Ph.D. Thesis

University College, London

# Algebraic properties of surface fibrations

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# Abstract

Algebraically, surface fibrations correspond to extensions of surface groups via their long homotopy exact sequences. First, it is proved that any group can be constructed by at most finitely many group extensions where the kernel and quotient correspond to finite free products of free groups and surface groups. This rigidity theorem has the important corollary that the group of all automorphisms of an extension of surface groups has finite index in the automorphism group of the fundamental group of a surface fibration.

The Baer-Nielsen theorem for surfaces is extended to show that the natural homomorphism from the homotopy classes of diffeomorphisms of surface fibrations maps surjectively onto the outer automorphism group of their fundamental group.

The virtual cohomological dimension of the outer automorphism groups of poly-surface and poly-free groups is calculated when the image of the operator homomorphism of the extension is finite. Using pure diffeomorphisms, this dimension is obtained when the image of the operator homomorphism is generated by Dehn twists about separating circles in a surface. A bound is also given on the virtual dimension of the automorphism group in all cases.

Finally, it is shown the mapping class group of a Stallings fibration M is not rigid in the sense that the automorphism group of the long homotopy exact sequence of M does not have finite index in the automorphism group of the fundamental group of M. The virtual cohomological dimension of the mapping class group of the trivial Stallings fibration is calculated to be 6g-5 where g is the genus of the fibre, whereas Stallings fibrations constructed from pseudo-Anosov diffeomorphisms are shown to have finite mapping class groups.

# Contents

# Introduction

The aim of this thesis is to consider how far results from the theory of surfaces can be extended to theorems concerning surface fibrations. The research was motivated by attempts to generalise the Baer-Nielsen theorem on the outer automorphisms group of surface groups and Harer's theorem which calculated the virtual cohomological dimension of the mapping class group of an orientable surface.

The closed 2–manifolds with genus $g \geq 2$ are known as hyperbolic surfaces since their universal cover is the hyperbolic disc. By considering fibrations with fibre and base space given by (hyperbolic) surfaces we construct the surface fibrations of the title. Much of the topology of surface fibrations can be derived from algebraic information, most notably, their long homotopy exact sequence which reduces to an extension of surface groups (fundamental groups of surfaces). In the course of writing this thesis, it became clear that many of the algebraic techniques for these poly-surface groups could be applied to poly-free groups. These results point towards the usefulness of our techniques for infinite group theorists.

The first chapter is an outline of some essential background material from group theory necessary for the algebraic aspects of this thesis. In Chapter 2 we shall consider group extensions where the kernel and quotient correspond to finite free products of free groups and surface groups (both orientable and nonorientable). Free groups and surface groups are brought together under the unifying framework of (torsion-free) Fuchsian groups. Our aim is to show that any group can be constructed by at most finitely many group extensions of this type. The approach taken is to consider a sufficient set of conditions for this rigidity theorem to hold and we then go on to prove that they are satisfied by the class of iterated extensions of finite free products of torsion-free Fuchsian groups. The rigidity theorem has the important corollary that the group of automorphisms of the extension leaving the kernel invariant has finite index in the automorphism group of the extension. We shall use this fact frequently in all other chapters.

In the 1920's, Baer and Nielsen proved that the natural homomorphism from the group of diffeomorphisms of a surface to the outer automorphism group of its fundamental group is surjective and has a kernel given by the diffeomorphisms homotopic to the identity. Waldhausen later generalised this theorem to the class of sufficiently large 3–manifolds. We will examine how far the Baer-Nielsen theorem extends to surface fibrations in Chapter 3. The chapter begins by expounding the elementary properties of fibrations and simplicial sets. The category of simplicial sets provides a naturally occurring category for homotopy theory which eases the exposition of the proofs in this chapter. We then create a theory of fibre smoothing which enables us to smooth our surface fibrations to smooth fibre bundles in order to prove our generalisation of the Baer-Nielsen theorem. In particular we show that for a certain class of group extensions known as characteristic extensions, the natural homomorphism

$$\pi_0(\mathrm{Diff}\ (\mathrm{X}_\Gamma)) \twoheadrightarrow \mathrm{Out}\ (\Gamma)$$

is surjective, where $X_\Gamma$ is a smooth manifold with $\pi_1(X_\Gamma) = \Gamma$. This is generalised to iterated surface fibrations before the last section of the chapter which considers non-characteristic extensions of surface groups. In this case we prove that the image of the above homomorphism is a subgroup of finite index in Out $(\Gamma)$.

Chapter 4 investigates the automorphism groups of certain poly-Fuchsian groups; in particular, we shall consider extensions of free groups and of orientable surface groups. This research was motivated by theorems due to Harer and Culler/Vogtmann who investigated the outer automorphism groups of surface groups and free groups respectively, and calculated the virtual cohomological dimension in each case. In this chapter, we extend their results to outer automorphism groups of poly-surface and poly-free groups in the case where the image of the operator homomorphism of the extension is finite. When the image of the operator homomorphism is infinite, the problem seems to be far more complex. However, we are able to give a bound on the virtual dimension of the *automorphism group* for all cases. The purpose of the final section is to calculate an exact sequence for the outer automorphism group of an extension consisting of *centreless* groups. This reduces the calculation of the v.c.d. of the outer automorphism group to the corresponding calculation for the ends of the exact sequence. In the next chapter we shall use the results of this section to calculate the v.c.d. in a particular case where the image of the operator homomorphism is generated by certain diffeomorphisms about separating circles in the surface.

The final chapter brings together much of the work from previous chapters and applies results from Thurston's classification of surface diffeomorphisms. We begin by collecting together various kinds of surface diffeomorphisms and quoting Thurston's classification. The second section studies certain subgroups of the mapping class group of a surface using Ivanov's work on pure diffeomorphisms which are variations upon pseudo-Anosov diffeomorphisms for disconnected surfaces. Using this work, we calculate the v.c.d. of the quotient of the exact sequence from the previous chapter. The final sections consider the applications of our techniques to the study of 3-manifolds; in particular, Stallings fibrations which are fibrations of surfaces over the circle. The aim of Section 3 is to demonstrate that, in general, the mapping class group of a Stallings fibration $M$ is not rigid in the sense that the automorphism group of the long homotopy exact sequence of $M$ does not have finite index in Aut $(\pi_1(\mathrm{M}))$. We also calculate the virtual cohomological dimension of the mapping class group of the trivial Stallings fibration. The final section then proves that when a Stallings fibration is constructed by a pseudo-Anosov diffeomorphism, then its mapping class group is finite.

I would like to thank my supervisor Dr. F.E.A. Johnson, for the help he has given me during the writing of this thesis.

# Chapter 1

# Preliminary results from group theory

## 1.1 Torsion-free Fuchsian groups

Let $\mathcal{D}$ denote the class of torsion-free Fuchsian groups, which consists of all torsion-free discrete lattices of finite covolume in $\mathrm{PGL}_2(R)$. $\mathcal{D}$ is the disjoint union $\mathcal{F} \sqcup \mathcal{S}^+ \sqcup \mathcal{S}^-$ where

(i) $\mathcal{F}$ is the class of free groups $F_n$ of finite rank $n \geq 2$;

(ii) $\mathcal{S}^+$ is the class of fundamental groups of closed *orientable* surfaces whose universal cover is given by the hyperbolic plane and hence have genus $g \geq 2$. These groups have the following presentation:

$$\Sigma_g^+ = \langle X_1, \ldots, X_g, Y_1, \ldots Y_g : \prod_{i=1}^{g} X_i Y_i X_i^{-1} Y_i^{-1} \rangle;$$

(iii) $\mathcal{S}^-$ is the class of fundamental groups of closed *nonorientable* surfaces with the following presentation:

$$\Sigma_g^- = \langle C_0, \ldots, C_g : \prod_{i=0}^{g} C_i^2 \rangle.$$

The following properties are well-known and can be found in the references by [Kat] and [Bea]:

**Proposition 1.1 (Properties of torsion-free Fuchsian groups)**
*Let $G$ be a torsion-free Fuchsian group so that $G$ belongs to $\mathcal{F} \sqcup \mathcal{S}^+ \sqcup \mathcal{S}^-$. Then*

*(I): $G$ is nontrivial and has trivial centre.*

*(II): The* rank *of $G$, denoted $rk(G)$, is given by $rk(\Sigma_n^+) = 2n$ for $\Sigma_n^+ \in \mathcal{S}^+$, $rk(\Sigma_n^-) = n+1$ for $\Sigma_n^- \in \mathcal{S}^-$ and $rk(F_n) = n$ for $F_n \in \mathcal{F}_n$.*

*(III): The* Euler characteristic *of $G$, denoted $\chi(G)$, is given by $\chi(\Sigma_n^+) = 2-2n$, $\chi(\Sigma_n^-) = 1-n$ and $\chi(F_n) = 1-n$. In particular $\chi(G) \neq 0$ for $G \in D$.*

*(IV): If $H \subset G$ is a subgroup of finite index then $H \in \mathcal{D}$.*

*(V): If $N \triangleleft G$ is a nontrivial normal subgroup of infinite index, then $N$ is a free group of infinite rank.*

An important result in the theory of Fuchsian groups is the Riemann-Hurwitz theorem which relates the rank of a subgroup of finite index in a Fuchsian group to its index. For the purposes of Chapter 2, it will be necessary to generalise the formula to free products of Fuchsian groups. Here we state the original result (see for example [LS] Chapter III):

**Theorem 1.2 (The Riemann-Hurwitz formula)** *If $G \in \mathcal{D}$ and $N$ is a finitely generated normal subgroup of $G$, then $N$ has finite index $j(N)$ in $G$ given by the Riemann-Hurwitz formula:*

$$j(N) = \frac{rk(N) - \delta}{rk(G) - \delta}$$

*where $\delta$ is equal to $1$ when $G$ is a free group, and equals $2$ when $G$ is a surface group ($\delta$ is the cohomological dimension of $G$; see Section 5).*

## 1.2 Free products

The *free product* of groups is the coproduct in the category of groups. Specifically; let $\{G_\alpha\}$be a family of groups, $G$ a group and let $i_\alpha : G_\alpha \to G$ be homomorphisms. Then $(G, \{i_\alpha\})$ is called a *free product* of the groups $\{G_\alpha\}$ if for every group $H$ and homomorphisms $f_\alpha : G_\alpha \to H$ there is a unique homomorphism $f : G \to H$ such that $f_\alpha = i_\alpha f$ for all $\alpha$.

An alternative way to view this definition is to write the groups in the product in terms of generators and relations:
Let $A$ and $B$ be groups with presentations $A = \langle a_1, \ldots; r_1, \ldots \rangle$ and $B = \langle b_1, \ldots; s_1, \ldots \rangle$ respectively, with disjoint sets of generators. The *free product* $A * B$ of $A$ and $B$ is the group

$$A * B = \langle a_1, \ldots b_1, \ldots; r_1, \ldots s_1, \ldots \rangle$$

Using this alternative definition it is clear that free groups - which have no relations - are closed under the free product operation, thus motivating the nomenclature. The following result is simple to prove:

**Proposition 1.3** *If $(G, \{i_\alpha\})$ and $(H, \{j_\alpha\})$ are both free products of the family of groups $\{G_\alpha\}$ then there is a unique isomorphism $f : G \to H$ such that $i_\alpha f = j_\alpha$ for all $\alpha$.*

We shall make use of the following well-known theorems in the proof of the rigidity theorem 2.11. The proofs can be found in [LS] Chapter III.

**Theorem 1.4 (Grushko-Neumann)** *Let $F$ be a free group, and let there be a homomorphism $\phi : F \to *A_i$ of $F$ onto $*A_i$. Then there is a factorisation of $F$ as a free product, $F = *F_i$, such that $\phi(F_i) = A_i$.*

In particular, the following corollary will be most useful:

**Corollary 1.5** *If $G = A_1 * \ldots * A_n$ and the rank of $A_i$ is $r_i$, then the rank of $G$ is $r_1 + \ldots + r_n$.*

**Theorem 1.6 (Kurosh Subgroup Theorem)** *Let $H$ be a subgroup of the free product $G = *_{\lambda \in \Lambda} G_\lambda$. Then $H$ is a free product of the form*

$$H = H_0 * (*_{\lambda, d_\lambda}(H \cap (d_\lambda G_\lambda d_\lambda^{-1})$$

*where $H_0$ is a free group, $d_\lambda$ varies over a set of $(H, G_\lambda)$-double coset representatives and $\lambda$ varies over $\Lambda$.*

*Furthermore, if $H$ has finite index in $G$, the rank of the free group $H_0$ is $\sum_{\lambda \in \Lambda}(m - m_\lambda) + 1 - m$ where $m_\lambda$ is the number of $(H, G_\lambda)$-double cosets in $G$.*

The concept of a free product may be considered as a generalisation of the familiar concept of a free group. As we have already observed in the first section, a finitely-generated normal subgroup in a free group has finite index. Thus we may expect a similar result to hold for free products. The following generalisation was proved by [Bau] and will be useful in the next chapter.

**Theorem 1.7 (B. Baumslag)** *Let $G$ be the free product of two nontrivial groups. If a finitely generated subgroup $H$ contains a nontrivial normal subgroup of $G$ then $H$ has finite index in $G$.*

## 1.3 Subgroups of finite index

Throughout this thesis we shall consider subgroups of finite index within groups. Recall first that a subgroup $H_0$ in $G$ is a *characteristic* (respectively, *fully-invariant*) subgroup of $G$ if $\alpha(G) = G$ for all $\alpha$ belonging to the automorphism group (resp. endomorphism group) of $G$. Note that fully-invariant subgroups are characteristic since Aut (G) $\subset$ End$(G)$.

The following lemmas will prove invaluable in several places and so we include proofs below (see [Neu] and [Iv2],p.80).

**Lemma 1.8** *Let $G$ be any group containing a subgroup of finite index $K$. Then $G$ has normal subgroup $H$ of finite index such that $H \subset K \subset G$*

**Proof :** Let $\{g_t\}_{t\in T}$ be an arbitrary set of representatives of left cosets of $K$. $T$ is a finite set since $K$ has finite index in $G$. Define

$$H = \bigcap_{t\in T} g_t K g_t^{-1}.$$

This has finite index in $G$ since it is a finite intersection of subgroups of finite index in $G$ which has finite index by Poincaré's lemma. Therefore, it suffices to show that it is a normal subgroup of $G$. For any $g$ in $G$, the cosets $gg_sK$ and $gg_tK$ derived from distinct left cosets of $K$ are equal if and only if $(gg_t)^{-1}gg_s \in K$; that is, $g_t^{-1}g_s \in K$ which is impossible since $g_s$ and $g_t$ are distinct coset representatives. Analogously the right cosets $K(gg_t)^{-1}$ are all distinct and the sets $\{g_tKg_t^{-1} : t \in T\}$ and $\{gg_tK(gg_t)^{-1} : t \in T\}$ are equivalent. Therefore

$$\begin{aligned} gHg^{-1} &= g\left(\bigcap g_tKg_t^{-1}\right)g^{-1} \\ &= \bigcap (gg_t)K(gg_t)^{-1} \\ &= H \end{aligned}$$

and $H$ is a normal subgroup of $G$. □

Let $F$ be a free group on the set $x_1, x_2, \ldots$ and let $W$ be a nonempty subset of $F$. If $w = x_{i_1}^{l_1} \ldots x_{i_r}^{l_r} \in W$ and $g_1, \ldots, g_r$ are elements of a group $G$, then the *value* of $w$ at $(g_1, \ldots g_r)$ is given by

$$w(g_1, \ldots g_r) = g_1^{l_1} \ldots g_r^{l_r}.$$

The class of all groups $G$ such that $W(G) = 1$ is the *variety* of groups determined by $W$. The subgroup of $G$ generated by all values of words in $W$ is the *verbal subgroup* of $G$ determined by $W$ and every verbal subgroup of a group is fully-invariant. The converse is not true in general, but for free groups, every fully-invariant subgroup is verbal.

**Lemma 1.9 (see [Neu], p.112)** *Let $G$ be a finitely generated group containing a normal subgroup $N$ of finite index in $G$. Then there exists a subgroup $L$ where $L$ contained in $N$ which is a fully-invariant subgroup of finite index in $G$.*

**Proof :** Let $G = \alpha(F_k)$ be a presentation of $G$ where $F_k$ denotes a free group of rank $k$, and let $S$ be the complete inverse image of $N$ in $F_k$ so that $\alpha(S) = N$. As $N$ is normal in $G$, it follows that $S$ is normal in $F_k$ and further, $[F_k : S] = [\alpha(F_k) : \alpha(S)] = [G : N]$ showing that $S$ has finite index in $F_k$. Consider the variety of groups generated by the finite group $F_k/S$. The free group of rank $k$ of this variety we shall denote by $\bar{F}_k$. This is finite since the free groups of finite rank of a variety generated by a finite group are finite and furthermore, $F_k/S$ is a factor group of $\bar{F}_k$. Therefore, putting $\bar{F}_k \cong F_k/V$, then $V$ is a verbal subgroup of $F_k$ of finite index in $F_k$, and $V$ is contained in S. Therefore, $\alpha(V)$ is a verbal subgroup, and hence a fully invariant subgroup of $\alpha(F_k)$ which is of finite index in $\alpha(F_k) = G$ and contained in $\alpha(S) = N$. Writing $L = \alpha(V)$ gives the result. $\Box$

Combining these results gives

**Lemma 1.10 (Finite index lemma)** *Let $G$ be a finitely generated group containing a subgroup $N$ of finite index. Then $G$ has a normal subgroup of finite index $L$ contained in $N$, and therefore, a characteristic subgroup $H$ such that $H \subset L \subset N \subset G$, where $H$ has finite index in $G$.*

## 1.4 Group extensions and the Eilenberg-Mac Lane Theorem

A group extension of $K$ by $Q$ is a short exact sequence

$$1 \to K \xrightarrow{i} G \xrightarrow{\pi} Q \to 1$$

where $i$ and $\pi$ are group homomorphisms. By a *morphism* of group extensions is meant a triple of homomorphisms $(\alpha, \beta, \gamma)$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 1 & \to & K & \to & G & \to & Q & \to & 1 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ 1 & \to & K' & \to & G' & \to & Q' & \to & 1 \end{array}$$

A morphism of the form $(1_K, \beta, 1_Q)$ is called a *congruence* and group extensions are normally classified up to congruence. By the 5-lemma, $\beta$ is an isomorphism. To each group extension there exists a unique homomorphism, called the *operator homomorphism*, constructed as follows:

Consider a *transversal function* $s : Q \to G$ with the property

$$s\pi = 1.$$

This is generally not a homomorphism but we can create a homomorphism by conjugating automorphisms of $K$. Suppose we have two automorphisms of $K$, $\alpha$ and $\alpha'$, defined by transversals $s$ and $s'$ respectively. Observe that since any two transversals differ by an element of $K$, we may put $s(t) = x's'(t)$ where $x' \in K$. By writing $\psi_t(x) = \alpha(x) = s(t)^{-1}xs(t)$ we obtain a function $\psi : Q \to$ Aut (K), and similarly $\alpha'(x) = \psi'_t(x) = s'(t)^{-1}xs'(t)$. Rearranging,

$$\begin{aligned} \psi'_t(x) &= s'(t)^{-1}s(t)\psi_t(x)s(t)^{-1}s'(t) \\ &= \{s(q)^{-1}x's(t)\}\phi_t(x)\{s(t)^{-1}x'^{-1}s(t)\} \\ &= g^{-1}\phi_t(x)g \qquad \text{where } g = s(t)^{-1}x'^{-1}s(t) \end{aligned}$$

proving that any two $\psi$'s differ by an inner automorphism of $K$. This we may write as

$$\psi_t(x)(\text{Inn (K)}) = \psi_t'(\text{x})(\text{Inn (K)})$$

and hence there is an unique homomorphism $\phi_t(x) = \psi_t(x)(\text{Inn (K)})$ called the *operator homomorphism* and

$$\phi : Q \to \text{Out (K)}.$$

Two fundamental questions we may ask about group extensions are whether there is an extension corresponding to a given operator homomorphism $\phi$, and, given at least one extension realising $\phi$, what other extensions realise $\phi$. These questions may be interpreted as questions about certain cohomology groups as below (for a detailed account of this see [Mac] Chapter IV).

By an *abstract kernel* is meant a triple $(K, Q, \phi)$ consisting of groups $K, Q$ and a homomorphism $\phi : Q \to \text{Out K}$.

**Theorem 1.11 (The Eilenberg-MacLane Theorem)** *An abstract kernel $(K, Q, \phi)$ corresponds to a group extension if and only if the obstruction belonging to*

$$H^3(Q, Z(K))$$

*vanishes where Z(K) denotes the centre of $K$.*

*Given that the abstract kernel $(K, Q, \phi)$ corresponds to an extension, the congruence classes of extensions are in $1-1$-correspondence with the elements of the second cohomology group*

$$H^2(Q, Z(K))$$

In later chapters we will also consider various properties of the automorphism group of an extension. This group is closely related to the group of congruences of an extension as we shall now describe:

Let Aut $(\mathcal{E})$ be the group of automorphisms which preserve $\mathcal{E}$; that is, the group of those automorphisms $\alpha : G \to G$ for which there exist automorphisms $\alpha_K, \alpha_Q$ making the following commute:

$$\begin{array}{ccccccccc} 1 & \to & K & \to & G & \to & Q & \to & 1 \\ & & \downarrow \alpha_K & & \downarrow \alpha & & \downarrow \alpha_Q & & \\ 1 & \to & K & \to & G & \to & Q & \to & 1 \end{array}$$

There is a homomorphism $\rho$ : Aut $(\mathcal{E}) \to$ Aut (K) $\times$ Aut (Q) given by $\rho(\alpha) = (\alpha_K, \alpha_Q)$ for which the kernel corresponds to the group of *self-congruences* of $\mathcal{E}$, denoted $C(\mathcal{E})$. This consists of automorphisms of $G$ making the following diagram commute:

$$\begin{array}{ccccccccc} 1 & \to & K & \to & G & \to & Q & \to & 1 \\ & & \downarrow 1_K & & \downarrow \alpha & & \downarrow 1_Q & & \\ 1 & \to & K & \to & G & \to & Q & \to & 1 \end{array}$$

It is clear that the homomorphism $\rho$ gives rise to an exact sequence

$$1 \to C(\mathcal{E}) \to \text{Aut } (\mathcal{E}) \xrightarrow{\rho} \text{Aut (K)} \times \text{Aut (Q)}. \tag{1.1}$$

**Proposition 1.12** *The group $\mathcal{C}(\mathcal{E})$ of self-congruences of an extension $\mathcal{E}$ is isomorphic to the group of 1-cocycles*

$$\mathcal{C}(\mathcal{E}) \cong Z^1(Q, Z(K))$$

*where $Z(K)$ denotes the centre of $K$.*

**Proof :** If $\alpha : G \to G$ is a self-congruence of $\mathcal{E}$, we can define a function $\bar{z}_\alpha$ by

$$\bar{z}_\alpha(g) = \alpha(g)g^{-1}$$

We will show that the values of this function belong to $Z(K)$. First, observe that $\alpha(g)g^{-1}$ is an element of $K$ if $\alpha$ is a self-congruence, since

$$\begin{aligned} \pi(\alpha(g)g^{-1}) &= \pi(\alpha(g))\pi(g^{-1}) \\ &= \pi(g)\pi(g^{-1}) = 1 \end{aligned}$$

where $\pi$ is the projection $\pi : G \to Q$ implying that $\alpha(g)g^{-1}$ belongs to $\ker(\pi) = K$. If we choose an element $k \in K$, then $g^{-1}kg$ also belongs to $K$ so that $\alpha(g^{-1}kg) = g^{-1}kg$ since the self-congruence $\alpha$ restricts to the identity on $K$. Upon rearrangement, this equation states that

$$k\alpha(g)g^{-1}k^{-1} = \alpha(g)g^{-1}$$

which proves that for any $k$ in $K$, $\bar{z}_\alpha(kg) = \bar{z}_\alpha(g)$. Therefore,

$$\begin{aligned} \bar{z}_\alpha(g) = \bar{z}_\alpha(kg) &= \alpha(k)\alpha(g)g^{-1}k^{-1} \\ &= k\bar{z}_\alpha(g)k^{-1} \end{aligned}$$

and so the function $\bar{z}_\alpha$ takes $G$ to the centre of $K$.

This in turn gives rise to a function $z_\alpha : Q \to Z(K)$ by letting $z_\alpha(q) = \bar{z}_\alpha(h)$ if $\pi(h) = q$. The 1-cocycle condition for a function $\phi$ is that

$$\phi(g_1g_2) = \phi(g_1)g_1\phi(g_2)g_1^{-1}.$$

The next step in the proof is to show that $z_\alpha$ is a 1-cocycle as is demonstrated below:

$$\begin{aligned} z_\alpha(pq) &= \alpha(pq)(pq)^{-1} \\ &= \alpha(p)\alpha(q)q^{-1}p^{-1} \\ &= \alpha(p)p^{-1}(p\alpha(q)q^{-1}p^{-1}) \\ &= z_\alpha(p)pz_\alpha(q)p^{-1} \end{aligned}$$

Thus, $z_\alpha$ is an element of the group of 1-cocycles of $Q$ with elements in $Z(K)$ denoted $Z^1(Q, Z(K))$. Moreover, the mapping from $C(\mathcal{E})$ to $Z^1(Q, Z(K))$ given by $\alpha \mapsto z_\alpha$ is an isomorphism of groups. □

Putting together all of the results on cohomology groups for centreless groups gives:

**Corollary 1.13 (Centreless groups)** *Suppose $K$ has trivial centre, then both of the cohomology groups $H^2(Q, Z(K))$ and $H^3(Q, Z(K))$ are trivial and hence there is a unique extension (up to congruence) realising the abstract kernel $(K, Q, \phi)$. Moreover the group of congruences $Z^1(Q, Z(K))$ is trivial and the exact sequence (1.1) simplifies to give an injection*

$$\mathrm{Aut}\ (\mathcal{E}) \overset{\rho}{\rightarrowtail} \mathrm{Aut}\ (\mathrm{K}) \times \mathrm{Aut}\ (\mathrm{Q})$$

Observe that torsion-free Fuchsian groups all have trivial centre and so the above corollary applies when we go on to consider extensions of surface groups and free groups in later chapters.

## 1.5 Cohomological dimension

A group $\Gamma$ is said to have *finite cohomological dimension* cd $(\Gamma)$ if, for all $\Gamma-$modules $M$ and all integers $i > n$, the cohomology group $H^i(\Gamma; M) = 0$. If the group has torsion then cd $(\Gamma) = \infty$; however, we may still obtain a meaningful invariant if it has a torsion-free subgroup $\Gamma_0$ of finite index. J-P. Serre has shown that all such subgroups have the same cohomological dimension and this dimension is called the *virtual cohomological dimension* of $\Gamma$ , vcd $(\Gamma)$.

Below we shall state many elementary properties for these concepts for which the main reference is [Ser].

**Proposition 1.14 (Properties of the cohomological dimension)**
*(i):* $0 \leq$ cd $(\Gamma) \leq \infty$ *and* cd $(\Gamma) = 0$ *if and only if* $\Gamma = \{1\}$. *If* $\Gamma$ *is a group of type FL (see Section 7) then* cd $(\Gamma) < \infty$
*(ii): Let* $\Gamma'$ *be a subgroup of* $\Gamma$. *Then* cd $(\Gamma') \leq$ cd $(\Gamma)$
*(iii): If* cd $(\Gamma) < \infty$ *and* $\Gamma'$ *is a subgroup of finite index in* $\Gamma$, *then* cd $(\Gamma') =$ cd $(\Gamma)$

*(iv): If* $\Gamma'$ *is a normal subgroup of* $\Gamma$*, then*

$$\text{cd}(\Gamma) \leq \text{cd}(\Gamma') + \text{cd}(\Gamma/\Gamma')$$

**Proposition 1.15 (Properties of the virtual c.d.)**

*(i):* $\text{vcd}(\Gamma) = 0$ *if and only if* $\Gamma$ *is a finite group.*

*(ii): If* $\Gamma'$ *is a subgroup of* $\Gamma$ *then* $\text{vcd}(\Gamma') \leq \text{vcd}(\Gamma)$.

*(iii):If* $\Gamma$ *is a group without torsion then* $\text{vcd}(\Gamma) = \text{cd}(\Gamma)$.

*(iv): If* $G$ *and* $H$ *are virtually torsion-free then* $\text{vcd}(\text{G} \times \text{H}) = \text{vcd}(\text{G}) + \text{vcd}(\text{H})$.

**Lemma 1.16** *Let* $K$ *be a torsion-free group and let* $Q$ *be a virtually torsion-free group giving rise to an extension* $1 \to K \to G \to Q \to 1$*. Then* $G$ *is also virtually torsion-free and further*

$$\text{vcd}(\text{G}) \leq \text{cd}(\text{K}) + \text{vcd}(\text{Q})$$

**Proof :** Since $Q$ is virtually torsion-free, it has a torsion-free subgroup of finite index denoted $Q^0$. Let $G^0$ be the extension of $Q^0$ by $K$. Note that we can guarantee that $G^0$ exists since the operator homomorphism $\phi' : Q^0 \to \text{Out}(\text{K})$ corresponding to the extension

$$1 \to K \to G^0 \to Q^0 \to 1$$

is just the restriction of the original operator homomorphism $\phi$. Using properties of the cohomological dimension, it follows that

$$\begin{aligned} \text{cd}(\text{G}^0) &\leq \text{cd}(\text{K}) + \text{cd}(\text{Q}^0) \\ &\leq \text{cd}(\text{K}) + \text{vcd}(\text{Q}) \end{aligned}$$

Finally, it is clear that $G^0$ has finite index in $G$ since $[G : G^0] = [Q : Q^0]$. $\square$

## 1.6 Duality groups

Bieri and Eckmann introduced the notion of a duality group in [BE]. $G$ is an *n-dimensional duality group* with respect to a right $G$–module $C$ if there is an element $e \in H_n(G;C)$ such that the cap-product with $e$ induces isomorphisms

$$H^k(G;A) \cong H_{n-k}(G;C \otimes A)$$

for all integers $k$ and all left $G$–modules $A$. Let $\mathcal{Z}$ denote the integers. A *Poincaré duality group* is a duality group with respect to $\mathcal{Z}$ (for more on these see [JW]). Surface groups and free groups are both examples of Poincaré duality groups. The following properties are demonstrated in [BE] and [Ser] respectively.

**Proposition 1.17 (Properties of duality groups)**

*(i) : If $G$ is torsion-free and $H$ is a subgroup of finite index, then $G$ is a duality group if and only if $H$ is a duality group.*

*(ii): If $G/K = Q$ and $Q$ and $K$ are duality groups, then $G$ is a duality group and* cd G = cd K + cd Q.

As before, there is a notion of a *virtual duality group* which has a subgroup of finite index which is a duality group. The following lemma strengthens the lemma in the previous section on virtually torsion-free groups:

**Lemma 1.18** *Let $K$ be a duality group and let $Q$ be a virtual duality group with finite virtual cohomological dimension. Let $G$ be formed by the extension $1 \to K \to G \to Q \to 1$. Then*

$$\text{vcd (G)} = \text{cd (K)} + \text{vcd (Q)}$$

**Proof :** Let $Q^0$ denote a duality group of finite index in $Q$. Write $G^0$ for the extension of $K$ by $Q^0$ (again we know that this extension exists since the corresponding operator homomorphism is just the restriction of the original operator homomorphism). Then $G^0$ is a subgroup of finite index in $G$ and furthermore it is a duality group with

$$\mathrm{cd}\ (\mathrm{G}^0) = \mathrm{cd}\ (\mathrm{K}) + \mathrm{cd}\ (\mathrm{Q}^0)$$

□

## 1.7 Groups of type FL

Let $R$ be any ring with identity. A (left) $R$-module $M$ is said to be *free* if it can be written as a direct sum of copies of $R$. $M$ is said to be a *projective* (left) $R$-module if it is a direct summand of a free $R$-module.
A *resolution* of a left $R$-module $M$ is a long exact sequence of left $R$-modules $\mathbf{C} = \{C_i\}_{i \geq 0}$ together with an epimorphism $\epsilon : C_0 \to M$:

$$\cdots \to C_2 \to C_1 \to C_0 \xrightarrow{\epsilon} M \to 0$$

If the resolution $\mathbf{C}$ consists of free (resp. projective) modules then it is said to be a *free* (resp. *projective*) resolution. A group $\Gamma$ is said to be *of type FL* if the $\Gamma$−module $\mathcal{Z}$ has a finite free resolution. Following Serre [Ser], we obtain the following result for groups of type FL:

**Lemma 1.19** *(i) If $\Gamma_1$ and $\Gamma_2$ are of type FL, then so is $\Gamma_1 * \Gamma_2$ and, moreover,* if $\chi(\Gamma) = \sum_i (-1)^i \mathrm{rg}_\Gamma(L_i)$ for $L_i$ a finite free resolution,

$$\chi(\Gamma_1 * \Gamma_2) = \chi(\Gamma_1) + \chi(\Gamma_2) - 1$$

*(ii) If $K$ and $G/K$ are both of type FL, then $G$ is also of type FL and*

$$\chi(G) = \chi(K)\chi(G/K)$$

*(iii) If $D$ is of type FL, and $D^0$ is a subgroup of finite index in $D$, then $D^0$ is also of type FL and moreover*

$$\chi(D^0) = [D : D^0]\chi(D)$$

The importance of groups of type FL to topologists is explained by the following correlation to Eilenberg-MacLane spaces:

Given a group $G$, a path-connected space $Y$ is said to be an *Eilenberg-MacLane space* $K(G,n)$ if

$$\pi_m(Y) = \begin{cases} G & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

As an example, observe that since $\pi_q(S^1) = 0$ for $q \geq 2$ and $\pi_1(S^1) = C_\infty$ it follows that the circle is a $K(C_\infty, 1)$. In the category of CW-complexes such a space exists and is unique up to homotopy equivalence. Wall proved that every finitely presented group $G$ of type FL has a finite $K(G,1)$ (see [Wal]). Similarly we may ask when there exists a *manifold* of type $K(G,1)$:

**Theorem 1.20 (F.E.A. Johnson [Joh1])** *Let $G$ be a group. Then there exists an $n$–manifold of type $K(G,1)$ for some $n$ if and only if $G$ is countable and has finite cohomological dimension.*

In particular it is shown that we may choose the Eilenberg-MacLane space to be locally compact.

# Chapter 2

# Extensions of free products of torsion-free Fuchsian groups

In this chapter we shall consider group extensions where the kernel and quotient correspond to free products of free groups and surface groups (both orientable and nonorientable). We shall assume throughout that the number of summands in the free product is finite. Our aim is to show that any group can be constructed by at most finitely many group extensions of this type. This rigidity theorem has the important corollary that the group of all automorphisms of the extension which leave the kernel invariant has finite index in the automorphism group of the extension. We shall make frequent use of this fact in later chapters.

The first result we prove is a Riemann-Hurwitz type formula for finite free products of torsion-free Fuchsian groups. This generalises the usual Riemann-Hurwitz formula 1.2 which corresponds to the case $n = 1$.

**Proposition 2.1** *Let $D$ be a finite free product of $n$ torsion-free Fuchsian groups and let $D^0$ be a subgroup of finite index in $D$ with $[D : D^0] = j$. Then*

$$j \leq \frac{rk(D^0) - \delta}{rk(D) - \delta}$$

*where $\delta = n$ if $D$ contains a free group as a summand, and $\delta = n+1$ if no summands of $D$ are free groups.*

**Proof:** Let $\{G_\alpha\}$ be a finite family of torsion-free Fuchsian groups, and define a free product structure $D = (G, \{i_\alpha\})$ on this family. We have stated in Chapter 1 that all free product structures on a family of groups are isomorphic, thus we may fix the order of the summands and take $D$ to be of the form

$$D = F_{k_0} * \Sigma^+_{k_1} * \ldots * \Sigma^+_{k_r} * \Sigma^-_{k_{r+1}} * \ldots * \Sigma^-_{k_{r+s}}$$

where $F_*, \Sigma^+_*, \Sigma^-_*$ denote free groups, orientable surface groups and nonorientable surface groups respectively. Note that since free groups are closed in the category of free products of groups, we may take a single free summand in the free product.

Let $n$ be the total number of summands in the free product $D$ and define $\delta$ to be $n$ if $k_0 \neq 0$ and $n+1$ otherwise. This $\delta$ will play the same role the cohomological dimension played in the Riemann-Hurwitz formula 1.2 . By repeated use of lemma 1.19 we have that

$$\begin{aligned}
\chi(D) &= \chi(F_{k_0}) + \chi(\Sigma^+_{k_1}) + \ldots + \chi(\Sigma^-_{k_{r+s}}) - (r+s) \\
&= (1-k_0) + (2-2k_1) + \ldots + (1-k_{r+s}) - (r+s) \\
&= r + 1 - k_0 - 2\sum_{i=1}^{r} k_i - \sum_{j=r+1}^{r+s} k_j
\end{aligned}$$

Similarly, using the corollary to the Grushko-Neumann theorem 1.4,

$$\begin{aligned}
rk(D) &= rk(F_{k_0}) + rk(\Sigma^+_{k_1}) + \ldots + rk(\Sigma^-_{k_{r+s}}) \\
&= k_0 + 2\sum_{i=1}^{r} k_i + \sum_{j=r+1}^{r+s} k_j + s
\end{aligned}$$

and together these yield that

$$rk(D) = -\chi(D) + r + s + 1 \tag{2.1}$$

This may be rewritten in terms of the constant $\delta$ defined earlier as follows

$$rk(D) = -\chi(D) + \delta$$

since if $D$ has no free summand then $n = r + s$ and $n = r + s + 1$ otherwise. Similarly we have that the subgroup $D^0$ with $n'$ summands, satisfies

$$rk(D^0) = -\chi(D^0) + \delta'$$

where $\delta' = n'$ or $n' + 1$.

By examining the Kurosh subgroup theorem (Proposition 1.6) for the finite index case, we observe that $n' \geq n$ since the summands of $D^0$ range over a set $(\Lambda)$ which has $n$ elements, therefore $\delta' \geq \delta$. Lemma 1.19 gives the relationship

$$\chi(D^0) = j(\chi(D))$$

where $j$ is the index of $D^0$ in $D$; from this we obtain

$$\frac{rk(D^0) - \delta'}{rk(D) - \delta} = j$$

The result follows by the observation that $\delta' \geq \delta$. □

## 2.1 $\mathcal{K} - \mathcal{Q}$-Factorisations

Let $\mathcal{K}, \mathcal{Q}$ be classes of groups and let $G$ be some finitely generated group. By a $\mathcal{K} - \mathcal{Q}$*-factorisation* of $G$, denoted $K$, we mean a normal subgroup $K$ of $G$ for which $K \in \mathcal{K}$ and $G/K \in \mathcal{Q}$. The isomorphism class of $G/K$ is the *quotient type* of the factorisation. Our aim is to show that with suitable restrictions on the classes $\mathcal{K}$ and $\mathcal{Q}$, a given group $G$ has at most finitely many $\mathcal{K} - \mathcal{Q}$-factorisations.

Let $\widehat{\mathcal{D}}$ denote the class of all finite free products of torsion-free Fuchsian groups, and let $\widehat{\mathcal{D}}^n$ denote the class of iterated extensions of finite free products of torsion-free Fuchsian groups; that is, $\widehat{\mathcal{D}}^n$ is the class of groups formed by extensions

$$1 \to D^{n-1} \to G \to D \to 1$$

where $D^{n-1} \in \widehat{\mathcal{D}}^{n-1}$ and $D \in \widehat{\mathcal{D}}$.

We shall show in particular that a group $G$ has at most finitely many $\widehat{\mathcal{D}}^{n-1} - \widehat{\mathcal{D}}$-factorisations. In this section we shall give a suitable set of conditions on the class of quotient groups $\mathcal{Q}$ and prove that these are satisfied by the class $\widehat{\mathcal{D}}$ of all finite free products of torsion-free Fuchsian groups.

Define $\rho(G) = rk(H_1(G; \mathcal{Z}))$. Clearly $\rho(G) \leq rk(G)$.

**Conditions on the class $\mathcal{Q}$**

**Q1:** If $Q \in \mathcal{Q}$ and $Q^0$ is a subgroup of finite index in $Q$, then $Q^0 \in \mathcal{Q}$.

**Q2:** Let $Q \in \mathcal{Q}$ and let $Q^0 \subset Q$ be a subgroup of finite index. Then $\rho(Q^0) \geq \rho(Q)$ with equality if and only if $Q^0$ is isomorphic to $Q$.

**Q3:** If $Q' \subset Q$ is a nontrivial normal subgroup of infinite index, then $Q'$ is infinitely generated.

**Q4:** Each $Q \in \mathcal{Q}$ is finitely generated and of type FL.

**Q5:** For all $Q \in \mathcal{Q}$, $\chi(Q) \neq 0$.

**Q6:** For all $Q \in \mathcal{Q}$, there exist subgroups of $Q$ with arbitrarily large index.

**Q7:** Given a finitely generated group $G$, then the number of distinct isomorphism types of groups in $\mathcal{Q}$ onto which $G$ can map epimorphically is finite.

We shall call a class of groups a *Riemann-Hurwitz class* if it satisfies conditions **Q1** to **Q7**. The name is derived from the fact that we require a

Riemann-Hurwitz type formula for this class in order to prove our finiteness results.

**Theorem 2.2** *The class $\widehat{\mathcal{D}}$ of finite free products of torsion-free Fuchsian groups satisfies conditions* **Q1** *to* **Q7** *and so is a Riemann-Hurwitz class.*

**Proof of Q1 :** We need to prove that $\widehat{\mathcal{D}}$ is closed with respect to subgroups of finite index. The proof relies heavily on the form of a subgroup of finite index $D^0$ in the free product $D = *_{\lambda \in \Lambda} D_\lambda$ as given by the Kurosh subgroup theorem 1.6; precisely,

$$D^0 = F_k * (*_{\lambda, d_\lambda}(D^0 \cap (d_\lambda D_\lambda d_\lambda^{-1})$$

where $F_k$ is a free group of *finite* rank and $d_\lambda$ ranges over a set of $(D^0, D_\lambda)$-double coset representatives. The total number of double cosets $D^0 x D_\lambda$ is finite since $D^0$ has finite index in $D$ and so $D^0$ is indeed a *finite* free product. Hence it suffices to show that each $D^0 \cap d_\lambda D_\lambda d_\lambda^{-1} \in \mathcal{D}$:

**Sub-lemma 2.3** *The summand $D^0 \cap d_\lambda D_\lambda d_\lambda^{-1}$ is a subgroup of finite index in $D_\lambda$.*

**Proof :** Since $d_\lambda D_\lambda d_\lambda^{-1} \cong D_\lambda$, we shall consider the isomorphic subgroup $D^0 \cap D_\lambda$ and show that this has finite index in $D_\lambda$. The finite index lemma 1.10 states that a subgroup $D^0$ of finite index in a finitely-generated group $D$ contains a subgroup $D_N^0$ which is normal in $D$. Thus the lemma will follow if we can show that $D_N^0 \cap D_\lambda$ has finite index in $D_\lambda$. Clearly

$$D_\lambda / D_\lambda \cap D_N^0 \cong D_\lambda D_N^0 / D_N^0$$

and this is a subgroup of $D/D_N^0$ which is finite and so $[D_\lambda : D_\lambda \cap D_N^0]$ is finite. □

Using these lemmas, we have shown that the subgroup $D^0$ of finite index in $D$ is a finite free product where each summand is a subgroup of finite index in a torsion-free Fuchsian group. As a finite index subgroup of a torsion-free Fuchsian group is again a torsion-free Fuchsian group, we have that $D^0$ is a member of $\widehat{\mathcal{D}}$. Hence $\widehat{\mathcal{D}}$ is closed under subgroups of finite index and so satisfies condition **Q1**. $\square$

**Proof of Q2 :** The proof essentially follows from Proposition 2.1. This generalised the Riemann-Hurwitz formula to subgroups $D^0$ of finite index $m$ in a free product of $n$ torsion-free Fuchsian groups, denoted $D$, giving the formula

$$m \leq \frac{rk(D^0) - \delta}{rk(D) - \delta}$$

where $\delta = n$ if $D$ contains a free group as a summand, and $n+1$ otherwise. Rearranging we obtain

$$rk(D^0) \geq rk(D) + (m-1)(rk(D) - \delta)$$

Each summand has rank $\geq 2$ and so $rk(D) \geq 2n \geq n+1 \geq \delta$. Also the index $[D : D^0] = m \geq 1$ and so the term $(m-1)(rk(D) - \delta)$ is non-negative. Hence

$$rk(D^0) \geq rk(D)$$

Observe further that equality is obtained either when $m = 1$ so that $D^0$ is not a proper subgroup, or when $rk(D) = 2n = n+1 = \delta$. These equalities imply that $n = 1$ and $\delta = 2$ so that $D$ is a surface group with rank $= 2$ which is impossible under our restrictions. Hence equality is only obtained if $D^0 = D$. Since $\rho(D) \leq rk(D)$ the condition **Q2** holds for the class $\widehat{\mathcal{D}}$. $\square$

**Proof of Q3 :** Let $Q \in \widehat{\mathcal{D}}$ and let $Q'$ be a nontrivial normal subgroup of infinite index in $Q$. Here the proof splits into two cases: suppose first that

$Q$ is a free product of at least two nontrivial groups. In this case we use a theorem of B. Baumslag [Bau]:

**Theorem 2.4 (B. Baumslag)** *Let $G$ be the free product of two nontrivial groups. If a finitely generated subgroup $H$ contains a nontrivial normal subgroup of $G$ then $H$ has finite index in $G$.*

It immediately follows that if $Q'$ is a finitely generated normal subgroup of $Q$ then $Q'$ has finite index in $Q$ or, in other words, normal subgroups of infinite index in $Q$ are infinitely generated as required.

At this point it is worth observing that Baumslag's theorem together with our generalised Riemann-Hurwitz formula 2.1 for free products yield the following generalisation of the original Riemann-Hurwitz theorem to free products of torsion-free Fuchsian groups:

**Theorem 2.5 (Generalised Riemann-Hurwitz theorem)** *If $G$ is a free product of finitely many torsion-free Fuchsian groups containing $N$, a finitely generated normal subgroup, then $N$ has finite index $j$ in $G$ satisfying*

$$j \leq \frac{rk(N) - \delta}{rk(G) - \delta}$$

*where $\delta = n$ if $G$ contains a free group as a summand, and $\delta = n + 1$ if no summands of $G$ are free groups.*

The second case occurs when $Q$ consists of a single torsion-free Fuchsian group. Then a normal subgroup of infinite index is a free group of infinite rank and so is clearly infinitely generated. Hence in either case we have that a nontrivial normal subgroup of infinite index in $Q$ is infinitely generated as required. This proves condition **Q3**. $\square$

**Proof of Q4 :** It is clear from the definition that finite free products of finitely generated groups are again finitely generated. The fact that free products of groups of type FL are also of type FL follows from Lemma 1.19. Since surface groups and free groups are both finitely generated and of type FL, the above observations show that all groups in the class $\widehat{\mathcal{D}}$ are also finitely generated and of type FL, giving condition **Q4**.

**Proof of Q5 :** Writing the general form of a group $D \in \widehat{\mathcal{D}}$ as

$$D = F_{k_0} * \Sigma^+_{k_1} * \ldots * \Sigma^+_{k_r} * \Sigma^-_{k_{r+1}} * \ldots * \Sigma^-_{k_{r+s}},$$

the Euler characteristic becomes

$$\chi(D) = r + 1 - k_0 - 2\sum_{i=1}^{r} k_i - \sum_{j=r+1}^{r+s} k_j.$$

As we are only considering Fuchsian groups for which $k_0, k_i$ and $k_j$ are all $\geq 2$, it follows that $\chi(D) \leq -1 - 3r - 2s$. Hence $\chi(D) \leq -1$ (since $r, s \geq 0$) for all $D \in \widehat{\mathcal{D}}$, giving condition **Q5**. $\Box$

**Proof of Q6 :** Let $G$ be an arbitrarily large finite group and let $\pi$ denote the projection $\pi : D \to G$ where $D \in \widehat{\mathcal{D}}$. Suppose that $D^0$ is the kernel of $\pi$. Then $D^0$ is a normal subgroup of $D$ such that $D/D^0 \cong G$. $\widehat{\mathcal{D}}$ is closed with respect to subgroups of finite index by condition **Q1**, and so $D^0 \in \widehat{\mathcal{D}}$ with $[D : D^0] = |G|$ which is arbitrarily large.

**Corollary 2.6** *A group $D \in \widehat{\mathcal{D}}$ has a subgroup $D^0$ with arbitrarily large Euler characteristic.*

**Proof of Q7 :** Given a finitely generated group $G$ of rank $r$, we must prove that the number of distinct isomorphism types of $\widehat{\mathcal{D}}$ onto which $G$ can map epimorphically is finite. Consider the number of groups $D \in \widehat{\mathcal{D}}$ with a given rank $r'$. Each summand of $D$ has rank at least 2 and so there are at most

$r'/2$ summands. Further there are three types of summands (free, surface orientable and nonorientable) and each summand has rank $\leq r'$. Hence the number of groups with rank $= r'$ is bounded above by $(3r')^{r'/2}$. Now, $G$ has rank $r$ and the number of epimorphic images of $G$ is given by the number of groups in $\widehat{\mathcal{D}}$ with rank $\leq r$ which is finite. This proves condition **Q7** and so we have proven that $\widehat{\mathcal{D}}$ is a Riemann-Hurwitz class. $\square$

A $\mathcal{K}-\mathcal{Q}$-factorisation of a group $G$ is said to be *stable* if $\rho(K) < \rho(G/K)$. Using the properties of the Riemann-Hurwitz class $\mathcal{Q}$, we now prove a uniqueness result for stable $\mathcal{K}-\mathcal{Q}$-factorisations. This will be essential in proving the Rigidity theorem in the next section.

**Lemma 2.7 (Stability Lemma)** *Let $G$ be a group and let $\mathcal{Q}$ be a Riemann-Hurwitz class of groups. Suppose that $\mathcal{K}$ is a class of finitely generated groups. Then given two stable $\mathcal{K}-\mathcal{Q}$-factorisations $K_1, K_2$ associated to the same quotient type $Q \in \mathcal{Q}$ so that*

$$\rho(K_i) < \rho(Q) \quad for\ i = 1, 2$$

*then $K_1 = K_2$.*

**Proof :** Given the projections $p_i : G \to G/K_i$ for $i = 1, 2$, we have that $p_1(K_2)$ is a normal subgroup of $G/K_1 \cong Q$; so, *a priori*, either
(A) $p_1(K_2)$ has finite index in $G/K_1$ ; or
(B) $p_1(K_2)$ is nontrivial with infinite index in $G/K_1$ ; or
(C) $p_1(K_2) = 1$.
Suppose (A) holds implying that $p_1(K_2) \in \mathcal{Q}$ by **Q1**. Note first that since abelianisation preserves surjectivity, the induced map in homology

$$(p_1)_* : H_1(K_2; \mathcal{Z})) \twoheadrightarrow H_1(p_1(K_2); \mathcal{Z})$$

is surjective so that $\rho(p_1(K_2)) \leq \rho(K_2))$. Since $p_1(K_2)$ has finite index in $G/K_1 \cong Q$ then by condition **Q2**, $\rho(Q) \leq \rho(p_1(K_2))$. Hence $\rho(Q) \leq \rho(K_2)$ contradicting the stability assumption.

If (B) holds then $p_1(K_2)$ is nontrivial with infinite index in $Q$. Using condition **Q3**, we have that $p_1(K_2)$ is infinitely generated and hence that $H_1(p_1(K_2; \mathcal{Z}))$ is infinitely generated. This gives a contradiction since $H_1(K_2; \mathcal{Z})$ is finitely generated and the induced map in homology $(p_1)_*$ is surjective.

By exhaustion, we have that $p_1(K_2) = 1$, which implies that

$$K_2 \subset \ker p_1 = K_1$$

By repeating the above argument with $p_1$ instead of $p_2$, and $K_2$ instead of $K_1$, we obtain the opposite inclusion. Hence $K_1 = K_2$. □

## 2.2 Rigidity for extensions of finite free products of torsion-free Fuchsian groups

We wish to show that any group $G$ has at most finitely many $\mathcal{K} - \mathcal{Q}$-factorisations given certain restrictions on the classes $\mathcal{K}$ and $\mathcal{Q}$. This section considers the problem of finding sufficient conditions on the class of groups $\mathcal{K}$ belonging to the kernel of the extension, in order for the result to hold. After stating the conditions, we go on to prove that they are satisfied by the class $\widehat{\mathcal{D}}^n$ of iterated extensions of finite free products of torsion-free Fuchsian groups. The Rigidity theorem which states that there are only finitely many $\mathcal{K} - \mathcal{Q}$-factorisations, is proved at the end of this section.

**Conditions on the class $\mathcal{K}$**

**K1**: Each $K \in \mathcal{K}$ is finitely generated and of type FL.

**K2**: For each integer m, $\rho(G) = rk(H_1(G; \mathcal{Z}))$ is bounded on the class

$$\mathcal{K}_m = \{K \in \mathcal{K} : \chi(K) = m\}.$$

**K3**: $\mathcal{K}$ is closed under isomorphism.

If a class of groups $\mathcal{K}$ satisfies conditions **K1** to **K3**, then we shall call $\mathcal{K}$ a *controlled class.* This name is derived from the fact that we need to control the number of elements in this class with a given rank in order to prove the Rigidity theorem.

Consider a sequence of subgroups $(G_r)_{0 \leq r \leq n}$ of a group $G$ satisfying:

(i) $\{1\} = G_0 \subset G_1 \subset \ldots \subset G_n = G$; and

(ii) $G_r \lhd G_{r+1}$ and $G_{r+1}/G_r \in \widehat{\mathcal{D}}$ for each r.

Then we say that $G$ has a *poly-$\widehat{\mathcal{D}}$ filtration* of length $n$ and we denote the class of all such groups by $\widehat{\mathcal{D}}^n$.

**Theorem 2.8** *The class $\widehat{\mathcal{D}}^n$ of iterated extensions of finite free products of torsion-free Fuchsian groups is a controlled class.*

**Proof :** The proof comes down to showing that the class $\widehat{\mathcal{D}}^n$ satisfies the above conditions. It is clear from the definition that finite free products and extensions of finitely generated groups are again finitely generated. The fact that free products and extensions of groups of type FL are also of type FL follows from Lemma 1.19. Since surface groups and free groups are both finitely generated and of type FL, the above observations show that all groups in the class $\widehat{\mathcal{D}}^n$ are also finitely generated and of type FL, giving condition **K1**. The class $\mathcal{D}$ is closed under isomorphism. Given a set of groups, there is a unique free product up to isomorphism and so it is evident that the class $\widehat{\mathcal{D}}^n$

is also closed under isomorphism. Hence it is enough to prove condition **K2**.

**Proof of K2 :** Since $\rho(D) = rk(H_1(D;\mathcal{Z})) \leq rk(D)$, it suffices to show that the class

$$\widehat{\mathcal{D}}^n_m = \{D \in \widehat{\mathcal{D}}^n : \chi(D) = -m\}$$

has bounded rank.

**Proposition 2.9** *If $D \in \widehat{\mathcal{D}}$ then*

$$rk(D) \leq \lfloor 3/2(1 + |\chi(D)|)\rfloor$$

*where $\lfloor x \rfloor$ denotes the integer part of $x$.*

**Proof :** As we observed in Proposition 2.1, any group $D \in \mathcal{D}$ may be written in the form:

$$D = F_{k_0} * \Sigma^+_{k_1} * \ldots * \Sigma^+_{k_r} * \Sigma^-_{k_{r+1}} * \ldots * \Sigma^-_{k_{r+s}}$$

and by comparing the rank and the Euler characteristic of $D$ we obtain the relation 2.1

$$rk(D) = -\chi(D) + r + s + 1.$$

In order to obtain the result, it is necessary to find a bound on the value of $r + s$ in terms of $\chi(D)$, observing that

$$\chi(D) = r + 1 - k_0 + 2\sum_{i=1}^{r} k_i + \sum_{i=r+1}^{s} k_j.$$

The surfaces we are considering satisfy $k_i, k_j \geq 2$ and so

$$\chi(D) \leq r + 1 - 2(2r) - 2s$$

and upon rearranging we have

$$2(r + s) \leq |\chi(D)| - r + 1.$$

Since $r \geq 0$ and the value of $r + s$ is an integer this gives

$$r + s \leq \lfloor 1/2(|\chi(D)| + 1) \rfloor$$

Combining with our earlier equation 2.1 gives the desired result. □

**Remarks**

(i) If $\chi(D) = -(2k - 1)$, then the bound becomes $rk(D) \leq 3k$ and this is attained by $D = \Sigma_2^- \underbrace{* \ldots *}_{k} \Sigma_2^-$.

(ii) If $\chi(D) = -2k$ then the bound becomes $rk(D) \leq 3k + 1$ and this is attained by $D = \Sigma_2^- \underbrace{* \ldots *}_{k-1} \Sigma_2^- * \Sigma_2^+$.

(iii) For convenience, we shall use the bound $rk(D) \leq 2 + 3/2|\chi(D)|$.

**Proposition 2.10** *If $G \in \hat{\mathcal{D}}^n$ then $rk(G) \leq (2 + 3/2|\chi(G)|)^n$.*

**Proof :** Let $(G_r)_{0 \leq r \leq n}$ be a poly-$\hat{\mathcal{D}}$ filtration on $G$ and write $Q_i = G_i/G_{i+1}$. Since $Q_i \in \hat{\mathcal{D}}$ we know that $rk(G) \leq 2 + 3/2|\chi(G)|$ from the above remarks. Given a group extension $(1 \to K \to H \to Q \to 1)$, then $rk(H) \leq rk(K)rk(Q)$ and so

$$\begin{aligned} rk(G) &\leq rk(Q_1) \ldots rk(Q_n) \\ &\leq (2 + 3/2|\chi(Q_1)|) \ldots (2 + 3/2|\chi(Q_n)|) \end{aligned}$$

Applying Lemma 1.19 (ii) repeatedly to our exact sequence, we have that $\chi(G) = \chi(Q_1) \ldots \chi(Q_n)$ and so, for each $i$, $|\chi(Q_i)| \leq |\chi(G)|$ which allows us to obtain the required inequality. □

Using the above proposition and the observation that for all $G$, $\rho(G) \leq rk(G)$, we see that $\rho$ is bounded on the class $\hat{\mathcal{D}}^n$ and hence it is clearly bounded on the class

$$\hat{\mathcal{D}}^n_m = \{D \in \hat{\mathcal{D}}^n : \chi(D) = -m\}.$$

This proves that the class $\widehat{\mathcal{D}}^n$ satisfies condition **K2** and hence we have shown that $\widehat{\mathcal{D}}^n$ is a controlled class. □

**Theorem 2.11 (The Rigidity theorem)** *Let $\mathcal{K}$ be a controlled class of groups and let $\mathcal{Q}$ be a Riemann-Hurwitz class of groups. Then every group admits at most a finite number of $\mathcal{K} - \mathcal{Q}$-factorisations.*

**Proof :** First, observe that if a group is not finitely generated then it admits no $\mathcal{K} - \mathcal{Q}$-factorisation, and so we may assume that all our groups are finitely generated. From now on, we shall fix a group $G$ and a quotient type $Q$ and let $(K_\lambda)_{\lambda\in\Lambda}$ be a collection of $\mathcal{K} - \mathcal{Q}$-factorisations of $G$ all with the quotient type $Q$, so that for all $\lambda \in \Lambda$, $G/K_\lambda \cong Q$.

For each $\lambda$ choose an isomorphism

$$h_\lambda : G/K_\lambda \to Q.$$

Since $K_\lambda$ and $Q$ are both finitely generated and of type FL using the conditions **K1** and **Q4** respectively, then by Lemma 1.19, the same is also true for $G$ and further, for all $\lambda$

$$\chi(G) = \chi(K_\lambda)\chi(Q).$$

Condition **Q5** ensures that $\chi(Q) \neq 0$ and so $\chi(K_\lambda)$ has a constant value for all $\lambda \in \Lambda$. Invoking condition **K2**, which states that the set $\{\rho(G) : G \in \mathcal{K}_m\}$ is bounded on the class $\mathcal{K}_m = \{K \in \mathcal{K} : \chi(K) = m, m \in \mathcal{Z}\}$; we see that $\rho$ is bounded on the class $(K_\lambda)_{\lambda\in\Lambda}$, and hence that

$$\{\rho(K_\lambda) : \lambda) \in \Lambda\} \quad \text{is a finite set.}$$

Write $R = \max\{\rho(K_\lambda : \lambda \in \Lambda\}$. Since $Q$ has subgroups of arbitrarily large index (**Q6**),and since for any proper subgroup of finite index $Q' \subset Q$ we have that $\rho(Q') > \rho(Q)$ (**Q2**), we may choose a subgroup of $Q$ with arbitrarily large $\rho$. Let $Q^0$ be a subgroup of some finite index $j$ in $Q$ for which $\rho(Q^0) > R$, so that for all $\lambda \in \Lambda$,

$$\rho(K_\lambda) < \rho(Q^0). \tag{2.2}$$

Let $p_\lambda$ be the projection $p_\lambda : G \to G/K_\lambda$ and define

$$G_\lambda = p_\lambda^{-1} h_\lambda^{-1}(Q^0).$$

Then $G_\lambda$ is a subgroup of index $j$ in $G$. As $G$ is finitely generated, it has only finitely many subgroups of index $j$, so we shall write $H_1, \ldots, H_M$ for the distinct subgroups arising from some $G_\lambda$. Partition $\Lambda$ into equivalence classes $\Lambda_1, \ldots, \Lambda_M$ by the requirement:

$$\lambda \in \Lambda_i \text{ if, and only if } G_\lambda = H_i$$

For each $\lambda$ belonging to some $\Lambda_i$, we have $G_\lambda/K_\lambda \cong Q^0$ and by equation 2.2 ,we see that $K_\lambda$ is a stable $\mathcal{K} - \mathcal{Q}$-factorisation of $G_\lambda$. Hence, using the Stability Lemma 2.7, $\Lambda_i$ consists of a single element and so the set $\Lambda = \{\Lambda_1, \ldots, \Lambda_M\}$ is finite. This proves that $G$ has only finitely many $\mathcal{K} - \mathcal{Q}$-factorisations *with a given quotient type* $Q$.

Now, given a finitely generated group $G$, condition **Q7** tells us that the number of distinct isomorphism types of groups in $\mathcal{Q}$ onto which $G$ can map epimorphically is finite, and so there are only finitely many quotient types for $G$. This proves the Rigidity theorem. □

## 2.3 Consequences of rigidity

In this section we shall show that the set of all poly-$\widehat{\mathcal{D}}$ filtrations of a group is finite. By applying the Rigidity theorem 2.11 we conclude that any group $G$ has at most a finite number of $\widehat{\mathcal{D}}^n - \widehat{\mathcal{D}}$-factorisations.

Given a group $G$, denote the set of poly-$\widehat{\mathcal{D}}$ filtrations of length $n$ by $F_n(G)$ and let $F(G)$ be the set of all poly-$\widehat{\mathcal{D}}$ filtrations of $G$. Clearly, $F(G) = \bigcup_{n\geq 1} F_n(G)$.

**Proposition 2.12** *For any group $G$, $F(G)$ is a finite set.*

**Proof :** We have demonstrated that the class of finite free products of torsion-free Fuchsian groups $\widehat{\mathcal{D}}$ is a Riemann-Hurwitz class and also that $\widehat{\mathcal{D}}^r$ is a controlled class. Hence the Rigidity theorem proves that there are at most finitely many $\widehat{\mathcal{D}}^r - \widehat{\mathcal{D}}$-factorisations of a group $G$. This proves that the set $F_{r+1}(G)$ is finite. If $G$ has infinite cohomological dimension, then it has no poly-$\widehat{\mathcal{D}}$ filtrations, so suppose that $G$ has finite cohomological dimension $cd(G) = k$. In this case $F_n(G) = \emptyset$ when $n \geq k$ and so $F(G) = \bigcup_{r=1}^{k} F_r(G)$ which we have shown is a finite set. □

The following set of corollaries will be used in the proofs of theorems in later chapters and served as the motivation for attempting to prove the Rigidity theorem for the class $\widehat{\mathcal{D}}$. First we require a further definition.

Let $\mathcal{G} = (G_r)_{0\leq r\leq n}$ be a poly-$\widehat{\mathcal{D}}$ filtration on a group $G$. Then the automorphism group of $G$, denoted Aut$(G)$, has a subgroup Aut$(\mathcal{G})$ consisting of all the automorphisms $\alpha$ $\in$Aut$(G)$ such that $\alpha(G_r) = G_r$ for each r, $0 \leq r \leq n$.

**Corollary 2.13** *Let $\mathcal{G} = (G_r)_{0\leq r\leq n}$ be a poly-$\widehat{\mathcal{D}}$ filtration on a group $G$. Then $Aut(\mathcal{G})$ is a subgroup of finite index in $Aut(G)$.*

**Proof :** Let $G$ have a poly-$\widehat{\mathcal{D}}$ filtration $\mathcal{G} = (G_r)_{0\leq r\leq n}$ and consider an automorphism $\alpha \in$ Aut$(G)$. The image of $\mathcal{G}$ under an automorphism of $G$ is again a poly-$\widehat{\mathcal{D}}$ filtration given by

$$\alpha(\mathcal{G}) = (\alpha(G_r))_{0\leq r\leq n}$$

and so the orbit of $\mathcal{G}$ under $\alpha$ is contained in $F(G)$ which by the above is a finite set. Hence

$$\begin{aligned} \mathrm{Stab}_{\mathrm{Aut}(G)}(\mathcal{G}) &= \{\alpha \in \mathrm{Aut}(G) : \alpha(\mathcal{G}) = (\mathcal{G})\} \\ &= \mathrm{Aut}(\mathcal{G}) \end{aligned}$$

has finite index in $\mathrm{Aut}(G)$. □

In particular we have the following rigidity theorems for poly-surface and poly-free groups:

**Theorem 2.14 (Rigidity of length 2 poly-surface/poly-free groups)** *Let $\mathcal{E}$ be the group extension*

$$\mathcal{E} = (1 \to K \to G \to Q \to 1)$$

*where $K$ and $Q$ are both either fundamental groups of orientable surfaces with genus $\geq 2$ or free groups with rank $\geq 2$. Then the group of automorphisms* $\mathrm{Aut}(\mathcal{E}) = \{\alpha \in \mathrm{Aut}(G)\colon \alpha(K) = K\}$ *is a subgroup of finite index in* $\mathrm{Aut}(G)$.

This gives us a useful corollary for the virtual cohomological dimension of the *outer automorphism group* of the extension which justifies the use of the word rigidity. Recall that the outer automorphism group of $G$ is the quotient group

$$\mathrm{Out\ (G)} = \frac{\mathrm{Aut\ (G)}}{\mathrm{Inn\ (G)}}$$

**Proposition 2.15** *Given an extension of torsion-free Fuchsian groups $K$ and $Q$, $\mathcal{E} = (1 \to K \to G \to Q \to 1)$, the outer automorphism group of the extension* Out ($\mathcal{E}$) *is a subgroup of finite index in* Out (G).

**Proof :** The Rigidity theorem states that for the above extension, Aut ($\mathcal{E}$) is a subgroup of finite index in Aut (G); thus, using the finite index lemma

1.10, Aut (G) has a characteristic subgroup $\mathrm{Aut}_0$ $(\mathcal{E})$ of finite index which is contained in Aut $(\mathcal{E})$. As $K$ is a normal subgroup of $G$, the group of inner automorphisms of the extension

$$\mathrm{Inn}\ (\mathcal{E}) = \{\alpha \in \mathrm{Inn}\ (\mathrm{G}) : \alpha(\mathrm{K}) = \mathrm{K}\}$$

is equal to Inn (G). Hence

$$\begin{aligned} \frac{\mathrm{Aut}\ (\mathrm{G})}{\mathrm{Aut}_0\ (\mathcal{E})} &\cong \frac{\mathrm{Aut}\ (\mathrm{G})/\mathrm{Inn}\ (\mathrm{G})}{\mathrm{Aut}_0\ (\mathcal{E})/\mathrm{Inn}\ (\mathrm{G})} \\ &\cong \frac{\mathrm{Out}\ (\mathrm{G})}{\mathrm{Out}_0\ (\mathcal{E})} \end{aligned}$$

which is a finite group proving that $\mathrm{Out}_0$ $(\mathcal{E})$ has finite index in Out (G) and therefore Out $(\mathcal{E})$ is a subgroup of finite index in $\mathrm{Out}_0$ $(G)$. □

**Corollary 2.16 (Rigidity)** *Given the above exact sequence $\mathcal{E}$, then*

$$\mathrm{vcd}\ (\mathrm{Out}\ (\mathcal{E})) = \mathrm{vcd}\ (\mathrm{Out}\ (\mathrm{G}))$$

**Proof :** If Out (G) is virtually torsion-free, then it has a torsion-free subgroup of finite index $\mathrm{Out}_0$ $(G)$. Let $\mathrm{Out}_0\ (\mathcal{E}) = \mathrm{Out}\ (\mathcal{E}) \cap \mathrm{Out}_0\ (\mathrm{G})$ so that $\mathrm{Out}_0$ $(\mathcal{E})$ is torsion-free. By Poincaré's lemma, the intersection of finitely many subgroups of finite index also has finite index, so $\mathrm{Out}_0$ $(\mathcal{E})$ has finite index in Out (G) and is torsion-free. Therefore, $\mathrm{cd}\ (\mathrm{Out}_0\ (\mathcal{E})) = \mathrm{cd}\ (\mathrm{Out}_0\ (\mathrm{G}))$.

If Out (G) is not virtually torsion-free then all subgroups of finite index have torsion so $\mathrm{vcd}\ (\mathrm{Out}\ (\mathrm{G})) = \mathrm{vcd}\ (\mathrm{Out}\ (\mathcal{E})) = \infty$. □.

# Chapter 3

# Extending the Baer-Nielsen theorem to surface fibrations

In this chapter we shall consider closed 2–manifolds with genus $g \geq 2$. These are known as hyperbolic surfaces since their universal cover is the hyperbolic disc and in many respects they represent generic surfaces, the only orientable exceptions to this class being the sphere and torus. By considering fibrations with fibre and base space given by (hyperbolic) surfaces we construct the surface fibrations of the title. In the 1920's, Baer and Nielsen proved that the natural homomorphism from the group of diffeomorphisms of a surface to the outer automorphism group of its fundamental group is surjective and has a kernel given by the diffeomorphisms homotopic to the identity. This may be written algebraically as

$$\pi_0(\text{Diff }(\Sigma^g)) \cong \text{Out }(\pi_1(\Sigma^g))$$

where $\Sigma^g$ denotes a surface of genus $g$ (see Epstein [Eps] for a modern treatment of this work). Waldhausen ([Wald]) later generalised this theorem to the class of sufficiently large 3–manifolds which includes Stallings fibrations (surface fibrations over the circle). These shall be considered further in the final chapter.

We will examine how far the Baer-Nielsen Theorem extends to surface fibrations. By considering group extensions where the kernel and quotient correspond to surface groups (i.e. fundamental groups of surfaces) denoted by $\Sigma_g$ and $\Sigma_h$:

$$1 \to \Sigma_g \to \Gamma \to \Sigma_h \to 1,$$

then it will be demonstrated that for a certain class of group extensions known as characteristic extensions, there is a surjective homomorphism

$$\pi_0(\text{Diff }(\mathrm{X}_\Gamma)) \twoheadrightarrow \text{Out }(\Gamma)$$

where $X_\Gamma$ is a smooth manifold with $\pi_1(X_\Gamma) = \Gamma$. The last section of this chapter considers non-characteristic extensions of surface groups. In this case we prove that the image of the above homomorphism is a subgroup of finite index in Out $(\Gamma)$.

Let $X$ be a smooth closed manifold and let $\mathcal{H}(X)$ denote the monoid of all homotopy equivalences of $X$. Below we demonstrate the natural homomorphism from this monoid to the outer automorphism group of $\pi_1(X)$:

**Proposition 3.1** *There exists a natural homomorphism*

$$\phi : \mathcal{H}(X) \to \text{Out }(\pi_1(\mathrm{X}, *))$$

**Proof** Given a homotopy equivalence $\alpha : X \to X$ , there is an induced map in homotopy:

$$\alpha_* : \pi_1(X, *) \to \pi_1(X, \alpha(*)).$$

Let $p_\alpha : * \to \alpha(*)$ be a path beginning at the base point and let $\lambda \in \pi_1(X, \alpha(*))$ be a loop based at $\alpha(*)$; then

$$\alpha^{[p]} = p_\alpha^{-1} \lambda p_\alpha : \pi_1(X, *) \to \pi_1(X, *)$$

is an automorphism. Now consider a different path $q_\alpha : * \to \alpha(*)$ such that

$$\alpha^{[q]} = q_\alpha^{-1} \lambda q_\alpha : \pi_1(X,*) \to \pi_1(X,*)$$

is also an automorphism of the fundamental group. It follows that

$$\begin{aligned} (p_\alpha^{-1} q_\alpha)^{-1} \alpha^{[p]} (p_\alpha^{-1} q_\alpha) &= q_\alpha^{-1} p_\alpha (p_\alpha^{-1} \lambda p_\alpha) p_\alpha^{-1} q_\alpha \\ &= q_\alpha^{-1} \lambda q_\alpha \\ &= \alpha^{[q]} \end{aligned}$$

and also $(p_\alpha^{-1} q_\alpha)(*) = *$, so that $p_\alpha^{-1} q_\alpha$ is a loop embedded in $X$ based at $*$. Therefore it belongs to $\pi_1(X,*)$ giving that conjugation by $p_\alpha^{-1} q_\alpha$ is an inner automorphism of $\pi_1(X,*)$. So, if we factor Aut $(\pi_1(\mathrm{X},*))$ by Inn $(\pi_1(\mathrm{X},*))$ then the image of $\alpha$ in Out $(\pi_1(\mathrm{X}))$ is independent of the path chosen. In this way we obtain a well-defined homomorphism

$$\phi : \mathcal{H}(X) \to \text{Out } (\pi_1(\mathrm{X},*))$$

as stated. □

The kernel of this natural homomorphism

$$\phi : \mathcal{H}(X) \to \text{Out } (\pi_1(\mathrm{X}))$$

is the path-component of homotopy equivalences homotopic to the identity which we shall denote by $\mathcal{H}_0(X)$. The group of diffeomorphisms of $X$, Diff (X), is contained in the monoid of self-homotopy equivalences $\mathcal{H}(X)$. The aim of this chapter is to strengthen the above proposition to show that the natural map from Diff (X) to Out (X) is surjective for a large class of surface fibrations. We shall analyse the problem algebraically in terms of extensions of surface groups by surface groups and consider necessary conditions on the kernel of the extension.

## 3.1 Fibrations and fibre bundles

A (Hurewicz) *fibration* is a map $p : E \to B$ which has the homotopy lifting property for every space; that is, given maps $f : X \to E$ and $F : X \times I \to B$ where for all $x$ in $X$, $F(x, 0) = pf(x)$ then there exists a map $F'$ making the following diagram commute:

$$\begin{array}{ccc} X \times 0 & \xrightarrow{f} & E \\ \downarrow & \overset{F'}{\nearrow} & \downarrow p \\ X \times I & \xrightarrow{F} & B \end{array}$$

$E$ is the *total space* of the fibration, $B$ is the *base space* and for $b \in B$, $p^{-1}(b)$ is the *fibre* over $b$.

A *surface fibration* $p : X_\Gamma \to \Sigma^2$ is a (Hurewicz) fibration where both the base space $\Sigma^2$ and the fibre $\Sigma^1 = p^{-1}(\sigma)$ (for $\sigma \in \Sigma^2$) are surfaces. To every fibration $p : E \to B$ with fibre $F$ there is associated a *long homotopy exact sequence* of homotopy groups:

$$\cdots \to \pi_q(F) \xrightarrow{i_*} \pi_q(E) \xrightarrow{p_*} \pi_q(B) \xrightarrow{\partial} \pi_{q-1}(F) \to \cdots$$
$$\cdots \to \pi_0(F) \to \pi_0(E) \to \pi_0(B) \to 1.$$

For surface fibrations, the long homotopy exact sequence reduces to a short exact sequence of surface groups:

$$1 \to \pi_1(\Sigma^1) \to \pi_1(X_\Gamma) \to \pi_1(\Sigma^2) \to 1.$$

More generally, a $2n$-dimensional surface fibration is a fibration where the base space is a surface and the fibres are $(2n-2)$-dimensional surface fibrations.

Important examples of fibrations are given by fibre bundles. The construction of fibre bundles is outlined below for which the standard reference is Steenrod [Ste] (see also Husemoller [Hus] for a different treatment):

By a *coordinate bundle* we mean a collection

$$\mathcal{B} = \begin{cases} F \rightarrow & E \\ & \downarrow p \\ & B \end{cases}$$

where $F$, $E$ and $B$ are the *fibre, bundle space* and *base space* respectively and the map $p : E \rightarrow B$ is a projection of $E$ onto $B$, together with an effective topological transformation group $G$ of $F$ called the *structure group* of the bundle satisfying the following relationships:

(i): there is a family $\{V_j\}$ of open sets covering $B$ called the *coordinate neighbourhoods* indexed by a set $J$, and

(ii): for each $j$ in $J$, there is a homeomorphism

$$\phi_j : V_j \times F \rightarrow p^{-1}(V_j)$$

called the *coordinate function.* These satisfy

(iii): for $x \in V_j$, $f \in F$,

$$p\phi_j(x, f) = x$$

(iv): if we define the map $\phi_{j,x} : F \rightarrow p^{-1}(x)$ by $\phi_{j,x}(f) = \phi_j(x, f)$, then for each $i, j$ in $J$ and each $x \in V_i \cap V_j$, the homeomorphism

$$\phi_{j,x}^{-1}\phi_{i,x} : F \rightarrow F$$

coincides with the operation of an (unique) element of $G$, and

(v): for each pair $i, j$ in $J$, the map

$$g_{ji} : V_i \cap V_j \rightarrow G$$

defined by $g_{ji}(x) = \phi_{j,x}^{-1}\phi_{i,x}$ is continuous. The functions $g_{ji}$ are the *coordinate transformations* of the bundle.

Two coordinate bundles $\mathcal{B}$ and $\mathcal{B}'$ are said to be *strictly equivalent* if they have the same $E, B, p, F, G$ and their coordinate functions $\{\phi_j\}$, $\{\phi'_k\}$ satisfy the condition that for $x \in V_j \cap V'_k$

$$\bar{g}_{kj}(x) = \phi'^{-1}_{k,x}\phi_{j,x}$$

coincides with the operation of an element of $G$, and the map we obtain

$$\bar{g}_{kj} : V_j \cap V'_k \to G$$

is continuous.

Now a *fibre bundle* is defined to be an equivalence class of coordinate bundles under this equivalence relation. The following weaker notion of equivalence is the one most suitable for the classification theorem used in Section 4 of this chapter:

Two coordinate bundles $\mathcal{B}$ and $\mathcal{B}'$ with the same $B, F, G$ are *equivalent* if there exists a map $\mathcal{B} \to \mathcal{B}'$ which induces the identity map on $B$. Fibre bundles having the same $X, F, G$ are *equivalent* if they have representative coordinate bundles which are equivalent.

## 3.2 Simplicial homotopy theory: constructing fibrations from group extensions

Let $\mathcal{E} = (1 \to \Gamma \to G \to Q \to 1)$ be a short exact sequence of groups. An *$\mathcal{H}$-realisation* of $\mathcal{E}$ is a Hurewicz fibration

$$\xi = \left\{ \begin{array}{cc} X_\Gamma \to & X_G \\ & \downarrow p \\ & X_Q \end{array} \right.$$

in which the base space, fibre space and total space are homotopy equivalent to CW-complexes and the long homotopy exact sequence of $\xi$ reduces to $\mathcal{E}$.

The framework for constructing $\mathcal{H}$-realisations is given by simplicial sets and Kan complexes as exemplified in the references [Cur] and [GZ]. The category of simplicial sets provides a naturally occurring category for homotopy theory which eases the exposition of the proofs in this chapter. The definitions of simplicial sets together with some necessary material is now outlined.

Let $\mathcal{O}$ be the category of finite ordered sets $[n] = \{0, 1, \ldots, n\}$ with morphisms given by order-preserving maps. A *simplicial set* $K$ is a contravariant functor from $\mathcal{O}$ to the category of sets where

$$\begin{aligned} K_n &= K(\{0,1,\ldots,n\}) = K([n]) \\ d_i &= K(\text{the map which skips i}) \\ s_i &= K(\text{the map which repeats i}). \end{aligned}$$

The maps $d_i$ and $s_i$ are called the *face maps* and *degeneracy maps* respectively and satisfy the following relations:

$$\begin{aligned} d_i d_j &= d_{j-1} d_i \quad \text{for } i < j \\ d_i s_j &= \begin{cases} s_{j-1} d_i & \text{for } i < j \\ Id & \text{for } i = j, j+1 \\ s_j d_{i-1} & \text{for i¿j+1} \end{cases} \\ s_i s_j &= s_{j+1} s_i \end{aligned}$$

More generally, if $\mathcal{C}$ is some category then a *simplicial $\mathcal{C}$-object* is a contravariant functor from the category of finite ordered sets to $\mathcal{C}$.

The *standard n-simplex* $\Delta[n]$, is the simplicial set with vertices $0, 1, \ldots, n$ where

$$(\Delta[n])_q = \{\langle v_0, \ldots, v_q \rangle : 0 \leq v_0 \leq \ldots < v_q \leq n\}$$

Let $i_n = \langle 0, 1, \ldots, n \rangle \in (\Delta[n])_n$ and let $\Lambda^k[n]$ be the subcomplex of $\Delta[n]$ generated by all $d_i(i_n)$ for $i \neq k$. A simplicial set is a *Kan complex* if every map $f : \Lambda^k[n] \to K$ has an extension $g : \Delta[n] \to K$. Observe that a simplicial group is a Kan complex.

A *simplicial map* $f : K \to L$ is a family of functions $f_n : K_n \to L_n$ that commute with the face and degeneracy maps. We may represent an element $x$ in $K_n$ by a map $f_x : \Delta[n] \to K$. Elements $x, y$ in $K_n$ are homotopic ($x \simeq y$) if their representing maps $f_x$ and $f_y$ are homotopic relative to the interior of $\Delta[n]$. We shall call a Kan complex *minimal* if $x \simeq y$ implies $x = y$.

A simplicial map $p : E \to B$ is a *fibre map* if given $f : \Lambda^k[n] \to E$ and $g : \Delta[n] \to B$ with $pf = g\,|_{\Lambda^k[n]}$, there exists an extension of $f$ to a map $f' : \Delta[n] \to E$ with $pf' = g$; that is, the following diagram commutes:

$$\begin{tikzcd} \Lambda^k[n] \arrow[r, "f"] \arrow[d] & E \arrow[d, "p"] \\ \Delta[n] \arrow[r, "g"] \arrow[ur, "f'"] & B \end{tikzcd}$$

The fibre map is said to be *minimal* if given two extensions $f', f''$ then $f'(d_k i_n) = f''(d_k i_n)$. A sequence of simplicial maps $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibration if $p$ is a surjective fibre map and $i$ maps $F$ bijectively to $p^{-1}(*)$. Thus a minimal Kan fibration is a (surjective) minimal fibre map between Kan complexes. A fibre map $p : E \to B$ is a fibre bundle map if $p$ is onto and for each $b \in B_n$, the representing map for $b, f_b : \Delta[n] \to B$ induces a fibration $p' : E' \to \Delta[n]$ which is isomorphic to the fibration $F \times \Delta[n] \to \Delta[n]$ with fibre $F$.

The relationship between minimal fibrations and fibre bundles is given by the following theorem ( [Cur] p.164):

**Theorem 3.2** *If $p : E \to B$ is a minimal fibration onto a connected base then $p$ is a fibre bundle.*

If $G$ is a simplicial group, define a simplicial set $\mathcal{W}G$ by

$$\begin{aligned}
(\mathcal{W}G)_n &= \{(g_{n-1},\ldots,g_0) : g_i \in G_i\} \\
d_0(g_{n-1},\ldots,g_0) &= (g_{n-2},\ldots,g_0) \\
d_i(g_{n-1},\ldots,g_0) &= (d_{i-1}g_{n-1}, ldots, d_0 g_{n-i}\cdot g_{n-i-1}, g_{n-i-2},\ldots g_0) \\
s_i(g_{n-1},\ldots,g_0) &= (s_{i-1}g_{n-1}, ldots, s_0 g_{n-i}, g_{n-i-1},\ldots g_0)
\end{aligned}$$

$\mathcal{W}G$ is a Kan complex and a $K(G,1)$. Furthermore, $\mathcal{W}G$ is a classifying space for $G$ and there is a classifying bundle called the *Eilenberg-MacLane principal simplicial G-bundle*

$$\begin{array}{rl}
G \rightarrow & \overline{W}G \\
& \downarrow \\
& \mathcal{W}G
\end{array}$$

where the total complex $\overline{W}G$ is defined by

$$\begin{aligned}
(\overline{W}G)_n &= G_n \times G_{n-1} \times \ldots \times G_0 \\
d_i(g_n,\ldots,g_0) &= (d_i g_n,\ldots, d_0 g_{n-i}\cdot g_{n-i-1}, g_{n-i-2},\ldots g_0) \\
s_i(g_n,\ldots,g_0) &= (s_i g_n,\ldots, s_0 g_{n-i}, g_{n-i-1},\ldots g_0)
\end{aligned}$$

and $G$ acts on the left of $\overline{W}G$ by

$$g\cdot(g_n,\ldots,g_0) = (g\cdot g_n,\ldots,g_0)$$

for $g$ in $G_n$, $(g_n,\ldots,g_0)$ in $\overline{W}G_n$. Note that $\overline{W}G$ is a contractible Kan complex.

Using the $\mathcal{W}$ functor there is the following classification of simplicial fibre bundles given in [Cur] p.162.

**Theorem 3.3** *There is a 1-1 correspondence between the homotopy classes of maps $[B,\mathcal{W}G]$ and G-equivalence classes of G-bundles with base $B$ and fibre $F$.*

**Corollary 3.4 (I)** *The set of fibre homotopy equivalence classes of fibrations with base $B$ and fibre $\Sigma$ are in 1-1 correspondence with homotopy classes of maps* $[B, \mathcal{WH}(\Sigma)]$.

**Corollary 3.5 (II)** *There is a 1-1 correspondence between smooth equivalence classes of locally trivial fibre bundles over $B$ with fibre $\Sigma$ and homotopy classes of maps* $[B, \mathcal{W}\mathrm{Diff}\ (\Sigma)]$.

So far we have considered objects in the category **SS** of simplicial set with morphisms given by simplicial maps. Now, define **TOP** to be the category of topological spaces $X, Y, \ldots$ with morphisms consisting of all continuous maps $f : X \to Y$. We shall define a subcategory of **TOP** that will prove to be more useful in the context of simplicial sets. A topological space is $X$ is *compactly generated* if every subset that intersects all compact subsets of $X$ in a closed set is itself closed. Let **CG** denote the category with objects all compactly generated Hausdorff spaces and morphisms consisting of continuous functions between them. As an example, all locally compact spaces are in **CG**.

In **TOP**, the inverse limit of two objects $X, Y$ is the direct product $X \times Y$. We shall denote the inverse limit in **CG** by $(X \times Y)_{CG}$. In certain circumstances these two notions coincide; in particular:

**Lemma 3.6** *If $X$ is a compactly-generated Hausdorff space and $Y$ is locally compact then*

$$X \times Y = (X \times Y)_{CG}$$

**Proof :** See [GZ], p.47.

We shall define an Euclidean simplex $\delta[n] \subset \mathcal{R}^{n+1}$ to be the topological space

$$\delta[n] = \{(x_0, \ldots, x_n) : \Sigma x_i = 1, x_i \geq 0\}$$

Define the maps $\epsilon : \delta[n-1] \to \delta[n]$ and $\eta : \delta[n+1] \to \delta[n]$ by

$$\begin{aligned}
\epsilon_i(x_0, \ldots, x_{n-1}) &= (x_0, \ldots, x_{i-1}, 0, x_{i+1}, \ldots x_{n-1}) \\
\eta_i(x_0, \ldots, x_{n+1}) &= (x_0, \ldots, x_i + x_{i+1}, \ldots, x_{n+1}).
\end{aligned}$$

Given $K$ a simplicial set, let $RK$ be the topological space

$$RK = \bigsqcup_{x \in K} (\delta[\mathrm{dim} x], x)$$

and define an equivalence relation on $RK$ by $(p, x) \sim (y, q)$ if either

$$\begin{array}{llll}
i): & d_i x = y & \text{and} & \epsilon_i(q) = p, \quad \text{or,} \\
ii): & s_i x = y & \text{and} & \eta_i(q) = p.
\end{array}$$

Then $|K| = RK/\sim$ is the *geometric realisation* of the simplicial set $K$. Note that geometric realisation is a functor $|-|$ from the category of simplicial sets **SS** to **CG** ( [GZ], p.49).

A morphism $Y \to X$ of **SS** is *trivial* if there exists a complex $F$ and an isomorphism $\alpha : X \times F \cong Y$ such that $f \cdot \alpha = p$, where $p$ is the canonical projection of $X \times F$ onto $X$. $F$ is called the fibre of $f$. $f$ is said to be *locally trivial* if for each simplex $\sigma : \Delta[n] \to X$, the projection of the pullback $\Delta[n] \times_{\sigma, f} Y$ onto $\Delta[n]$ is trivial.

Similarly, a morphism $u : L \to K$ of **CG** is *trivial* with fibre $T$ if there exists an isomorphism $\beta : (K \times T)_{CG} \cong L$ such that $u \cdot \beta$ is the canonical projection. $u$ is *locally trivial* if every point $x$ in $K$ has an open neighbourhood $U$ such that $u$ induces a trivial morphism from $u^{-1}(U)$ into $U$. If $K$ is connected then all fibres of $u$ are isomorphic and we say that $u$ is *locally trivial with fibre $F$.*

These definitions of local triviality are compatible with respect to the geometric realisation functor as shown by the following ([GZ], p.55):

**Theorem 3.7** *The geometric realisation functor* $|-|: \mathbf{SS} \to \mathbf{CG}$ *takes a locally trivial morphism with fibre* $F$ *into a locally trivial morphism with fibre* $|F|$.

The fact that locally trivial morphisms occur frequently is demonstrated by the following theorem ([GZ],p.127):

**Theorem 3.8** *Every minimal fibration is locally trivial.*

Using the theory of simplicial sets we are now in a position to construct Hurewicz fibrations from group extensions.

**Theorem 3.9** *If* $\mathcal{E} = (1 \to K \to G \xrightarrow{p} Q \to 1)$ *is a short exact sequence of discrete groups with* $Q$ *countable of finite cohomological dimension, then* $\mathcal{E}$ *has an* $\mathcal{H}$*-realisation. That is, there is an Hurewicz fibration* $\xi = (X_G \xrightarrow{p} X_Q)$ *with fibre* $X_K$ *where* $X_K, X_G, X_Q$ *are homotopy equivalent to CW-complexes and such that the long homotopy exact sequence of* $\xi$ *is* $\mathcal{E}$.

**Proof :** Given a group homomorphism $p : G \to Q$, there is a simplicial map $p' : \mathcal{W}G \to \mathcal{W}Q$ defined by

$$p'(g_{n-1}, \ldots, g_0) = (pg_{n-1}, \ldots pg_0)$$

and furthermore, the induced map $p'_* : \pi_1(\mathcal{W}G) \to \pi_1(\mathcal{W}Q)$ is $p$ (see R.O. Hill Jr. [Hil], p.410). If $\mathcal{E} = (1 \to K \to G \xrightarrow{p} Q \to 1)$ is exact then $p' : \mathcal{W}G \to \mathcal{W}Q$ gives a minimal Kan fibration

$$\mathcal{W}\mathcal{E} = \left\{ \begin{array}{ll} \mathcal{W}K \to & \mathcal{W}G \\ & \downarrow p' \\ & \mathcal{W}Q \end{array} \right.$$

which is locally trivial by Theorem 3.8 . Taking the geometric realisation of $\mathcal{WE}$ gives a fibration in the category **CG**

$$|\mathcal{WE}| = \begin{cases} |\mathcal{W}K| \to & |\mathcal{W}G| \\ & \downarrow p_0 \\ & |\mathcal{W}Q| \end{cases}$$

which is locally trivial since the geometric realisation of a locally trivial fibration is locally trivial (Theorem 3.7). As $Q$ is countable of finite cohomological dimension, we may choose a locally compact CW-complex $X_G$ of homotopy type $K(G,1)$ by F.E.A. Johnson's Theorem 1.20 and the homotopy equivalence $f : X_Q \to |\mathcal{W}Q|$ induces a (locally trivial) fibration

$$\xi = \begin{cases} |\mathcal{W}Q| \to & E \\ & \downarrow p_1 \\ & X_Q. \end{cases}$$

The fact that $\xi$ is locally trivial in the category **CG** implies that for all neighbourhoods $U \subset X_Q$, there exists a map

$$q : p_1^{-1}(U) \to (U \times |\mathcal{W}K|)_{CG}.$$

Furthermore, $(U \times |\mathcal{W}K|)_{CG} \cong U \times |\mathcal{W}Q|$ because $X_Q$ is locally compact and $|\mathcal{W}K|$ belongs to **CG** (Lemma 3.6). Thus $\xi$ is locally trivial in the category of topological spaces **TOP** and is a Hurewicz fibration. □

## 3.3 Fibre smoothing

A discrete group $\Gamma$ is said to be *smoothable* if the Eilenberg-MacLane space of the group, $K(\Gamma, 1)$ is homotopy equivalent to a smooth closed manifold

$X_\Gamma$ called a smooth model for $\Gamma$. Given an exact sequence of groups $\mathcal{E} = (1 \to \Gamma \to G \to Q \to 1)$ with $Q$ countable of finite cohomological dimension , we shall choose the canonical $\mathcal{H}$-realisation as constructed in the previous section:

$$\xi = \begin{cases} K(\Gamma,1) \to & K(G,1) \\ & \downarrow \\ & K(Q,1). \end{cases}$$

We shall say that $\Gamma$ has the *fibre smoothing property* if for all extensions of the form $\mathcal{E}$ where $Q$ is smoothable with a smooth model $X_Q$, the associated $\mathcal{H}$-realisation $\xi$ is fibre homotopy equivalent to a smooth fibre bundle:

$$\xi = \begin{cases} X_\Gamma \to & E \\ & \downarrow \\ & X_Q \end{cases}$$

where the fibre is a smooth finite dimensional manifold of homotopy type $K(\Gamma,1)$.

In the course of our proof of the main theorem it will be necessary to prove that certain extensions of surface groups have the fibre smoothing property. The proof of this fact relies on the important result that surface groups have the fibre smoothing property which is due to F.E.A. Johnson (e.g. [Joh4]).

First, let $B_F^0(X)$ ($B_F^\infty(X)$) denote the equivalence classes of continuous (smooth) fibre bundles over $X$ with fibre $F$. The following is standard (see e.g. [BL]):

**Proposition 3.10** *Let $\Sigma$ be a smooth closed surface and let $X$ be a smooth manifold. Then*

$$B_\Sigma^0(X) \cong B_\Sigma^\infty(X)$$

*That is, any continuous fibre bundle over $X$ with fibre $\Sigma$ is smoothly equivalent to a smooth fibre bundle over $X$ with fibre $\Sigma$.*

**Theorem 3.11 (F.E.A. Johnson)** *The fundamental groups of surfaces of genus $\geq 2$ possess the fibre smoothing property.*

**Sketch Proof :** Let $\Sigma$ be a fixed smooth closed surface and denote its fundamental group by $\pi_1(\Sigma) = G$. The monoid of homotopy equivalences of $\Sigma$, $\mathcal{H}(\Sigma)$, maps to Out (G) by Proposition 3.1 and has kernel given by the path-component $\mathcal{H}_0(X)$ containing the identity. In [Got] the following theorem is proved:

> **Theorem (Gottlieb):** *If $X$ is a path-connected aspherical manifold with centre $Z(\pi_1(X)) = 1$ then the path-component of the space of continuous mappings from $X \to X$ containing the identity is contractible.*

Therefore it follows that $\mathcal{H}_0(X)$ is contractible since $Z(G) = 1$ when $G$ is a surface group. Together with the fact that the Eilenberg-MacLane classifying space functor $\mathcal{W}$ preserves homotopy equivalences (see [Cur] p.114) this proves that there is a homotopy equivalence

$$\mathcal{W}\mathcal{H}(\Sigma) \simeq \mathcal{W}\mathrm{Out}\ (\mathrm{G}).$$

The set of fibre homotopy classes of fibrations with fibre $\Sigma$ over a CW-complex $X$ we shall denote by $F_\Sigma(X)$. This set is naturally equivalent to $[X, \mathcal{W}\mathcal{H}(\Sigma)]$ by Corollary 3.4 which is isomorphic to $[X, \mathcal{W}\mathrm{Out}\ (\mathrm{G})]$ by the above.

The Baer-Nielsen Theorem states that the map from Diff ($\Sigma$) to Out (G) is surjective and has kernel given by the diffeomorphisms homotopic to the identity Diff $_0(\Sigma)$. Applying the classifying space functor to the resulting exact sequence gives a fibration

$$\begin{array}{ccc} \mathcal{W}\mathrm{Diff}\ _0(\Sigma) \to & \mathcal{W}\mathrm{Diff}\ (\Sigma) & \\ & \downarrow & . \\ & \mathcal{W}\mathrm{Out}\ (\mathrm{G}) & \end{array}$$

In Earle and Eells paper [EE] a fibre bundle description of Teichmuller theory is given and it is shown that Diff $_0(\Sigma)$ is contractible and therefore $\mathcal{W}$Diff $(\Sigma)_0$ is contractible since $\mathcal{W}$ preserves homotopy equivalences. Hence there is a homotopy equivalence $\mathcal{W}$Diff $(\Sigma) \simeq \mathcal{W}$Out (G). From the above proposition, we know that the set of smooth equivalence classes of smooth fibre bundles with fibre $\Sigma$ over a smooth connected manifold $X$, denoted $B^{\infty}_{\Sigma}(X)$, is naturally equivalent to the set of based homotopy classes $[X, B\text{Diff }(\Sigma)]$. Thus we have a chain of equivalences:

$$F_{\Sigma}(X) \cong [X, \mathcal{W}\mathcal{H}(\Sigma)] = [X, \mathcal{W}\text{Out (G)}] = [\text{X}, \mathcal{W}\text{Diff }(\Sigma)] \cong \text{B}^{\infty}_{\Sigma}(\text{X})$$

These equivalences give us that the category of smooth fibre bundles with fibre $\Sigma$ coincides with the category of fibrations with fibre $\Sigma$. In particular, the space $\mathcal{H}(\Sigma)/\text{Diff }(\Sigma)$ is contractible. Hence, given a smooth manifold $X$ and a fibration with fibre $\Sigma$

$$\xi = \begin{cases} \Sigma \to & E \\ & \downarrow \\ & X \end{cases}$$

then $\xi$ is fibre homotopy equivalent to a smooth locally trivial fibre bundle with fibre diffeomorphic to $\Sigma$ and moreover, this fibration is unique up to smooth equivalence. Hence we have proved that surface groups possess the fibre smoothing property. $\square$

In order to prove that surface fibrations possess the fibre smoothing property we require an extra condition on the kernel of the associated group extension. A subgroup $K$ of a group $G$ is said to be *characteristic* in $G$ if given any automorphism of $G$, $\alpha : G \to G$, then $\alpha(K) = K$. Observe the following properties of characteristic subgroups: If $G_1$ is characteristic in $G_2$ and $G_2$

is a normal subgroup of $G_3$, then $G_1$ is also a normal subgroup of $G_3$. Further, let $\mathcal{E}$ be the extension $1 \to \Gamma \to G \to Q \to 1$. If $\Sigma$ is a characteristic subgroup of $\Gamma$ then there is a factorisation of the extension $\mathcal{E}$ into two group extensions:

$$\mathcal{E}_1 = (1 \to \Sigma \to G \to \Delta \to 1)$$

$$\mathcal{E}_2 = (1 \to \Gamma/\Sigma \to \Delta \to Q \to 1)$$

where $\Delta = G/\Sigma$.

**Theorem 3.12** *Let $\Gamma$ be a group extension $1 \to \Sigma_1 \to \Gamma \to \Sigma_2 \to 1$ where $\Sigma_1$, $\Sigma_2$ are surface groups and $\Sigma_1$ is characteristic in $\Gamma$. Let $G$ be the semi-direct product $G = \Gamma \rtimes_\alpha C_\infty$ where $\alpha : \Gamma \to \Gamma$ is an automorphism of $\Gamma$, so that $G$ is a split extension*

$$\mathcal{E} = (1 \to \Gamma \to G \rightleftharpoons C_\infty \to 1)$$

*Then the canonical fibration of the extension $\mathcal{E}$*

$$\xi = \begin{cases} K(\Gamma, 1) \to & K(G, 1) \\ & \downarrow \\ & K(C_\infty, 1) \end{cases}$$

*is fibre homotopy equivalent to a smooth fibre bundle*

$$\hat{\xi} = \begin{cases} X_\Gamma \to & X_G \\ & \downarrow \\ & S^1 \end{cases}$$

*where the fibre $X_\Gamma$ is a smooth finite dimensional manifold of homotopy type $K(\Gamma, 1)$.*

**Proof :** First, construct the semi-direct product $G = \Gamma \rtimes_\alpha C_\infty$ as below: Let $\alpha$ be an automorphism of $\Gamma$ and construct the split extension

$$1 \to \Gamma \to G \overset{s}{\rightleftharpoons} C_\infty \to 1$$

where $s$ is a splitting homomorphism $s : C_\infty \to G$. Thus we may write for all $x$ in $\Gamma$, $\alpha(x) = s(t)xs(t)^{-1}$ where $C_\infty$ is generated by $t$. Since $\Sigma_1$ is characteristic in $\Gamma$, the extension $\mathcal{E}$ factorises to give extensions

$$\mathcal{E}_1 = (1 \to \Sigma_1 \to G \to G/\Sigma_1 \to 1)$$

$$\begin{array}{c} \mathcal{E}_2 = (1 \to \underset{\displaystyle \overset{\|}{\Sigma_2}}{\Gamma/\Sigma_1} \to G/\Sigma_1 \to C_\infty \to 1) \end{array}$$

Note that the cohomological dimension of $C_\infty$ is 1 whilst $G$ has cohomological dimension 5. To see this, observe that $\Gamma$ is an extension of surface groups which are duality groups of dimension 2 and thus $\Gamma$ has dimension 4 and $G$ is an extension of $\Gamma$ by $C_\infty$. This means that $G/\Sigma_1$ has cohomological dimension 3. Hence the quotients of these extensions are both countable groups with finite cohomological dimension and they have $\mathcal{H}$-realisations by Theorem 3.9. Applying the Eilenberg-MacLane classifying space functor we obtain fibrations

$$\xi_1 = \left\{ \begin{array}{ccc} K(\Sigma_1, 1) & \to & K(G, 1) \\ & & \downarrow \\ & & K(G/\Sigma_1, 1) \end{array} \right.$$

$$\xi_2 = \left\{ \begin{array}{ccc} K(\Sigma_2, 1) & \to & K(G/\Sigma_1, 1) \\ & & \downarrow \\ & & K(C_\infty, 1) \end{array} \right.$$

Since both of these fibrations have fibres corresponding to Eilenberg-MacLane spaces of surface groups, we may invoke the fibre smoothing property for surfaces and obtain *smooth* fibre bundles

$$\widehat{\xi_1} = \left\{ \begin{array}{ll} Y \rightarrow & X_G \\ & \downarrow p_1 \\ & E \end{array} \right.$$

$$\widehat{\xi_2} = \left\{ \begin{array}{ll} Z \rightarrow & E \\ & \downarrow p_2 \\ & S^1 \end{array} \right.$$

where $\pi_1(Y) = \Sigma_1$, $\pi_1(Z) = \Sigma_2$, $E$ is homotopy equivalent to $K(G/\Sigma_1, 1)$ and $p_1$, $p_2$ denote the bundle projections. It is well known that $K(C_\infty, 1) \simeq S^1$. Since $p_1 : X_G \rightarrow E$ and $p_2 : E \rightarrow S^1$ both have compact fibres, put $p = p_1 \circ p_2 : X_G \rightarrow S^1$ and then $p^{-1}(x) = (p_2 p_1)^{-1}(x)$ where $x \in S^1$ also has compact fibre. Furthermore, since the tangent map $T_p : TX_G \mid_x \rightarrow TS^1 \mid_{p(x)}$ is surjective, then the map $p$ gives rise to a smooth fibre bundle:

$$\zeta = \left\{ \begin{array}{ll} X \rightarrow & X_G \\ & \downarrow \\ & S^1 \end{array} \right.$$

The fibre $X$ is itself fibred over $Z$ with fibre $Y$:

$$\begin{array}{ll} Y \rightarrow & X \\ & \downarrow \\ & Z \end{array}$$

If we examine our original extension $\mathcal{E} = (1 \rightarrow \Sigma_1 \rightarrow \Gamma \rightarrow \Sigma_2 \rightarrow 1)$ then again we have an extension whose kernel is given by a surface group and we can invoke the fibre smoothing operation to obtain a smooth fibre bundle over $\Sigma^2$ with fibre $\Sigma^1$ as above. Moreover, this fibre bundle is unique up to smooth equivalence since surfaces have a unique differentiable structure. Hence the manifold $X$ is smoothly equivalent to the surface fibration $X_\Gamma$. $\square$

## 3.4 Surjectivity of the natural homomorphism

This section is devoted to proving that the natural homomorphism from the group of diffeomorphisms of a surface fibration maps surjectively onto the outer automorphism group of its fundamental group. The proof brings together work from the previous sections in order to construct a fibration over the circle from a group extension and then to smooth this to a smooth fibre bundle. The bulk of the remaining work is then to classify such fibre bundles using Steenrod's work on the classification of bundles over spheres.

**Theorem 3.13** *Let $\mathcal{E}$ be the group extension* $(1 \to \Sigma_1 \to \Gamma \to \Sigma_2 \to 1)$ *where $\Sigma_1$ and $\Sigma_2$ are surface groups and $\Sigma_1$ is a characteristic subgroup of $\Gamma$. Suppose $X_\Gamma$ is a closed manifold with $\Gamma$ as its fundamental group. Then the natural homomorphism*

$$\pi_0(\text{Diff }(\text{X}_\Gamma)) \twoheadrightarrow \text{Out }(\Gamma)$$

*is surjective.*

**Proof :** The idea of the proof is to construct a homotopy class of diffeomorphisms from an outer automorphism of $\Gamma$. Let $\alpha$ be an automorphism of $\Gamma$ and construct the split extension

$$1 \to \Gamma \to G \overset{s}{\rightleftharpoons} C_\infty \to 1$$

where $s$ is a splitting homomorphism $s : C_\infty \to G$. Thus we may write for all $x$ in $\Gamma$, $\alpha(x) = s(t)xs(t)^{-1}$ where $C_\infty$ is generated by $t$. The quotient of this extensions $C_\infty$ is a countable group with finite cohomological dimension and thus has $\mathcal{H}$-realisations by Theorem 3.9. So we may apply the Eilenberg-MacLane functor $K(-,1)$ to this extension to obtain a fibration

$$\xi = \begin{cases} K(\Gamma,1) \to & K(G,1) \\ & \downarrow \\ & S^1 \end{cases}$$

which by the previous theorem on fibre smoothing is fibre homotopy equivalent to a smooth fibre bundle

$$\hat{\xi} = \begin{cases} X_\Gamma \to & X_G \\ & \downarrow \\ & S^1 \end{cases}$$

where $X_\Gamma$ is a smooth model for $\Gamma$ with the homotopy type of a $K(\Gamma, 1)$. This fibre bundle has a structure group given by Diff ($X_\Gamma$). It is now necessary to classify such fibre bundles and the following theorem is adapted from Steenrod [Ste] p.99:

**Theorem 3.14 (Classification of bundles over the circle)** *The equivalence classes of bundles over $S^1$ with structure group $G$ are in $1-1$ correspondence with $\pi_0(G)$.*

**Sketch Proof :** Let $S^0$ be the end-points of a diameter of $S^1$ and let $E_1$, $E_2$ be the closed semicircles of $S^1$ determined by $S^0$. For $i = 1, 2$, let $V_i$ be an open 1-cell on $S^1$ containing $E_i$. These $V_i$ cover $S^1$ and their intersection is an equatorial band containing $S^0$. Mark a point $x_0$ on $S^0$. We shall say that a coordinate bundle $\mathcal{B}$ over $S^1$ is in *normal form* if its coordinate neighbourhoods are $V_1$, $V_2$, and $g_{12}(x_0) = e$, the base point of $S^1$. Now, any bundle $\mathcal{B}$ is strictly equivalent to a bundle in normal form. Assuming that $\mathcal{B}$ is a bundle in normal form we shall consider the restriction of the coordinate transformation which maps $S^0$ to $G$:

$$T = g_{12} \mid_{S^{n-1}} .$$

$T$ is called the *characteristic map* of $\mathcal{B}$ and any map $T : (S^0, x_0) \to (G, e)$ is the characteristic map of some bundle over $S^1$ in normal form.

The classification theorem will follow from the following claim:

*Let $\mathcal{B}$, $\mathcal{B}'$ be bundles over $S^1$ in normal form with the same fibre and structure group and let $T$ and $T'$ be their characteristic maps. Then $\mathcal{B}$ and $\mathcal{B}'$ are equivalent if and only if there exists an element $a \in G$ and a homotopy $T' \simeq a^{-1}Ta$.*

This is proved in Steenrod [Ste] p.97-8. Now, given elements in the same component of $G$, $g_0, g_1$, joined by a curve $g_t$, say, $0 \leq t \leq 1$, then $h(g,t) = g_t^{-1} g g_t$ is a homotopy of the inner automorphism corresponding to $g_0$ into the inner automorphism corresponding to $g_1$ keeping $e$ fixed. Therefore $g_0$ and $g_1$ give rise to equivalent bundles. Hence the group of path components $\pi_0(G)$ is in $1-1$ correspondence with equivalence classes of bundles over the circle. $\square$

By using this classification theorem we may deduce that the equivalence classes of smooth fibre bundles of the form

$$\hat{\xi} = \left\{ \begin{array}{ccc} X_\Gamma & \rightarrow & X_G \\ & & \downarrow \\ & & S^1 \end{array} \right.$$

are in $1-1$correspondence with elements of the *group* of path components

$$\pi_0(\text{Structure group}) = \pi_0(\text{Diff } \mathrm{X}_\Gamma).$$

The final step is to show that inner automorphisms of $\Gamma$ give rise to equivalent extensions and thus correspond to the same element of $\pi_0(\text{Diff } \mathrm{X}_\Gamma)$. Suppose we have two automorphisms of $\Gamma$, $\alpha$ and $\alpha'$, defined by transversals $s$ and $s'$ respectively. Observe that since any two transversals differ by an element of $\Gamma$, we may put $s(t) = x's'(t)$ where $x' \in \Gamma$. By writing $\phi_t(x) = \alpha(x) = s(t)^{-1}xs(t)$ we obtain a function $\phi : C_\infty \rightarrow \text{Aut }(\Gamma)$, and similarly $\alpha'(x) = \phi'_t(x) = s'(t)^{-1}xs'(t)$. Rearranging,

$$\phi'_t(x) \quad = \quad s'(t)^{-1}s(t)\phi_t(x)s(t)^{-1}s'(t)$$

$$= \{s(t)^{-1}x's(t)\}\phi_t(x)\{s(t)^{-1}x'^{-1}s(t)\}$$
$$= g^{-1}\phi_t(x)g \qquad \text{where } g = s(t)^{-1}x'^{-1}s(t)$$

proving that any two $\phi$'s differ by an inner automorphism of $\Gamma$. This we may write as

$$\phi_t(x)(\text{Inn }(\Gamma)) = \phi_t'(\text{x})(\text{Inn }(\Gamma))$$

and hence there is a well-defined homomorphism $\hat{\phi}_t(x) = \phi_t(x)(\text{Inn }(\Gamma))$ (the operator homomorphism). From this it is clear that inner automorphisms give rise to the same operator homomorphism and moreover, as $\Gamma$ has trivial centre, each congruence class of extensions corresponds to a unique operator homomorphism. Conversely, there exists an extension corresponding to each operator homomorphism (these statements follow from the corollary to the Eilenberg-MacLane theorem 1.11). Therefore, inner automorphisms give rise to equivalent extensions as required.

And so we have shown that (up to conjugacy) every automorphism of $\Gamma$ gives rise to a fibre bundle over $S^1$ which in turn corresponds a homotopy class of diffeomorphisms $\pi_0(\text{Diff }(\text{X}_\Gamma))$. Hence the natural homomorphism from $\pi_0(\text{Diff }(\text{X}_\Gamma))$ to Out $(\Gamma)$ is surjective. □

## 3.5 Generalisation to poly-surface groups

By taking the extension of a surface group by a surface group we obtain a poly-surface group of length 2. Iterating these extensions with surface groups as quotients and kernels given by poly-surface groups of length $n-1$ gives a poly-surface group of length $n$.

In this section, the previous theorem will be generalised to poly-surface groups and thus to iterated surface fibrations. As before, we shall need extra conditions on the kernels of the extensions:

Given a class of groups $\mathcal{C}$, we may write poly-$\mathcal{C}$ groups of length $n$ as *filtrations* of length $n$. By this we mean a sequence of

$$\begin{array}{ll} (i) & 1 = G_0 \lhd G_1 \lhd \ldots \lhd G_n = G \\ (ii) & \text{For all } \; 0 \leq r \leq n-1, \; \dfrac{G_{r+1}}{G_r} \in \mathcal{C}. \end{array}$$

Such a filtration is called *characteristic* if in addition, each $G_r$ is a characteristic subgroup of $G_{r+1}$.

**Theorem 3.15** *Let $\Gamma$ be a poly-surface group of length $n$ constructed from a characteristic filtration of surface groups and let $Q$ be a smoothable group with smooth model $X_Q$. Construct the semi-direct product extension $\mathcal{E} = (1 \to \Gamma \to G \to Q \to 1)$ from an automorphism $\alpha : G \to G$ so that $G = \Gamma \rtimes_\alpha Q$. Then the canonical fibration of the extension $\mathcal{E}$*

$$\xi = \left\{ \begin{array}{ccc} K(\Gamma,1) & \to & K(G,1) \\ & & \downarrow \\ & & K(Q,1) \end{array} \right.$$

*is fibre homotopy equivalent to a smooth fibre bundle*

$$\hat{\xi} = \left\{ \begin{array}{ccc} X_\Gamma & \to & X_G \\ & & \downarrow \\ & & X_Q \end{array} \right.$$

*where the fibre $X_\Gamma$ is a smooth finite dimensional manifold of homotopy type $K(\Gamma,1)$.*

*In other words, $\Gamma$ has the fibre smoothing property.*

**Proof :** The proof is by induction on the length of the poly-surface filtration. It is important to note that the earlier proof for the case $n = 2$ actually showed that $\Gamma$ has the fibre smoothing property since interchanging the quotient group $C_\infty$ with any other smoothable group will not alter the outcome.

Suppose that all (characteristic) poly-surface groups of length $k-1$ have the fibre smoothing property and let $\Gamma$ be a poly-surface group of length $k$ derived from a characteristic filtration. Thus $\Gamma$ belongs to an extension of the form

$$1 \to \Gamma_1 \to \Gamma \to \Sigma \to 1$$

where $\Sigma$ is a surface group and $\Gamma_1$ is a poly-surface group of length $k-1$ which is characteristic in $\Gamma$. This implies that the extension $\mathcal{E}$ may be factorised into two extensions:

$$\mathcal{E}_1 = (1 \to \Gamma_1 \to G \to G/\Gamma_1 \to 1)$$

$$\mathcal{E}_2 = (1 \to \Sigma \to G/\Gamma_1 \to Q \to 1).$$

In order to obtain fibrations corresponding to these extensions we require their quotients to be countable groups of finite cohomological dimension. This is automatically true for $Q$ since we assumed that $Q$ was a smoothable group. The cohomological dimension of $C_\infty$ is 1 whilst $G$ has cohomological dimension $2k+1$ since it is an iterated extension of $k$ surface groups which are duality groups of dimension 2 by $C_\infty$. Therefore $G/\Gamma_1$ has dimension 3. Hence the quotients of these extensions are both countable groups with finite cohomological dimension and thus they have $\mathcal{H}$-realisations by theorem 3.9. Applying the Eilenberg-MacLane classifying space functor to both of these extensions we obtain fibrations

$$\xi_1 = \left\{ \begin{array}{ccc} K(\Gamma_1, 1) \to & K(G,1) \\ & \downarrow \\ & K(G/\Gamma_1, 1) \end{array} \right.$$

$$\xi_2 = \begin{cases} K(\Sigma,1) \to & K(G/\Gamma_1,1) \\ & \downarrow \\ & K(Q,1) \end{cases}$$

The fibration $\xi_2$ has fibre corresponding to the Eilenberg-MacLane space of a surface, and so we may invoke the fibre smoothing property for surfaces and obtain a *smooth* fibre bundle $\widehat{\xi_2}$ as before. The fibre of $\xi_1$ is a poly-surface group of length $k-1$ which has the fibre smoothing property by the induction hypothesis. Therefore, $\xi_1$ may be smoothed to give a smooth fibre bundle $\widehat{\xi_1}$ as below.

$$\widehat{\xi_1} = \begin{cases} X \to & X_G \\ & \downarrow p_1 \\ & E_2 \end{cases}$$

$$\widehat{\xi_2} = \begin{cases} Y \to & E_2 \\ & \downarrow p_2 \\ & X_Q \end{cases}$$

where $\pi_1(X) = \Gamma_1$ and $\pi_1(Y) = \Sigma$. In this way, by a similar method to the case for a poly-surface group of length 2, we may construct a smooth fibre bundle

$$\begin{cases} X_\Gamma \to & X_G \\ & \downarrow p_1 \\ & X_Q \end{cases}$$

where $X_\Gamma$ is fibred over $\Sigma$ with fibre given by $X_{\Gamma_1}$. This argument demonstrates that the poly-surface group $\Gamma$ constructed from a characteristic filtration of length $k$ possesses the fibre smoothing property. The theorem follows by induction and our earlier proof for a (characteristic) poly-surface group of length 2. □

**Corollary 3.16** *Let $X_\Gamma$ be an iterated surface fibration of dimension $2n$ with $\pi_1(X_\Gamma) = \Gamma$ where $\Gamma$ is a poly-surface group constructed from a characteristic filtration of length $n$. Then the natural homomorphism*

$$\pi_0(\text{Diff }(\mathrm{X}_\Gamma)) \twoheadrightarrow \text{Out }(\Gamma)$$

*is surjective.*

**Proof :** In the above theorem, consider the case where $Q = C_\infty$ and construct from an automorphism of $\Gamma$ the semi-direct product $G = \Gamma \rtimes_\alpha C_\infty$ in the following way: let $s$ be a splitting homomorphism $s : C_\infty \to G$ and write for all $x$ in $\Gamma$, $\alpha(x) = s(t)xs(t)^{-1}$ where $C_\infty$ is generated by $t$. Referring to the work in Section 2, we see that this (split) extension $\mathcal{E} = (1 \to \Gamma \to G \to C_\infty \to 1)$ has a $\mathcal{H}$-realisation corresponding to a fibration

$$\xi = \begin{cases} K(\Gamma,1) \to & K(G,1) \\ & \downarrow \\ & K(C_\infty,1) \end{cases}$$

whose long homotopy exact sequence coincides with $\mathcal{E}$. Now apply the fibre smoothing property proved in the above theorem to obtain a smooth fibre bundle $\hat{\xi}$ fibre homotopy equivalent to $\xi$:

$$\hat{\xi} = \begin{cases} X_\Gamma \to & X_G \\ & \downarrow \\ & S^1 \end{cases}$$

where $X_\Gamma$ is a smooth model for $\Gamma$ with the homotopy type of a $K(\Gamma,1)$. By the classification theorem for bundles over the circle, the class of such fibre bundles is in $1-1$ correspondence with elements of $\pi_0(\text{Diff }(\mathrm{X}_\Gamma)$. Hence, given an automorphism of $\Gamma$ we have constructed an element of the homotopy class of diffeomorphisms of $X_\Gamma$. Furthermore, inner automorphisms of $\Gamma$ give rise to the same homotopy class of diffeomorphisms as before. Thus, we have proved that if $\Gamma$ is a poly-surface group constructed from a characteristic filtration then the natural homomorphism

$$\pi_0(\mathrm{Diff}\ (\mathrm{X}_\Gamma)) \twoheadrightarrow \mathrm{Out}\ (\mathcal{E})$$

is surjective. □

## 3.6 Non-characteristic extensions of surface groups

The above theorems only apply to extensions $\mathcal{E}$ of surface groups for which the kernel $\Sigma_1$ is a characteristic subgroup of $\Gamma$. However we may still obtain a similar result for non-characteristic extensions.

Let the automorphism group of the extension Aut $(\mathcal{E})$ be the subgroup of Aut $(\Gamma)$ such that

$$\mathrm{Aut}\ (\mathcal{E}) = \{\alpha \in \mathrm{Aut}\ (\Gamma) : \alpha(\Sigma_1) = \Sigma_1\}$$

In Chapter 2 it was shown that this subgroup has finite index in Aut $(\Gamma)$ (Theorem 2.14) and similarly for Out $(\mathcal{E})$. In this section we shall prove the following:

**Theorem 3.17** *Let $\mathcal{E}$ be the extension $(1 \to \Sigma_1 \to \Gamma \to \Sigma_2 \to 1)$ where $\Sigma_1$ is not necessarily characteristic in $\Gamma$. Let $X_\Gamma$ be a smooth closed connected* *manifold with fundamental group $\pi_1(X_\Gamma) = \Gamma$. Then the natural homomorphism*

$$\pi_0(\text{Diff } (\text{X}_\Gamma)) \longrightarrow \text{Out } (\mathcal{E})$$

*is surjective.*

**Proof :** Let $\alpha$ be an automorphism of the extension so that $\alpha \in \text{Aut } (\mathcal{E})$. Consider the semi-direct product extension constructed by $\alpha$

$$1 \to \Gamma \to G \to C_\infty \to 1$$

and denote this by $\Gamma \rtimes_\alpha C_\infty$. Explicitly we write for all $x \in \Gamma$, $\alpha(x) = s(t)xs(t^{-1})$ where $s$ is the splitting homomorphism and $t$ is a generator of the infinite cyclic group $C_\infty$. The proof will follow from earlier results if we can show that $\Sigma_1$ is a normal subgroup of the semi-direct product $\Gamma \rtimes_\alpha C_\infty$.

**Sub-lemma 3.18** *The surface group $\Sigma_1$ is a normal subgroup of the semi-direct product $\Gamma \rtimes_\alpha C_\infty$ where $\alpha$ is an automorphism of the extension $\mathcal{E}$.*

**Proof :**
Denote the elements of the semi-direct product $\Gamma \rtimes_\alpha C_\infty$ by $(\gamma, t)$ where $\gamma \in \Gamma$ and $t \in C_\infty$. The normal subgroup $\Gamma$ is naturally included in $\Gamma \rtimes_\alpha C_\infty$ by the mapping

$$\gamma \mapsto (\gamma, 1)$$

and since $\Sigma_1 \subset \Gamma$, we may write all elements of $\Sigma_1$ in the form $(\sigma, 1)$. For brevity we shall denote the conjugating homomorphism of the semi-direct product by $\phi$. Then

$$\begin{aligned}
(\gamma,t)(\sigma,1)(\gamma,t)^{-1} &= (\gamma\phi(t)(\sigma),t)(\phi(t^{-1})(\gamma^{-1}),t^{-1}) \\
&= (\gamma\phi(t)(\sigma)\phi(t)\phi(t^{-1})(\gamma^{-1}),tt^{-1}) \\
&= (\gamma\phi(t)(\sigma)\gamma^{-1},1)
\end{aligned}$$

To complete the proof that $\Sigma_1$ is a normal subgroup of $\Gamma \rtimes_\alpha C_\infty$ it suffices to show that $\gamma\phi(t)(\sigma)\gamma^{-1}$ is an element of $\Sigma_1$. But $\phi(t)(\sigma) = s(t)\sigma s(t^{-1}) = \alpha(\sigma)$ and since $\alpha \in$ Aut $(\mathcal{E})$ we have that $\alpha(\sigma) \in \Sigma_1$. Hence $\gamma\phi(t)(\sigma)\gamma^{-1} = \gamma\sigma'\gamma^{-1}$ where $\sigma' \in \Sigma_1$. However, this is an inner automorphism of $\Sigma_1$ by elements in $\Gamma$ and $\Sigma_1$ is invariant under all inner automorphisms of $\Gamma$ since it is a normal subgroup of $\Gamma$. This proves the sub-lemma. □

By using this lemma we may see clearly that the extension $\mathcal{E} = (1 \to \Gamma \to G \to C_\infty \to 1)$ factorises to give the extensions

$$\mathcal{E}_1 = (1 \to \Sigma_1 \to G \to G/\Sigma_1 \to 1)$$

$$\mathcal{E}_2 = (1 \to \Sigma_2 \to G/\Sigma_1 \to C_\infty \to 1)$$

The kernels of both of these extensions are given by surface groups and so we may invoke the fibre smoothing theorem as before. The rest of the proof is identical to the earlier proof in the characteristic case except for the fact that we are now considering Aut $(\mathcal{E})$ instead of Aut (G). □

Now we shall generalise this theorem to poly-surface groups and thus to iterated surface fibrations to obtain the following result:

**Proposition 3.19** *An iterated surface fibration X, corresponding to an iterated extension of surface groups $\mathcal{E}$, gives rise to a surjective homomorphism*

$$\pi_0(\text{Diff X}) \twoheadrightarrow \text{Out }(\mathcal{E})$$

*where* Out $(\mathcal{E})$ *is a subgroup of finite index in* Out $(\pi_1(\text{X}))$.

**Proof :** The proof is inductive on the length of the poly-surface filtration. $\mathcal{E}$ is an extension of the form

$$1 \to \Gamma_1 \to \Gamma \to \Sigma \to 1$$

where $\Gamma_1$ is a poly-surface group of length $k-1$ and $\Sigma$ is a surface group. Now, let $\alpha$ be an automorphism of this extension so that $\alpha \in \text{Aut }(\mathcal{E})$ and consider the semi-direct product extension constructed by $\alpha$, $\Gamma \rtimes_\alpha C_\infty$ as before. We need to show that $\Gamma_1$ is a normal subgroup of the semi-direct product $\Gamma \rtimes_\alpha C_\infty$:

**Lemma 3.20** *The surface group $\Gamma_1$ is a normal subgroup of the semi-direct product $\Gamma \rtimes_\alpha C_\infty$ where $\alpha$ is an automorphism of the extension $\mathcal{E}$.*

The proof of this result is identical to the case above where $\Gamma$ is a poly-surface group of length 2. This implies that the extension $\mathcal{E}$ may be factorised into two extensions:

$$\mathcal{E}_1 = (1 \to \Gamma_1 \to G \to G/\Gamma_1 \to 1)$$

$$\mathcal{E}_2 = (1 \to \Sigma \to G/\Gamma_1 \to C_\infty \to 1).$$

The cohomological dimension of $C_\infty$ is 1 whilst $G$ has cohomological dimension $2k+1$ since it is an iterated extension of $k$ surface groups which are duality groups of dimension 2 by $C_\infty$. Therefore $G/\Gamma_1$ has dimension 3. Hence the quotients of these extensions are both countable groups with finite cohomological dimension and thus they have $\mathcal{H}$-realisations by theorem 3.9. Applying the Eilenberg-MacLane classifying space functor to both of these extensions we obtain fibrations

$$\xi_1 = \begin{cases} K(\Gamma_1, 1) \to & K(G,1) \\ & \downarrow \\ & K(G/\Gamma_1, 1) \end{cases}$$

$$\xi_2 = \begin{cases} K(\Sigma, 1) \to & K(G/\Gamma_1,1) \\ & \downarrow \\ & S^1 \end{cases}$$

The fibration $\xi_2$ has fibre corresponding to the Eilenberg-MacLane space of a surface, and so we may invoke the fibre smoothing property for surfaces and obtain a *smooth* fibre bundle $\widehat{\xi_2}$ as before. The fibre of $\xi_1$ is a poly-surface group of length $k-1$ which has the fibre smoothing property by the induction hypothesis. Therefore, $\xi_1$ may be smoothed to give a smooth fibre bundle $\widehat{\xi_1}$ as below.

$$\widehat{\xi_1} = \begin{cases} X \rightarrow & X_G \\ & \downarrow p_1 \\ & E_2 \end{cases}$$

$$\widehat{\xi_2} = \begin{cases} Y \rightarrow & E_2 \\ & \downarrow p_2 \\ & X_Q \end{cases}$$

where $\pi_1(X) = \Gamma_1$ and $\pi_1(Y) = \Sigma$. Now construct a smooth fibre bundle

$$\begin{cases} X_\Gamma \rightarrow & X_G \\ & \downarrow p_1 \\ & X_Q \end{cases}$$

where $X_\Gamma$ is fibred over $\Sigma$ with fibre given by $X_{\Gamma_1}$. The theorem follows by induction and the proof for a non-characteristic poly-surface group of length 2. $\square$

# Chapter 4

# The virtual cohomological dimensions of poly-Fuchsian automorphism groups

This chapter shall investigate the automorphism groups of certain poly-Fuchsian groups; in particular, we shall consider extensions of free groups and of orientable surface groups. This research was motivated by theorems due to Harer and Culler/Vogtmann who investigated the outer automorphism groups of surface groups and free groups respectively.

In 1986, Harer calculated the virtual cohomological dimension of the mapping class group of an orientable surface in his paper [Har]. The proof considered equivariant actions of the mapping class group on the Teichmüller space of markings on the associated Riemann surface.

**Theorem 4.1 (Harer, 1986)** *Let $\Sigma^{g,r}$ be a closed orientable surface of genus $g$ with $r$ boundary components. Write $\pi_1(\Sigma^{g,r}) = \Sigma_{g,r}$ and denote the mapping class group of the surface by* Out $(\Sigma_{\mathrm{g,r}})$. *Then* Out $(\Sigma_{\mathrm{g,r}})$ *is a virtual duality group in the sense of Bieri and Eckmann and furthermore, its virtual* *cohomological dimension satisfies*

$$\begin{aligned} \mathrm{vcd}\,(\mathrm{Out}\,(\Sigma_{\mathrm{g}})) &= 4g-5 && \text{when } \mathrm{r}=0, \text{ and} \\ \mathrm{vcd}\,(\mathrm{Out}\,(\Sigma_{\mathrm{g,r}})) &= 4g+2r-4 && \text{otherwise} \end{aligned}$$

An analogous theorem was proved in the same year for the outer automorphism group of a free group (see [CV]):

**Theorem 4.2 (Culler and Vogtmann)** *Let $F_n$ denote a free group of rank $n \geq 2$. Then the outer automorphism group* Out $(\mathrm{F_n})$ *is virtually torsion-free and its virtual cohomological dimension satisfies*

$$\mathrm{vcd}\,(\mathrm{Out}\,(\mathrm{F_n})) = 2\mathrm{n} - 3$$

*(Note that in the case of free groups, it is as yet unknown whether or not* Out $(\mathrm{F_n})$ *is a virtual duality group).*

In this chapter, we shall extend these results to poly-surface and poly-free groups in the case where the image of the operator homomorphism of the extension is finite. When the image of the operator homomorphism is infinite, the problem seems to be far more complex. However, we are still able to achieve some results in this situation using Thurston's theory of surface diffeomorphisms. This will be explored further in the next chapter.

## 4.1 Automorphisms of direct products

The first result concerns the nature of the direct product of automorphism groups of surface groups and free groups. Recall that the *wreath product* $H \int \mathcal{O}_n$ is defined to be the semi-direct product $H^{(n)} \rtimes \mathcal{O}_n$ where $\mathcal{O}_n$ is the symmetric group on $n$ elements, $H^{(n)}$ denotes the $n$-fold direct product of $H$ and the action of the symmetric group $\mathcal{O}_n$ on $H$ is by

$$\sigma(h_1, \ldots, h_n) = (h_{\sigma^{-1}(1)}, \ldots, h_{\sigma^{-1}(n)}).$$

We shall make use of the following result from [Joh3].

**Theorem 4.3 (F.E.A. Johnson)** *Let $K, Q$ be distinct surface groups or distinct free groups. Then the injective group homomorphism*

$$\natural : \mathrm{Aut}\ (\mathrm{K}) \times \mathrm{Aut}\ (\mathrm{Q}) \rightarrow \mathrm{Aut}\ (\mathrm{K} \times \mathrm{Q})$$

*given by $(\natural(a_1, a_2))(s_1, s_2) = (a_1(s_1), a_2(s_2))$ is in fact a group isomorphism. If $K = Q$, then the injective group homomorphism*

$$\natural : \mathrm{Aut}\ (\mathrm{K}) \int \sigma_2 \rightarrow \mathrm{Aut}\ (\mathrm{K}^{(2)})$$

*defined by $(\natural(a_1, a_2))(s_1, s_2) = (a_1(s_{\sigma^{-1}(1)}), a_2(s_{\sigma^{-1}(2)}))$ is a group isomorphism.*

**Corollary 4.4** *If $g \neq h$, then* $\mathrm{Out}\ (\Sigma_\mathrm{g}) \times \mathrm{Out}\ (\Sigma_\mathrm{h}) \cong \mathrm{Out}\ (\Sigma_\mathrm{g} \times \Sigma_\mathrm{h})$. *If $g = h$, then* $\mathrm{Out}\ (\Sigma_\mathrm{g}) \int \sigma_2 \cong \mathrm{Out}\ (\Sigma_\mathrm{g} \times \Sigma_\mathrm{g})$

In fact we shall give a new proof of this corollary for surface groups $\Sigma_g, \Sigma_h$ when $(g \neq h) \geq 3$. Recall that the class of *complete* groups consists of all centreless groups for which the outer automorphism group is trivial. A theorem by Ivanov proves that if $\Sigma_g \geq 3$ then

$$\mathrm{Out}\ (\mathrm{Out}\ (\Sigma_\mathrm{g})) = 1.$$

For a thorough analysis of this result see John McCarthy's paper [McC]. This theorem together with the well-known fact that the centre of $\mathrm{Out}\ (\Sigma_\mathrm{g})$ is trivial (e.g. [Iv1]) show that the mapping class group of a surface is a complete group. Dyer and Formanek demonstrated analogously that $\mathrm{Aut}\ (\mathrm{F_n})$ is a complete group in [DF] (and thus an analogous proof could be performed for free groups).

For complete groups there is the following characterisation of Hölder and Baer ( [Rob] p.398): a group $G$ is complete if and only if whenever $G \cong N$

and $N \triangleleft H$ then it follows that $N$ is a direct factor of $H$. Hence it is sufficient for us to prove that Out $(\Sigma_g)$ and Out $(\Sigma_h)$ are normal subgroups of Out $(\Sigma_g \times \Sigma_h)$. This is elementary to show and completes the proof. □

**Corollary 4.5** *For a direct product of two (not necessarily distinct) surface groups* $(\Sigma_g, \Sigma_h)$, *or free groups* $(F_m, F_n)$,

$$\mathrm{vcd}\ (\mathrm{Out}\ (\Sigma_g \times \Sigma_h)) = 4(g + h) - 10$$
$$\mathrm{vcd}\ (\mathrm{Out}\ (F_m \times F_n)) = 2(m + n) - 6$$

*Furthermore,* Out $(\Sigma_g \times \Sigma_h)$ *is a virtual duality group.*

**Proof :** First, let us suppose that $g \neq h$. Then we have that Out $(\Sigma_g \times \Sigma_h) \cong$ Out $(\Sigma_g) \times$ Out $(\Sigma_h)$. By Harer's theorem 4.1 Out $(\Sigma_g)$ is virtually torsion-free and so it has a torsion-free subgroup of finite index which we shall call $\mathrm{Out}_0$ $(\Sigma_g)$. Furthermore vcd (Out $(\Sigma_g)$) = cd ($\mathrm{Out}_0$ $(\Sigma_g)$). Putting these results together we have

$$\begin{aligned}
\mathrm{vcd}\ (\mathrm{Out}\ (\Sigma_g \times \Sigma_h)) &= \mathrm{vcd}\ (\mathrm{Out}\ (\Sigma_g) \times \mathrm{Out}\ (\Sigma_h)) \\
&= \mathrm{cd}\ (\mathrm{Out}_0\ (\Sigma_g)) + \mathrm{cd}\ (\mathrm{Out}_0\ (\Sigma_h)) \\
&= \mathrm{vcd}\ (\mathrm{Out}\ (\Sigma_g)) + \mathrm{vcd}\ (\mathrm{Out}\ (\Sigma_h)) \\
&= (4g - 5) + (4h - 5)
\end{aligned}$$

Analogously, when $m \neq n$, Out $(F_m \times F_n) \cong$ Out $(F_m) \times$ Out $(F_n)$. The theorem of Culler and Vogtmann 4.2 states that Out $(F_m)$ has a torsion-free subgroup of finite index denoted $\mathrm{Out}_0$ $(F_m)$ such that vcd (out$(F_m)$ = cd ($\mathrm{Out}_0$ $(F_m)$). Hence

$$\begin{aligned}
\mathrm{vcd}\ (\mathrm{Out}\ (F_m \times F_n)) &= \mathrm{vcd}\ (\mathrm{Out}\ (F_m) \times \mathrm{Out}\ (F_n)) \\
&= \mathrm{cd}\ (\mathrm{Out}_0\ (F_m)) + \mathrm{cd}\ (\mathrm{Out}_0\ (F_n)) \\
&= (2m - 3) + (2n - 3)
\end{aligned}$$

When $g = h$, by the above corollary Out $(\Sigma_g \times \Sigma_g) \cong$ Out $(\Sigma_g) \int \sigma_2$. But Out $(\Sigma_g) \times$ Out $(\Sigma_g)$ is a subgroup of index 2 of the group Out $(\Sigma_g) \int \sigma_2$ and so both of these groups have the same virtual cohomological dimension; that is,

$$\begin{aligned}
\text{vcd}\,(\text{Out}\,(\Sigma_g \times \Sigma_g)) &= \text{vcd}\,(\text{Out}\,(\Sigma_g) \int \sigma_2) \\
&= \text{vcd}\,(\text{Out}\,(\Sigma_g) \times \text{Out}\,(\Sigma_g)) \\
&= \text{cd}\,(\text{Out}_0\,(\Sigma_g)) + \text{cd}\,(\text{Out}_0\,(\Sigma_g)) \\
&= 8g - 10
\end{aligned}$$

Similarly, when $m = n$, Out $(F_m \times F_m) \cong$ Out $(F_m) \int \sigma_2$ and Out $(F_m) \times$ Out $(F_m)$ is a subgroup of index 2 of the group Out $(F_m) \int \sigma_2$. Therefore,

$$\begin{aligned}
\text{vcd}\,(\text{Out}\,(F_m \times F_m)) &= \text{vcd}\,(\text{Out}\,(F_m) \int \sigma_2) \\
&= \text{vcd}\,(\text{Out}\,(F_m) \times \text{Out}\,(F_m)) \\
&= 4m - 6
\end{aligned}$$

Furthermore, $\text{Out}_0\,(\Sigma_g) \times \text{Out}_0\,(\Sigma_h)$ is a (trivial) extension of two duality groups and so is again a duality group by Bieri and Eckmann with finite index in Out $(\Sigma_g \times \Sigma_h)$. This implies that Out $(\Sigma_g \times \Sigma_h)$ is a virtual duality group. □

By the Baer-Nielsen Theorem for surfaces, there is an isomorphism

$$\pi_0(\text{Diff}\,(\Sigma^g)) \cong \text{Out}\,(\Sigma_g)$$

and so both of these groups are referred to as the mapping class group of a surface. However, for a surface fibration $X_\Gamma$ with $\pi_1(X_\Gamma) = \Gamma$ the mapping class group is defined as the homotopy classes of self-diffeomorphisms and we have only demonstrated that the natural homomorphism between this group and the outer automorphism group of $\Gamma$ is surjective, not isomorphic. Since the majority of techniques used in this chapter are algebraic in nature, we shall focus our attention upon the group Out $(\Gamma)$. First however, from the above result, we can make the following deduction about the mapping class group of the direct product of two surfaces (cf. [Joh3] p.357):

**Proposition 4.6** *Let $\Sigma^i$ denote the closed surface of genus $i$, with fundamental group $\Sigma_i$. Then*

$$\text{vcd }(\pi_0(\text{Diff }(\Sigma^{\mathrm{g}} \times \Sigma^{\mathrm{h}})) = 4(\mathrm{g}+\mathrm{h}) - 10$$

**Proof :** The inclusion $j : \text{Diff }(\Sigma^{\mathrm{g}}) \times \text{Diff }(\Sigma^{\mathrm{h}}) \hookrightarrow \text{Diff }(\Sigma^{\mathrm{g}} \times \Sigma^{\mathrm{h}})$ induces a corresponding map between classifying spaces

$$j : \mathcal{W}(\text{Diff }(\Sigma^{\mathrm{g}}) \times \text{Diff }(\Sigma^{\mathrm{h}})) \hookrightarrow \mathcal{W}(\text{Diff }(\Sigma^{\mathrm{g}} \times \Sigma^{\mathrm{h}}))$$

(here $\mathcal{W}$ is the Eilenberg-MacLane classifying space functor - see Chapter 3). This together with the induced natural maps $\lambda_r : \text{Diff }(\mathrm{Y}) \to \text{Out }(\pi_1(\mathrm{Y}))$ gives rise to the following homotopy commutative diagram:

$$\begin{array}{ccccc}
\mathcal{W}\text{Diff }(\Sigma^{\mathrm{g}}) \times \mathcal{W}\text{Diff }(\Sigma^{\mathrm{h}}) & \xrightarrow{i} & \mathcal{W}(\text{Diff }(\Sigma^{\mathrm{g}}) \times \text{Diff }(\Sigma^{\mathrm{h}})) & \xrightarrow{j} & \mathcal{W}\text{Diff }(\Sigma^{\mathrm{g}} \times \Sigma^{\mathrm{h}}) \\
\downarrow \lambda_1 & & \downarrow \lambda_2 & & \downarrow \lambda_3 \\
\mathcal{W}\text{Out }(\Sigma_{\mathrm{g}}) \times \mathcal{W}\text{Out }(\Sigma_{\mathrm{h}}) & \xrightarrow{h} & \mathcal{W}(\text{Out }(\Sigma_{\mathrm{g}}) \times \text{Out }(\Sigma_{\mathrm{h}})) & \xrightarrow{k} & \mathcal{W}\text{Out }(\Sigma_{\mathrm{g}} \times \Sigma_{\mathrm{h}})
\end{array}$$

where $h$ and $i$ are homotopy equivalences and $k = \mathcal{W}(\natural\quad)$. In the proof of Theorem 3.11, it was shown that there is a homotopy equivalence $\mathcal{W}\text{Diff }(\Sigma^{\mathrm{g}}) \simeq \mathcal{W}\text{Out }(\Sigma_{\mathrm{g}})$ and so $\lambda_1$ has a homotopy right inverse. Hence $\lambda_2$ has a homotopy right inverse that we shall call $\mu$. Then

$$\nu = j \circ \mu \circ \mathcal{W}(\natural\quad)$$

is a homotopy right inverse for $\lambda_3$. This proves that

$$\text{vcd }(\pi_0(\text{Diff }(\Sigma^{\mathrm{g}} \times \Sigma^{\mathrm{h}})) = \text{vcd }(\text{Out }(\Sigma_{\mathrm{g}} \times \Sigma_{\mathrm{h}}))$$

as required. $\square$

## 4.2 The v.c.d. of the automorphism group of a poly-Fuchsian group

The following is a corollary to the theorems by Harer (4.1) and Culler/Vogtmann (4.2) which shall prove useful in this section:

**Proposition 4.7** *Given a surface group $\Sigma_g$ of genus $g \geq 2$ or a free group $F_n$ of rank $n \geq 2$, then the virtual cohomological dimensions of their automorphism groups are*

$$\begin{aligned} \text{vcd}\ (\text{Aut}\ (\Sigma_{\text{g}})) &= 4g - 3 && (4.1) \\ \text{vcd}\ (\text{Aut}\ (\text{F}_{\text{n}})) &\leq 2n - 2 && (4.2) \end{aligned}$$

*respectively.*

**Proof :** Given a group $K$ there is a natural exact sequence $1 \to Z(K) \to K \to \text{Inn (K)} \to 1$ from which we deduce that when the centre of $K$ is trivial (as is true for the surface groups and free groups under consideration), there is an isomorphism Inn (K) $\cong$ K. In this case, the automorphism group of $K$ may be written as an extension

$$1 \to K \to \text{Aut (K)} \to \text{Out (K)} \to 1.$$

When $K$ is given by a surface group $\Sigma_g$ then we know that Out $(\Sigma_{\text{g}})$ has a torsion-free subgroup $\text{Out}_0$ $(\Sigma_g)$ of finite index which is a duality group and has cohomological dimension $4g - 5$. Also $\Sigma_g$ has cohomological dimension 2 and is also a duality group (indeed a Poincaré duality group). Hence the extension of $\Sigma_g$ and Out $(\Sigma_{\text{g}})$ is again a duality group which we shall denote by $\text{Aut}_0$ $(\Sigma_g)$ and its cohomological dimension is given by

$$\begin{aligned} \text{cd}\ (\text{Aut}_0\ (\Sigma_{\text{g}})) &= \text{cd}\ (\Sigma_{\text{g}}) + \text{cd}\ (\text{Out}_0\ (\Sigma_{\text{g}})) \\ &= 2 + 4g - 5 \end{aligned}$$

(this equation follows from Bieri-Eckmann - see Theorem 1.17). $\mathrm{Aut}_0\,(\Sigma_g)$ is a subgroup of finite index in $\mathrm{Aut}\,(\Sigma_g)$ and so this shows that $\mathrm{vcd}\,(\mathrm{Aut}\,(\Sigma_g)) = 4g - 3$.

Now consider the case where $K$ is a free group $F_n$ of rank $n \geq 2$. This group is centreless so the exact sequence above still holds so that

$$1 \to F_n \to \mathrm{Aut}\,(\mathrm{F_n}) \to \mathrm{Out}\,(\mathrm{F_n}) \to 1.$$

The proof differs here from the surface case because it is unknown whether or not $\mathrm{Out}\,(\mathrm{F_n})$ is a virtual duality group. However, it is virtually torsion free by Theorem 4.2 and so we may form the extension of $F_n$ with the torsion-free subgroup of finite index $\mathrm{Out}_0\,(F_n)$ to give

$$1 \to F_n \to \mathrm{Aut}_0\,(F_n) \to \mathrm{Out}_0\,(F_n) \to 1$$

where $\mathrm{Aut}_0\,(F_n)$ is also torsion-free and has finite index in $\mathrm{Aut}\,(\mathrm{F_n})$. Then by a result of Serre [Ser] (Theorem 1.14:

$$\mathrm{cd}\,(\mathrm{Aut}_0\,(\mathrm{F_n})) \leq \mathrm{cd}\,(\mathrm{F_n}) + \mathrm{cd}\,(\mathrm{Out}_0\,(\mathrm{F_n}))$$

Free groups are Poincaré duality groups and have cohomological dimension equal to 1. Also $\mathrm{cd}\,(\mathrm{Out}_0\,(\mathrm{F_n})) = 2\mathrm{n} - 3$ by Culler and Vogtmann [CV] giving that

$$\begin{aligned} \mathrm{vcd}\,(\mathrm{Aut}\,(\mathrm{F_n})) &= \mathrm{cd}\,(\mathrm{Aut}_0\,(\mathrm{F_n})) \\ &\leq \mathrm{cd}\,(\mathrm{F_n}) + \mathrm{cd}\,(\mathrm{Out}_0\,(\mathrm{F_n})) \\ &= 1 + 2n - 3 \end{aligned}$$

as stated. □

**Proposition 4.8 (vcd (Aut (poly-Fuchsian group)))** *Let $\mathcal{E}$ be an extension $\mathcal{E} = (1 \to K \to G \to Q \to 1)$ where $K, Q$ are either (orientable) surface groups with genus $\geq 2$ or free groups of rank $\geq 2$.*

*(I): If $K, Q$ are surface groups of genus $g, h$ respectively, then*

$$\mathrm{vcd}\ (\mathrm{Aut}\ (\mathrm{G})) \leq 4(\mathrm{g}+\mathrm{h}) - 6.$$

*(II): If $K, Q$ are free groups of rank $m, n$ respectively, then*

$$\mathrm{vcd}\ (\mathrm{Aut}\ (\mathrm{G})) \leq 2(\mathrm{m}+\mathrm{n}) - 4.$$

*(III): If $K = \Sigma_g$ and $Q = F_n$ or vice versa, then*

$$\mathrm{vcd}\ (\mathrm{Aut}\ (\mathrm{G})) \leq 4\mathrm{g} + 2\mathrm{n} - 5.$$

**Proof :** Using the corollary to the Rigidity theorem 2.16, it suffices to calculate vcd (Aut ($\mathcal{E}$)) in each case. Since $K$ has trivial centre whether it is a surface group or a free group there is an injection

$$\mathrm{Aut}\ \mathcal{E} \rightarrowtail \mathrm{Aut}\ (\mathrm{K}) \times \mathrm{Aut}\ (\mathrm{Q})$$

and so Aut ($\mathcal{E}$) = Aut ($\mathcal{E}$) $\cap$ (Aut (K) $\times$ Aut (Q)) (see Chapter 1, Section 4). By the above proposition, in either case, Aut (K) has a torsion-free subgroup of finite index we shall denote by $\mathrm{Aut}_0$ ($K$). Hence Aut ($\mathcal{E}$) has a torsion-free subgroup given by

$$\mathrm{Aut}_0\ (\mathcal{E}) = \mathrm{Aut}\ (\mathcal{E}) \cap (\mathrm{Aut}_0\ (\mathrm{K}) \times \mathrm{Aut}_0\ (\mathrm{Q})$$

We need to show that $\mathrm{Aut}_0$ ($\mathcal{E}$) has finite index in Aut ($\mathcal{E}$). By the finite index lemma 1.10, if a group has a subgroup of finite index then this subgroup contains a finite index subgroup which is normal in the whole group. For Aut (K) and Aut (Q) we shall call these normal subgroups Aut $_1$(K) and Aut $_1$(Q) respectively. Consider the group Aut $_1(\mathcal{E})$ of the form Aut ($\mathcal{E}$) $\cap$

$(\mathrm{Aut}\ _1(K) \times \mathrm{Aut}\ _1(Q))$. This group is normal in Aut $(\mathcal{E})$ since normality is preserved by direct products and so we may take the quotient group:

$$\begin{aligned}\frac{\mathrm{Aut}\ (\mathcal{E})}{\mathrm{Aut}\ _1(\mathcal{E})} &= \frac{\mathrm{Aut}\ (\mathcal{E})}{\mathrm{Aut}\ (\mathcal{E}) \cap \mathrm{Aut}\ _1(K) \times \mathrm{Aut}\ _1(Q))} \\ &\cong \frac{(\mathrm{Aut}\ _1(K) \times \mathrm{Aut}\ _1(Q)) \cdot (\mathrm{Aut}\ (\mathcal{E}))}{\mathrm{Aut}\ _1(K) \times \mathrm{Aut}\ _1(Q)}\end{aligned}$$

This quotient group is a subgroup of $(\mathrm{Aut}\ (K) \times \mathrm{Aut}\ (Q))/(\mathrm{Aut}\ _1(K) \times \mathrm{Aut}\ _1(Q))$ which is a finite group since Aut $_1(K)$ and Aut $_1(Q)$ have finite index in Aut (K) and Aut (Q) respectively. Therefore Aut $_1(\mathcal{E})$ has finite index in Aut $(\mathcal{E})$ implying that $\mathrm{Aut}_0$ $(\mathcal{E})$ has finite index in Aut $(\mathcal{E})$ as required. Note also that $\mathrm{Aut}_0$ $(\mathcal{E})$ is clearly torsion-free. Now

$$\begin{aligned}\mathrm{vcd}\ (\mathrm{Aut}\ (G)) &= \mathrm{cd}\ (\mathrm{Aut}_0\ (\mathcal{E})) \\ &= \mathrm{cd}\ (\mathrm{Aut}\ (\mathcal{E}) \cap (\mathrm{Aut}_0\ (K) \times \mathrm{Aut}_0\ (Q))) \\ &\leq \mathrm{cd}\ (\mathrm{Aut}_0\ (K) \times \mathrm{Aut}_0\ (Q)) \\ &= \mathrm{cd}\ (\mathrm{Aut}_0\ (K)) + \mathrm{cd}\ (\mathrm{Aut}_0\ (Q)) \\ &= \mathrm{vcd}\ (\mathrm{Aut}\ (K)) + \mathrm{vcd}\ (\mathrm{Aut}\ (Q)).\end{aligned}$$

By substituting the v.c.d.'s of Aut $(\Sigma_g)$ and Aut $(F_n)$ using the previous proposition in each case we obtain the desired results. □

## 4.3 Calculating vcd (Out (G)) when the image of the operator homomorphism is finite

In this section, we shall consider an extension of either surface groups $\Sigma_g$ with genus $g \geq 2$ or free groups $F_n$ of rank $n \geq 2$:

$$1 \to K \to G \xrightarrow{\pi} Q \to 1$$

for which the operator homomorphism $\phi : Q \to \text{Out (K)}$ has finite image.

**Proposition 4.9** *Given an extension of surface groups* $1 \to \Sigma_g \to G \to \Sigma_h \to 1$ *where the image of the operator homomorphism is a finite group of order* $j$*, then* $G$ *has a subgroup of finite index* $G_0$*, which is of the form*

$$G_0 \cong \Sigma_g \times \Sigma_k \quad \textit{where } k = 1 + j(h-1).$$

*Similarly, given an extension of free groups* $1 \to F_m \to G \to F_n \to 1$ *where the image of the operator homomorphism is a finite group of order* $j$*, then* $G$ *has a subgroup of finite index* $G_0$*, which is of the form*

$$G_0 \cong F_m \times F_k \quad \textit{where } k = 1 + j(n-1)$$

**Proof :** As far as possible, we shall prove these two statements simultaneously by considering the exact sequence

$$1 \to K \to G \to Q \to 1$$

with operator homomorphism $\phi : Q \to \text{Out (K)}$. As the image of $\phi$ is finite, the kernel of $\phi$ has finite index in $Q$ and this implies that $\pi^{-1}(\text{ker } (\phi))$ is a subgroup of finite index in $G$. Define

$$G_0 = \pi^{-1}(\text{ker } (\phi)) \subset G$$

to be this subgroup of index $j$ in $G$ so that $G_0$ is an extension of the form

$$1 \to K \to G_0 \xrightarrow{\pi} \ker(\phi) \to 1.$$

The operator homomorphism of this extension $\phi' : \ker(\phi) \to \mathrm{Out}(K)$ is clearly trivial since it is a restriction of $\phi$ and so $G_0$ is isomorphic to a direct product

$$G_0 \cong K \times \ker(\phi)$$

When $Q$ is a surface group $\Sigma_h$ of genus $h$, $\ker(\phi)$ is a subgroup of index $j$ in $\Sigma_h$ and so must also be a surface group $\Sigma_k$, say. Moreover, the genus $k$ is given by the Riemann-Hurwitz formula:

$$j = \frac{rk(\Sigma_k) - \delta}{rk(\Sigma_h) - \delta}$$

where $\delta$ is equal to the cohomological dimension of $\Sigma_h$. This implies that $2 - 2k = j(2 - 2h)$ giving the result in this case.

Now consider the case where $Q$ is a free group $F_n$ of rank $n$. Again $\ker(\phi)$ is a subgroup of index $j$ in $\Sigma_h$ and therefore is a free group $F_k$, say. The rank $k$ is given by the Riemann-Hurwitz formula:

$$j = \frac{rk(F_k) - \delta}{rk(F_n) - \delta}$$

where $\delta$ is equal to 1 since $F_n$ is a free group. Hence $1 - k = j(1 - n)$ as required. □

**Theorem 4.10** *Let $K$ and $Q$ be either both surface groups $\Sigma_g$ or both free groups $F_n$. Given an exact sequence $1 \to K \to G \to Q \to 1$ where the image of the operator homomorphism is a finite group of order $j$, then*

$$\mathrm{vcd}(\mathrm{Out}(G)) = \mathrm{cd}\left(\frac{\mathrm{Aut}(G)}{\mathrm{H}} \cap \mathrm{Out}_0(\mathrm{H})\right)$$

*where $H$ is a subgroup of finite index in $G$ and* $\mathrm{Out}_0$ *$(H)$ is a subgroup of finite index in* Out (H).

*In particular, the virtual cohomological dimension of* Out (G) *is finite.*

**Proof :** By the above proposition, $G$ has a subgroup of finite index $G_0$ which is a direct product of surface groups or free groups and so we may invoke the finite index lemma 1.10 to show that $G$ has a characteristic subgroup of finite index $H$ which is contained in $G_0$. It is clear that $H$ is also a direct product of either surface groups or free groups and so we may write $H = H_s \times H_t$. Furthermore, $H$ has trivial centre since surface groups and free groups have no nontrivial abelian normal subgroups. The extension $\mathcal{E} = (1 \to H \to G \to G/H \to 1)$ gives rise to an exact sequence as in Section 2:

$$1 \to C(\mathcal{E}) \to \mathrm{Aut}\ (\mathcal{E}) \xrightarrow{\rho} \mathrm{Aut}\ (\mathrm{H}) \times \mathrm{Aut}\ (\mathrm{G/H})$$

and as before, this simplifies to an injection Aut $(\mathcal{E}) \rightarrowtail$ Aut (H)$\times$Aut (G/H). The fact that $H$ is a characteristic subgroup of $G$ implies that Aut $(\mathcal{E})$ = Aut (G) and so Aut (G) injects as

$$\rho : \mathrm{Aut}\ (\mathrm{G}) \rightarrowtail \mathrm{Aut}\ (\mathrm{H}) \times \mathrm{Aut}\ (\mathrm{G/H}). \tag{4.3}$$

We claim that vcd (Aut (G)) $\leq$ vcd (Aut (H)) in the following way: first observe that vcd (Aut (G)) $\leq$ vcd (Aut (H))$+$vcd (Aut (G/H)) using Proposition 1.15. Now, $H$ has finite index in $G$ so that $G/H$ is a finite group and hence has a finite automorphism group; that is, vcd (Aut (G/H)) $= 0$ proving the claim.

To continue our proof, take the induced homomorphism

$$\rho_* : \mathrm{Aut}\ (\mathrm{G})/\mathrm{H} \rightarrowtail \mathrm{Aut}\ (\mathrm{H})/\mathrm{H} \times \mathrm{Aut}\ (\mathrm{G/H}). \tag{4.4}$$

Because $H$ has trivial centre, Inn (H) $\cong$ H which implies that Aut (H)/H $\cong$ Aut (H)/Inn (H) $\cong$ Out (H). Also by Theorem 4.3, Out (H) $\cong$ Out ($\mathrm{H_s}$) $\times$

Out ($H_t$) and the theorems by Harer 4.1 and Culler/Vogtmann 4.2 show that Out (H) has a torsion-free subgroup of finite index $\mathrm{Out}_0\ (H_s) \times \mathrm{Out}_0\ (H_t)$. We shall denote this subgroup by $\mathrm{Out}_0\ (H)$.

The rest of the proof involves finding a torsion-free subgroup of finite index in Aut (G)/H and projecting this onto Out (G). Consider the group

$$\frac{\mathrm{Aut}\ (\mathrm{G})}{H} \cap \mathrm{Out}_0\ (H).$$

This is clearly torsion-free so it suffices to show that it has finite index in Aut (G)/H. Using the induced homomorphism 4.4, let

$$\mathrm{Aut}\ (\mathrm{G})/\mathrm{H} = (\mathrm{Aut}\ (\mathrm{G})/\mathrm{H}) \cap (\mathrm{Out}\ (\mathrm{H}) \times \mathrm{Aut}\ (\mathrm{G}/\mathrm{H})).$$

Then, as $\mathrm{Out}_0\ (H)$ has finite index in Out (H) and Aut (G/H) is a finite group, $\mathrm{Out}_0\ (H)$ also has finite index in Out (H) $\times$ Aut (G/H). This shows that (Aut (G)/H) $\cap$ ($\mathrm{Out}_0$ (H)) is a subgroup of finite index of Aut (G)/H $\cap$ (Out (H) $\times$ Aut (G/H)) = Aut (G)/H.

As $G$ has trivial centre, there is an exact sequence $1 \to G \to \mathrm{Aut}\ (\mathrm{G}) \to \mathrm{Out}\ (\mathrm{G}) \to 1$ which gives rise to the exact sequence

$$1 \to \frac{G}{H} \to \frac{\mathrm{Aut}\ (\mathrm{G})}{H} \xrightarrow{p} \mathrm{Out}\ (\mathrm{G}) \to 1$$

The kernel of the projection $p$ is given by the image of $G/H$ in Aut (G)/H which is finite since $H$ is a subgroup of finite index in $G$. Therefore, by projecting a torsion-free subgroup, the kernel must be trivial and hence $p$ becomes an isomorphism of groups. By considering the torsion-free subgroup (Aut (G)/H) $\cap$ ($\mathrm{Out}_0$ (H)) which has finite index in Aut (G)/H, it is clear that

$$p\left(\frac{\mathrm{Aut}\ (\mathrm{G})}{H} \cap \mathrm{Out}_0\ (H)\right)$$

is also a torsion-free subgroup of finite index in Out (G) as required.

In particular, this proof shows that vcd (Out (G)) $\leq$ cd ($\mathrm{Out}_0$ (H)). Furthermore, cd ($\mathrm{Out}_0$ (H)) = cd ($\mathrm{Out}_0$ ($\mathrm{H_s}$)) + cd ($\mathrm{Out}_0$ ($\mathrm{H_t}$)) = $4(\mathrm{s}+\mathrm{t}) - 10$

if $H_*$ are surface groups and $= 2(s+t)-6$ if $H_*$ are free groups. Also, $s \geq h$ and $t \geq 1 + j(g-1)$ and hence vcd (Out (G)) is finite. □

## 4.4 An exact sequence for Out ($\mathcal{E}$)

The purpose of this section is to calculate an exact sequence for the outer automorphism group of an extension consisting of *centreless* groups. This reduces the calculation of the v.c.d. of the outer automorphism group to the corresponding calculation for the ends of the exact sequence. Although we have been able to find the virtual cohomological dimension for the outer automorphism group of poly-Fuchsian groups when the image of the operator homomorphism is finite, these methods do not suffice when the operator homomorphism has infinite image. In the next chapter we shall use the results of this section to calculate the v.c.d. in the case where the image of the operator homomorphism is an infinite group generated by certain surface diffeomorphisms.

From now on, we shall consider the *centreless* groups $K, Q$ belonging to the extension

$$\mathcal{E} = (1 \to K \to G \to Q \to 1).$$

In this case we have the following lemma:

**Lemma 4.11** *For the above extension where $Z(K) = Z(Q) = 1$,*

$$\text{Out } (\mathcal{E}) \cong ((\text{Aut } (\mathcal{E}))/\text{K})/\text{Q}$$

**Proof :** First, observe that $G$ must also have trivial centre so that $G \cong$ Inn (G). Thus there is an exact sequence $1 \to G \to \text{Aut } (\mathcal{E}) \to \text{Out } (\mathcal{E}) \to 1$. Quotienting out the kernel by $K$ gives the exact sequence $1 \to Q \to$

Aut $(\mathcal{E})/\mathrm{K} \to \mathrm{Out}\ (\mathcal{E}) \to 1$ as required. $\square$

**Theorem 4.12 (Exact sequence for Out ($\mathcal{E}$))** *Let $G$ be a split extension of centreless groups $K, Q$ in the exact sequence: $\mathcal{E} = (1 \to K \to G \to Q \to 1)$; then* Out ($\mathcal{E}$) *is constructed by an exact sequence:*

$$1 \to \frac{\mathrm{Stab}_{\mathrm{Aut}(Q)}(\phi)}{\ker\phi} \to \mathrm{Out}\ (\mathcal{E}) \to \frac{\mathrm{Im}\ (\mathrm{proj}_1)}{\mathrm{Im}\ (\phi)} \to 1$$

*where $\phi$ is the operator homomorphism of the extension,* $\mathrm{Stab}_{\mathrm{Aut}(Q)}(\phi)$ *denotes the stabiliser of $\phi$ in* Aut (Q)*, and*

$$\mathrm{C}_{\mathrm{Out}(K)}(\mathrm{Im}(\phi)) \leq \mathrm{Im}\ (\mathrm{proj}_1) \leq \mathrm{N}_{\mathrm{Out(K)}}(\mathrm{Im}(\phi))$$

The proof of this theorem is derived from the following sequence of propositions. Observe that, since we have constructed $\mathcal{E}$ as a split extension we may take $G$ to be the semi-direct product $K \rtimes_{\hat{\phi}} Q$ where $\hat{\phi} : Q \to \mathrm{Aut}\ (\mathrm{K})$ is the conjugating homomorphism. When $Q$ is a free group then the exact sequence $\mathcal{E}$ is always split.

**Proposition 4.13** *Let $\alpha \in$ Aut (K) and $\beta \in$ Aut (Q) be automorphisms of $K$ and $Q$ respectively. Then $(\alpha, \beta) \in$ Aut (K) $\times$ Aut (Q) is an automorphism of $G$ if the following condition holds:*

$$\hat{\phi}(q) = \alpha^{-1} \cdot \hat{\phi}(\beta(q)) \cdot \alpha$$

**Proof :** $G = \{(k, q) \in K \times Q : (k_1, q_1)(k_2, q_2) = (k_1\hat{\phi}(q_1)(k_2), q_1q_2)\}$

$$\begin{aligned}(\alpha,\beta)((k_1,q_1)(k_2,q_2)) &= (\alpha,\beta)(k_1,q_1).(\alpha,\beta)(k_2,q_2)\\ &= (\alpha(k_1),\beta(q_1))(\alpha(k_2),\beta(q_2))\\ &= (\alpha(k_1)\hat{\phi}(q_1)(k_2),\beta(q_1)\beta(q_2))\end{aligned}$$

Also,

$$\begin{aligned}
(\alpha,\beta)((k_1,q_1)(k_2,q_2)) &= (\alpha,\beta)(k_1\hat{\phi}(q_1)(k_2),q_1,q_2) \\
&= (\alpha(k_1\hat{\phi}(q_1)(k_2),\beta(q_1q_2))
\end{aligned}$$

Comparing terms, it is clear that

$$\begin{aligned}
\alpha(k_1)\hat{\phi}(\beta(q_1)) &= \alpha(k_1)\alpha(\hat{\phi}(q_1)(k_2)) \\
\Longrightarrow \qquad \hat{\phi}(\beta(q_1))(\alpha(k_2)) &= \alpha(\hat{\phi}(q_1)(k_2)) \\
\Longrightarrow \qquad \hat{\phi}(q_1)(k_2) &= \alpha^{-1}(\hat{\phi}(\beta(q_1))(\alpha(k_2)))
\end{aligned}$$

Therefore, for all $k \in K$, given $q \in Q$ the following formula holds:

$$\hat{\phi}(q) = \alpha^{-1} \cdot (\hat{\phi}(\beta(q))) \cdot \alpha.$$

□

**Proposition 4.14** Aut (K)×Aut (Q) *has a right action on* Hom$(Q, \text{Aut (K)})$ *given by*

$$(\hat{\phi} \cdot (\alpha,\beta))(q) = \alpha^{-1}(\hat{\phi}\beta)(q)\alpha$$

**Proof :** First, observe that $(\hat{\phi} \cdot (e,e))(q) = e^{-1}(\hat{\phi}e)(q)e = \hat{\phi}(q)$ where $(e,e)$ is the identity of Aut (K) × Aut (Q). Now,

$$\begin{aligned}
(\hat{\phi} \cdot (\alpha_1,\beta_1)(\alpha_2,\beta_2))(q) &= (\hat{\phi} \cdot (\alpha_1\alpha_2,\beta_1\beta_2))(q) \\
&= (\alpha_1\alpha_2)^{-1}(\hat{\phi}\beta_1\beta_2)(q)\alpha_1\alpha_2 \\
&= \alpha_2^{-1}(\hat{\phi}'\beta_2)(q)\alpha_2 \\
&= (\hat{\phi}' \cdot (\alpha_2,\beta_2))(q)
\end{aligned}$$

where $\hat{\phi}' = (\hat{\phi} \cdot (\alpha_1,\beta_1))$, proving that this is indeed a right action. □

From the section on automorphisms of extensions in Chapter 1, we know that since $K$ has trivial centre, there is an injection

$$\text{Aut }(\mathcal{E}) \rightarrowtail \text{Aut (K)} \times \text{Aut (Q)}$$

Also, we know that $(\alpha, \beta) \in$ Aut (G) must satisfy $\hat{\phi}(q) = \alpha^{-1} \cdot \hat{\phi}(\beta(q)) \cdot \alpha$. Combining these conditions we see that

$$\text{Aut }(\mathcal{E}) = \{(\alpha, \beta) \in \text{Aut (K)} \times \text{Aut (Q)} : \hat{\phi}(\text{q}) = \alpha^{-1} \cdot \hat{\phi}(\beta(\text{q})) \cdot \alpha\}$$

**Proposition 4.15** *Given the extension $\mathcal{E} = (1 \to K \to G \to Q \to 1)$ where $K$ has trivial centre, then the automorphism group of the extension* Aut ($\mathcal{E}$) *has $K$ as a normal subgroup.*

**Proof :** Define $i_k$ to be an inner automorphism of $K$ of the form $i_k(x) = kxk^{-1}$ for $x \in K$. As before, let $\alpha, \beta$ be automorphisms of $K, Q$ respectively. Then there is a series of mappings between exact sequences:

$$\begin{array}{ccccccccc}
1 & \to & K & \to & G & \to & Q & \to & 1 \\
 & & \downarrow \alpha^{-1} & & \downarrow (\alpha,\beta)^{-1} & & \downarrow \beta^{-1} & & \\
1 & \to & K & \to & G & \to & Q & \to & 1 \\
 & & \downarrow i_k & & \downarrow i_k & & \| & & \\
1 & \to & K & \to & G & \to & Q & \to & 1 \\
 & & \downarrow \alpha & & \downarrow (\alpha,\beta) & & \downarrow \beta & & \\
1 & \to & K & \to & G & \to & Q & \to & 1
\end{array}$$

Concertinaing, we obtain homomorphisms

$$\begin{array}{ccccccccc}
1 & \to & K & \to & G & \to & Q & \to & 1 \\
 & & \downarrow i_{\alpha^{-1}(k)} & & \downarrow i_{\alpha^{-1}(k)} & & \| & & \\
1 & \to & K & \to & G & \to & Q & \to & 1
\end{array}$$

To see this, we just need to note that for $k_1, k_2 \in K$ with $\alpha(k_1) = k_2$:

$$(\alpha^{-1} i_k \alpha)(k_1) \quad = \quad (\alpha^{-1} i_k)(k_2)$$

$$
\begin{aligned}
&= \alpha^{-1}(kk_2k^{-1}) \\
&= \alpha^{-1}(k)k_1\alpha(k) \\
&= i_{\alpha^{-1}(k)}
\end{aligned}
$$

Therefore $(\alpha, \beta)^{-1} i_k(\alpha, \beta) = i_{\alpha^{-1}(k)}$ and so the group of inner automorphisms of $K$ which is isomorphic to $K$ is closed under conjugation by elements of Aut $(\mathcal{E})$. $\square$

**Proposition 4.16** *The group $K$ is isomorphic to the subgroup of* Aut $(\mathcal{E})$ *given by*

$$\{(c_k, 1) \in \text{Aut (K)} \times \text{Aut (Q)} : \text{k} \in \text{K}\}$$

*where $c_k$ denotes conjugation by the element $k$.*

**Proof :** There is a map $\sigma : K \to$ Aut $(\mathcal{E})$ given by $k \mapsto \{g \mapsto kgk^{-1}\}$ for $k \in K, g \in G$. If $c_k$ denotes conjugation by $k$ (so that $c_k(g) = kgk^{-1}$), then it is clear that $\{c_k : k \in K\} =$ Inn (K) $=$ K as $K$ has trivial centre. Therefore it suffices to prove that $K$ is contained in the kernel of the map $\tau :$ Aut $(\mathcal{E}) \to$ Aut (Q) given by $\tau(\{g \mapsto kgk^{-1}\}) = \{\pi(g) \mapsto \pi(kgk^{-1})\}$ where $\pi : G \to Q$ is the quotient map. Now, $\pi(kgk^{-1}) = \pi(g)$ because $\pi(k) = \pi(k^{-1}) = 1$ and so the image of $\tau$ is the identity automorphism and this proves our claim. $\square$

With this proposition it is clear that

$$\frac{\text{Aut }(\mathcal{E})}{K} = \left\{([\alpha], \beta) \in \frac{\text{Aut (K)}}{K} \times \text{Aut (Q)} : \phi(\text{q}) = [\alpha]^{-1}\phi(\beta(\text{q}))[\alpha]\right\}$$

where $[\alpha]$ denotes the conjugacy class of $\alpha$ in Aut (K) and $\phi = [.]\hat{\phi}$ so that Im $(\phi) \in$ Out (K). What is $\phi$? The conjugating homomorphism of an extension depends on the choice of the transversal function from $Q$ to

$G$. Two transversal functions differ by an element of $K$ and further, the associated conjugating homomorphism are equal up to inner automorphisms of $K$. However, the operator homomorphism is only defined up to inner automorphisms of $K$ and since $K$ has trivial centre, Inn (K) = K. This shows that the conjugacy class of the conjugating homomorphism is the operator homomorphism which we shall identify with $\phi$.

**Proposition 4.17** *$Q$ is contained in* Aut $(\mathcal{E})$/K *by the map $q \mapsto (\phi(q), c_q)$, as a normal subgroup.*

**Proof :** It is obvious that $(\phi(q), c_q) \in$ Out (K) $\times$ Aut (Q) and hence we just need to show that $\phi(\beta(y)) = [\alpha]\phi(y)[\alpha]^{-1}$ where $([\alpha], \beta) = (\phi(q), c_q)$ as follows:

$$\begin{aligned} \phi(\beta(y)) &= \phi(c_q(y)) \\ &= \phi(q)\phi(y)\phi(q)^{-1} \\ &= [\alpha]\phi(y)[\alpha]^{-1} \end{aligned}$$

To show that $Q$ is a normal subgroup of Aut $(\mathcal{E})$/K, consider the following:

$$\begin{aligned} ([\alpha], \beta)(\phi(q), c_q)([\alpha], \beta)^{-1} &= ([\alpha]\phi(q)[\alpha]^{-1}, \beta c_q \beta^{-1}) \\ &= (\phi(\beta(q)), \beta c_q \beta^{-1}) \end{aligned}$$

But $(\beta c_q \beta^{-1})(y) = \beta(q\beta^{-1}(y)q^{-1}) = \beta(q)y\beta(q)^{-1} = c_{\beta(q)}$, proving that Q is invariant under inner automorphisms of Aut $(\mathcal{E})$/K. □

Consider the projection $\text{proj}_1$ : Aut $(\mathcal{E})$/K $\to$ Out (K) given by

$$\text{proj}_1([\alpha], \beta) = [\alpha].$$

This will correspond to the projection map of the exact sequence for Out $(\mathcal{E})$.

**Proposition 4.18** $\ker(\mathrm{proj}_1) = \mathrm{Stab}_{\mathrm{Aut}(Q)}(\phi)$, *and*

$$C_{\mathrm{Out}(K)}(\mathrm{Im}(\phi)) \leq \mathrm{Im}\ (\mathrm{proj}_1) \leq N_{\mathrm{Out(K)}}(\mathrm{Im}(\phi))$$

**Proof :**

$$\begin{aligned}
\ker(\mathrm{proj}_1) &= \{([1], \beta) \in \mathrm{Out\ (K)} \times \mathrm{Aut\ (Q)} : \phi\beta = [1]\phi[1]^{-1}\} \\
&= \{\beta \in \mathrm{Aut\ (Q)} : \phi\beta = \phi\} \\
&= \mathrm{Stab}_{\mathrm{Aut}(Q)}(\phi)
\end{aligned}$$

where the stabiliser acts on $\phi$ by the induced right action of Aut (Q) upon Hom $(Q, \mathrm{Out\ (K)})$, given by $(\phi \cdot \beta)(q) = \phi(\beta(q))$. Now take $x \in \mathrm{Im}\ (\phi)$ and write $x = \phi(q)$ for some $q \in Q$. We have that $[\alpha]x[\alpha]^{-1} = \phi(\beta(q)) \in \mathrm{Im}\ (\phi)$ by the condition on Aut $(\mathcal{E})$/K and this implies that $[\alpha]$ normalises the elements of Im $(\phi)$. Conversely, if $[\alpha] \in C_{\mathrm{Out}(K)}(\mathrm{Im}(\phi))$, then for all $q$ in $Q$,

$$[\alpha]\phi(q)[\alpha]^{-1} = \phi(q)$$

which is derived from $\mathrm{proj}_1([\alpha], 1) = [\alpha]$. These results show that

$$C_{\mathrm{Out}(K)}(\mathrm{Im}(\phi)) \leq \mathrm{Im}\ (\mathrm{proj}_1) \leq N_{\mathrm{Out(K)}}(\mathrm{Im}(\phi)).$$

□

Hence we have constructed an exact sequence

$$1 \to \mathrm{Stab}_{\mathrm{Aut}(Q)}(\phi) \to \mathrm{Aut}\ (\mathcal{E})/\mathrm{K} \to \mathrm{Im}\ (\mathrm{proj}_1) \to 1$$

In order to obtain an exact sequence involving Out $(\mathcal{E})$, it is necessary to quotient Aut $(\mathcal{E})$ by $Q$, and thus, we must establish the image of $Q$ under $\mathrm{proj}_1$. $Q = \{(\phi(q), c_q) : q \in Q\}$, so $\mathrm{proj}_1(Q) = \{\phi(q) : q \in Q\} = \mathrm{Im}\ (\phi)$. Similarly, $\ker(\mathrm{proj}_1(Q)) = \{q \in Q : \phi(q) = q\} = \ker(\phi)$. Therefore, by quotienting out by the exact sequence

$$1 \to \ker(\phi) \to Q \xrightarrow{\mathrm{proj}_1} \mathrm{Im}\ (\phi) \to 1$$

we obtain the required exact sequence from the statement of the Theorem:

$$1 \to \frac{\mathrm{Stab}_{\mathrm{Aut}(Q)}(\phi)}{\ker\phi} \to \mathrm{Out}\ (\mathcal{E}) \to \frac{\mathrm{Im}\ (\mathrm{proj}_1)}{\mathrm{Im}\ (\phi)} \to 1$$

where $\phi$ is the operator homomorphism of the extension, and

$$\mathrm{C}_{\mathrm{Out}(K)}(\mathrm{Im}(\phi)) \leq \mathrm{Im}\ (\mathrm{proj}_1) \leq \mathrm{N}_{\mathrm{Out(K)}}(\mathrm{Im}(\phi))$$

as required. □

This exact sequence reduces the problem of finding the virtual cohomological dimension for the outer automorphism group of a poly-Fuchsian group to finding the v.c.d.'s of the kernel and quotient of the sequence. In the next chapter we shall calculate the v.c.d. of

$$\frac{\mathrm{Im}\ (\mathrm{proj}_1)}{\mathrm{Im}\ (\phi)}$$

in the case where the image of the operator homomorphism consists of certain diffeomorphisms about separating circles in a surface. This result will require background work in Thurston's theory of surface diffeomorphisms and shows the breadth of the problem when the operator homomorphism has infinite image.

# Chapter 5

# Pseudo-Anosov diffeomorphisms and Stallings fibrations

## 5.1 A menagerie of surface diffeomorphisms

Given a simple closed curve $C$ on a surface $\Sigma$ we may construct a homeomorphism of the surface in the following way. Parametrise an annulus in the plane by $(r, \theta)$ where $1 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$. Then a homeomorphism $\tau$ of the annulus may be defined by

$$\tau(r, \theta) = (r, \theta - 2\pi r)$$

Embed the annulus as a neighbourhood of the curve $C$ and extend the homeomorphism by the identity outside the embedded annulus. This gives a homeomorphism of the surface known as the *Dehn twist homeomorphism* about $C$ (see figure 5.1).

These homeomorphisms epitomise surface homeomorphisms in the sense that all homeomorphisms of a surface can be created by composing a finite

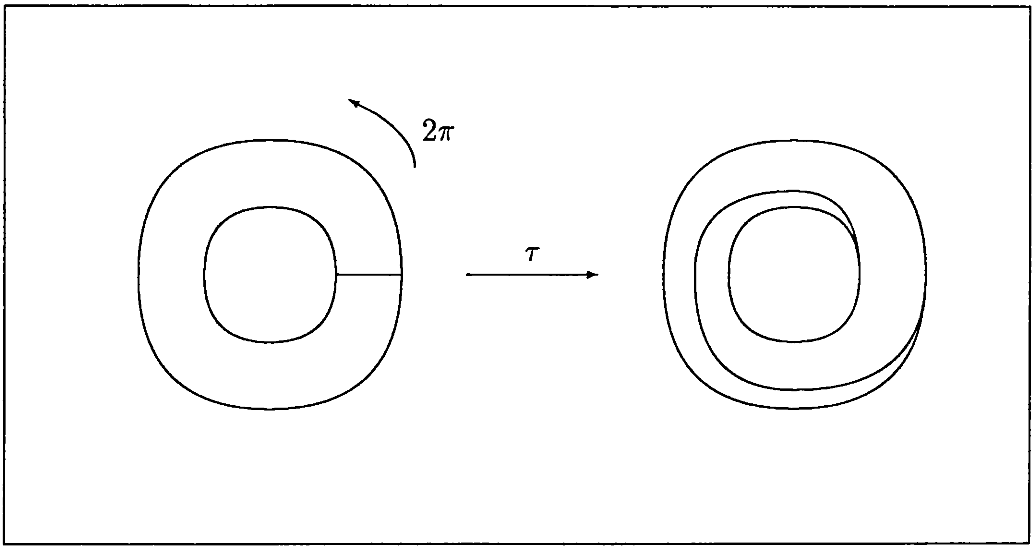


Figure 5.1: The Dehn twist homeomorphism

number of Dehn twist homeomorphisms. This was proved by Lickorish ( [Lic]) who also gave a canonical representation for the generating Dehn twists about non-separating circles in the surface.

Here, we shall describe the classification of surface diffeomorphisms due to Thurston: The elements of the mapping class group Out $(\pi_1(\Sigma)) \cong \pi_0(\text{Diff }(\Sigma))$ consist of isotopy classes of surface diffeomorphisms. We shall call an element of Out $(\pi_1(\Sigma))$ *periodic* if it has finite order in the group. For these isotopy classes there is the classical theorem due to Nielsen (see, for example, Birman's article in [Harv]) :

**Theorem 5.1 (Nielsen)** *An element $f \in$ Out $(\pi_1(\Sigma))$ is finite if and only if the isotopy class contains a periodic diffeomorphism $F : \Sigma \to \Sigma$ such that*

$$F^n = Id_\Sigma$$

*for some $n$.*

A circle on $\Sigma$ is nontrivial if it does not bound a disc in the surface and cannot be deformed into a boundary component. In this situation, a one-dimensional submanifold $\mathcal{C}$ of $\Sigma$ consists of several pairwise nonisotopic circles. If the diffeomorphism $F : \Sigma \to \Sigma$ satisfies

$$F(\mathcal{C}) = \mathcal{C}$$

for some non-empty one-dimensional submanifold of $\Sigma$ consisting of non-trivial circles then we say that $F$ is a *reducible diffeomorphism.* An element of Out $(\pi_1(\Sigma))$; that is , an isotopy class of diffeomorphisms of $\Sigma$ is called *reducible* if it contains a reducible diffeomorphism and *irreducible* otherwise.

The main achievement of Thurston's theory is the existence of *pseudo-Anosov* diffeomorphisms in nonperiodic irreducible isotopy classes (see [Thu]). The following description of pseudo-Anosov diffeomorphisms is adapted from the book by Casson and Bleiler [CB]:

Recall that a *geodesic* in the hyperbolic plane $\mathcal{H}^2$ is a circle meeting the boundary of the hyperbolic plane orthogonally. In the surface $\Sigma$, a geodesic is the image of a geodesic in its universal cover. This geodesic is said to be *simple* if it has no transverse self-intersections. By taking a disjoint union $L$ of simple geodesics in a surface we obtain a geodesic *lamination*; the geodesics are the *leaves* of the lamination. The surface $\Sigma$ can be decomposed into a disjoint union of leaves together with a singular set of points. This decomposition is called a *singular foliation* $\mathcal{F}$. Two foliations are *transverse* if they have the same singular set and at all other points the leaves are transverse. A *transverse measure* $\mu$ to a singular foliation $\mathcal{F}$ assigns to each arc $\alpha$ transverse to $\mathcal{F}$ a non-positive Borel measure $\mu|_\alpha$ such that:

(i): for a subarc $\beta$ of $\alpha$, $\mu|_\beta$ is the restriction of $\mu|_\alpha$.

(ii): If $\alpha_0, \alpha_1$ are arcs transverse to $\mathcal{F}$ related by a homotopy $\alpha : I \times I \to \Sigma$ such that $\alpha(I \times 0) = \alpha_0$, $\alpha(I \times 1) = \alpha_1$ and $\alpha(a \times I)$ is contained in a leaf of $\mathcal{F}$ for all $a \in I$, then $\mu|_{\alpha_0} = \mu|_{\alpha_1}$.

We are now in a position to give a formal definition of a pseudo-Anosov diffeomorphism:

A diffeomorphism $h$ of a closed orientable surface is *pseudo-Anosov* if there exist transverse singular foliations $\mathcal{F}^s, \mathcal{F}^u$ equipped with transverse measures $\mu^s, \mu^u$ such that:

$$\begin{aligned} h(\mathcal{F}^s, \mu^s) &= (\mathcal{F}^s, \lambda\mu^s) \\ h(\mathcal{F}^u, \mu^u) &= (\mathcal{F}^u, \lambda^{-1}\mu^u) \end{aligned}$$

for some $\lambda > 1$.

**Theorem 5.2 (Thurston [Thu])**
*Every nonperiodic irreducible diffeomorphism of a closed orientable surface of genus greater than* 2 *is isotopic to a pseudo-Anosov diffeomorphism. Irreducible and nonperiodic isotopy classes in* Out $(\pi_1(\Sigma))$ *are called pseudo-Anosov isotopy classes.*

Related to the concept of a pseudo-Anosov diffeomorphism of a surface are pure diffeomorphisms. These were first used by Ivanov (see [Iv2]) in connection with his classification of subgroups of the mapping class group of a surface. We shall make use of pure diffeomorphisms in the next section; first, here is the definition:

By cutting the surface along the 1-submanifold $\mathcal{C}$ we obtain a new (possibly disconnected) surface $\Sigma_{\mathcal{C}}$. We will call a diffeomorphism $F$ of the surface, a *pure* diffeomorphism if for some system of circles $\mathcal{C}$ the following condition is fulfilled:

> (Pure Diffeomorphism): All points of $\mathcal{C}$ and the boundary of $\Sigma$ are fixed by $F$; $F$ does not permute the components of $\Sigma \setminus \mathcal{C}$ and induces on every component of $\Sigma_{\mathcal{C}}$ a diffeomorphism isotopic either to a pseudo-Anosov diffeomorphism or to the identity.

The isotopy classes of pure diffeomorphisms are *pure* elements of the mapping class group of the surface.

## 5.2 Subgroups of the mapping class group

The importance of the concept of pure diffeomorphisms described in the previous section lies in the fact that Out $(\Sigma)$ contains a subgroup of finite index consisting entirely of pure elements. Namely, let $I_\Sigma(m)$, $m \in \mathcal{Z}$ be the kernel of the natural homomorphism:

$$\text{Out }(\Sigma) \to \text{Aut }(\text{H}_1(\Sigma; \mathcal{Z}_\text{m}))$$

This subgroup of the mapping class group is known as the *congruence subgroup of level* $m$ and, if $m \neq 0$, it clearly has finite index in Out $(\Sigma)$. Serre proved that the congruence subgroups of level $\geq 3$ are torsion-free. This result was sharpened by Ivanov in [Iv2]:

**Theorem 5.3** *If* $m \geq 3$*, then all the elements of* $I_\Sigma(m)$ *are pure.*

Ivanov used the result to classify subgroups of the mapping class group analogously to Thurston's classification of surface diffeomorphisms. From now on, we shall tacitly assume that $m \geq 3$ and denote the congruence subgroup by $I_\Sigma$.

The theory of pure diffeomorphisms links into the problem of calculating virtual cohomological dimensions by virtue of the following result on centralisers of the mapping class group. Ivanov proved a similar theorem in [Iv1] and our method of proof is loosely based on his approach. Note first, that two groups $G, H$ are called *commensurable* if they have isomorphic subgroups of finite index. We shall notate this by $G \sim H$.

**Theorem 5.4** *Let* $C$ *consist of several pairwise nonisotopic nontrivial circles on* $\Sigma$ *and let* $A(C)$ *be the group generated by Dehn twist homeomorphisms* *about the components of $\mathcal{C}$. Cut $\Sigma$ along $\mathcal{C}$ and call the resulting (t, say) components $\Sigma_{g_i,r_i}$ where $g_i$ is the genus and $r_i$ is the number of boundary components of the surface. Denote the mapping class group of $\Sigma_{g_i,r_i}$ by $\Gamma_{g_i,r_i}$. Let $I_\Sigma$ be the kernel of the homomorphism $h$ : Out $(\Sigma) \to$ Aut $(\mathrm{H}_1(\Sigma;\mathcal{Z}_{\mathrm{m}}))$ and write $A = A(\mathcal{C}) \cap I_\Sigma(m)$. Then*

$$\frac{C_{I_\Sigma(m)}(A)}{A} \sim \Gamma_{g_1,r_1} \times \Gamma_{g_2,r_2} \times \ldots \times \Gamma_{g_t,r_t}$$

**Proof :** For convenience, we shall assume that the group $A(\mathcal{C})$ generated by Dehn twist homeomorphisms about components of $\mathcal{C}$ is a subgroup of $I_\Sigma$ which we denote by $A$ (note that this is true for all of our applications).

First, let us consider the structure of abelian subgroups of $I_\Sigma$. Let $\mathcal{C}$ be a system of circles on $\Sigma$. Given some components $R$ of $\Sigma \setminus \mathcal{C}$, suppose we have a diffeomorphism $F_R : R \to R$, fixed on the boundary of $R$ which is isotopic to a pseudo-Anosov diffeomorphism. Extend $F_R$ to a diffeomorphism of the surface $\Sigma$ by the identity and denote the isotopy class of this diffeomorphism by $f^R$. Denote the subgroup of Out $(\Sigma)$ generated by all $f^R$ and all Dehn twists about components of $\mathcal{C}$ by $\Pi$. Then $\Pi$ is abelian and conversely, every abelian subgroup of $I_\Sigma$ is contained in some subgroup of this type (the converse is shown in [Iv2] p.78). However, $\Pi$ is not in general a subgroup of $I_\Sigma$ as is clear when we consider Dehn twists about nonseparating circles.

Consider an element $d$ in the centraliser of $A$ in $I_\Sigma$, $C_{I_\Sigma}(A)$. Then the group

$$\langle d, A\rangle$$

generated by $d$ and $A$ is an abelian subgroup of $I_\Sigma$ (since $A$ is abelian) and is contained in an abelian group $\Pi$ constructed in the above manner. Therefore, the isotopy class $d \in C_{I_\Sigma}(A)$ contains a diffeomorphism $D : \Sigma \to \Sigma$ which is fixed on $\mathcal{C}$ and does not permute the components of $\Sigma \setminus \mathcal{C}$ (recall that all the elements of $I_\Sigma$ are pure). This diffeomorphism induces a diffeomorphism on the components of $\Sigma$ cut along the system of circles $D_{\mathcal{C}} : \Sigma_{\mathcal{C}} \to \Sigma_{\mathcal{C}}$ and if we take isotopy classes of such diffeomorphisms, then we obtain a homomorphism:

$$\begin{aligned} \alpha_0 : C_{I_\Sigma}(A) &\to \text{Out } (\Sigma \setminus \mathcal{C}) \\ d &\mapsto [D_{\mathcal{C}}] \end{aligned}$$

where [.] denotes the isotopy class of the diffeomorphism. The kernel of $\alpha_0$ is contained in $A$ because the isotopy classes mapping to the boundary of $\Sigma_{\mathcal{C}}$ correspond to powers of Dehn twist homeomorphisms about $\mathcal{C}$. Also since $D$ does not permute the components of $\Sigma \setminus \mathcal{C}$, the image is contained in $\Gamma_{g_1,r_1} \times \ldots \times \Gamma_{g_t,r_t}$. Therefore $\alpha_0$ induces a homomorphism

$$\alpha : \frac{C_{I_\Sigma}(A)}{A} \to \Gamma_{g_1,r_1} \times \ldots \times \Gamma_{g_t,r_t}.$$

In order to prove the theorem, it is sufficient to show that $\alpha$ is injective and its image has finite index in $\Gamma_{g_1,r_1} \times \ldots \times \Gamma_{g_t,r_t}$.

To prove that $\alpha$ is injective, consider an element $d'$ which is in the kernel of $\alpha_0$. This isotopy class may be represented by a diffeomorphism $D'$ which is fixed on $\mathcal{C}$ and such that $D'_{\mathcal{C}}$ is isotopic to the identity diffeomorphism on $\Sigma \setminus \mathcal{C}$. Furthermore, $D'$ is isotopic to a diffeomorphism supported in a small neighbourhood of $\mathcal{C}$ and thus is isotopic to the composition of several Dehn twists about components of $\mathcal{C}$. We have shown that the element $d'$ must be an element of the group $A$ and hence the kernel of $\alpha_0 = A$ proving injectivity.

By construction, the image of $\alpha$ is given by

$$\text{Im } (\alpha) = \mathrm{I}_{\Sigma_{g_1,r_1}} \times \cdots \times \mathrm{I}_{\Sigma_{g_t,r_t}}$$

where the $\Sigma_{g_i,r_i}$ are the components of $\Sigma \setminus \mathcal{C}$. However, each $\mathrm{I}_{\Sigma_{g_i,r_i}}$ is a subgroup of finite index in $\Gamma_{g_i,r_i}$ and so the image of $\alpha$ is a subgroup of finite index in $\Gamma_{g_1,r_1} \times \ldots \times \Gamma_{g_t,r_t}$ proving the theorem. □

Recall that in Chapter 4 we constructed an exact sequence for the outer automorphism group of the extension of surface groups

$$\mathcal{E} = (1 \to K \to G \to Q \to 1).$$

This took the form

$$1 \to \frac{\mathrm{Stab}_{\mathrm{Aut}(Q)}(\phi)}{\ker \phi} \to \mathrm{Out}\ (\mathcal{E}) \to \frac{\mathrm{Im}\ (\mathrm{proj}_1)}{\mathrm{Im}\ (\phi)} \to 1 \qquad (5.1)$$

where $\phi$ is the operator homomorphism of the extension, and

$$\mathrm{C}_{\mathrm{Out}(K)}(\mathrm{Im}(\phi)) \leq \mathrm{Im}\ (\mathrm{proj}_1) \leq \mathrm{N}_{\mathrm{Out}(\mathrm{K})}(\mathrm{Im}(\phi))$$

The above theorem now enables us to calculate the virtual cohomological dimension of the quotient of this exact sequence for certain situations as follows:

Let the image of the operator homomorphism Im $(\phi)$ be generated by Dehn twist homeomorphisms about a system $\mathcal{C}$ of *separating* circles in $\Sigma$. In this situation $A(\mathcal{C}) \subset I_\Sigma$ and we may identify Im $(\phi) \cong \mathrm{A}(\mathcal{C}) \cong \mathcal{Z}^{\mathrm{n}}$ with the group $A$.

**Theorem 5.5** *Let* Im $(\phi)$ *be generated by a single Dehn twist about a separating circle in* $\Sigma_g$. *Then there is a commensuration*

$$\mathrm{Im}\ (\mathrm{proj}_1)/\mathrm{Im}\ (\phi) \sim \frac{\mathrm{C}_{\mathrm{I}_{\Sigma_g}}(\mathrm{A})}{\mathrm{A}}.$$

*Moreover, if* $\Sigma_{g_1,r_1}, \Sigma_{g_2,r_2}$ *are the connected components of* $\Sigma_g$ *cut along the separating circle, then*

$$\begin{aligned} \mathrm{vcd}\ (\mathrm{Im}\ (\mathrm{proj}_1)/\mathrm{Im}\ (\phi)) &= \mathrm{vcd}\ (\Gamma_{g_1,r_1} \times \Gamma_{g_2,r_2}) \\ &= 4g-4 \end{aligned}$$

**Proof :** The group generated by a Dehn twist about a separating circle is isomorphic to $\mathcal{Z}$ and so

$$\mathrm{C}_{\mathrm{Out}(\Sigma_g)}(\mathcal{Z}) \leq \mathrm{Im}\ (\mathrm{proj}_1) \leq \mathrm{N}_{\mathrm{Out}(\Sigma_g)}(\mathcal{Z}).$$

Suppose $g$ belongs to $\mathrm{N}_{\mathrm{Out}(\Sigma_g)}(\mathcal{Z})$ and consider the homomorphism

$$\tau : \mathrm{N}_{\mathrm{Out}(\Sigma_g)}(\mathcal{Z}) \to \mathrm{Aut}\ (\mathcal{Z})$$

given by $\tau(z) = gzg^{-1}$. The kernel of this homomorphism is given by the centraliser subgroup $\mathrm{C}_{\mathrm{Out}(\Sigma_g)}(\mathcal{Z})$ and so there is a natural injection

$$\frac{\mathrm{N}_{\mathrm{Out}(\Sigma_g)}(\mathcal{Z})}{\mathrm{C}_{\mathrm{Out}(\Sigma_g)}(\mathcal{Z})} \rightarrowtail \mathrm{Aut}\ (\mathcal{Z}) \cong \mathcal{Z}_2.$$

From this it is clear that $\mathrm{C}_{\mathrm{Out}(\Sigma_g)}(\mathcal{Z})/\mathcal{Z}$ is a subgroup of $\mathrm{Im}\ (\mathrm{proj}_1)/\mathrm{Im}\ (\phi)$ of index at most 2. Hence it is sufficient to prove that $\mathrm{C}_{\mathrm{Out}(\Sigma_g)}(\mathcal{Z})/\mathcal{Z}$ is commensurable with $\mathrm{C}_{I_{\Sigma_g}}(A)/A$. This will be achieved with the help of the following lemma:

**Sub-lemma 5.6** *Let $B$ be a finitely generated group, $A$ a subgroup of finite index. Then for any subgroup $S$ of $B$, the centraliser of $S$ in $A$, $C_A(S)$, is a subgroup of finite index in $C_B(S)$.*

**Proof :** First, using the finite index lemma 1.10, we may consider a normal subgroup $D$ of finite index in $B$ such that $D \subset A \subset B$. If $x \in D$ and $x$ centralises $S$, then $x \in B$ (and $x$ centralises $S$); in other words, $C_D(S) \subset C_B(S)$. Moreover, $C_D(S) \subset D$ implies that $C_D(S) \subset C_B(S) \cap D$ and conversely, if $x \in C_B(S) \cap D$ then $x \in D$ and $x$ centralises $S$. Hence $C_D(S) = C_B(S) \cap D$. Now, $C_D(S)$ is a normal subgroup of $C_B(S)$ since for all $b$ in $C_B(S)$, $bC_D(S)b^{-1} = C_D(bSb^{-1}) = C_D(S)$. Therefore

$$\begin{aligned} \frac{C_B(S)}{C_D(S)} &= \frac{C_B(S)}{C_B(S) \cap D} \\ &\cong \frac{C_B(S) \cdot D}{D} \subset \frac{B}{D} \end{aligned}$$

which is finite. But $C_D(S) \subset C_A(S) \subset C_B(S)$ and so $C_A(S)$ has finite index in $C_B(S)$ proving the sub-lemma. $\square$

The congruence subgroup $I_\Sigma(m)$ is a subgroup of finite index in Out $(\Sigma)$ and hence the above lemma proves $\mathrm{C}_{I_{\Sigma_g}}(A)$ is a subgroup of finite index in $\mathrm{C}_{\mathrm{Out}(\Sigma_g)}(\mathcal{Z})$. Thus

$$\mathrm{Im}\ (\mathrm{proj}_1)/\mathrm{Im}\ (\phi) \sim \frac{\mathrm{C}_{\mathrm{Out}(\Sigma_g)}(\mathcal{Z})}{\mathcal{Z}}$$

proving the first statement.

By Harer's Theorem 4.1, vcd $(\Gamma_{\mathrm{g,r}}) = 4\mathrm{g} + 2\mathrm{r} - 4$ and so

$$\mathrm{vcd}\ (\Gamma_{\mathrm{g}_1,\mathrm{r}_1} \times \Gamma_{\mathrm{g}_2,\mathrm{r}_2}) = 4(\mathrm{g}_1 + \mathrm{g}_2) + 2(\mathrm{r}_1 + \mathrm{r}_2) - 8.$$

If $\Sigma_g$ is a surface of genus $g$ cut along the single separating curve $C$, then $g_1 + g_2 = g$ and $r_1 + r_2 = 2$ and so

$$\begin{aligned} \mathrm{vcd}\ (\mathrm{Im}\ (\mathrm{proj}_1)/\mathrm{Im}\ (\phi)) &= \mathrm{vcd}\ \left(\frac{\mathrm{C}_{\mathrm{Out}(\Sigma_g)}(\mathcal{Z})}{\mathcal{Z}}\right) \\ &= \mathrm{vcd}\ (\Gamma_{\mathrm{g}_1,\mathrm{r}_1} \times \Gamma_{\mathrm{g}_2,\mathrm{r}_2}) \\ &= 4(g_1 + g_2) + 2(r_1 + r_2) - 8 \\ &= 4g - 4 \end{aligned}$$

giving the result. $\square$

**Corollary 5.7** *If* Im $(\phi)$ *is generated by Dehn twists about a system of $n$ separating circles $C$ on $\Sigma_g$ , then*

$$\mathrm{vcd}\ \left(\frac{\mathrm{Im}\ (\mathrm{proj}_1)}{\mathrm{Im}\ (\phi)}\right) \leq (4\mathrm{g} - 4) + \frac{\mathrm{n}(\mathrm{n}+1)}{2}.$$

**Proof :** In order to generalise the above theorem to a system of $n$ separating circles in the surface $\Sigma_g$, we may mimic the above proof. The main difference is that we no longer know that Im $(\mathrm{proj}_1)$ is commensurable with $\mathrm{C}_{\mathrm{Out}(\Sigma_g)}(\mathcal{Z}^n)$ since in this case, the natural injection

$$\frac{\mathrm{N}_{\mathrm{Out}(\Sigma_g)}(\mathcal{Z}^n)}{\mathrm{C}_{\mathrm{Out}(\Sigma_g)}(\mathcal{Z}^n)} \rightarrowtail \mathrm{Aut}\ (\mathcal{Z}^{\mathrm{n}}) = \mathrm{GL}_{\mathrm{n}}(\mathcal{Z})$$

does not have finite image. However, $\mathrm{GL}_n(\mathcal{Z})$ does have finite cohomological dimension given by $n(n+1)/2$ and so we may deduce that

$$\mathrm{vcd}\ \left(\frac{\mathrm{Im}\ (\mathrm{proj}_1)}{\mathcal{Z}^{\mathrm{n}}}\right) \leq \mathrm{vcd}\ \left(\frac{\mathrm{C}_{\mathrm{Out}(\Sigma_{\mathrm{g}})}(\mathcal{Z}^{\mathrm{n}})}{\mathcal{Z}^{\mathrm{n}}}\right) + \frac{\mathrm{n(n+1)}}{2}.$$

Now $\mathrm{C}_{\mathrm{Out}(\Sigma_g)}(\mathcal{Z}^n)/\mathcal{Z}^n$ is commensurable with $\mathrm{C}_{I_{\Sigma_g}}(A)/A$ where $A = \mathcal{Z}^n$ by the above sub-lemma and this is commensurable with

$$\frac{C_{I_\Sigma}(\mathcal{Z}^n)}{\mathcal{Z}^n} \sim \Gamma_{g_1,r_1} \times \Gamma_{g_2,r_2} \times \ldots \times \Gamma_{g_t,r_t}$$

by Theorem 5.4, where $\Gamma_{g_i,r_i}$ is the mapping class group of $\Sigma_{g_i,r_i}$ formed by cutting $\Sigma_g$ along the $n$ separating curves $\mathcal{C}$. The sums

$$\sum_i g_i = g, \quad \sum_i r_i = 2(t-1)$$

together with Harer's theorem 4.1 imply that

$$\begin{aligned} \mathrm{vcd}\ (\Gamma_{\mathrm{g}_2,\mathrm{r}_2} \times \ldots \times \Gamma_{\mathrm{g}_t,\mathrm{r}_t}) &= \sum_i^t 4g_i + 2r_i - 4 \\ &= 4g + 4(t-1) - 4t = 4g - 4. \end{aligned}$$

Therefore

$$\mathrm{vcd}\ (\mathrm{Im}\ (\mathrm{proj}_1)/\mathrm{Im}\ (\phi)) \leq 4\mathrm{g} - 4 + \mathrm{n(n+1)}/2.$$

□

In fact, since $n \leq g-1$, the virtual cohomological dimension is bounded by

$$\mathrm{vcd}\ \left(\frac{\mathrm{Im}\ (\mathrm{proj}_1)}{\mathrm{Im}\ (\phi)}\right) \leq \frac{(\mathrm{g}-1)(\mathrm{g}+8)}{2}.$$

**Remark :** This work can be similarly applied to calculate the v.c.d. of Im $(\mathrm{proj}_1)/\mathrm{Im}\ (\phi)$ when Im $(\phi)$ is generated by any Dehn twists which act trivially on homology.

## 5.3 Non-rigidity of Stallings fibrations

Given a diffeomorphism $\phi$ of the surface $\Sigma$, we may obtain an oriented 3-manifold from the cylinder $\Sigma \times I$ by identifying

$$(x,0) \sim (\phi(x),1)$$

for every $x \in \Sigma$. This 3-manifold $M$ is called a *Stallings fibration* or *mapping torus.* There is a natural fibration $M \to S^1$ with fibre $\Sigma$ and the long homotopy exact sequence of this fibration reduces to the following split exact sequence:

$$1 \to \Sigma \to M \to C_\infty \to 1.$$

Conversely, Stallings [Sta] proved that when a compact 3-manifold $M$ has fundamental group containing a finitely generated normal subgroup $\Sigma$ whose quotient group is $C_\infty$, then $\Sigma$ is the fundamental group of a surface embedded in $M$. By a theorem of Waldhausen ([Wald]), the mapping class group $\pi_0(\mathrm{Diff}\ (\mathrm{M}))$ of a Stallings fibration formed from a closed surface of negative Euler characteristic satisfies

$$\pi_0(\mathrm{Diff\ M}) \cong \mathrm{Out}\ (\pi_1(\mathrm{M})).$$

The aim of this section is to demonstrate that, in general, the mapping class group Out (G) of a Stallings fibration $M$ is not rigid in the sense that the automorphism group of the long homotopy exact sequence of $M$ does not have finite index in Aut (G). In particular we show:

**Theorem 5.8** *Let $M$ be the Stallings fibration constructed from the trivial diffeomorphism $\phi = Id$ so that the long homotopy exact sequence $\mathcal{E}$ corresponds to the direct product $G = \Sigma_g \times C_\infty$ where $\Sigma_g$ denotes a surface of genus $g \geq 2$. Then*

$$\begin{aligned} \text{vcd (Aut (G))} &= 6g-3 \\ \text{vcd (Aut } (\mathcal{E})) &= 4g-3 \end{aligned}$$

*In particular* Aut $(\mathcal{E})$ *is* not *a subgroup of finite index in* Aut (G).

**Proof :** The method of proof is to analyse the automorphisms of the exact sequences

$$\begin{aligned} \mathcal{E}_1 &= (1 \to C_\infty \to C_\infty \times \Sigma_g \to \Sigma_g \to 1) \\ \mathcal{E}_2 &= (1 \to \Sigma_g \to \Sigma_g \times C_\infty \to C_\infty \to 1). \end{aligned}$$

The automorphism group of $\mathcal{E}_1 = \{\alpha \in \text{Aut (G)} : \alpha(\mathrm{C}_\infty) = \mathrm{C}_\infty\}$ fits inside a split exact sequence

$$1 \to C(\mathcal{E}_1) \to \text{Aut } (\mathcal{E}_1) \to \text{Aut } (\Sigma_g) \times \text{Aut } (\mathrm{C}_\infty) \to 1 \tag{5.2}$$

where $C(\mathcal{E}_1)$ denote the set of congruences of the extension. From the work in Chapter 1 we know there is an isomorphism

$$\begin{aligned} C(\mathcal{E}_1) &\cong Z^1(\Sigma_g, Z(C_\infty)) \\ &= H^1(\Sigma_g; \mathcal{Z})) \\ &= \mathcal{Z}^{2g} \end{aligned}$$

The next step is to show that the subgroup $C_\infty$ is characteristic in $G$ so that Aut $(\mathcal{E}_1) =$ Aut (G). Consider the automorphism $\alpha$ of $C_\infty \times \Sigma_g$ in the diagram

$$\begin{array}{ccccccccc} 1 & \to & C_\infty & \to & C_\infty \times \Sigma_g & \to & \Sigma_g & \to & 1 \\ & & \downarrow & & \downarrow \alpha & & \downarrow & & \\ 1 & \to & C_\infty & \to & C_\infty \times \Sigma_g & \xrightarrow{\pi} & \Sigma_g & \to & 1 \end{array}$$

From this we can deduce that $\pi\alpha(C_\infty) \triangleleft \Sigma_g$. However, surface groups have no nontrivial abelian normal subgroups and so

$$\pi\alpha(C_\infty) = 1$$

giving that $\alpha(C_\infty) \subset C_\infty$. By taking the automorphism $\alpha^{-1}$ we obtain the opposite inclusion and since the automorphism of $G$ was chosen arbitrarily, $C_\infty$ is characteristic in $G$. Thus the exact sequence 5.2 becomes

$$1 \to \mathcal{Z}^{2g} \to \text{Aut (G)} \to \text{Aut } (\Sigma_g) \times \mathcal{Z}/2 \to 1$$

using the fact that the automorphism group of the infinite cyclic group $C_\infty$ is $\mathcal{Z}/2$. In Chapter 4, it was shown that the automorphism group of a surface group is a virtual duality group and vcd (Aut $(\Sigma_g)$) $= 4g-3$ (see Proposition 4.7). Hence the quotient of the above exact sequence contains a torsion-free subgroup of finite index $\text{Aut}_0$ $(\Sigma_g)$ that is a duality group. By Bieri and Eckmann 1.17 the extension of $\mathcal{Z}^{2g}$ by $\text{Aut}_0$ $(\Sigma_g)$ is also a duality group denoted $\text{Aut}_0$ $(G)$ and its cohomological dimension satisfies

$$\begin{aligned}\text{cd } (\text{Aut}_0 \text{ (G)}) &= \text{cd } (\mathcal{Z}^{2g}) + \text{cd } (\text{Aut}_0 \ (\Sigma_g)) \\ &= 2g + 4g - 3.\end{aligned}$$

Therefore vcd (Aut (G)) $= 6g - 3$ as stated.

Now consider the exact sequence $\mathcal{E}_2 = (1 \to \Sigma_g \to \Sigma_g \times C_\infty \to C_\infty \to 1)$. The automorphism group of the extension is again in an exact sequence

$$1 \to C(\mathcal{E}_2) \to \text{Aut } (\mathcal{E}_2) \to \text{Aut } (\Sigma_g) \times \text{Aut } (C_\infty) \to 1.$$

However, in this case the kernel of $\mathcal{E}_2$ has trivial centre meaning that the group of congruences of $\mathcal{E}_2$ is trivial. So the exact sequence above reduces to an isomorphism,

$$\text{Aut } (\mathcal{E}_2) \cong \text{Aut } (\Sigma_g) \times \text{Aut } (C_\infty).$$

As before, the right-hand side has a duality group $\mathrm{Aut}_0\ (\Sigma_g)$ as a subgroup of finite index and this has cohomological dimension equal to $4g-3$. Therefore the virtual cohomological dimension of Aut $(\mathcal{E}_2)$ is $4g-3$, proving the theorem. Observe also that Aut $(\mathcal{E}_2)$ is a virtual duality group. □

**Corollary 5.9** *The mapping class group of the trivial Stallings fibration $\Sigma^g \times S^1$ is virtually torsion-free and has virtual cohomological dimension*

$$\mathrm{vcd}\ (\mathrm{Out}\ (\Sigma_g \times C_\infty)) = 6g-5$$

*where $\Sigma_g$ denotes the surface of genus $g$.*
*By contrast, given the extension $\mathcal{E} = (1 \to \Sigma_g \to \Sigma_g \times C_\infty \to C_\infty \to 1)$,*

$$\mathrm{vcd}\ (\mathrm{Out}\ (\mathcal{E})) = 4g-5$$

**Proof :** First observe that the inner automorphisms of $\mathcal{E}$

$$\begin{aligned} \mathrm{Inn}\ \mathcal{E} &= \{\alpha \in \mathrm{Inn}\ (G) : \alpha(\Sigma_g) = \Sigma_g\} \\ &= \mathrm{Inn}\ (G) \end{aligned}$$

since $\Sigma_g$ is normal in $G$. Given an element $(g,t)$ in $\Sigma_g \times C_\infty$ then under conjugation:

$$\begin{aligned} (h,s)(g,t)(h,s)^{-1} &= (hgh^{-1}, sts^{-1}) \\ &= (hgh^{-1}, t) \end{aligned}$$

since $sts^{-1}$ is in the abelian group $C_\infty$, proving that

$$\mathrm{Inn}\ (\Sigma_g \times C_\infty) = \mathrm{Inn}\ (\Sigma_g).$$

In addition the centre of $\Sigma_g$ is trivial and so Inn $(\Sigma_g) \cong \Sigma_g$. Using this result, the natural homomorphism from Aut to Out gives rise to the exact sequences:

$$1 \to \Sigma_g \to \ \text{Aut (G)} \ \to \text{Out (G)} \to 1$$
$$1 \to \Sigma_g \to \ \text{Aut} \ (\mathcal{E}) \ \to \text{Out} \ (\mathcal{E}) \to 1$$

Since Aut (G) and Aut ($\mathcal{E}$) are both virtually torsion-free, from these sequences we can construct torsion-free sequences

$$1 \to \Sigma_g \to \ \text{Aut}_0 \ (G) \ \to \text{Out}_0 \ (G) \to 1 \tag{5.3}$$
$$1 \to \Sigma_g \to \ \text{Aut}_0 \ (\mathcal{E}) \ \to \text{Out}_0 \ (\mathcal{E}) \to 1 \tag{5.4}$$

where $\text{Aut}_0$ $(G)$ and $\text{Aut}_0$ $(\mathcal{E})$ are torsion-free subgroups of finite index in the automorphism groups. Applying Serre's theorem 1.14 to the first sequence and using the above result we deduce that

$$\begin{aligned} \text{cd (Aut}_0 \text{ (G))} &\leq \text{cd } (\Sigma_g) + \text{cd (Out}_0 \text{ (G))} \\ 6g - 3 &\leq 2 + \text{cd (Out}_0 \text{ (G))} \end{aligned}$$

which proves that cd ($\text{Out}_0$ (G)) $\geq$ 6g $-$ 5. In order to obtain the opposite inequality we shall use the Lyndon-Hochschild-Serre spectral sequence associated to the exact sequence 5.3 (see [Bro] p.171):

**Theorem 5.10 (Lyndon-Hochschild-Serre)**
*For any group extension $1 \to K \to G \to Q \to 1$ and any G-module $M$, there is a spectral sequence of the form*

$$E_2^{p,q} = H^p(Q; H^q(K; M)) \Rightarrow H^{p+q}(G; M).$$

Let $M$ be a $\mathcal{Z}(\text{Out}_0\ (G))$-module with $H^d(\text{Out}_0\ (G); M) \neq 0$ where $d$ is the virtual cohomological dimension of Out (G) (cf. [Har] p.174). Then from the spectral sequence associated to the extension 5.3 we obtain

$$\begin{aligned} H^{n+2}(\text{Aut}_0\ (G); M) &\cong H^n(\text{Out}_0\ (G); H^2(\Sigma_g; M)) \\ &\cong H^n(\text{Out}_0\ (G); M) \end{aligned}$$

using the fact that $H^2(\Sigma_g; M) = M$. The above result calculated that cd $(\mathrm{Aut}_0\ (\mathrm{G})) = 6\mathrm{g} - 3$ and so we may infer that vcd (Out (G)) is precisely $6g - 5$.

In the same manner we may use the spectral sequence associated to the exact sequence 5.4 to prove that vcd $(\mathrm{Out}\ (\mathcal{E})) = 4\mathrm{g} - 5$ as stated. □

## 5.4 Pseudo-Anosov Stallings fibrations

In this final section we shall calculate the mapping class groups of Stallings fibrations constructed from pseudo-Anosov isotopy classes of diffeomorphisms. By using Mostow rigidity and a theorem by Thurston we are able to show that these groups are finite and so have zero virtual cohomological dimension.

**Theorem 5.11 (Thurston)** *The Stallings fibration constructed via the diffeomorphism $\phi$ admits a complete hyperbolic structure if and only if $\phi$ is isotopic to a pseudo-Anosov diffeomorphism.*

For a fairly thorough proof of this result see [McM] pp.50-53. To show that the automorphism group of a hyperbolic manifold is finite, we shall need to invoke the Rigidity theorem by Mostow which was first demonstrated in [Mos] (see p.189):

**Theorem 5.12 (Mostow Rigidity Theorem)** *Let $G$ and $G'$ be semisimple analytic centreless groups with no compact factors and let $\Gamma$ and $\Gamma'$ be discrete subgroups of $G$ and $G'$ respectively such that $G/\Gamma$ and $G'/\Gamma'$ have finite volume. Let $\theta$ be an isomorphism $\theta : \Gamma \to \Gamma'$. Then $\theta$ extends to an analytic isomorphism $\hat{\theta} : G \to G'$ provided that there is no analytic homomorphism*

$$\pi : G \to \mathrm{PSL}_2(\mathcal{R})$$

*with $\pi(\Gamma)$ discrete.*

**Corollary 5.13 (Automorphism extension property)** *Let $\Gamma$ be a discrete subgroup in an analytic semisimple centreless group $G$ having no compact factors and suppose that $G/\Gamma$ has finite volume. Suppose further that given any epimorphism $\pi : G \to \mathrm{PSL}_2(\mathcal{R})$ then $\pi(\Gamma)$ is non-discrete in $\mathrm{PSL}_2(\mathcal{R})$.*

*Then any automorphism of $\Gamma$ extends to an automorphism of $G$.*

Now let $G$ be a centreless semisimple Lie group with $\Gamma$ a discrete subgroup in $G$. Then the pair of groups $(\Gamma, G)$ is called a *Mostow rigid pair.* The automorphism extension property implies that the outer automorphism groups of a Mostow rigid pair are finite. This was originally proved without using Mostow rigidity by Borel [Bor]. The proof of this fact using rigidity is outlined below (cf. [Joh2]):

**Proposition 5.14** *If $\Gamma$ and $G$ are a Mostow rigid pair then* Out $(\Gamma)$ *and* Out (G) *are finite groups.*

**Proof :** As $G$ is a centreless group, we may apply Mostow's rigidity theorem to give an homomorphism

$$\begin{array}{ccc} \mathrm{Aut}\ (\Gamma) & \to & \mathrm{Aut}\ (\mathrm{G}) \\ \alpha & \mapsto & \widehat{\alpha} \end{array}$$

extending the automorphism $\alpha$ of $\Gamma$ to an automorphism of $G$. Hence we have an homomorphism Aut $(\Gamma) \to$ Out (G) given by $\alpha \mapsto [\widehat{\alpha}]$. The kernel of this homomorphism consists of the automorphisms of $\Gamma$ which extend to inner automorphisms of $G$:

$$\widehat{\alpha}(g) = xgx^{-1} \quad \text{for some } x \in G.$$

In this case $\widehat{\alpha}(\gamma) = x\gamma x^{-1}$ for $\gamma$ in $\Gamma$ and so $x$ normalises $\Gamma$ (i.e. $x \in \mathrm{N}_G(\Gamma)$). Therefore there is an exact sequence

$$1 \to \mathrm{N}_G(\Gamma) \to \mathrm{Aut}\ (\Gamma) \to \mathrm{Out}\ (\mathrm{G}) \to 1.$$

Furthermore, $\Gamma$ is a normal subgroup of $\mathrm{N}_G(\Gamma)$ allowing us to factor through by $\Gamma$ to give the exact sequence

$$1 \to \frac{\mathrm{N}_G(\Gamma)}{\Gamma} \to \mathrm{Out}\ (\Gamma) \to \mathrm{Out}\ (\mathrm{G}) \to 1 \tag{5.5}$$

The finiteness of Out $(\Gamma)$ will follow from the finiteness of the ends of this exact sequence.

Since $G$ is semisimple, the group of Lie automorphisms $\mathrm{Aut}_{\mathrm{Lie}}(\mathrm{G})$ is a real algebraic group with Inn (G) as identity component. However, real algebraic groups have only finitely many connected components (see [Rag] p.10) and thus

$$\mathrm{Out}\ (\mathrm{G}) = \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{G})/\mathrm{Inn}\ (\mathrm{G})$$

is finite.

Now consider the normaliser $N_G(\Gamma)$ in the kernel of the exact sequence 5.5. Let $n$ be an element of the identity component $N_G(\Gamma)_0$ and choose a path $p(t)$ contained in this identity component from $n = p(0)$ to the identity $1 = p(1)$. This gives rise to a path in $\Gamma$,

$$p(t)\gamma p(t)^{-1}$$

which starts at $n\gamma n^{-1}$ and ends at $\gamma$ for any $\gamma$ in $\Gamma$. However, $\Gamma$ is discrete and hence this is a constant path. This implies that the identity component $N_G(\Gamma)_0$ is trivial and so $N_G(\Gamma)$ is discrete. Since

$$\frac{N_G(\Gamma)}{\Gamma} \subset \frac{G}{\Gamma}$$

and $G/\Gamma$ has finite volume, it follows that $N_G(\Gamma)$ must be finite. Therefore Out $(\Gamma)$ is the extension of two finite groups, and so is itself finite. □

**Corollary 5.15** *The mapping class group of a Stallings fibration constructed via a diffeomorphism from a pseudo-Anosov isotopy class is finite.*

**Proof :** By Thurston's theorem, a Stallings fibration constructed from a diffeomorphism isotopic to a pseudo-Anosov diffeomorphism has a hyperbolic structure and so it satisfies Mostow rigidity. Hence its outer automorphism group is finite as claimed. □.

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