\rceil$ and since $|E_k|$ is an integer it is sufficient to prove that

$$\frac{b|C|}{k-2} \leq |E_k|. \tag{2.8}$$

Since $bk-1k \in E$ and E is left compressed we know that $[b]^{(2)} \cup \{1, \ldots, b\} \times \{b+1, \ldots, k-1\} \subseteq E_k$. Hence

$$|E_k| \geq \frac{b(2k-(b+3))}{2} \geq \frac{b(k-1)}{2}. \tag{2.9}$$

Also, since $C \subset [k-2]^{(2)}$, we have

$$|C| \leq \binom{k-2}{2} < \binom{k-1}{2}. \tag{2.10}$$

So using (2.9) and (2.10) we obtain

$$\frac{b|C|}{k-2} \leq \frac{b(k-1)}{2} \leq |E_k|.$$

Hence (2.8) holds and so both (2.4) and (2.5) are satisfied. Thus we may construct the new 3-graph $G' = (V, E')$ as described above with $|E'| \leq |E|$ and $\lambda(G') > \lambda(G)$, contradicting the maximality of $\lambda(G)$.

Case 2 $\alpha \leq \beta$

Suppose that Lemma 2.6 fails. So $|[k-1]^{(3)} \backslash E| \geq \beta + 1$. Let F_2 consist of any $\beta + 1 - \alpha$ edges in $[k-1]^{(3)} \backslash (E \cup F_1)$ and define $F = F_1 \cup F_2$. Then, using (2.7),

$$\begin{aligned} w(F, \mathbf{y}) - bx_1x_k^2 &= w(F_1, \mathbf{y}) - bx_1x_k^2 + w(F_2, \mathbf{y}) \\ &\geq -x_{k-1}x_k^2(\beta - \alpha) + (\beta - \alpha + 1)x_{k-1}^3 \\ &> 0. \end{aligned}$$

So F satisfies (2.4) and we again need to check that $|F| \leq |E_k|$.

$$|F| = |F_1| + |F_2| = \beta + 1 = \left\lceil b\left(1 + \frac{k-(b+2)}{k-3}\right)\right\rceil + 1.$$

Since $|E_k|$ is an integer it is sufficient to prove that

$$b\left(1 + \frac{k-(b+2)}{k-3}\right) + 1 \leq |E_k|.$$

Using (2.9) it is sufficient to show that

$$b\left(\frac{2k-(b+5)}{k-3}\right) + 1 \leq b\left(\frac{2k-(b+3)}{2}\right).$$

This is easily seen to be true for $k \geq 5$. Note that if $k = 3$ then there is nothing to prove and if $k = 4$ then $|[3]^{(3)} \backslash E| \leq 1$ trivially implies that Lemma 2.6 holds.

Hence $|F| \leq |E_k|$ and so (2.5) holds in this case as well. Since (2.4) and (2.5) both hold we can again construct the new 3-graph G' as described above, contradicting the maximality of $\lambda(G)$. This completes the proof of Lemma 2.6. □

Proof of Lemma 2.7: This proceeds in a very similar way to the previous proof and we assume some of the notation from there.

If Lemma 2.7 fails, then $|C| = |[k-2]^{(2)} \backslash E_{k-1}| \geq b+1$. We again construct a new set of edges $F \subseteq [k-1]^{(3)} \backslash E$ and need to check that F satisfies (2.4) and (2.5). Let F consist of all edges in $[k-1]^{(3)} \backslash E$ containing the vertex $k-1$ (so $F = C \times \{k-1\}$). Then, since $y_{k-1} = x_{k-1} + x_k$,

$$w(F, \mathbf{y}) = (x_{k-1} + x_k) w(C, \mathbf{x}) \geq 2x_k w(C, \mathbf{x}).$$

Using (2.6) we obtain

$$w(F, \mathbf{y}) - bx_1 x_k^2 \geq -bx_{k-1} x_k^2 \left(1 + \frac{k-(b+2)}{k-3}\right) + x_k w(C, \mathbf{x}) \left(2 - \frac{b}{k-2}\right).$$

In order to show that (2.4) holds it is sufficient to prove that

$$|C| \left(2 - \frac{b}{k-2}\right) > b\left(1 + \frac{k-(b+2)}{k-3}\right).$$

This follows simply from $|C| \geq b+1$. To show that (2.5) holds we note that by Lemma 2.6 we have $|F| \leq |[k-1]^{(3)} \backslash E| \leq \beta$. So we simply need to check that $\beta \leq |E_k|$. However, in the proof of Lemma 2.6 we have already shown that $\beta + 1 \leq |E_k|$. Hence F satisfies (2.5). So as in the proof of the previous lemma we may construct a new 3-graph G' with the same number of edges as G but with a larger Lagrangian. This contradiction completes the proof of Lemma 2.7. $\square$

Proof of Lemma 2.8: This proof is again almost identical to that of Lemma 2.6, the main difference being that this time the new legal weighting for G is given by moving weight from vertex $k-1$ to vertex k.

Consider a new legal weighting for G, $\mathbf{z} = (z_1, \ldots, z_k)$, given by $z_i = x_i$ for $i \neq k-1, k$, $z_{k-1} = 0$ and $z_k = x_{k-1} + x_k$. (We think of this weighting as being given by taking the weight from vertex $k-1$ and placing it at vertex k.) By Lemma 2.3(a) $w(E_{k-1}, \mathbf{x}) = w(E_k, \mathbf{x})$, so

$$\begin{aligned} w(G,\mathbf{z}) - w(G,\mathbf{x}) &= x_{k-1}\left(w(E_k,\mathbf{x}) - w(E_{k-1},\mathbf{x})\right) - x_{k-1}^2 \sum_{i=1}^{b} x_i \\ &= -b x_1 x_{k-1}^2. \end{aligned} \tag{2.11}$$

Since $z_{k-1} = 0$ we may remove all edges containing $k-1$ from E to give a new 3-graph $G^* = (V, E^*)$ with $w(G^*, \mathbf{z}) = w(G, \mathbf{z})$ and $|E^*| = |E| - |E_{k-1}|$.

By Lemma 2.7 we know that

$$\begin{aligned}|E_{k-1}| &= |[k-2]^{(2)} \cap E_{k-1}| + b \\ &\geq \binom{k-2}{2}.\end{aligned}$$

We will show that if Lemma 2.8 fails, then there exists a set of edges $H \subseteq \{1, \ldots, k-2, k\}^{(3)} \backslash E$ satisfying

$$w(H, \mathbf{z}) > bx_1 x_{k-1}^2 \tag{2.12}$$

and

$$|H| \leq \binom{k-2}{2}. \tag{2.13}$$

Then using (2.11), (2.12) and (2.13) the graph $G'' = (V, E'')$, where $E'' = E^* \cup H$, satisfies $|E''| \leq |E|$ and $\lambda(G'') \geq w(G'', \mathbf{z}) > \lambda(G)$, contradicting the maximality of $\lambda(G)$. We must now construct the set of edges H.

Consider the equality given by Lemma 2.3(a), $w(E_1, \mathbf{x}) = w(E_k, \mathbf{x})$. Since E is left compressed this implies that

$$x_1 = x_k + \frac{w(E_1 \cap E_k^c, \mathbf{x})}{w(E_{1k}, \mathbf{x})}.$$

Hence

$$bx_1 x_{k-1}^2 = bx_{k-1}^2 x_k + \frac{bx_{k-1}^3 \sum\limits_{i=b+1}^{k-2} x_i}{\sum\limits_{i=2}^{k-1} x_i} + \frac{bx_{k-1}^2 w(D, \mathbf{x})}{\sum\limits_{i=2}^{k-1} x_i},$$

where $D = [k-2]^{(2)} \backslash E_k$. Then, since $x_1 \geq x_2 \geq \cdots \geq x_k$,

$$bx_1 x_{k-1}^2 \leq bx_{k-1}^2 x_k + \frac{bx_{k-1}^3(k-(b+2))}{k-3} + \frac{bx_{k-1} w(D, \mathbf{x})}{k-2}. \tag{2.14}$$

Let H consist of those edges in $\{1, \ldots, k-2, k\}^{(3)} \backslash E$ containing the vertex k. Then

$$w(H, \mathbf{z}) = (x_{k-1} + x_k) w(D, \mathbf{x}).$$

Suppose now that Lemma 2.8 fails. So $|D| \geq b+1$. Using (2.14) and the fact that $w(D, \mathbf{x}) \geq x_{k-1}^2 |D|$ we obtain

$$w(H, \mathbf{z}) - bx_1 x_{k-1}^2 \geq x_k x_{k-1}^2 + |D| x_{k-1}^3 \left(1 - \frac{b}{k-2}\right) - \frac{bx_{k-1}^3(k-(b+2))}{k-3}.$$

Then, since $|D| \left(1 - \frac{b}{k-2}\right) \geq \frac{b(k-(b+2))}{k-3}$, we have $w(H, \mathbf{z}) > bx_1 x_{k-1}^2$ and so (2.12) holds. Finally, $D \subset [k-2]^{(2)}$ implies that $|H| = |D| \leq \binom{k-2}{2}$ and hence (2.13) holds. Therefore we may construct the 3-graph G'' as described above, contradicting the maximality of $\lambda(G)$. This completes the proof of Lemma 2.8. $\square$

2.3 The remaining cases for $r = 3$

Despite the fact that Theorem 2.2 deals with the very natural case of $m = \binom{k}{3}$, the remaining values of m for which Theorem 2.2 does not apply include what we feel are perhaps the most interesting cases of the problem.

If m satisfies

$$\binom{k-1}{3} + \binom{k-2}{2} + 1 \leq m \leq \binom{k}{3} - 1$$

it is easy to construct examples of 3-graphs with m edges whose Lagrangians are strictly larger than $\lambda([k-1]^{(3)})$. Indeed if Conjecture 2.1 is true for all values of m then λ_m^3 jumps at each m in the range given above.

In fact Conjecture 2.1 is true in two cases which are jumps for λ_m^3 namely:

$$m = \binom{k}{3} - 1 \qquad \text{and} \qquad m = \binom{k}{3} - 2.$$

This follows from the proof of Theorem 2.2, by noting that $m \leq \binom{k}{3}$ implies $E \subseteq [k]^{(3)}$ and then recalling that we may suppose that E is left compressed.

For the remaining values of m we have the following approximate result. This tells us that any counterexample to Conjecture 2.1 for $r = 3$ cannot differ greatly from $C_{3,m}$.

Theorem 2.9 *Let m, k and a be integers satisfying*

$$m = \binom{k-1}{3} + \binom{k-2}{2} + a$$

and $-(k-2) \leq a \leq (k-5)$.

Suppose $G = (V, E)$ is a 3-graph with m edges satisfying $\lambda(G) = \lambda_m^3$ and that $|V|$ is minimal in the sense that any other 3-graph H satisfying $\lambda(H) = \lambda_m^3$ has at least $|V|$ vertices.

If Conjecture 2.1 fails to hold for $r = 3$ and this value of m then G and $C_{3,m}$ differ in at most $2(k - a - 2)$ edges, i.e. $|E \triangle E(C_{3,m})| \leq 2(k - a - 2)$.

This follows simply from noting that the proof of Theorem 2.2 implies that $E \subset [k]^{(3)}$.

2.4 Results for general r

We have also considered Conjecture 2.1 for $r > 3$. For such values of r, indeed for simply the next case of $r = 4$, it seems very difficult to generalize the ideas used in the proof of Theorem 2.2.

The main argument used to prove Theorem 2.2 requires two conditions to be satisfied. Firstly there must exist edges, not already present in the r-graph, which are "reasonably heavy". Secondly the number of these edges we need to insert must not exceed the number of edges previously removed. The proof of Theorem 2.2 can be adapted for $r \geq 4$ so that the former condition holds.

However, the later condition has so far escaped our attempts at verification, although there is no obvious reason why it should fail.

Frankl and Füredi [12] originally asked how large the Lagrangian of an r-graph of order k and size m can be, where $m \leq \binom{k}{r}$. Define

$$\lambda(k, r, m) = \max\{\lambda(G) : G = (V, E) \text{ is an } r\text{-graph}, |V| = k, |E| = m\}.$$

If Conjecture 2.1 is true for a given value of r and m then clearly $\lambda(k, r, m) = \lambda(l, r, m)$ whenever k and l satisfy $m \leq \binom{k}{r} \leq \binom{l}{r}$. (In other words it does not matter how many vertices we are allowed to use, the r-graph with m edges and largest Lagrangian uses the smallest number of vertices possible.)

Given that we have been unable to verify Conjecture 2.1 for any values of m with $r \geq 4$ the following weaker result may be of interest.

Theorem 2.10 *For any $r \geq 4$ there exist constants γ_r and $k_0(r)$ such that if m satisfies*

$$\binom{k-1}{r} \leq m \leq \binom{k-1}{r} + \binom{k-2}{r-1} - \gamma_r k^{r-2},$$

with $k \geq k_0(r)$, then $\lambda(k, r, m) = \lambda(C_{r,m})$.

In particular if $k \geq k_0(r)$ then the r-graph of order $k+1$ with $\binom{k}{r}$ edges and largest Lagrangian is $[k]^{(r)}$.

A proof of this result follows along similar lines to the proof of Theorem 2.2 although the details are rather more involved. The parts of the earlier proof checking that when we insert edges $F \subseteq [k-1]^{(r)} \backslash E$ we have $|F| \leq |E_k|$ are now redundant.

Proof of Theorem 2.10: We will take $\gamma_r = 2^{2^r}$. This is not a best possible constant, simply a convenient value.

Suppose m satisfies

$$\binom{k-1}{r} \leq m \leq \binom{k-1}{r} + \binom{k-2}{r-1} - \gamma_r k^{r-2}. \qquad (2.15)$$

Let G be an r-graph of order k and size m satisfying $\lambda(G) = \lambda(k, r, m)$. Choose $\mathbf{x} = (x_1, \ldots, x_k)$, an optimal legal weighting for G using a minimal number of non-zero weights. If $x_k = 0$ then $\lambda(G) \leq \lambda([k-1]^{(r)})$, so by Lemma 2.5 there is nothing to prove. Therefore we may suppose, for a contradiction, that $x_k > 0$.

We know, by Lemma 2.3(b), that the vertices $k-1$ and k appear in some common edge $e \in E$. Also, by Lemma 2.4, we may suppose that E is left compressed and hence $12 \ldots r-2k-1k \in E$. Recall that $E_{ij} = \{A \in V^{(r-2)} : A \cup \{i, j\} \in E\}$. Let $b = |E_{k-1k}|$. So $1 \leq b < \binom{k-2}{r-2}$.

Our aim is to show that G must contain more than m edges, a contradiction.

In order to achieve this we need to generalize Lemmas 2.6 and 2.8. First we require two rather technical lemmas.

Lemma 2.11 *Let $D_{k-i} = [k-(i+1)]^{(r-1)} \cap E^c_{k-i}$, for $i \in [k-1]$, and $D_k = [k-2]^{(r-1)} \cap E^c_k$. Then for any $j < k-i$*

$$x_j \leq \frac{w(D_{k-i}, \mathbf{x})}{w(E_{jk-i}, \mathbf{x})} + \sum_{l=k-i}^{k} x_l,$$

and

$$x_{k-1} \leq \frac{w(D_k, \mathbf{x})}{w(E_{k-1k}, \mathbf{x})} + x_k.$$

Proof: We will prove the first part of this lemma, the second part follows identically. Suppose $i \in [k-1]$ and $j < k-i$, then by Lemma 2.3(a) we know that $w(E_j, \mathbf{x}) = w(E_{k-i}, \mathbf{x})$. Since E is left compressed, we have

$$\begin{aligned} x_j &= x_{k-i} + \frac{w(E_j \cap E^c_{k-i}, \mathbf{x})}{w(E_{jk-i}, \mathbf{x})} \\ &\leq x_{k-i} + \frac{\sum_{l=k-i+1}^{k} x_l w(E_{jl} \cap E^c_{k-il}, \mathbf{x})}{w(E_{jk-i}, \mathbf{x})} + \frac{w(D_{k-i}, \mathbf{x})}{w(E_{jk-i}, \mathbf{x})} \\ &\leq \frac{w(D_{k-i}, \mathbf{x})}{w(E_{jk-i}, \mathbf{x})} + \sum_{l=k-i}^{k} x_l. \end{aligned}$$

The first inequality follows from expanding $w(E_j \cap E^c_{k-i}, \mathbf{x})$. The second inequality follows from $w(E_{jl} \cap E^c_{k-il}, \mathbf{x}) \leq w(E_{jk-i}, \mathbf{x})$ for $k-i+1 \leq l \leq k$.□

Note that whenever the lower limit of a sum or product is greater than the upper limit we take this to be the empty sum or product. These are defined to be equal to zero and one respectively.

Lemma 2.12 *For* $0 \leq t \leq r-3$ *and* $i < j < k-d_t < \cdots < k-d_1 \leq k-2$ *we have*

$$\sum_{\substack{i_1 \dots i_{r-2} \in E_{k-1k} \\ i_{t+1} < j}} x_{k-1} x_k \prod_{m=t+2}^{r-2} x_{i_m} \prod_{p=1}^{t} x_{k-d_p} \leq x_j w(E_{ij}, \mathbf{x}). \tag{2.16}$$

Proof: Let $A = \{i_1, \dots, i_{r-2}\} \in E_{k-1k}$ with $i_{t+1} < j$. For each such set A we need to find a set $B \in E_{ij}$, uniquely determined by A, such that the contribution of A to the left hand side of (2.16) is less than or equal to the contribution of B to the right hand side. The contribution of A is always $x_{k-1} x_k \prod_{m=t+2}^{r-2} x_{i_m} \prod_{p=1}^{t} x_{k-d_p}$.

If $A \cap \{i, j\} = \emptyset$ then $A \in E_{ij}$ so let $B = A$. Then the contribution of B to (2.16) is $x_j \prod_{m=1}^{r-2} x_{i_m}$ and since $x_{k-1} x_k \prod_{p=1}^{t} x_{k-d_p} \leq x_j \prod_{m=1}^{t+1} x_{i_m}$ these terms satisfy (2.16).

If $i \in A$ but $j \notin A$ then there exists $1 \leq s \leq r-2$ such that $i = i_s$. Let $B = (A \cup \{k\}) \setminus \{i\}$ then $B \in E_{ij}$ and $k \in B$ but $k-1 \notin B$. This time the contribution of B to (2.16) is $x_j x_k \prod_{m=1,\ m \neq s}^{r-2} x_{i_m}$. We have two cases: if

$s \leq t+1$ then $x_{k-1} \prod_{p=1}^{t} x_{k-d_p} \leq x_j \prod_{m=1,\ m\neq s}^{t+1} x_{i_m}$ so (2.16) is satisfied. Otherwise we have $s \geq t+2$ so $x_{i_s} \leq x_{i_{t+1}}$ and $x_{k-1} \prod_{p=1}^{t} x_{k-d_p} \leq x_j \prod_{m=1}^{t} x_{i_m}$. Hence (2.16) is satisfied.

If $j \in A$ but $i \notin A$ then there exists $t+2 \leq s \leq r-2$ such that $j = i_s$. Let $B = (A \cup \{k-1\}) \backslash \{j\}$ then $B \in E_{ij}$ and $k-1 \in B$ but $k \notin B$. This time the contribution of B to (2.16) is $x_j x_{k-1} \prod_{m=1,\ m\neq s}^{r-2} x_{i_m}$. Then, since $x_k \prod_{p=1}^{t} x_{k-d_p} \leq x_j \prod_{m=1}^{t} x_{i_m}$ and $x_{i_s} \leq x_{i_{t+1}}$, (2.16) is satisfied.

Finally, if $i, j \in A$ then there exist $1 \leq s < v \leq r-2$ such that $i = i_s$ and $j = i_v$. Let $B = (A \cup \{k-1, k\}) \backslash \{i, j\}$ then $B \in E_{ij}$ and $k-1, k \in B$. This time the contribution of B to (2.16) is $x_j x_{k-1} x_k \prod_{m=1,\ m\neq s,v}^{r-2} x_{i_m}$. We know that $v \geq t+2$ but we must distinguish two cases depending on the value of s. First suppose that $s \leq t+1$ then $x_{i_v} \prod_{p=1}^{t} x_{k-d_p} \leq x_j \prod_{m=1,\ m\neq s}^{t+1} x_{i_m}$ implies that (2.16) holds. Now suppose $s \geq t+2$ then $x_{i_v} x_{i_s} \prod_{p=1}^{t} x_{k-d_p} \leq x_j \prod_{m=1}^{t+1} x_{i_m}$ so (2.16) holds in this case also.

Note that in each case the set B is uniquely determined by A. (We can see this by considering whether or not $k-1$ and k belong to B in each case.) This completes the proof of the lemma. $\square$

Let c_t and d_t be defined as follows for $t \geq 0$,

$$\begin{aligned} c_0 &= 1, \qquad c_{t+1} = \sum_{i=1}^{c_t} (d_t + i + 1), \\ d_0 &= 1, \qquad d_{t+1} = c_t + d_t. \end{aligned}$$

We have the following generalizations of Lemmas 2.6 and 2.8.

Lemma 2.13

$$|[k - (d_{r-2} + 1)]^{(r)} \backslash E| \leq c_{r-2} |E_{k-1k}|.$$

Lemma 2.14

$$|[k - (d_{r-2} + 1)]^{(r-1)} \backslash E_k| \leq c_{r-2} |E_{k-1k}|.$$

Once these lemmas are verified we may complete the proof of Theorem 2.10 as follows. Since $|E_{k-1k}| < \binom{k-2}{r-2}$ Lemma 2.13 implies that

$$|E \cap [k - (d_{r-2} + 1)]^{(r)}| > \binom{k - (d_{r-2} + 1)}{r} - c_{r-2} \binom{k-2}{r-2}. \tag{2.17}$$

Since E is left compressed, if i satisfies $k - d_{r-2} \leq i \leq k$ then

$$|E_i \cap [k - (d_{r-2} + 1)]^{(r-1)}| \geq |E_k \cap [k - (d_{r-2} + 1)]^{(r-1)}|.$$

Hence Lemma 2.14 implies that

$$|E_i \cap [k - (d_{r-2} + 1)]^{(r-1)}| > \binom{k - (d_{r-2} + 1)}{r-1} - c_{r-2} \binom{k-2}{r-2} \tag{2.18}$$

for i satisfying $k - d_{r-2} \le i \le k$. So using (2.17) and (2.18) we have the following lower bound for $|E|$

$$|E| > \binom{k-(d_{r-2}+1)}{r} + (d_{r-2}+1)\binom{k-(d_{r-2}+1)}{r-1} - (d_{r-2}+2)c_{r-2}\binom{k-2}{r-2}.$$

Since $d_r < 2c_{r-1}$ and $c_r < 2^{2^r}$, a tedious but straightforward calculation yields

$$|E| > \binom{k-1}{r} + \binom{k-2}{r-1} - \gamma_r k^{r-2},$$

for k sufficiently large. This contradicts our assumption that $|E|$ lies in the range given by (2.15) and so completes the proof of Theorem 2.10. □

Proof of Lemma 2.13: As in Lemma 2.11 let $D_{k-i} = [k-(i+1)]^{(r-1)} \cap E^c_{k-i}$, for $i \in [k-1]$, and $D_k = [k-2]^{(r-1)} \cap E^c_k$. We claim that the following inequality holds for every $0 \le t \le r-2$,

$$x_{k-1}x_k w(E_{k-1k}, \mathbf{x}) \le \sum_{i=2}^{d_t} x_{k-i} w(D_{k-i}, \mathbf{x}) \quad + \\ \sum_{i_1 \dots i_{r-2} \in E_{k-1k}} c_t x_{k-1} x_k \prod_{m=t+1}^{r-2} x_{i_m} \prod_{p=1}^{t} x_{k-d_p}. \quad (2.19)$$

We prove this by induction on t. Since $c_0 = 1$ and $d_0 = 1$ (2.19) holds for $t = 0$ with a simple equality.

Now let us suppose (2.19) holds for some $0 \le t \le r-3$. We will show that (2.19) also holds for $t+1$. Let l satisfy $d_t + 1 \le l \le d_{t+1}$. Consider $i_{t+1} \in [k]$.

If $k-l \leq i_{t+1} \leq k$ then $x_{i_{t+1}} \leq x_{k-l}$. Otherwise $i_{t+1} < k-l$, so Lemma 2.11 implies that

$$x_{i_{t+1}} \leq \frac{w(D_{k-l},\mathbf{x})}{w(E_{i_{t+1}k-l},\mathbf{x})} + \sum_{j=k-l}^{k} x_j.$$

Hence

$$\sum_{i_1\ldots i_{r-2}\in E_{k-1k}} \prod_{m=t+1}^{r-2} x_{i_m} \leq \sum_{\substack{i_1\ldots i_{r-2}\in E_{k-1k}\\ i_{t+1}\geq k-l}} x_{k-l} \prod_{m=t+2}^{r-2} x_{i_m} \quad + \\ \sum_{\substack{i_1\ldots i_{r-2}\in E_{k-1k}\\ i_{t+1}<k-l}} \left(\frac{w(D_{k-l},\mathbf{x})}{w(E_{i_{t+1}k-l},\mathbf{x})} + (l+1)x_{k-l}\right) \prod_{m=t+2}^{r-2} x_{i_m}.$$

So

$$\sum_{i_1\ldots i_{r-2}\in E_{k-1k}} \prod_{m=t+1}^{r-2} x_{i_m} \leq \sum_{i_1\ldots i_{r-2}\in E_{k-1k}} (l+1)x_{k-l} \prod_{m=t+2}^{r-2} x_{i_m} \quad + \\ \sum_{\substack{i_1\ldots i_{r-2}\in E_{k-1k}\\ i_{t+1}<k-l}} \frac{w(D_{k-l},\mathbf{x}) \prod_{m=t+2}^{r-2} x_{i_m}}{w(E_{i_{t+1}k-l},\mathbf{x})}. \tag{2.20}$$

Also, by Lemma 2.12, we have

$$\sum_{\substack{i_1\ldots i_{r-2}\in E_{k-1k}\\ i_{t+1}<k-l}} \frac{x_{k-1}x_k \prod_{m=t+2}^{r-2} x_{i_m} \prod_{p=1}^{t} x_{k-d_p}}{w(E_{i_{t+1}k-l},\mathbf{x})} \leq x_{k-l}. \tag{2.21}$$

Using (2.20) and (2.21) we obtain

$$\sum_{i_1\ldots i_{r-2}\in E_{k-1k}} x_{k-1}x_k \prod_{m=t+1}^{r-2} x_{i_m} \prod_{p=1}^{t} x_{k-d_p} \leq x_{k-l}w(D_{k-l},\mathbf{x}) \quad + \sum_{i_1\ldots i_{r-2}\in E_{k-1k}} (l+1)x_{k-l}x_{k-1}x_k \prod_{m=t+2}^{r-2} x_{i_m} \prod_{p=1}^{t} x_{k-d_p}. \quad (2.22)$$

This last inequality holds for each l satisfying $d_t + 1 \leq l \leq d_{t+1}$.

Finally, since $d_{t+1} = c_t + d_t$, $c_{t+1} = \sum_{i=1}^{c_t}(d_t + i + 1)$ and $x_{k-l} \leq x_{k-d_{t+1}}$, we can use (2.22) repeatedly to obtain

$$\sum_{i_1\ldots i_{r-2}\in E_{k-1k}} c_t x_{k-1}x_k \prod_{m=t+1}^{r-2} x_{i_m} \prod_{p=1}^{t} x_{k-d_p} \leq \sum_{l=d_t+1}^{d_{t+1}} x_{k-l}w(D_{k-l},\mathbf{x}) \quad + \sum_{i_1\ldots i_{r-2}\in E_{k-1k}} c_{t+1}x_{k-1}x_k \prod_{m=t+2}^{r-2} x_{i_m} \prod_{p=1}^{t+1} x_{k-d_p}.$$

Hence (2.19) holds for $t+1$ and the induction is complete.

Setting $t = r-2$ in (2.19) we obtain

$$x_{k-1}x_k w(E_{k-1k},\mathbf{x}) \leq \sum_{i=2}^{d_{r-2}} x_{k-i}w(D_{k-i},\mathbf{x}) + c_{r-2}x_{k-1}x_k|E_{k-1k}| \prod_{p=1}^{r-2} x_{k-d_p}. \quad (2.23)$$

Now suppose that Lemma 2.13 fails. Then we may proceed as in the proof of Lemma 2.6 to give a new weighting, $\mathbf{y}$, for our r-graph G by moving the weight from vertex k to vertex $k-1$. Let $y_i = x_i$, for $i \neq k-1, k$, $y_{k-1} = x_{k-1} + x_k$ and $y_k = 0$. Clearly $\mathbf{y}$ is a legal weighting for G. Let G' be formed from G by removing all of the edges in G containing k. Then

$\lambda(G') \geq w(G', \mathbf{y}) = w(G, \mathbf{y}) = w(G, \mathbf{x}) - x_k^2 w(E_{k-1k}, \mathbf{x})$. Let $G^* = [k-1]^{(r)}$. Note that G^* cannot contain more edges than G.

Since we are assuming that Lemma 2.13 fails we have

$$w(G^*, \mathbf{y}) > w(G', \mathbf{y}) + \sum_{i=2}^{d_{r-2}} x_{k-i} w(D_{k-i}, \mathbf{x}) + c_{r-2}|E_{k-1k}| \prod_{j=1}^{r} x_{k-d_{r-2}-j}.$$

Hence by (2.23) $w(G^*, \mathbf{y}) > w(G, \mathbf{x})$. This contradicts the assumption that $\lambda(k, r, m) = \lambda(G) = w(G, \mathbf{x})$ and completes the proof of Lemma 2.13. □

Proof of Lemma 2.14: This is easy given the work we have already done. We proceed as in the proof of Lemma 2.8 to give a new weighting, $\mathbf{z}$, for G by moving the weight from vertex $k-1$ to vertex k. So $z_i = x_i$, for $i \neq k-1, k$, $z_{k-1} = 0$ and $z_k = x_{k-1} + x_k$. Then $\mathbf{z}$ is clearly a legal weighting for G. Now let G' be formed from G by removing all of the edges in G containing $k-1$. Then $\lambda(G') \geq w(G', \mathbf{z}) = w(G, \mathbf{z}) = w(G, \mathbf{x}) - x_{k-1}^2 w(E_{k-1k}, \mathbf{x})$. By Lemma 2.11 we have

$$x_{k-1} \leq \frac{w(D_k, \mathbf{x})}{w(E_{k-1k}, \mathbf{x})} + x_k.$$

So $w(G', \mathbf{z}) \geq w(G, \mathbf{x}) - x_{k-1} w(D_k, \mathbf{x}) - x_{k-1} x_k w(E_{k-1k}, \mathbf{x})$. Now let $G^* = \{1, \ldots, k-2, k\}^{(r)}$. As before G^* cannot contain more edges than G. Suppose, for a contradiction, that Lemma 2.14 fails, then we have

$$w(G^*, \mathbf{z}) \quad > \quad w(G', \mathbf{z}) + x_{k-1} w(D_k, \mathbf{x}) + \sum_{i=2}^{d_{r-2}} x_{k-i} w(D_{k-i}, \mathbf{x}) \quad +$$

$$c_{r-2}|E_{k-1k}|x_k \prod_{j=1}^{r-1} x_{k-d_{r-2}-j}.$$

So again using (2.23) we obtain $w(G^*, \mathbf{z}) > w(G, \mathbf{x})$. This contradicts our assumption that $\lambda(k,r,m) = \lambda(G) = w(G,\mathbf{x})$ and completes the proof of Lemma 2.14. $\square$

2.5 Further remarks

It would obviously be nice to settle Frankl and Füredi's conjecture in general. However, as we mentioned at the beginning of the previous section, there are problems which we have been unable to overcome in generalizing the methods used in the proof of Theorem 2.2. We can perhaps claim the main result of the last section (Theorem 2.10) as intuitive evidence of the truth of the conjecture for r-graphs with $\binom{k}{r}$ edges. This is because it says essentially that *if* there exists a counterexample to the conjecture then it must use at least $k+2$ positively weighted vertices and so there is no r-graph whose set of edges is "similar" to $[k]^{(r)}$ with a larger Lagrangian. Hence a counterexample, should one exist, would contain lots of "gaps" - this seems a little implausible.

Chapter 3

Erdős' jumping constant conjecture

3.1 Introduction

Recall that for an r-graph $G = (V, E)$ of order n the *density* of G is the proportion of all possible edges G contains,

$$d(G) = \frac{|E|}{\binom{n}{r}}.$$

For an r-graph H the *extremal density* of H is the limit, as the number of vertices tends to infinity, of the maximum density of an r-graph of order n

not containing a copy of H:

$$\gamma(H) = \lim_{n\to\infty} \frac{\max\{|E| : G = (V, E) \text{ is an } H\text{-free } r\text{-graph of order } n\}}{\binom{n}{r}}.$$

As before $K_l^{(r)}(t)$ denotes the complete l-partite r-graph with t vertices in each vertex class.

The Erdős-Stone theorem is a fundamental result of extremal graph theory. Turán's theorem told us that if a graph of order n has more than $\left(1 - \frac{1}{l}\right)\binom{n}{2}$ edges then it contains a copy of $K_{l+1}^{(2)}$. The Erdős-Stone theorem says that for any integer t and $\epsilon > 0$ there exists $n_0(l, t, \epsilon)$ such that any graph of order $n \geq n_0$, with more than $\left(1 - \frac{1}{l}\right)\binom{n}{2} + \epsilon n^2$ edges contains a copy of $K_{l+1}^{(2)}(t)$. We state the extremal density version of this result.

Theorem 3.1 (Erdős and Stone [10]) *For $l \geq 2$ and $t \geq 2$*

$$\gamma(K_l^{(2)}(t)) = 1 - \frac{1}{l-1}.$$

This result allows us to determine the extremal density of any given 2-graph using the following simple corollary.

Corollary 3.2 (Erdős and Simonovits [9]) *For a 2-graph G, with chromatic number $\chi(G)$,*

$$\gamma(G) = 1 - \frac{1}{\chi(G) - 1}.$$

Recall that for a family of r-graphs $\mathcal{G}$ the maximal size of a $\mathcal{G}$-free r-graph of order n is denoted by $ex(n,\mathcal{G})$. The *extremal density* of such a family is denoted by

$$\gamma(\mathcal{G}) = \lim_{n\to\infty} \frac{ex(n,\mathcal{G})}{\binom{n}{r}}.$$

If we define $\gamma_r = \{\gamma(\mathcal{G}) : \mathcal{G} \text{ is a family of } r\text{-graphs}\}$ then

$$\gamma_2 = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\}.$$

For 2-graphs we have a complete characterisation of γ_2 which is in sharp contrast to the situation for $r \geq 3$. Erdős' *jumping constant conjecture* concerns the structure of the set γ_r for $r \geq 2$.

Given $r \geq 2$ and $\alpha \in [0,1]$, we say that α is a *jump* for r if there exists a constant $c(r,\alpha) > 0$ such that for every $\epsilon > 0$ and every integer $m \geq r$ there exists $n_0(r,\alpha,\epsilon,m)$ such that every r-graph of order $n \geq n_0$ and density at least $\alpha + \epsilon$ contains a subhypergraph of order m with density at least $\alpha + c(r,\alpha)$. Note that if α is a jump for r then $\gamma_r \cap (\alpha, \alpha + c(r,\alpha)) = \emptyset$.

For $r = 2$ every $\alpha \in [0,1]$ is a jump. Clearly the set γ_r is well-ordered iff every α is a jump for r.

Conjecture 3.3 (Erdős [7])

γ_r is well-ordered for every $r \geq 2$.

In 1984 Frankl and Rödl [13] showed that this conjecture is false with the following result.

Theorem 3.4 (Frankl and Rödl [13])

For $r \geq 3$ and $l > 2r$ the value $1 - \frac{1}{l^{r-1}}$ is not a jump for r.

However, this result still leaves us with very little information about the structure of the set γ_r for $r \geq 3$.

For example is $\frac{r!}{r^r}$ a jump for all $r \geq 2$? This was of particular interest to Erdős since the next result tells us that $[0, \frac{r!}{r^r}) \cap \gamma_r = \{0\}$, for $r \geq 2$. Also, it is easy to see that $\frac{r!}{r^r} \in \gamma_r$ for all $r \geq 2$. Consider the family of r-graphs $\mathcal{G} = \bigcup_{t=1}^{\infty} G_t$, where G_t consists of those r-graphs of order rt with more edges than $K_r^{(r)}(t)$. Then, for each $t \geq 1$, $K_r^{(r)}(t)$ is a $\mathcal{G}$-free r-graph of order rt and maximal size. Hence

$$\frac{ex(rt, \mathcal{G})}{\binom{rt}{r}} = d(K_r^{(r)}(t)) \to \frac{r!}{r^r},$$

as $t \to \infty$.

Theorem 3.5 (Erdős [6])

For $r \geq 2$ *and* $t \geq 3$

$$\gamma(K_r^{(r)}(t)) = 0.$$

For $l \geq r \geq 2$ we define the *limiting density* of the complete l-partite r-graph to be

$$d_{r,l} = \lim_{n \to \infty} d(K_l^{(r)}(n)) = \binom{l}{r} \frac{r!}{l^r}.$$

So for $r = 2$ we have

$$\gamma_2 = \{d_{2,1}, d_{2,2}, d_{2,3}, \dots\}.$$

Sarkar [24] noted that the method employed by Frankl and Rödl to prove Theorem 3.4 cannot be used directly to show that the value $d_{r,l}$ is not a jump for any $l \geq r \geq 3$. This is because their construction is based on the r-graph with l vertex classes each of order t and all possible edges except those contained entirely within a single class. Their proof relies on the fact that we may insert new edges into such an r-graph without affecting its Lagrangian too much.

The obvious r-graph to consider if we are to attempt to show that $d_{r,l}$ is not a jump using Frankl and Rödl's method is $K_l^{(r)}(t)$. However, adding just a single edge to $K_l^{(r)}(t)$ increases its Lagrangian by at least $\frac{1}{(lr)^r}$ and so this approach fails.

In the next section we will examine the role of Lagrangians in relation to Conjecture 3.3. Our main result, Theorem 3.7, adapts Frankl and Rödl's method to give an example of a limiting density which is not a jump: $d_{3,6}$. The proof relies on the use of a different hypergraph as the starting point for the construction. We also describe a 5-graph that can be used in a similar way to show that $d_{5,10}$ is not a jump for $r = 5$.

In the final section of this chapter we will briefly consider the question of whether $\frac{r!}{r^r}$ is a jump for all $r \geq 2$.

3.2 Limiting densities which are not jumps

We say that $\alpha \in [0, 1]$ is *threshold* for a family of r-graphs $\mathcal{F}$ if for every $\epsilon > 0$ there exists $n_0(\epsilon, \alpha, \mathcal{F})$ such that any r-graph G of order at least n_0 and with density $d(G) \geq \alpha + \epsilon$ contains some $F \in \mathcal{F}$ as a subhypergraph.

Theorem 3.6 (Frankl and Rödl [13]) *For $r \geq 2$ and $\alpha \in [0, 1]$ the following are equivalent:*

(i) α *is a jump for* r.

(ii) α *is threshold for some finite family* $\mathcal{F}$ *of* r*-graphs satisfying* $\lambda(F) > \frac{\alpha}{r!}$, *for all* $F \in \mathcal{F}$.

This result allows us to reduce the problem of deciding whether a given $\alpha \in [0,1]$ is a jump for some value of r to a problem concerning Lagrangians. Our next theorem gives a new value $d_{3,6}$ (that is the limiting density of $K_6^{(3)}(t)$) which is not a jump for $r = 3$. The proof follows Frankl and Rödl's method [13] but the original 3-graph we consider and the final calculation are new.

Theorem 3.7 $\binom{6}{3}\frac{3!}{6^3} = \frac{5}{9}$ *is not a jump for* $r = 3$.

Proof: Let V_1, V_2 and V_3 be disjoint sets of vertices each of order t. Write $G(t)$ for the 3-graph with vertex set $V = \bigcup_{i=1}^{3} V_i$ and edges consisting of all triples from V with either one vertex from each V_i or one vertex from V_i and two vertices from V_{i+1} for $i = 1, 2$ or 3, where $V_4 = V_1$ (see Figure 3.1). It is not difficult to see that the Lagrangian of $G(t)$ is achieved by taking each vertex to be equally weighted and so

$$\lambda(G(t)) = \left(t^3 + 3t\binom{t}{2}\right)\frac{1}{(3t)^3} = \frac{5}{54}\left(1 - \frac{3}{5t}\right).$$

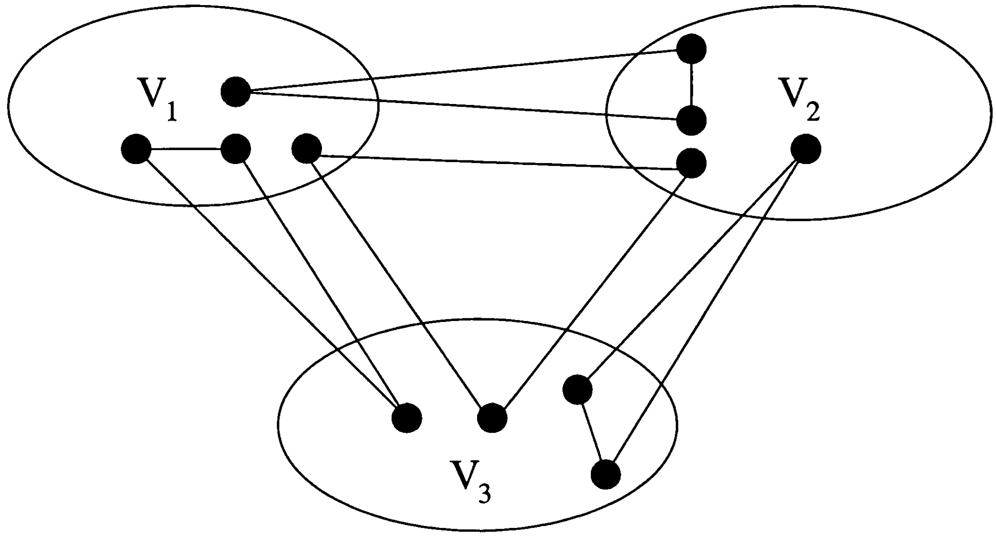

Figure 3.1: The 3-graph $G(t)$

Suppose, for a contradiction, that $\frac{5}{9}$ is a jump for $r = 3$. Then by Theorem 3.6 there exists a finite family $\mathcal{F}$ of 3-graphs such that $\frac{5}{9}$ is threshold for $\mathcal{F}$ and $\min_{F\in\mathcal{F}} \lambda(F) > \frac{5}{54}$. Let $k = \max\{|V(F)| : F \in \mathcal{F}\}$. We require the following lemma.

Lemma 3.8 (Frankl and Rödl [13]) *If k and c are fixed then there exists $t_0(k, c)$ such that for all $t \geq t_0$ there is a 3-graph H satisfying:*

(i) $|V(H)| = t$,

(ii) $|E(H)| = \lceil ct^2 \rceil$,

(iii) if $V_0 \subset V(H)$ *with* $3 \le |V_0| \le k$ *then* $|E(H) \cap [V_0]^{(3)}| \le |V_0| - 2$.

Proof of Lemma: (See [13] for details). Consider a random 3-graph H_0 of order t in which edges are inserted independently with probability $\frac{12c}{t}$. The expected number of edges in H_0 is at least $\frac{3t^2c}{2}$. Further, the expected number of subsets $V_0 \subset V(H_0)$ which do not satisfy condition (iii) is less than $\frac{t^2c}{2\binom{k}{3}}$, for t sufficiently large. Hence, the probability that H_0 contains a subhypergraph H with the required properties is strictly positive. □

In order to obtain a contradiction we need to show that for some $\epsilon > 0$ there exist $\mathcal{F}$-free 3-graphs with arbitrarily many vertices satisfying $d(G) > \frac{5}{9} + \epsilon$.

Let H be the 3-graph on $t \ge t_0(k, c)$ vertices given by the previous lemma, with k as defined above and $c = 1$. Define $G^*(t)$ to be the 3-graph formed from $G(t)$ by inserting the edges of H into each vertex class. Then let $G^*(t, n)$ be the *n-blow-up* of $G^*(t)$. That is the 3-graph formed from $G^*(t)$ by replacing each vertex v by a collection of n new vertices W_v and taking the edges in $G^*(t, n)$ to be $E(G^*(t, n)) = \{abc : a \in W_i, b \in W_j, c \in W_k, ijk \in E(G^*(t))\}$.

Lemma 3.9 *If* $G^*(t, n)$ *is defined as above then*

$$d(G^*(t,n)) \ge \frac{5}{9}\left(1 + \frac{3}{5t}\right).$$

Proof of Lemma: Counting edges in $G^*(t,n)$ we obtain

$$\begin{aligned} d(G^*(t,n)) &\geq \frac{t^3n^3 + 3tn^3\binom{t}{2} + 3t^2n^3}{\binom{3tn}{3}} \\ &\geq \frac{5}{9}\left(1 + \frac{3}{5t}\right). \end{aligned}$$

□

Since we are assuming that $\frac{5}{9}$ is threshold for $\mathcal{F}$, Lemma 3.9 implies that for n sufficently large $G^*(t,n)$ contains some $F \in \mathcal{F}$. So for large enough n, $G^*(t,n)$ should contain a subhypergraph P of order k with $\lambda(P) \geq \min_{F\in\mathcal{F}} \lambda(F) > \frac{5}{54}$. The next lemma will provide the desired contradiction.

Lemma 3.10 *Regardless of the value of n, any subhypergraph P of $G^*(t,n)$ of order k satisfies*

$$\lambda(P) \leq \frac{5}{54}.$$

Proof of Lemma: Let $P \subset G^*(t,n)$ be a subhypergraph of order k then Lemma 2.3(b) implies that when evaluating $\lambda(P)$ we may suppose that $P \subset G^*(t)$. For $i = 1,2,3$ let $P_i = V(P) \cap V_i$. Then, by adding vertices if needed, we may suppose that $|P_i| = k$ for each $i = 1,2,3$. Let $P[V_i]$ denote the subhypergraph of P induced by P_i. Note that each $P[V_i]$ is a subhypergraph of the 3-graph H given by Lemma 3.8. For fixed i let $\mathbf{x}$ be a legal weighting for $P[V_i]$ with $x_1 \geq x_2 \geq \cdots \geq x_k$. We will show that we may replace $P[V_i]$ by a new 3-graph P^*.

Let P^* be the 3-graph of order k with edges $E(P^*) = \{12j : j = 3, \ldots, k\}$. We would like to show that

$$w(P[V_i], \mathbf{x}) \leq w(P^*, \mathbf{x}).$$

Suppose $P[V_i]$ has s edges $e_1, \ldots, e_s$, listed in order of decreasing weight. Since $P[V_i]$ is a subhypergraph of H of order k Lemma 3.8(iii) implies that $s \leq k-2$. We claim that $|\bigcup_{j=1}^{l} e_j| \geq l+2$ for $l = 1, 2, \ldots, s$. If this is not true, say $|\bigcup_{j=1}^{l} e_j| \leq l+1$, then there exists $V_0 \subset P[V_i] \subset H$ with $|V_0| \leq l+1$ and $|E(H) \cap [V_0]^{(3)}| \geq l > |V_0| - 2$ contradicting property (iii) of H as given by Lemma 3.8. Hence there must exist a vertex $m \in \bigcup_{j=1}^{l} e_j$ with $m \geq l+2$ and so $w(e, \mathbf{x}) \leq x_1 x_2 x_{l+2}$ for some $e = e_j$ with $j \leq l$. Therefore $w(e_l, \mathbf{x}) \leq x_1 x_2 x_{l+2}$ and so $w(P[V_i], \mathbf{x}) = w\left(\bigcup_{j=1}^{s} e_j, \mathbf{x}\right) \leq \sum_{j=3}^{s+2} x_1 x_2 x_j \leq w(P^*, \mathbf{x})$. So we may suppose that for each $i = 1, 2, 3$ we have $P[V_i] = P^*$.

We must now calculate the Lagrangian of P directly. We will take the k vertices of P_i to be $\{v_1^i, \ldots, v_k^i\}$. Let $\mathbf{y}$ be an optimal legal weighting for P, so $w(P, \mathbf{y}) = \lambda(P)$. Lemma 2.3(a) tells us that for any two vertices v and w we have $w(E_v, \mathbf{y}) = w(E_w, \mathbf{y})$. So if vertex v_j^i has weight y_j^i then it is easy to see that we may take

$$y_1^i = y_2^i = a_i, \qquad y_3^i = \cdots = y_k^i = b_i,$$

for $i = 1, 2, 3$.

Since $w(E_v, \mathbf{y})$ is constant over all vertices of P then for any vertex v

$$\frac{w(E_v, \mathbf{y})}{3} = \sum_{w \in P} \frac{y_w w(E_w, \mathbf{y})}{3} = w(P, \mathbf{y}) = \lambda(P). \tag{3.1}$$

Now consider three vertices, u, v and w, one from each V_i receiving weights b_1, b_2 and b_3 respectively. We know by (3.1) that

$$\frac{w(E_u, \mathbf{y}) + w(E_v, \mathbf{y}) + w(E_w, \mathbf{y})}{9} = \lambda(P).$$

So in order to obtain $\lambda(P) \leq \frac{5}{54}$ it is sufficent to show that

$$w(E_u, \mathbf{y}) + w(E_v, \mathbf{y}) + w(E_w, \mathbf{y}) \leq \frac{5}{6}.$$

Let $w_i = \sum\limits_{v \in P_i} y_v$. So $w_1 + w_2 + w_3 = 1$. Then

$$\begin{aligned}
w(E_u, \mathbf{y}) &= a_1^2 + w_1 w_3 - b_1 w_3 + w_2 w_3 + \sum_{\{i,j\} \in P_2^{(2)}} y_i y_j \\
w(E_v, \mathbf{y}) &= a_2^2 + w_1 w_2 - b_2 w_1 + w_1 w_3 + \sum_{\{i,j\} \in P_3^{(2)}} y_i y_j \\
w(E_w, \mathbf{y}) &= a_3^2 + w_2 w_3 - b_3 w_2 + w_1 w_2 + \sum_{\{i,j\} \in P_1^{(2)}} y_i y_j.
\end{aligned}$$

Hence

$$w(E_u, \mathbf{y}) + w(E_v, \mathbf{y}) + w(E_w, \mathbf{y}) \leq \sum_{l=1}^{3} \left(a_l^2 + w_l(1 - w_l) + \sum_{\{i,j\} \in P_l^{(2)}} y_i y_j \right).$$

We now wish to show that for $l = 1, 2, 3$ we have

$$a_l^2 + \sum_{\{i,j\} \in P_l^{(2)}} y_i y_j \leq \frac{w_l^2}{2}. \tag{3.2}$$

Assuming this we obtain

$$\begin{aligned} w(E_u, \mathbf{y}) + w(E_v, \mathbf{y}) + w(E_w, \mathbf{y}) &\leq \sum_{l=1}^{3} \frac{w_l^2}{2} + w_l(1 - w_l) \\ &= 1 - \frac{(w_1^2 + w_2^2 + w_3^2)}{2} \\ &\leq \frac{5}{6}, \end{aligned}$$

as required.

So we simply need to show that (3.2) holds. We have $w_l = 2a_l + (k-2)b_l$ so $b_l = \frac{w_l - 2a_l}{k-2}$. Hence

$$\begin{aligned} a_l^2 + \sum_{\{i,j\} \in P_l^{(2)}} y_i y_j &= 2a_l^2 + 2(k-2)a_l b_l + \binom{k-2}{2} b_l^2 \\ &< 2a_l^2 + 2a_l(w_l - 2a_l) + \frac{(w_l - 2a_l)^2}{2} \\ &= \frac{w_l^2}{2}. \end{aligned}$$

This completes the proof of Lemma 3.10 and hence of Theorem 3.7. □

Having shown that one value $d_{r,l}$ is not a jump our next objective is to generalize this to other values of r and l. To date we can only give one other example which is not a jump: $d_{5,10}$. This can be proved in a similar way to Theorem 3.7. However we first need to find a suitable 5-graph on which to base our construction.

In general, if we wish to adapt Frankl and Rödl's method to show that some value α is not a jump for a given $r \geq 3$, then we need to find a sequence of r-graphs, $\{G_n\}_{n=1}^{\infty}$, with the following properties:

(i) G_n has order at least n,

(ii) $\lambda(G_n) = w\left(G_n, \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)\right) = \frac{|E(G_n)|}{n^r}$,

(iii) $\lambda(G_n)$ is not attained on any proper subhypergraph,

(iv) $\lambda(G_n) \uparrow \frac{\alpha}{r!}$ as $n \to \infty$.

In the case of $d_{5,10}$ the following 5-graphs have the required properties. Let G_n be the 5-graph of order $5n$ given by taking five disjoint vertex classes each of order n: V_1, V_2, V_3, V_4, V_5. Define the edges in G_n by specifying an ordered partition of five such as $(2, 1, 1, 1, 0)$ and taking as edges those subsets of size five formed by taking two vertices from V_i and one vertex from each of

V_{i+1}, V_{i+2} and V_{i+3} for $i = 1, 2 \ldots, 5$, where $V_j = V_{j-5}$ for $j \geq 6$. The ordered partitions we use are: $(1,1,1,1,1)$, $(2,1,1,1,0)$, $(2,0,1,1,1)$, $(3,1,1,0,0)$, $(3,0,0,1,1)$ and $(1,4,0,0,0)$. Figure 3.2 gives an idea of what this 5-graph looks like.

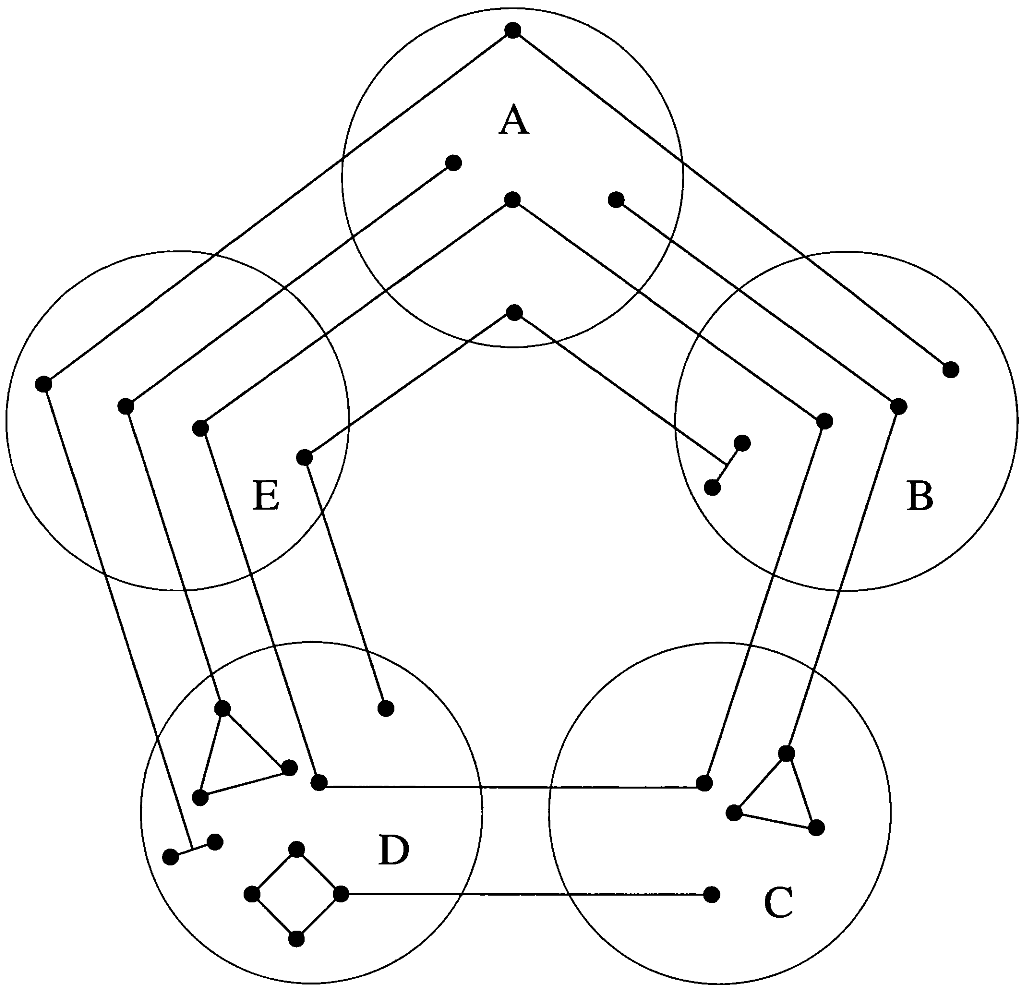

Figure 3.2: The 5-graph used to show that $d_{5,10}$ is not a jump for $r = 5$

We cannot give any other examples of limiting densities which are not jumps. This is because it becomes increasingly difficult to find suitable r-graphs on which to base our construction. It seems likely that for each $r \geq 3$ there exists some value $d_{r,l}$ which is not a jump for r. Indeed it would perhaps be more surprising to find a value $d_{r,l}$ with $l > r \geq 3$ which *is* a jump for r. The case of $l = r$ will be examined in the next section.

3.3 Is $\frac{r!}{r^r}$ a jump?

As we mentioned earlier Erdős was particularly interested in the question of whether or not $\frac{r!}{r^r}$ is a jump for every $r \geq 2$. We cannot at present resolve this problem but we have a few observations.

Firstly we note that Frankl and Rödl's method cannot be extended to show that $\frac{r!}{r^r}$ is not a jump. This is because we would require a sequence of r-graphs satisfying $\lambda(G_n) \uparrow \frac{1}{r^r}$. However, since $\lambda(K_r^{(r)}) = \frac{1}{r^r}$, this is impossible.

We also have a result due to Sarkar.

Theorem 3.11 (Sarkar [24]) *Suppose $\frac{r!}{r^r}$ is not a jump for some r. Then there is a sequence of r-graphs $G_1, G_2, \ldots$ of orders $t_1 < t_2 < \ldots$ such that*

$$\lambda(G_n) \downarrow \frac{1}{r^r}$$

and, for all n,

$$\lambda(G'_n) = \frac{1}{r^r}$$

for any proper subhypergraph G'_n of G_n.

This seems to provide evidence that $\frac{r!}{r^r}$ is indeed a jump for all r since the existence of such r-graphs would be surprising. For large n, G_n must in some sense be symmetric since $\lambda(G_n)$ is achieved only by giving each vertex a strictly positive weight. However, all their proper subhypergraphs achieve their Lagrangians on a single edge.

If $\frac{r!}{r^r}$ is a jump for some $r \geq 3$ then by Theorem 3.6 there exists a finite family of r-graphs, $\mathcal{F}^*$, for which $\frac{r!}{r^r}$ is threshold satisfying $\min\{\lambda(F) : F \in \mathcal{F}^*\} > \frac{1}{r^r}$. We have examined the 3-graph case of this problem in detail.

Looking only at small 3-graphs the following three appear to be obvious candidates to belong to the family $\mathcal{F}^*$:

$$G_1 = \{123, 124, 134\} \qquad \lambda(G_1) = \tfrac{4}{81} = 0.0493827,$$

$$G_2 = \{123, 124, 125, 345\} \qquad \lambda(G_2) = \tfrac{63+5\sqrt{5}}{1922} = 0.0385954,$$

$$G_3 = \{123, 124, 235, 145, 345\} \quad \lambda(G_3) = \tfrac{1}{25} = 0.04.$$

Indeed any 3-graph with at most twelve edges and Lagrangian larger than $\frac{1}{27}$ contains G_1, G_2 or G_3.

If $\frac{3!}{3^3}$ is a jump for $r = 3$ then it is possible that there are no 3-graphs whose Lagrangian lies in the interval $\left(\frac{1}{27}, \lambda(G_2)\right)$, although this is pure speculation based only on our rather limited knowledge of the values occurring as the Lagrangians of small 3-graphs.

However, given Theorem 3.11 and the fact that Frankl and Rödl's method for showing that values are not jumps cannot possibly be extended to work for the case $\frac{r!}{r^r}$, it seems quite plausible that $\frac{r!}{r^r}$ is a jump.

We will return to the family $\mathcal{F} = \{G_1, G_2, G_3\}$ when we consider the related Turán-type problem in the next chapter.

Chapter 4

Turán-type problems

4.1 Introduction

The new proof of Turán's theorem given by Motzkin and Straus, using a characterization of the Lagrangians of all 2-graphs (Theorem 1.1), was the original catalyst for the study of Lagrangians of hypergraphs. In this chapter we will briefly examine Turán's original conjectures and discuss the possible applications of Lagrangians to these problems. However our main result, given in the final section of this chapter, is a bound for a different Turán-type problem related to Erdős' jumping constant conjecture.

Recall that $K_t^{(r)}$ denotes the complete r-graph of order t,

$$\mathrm{ex}(n, K_t^{(r)}) = \max\{|E| : G = (V, E) \text{ is a } K_t^{(r)}\text{-free } r\text{-graph}, |V| = n\}$$

and

$$\gamma(K_t^{(r)}) = \lim_{n\to\infty} \frac{\mathrm{ex}(n, K_t^{(r)})}{\binom{n}{r}}.$$

Turán's problem

For $t > r \geq 3$ determine $ex(n, K_t^{(r)})$ and $\gamma(K_t^{(r)})$.

There are two old conjectures due to Turán for the 3-graph case of this problem [28], [29].

Conjecture 4.1 *Consider the 3-graph of order n given by dividing the vertices into three almost equal classes: V_1, V_2 and V_3. Take as edges in G all triples consisting either of one vertex from each V_i or one vertex from V_i and two vertices from V_{i+1} for $i = 1, 2, 3$, where $V_4 = V_1$ (see Figure 4.1). Then* $\mathrm{ex}(n, K_4^{(3)}) = |E|$ *and hence* $\gamma(K_4^{(3)}) = \frac{5}{9}$.

Conjecture 4.2 *Consider the 3-graph of order n given by dividing the vertices into two almost equal classes: V_1 and V_2. Take as edges in G all triples not contained entirely in one vertex class (see Figure 4.2). Then* $\mathrm{ex}(n, K_5^{(3)}) = |E|$ *and hence* $\gamma(K_5^{(3)}) = \frac{3}{4}$.

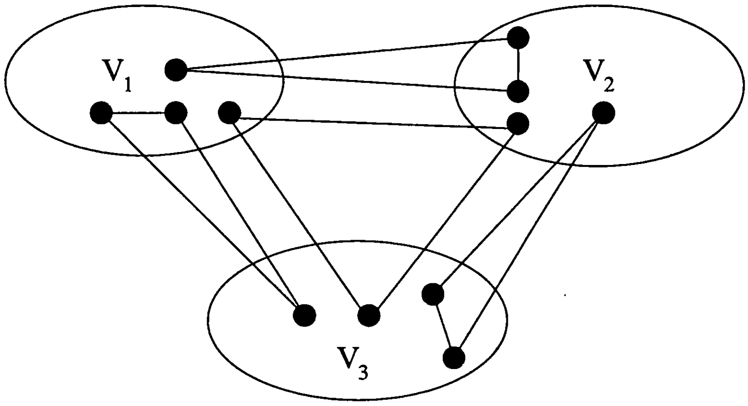

Figure 4.1: Turán's $K_4^{(3)}$-free example

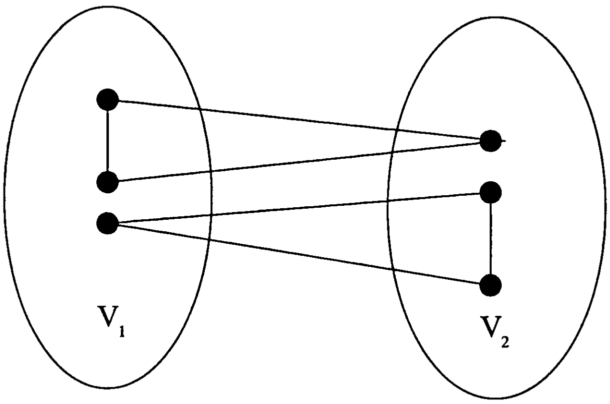

Figure 4.2: Turán's $K_5^{(3)}$-free example

Currently Conjecture 4.1 remains unsolved and no larger example of a $K_4^{(3)}$-free 3-graph has been found. However, if this conjecture is true the optimal $K_4^{(3)}$-free 3-graph is not unique. Kostochka [19] has described 2^{m-2} non-isomorphic 3-graphs of order $3m$ which are $K_4^{(3)}$-free and have the same number of edges as Turán's example.

The best known upper bound for $\mathrm{ex}(n, K_4^{(3)})$ is due to Chung and Lu [4]. They showed that

$$\frac{\mathrm{ex}(n, K_4^{(3)})}{\binom{n}{r}} \leq \frac{3+\sqrt{17}}{12}.$$

This implies that $\gamma(K_4^{(3)}) \leq \frac{3+\sqrt{17}}{12} = 0.593592\ldots$, while Turán's example gives $\gamma(K_4^{(3)}) \geq \frac{5}{9} = 0.5555\ldots$.

Turán's conjecture for $K_5^{(3)}$ (Conjecture 4.2) has been shown to be false by Kostochka and Sidorenko [20]. They give examples for all odd $n \geq 9$ of $K_5^{(3)}$-free 3-graphs with more edges than Turán's example. However, for $n = 2m+1$ they contain at most $\frac{m}{2}$ extra edges and so the second part of Conjecture 4.2, $\gamma(K_5^{(3)}) = \frac{3}{4}$, is still unresolved. The best known upper bound for $\gamma(K_5^{(3)})$ is due to de Caen [2]. He showed that $\gamma(K_5^{(3)}) \leq \frac{5}{6}$, while Turán's example gives $\gamma(K_5^{(3)}) \geq \frac{3}{4}$.

4.2 Lagrangians and Turán's problem

For a 2-graph G the value of $\lambda(G)$ is determined solely by the order of the largest clique in G. For general r-graphs, with $r \geq 3$, this is no longer the case. There are many examples of r-graphs for which $K_t^{(r)} \not\subseteq G$ and yet $\lambda(G) > \lambda(K_t^{(r)})$. For instance the example given in Conjecture 4.1 satisfies $K_4^{(3)} \not\subseteq G$ and $\lambda(G) > \lambda(K_5^{(3)})$. The simple reason for this is that the Lagrangian of a general r-graph need not be attained on a small subhypergraph. Indeed for many r-graphs $\lambda(G)$ is only attained on the entire vertex set.

One observation we have concerning Conjecture 4.1 and Lagrangians is that the asymptotic version of this conjecture is equivalent to the statement that every $K_4^{(3)}$-free 3-graph G satisfies $\lambda(G) \leq \lambda(K_6^{(3)})$. It is easy to show that this is true for Turán's $K_4^{(3)}$-free example directly (that is not simply by counting edges). However, we have been unsuccessful in finding a direct proof of this fact for Kostochka's examples of $K_4^{(3)}$-free 3-graphs.

Although Lagrangians have not provided any real insights into Turán's problem for hypergraphs they have proved very useful when considering other Turán-type problems. For example consider the following problem due to Katona.

Problem

Determine the maximal number of edges in an r-graph of order n with the property that the symmetric difference of any two edges is not contained in a *third.*

This problem was first answered for 3-graphs by Bollobás [1]. He showed that the unique extremal 3-graph of order n with this property is the tripartite 3-graph whose three vertex classes are as equal as possible in size and whose edges are all triples meeting each vertex class exactly once.

Later Sidorenko [26] showed how useful Lagrangians could be in this context. He provided a new proof of the asymptotic version of Bollobás' result, and gave a similar result for 4-graphs. His proof uses the fact that if G is a 3-graph or 4-graph satisfying the conditions of Katona's problem then the Lagrangian of G is given by weighting a single edge. Hence

$$\frac{|E|}{n^r} \leq \lambda(G) = \frac{1}{r^r},$$

for such an r-graph (where $r = 3$ or 4).

4.3 A bound for a Turán-type problem

While considering the question of whether $\frac{r!}{r^r}$ is a jump for $r = 3$ in the previous chapter we saw that the family of 3-graphs $\mathcal{F} = \{G_1, G_2, G_3\}$ was of particular interest, where

$$G_1 = \{123, 124, 134\}, \quad G_2 = \{123, 124, 125, 345\}$$

and

$$G_3 = \{123, 124, 235, 145, 345\}.$$

This was because we have no examples of 3-graphs whose Lagrangian is strictly larger than $\frac{1}{27}$ not containing a copy of some member of $\mathcal{F}$. For this reason it is interesting to consider the related Turán-type problem. That is, how dense may an $\mathcal{F}$-free 3-graph be?

We would really like to show that there is a finite family $\mathcal{F}^* \subset \{G : \lambda(G) > \frac{1}{27}\}$ satisfying $\gamma(\mathcal{F}^*) = \frac{2}{9}$. This would imply (by Theorem 3.6) that $\frac{2}{9}$ is a jump for $r = 3$. However, for any such family $\mathcal{F}^*$ we may suppose that $\mathcal{F} \subseteq \mathcal{F}^*$ and so any upper bound for $\gamma(\mathcal{F})$ is equally an upper bound for $\gamma(\mathcal{F}^*)$.

Before considering this problem we note the following bounds for $\gamma(G_1)$ due to Frankl and Füredi [11] and de Caen [2] respectively

$$\frac{2}{7} \leq \gamma(G_1) \leq \frac{1}{3}.$$

Since $G_1 \in \mathcal{F}$ and $\gamma(G_1) \leq \frac{1}{3}$ we would like, at the very least, to show that $\gamma(\mathcal{F}) < \frac{1}{3}$.

We will make use of the following averaging lemma due to Katona, Nemetz and Simonovits [17].

Lemma 4.3 (Katona, Nemetz and Simonovits [17])

If $\mathcal{G}$ is a family of r-graphs then for any integer $n \geq r$

$$\frac{ex(n+1,\mathcal{G})}{\binom{n+1}{r}} \leq \frac{ex(n,\mathcal{G})}{\binom{n}{r}}.$$

Our main result is the following.

Theorem 4.4 *If $\mathcal{F}$ is the family of 3-graphs given above then*

$$\gamma(\mathcal{F}) \leq 0.3103\cdots < \frac{1}{3},$$

where $0.3103\ldots$ is the real root of $5x(1+3x^2)-2=0$.

The proof of this result involves an averaging argument over the subgraphs of order six of an $\mathcal{F}$-free 3-graph. We first need to examine $\mathcal{F}$-free 3-graphs of small order and we have a series of lemmas.

Lemma 4.5 *There are three maximal $\mathcal{F}$-free 3-graphs of order 5, $H_1 = \{123, 124, 125\}$, $H_2 = \{123, 134, 125, 145\}$, and $H_3 = \{123, 234, 125, 145\}$.*

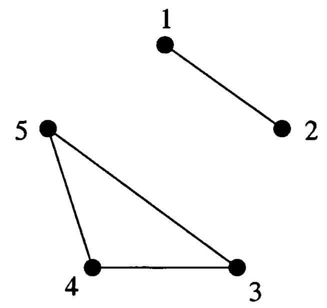

Figure 4.3: The 2-graph representation of $G_2 = \{123, 124, 125, 345\}$

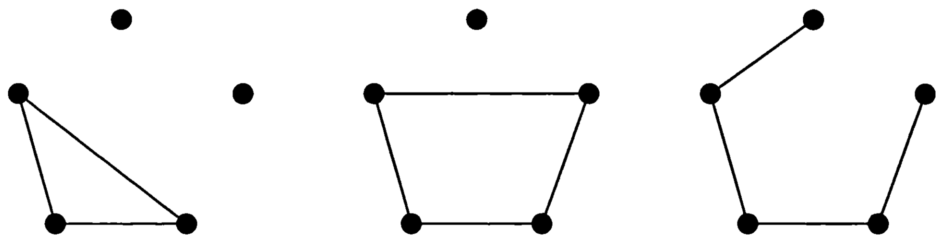

Figure 4.4: The three maximal $\mathcal{F}$-free 3-graphs of order 5.

Proof: We note that any 3-graph of order 5 may be represented by a 2-graph of order 5 where edges are given by complements. (See Figure 4.3 for such a representation of G_2.) So if H is a maximal $\mathcal{F}$-free 3-graph of order 5 and $v \in V(H)$ we can define $d(v)$ to be the degree of the corresponding vertex in the 2-graph representing H.

Since $G_1 \not\subseteq H$ we have $d(v) \leq 2$ for every $v \in V(H)$. Hence H contains at most 5 edges with equality iff H is a pentagon (as a 2-graph). But this is G_3. Hence H contains at most 4 edges. Further, since H is G_2-free, if it contains a triangle then it contains no other edges. Hence H is one of the three 3-graphs H_1, H_2 or H_3 (see Figure 4.4). $\square$

For an r-graph $G = (V, E)$ we say that $U \subseteq V$ is an *independent* set if $E[U] = \emptyset$. Our next lemma is interesting in its own right since it implies that if H is an $\mathcal{F}$-free 3-graph then it contains a large independent set of vertices.

Lemma 4.6 *Let $t \geq 5$ be an integer. If H is an $\mathcal{F}$-free 3-graph and $\{123,$ $124, 125, \ldots, 12t\} \subset E(H)$ then*

$$E(H) \cap [t]^{(3)} = \{123, 124, 125, \ldots, 12t\}$$

and hence

$$E[\{3, \ldots, t\}] = \emptyset.$$

Proof: Suppose there is an edge $1ij \in E$ with $i, j \in \{3, 4, \ldots, t\}$ then $\{1ij, 12i, 12j\} = G_1 \subset H$. Similarly there can be no edge of the form $2ij$.

Also if there is an edge $ijk \in E$ with $i, j, k \in \{3, 4, \ldots, t\}$ then we have $\{12i, 12j, 12k, ijk\} = G_2 \subset H$. $\square$

Corollary 4.7 *If H is an $\mathcal{F}$-free 3-graph with density γ then there exists an independent set of vertices U of order $\lceil \gamma(n-2) \rceil$.*

Proof: The identity

$$3|E| = \sum_{\{i,j\} \in V^{(2)}} |E_{ij}|,$$

implies that there exist vertices $i, j \in V$ with $|E_{ij}| \geq \frac{3|E|}{\binom{n}{2}} = \gamma(n-2)$. By the previous lemma E_{ij} is an independent set in H. □

Our next two lemmas are slightly more technical. Lemma 4.8 provides useful information concerning the number of edges in $\mathcal{F}$-free 3-graphs of order 6, while Lemma 4.9 enables us to count special subhypergraphs of order 6 in a general $\mathcal{F}$-free 3-graph.

Lemma 4.8 *If $H = (V, E)$ is an $\mathcal{F}$-free 3-graph of order 6 and G_0 is the 3-graph $\{123, 124, 125\}$ then*

$$|E| \leq 8 - |\{G_0 : G_0 \subset H\}|.$$

Proof: Note that by Lemma 4.5 we have $ex(5, \mathcal{F}) = 4$ so Lemma 4.3 implies that

$$\frac{ex(6, \mathcal{F})}{\binom{6}{3}} \leq \frac{4}{\binom{5}{3}} = \frac{2}{5}.$$

Hence if $H = (V, E)$ is an $\mathcal{F}$-free 3-graph of order 6 then $|E| \leq 8$.

If there exist vertices i and j for which $|E_{ij}| = 4$ then $\{123, 124, 125, 126\} \subseteq H$. So, by Lemma 4.6, H contains no other edges and hence $|E(H)| = 4 = 8 - |\{G_0 : G_0 \subset H\}|$, as required.

For the remainder of this proof we may suppose that $|E| \leq 8$ and $|E_{ij}| \leq 3$ for all $\{i, j\} \in V^{(2)}$. We may also assume that there is at least one copy of G_0 contained in H (otherwise $|E| \leq 8$ implies the result).

If there is exactly one copy of G_0 contained in H we may suppose (by relabelling if necessary) that it is $\{123, 124, 125\}$. Lemma 4.6 implies that $E[\{1, 2, 3, 4, 5\}] = \{123, 124, 125\}$. Also, since $ex(5, \mathcal{F}) = 4$, any other 5-set in H contains at most four edges so summing over all 5-sets in H we obtain

$$3|E| = \sum_{B \in V^{(5)}} |E[B]| \leq 3 + 5 \cdot 4 = 23.$$

Hence $|E| \leq 7$. So the result holds when $|\{G_0 : G_0 \subset H\}| = 1$.

Now suppose there are at least two copies of G_0 contained in H. Without loss of generality one of them is $\{123, 124, 125\}$. If the other copy of G_0 is $\{abc, abd, abe\}$ then by Lemma 4.6 and symmetry we may take $b = 6$ and either $a = 1$ or $a = 3$. Using the fact that $|E_{ij}| \leq 3$ for all $\{i, j\} \in V^{(2)}$ and remembering that H is $\mathcal{F}$-free we have two cases to consider.

Either H contains a copy of $\{123, 124, 125, 136, 146, 156\}$ or H contains a copy of $\{123, 124, 125, 136, 346, 356\}$. It is easy to check that these are both maximal $\mathcal{F}$-free 3-graphs. (In both cases any new edge we try to insert must contain 2 and 6, but adding any such edge would mean that H is no longer $\mathcal{F}$-free.) Hence we see that H cannot contain exactly three copies of G_0 and if it contains exactly two copies of G_0 then $|E| \leq 6 = 8 - |\{G_0 : G_0 \subset H\}|$ as required. $\square$

We require one final lemma before proving Theorem 4.4.

Lemma 4.9 *If $G = (V, E)$ is a 3-graph of order n and size m then*

$$4 \sum_{\{i,j\} \in V^{(2)}} \binom{|E_{ij}|}{4} + \sum_{\{i,j\} \in V^{(2)}} \binom{|E_{ij}|}{3} |E_{ij}^c| \geq \frac{9(n-5)m^3}{2\binom{n}{2}^2} + O(n^5), \qquad (4.1)$$

where $E_{ij} = \{k : ijk \in E\}$ and $E_{ij}^c = \{k : ijk \in [n]^{(3)} \backslash E\}$.

Proof: Let us denote the left hand side of (4.1) by S and $|E_{ij}|$ and $|E_{ij}^c|$ by e_{ij} and $\overline{e}_{ij}$ respectively. Then $\overline{e}_{ij} = n - 2 - e_{ij}$ so

$$\begin{aligned} 6S &= \sum_{\{i,j\} \in V^{(2)}} e_{ij}(e_{ij}-1)(e_{ij}-2)(e_{ij}-3+\overline{e}_{ij}). \\ &= \sum_{\{i,j\} \in V^{(2)}} e_{ij}(e_{ij}-1)(e_{ij}-2)(n-5). \end{aligned}$$

Now $e_{ij} \leq n - 2$ so

$$\sum_{\{i,j\} \in V^{(2)}} e_{ij}^2 \leq \binom{n}{2} n^2 = O(n^4).$$

Hence

$$\frac{6S}{n-5} = \sum_{\{i,j\}\in V^{(2)}} e_{ij}^3 + O(n^4). \tag{4.2}$$

We now require Hölder's inequality which says that for $p, q > 0$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$ and sequences a_t, b_t of non-negative real numbers

$$\sum_{t=1}^{k} a_t b_t \leq \left(\sum_{t=1}^{k} a_t^p\right)^{\frac{1}{p}} \left(\sum_{t=1}^{k} b_t^q\right)^{\frac{1}{q}}.$$

Applying this with $p = 3$, $q = \frac{3}{2}$, $k = \binom{n}{2}$, $a_t = e_{ij}$ and $b_t = \frac{1}{\binom{n}{2}^{\frac{2}{3}}}$ we have $\sum_{t=1}^{k} b_t^q = 1$ so

$$\begin{aligned} \sum_{\{i,j\}\in V^{(2)}} e_{ij}^3 &\geq \frac{\left(\sum_{\{i,j\}\in V^{(2)}} e_{ij}\right)^3}{\binom{n}{2}^2} \\ &= \frac{27m^3}{\binom{n}{2}^2}. \end{aligned} \tag{4.3}$$

Combining (4.2) and (4.3) we obtain

$$S \geq \frac{9(n-5)m^3}{2\binom{n}{2}^2} + O(n^5),$$

which is (4.1) as required. □

We are now ready to prove Theorem 4.4. Let $G = (V, E)$ be an $\mathcal{F}$-free 3-graph of order n. We will give an upper bound for $|E| = m$ by carefully counting the number of edges present in subsets of V of order 6 in two different ways. Firstly we have the simple identity

$$\sum_{A \in V^{(6)}} |E[A]| = m\binom{n-3}{3}. \tag{4.4}$$

We now want to count the number of edges in those induced subhypergraphs of order 6 containing a copy of $G_0 = \{123, 124, 125\}$. We claim that the following inequality holds

$$4\sum_{\{i,j\}\in V^{(2)}} \binom{|E_{ij}|}{4} + \sum_{\{i,j\}\in V^{(2)}} \binom{|E_{ij}|}{3} |E_{ij}^c| \leq \sum_{A \in V^{(6)}, G_0 \subseteq E[A]} (8 - |E[A]|). \tag{4.5}$$

To see this consider $A \in V^{(6)}$ with $G_0 \subseteq E[A]$. We saw in the proof of Lemma 4.8 that A can contain either $1, 2$ or 4 copies of G_0. Lemma 4.8 also implies that in each case $8 - |E[A]| \geq |\{G_0 : G_0 \subseteq E[A]\}|$. So we need to check that the left hand side of (4.5) counts those $A \in V^{(6)}$ containing $1, 2$ or 4 copies of G_0 at most once, twice or four times respectively. We must also check that any $A \in V^{(6)}$ that is G_0-free is never counted in the left hand side of (4.5).

Suppose $A \in V^{(6)}$ contains exactly one copy of G_0. Then $|E_{ab} \cap A| = 3$ for some $\{a, b\} \in A^{(2)}$ and $|E_{ij} \cap A| \leq 2$ for all other choices of $\{i, j\}$ from $A^{(2)}$. So the first term in (4.5) does not count A and the second term counts it exactly once (when $\{i, j\} = \{a, b\}$).

Now suppose $A \in V^{(6)}$ contains exactly two copies of G_0. Then $|E_{ij} \cap A| = 3$ for two choices of $\{i,j\} \in A^{(2)}$ and for all other choices of $\{i,j\} \in A^{(2)}$ we have $|E_{ij} \cap A| \leq 2$. So the first term in (4.5) does not count A and the second term counts it exactly twice.

If $A \in V^{(6)}$ contains four copies of G_0 then from the proof of Lemma 4.8 we know that $E[A] = \{123, 124, 125, 126\}$ and so A is counted four times by the first term in (4.5) and never by the second term.

Finally, if $A \in V^{(6)}$ does not contain a copy of G_0 then $|E_{ij} \cap A| \leq 2$ for all $\{i,j\} \in A^{(2)}$ and so A is not counted in the left hand side of (4.5). Hence inequality (4.5) holds.

Since $ex(6, \mathcal{F}) = 8$ we can combine (4.4) and (4.5) to obtain

$$\begin{aligned} m\binom{n-3}{3} &\leq 8\left(\binom{n}{6} - |\{A \in V^{(6)} : G_0 \subseteq E[A]\}|\right) + \sum_{A \in V^{(6)}, G_0 \subseteq E[A]} |E[A]| \\ &\leq 8\binom{n}{6} - \left(4\sum_{\{i,j\} \in V^{(2)}} \binom{|E_{ij}|}{4} + \sum_{\{i,j\} \in V^{(2)}} \binom{|E_{ij}|}{3}|E_{ij}^c|\right). \end{aligned}$$

Applying Lemma 4.9 we have

$$m\binom{n-3}{3} \leq 8\binom{n}{6} - \frac{9(n-5)m^3}{2\binom{n}{2}^2} + O(n^5).$$

Dividing by $\frac{n^6}{36}$ we obtain

$$d(G) \leq \frac{2}{5} - 3(d(G))^3 + O(\frac{1}{n}).$$

Then letting n tend to infinity we see that $\gamma(\mathcal{F})$ satisfies

$$\gamma(\mathcal{F})(1+3(\gamma(\mathcal{F}))^2) \leq \frac{2}{5}.$$

This completes the proof of Theorem 4.4. $\square$

Chapter 5

Intersecting Families of Separated Sets

5.1 Introduction

We say that a family of sets is *intersecting* if the intersection of any two sets from the family is non-empty. How large can an intersecting family of sets from $[n]^{(r)}$ be? If $n < 2r$ this is easy to answer since $[n]^{(r)}$ is intersecting. However, for $n \geq 2r$ this question is more difficult. It was answered by Erdős, Ko and Rado in 1961 [8].

Theorem 5.1 (Erdős, Ko and Rado [8])

Let $n \geq 2r$, $x \in [n]$ and $\mathcal{A} \subset [n]^{(r)}$ be intersecting. Then $|\mathcal{A}| \leq |\mathcal{A}_x|$, where $\mathcal{A}_x = \{A \in [n]^{(r)} : x \in A\}$. If $n > 2r$ then $|\mathcal{A}| = |\mathcal{A}_x|$ iff $\mathcal{A} \simeq \mathcal{A}_x$.

For completeness, and for later reference, we will give two elegant proofs of this theorem. The first is due to Katona [16] and the second is due to Daykin [5].

Proof 1 of Theorem 5.1 (Katona [16])

Let $\mathcal{A}$ be an intersecting family in $[n]^{(r)}$. Consider the bipartite graph G whose two vertex classes are the collection $\mathcal{C}$ of all cyclic orderings of $[n]$ and $[n]^{(r)}$. For an r-set B and a cyclic ordering C we say that B is an *interval* in C if B appears as r consecutive elements in C. The edges of G join an r-set B to any cyclic ordering C containing B as an interval. Every r-set is adjacent to $r!(n-r)!$ cyclic orderings. So the number of edges from $\mathcal{A}$ to $\mathcal{C}$ is $|\mathcal{A}|r!(n-r)!$. Given a particular cyclic ordering C we know that C contains at most r sets from $\mathcal{A}$ as intervals since if it contained more then at least two would be disjoint. Hence the number of edges from $\mathcal{C}$ to $\mathcal{A}$ is at most $r|\mathcal{C}|$. Then, as the total number of cyclic orderings of $[n]$ is $(n-1)!$ we have

$$|\mathcal{A}| \leq \frac{(n-1)!r}{r!(n-r)!} = \frac{r}{n}\binom{n}{r} = |\mathcal{A}_x|.$$

The case of equality is easy to deduce. □

Given a family of r-sets $\mathcal{A} \subseteq [n]^{(r)}$ the *lower shadow* of $\mathcal{A}$ is defined to be

$$\partial(\mathcal{A}) = \{B \in [n]^{(r-1)} : B \subset A \text{ for some } A \in \mathcal{A}\}.$$

Recall the definition of the colex ordering of $\mathbf{N}^{(r)}$ from page 16. For the second proof of Theorem 5.1 we will require the following result.

Theorem 5.2 (Kruskal and Katona [21])

Let $r \geq 1$ and $\mathcal{A} \subseteq [n]^{(r)}$. Then the lower shadow of $\mathcal{A}$ is at least as large as the lower shadow of the first $|\mathcal{A}|$ r-sets in the colex order. If $|\mathcal{A}| = \binom{m}{r}$, for some $m \geq r$, then equality holds iff $\mathcal{A} \simeq [m]^{(r)}$.

Proof 2 of Theorem 5.1 (Daykin [5])

First suppose that $n = 2r$. Then, as the complement of an r-set is also an r-set, an intersecting family $\mathcal{A}$ may contain at most a half of the sets from $[n]^{(r)}$. This is the required result for $n = 2r$.

For the remainder of this proof we will assume that $n > 2r$.

Let ∂^t denote the operation of taking the lower shadow t times. Given an intersecting family of r-sets, $\mathcal{A}$, let $\mathcal{B} = \{[n] \backslash A : A \in \mathcal{A}\}$. If $A \in \mathcal{A}$ and $B \in \mathcal{B}$ then $A \not\subset B$. So $(\partial^{n-2r}\mathcal{B}) \cap \mathcal{A} = \emptyset$.

As $\mathcal{A}_x$ is a maximal intersecting family we may suppose that $|\mathcal{A}| = |\mathcal{A}_x|$ then it is sufficient to prove that $\mathcal{A} \simeq \mathcal{A}_x$. So suppose $|\mathcal{A}| = |\mathcal{B}| = \binom{n-1}{r-1}$. Applying Theorem 5.2 we have $|\partial\mathcal{B}| \geq \binom{n-1}{n-r-1}, |\partial^2\mathcal{B}| \geq \binom{n-1}{n-r-2}, \ldots, |\partial^{n-2r}\mathcal{B}| \geq \binom{n-1}{r}$. Then $\mathcal{A} \cup \partial^{n-2r}\mathcal{B} \subseteq [n]^{(r)}$ and, as noted above, this is a union of disjoint families of sets. Hence we have $|\partial^{n-2r}\mathcal{B}| = \binom{n-1}{r}$. Theorem 5.2 then implies that $\mathcal{B} \simeq [n-1]^{(n-r)}$ and so $\mathcal{A} \simeq \mathcal{A}_x$. □

Many questions concerning families of sets from $[n]^{(r)}$ can be framed in the language of graphs. Consider the graph the *Kneser graph*, $K_{n,r}$, with vertex set $[n]^{(r)}$ and edges between any two vertices corresponding to disjoint r-sets. The Erdős-Ko-Rado theorem (Theorem 5.1) can be restated as: the largest independent set of vertices in $K_{n,r}$ has order $\binom{n-1}{r-1}$.

One of the most fundamental properties of a graph is its chromatic number. For the Kneser graph it is clear that this must be at least $\lceil \frac{n}{r} \rceil$, since any monochromatic set of vertices corresponds to an intersecting family of sets and so by Theorem 5.1 has order at most $\binom{n-1}{r-1}$. In the other direction, the Kneser graph cannot have chromatic number larger than $n - 2r + 2$ since we may colour it with $n - 2r + 2$ colours as follows. For $i \in [n]$ let $\mathcal{S}_i$ denote those sets in $[n]^{(r)}$ whose smallest element is i. Colour those sets in $\mathcal{S}_i$ with colour i for $i = 1, \ldots, n - 2r + 1$. Then the remaining sets which have not been coloured form an intersecting family (since they are a collection of r-sets from a set of size $2r-1$). Hence they form an independent set in the Kneser graph and so may all be coloured with colour $n-2r+2$.

A longstanding conjecture due to Kneser [18] was that the chromatic number of $K_{n,r}$ is $n-2r+2$. This was answered in the affirmative by Lovász in 1977 [22]. Later Schrijver [25] identified a vertex-critical subgraph of $K_{n,r}$, in other words a minimal subgraph of $K_{n,r}$ with chromatic number $n-2r+2$. In order to describe this subgraph we require the following definition. We say that a set $A \in [n]^{(r)}$ is *separated* if, when considered as a subset of $[n]$ arranged around a circle in the usual ordering, A does not contain any two adjacent points. Schrijver's vertex-critical subgraph of the Kneser graph is the subgraph induced by those vertices corresponding to the collection of all separated sets in $[n]^{(r)}$.

Let us denote the collection of all separated sets in $[n]^{(r)}$ by $[n]^{(r)}_*$. Then the corresponding subgraph of the Kneser graph has (by Schrijver's result) chromatic number $n-2r+2$. However, the independence number of this subgraph is not known. This is a rather strange situation in that generally determining the independence number of a graph is "easier" than determining its chromatic number. For the remainder of this chapter we will consider a well-known conjecture of Holroyd and Johnson on this problem, namely that an analogue of the Erdős-Ko-Rado theorem holds for intersecting families of separated sets. In fact their conjecture is more general. They define the collection of *k-separated* r-sets in $[n]^{(r)}$ to be those r-sets $A = \{a_1, \ldots, a_r\}$ satisfying $a_{i+1} - a_i > k$ for $i = 1, \ldots, r$, where $a_{r+1} = a_1 + n$. We will denote this family by $[n]_k^{(r)}$. Note that a 1-separated set is simply a separated set.

Conjecture 5.3 (Holroyd and Johnson [15])

Let n, k and r be positive integers satisfying $n \geq (k+1)r$. Suppose $x \in [n]$ and $\mathcal{A} \subset [n]_k^{(r)}$ is intersecting. Then $|\mathcal{A}| \leq |\mathcal{A}_x^*|$, where $\mathcal{A}_x^* = \{A \in [n]_k^{(r)} : x \in A\}$.

Our main results are Theorems 5.5, 5.6 and 5.7. Theorem 5.5 shows that when $n = 2r + 2$ and $k = 1$ (the first non-trivial case) Conjecture 5.3 is true. Theorem 5.6 gives a version of the Erdős-Ko-Rado theorem for weighted k-separated sets. Finally, Theorem 5.7 gives an extension of Theorem 5.6 along the same lines as the Hilton-Milner theorem (Theorem 5.4). These last two results allow us to give non-trivial bounds on the size of intersecting families of k-separated sets.

Except in the final section of this chapter we will only consider separated sets, that is the case $k = 1$ of Conjecture 5.3.

Let us first note, following Holroyd and Johnson [15], that Conjecture 5.3 is true when n is large compared to r and k. This follows trivially from a result of Hilton and Milner [14] on intersecting families of r-sets which are not fixed by single element (that is families $\mathcal{A}$ for which there does not exist $x \in [n]$ such that every $A \in \mathcal{A}$ contains x).

Theorem 5.4 (Hilton and Milner [14]) *If $\mathcal{A}$ is an intersecting family of r-sets from $[n]^{(r)}$ not fixed by a single element then $|\mathcal{A}| \leq |\hat{\mathcal{A}}|$ where $\hat{\mathcal{A}}$ is formed by taking all r-sets containing both the element $r+1$ and at least one of the elements $1, 2, \ldots, r$ together with the set $\{1, 2, \ldots, r\}$.*

We note that the family $\hat{\mathcal{A}}$ has size $\binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1$. A simple calculation then implies that Conjecture 5.3 is true for $n \geq 2kr^2$.

5.2 Some negative results

In this section we will explore the first "obvious" ideas for a proof of Conjecture 5.3. We will consider ways of adapting the two proofs of the Erdős-Ko-Rado theorem given in the previous section and show that there are simple reasons why neither proof is easily modified to give a proof of this new problem.

5.2.1 Cyclic orderings

Recall the first proof of Theorem 5.1. When considering separated r-sets we need a corresponding idea of a "separated" cyclic ordering. We will call a cyclic ordering of $[n]$ an *r-legal cyclic ordering* if every interval of length r forms a separated r-set. So for example 1352746 is a 2-legal cyclic ordering of [7] but it is not a 3-legal cyclic ordering of [7] since 352 is not a separated 3-set.

If all separated r-sets were contained as intervals in the same number of r-legal cyclic orderings of $[n]$ then we could adapt the first proof of Theorem 5.1 as follows.

Suppose each separated r-set belonged to exactly α r-legal cyclic orderings. Let $\mathcal{C}_*$ denote the collection of all r-legal cyclic orderings of $[n]$. Then given an intersecting family of separated r-sets, $\mathcal{A}$, we could form a bipartite graph with vertex classes $\mathcal{C}_*$ and $[n]^{(r)}_*$, and edges joining an r-set to any cyclic ordering in which it appeared as an interval. As before, since $\mathcal{A}$ is intersecting, any $C \in \mathcal{C}_*$ would be adjacent to at most r sets in $\mathcal{A}$. Hence by counting the number of edges between $\mathcal{A}$ and $\mathcal{C}_*$ we would obtain $\alpha|\mathcal{A}| \leq r|\mathcal{C}_*|$. Counting the total number of edges in this graph we would also have $\alpha|[n]^{(r)}_*| = n|\mathcal{C}_*|$

and hence

$$|\mathcal{A}| \leq \frac{r|\mathcal{C}_*|}{\alpha} = \frac{r|[n]_*^{(r)}|}{n} = |\mathcal{A}_x^*|.$$

However, the problem with this argument is that it is simply not true in general that every separated r-set is contained as an interval in the same number of r-legal cyclic orderings. For example if we consider $[7]_*^{(2)}$ the 2-set 13 belongs to fewer 2-legal cyclic orderings of [7] than the 2-set 14. Hence the most simple attempt to modify the first proof of Theorem 5.1 fails.

We have considered trying to overcome this problem by giving weights to the cyclic orderings but have found no consistent way of doing this.

5.2.2 Ordering of separated sets

Given an r-set A and a family of r-sets $\mathcal{B}$ we say that A is *disjoint* from $\mathcal{B}$ if there exists $B \in \mathcal{B}$ such that $A \cap B = \emptyset$.

For $\mathcal{A} \subseteq [n]^{(r)}$ define

$$D(\mathcal{A}) = |\{B \in [n]^{(r)} : B \text{ is disjoint from } \mathcal{A}\}|.$$

The second proof of Theorem 5.1 used a special case of the Kruskal-Katona theorem. Essentially it made use of the following result.

If $\mathcal{A} \subset [n]^{(r)}$ then $D(\mathcal{A}) \geq D(\mathcal{C})$, where $\mathcal{C}$ is the family of r-sets from $[n]^{(r)}$ of size $|\mathcal{A}|$ formed by taking the complements of the first $|\mathcal{A}|$ sets in the colex ordering of $[n]^{(n-r)}$.

In order to adapt this argument to give a proof of Conjecture 5.3 we would require an ordering of $[n]_*^{(r)}$ with the property that any collection of separated r-sets of size m should be disjoint from at least as many other separated r-sets as the first m elements in our ordering of $[n]_*^{(r)}$.

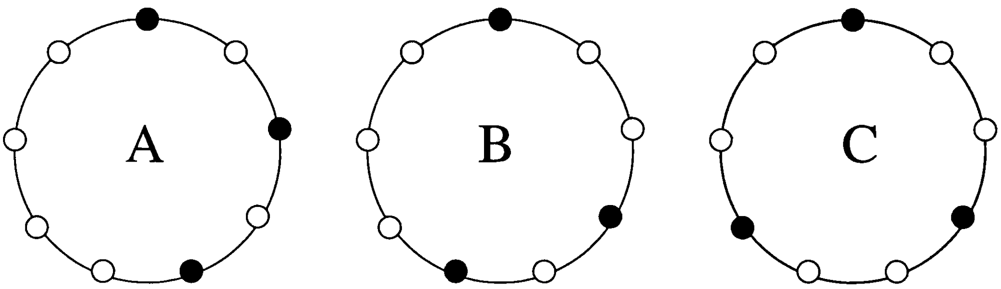

Figure 5.1: The three types of set in $[9]_*^{(3)}$.

Unfortunately such an ordering does not (in general) exist. We give the following example. In $[9]_*^{(3)}$ there are three basic types of sets. We will call these types A, B and C. Typical sets of these three types are $135, 146$ and 147 respectively (see Figure 5.1). A set of type A is disjoint from 10 other sets in $[9]_*^{(3)}$, a set of type B is disjoint from 9 other sets in $[9]_*^{(3)}$ and a set of type C is disjoint from 8 other sets in $[9]_*^{(3)}$. Hence, if an ordering as described above existed for $[9]_*^{(3)}$ its first set would be of type C. Now consider which pair of sets in $[9]_*^{(3)}$ are disjoint from the smallest number of other sets in $[9]_*^{(3)}$. It is easy to see that if we take the two sets of type B, 137 and 157, then they are disjoint from 11 other sets in $[9]_*^{(3)}$. So if an ordering as described above exists we must be able to find a pair of sets, at least one of which is of type C, that are disjoint from at most 11 other sets in $[9]_*^{(3)}$. However, any such pair of sets is disjoint from at least 13 other sets in $[9]_*^{(3)}$ so no such ordering can exist.

5.3 Result for $n = 2r + 2$

The first non-trivial case of Conjecture 5.3 that we can prove is the case $n = 2r + 2$. (Note that for $n = 2r$ there are only two separated r-sets and these are disjoint. For $n = 2r + 1$ there are n separated r-sets all of which are rotations of each other. So clearly in both cases Conjecture 5.3 is true.)

Theorem 5.5 *If $\mathcal{A} \subset [2r+2]_*^{(r)}$ is intersecting then $|\mathcal{A}| \leq \frac{r(r+1)}{2}$ and hence Conjecture 5.3 is true for $n = 2r + 2$ and $k = 1$.*

Proof: We prove this from first principles by examining in detail the subgraph, G, of the Kneser graph $K_{2r+2,r}$ induced by the collection of all separated r-sets. G has vertex set $[n]_*^{(r)}$ and edges between any two sets which are disjoint. We need to show that the largest independent set in G has order $|\mathcal{A}_x^*| = \frac{r(r+1)}{2}$. We will assume that r is odd, say $r = 2d+1$. The case when r is even may be proved in a similar fashion.

We partition $[n]_*^{(r)}$ into classes of size $r+1$ as follows. Let $\mathcal{A}_{\mathrm{o}}$ ($\mathcal{A}_{\mathrm{e}}$) be the class of all separated r-sets consisting of only odd (even) numbers. All other sets in $[n]_*^{(r)}$ contain two gaps of size 2 and $r-2$ gaps of size 1. We partition these into $r-1$ classes each containing $r+1$ sets by defining for $i = 0, \ldots, d-1$ $\mathcal{B}(i, \mathrm{o})$ ($\mathcal{B}(i, \mathrm{e})$) to be the class of all r-sets in which the two gaps of size 2 are separated by i gaps of size 1 and the numbers outside this part of the set are all odd (even).

Having partitioned $[n]_*^{(r)}$ we note that $\mathcal{A}_{\mathrm{o}} \cup \mathcal{A}_{\mathrm{e}}$ induces a complete bipartite graph of order n in G. This is because every r-set containing only odd numbers is obviously disjoint from every r-set containing only even numbers. Also there exist matchings between $\mathcal{A}_{\mathrm{o}}$ and $\mathcal{B}(0, \mathrm{e})$ and between $\mathcal{A}_{\mathrm{e}}$ and $\mathcal{B}(0, \mathrm{o})$ (see Figure 5.2).

If we wish to choose an independent set in G we may assume without loss of generality that we choose sets in $\mathcal{A}_{\mathrm{o}}$ rather than in $\mathcal{A}_{\mathrm{e}}$. The complete bipartite graph induced by $\mathcal{A}_o \cup \mathcal{A}_e$ implies that we cannot choose any sets

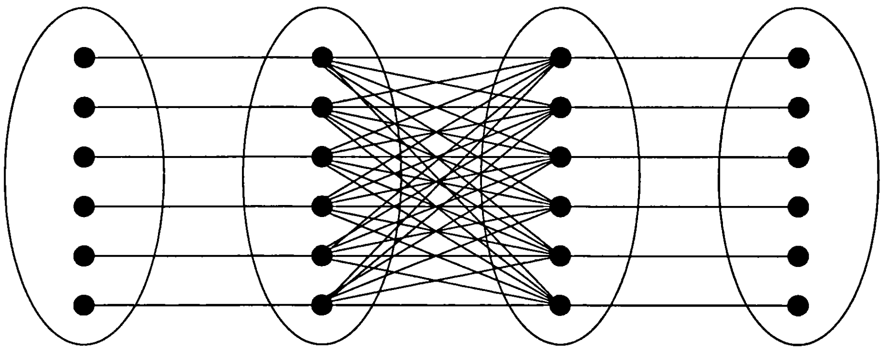

Figure 5.2: Part of the Kneser graph $K_{2r+2,r}$

from $\mathcal{A}_{\mathrm{e}}$. Further, the matching between $\mathcal{A}_{\mathrm{o}}$ and $\mathcal{B}(0, \mathrm{e})$ means that we can choose in total at most $r+1$ sets from $\mathcal{A}_{\mathrm{o}} \cup \mathcal{A}_{\mathrm{e}} \cup \mathcal{B}(0, \mathrm{e})$.

There also exist matchings between $\mathcal{B}(i, \mathrm{o})$ and $\mathcal{B}(i+1, \mathrm{e})$ for $i = 0, \ldots, d-2$. Hence we may choose at most a half of the r-sets from $\mathcal{B}(i, \mathrm{o}) \cup \mathcal{B}(i+1, \mathrm{e})$ for $i = 0, 1, \ldots, d-2$.

This leaves only $\mathcal{B}(d-1, \mathrm{o})$ which we can spilt into two equal halves. There is a matching between these two halves and so we may choose at most a half of the r-sets in this class.

Hence we have at most $(r+1) + (d-1)(r+1) + \frac{(r+1)}{2} = \frac{r(r+1)}{2}$ r-sets in our independent set as required. $\square$

Theorem 5.5 says nothing about the uniqueness of intersecting families of maximal size. In general there is not a unique family. For example if $n = 8$ and $r = 3$ then Theorem 5.5 tells us that any intersecting family has size at most 6. Obviously this is attained by $\mathcal{A}_1^* = \{135, 136, 137, 146, 147, 157\}$. However, another non-isomorphic family of maximal size is $\{A \in [8]_*^{(3)} : |A \cap \{1,3,5\}| \geq 2\} = \{135, 136, 137, 157, 357, 358\}$.

5.4 Results for weighted sets

For $A = \{a_1, \ldots, a_r\} \in [n]_k^{(r)}$ we define the *weight* of A to be

$$w(A) = \prod_{i=1}^{r} \binom{a_{i+1} - a_i - 1}{k},$$

where $a_{r+1} = a_1 + n$. So the weight of a k-separated set $A \in [n]_k^{(r)}$ is simply the number of different ways A may be extended to form a set $B \in [n]^{((k+1)r)}$ by inserting exactly k new elements into each gap in A. We then define the *weight* of a family of sets $\mathcal{A} \subset [n]_k^{(r)}$ to be

$$w(\mathcal{A}) = \sum_{A \in \mathcal{A}} w(A).$$

The following result says that an analogue of the Erdős-Ko-Rado theorem holds for weighted k-separated sets.

Theorem 5.6 *Let $n \geq 2(k+1)r$. Suppose $x \in [n]$ and $\mathcal{A} \subset [n]_k^{(r)}$ is intersecting. Then $w(\mathcal{A}) \leq w(\mathcal{A}_x^*)$, where $\mathcal{A}_x^* = \{A \in [n]_k^{(r)} : x \in A\}$.*

Proof: Consider the bipartite graph $G = (V \cup W, E)$ with vertex classes $V = [n]_k^{(r)}$ and $W = [n]^{((k+1)r)}$. We define E as follows: let $A \in V$ be adjacent to $B \in W$ if we can construct B from A by inserting exactly k elements into each gap in A.

Let $\mathcal{A} \subset [n]_k^{(r)}$ be intersecting then

$$\Gamma(\mathcal{A}) = \{B \in [n]^{((k+1)r)} : (A, B) \in E, \text{ for some } A \in \mathcal{A}\}$$

is also intersecting. Since $\Gamma(\mathcal{A})$ is an intersecting family of $(k+1)r$-sets from $[n]^{((k+1)r)}$ and $n \geq 2(k+1)r$ Theorem 5.1 implies that

$$|\Gamma(\mathcal{A})| \leq \binom{n-1}{(k+1)r-1}.$$

For distinct $A_1, A_2 \in \mathcal{A}$ we have $\Gamma(A_1) \cap \Gamma(A_2) = \emptyset$. To see this, suppose we had $B \in \Gamma(A_1) \cap \Gamma(A_2)$ with $B = \{b_1, \ldots, b_{(k+1)r}\}$. Without loss of generality we may suppose that $A_1 = \{b_1, b_{k+2}, \ldots, b_{(k+1)r-k}\}$ and $A_2 = \{b_i, b_{k+i+1}, \ldots, b_{(k+1)(r-1)+i}\}$, for some $2 \leq i \leq k+1$. Hence $A_1 \cap A_2 = \emptyset$. This contradicts the fact that $\mathcal{A}$ is intersecting.

Then, since $|\Gamma(A)| = w(A)$, we have

$$w(\mathcal{A}) = \sum_{A \in \mathcal{A}} w(A) = \sum_{A \in \mathcal{A}} |\Gamma(A)| = |\Gamma(\mathcal{A})| \leq \binom{n-1}{(k+1)r-1} = w(\mathcal{A}_x^*).$$

□

In fact we can prove a stronger result by using the Hilton-Milner theorem (Theorem 5.4).

Theorem 5.7 *Let $n > 2(k+1)r$. Suppose $x \in [n]$ and $\mathcal{A} \subset [n]_k^{(r)}$ is intersecting. Then at least one of the following holds:*

$$\text{(i)} \qquad w(\mathcal{A}) \leq \binom{n-1}{(k+1)r-1} - \binom{n-(k+1)r-1}{(k+1)r-1} + 1.$$

$$\text{(ii)} \qquad |\mathcal{A}| \leq |\mathcal{A}_x^*|, \tag{5.1}$$

Proof: Consider $\Gamma(\mathcal{A})$ as defined in the proof of the previous theorem. Either $\Gamma(\mathcal{A})$ is fixed by a single element, so there exists $y \in [n]$ such that $\Gamma(\mathcal{A}) \subseteq \mathcal{B}_y = \{B \in [n]^{((k+1)r)} : y \in B\}$, or Theorem 5.4 implies that

$$w(\mathcal{A}) = |\Gamma(\mathcal{A})| \leq \binom{n-1}{(k+1)r-1} - \binom{n-(k+1)r-1}{(k+1)r-1} + 1.$$

So either *(i)* holds or their exists $y \in [n]$ such that $\Gamma(\mathcal{A}) \subseteq \mathcal{B}_y$.

Suppose that $\Gamma(\mathcal{A}) \subseteq \mathcal{B}_y$. Since every $B \in \Gamma(\mathcal{A})$ contains y we see from the construction of $\Gamma(\mathcal{A})$ that for every $A \in \mathcal{A}$ either $y \in A$ or there exists $i \in [k]$ such that $\{y-i, y-i+k+1\} \subset A$.

Suppose $\{y-i, y-i+k+1\} \subset A \in \mathcal{A}$ for some $i \in [k]$. If we rotate A clockwise by i positions then we obtain a disjoint set $A^{+i} \in [n]_k^{(r)}$ containing y. This new set is uniquely determined by A. (If two distinct sets $A_1, A_2 \in \mathcal{A}$ gave rise to the same set then it is easy to see that $A_1 \cap A_2 = \emptyset$.) Further, $A^{+i} \notin \mathcal{A}$ since $A^{+i} \cap A = \emptyset$. So for every set $A \in \mathcal{A}$, either $y \in A$ or there exists a unique set A^{+i} containing y which does not belong to $\mathcal{A}$. Hence $|\mathcal{A}| \leq |\mathcal{A}_y^*| = |\mathcal{A}_x^*|$, which is *(ii)*. □

Both of the previous two theorems enable us to give bounds on the size of intersecting families in $[n]_k^{(r)}$, for given values of r, k and n, using simple information about the weights of these sets. Of course such bounds are not as good as the conjectured exact result but they are a significant improvement on the obvious trivial bounds. We will give an example to illustrate how we may do this.

If $\mathcal{A} \subset [n]_k^{(r)}$ is intersecting then what upper bounds can we give for $|\mathcal{A}|$? Consider a set $A \in \mathcal{A}$ and its rotation clockwise by i positions, for some $i \in [k]$. These sets are disjoint and so $\mathcal{A}$ contains at most $\frac{1}{k+1}$ of all sets in $[n]_k^{(r)}$.

Another upper bound for $|\mathcal{A}|$ is given by the Hilton-Milner theorem: either $\mathcal{A}$ is fixed by a single element, so $|\mathcal{A}| \leq |\mathcal{A}_x^*|$, or $|\mathcal{A}| \leq \binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1$. Combining these two observations we obtain the following upper bound,

$$|\mathcal{A}| \leq \max\left\{|\mathcal{A}_x^*|, \min\left\{\frac{n}{r(k+1)}\binom{n-kr-1}{r-1}, \binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1\right\}\right\}.$$

For our example we will take $k = 1$, $n = 22$ and $r = 5$. Then $|[22]_*^{(5)}| = 8008$. If $\mathcal{A} \subset [22]_*^{(5)}$ is intersecting then the best trivial bound is $|\mathcal{A}| \leq 4004$. We will use Theorem 5.7 to show that $|\mathcal{A}| \leq 2651$. We partition $[22]_*^{(5)}$ into classes according to weight. There are 41 different weights which occur, ranging from 13 upto 432. Since a set and its rotation by one are disjoint $\mathcal{A}$ contains at most a half of all sets of a given weight. Assuming that $|A|$ is larger than the conjectured bound, Theorem 5.7 tells us that $w(\mathcal{A}) \leq \binom{21}{9} - \binom{11}{9} + 1$. Hence, a simple calculation shows that even if we always pick the "lightest" possible sets we must have $|\mathcal{A}| \leq 2651$.

We note that this is a significant improvement on the trivial upper bound of 4004 but still far from the conjectured upper bound of 1820.

In fact we could have used Theorem 5.6 to give the same bound. In order to see that Theorem 5.7 can provide improved bounds we need to look at larger examples. If $\mathcal{A} \subset [40]_*^{(8)}$ is intersecting then the difference between the bounds on $w(\mathcal{A})$ given by Theorems 5.6 and 5.7 is $\binom{23}{15} - 1$. Then, since the heaviest set in $[40]_*^{(8)}$ has weight 4^8, we see that the bound on $|\mathcal{A}|$ given by Theorem 5.7 would improve the bound given by Theorem 5.6 by at least 7. In practice the improvement would probably be a lot more.

We conclude this chapter with the following easy corollary to Theorem 5.6. This corollary tells us that if an intersecting family of sets $\mathcal{A} \subset [n]_k^{(r)}$ "looks like" a random family of sets, in the sense that the average weight of a set in $\mathcal{A}$ is equal to the average weight of a set in $[n]_k^{(r)}$, then $\mathcal{A}$ satisfies the claim of Conjecture 5.3.

Corollary 5.8 *Let $n \geq 2(k+1)r$. Suppose $x \in [n]$ and $\mathcal{A} \subset [n]_k^{(r)}$ is intersecting. If the average weight of a set in $\mathcal{A}$ is at least as large as the average weight of a set in $[n]_k^{(r)}$ then $|\mathcal{A}| \leq |\mathcal{A}_x^*|$.*

In particular, if the average weight of a set in $\mathcal{A}$ is equal to the average weight of a set in $[n]_k^{(r)}$ then $|\mathcal{A}| \leq \mathcal{A}_x^*|$.

Proof: Consider the graph G used in the proof of Theorem 5.6. If $A \in V$ then $|\Gamma(A)| = w(A)$. Hence $|E| = w([n]_k^{(r)})$. Also, if $B \in W$ then $|\Gamma(B)| = k+1$. Hence $|E| = (k+1)|[n]^{((k+1)r)}|$ and so

$$\text{average weight of a set in } [n]_k^{(r)} = \frac{(k+1)|[n]^{((k+1)r)}|}{|[n]_k^{(r)}|}.$$

Suppose $\mathcal{A} \subset [n]_k^{(r)}$ is intersecting. Theorem 5.6 implies that

$$|\mathcal{A}|(\text{average weight of a set in } \mathcal{A}) = w(\mathcal{A}) \leq \frac{(k+1)r|[n]^{((k+1)r)}|}{n}.$$

Hence, if the average weight of a set in $\mathcal{A}$ is at least as large as the average weight of a set in $[n]_k^{(r)}$, then

$$|\mathcal{A}| \leq \frac{r}{n}|[n]_k^{(r)}| = |\mathcal{A}_x^*|$$

as required. □

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