Pricing and Hedging European Options in the Presence of Taxes

Samuel Barnaby Dickson

A thesis submitted for the degree of
Doctor of Philosophy
of the University of London

Centre for Quantitative Finance
Imperial College

London, November 2002

ProQuest Number: U174096

All rights reserved

INFORMATION TO ALL USERS
The quality of this reproduction is dependent upon the quality of the copy submitted.

In the unlikely event that the author did not send a complete manuscript
and there are missing pages, these will be noted. Also, if material had to be removed,
a note will indicate the deletion.
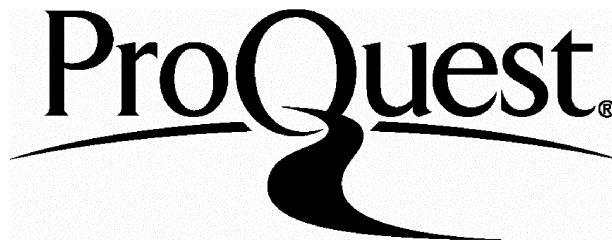
ProQuest U174096

Abstract

Following the work of Milevsky and Prisman (Milevsky (1997a)) we develop a tax-adjusted hedging algorithm that finds the tax-adjusted hedging price of a European option. We use a non-recombining binomial tree framework because the tax liability on the stock transactions is path dependent. In general, we find that in a Cox Ross Rubinstein environment (Cox(1979)) the hedging portfolio (of stock and bond) does not equal the option payoff, on a post-tax basis; there is a tax-mismatch. The algorithm uses an iterative procedure to force the tax-mismatch to zero across all the final nodes on the tree, thereby ensuring that the option writer is fully hedged on an after-tax basis.

We can consider the non-recombining binomial tree framework as being one in which we have an equal number of unknowns (the deltas and bond amounts at each node on the tree) as linearly independent equations. We can find the general forms for the deltas and bonds by partially solving the system of equations. The simultaneous equation algorithm uses these general forms to find the tax-adjusted price of the option. This algorithm is less demanding on memory and computationally faster than the tax-adjusted hedging algorithm.

The simultaneous equation approach allows us to derive an analytic formula for the tax-adjusted hedging price of the option, and in the one-period case we can use this to prove some of the empirical results found using the tax-adjusted hedging algorithm.

We relax one of the assumptions made in the original framework – the tax year-end coincides with the option's maturity – and allow a tax year-end to occur during the life of the option. This requires us to consider two tax charges: one paid during the life of the option at the first tax year-end, and one paid after the option expires at the second tax year-end. The simultaneous equation approach is used again and we develop the tax year-adjusted simultaneous equation algorithm that finds the tax-adjusted price of the option when the tax year-end can occur during the option's life.

Scholes has derived a modification to Black-Scholes, termed the tax-adjusted Black-Scholes equation (Scholes (1976)). We form a tax-adjusted risk neutral probability in the Cox Ross Rubinstein environment and use this to form the tax-adjusted binomial option pricing model. This is shown to be the discrete-time precursor to the tax-adjusted Black-Scholes equation. The tax-adjusted Black-Scholes equation is generalised to relax the assumption in the original derivation that the derivative is taxed as income. A martingale derivation is given for this equation, as for Milevsky and Prisman's tax-adjusted Black-Scholes equation with dividends (Milevksy (1997a)).

Contents

Acknowledgments

0.1 Notation and abbreviations

A guide to the main notation and abbreviations used in this thesis is given below.

Notation

VARIABLE/NOTATION	DESCRIPTION
$ACB_{<i,j>}$	Adjusted cost base at node $\langle i,j \rangle$.
$ATB_{<N,j>}$	After-tax bond position at $\langle N,j \rangle$.
$ATO_{<N,j>}$	After-tax option position at $\langle N,j \rangle$.
$ATS_{<N,j>}$	After-tax stock position at $\langle N,j \rangle$.
$B_{<i,j>}$	Bond amount at node $\langle i,j \rangle$.
$B^*_{<i,j>}$	Tax-adjusted bond amount at node $\langle i,j \rangle$.
$B^{CRR}_{<i,j>}$	Cox Ross Rubinstein bond amount at node $\langle i,j \rangle$.
$B^{call}_{<i,j>}$	Cox Ross Rubinstein bond amount at node $\langle i,j \rangle$ for a call.
$B^{put}_{<i,j>}$	Cox Ross Rubinstein bond amount at node $\langle i,j \rangle$ for a put.
$_N B_{<i,j>}$	Bond amount at node $\langle i,j \rangle$ on an N-period tree.
$_N^m B_{<i,j>}$	Bonds at node $\langle i,j \rangle$ on when a tax year-end occurs at period m.
$B^g_{<i,j>}$	Tax-adjusted bonds on gth iteration of tax-adjusted hedging algorithm.
$BI_{\langle N,j \rangle}$	Total bond interest accumulated up to and including node $\langle N,j \rangle$.
c	European call price.
$\Delta_{\langle i,j \rangle}$	Tree-delta at node $\langle i,j \rangle$.
d	Total return if stock moves down.
dW_t^P	An increment of Brownian Motion under the real-world measure P.
dW_t^Q	An increment of Brownian Motion under the risk-neutral measure Q.
$E^Q[\cdot]$	Expectation operator under the risk neutral measure Q.
h	Discretisation interval $(= T/N)$

Table 0.1: Notation

Variable/Notation	Description
$\langle i, j, -k \rangle$	Unique node k periods before node $\langle i, j \rangle$.
K	Strike price of the option.
m	Period when a tax year ends on the tree.
$\mathfrak{M}_{\langle N,j \rangle}$	Tax-mismatch at node $\langle N, j \rangle$.
μ	Expected rate of return on the underlying (drift).
N	Number of periods on the tree (maturity period of the option).
p	European put price.
q	Dividend yield rate or real-world probability of an up move in the stock.
π	Risk-neutral probability of an up-move in the stock price.
r	Annual risk-free rate.
R	One plus the risk-free rate over one period.
$RNG_{<N,j>}$	Realised net gain at node $\langle N, j \rangle$.
S	Price of asset underlying the derivative.
$S_{<i,j>}$	Stock price at node $\langle i, j \rangle$.
σ	Volatility.
T	Time at maturity of derivative.
τ_b	Tax rate that applies to bond transactions.
τ_s	Tax rate that applies to stock transactions
τ_o	Tax rate that applies to option transactions
τ_i	Income tax rate.
τ_{cg}	Captial gains tax rate.
u	Total return if stock moves up.
V	Value of derivative security.
$X_{<i,j>}$	Generic European put or call price.
$_nX^{cnts}$	Continuous approximation to option price from tree of n-periods.
$^{mkt}X_{<m,j>}$	Market price of the option at node $\langle m, j \rangle$.
$\mathfrak{X}_{\langle N,j \rangle}$	Synthetic tax-adjusted payoff from the option.

Table 0.1 ctd.

Abbreviations

ABBREVIAITON	MEANING
ACB	Adjusted cost basis.
ATB	After-tax bond.
ATO	After-tax option.
ATS	After-tax stock.
BOPM	Binomial option pricing model
BI	Bond interest.
CRR	Cox, Ross, Rubinstein
RNG	Realised net gain
taBOPM	Tax-adjusted binomial option pricing model.

Table 0.2: Abbreviations

Chapter 1

Introduction and Literature Review

1.1 Introduction

1.1.1 Overview

The aim of the thesis is to take the two approaches that are identified in the literature review, namely Scholes's tax-adjusted Black-Scholes equation (Scholes (1976)) and Milevsky and Prisman's (M&P) discrete-time tax-efficient hedging algorithm (Milevsky (1997a)), and develop them.

Chapter 2

A tax-adjusted hedging algorithm is developed that finds the tax-adjusted hedging price of European equity call and put options. This is an iterative procedure that uses the same backward recursion as in the Cos, Ross, Rubinstein environment (CRR) (Cox (1979)).

Chapter 3

Using the framework of Chapter 2, a simultaneous equation approach is proposed. This allows us to develop a simultaneous equation algorithm to find the tax-adjusted price of the option, which is computationally faster and less demanding on computer memory than the tax-adjusted hedging algorithm of Chapter 2.

Chapter 4

Using the approach of Chapter 3, we relax the assumption, made in Chapters 2 and 3, that the tax year-end coincides with the maturity of the option. Here we allow the tax year-end to occur during the life of the option, and develop a tax year-adjusted simultaneous equation algorithm to find the tax-adjusted price of the option under these circumstances.

Chapter 5

A tax-adjusted binomial option pricing model is developed that uses tax-adjusted risk-neutral probabilities to value European equity call and put options. This model is shown to be the discrete time precursor to the tax-adjusted Black-Scholes equation (Equation (1.2) derived in Scholes (1976)). A generalised version of the tax-adjusted Black-Scholes equation is derived and the solutions for a European put and call are given.

Chapter 6

The conclusions of this research are presented.

1.2 Literature review

The use of options before 1973 was limited, mainly because there was no generally agreed method for valuing them. Two events occurred that year which revolutionised the options industry: Fischer Black and Myron Scholes published the paper containing their option valuation model (Black (1973)), and the first exchange-traded options were launched on the Chicago Board Options Exchange. With the publication of the Black-Scholes model, option pricing was at last given a rigorous mathematical framework that was accepted by academics and practitioners alike.

In the intervening twenty-six years, the original Black-Scholes model has been modified in several important ways to relax some of the assumptions implicit in the original formula. The emphasis here is the pricing of options in the presence of taxes, an area which has until recently been largely neglected.

Scholes himself extended the original Black-Scholes model to include taxes (Scholes (1976)). The original Black-Scholes partial differential equation, together with Scholes' tax adjusted version (see Appendix A for an alternative martingale derivation of this equation), are shown below.

V is the value of the derivative, t is the time, σ is the volatility of the underlying security, S, and r is the risk-free rate.

The **Black-Scholes partial differential equation**:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{1.1}$$

The **tax-adjusted Black-Scholes partial differential equation**:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \left[r - \frac{(\tau_i - \tau_{cg})r}{(1 - \tau_{cg})} \right] S\frac{\partial V}{\partial S} - rV = 0, \tag{1.2}$$

where τ_i and τ_{cg} are the tax rates for income and capital gains respectively.

Equation (1.2) has been analysed by Merton when the underlying is common stock that pays dividends proportional to the stock's price (Merton (1973)). The constant dividend yield in this context would be equal to $r(\tau_i - \tau_{cg})/(1 - \tau_{cg})$.

Subject to the assumptions made in the derivation of (1.2), the effect of taxes is to increase the price of a put and decrease the price of a call, compared with the classical Black-Scholes values, provided that $\tau_i > \tau_{cg}$. The longer the life of the option, the more pronounced these price differences become.

It can be seen from (1.2) that when $\tau_i = \tau_{cg}$, the tax-adjusted Black-Scholes equation collapses to (1.1), the original Black-Scholes equation. This result seems to imply that for market participants whose marginal tax rates are equal, taxes have no effect on option valuation. M&P believe that this result is responsible for the lack of research into pricing options in the presence of taxes (Milevsky (1997a)). The result is borne out by M&P's work with a one-period discrete model with taxes (Milevsky (1996c)).

M&P have extended Scholes' analysis to include the case where the underlying asset pays dividends (Milevsky (1997a)). The partial differential equation they have derived is given

13

below (Appendix B contains an alternative martingale derivation):

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + [\hat{r} - \hat{q}]\, S \frac{\partial V}{\partial S} - rV = 0 \,, \qquad (1.3)$$

where,

$$\hat{r} = r\frac{(1 - \tau_i)}{(1 - \tau_d)} \,, \quad \hat{q} = \frac{r\,(\tau_d - \tau_u)\,(1 - \tau_i)}{(1 - \tau_u)\,(1 - \tau_d)} + q\frac{(1 - \tau_q)}{(1 - \tau_u)} \,,$$

τ_i is the marginal tax rate on interest income (usually ordinary income), τ_d is the marginal tax rate on the derivative security, τ_u is the marginal tax rate on the underlying security, and τ_q is the marginal tax rate on the constant dividend yield q. Equation (1.3) collapses to the Black-Scholes equation with dividends if all the marginal tax rates are set equal to each other.

The effect of taxes is to increase the price of a put and decrease the price of a call, compared with values obtained with Merton's extension of Black-Scholes to include dividends. As before, the longer the life of the option, the more pronounced these price differences become.

M&P argue that the tax-adjusted Black-Scholes equation and the tax-adjusted Black-Scholes equation with dividends, Equations (1.2) and (1.3), are derived using assumptions that ignore some fundamental features of tax systems (Milevsky (1997a)). The most obvious example is the assumption that the tax liability is paid on a continuous basis. As M&P point out, this may be a justified assumption when dealing with dividend payments on a large stock index, but in the case of taxes this does not distinguish between the timing of the tax liabilities and ignores the time-value of money. This is because tax is generally due for payment on one date in the year, and not spread continuously over the course of the year.

If the underlying or the derivative security is tax-marked-to-market at the end of the tax year (i.e., the tax is calculated from the marked-to-market profit or loss at the tax year-end), M&P show that it is possible for generic European options to become path-dependant (Milevsky (1997a)). This feature is of course completely ignored in the derivation of Equations (1.2) and (1.3), as is the distinction between the hedging and replicating price of an option. The hedging price and replicating price of an option are defined below.

The technique for modern derivative pricing in a frictionless market involves building a portfolio of simple instruments that replicates the exact payoff of the derivative. Applying

14

the no-arbitrage principle, the price of the derivative is the minimum cost of building such a portfolio. If this is not so, arbitrage profits are possible if the cheaper of the two (the derivative and the replicating portfolio) is bought, while the expensive one is sold. The minimum cost of the replicating portfolio is termed the replicating cost of the derivative.

M&P maintain that extending this to a taxable world is a case of simply applying the no-arbitrage principle on an after-tax basis, i.e., the price of a derivative is the minimum cost of building a portfolio of primitive securities that replicates the payoff from the derivative on an after-tax basis (Milevsky (1996c)). If the no-arbitrage principle holds, combining the derivative and the portfolio together in a composite portfolio, providing there is a short position in one and a long position in the other, should not yield arbitrage profits on an after-tax basis - the positions should neutralise each other exactly. However, although the overall cashflow from a portfolio of securities, in the absence of taxes, is simply a linear sum of the individual cashflows in that portfolio, with taxes this linear relationship often does not hold. M&P call this a non-additive tax system, and in these circumstances the tax treatment of the composite portfolio may be such that arbitrage profits can be made (Milevsky (1997a)).

In order to tackle the portfolio problem, M&P define another price for the derivative in addition to the replicating cost (Milevsky (1997a)). This is the hedging cost, and is the cost of the primitive underlying securities that replicate the exact opposite of the after-tax payoff from the derivative, when they are all held in one portfolio. The objective is to create a portfolio of securities that negate each other at payoff time. The advantage of the hedging cost lies in the fact that the tax laws used are those which relate to assets held in portfolios and not held individually. This is important because the tax-treatment of a portfolio of securities can be different from the tax-treatment of the same securities held individually. The tax treatment is therefore built into the no-arbitrage assumption at the outset, which is not the case when defining the replicating cost.

M&P provide a tax-efficient algorithm to find the hedging price of a European call option with taxes (Milevsky (1997)). A non-recombining binomial tree approach is used, programmed in MAPLE to take advantage of its symbolic properties. This algorithm is examined in more detail in Section 1.4.

M&P have also looked at tax arbitrage with options (Milevsky (1997a)). With the no-

15

arbitrage condition applied to put-call parity on a post-tax basis, it is possible for investors to exploit arbitrage opportunities if their marginal tax rate on capital gains is higher or lower than the "market rate" (i.e., the rate that prevails in the market to give the no-arbitrage condition). M&P also argue that, providing the income tax rate is higher than the capital gains tax rate, the price of a put with taxes will be more than a put without taxes, while the price of a call with taxes will be lower than a call without taxes. This result is consistent with the tax-adjusted Black-Scholes equation.

M&P show that it is optimal for Canadian investors to liquidate their short equity option positions prior to the tax year-end, and reopen them in the new tax year (Milevsky (1996a)). They also show that in some circumstances American style options may sell for less than European style options, because of the different Canadian tax treatment of naked and covered positions.

M&P have looked into hedging using derivatives when the tax treatment is uncertain (Milevsky (1996b)). The cost of the hedge is increased in these circumstances, but M&P argue that this is due to the tax uncertainty, and should be compared to other legal expenses involved in resolving tax issues.

1.3   Motivation

It is useful at this stage to look at a simple numerical example, using a one-period binomial tree, and show why we need to develop a tax-adjusted scheme for pricing and hedging European options in the presence of taxes. We will look at the CRR situation, which assumes there are no taxes, and then show that when we look at this classical hedge on an after-tax basis, we are no longer perfectly hedged.

Let us assume that the stock price when we write the call option is £100, and its strike price is also £100. When the option expires one year later we will assume that the stock can be either £110 or £90 (so the payoffs are £10 and £0, respectively). We will assume a constant risk-free rate of 5% for the life of the option.

In the CRR environment we compute the amount of stock (the delta) and holding in the riskless asset (the bond) that we need at the start of the option's life, such that we will be

perfectly hedged when the option matures, whether or not the stock moves up to £110 or down to £90. The situation is illustrated in Figure 1.1.

Stock=110

/ Payoff=10

Stock=100

Delta=0.5

Bond=-42.857

Option=7.143

\ Stock=90

Payoff=0
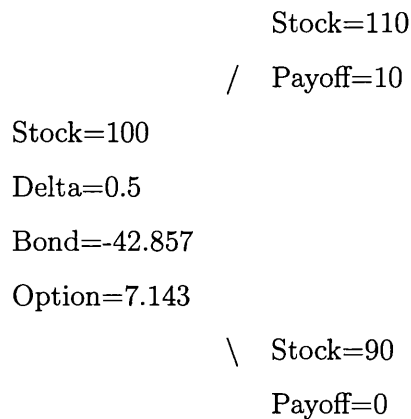
Figure 1.1: One-period CRR binomial tree

Tax due to the stock transactions

Let us assume that the stock transactions are taxed at a rate of 25%.

In the scheme identified in Figure 1.1 we should hold 0.5 units of stock for the year. If the stock moves up we have a gain of £5 (=0.5×(110-100)). If the stock moves down we have a gain of -£5 (=0.5×(90-100)). The resulting tax charges are £1.25 (0.25×5) and -£1.25 (0.25×(-5)) respectively.

Tax due on the bond transactions

Let us assume that the bond transactions are taxed at a rate of 40%.

In the scheme identified in Figure 1.1, we sell £42.857 in bonds. These grow at the risk-free rate to £45, which is the amount we have to repay at the option's expiry. The resulting gain is -£2.143 (=42.857-45). The resulting tax charge is -£0.8572 (= 0.4×(-2.143)).

Tax due on the option

Let us assume that the option is taxed at a rate of 25%.

The gain to the option holder if the stock moves up is £2.587 (=10-7.143). The gain if the stock moves down is -£7.143 (=0-7.143). The resulting tax charges are £0.647 (=0.25×2.587

) and -£1.786 (=0.25×(-7.143)) respectively.

Overall

For the CRR hedge to work on an after-tax basis, for each state of the world (either an up move in the stock price or a down move), the tax charge on the hedge (the sum of the tax charges on stock and the bond) must equal the tax charge on the option.

	Stock=£110	Stock=£90
Tax charge stock (£)	1.25	-£1.25
Tax charge bond (£)	-£0.86	-£0.86
Tax charge hedge (£)	£0.39	-£2.11
Tax charge option (£)	£0.65	-£1.77

Table 1.1: Tax charges for the hedge and option

We see from Table 1.1 that the tax-charge for the hedge and the option are not equal for either the up state or the down state. Therefore, we require a tax-adjusted hedge to ensure that the two are equal for all states of the world.

We will now look at M&P's work in more detail to see how they have approached the problem.

1.4 The Milevsky and Prisman algorithm

This section describes the tax-efficient hedging algorithm advocated by M&P in Milevsky (1997), Milevsky (1997a), Milevsky (1996c), and Milevsky (1996).

The price defined is the hedging price of a European call option (see Section 1.2 for a definition of the hedging price). The analysis starts in a single period setting and is then extended, as with CRR, to a multiperiod setting. However, in contrast to the classical case the development of a multiperiod model with taxes is quite complicated. This is because the tax liability in the multiperiod model is path dependent. The main culprit is the stock, since the tax liability on the stock at a particular node at the end of the tree is calculated from the

18

history of sales and purchases of the stock up to the final node, all made at various prices. The path-dependent nature means a non-recombining binomial tree has to be used.

Indeed, to demonstrate the fact that a classical delta hedge does not work with taxes, M&P show that the after-tax payoff from a call option at the final nodes in the classical tree, is generally not equal to the after-tax payoff from delta hedging with stock and bonds. In other words, a tax-mismatch arises as we saw in Section 1.3. The concept of the tax-mismatch forms the basis for the tax-adjusted hedging algorithm given in Chapter 2.

Definition 1.1: The tax-mismatch at the final nodes of the tree

$$\text{tax-mismatch} = \text{after-tax stock} + \text{after-tax bond} - \text{after-tax call}. \qquad (1.4)$$

At the moment it is not important to concern ourselves with how the individual quantities in (1.4) are calculated (although we looked at a numerical example in the single-period case in Section 1.3), but instead to recognise that the need for a tax-efficient hedging algorithm is motivated by the argument that the no-arbitrage condition at the final nodes on the tree, generally, does not hold on an after-tax basis.

1.4.1   Notation for the non-recombining tree

Consider a non-recombining binomial tree of N-periods. The Cartesian coordinates for each of the nodes on the tree are represented by a vector $\langle i, j \rangle \in \Re^2$. Here $i \in \{0, 1, ..., N\}$, represents the period, and $j \in \{0, 1, ..., 2^i - 1\}$ gives the specific node in the current period (see Figure 1.2). We can use this notation to identify the stock, bond, and delta at node $\langle i, j \rangle$ as $S_{\langle i,j \rangle}$, $B_{\langle i,j \rangle}$, and $\Delta_{\langle i,j \rangle}$ respectively.

The unique k^{th} node, prior to node $\langle i, j \rangle$, is denoted by $\langle i, j, -k \rangle$. As an example, the node immediately prior to $\langle 3, 0 \rangle$ in the tree above, is denoted $\langle 3, 0, -1 \rangle = \langle 2, 0 \rangle$. If $k = -2$, this would take us back two periods, and $\langle 3, 0, -2 \rangle = \langle 1, 0 \rangle$. This notation makes it easy to describe the entire path taken to a particular node.

19
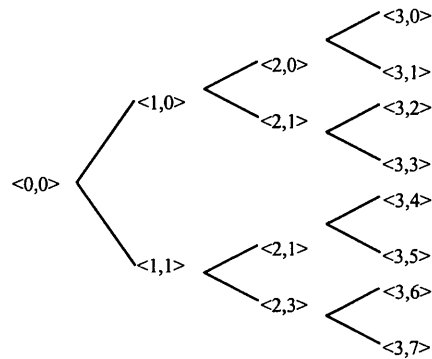
Figure 1.2: Co-ordinates for the non-recombining tree

1.4.2 Assumptions

1. No transaction costs.

2. Short selling is allowed.

3. The stock pays no dividends.

4. One plus the interest rate, R, stays constant over the life of the option.

5. The tax falls due at the final nodes of the tree.

6. The period does not traverse a tax year end, i.e., all transactions take place in the same tax year.

7. Losses can be used to offset other gains. This means that even if a long call option expires worthless, an effective cash flow equal to the tax rate multiplied by the option premium (its price at $t = 0$) is received by the option holder, as he or she can offset this against other gains for tax purposes.

1.4.3 Tax regulations

Delta-hedging the short call option means that the underlying security is being purchased and sold many times at a variety of prices which creates a path dependent tax liability. In order to

calculate the tax liability at the year-end, the total profit or loss must be calculated based on the adjusted cost basis (ACB) of the securities. When the security is bought or sold, the adjusted cost basis of the entire position must be recalculated to incorporate the new transaction price.

Definition 1.2: The Adjusted Cost Basis (from Milevsky 1996c), $ACB_{\langle i,j \rangle}$

$$ACB_{\langle i,j \rangle} = \frac{\Delta_{\langle i,j,-1 \rangle}ACB_{\langle i,j,-1 \rangle} + (\Delta_{\langle i,j \rangle} - \Delta_{\langle i,j,-1 \rangle})S_{\langle i,j \rangle}}{\Delta_{\langle i,j \rangle}} \quad \forall \Delta_{\langle i,j \rangle} > 0. \qquad (1.5)$$

$ACB_{\langle i,j \rangle}$ is the adjusted cost basis at the current node, $ACB_{\langle i,j,-1 \rangle}$ is the adjusted cost basis at the previous node, $\Delta_{\langle i,j \rangle}$ is the delta at the current node, $\Delta_{\langle i,j,-1 \rangle}$ is the delta at the previous node and $S_{\langle i,j \rangle}$ is the stock price at the current node.

Interpretation of the ACB

Equation (1.5) is given by M&P as the way to calculate the changing cost basis of the stock over the life of the hedge. The accounting treatment that is used to derive that equation is analogous to accounting for stock ("stock" as in items held in a warehouse) on a first-in-first-out basis. However, it is not clear to which countries' tax laws the accounting treatment relates.

The ACB is a linear sum of all the past purchase and sales prices divided by the current holding (delta) in the security. When a sale takes place there will be either a profit or a loss on the position. A profit will result if a sale takes place and the current level of the security is greater than the current ACB. This will generate a tax liability that falls due at the end of the current tax year, which we assume coincides with the final node of the tree. If a sale takes place and the current level of the security is less than the current ACB, a loss is incurred, which can be used as a deduction against other taxable gains. The profit or loss incurred on each transaction is called the realised net gain (RNG), and it is defined more formally below.

Definition 1.3: The Realised Net Gain (from Milevsky 1996c), $RNG_{\langle i,j \rangle}$

$$RNG_{\langle i,j \rangle} = \min[\Delta_{\langle i,j \rangle} - \Delta_{\langle i,j,-1 \rangle}, 0](ACB_{\langle i,j,-1 \rangle} - S_{\langle i,j \rangle}) + RNG_{\langle i,j,-1 \rangle}. \qquad (1.6)$$

The tax liability at the final nodes in the tree

The tax liability at the end of the final time period, which is also the tax due date, on the hedged short call position is comprised of three elements:

1. The tax on the stock transactions which is represented by the realised net gain,

$$\tau_s(RNG_{\langle N,j \rangle});$$

2. the tax on the premium received from the sale of the option less the payoff,

$$\tau_o(c_{\langle 0,0 \rangle} - c_{\langle N,j \rangle});$$

3. the reduction in the tax liability from the interest paid on the short bonds (this is a negative liability since the B_is are negative),

$$\tau_b \left((R-1) \sum_{k=1}^{N} B_{\langle N,j,-k \rangle} \right).$$

This gives a total tax liability of:

$$\tau_s(RNG_{\langle N,j \rangle}) + \tau_o(c_{\langle 0,0 \rangle} - c_{\langle N,j \rangle}) + \tau_b \left((R-1) \sum_{k=1}^{N} B_{\langle N,j,-k \rangle} \right). \tag{1.7}$$

This tax liability (or rebate) is responsible for the tax mismatch at the end of the classical binomial tree, which was given in Definition 1.1.

1.4.4   The tax-efficient algorithm of Milevsky and Prisman

The classical hedge does not work with taxes when the RNG is given by Equation (1.6) and the ACB by Equation (1.5), and so M&P modify the binomial tree by adjusting the payoffs to incorporate the tax liability. The price defined is the hedging price of a short call option. All tax rates are assumed to be equal (i.e., the writer of the call pays his marginal tax rate on all transactions), so $\tau_s = \tau_o = \tau_b = \tau$.

Note that the following equations use the same node subscripts as found in Milevsky (1996c).

This is done to ensure consistency between the exposition here and M&P's work. It would be clearer if, for example, $B^*_{\langle N-1,j\rangle}$ were written as $B^*_{\langle N,j,-1\rangle}$.

At all nodes prior to maturity we require:

$$\Delta^*_{\langle i,j\rangle}S_{\langle i,j\rangle} + B^*_{\langle i,j\rangle} - c^*_{\langle i,j\rangle} = 0 \quad \text{where } i \in \{0,1,...,N-1\}. \tag{1.8}$$

The asterisk denotes the variable or security as being tax-adjusted.

At maturity the tax liability is explicitly incorporated into the system.

For an "up" movement in the stock price:

$$\Delta^*_{\langle N-1,j\rangle}S_{\langle N-1,j\rangle}u - \tau(RNG_{\langle N,j\rangle}) + B^*_{\langle N-1,j\rangle}$$
$$-\tau\left((R-1)\left(\sum_{k=1}^{N-1} B^*_{\langle N-1,j,-k\rangle} + B^*_{\langle N-1,j\rangle}\right)\right) \tag{1.9}$$
$$= \max\left(S_{\langle N-1,j\rangle}u - K, 0\right)(1-\tau) + c^*_{\langle 0,0\rangle}\tau;$$

and for a "down" movement in the stock price:

$$\Delta^*_{\langle N-1,j\rangle}S_{\langle N-1,j\rangle}d - \tau(RNG_{\langle N,j\rangle}) + B^*_{\langle N-1,j\rangle}$$
$$-\tau\left((R-1)\left(\sum_{k=1}^{N-1} B^*_{\langle N-1,j,-k\rangle} + B^*_{\langle N-1,j\rangle}\right)\right) \tag{1.10}$$
$$= \max\left(S_{\langle N-1,j\rangle}d - K, 0\right)(1-\tau) + c^*_{\langle 0,0\rangle}\tau.$$

At maturity, $\Delta^*_{\langle N,j\rangle} = 0$ for $0 \le j \le 2^N - 1$, and so using Equation (1.6) M&P get:

$$RNG_{\langle N,j\rangle} = -\Delta^*_{\langle N-1,j\rangle}(ACB_{\langle N-1,j\rangle} - S_{\langle N,j\rangle}) + RNG_{\langle N-1,j\rangle}.$$

Equations (1.9) and (1.10) become, with this substitution:

$$\Delta^*_{\langle N-1,j\rangle}S_{\langle N-1,j\rangle}u + \tau\left(\Delta^*_{\langle N-1,j\rangle}(ACB_{\langle N-1,j\rangle} - S_{\langle N-1,j\rangle}u) - RNG_{\langle N-1,j\rangle}\right)$$

$$+B^*_{\langle N-1,j\rangle} - \tau\left((R-1)\left(\sum_{k=1}^{N-1}B^*_{\langle N-1,j,-k\rangle} + B^*_{\langle N-1,j\rangle}\right)\right) \tag{1.11}$$

$$= \max\left(S_{\langle N-1,j\rangle}u - K, 0\right)(1-\tau) + c^*_{\langle 0,0\rangle}\tau,$$

and,

$$\Delta^*_{\langle N-1,j\rangle}S_{\langle N-1,j\rangle}d + \tau\left(\Delta^*_{\langle N-1,j\rangle}(ACB_{\langle N-1,j\rangle} - S_{\langle N-1,j\rangle}d) - RNG_{\langle N-1,j\rangle}\right)$$

$$+B^*_{\langle N-1,j\rangle} - \tau\left((R-1)\left(\sum_{k=1}^{N-1}B^*_{\langle N-1,j,-k\rangle} + B^*_{\langle N-1,j\rangle}\right)\right) \tag{1.12}$$

$$= \max\left(S_{\langle N-1,j\rangle}d - K, 0\right)(1-\tau) + c^*_{\langle 0,0\rangle}\tau.$$

Solving for $\Delta^*_{\langle N-1,j\rangle}$ they establish that:

$$\Delta^*_{\langle N-1,j\rangle} = \frac{\max\left(S_{\langle N-1,j\rangle}u - K, 0\right) - \max\left(S_{\langle N-1,j\rangle}d - K, 0\right)}{S_{\langle N-1,j\rangle}(u-d)}. \tag{1.13}$$

M&P point out that Equation (1.13) is the result found by Scholes: the delta does not depend on the tax position of the investor (Scholes (1996)). This only works in a one-period setting since the bond terms contain the tax bracket of the investor and this feeds into the deltas when moving back through the tree in the multiperiod model.

$B^*_{\langle N-1,j\rangle}$ is found from substituting (1.13) into (1.12) to give:

$$B^*_{\langle N-1,j\rangle} = \frac{\left(\max(S_{\langle N-1,j\rangle}d - K, 0)(1-\tau) + c^*_{\langle 0,0\rangle}\tau\right) - \sum_{k=1}^{N-1}B^*_{\langle N-1,j,-k\rangle}}{(\tau - \tau R + 1)} \tag{1.14}$$

$$-\frac{\left(\Delta^*_{\langle N-1,j\rangle}S_{\langle N-1,j\rangle}d + \tau\left(\Delta^*_{\langle N-1,j\rangle}(ACB_{\langle N-1,j\rangle} - S_{\langle N-1,j\rangle}d) - RNG_{\langle N-1,j\rangle}\right)\right)}{(\tau - \tau R + 1)}.$$

The price of the call option at nodes $\langle N-1,j\rangle$ is given by the hedge portfolio:

$$c^*_{\langle N-1,j\rangle} = \Delta^*_{\langle N-1,j\rangle}S_{\langle N-1,j\rangle} + B^*_{\langle N-1,j\rangle}. \tag{1.15}$$

24

Problems to overcome

There are two problems to overcome:

1. The initial price of the call option, $c^*_{\langle 0,0 \rangle}$, is contained in the above expressions.

2. The $ACB_{\langle N-1,j \rangle}$ and $RNG_{\langle N-1,j \rangle}$ are functions of not only the current delta $\Delta^*_{\langle N-1,j \rangle}$, but also of the previous deltas in each path. However these deltas are not known at the current time, and to find them we need to evaluate the current delta. This cannot be done because the current delta is a function of the previous deltas. The problem is therefore a circular one involving an unknown feedback mechanism.

The first problem is relatively easy to solve. The expression for $c^*_{\langle 0,0 \rangle}$, which is found by working back through the tree, will be a function of itself and other variables (namely $S_{\langle N-1,j \rangle}$, R, u, d, and N). Consequently, finding $c^*_{\langle 0,0 \rangle}$ is simply a question of solving for the "fixed point".

The second problem is a little more difficult to overcome. M&P advocate using the following dynamic routine:

1. Compute $\Delta^*_{\langle i,j \rangle}$ and $B^*_{\langle i,j \rangle}$ by solving a single-period system of two equations and two unknowns, expressing $\Delta^*_{\langle i,j \rangle}$ and $B^*_{\langle i,j \rangle}$ in terms of the current $RNG_{\langle i,j \rangle}$, $ACB_{\langle i,j \rangle}$, $S_{\langle i,j \rangle}$, $B^*_{\langle i,j,\cdot \rangle}$ and the yet to be determined $c^*_{\langle 0,0 \rangle}$.

2. In the nodes $\langle N-1,j \rangle$, where $j = 0 \ldots 2^{N-1} - 1$, the next period's tax liability is explicitly incorporated into the system of equations. In all the nodes prior to $\langle N-1,j \rangle$, the tax liability is implicitly contained in the variables.

3. Write $RNG_{\langle i,j \rangle}$ and $ACB_{\langle i,j \rangle}$ in terms of $\Delta^*_{\langle i,j,-1 \rangle}$, $RNG_{\langle i,j,-1 \rangle}$ and $ACB_{\langle i,j,-1 \rangle}$.

4. Re-solve for $\Delta^*_{\langle i,j \rangle}$ and $B^*_{\langle i,j \rangle}$ in terms of the past $RNG_{\langle i,j,-1 \rangle}$, $ACB_{\langle i,j,-1 \rangle}$, $S_{\langle i,j \rangle}$, $B^*_{\langle i,j,\cdot \rangle}$ and the yet to be determined $c^*_{\langle 0,0 \rangle}$.

5. Express the current $c^*_{\langle i,j \rangle} = \Delta^*_{\langle i,j \rangle} S_{\langle i,j \rangle} + B^*_{\langle i,j \rangle}$.

6. Proceed backwards through the tree until the first node, at which we finally obtain that $c^*_{\langle 0,0 \rangle} = F(c^*_{\langle 0,0 \rangle})$ where the function $F(\cdot)$ should not contain any terms besides $S_{\langle 0,0 \rangle}$, R, u, d, N, and $c^*_{\langle 0,0 \rangle}$.

7. The final stage is to solve for the fixed point of the function $F(\cdot)$ and extract the tax-adjusted no-arbitrage equilibrium price of the option.

1.5 Analysis of the Milevsky and Prisman algorithm

1.5.1 The bond at the final nodes

The third term in Equations (1.9) and (1.10) is $B^*_{\langle N-1,j \rangle}$. In the taxless binomial option pricing model (Cox (1979)), this term is multiplied by R (one plus the risk-free interest rate over a single period). Therefore, the bond term in the M&P algorithm should be multiplied by R, as the only other bond term in these equations, the fourth term, is the tax on the bond interest. This gives Equations (1.9) and (1.10) as:

$$\Delta^*_{\langle N-1,j \rangle} S^*_{\langle N-1,j \rangle} u - \tau(RNG_{\langle N-1,j \rangle}) + RB^*_{\langle N-1,j \rangle}$$
$$-\tau\left((R-1)\left(\sum_{k=1}^{N-1} B^*_{\langle N-1,j,-k \rangle} + B^*_{\langle N-1,j \rangle}\right)\right) \qquad (1.16)$$
$$= \max\left(S_{\langle N-1,j \rangle} u - K, 0\right)(1-\tau) + c^*_{\langle 0,0 \rangle}\tau,$$

and,

$$\Delta^*_{\langle N-1,j \rangle} S_{\langle N-1,j \rangle} d - \tau(RNG_{\langle N-1,j \rangle}) + RB^*_{\langle N-1,j \rangle}$$
$$-\tau\left((R-1)\left(\sum_{k=1}^{N-1} B^*_{\langle N-1,j,-k \rangle} + B^*_{\langle N-1,j \rangle}\right)\right) \qquad (1.17)$$
$$= \max\left(S_{\langle N-1,j \rangle} d - K, 0\right)(1-\tau) + c^*_{\langle 0,0 \rangle}\tau.$$

Again substituting for $RNG_{\langle N-1,j \rangle}$ and solving for $B^*_{\langle N-1,j \rangle}$ we get:

$$B^*_{\langle N-1,j \rangle} = \frac{\left(\max(S_{\langle N-1,j \rangle} d - K, 0)(1-\tau) + c^*_{\langle 0,0 \rangle}\tau\right) - \sum\limits_{k=1}^{N-1} B^*_{\langle N-1,j,-k \rangle}}{(\tau - \tau R + R)} \qquad (1.18)$$
$$-\frac{\left(\Delta^*_{\langle N-1,j \rangle} S_{\langle N-1,j \rangle} d + \tau\left(\Delta^*_{\langle N-1,j \rangle}(ACB_{\langle N-1,j \rangle} - S_{\langle N-1,j \rangle} d) - RNG_{\langle N-1,j \rangle}\right)\right)}{(\tau - \tau R + R)}.$$

The only difference between Equation (1.14) and Equation (1.18) is the denominator, which is changed from $(\tau - \tau R + 1)$ to $(\tau - \tau R + R)$. The delta remains unaffected and is given in Equation (1.13).

1.5.2   The adjusted cost base and realised net gain

The calculation of the tax on the bond and the call is relatively straightforward, but the tax on the stock is more complicated, and where the tax treatment is most likely to vary, say between different countries. The RNG and ACB given by Equation (1.5), are therefore the key features that incorporate the tax laws into the model.

Is there another way to calculate the profit (or loss) on the stock transactions? In Chapter 2 we will look at another version of the RNG which is simpler to handle.

1.5.3   The symbolic nature of MAPLE

M&P have implemented their algorithm using the symbolic computational language, MAPLE, in order to find a symbolic expression for the option price. The symbolic nature of this language has two advantages over a numerical one: the final expression for the call price can be obtained exactly, as opposed to the approximate solution that a numerical system produces; and the ability to represent values symbolically enables easy investigation into the sensitivity of the call price with respect to changes in its arguments. However, using a numerical routine does have three distinct advantages over a symbolic one: it is easier to implement; results are obtained much more quickly (in Chapter 2 we will see at worst a few minutes, whereas the symbolic approach can take hours); and being able to "see" the numerical values for the option delta, bond and call at each node on the tree, helps the user understand how the algorithm reaches the tax-adjusted option price.

1.6   Summary

There have been two attempts to value options in the presence of taxes: Scholes' (Scholes (1976)) modification to the Black-Scholes equation, later extended by M&P to include dividends (Milevsky (1997a)), and M&P's tax-efficient hedging algorithm (Milevsky (1997)). Scholes'

analysis indicates that the effect of taxes is to increase the price of a put and decrease the price of a call, provided the income tax rate is greater than the capital gains tax rate. M&P do not give any indication of the effect of taxes on option prices in their approach, although they point out that the assumptions made by Scholes in developing his tax-adjusted Black-Scholes model are unrealistic. Indeed, it is one of the implications from Scholes' work (Scholes (1996)), in particular that option values are unchanged by taxes if all marginal tax rates are equal, that M&P believe is responsible for the lack of research in the field.

M&P's algorithm for valuing a call with taxes (Milevsky (1997)) uses a symbolic approach to overcome the two major hurdles that taxes introduce into option valuation in a binomial setting: the inclusion of the initial price of the option in the payoffs, and the feedback problem involving the calculation of the delta and bond amounts. The discussion of their algorithm has identified some issues that warrant further attention, as does the choice of a symbolic language to code the algorithm. In view of this, the next chapter presents an alternative tax-adjusted hedging algorithm implemented in C++, which values European calls and puts in the presence of taxes.

Chapter 2

The Tax-Adjusted Hedging Price of Options: A Dynamic Programming Approach

2.1 Introduction

The previous chapter looked at M&P's tax-efficient hedging algorithm. In this chapter we will develop an alternative tax-adjusted hedging algorithm. Before that, in Section 2.2, we discuss the aims of the thesis and define the option price that we are trying to find, being "the tax-adjusted hedging price", and look at how this relates to the concept of a "tax-adjusted no-arbitrage price". In Section 2.3 the tax-adjusted hedging algorithm is developed and results and numerical examples obtained using this algorithm form Sections 2.4 and 2.5. A method to obtain an approximation to the continuous-time option price, from discrete-time binomial tree prices, is presented in Section 2.6, and this allows tax-adjusted put-call parity to be examined in the context of prices obtained using the tax-adjusted hedging algorithm. Section 2.7 is a summary of the chapter.

The remainder of this section deals with some basic background information regarding options, which is included for completeness, and a basic guide to taxation.

2.1.1 Assumed knowledge

Options

The fundamental principle which underpins the valuation of options and derivative securities, in general, is the no-arbitrage condition. If the payoff from a derivative security can be replicated by a portfolio of simple instruments, then the initial price of the derivative must be the same as the minimum cost of building the portfolio of simple instruments. If this were not the case, arbitrage profits could be made by buying the cheaper of the two (the derivative and the portfolio) and selling the other, since at payoff the two will give a combined profit of zero. The no-arbitrage principle states that such profits cannot be made.

The binomial model, CRR, is a simple, discrete-time approach to valuing derivative securities. The life of the option is broken down into discrete time-steps, and the possible stock prices over this time form a binomial tree. At the end of the tree, the payoff from the option can be calculated. At the nodes prior to payoff the holdings in the underlying and a riskless security (a discount bond) can be found which replicate the payoff of the option. Using the principle of no-arbitrage, the price of the option at each node must be the same as the cost of the replicating portfolio at that node. By working back through the tree, the holdings in the underlying and bond at the first node can be found. The initial price of the option is equal to the initial value of this portfolio.

A basic guide to taxation

The following guide to taxation is not intended to be a comprehensive one, but instead aims to capture the general flavour of a generic tax system. The motivation behind this approach is to start from a relatively simple mathematical treatment of the tax laws, and to build in more complexity as we progress through the chapters. Even the simplified treatment of the tax laws adopted in this thesis proves demanding.

Taxation of stock and bonds. When a profit is made on a transaction it is subject to taxation. That is, a proportion of the gain is paid to the government. The proportion of the gain that is paid is called the tax rate. For example, if a person buys some stock for £100 and then sells it one year later for £120, they have made a gain of £20. If the tax rate is 25%, the

tax on that transaction is £5 (0.25×£20). This is an example of a capital gain, because stock transactions are of a capital nature. The gain here is uncertain; the stock could quite easily have decreased in value and the investor would have lost money on the transaction. Losses also have a tax implication (see below for more details).

Now consider if the person had instead used their £100 to buy a default-free bond with a face value of £100, a maturity of one year, and an annual coupon of 10% paid semi-annually. After one year the investor will have received £10 (two payments of £5) plus his/her initial £100 back as the bond is redeemed at its face value. This means the investor has made a gain of £10 which is subject to tax. This time the gain is income, since the investor was guaranteed to receive this amount as long as he/she held on to the bond until its maturity. The tax rate on this transaction may be 40%, which is higher than the capital gains tax rate of 25% because this is an income transaction, and this gives a tax charge of £4 (0.4×£10).

The two examples of stock and bond transactions here are very simple, but relevant to pricing options in the presence of taxes since we hold a portfolio of stock and bond when hedging a short option position. However, because dynamically hedging a short option with a portfolio of stock and bond means rebalancing the amounts of stock and bond held in the portfolio, the tax charge on this portfolio becomes more difficult to calculate than our simple examples above.

The tax on the bond simply relates to the interest paid on the bond, and this remains straightforward. The tax calculations involving the stock are more complicated, since when performing the hedge we are buying and selling different amounts of stock at a variety of different prices. This is why we need to define the Realised Net Gain, which was discussed in Chapter 1 and will be discussed further in this chapter, and why we need to value options using a non-recombining tree: the Realised Net Gain is path dependant.

Taxation of options   We will be pricing European cash-settled equity options in the presence of taxes. We need to know not only the tax involved in hedging the option, but also the tax charges that relate to the option itself. For the writer of the option we have a gain, which is the premium received at the inception of the option, and a cost, which is the amount the writer has to pay to the option buyer at maturity. From these two amounts we can calculate whether

31

the writer has made a profit or loss on the option, and hence determine the tax charge relating to it.

We will assume that the option is taxed as capital gains.

The end of the tax year. The tax year-end is the date in the year when the tax charge for that year is determined. We will assume throughout this thesis for simplicity that the tax charge is also payable on this date, although in reality it is usually due some time after the tax year-end.

To start with we will assume that the option is written in one tax year and matures at the tax year-end (when the tax is payable). In Chapter 4 we will relax this assumption and allow the option to be written in the first tax year and to expire in the second tax year. In this case we have to look at two tax charges, one for each of the two tax years.

Losses Generally speaking, when a loss is made on a transaction it can be used to offset gains on other transactions that the writer may have made. This means that the tax charge on the profits of the writer is reduced by an amount equal to the loss multiplied by the tax rate. In effect, the writer receives a rebate equal to the loss multiplied by the tax rate.

Again, it is worth stressing that this is an extremely simplified treatment of the tax laws, and is certainly not intended to capture every nuance of any particular country's tax system.

2.2 The tax-adjusted no-arbitrage price and the tax-adjusted hedging price

It is important to establish what we mean by the phrase "tax-adjusted no-arbitrage" and to determine whether it actually makes any sense to define the tax-adjusted no-arbitrage price of an option. To do this let us first consider the no-tax CRR situation.

No-tax case

In the CRR environment we hedge the option by dynamically rebalancing a portfolio of stock and bond. No-arbitrage under these circumstances means that the value of the portfolio at

each node on the tree is equal to the value of the option at that node. At maturity of the option we can write:

$$\underbrace{\text{value of the hedging porfolio at maturity}}_{\text{Long portfolio from perspective of writer}} = \underbrace{\text{payoff from the option}}_{\text{Long option from perspective of buyer}}. \qquad (2.1)$$

We could also argue that the option writer should be perfectly hedged at maturity. We write this as:

$$\underbrace{\text{value of the hedging porfolio at maturity} - \text{payoff from the option}}_{\text{Long portfolio and short option from perspective of the writer}} = 0. \qquad (2.2)$$

It is obvious that the two equations above are equivalent, just written from differing perspectives: (2.1) is written by considering the writer as being long the hedging portfolio and the buyer as being long the option, whereas (2.2) is written by considering the writer to be long the hedging portfolio and short the option. The two are equivalent because the writer's short position in the option is exactly the same as the buyer's long position.

The tax-adjusted case

If we are looking at the situation on an after tax-basis then the "after-tax no-arbitrage condition" is:

$$\underbrace{\text{after-tax value of the hedging porfolio at maturity}}_{\text{Long portfolio from perspective of writer}} = \underbrace{\text{after-tax payoff from the option}}_{\text{Long option from perspective of buyer}}.$$

$$(2.3)$$

The "after-tax hedging condition" is:

$$\underbrace{\text{after-tax value of the hedging porfolio at maturity} - \text{after-tax payoff from the option}}_{\text{Long portfolio and short option from perspective of the writer}} = 0.$$

$$(2.4)$$

Now the long after-tax payoff from the option from the buyer's perspective may not, in general, be the same as the short after-tax payoff from the writer's perspective because the

writer and the buyer may have different tax positions. (2.3) and (2.4) are equivalent only if we have the following condition:

Condition 2.1: for equivalence of the after-tax no-arbitrage and after-tax hedging conditions The tax position of the buyer is the same as the tax position of the writer. This means that the writer and the buyer pay the same tax rates, have the same ability to use losses to offset gains and pay their tax on the same date.

If market participants have different tax positions it doesn't make any sense to talk about "after-tax no-arbitrage" because we can't define a price that precludes after-tax arbitrage opportunities for all market participants. In view of this, to have an "after-tax no-arbitrage condition" and to define an "after-tax no-arbitrage price" we need a stronger condition than the one given above. This is the following condition:

Condition 2.2: to define an "after-tax no-arbitrage price" or "after-tax no-arbitrage condition" All market participants (buyers and sellers) have the same tax position.

Consequences

Throughout this thesis we will be looking at the after-tax position of the long hedge and short option from the writer's perspective. We will therefore be finding the "tax-adjusted hedging price" of the option. This is the price that the seller assigns to the option such that he/she is perfectly hedged on an after-tax basis.

We can argue that this is the "tax-adjusted no-arbitrage price" of the option only if we make the assumption that all market participants have exactly the same tax position. With this assumption, using the phrase "after-tax no-arbitrage" is valid. However, is this a plausible assumption to make? It is perhaps unrealistic to assume that all market participants have the same tax position, especially when we consider whether losses can be used to offset other gains, as this is an extremely complex issue and likely to be market participant specific. That said, the original assumptions in Black-Scholes, including the assumption of no taxes, are also unrealistic.

As is stated above, the price that is defined in this thesis is the "tax-adjusted hedging price" of an option. Whether we can call this the "tax-adjusted no-arbitrage price" is dependent on

whether we can make the assumption that all market participants have the same tax position. Throughout this thesis the phrases "tax-adjusted hedging price" and "after-tax hedging condition" will be used in most cases. If the reader wishes to make the assumption about the equivalence of the tax position of all market participants, then the reader can substitute "tax-adjusted no-arbitrage price" and "after-tax no-arbitrage condition" in their place.

On specific occasions it will be pointed out in the text the price that is being defined.

2.3 The tax-adjusted hedging algorithm for European equity call and put options

The aim is to develop a tax-adjusted hedging algorithm that gives us the amount of stock (delta) and holding in bonds that we require at each node on the non-recombining tree (the tax on the stock transaction is path dependent) such that the option writer is fully hedged, on an after-tax basis, at the maturity of the option. That is, we aim to find the tax-adjusted self-financing hedging portfolio of stock and bonds at each node on the tree, where the tax-adjusted hedging price is the value of this portfolio at the initial node on the tree.

When the option writer is fully hedged at maturity, the following equation, which is the after-tax hedging condition as given by (2.4), is true for $j = 0, 1, ..., 2^N - 1$:

$$ATS_{\langle N,j \rangle} + ATB_{\langle N,j \rangle} - ATO_{\langle N,j \rangle} = 0, \qquad (2.5)$$

where $ATS_{\langle N,j \rangle}$ is the after-tax stock position, $ATB_{\langle N,j \rangle}$ is the after-tax bond position and $ATO_{\langle N,j \rangle}$ is the after-tax option position, at node $\langle N, j \rangle$. Now Equation (1.4) in Definition 1.1 is:

$$\mathfrak{M}_{\langle N,j \rangle} = ATS_{\langle N,j \rangle} + ATB_{\langle N,j \rangle} - ATO_{\langle N,j \rangle}. \qquad (2.6)$$

where $\mathfrak{M}_{\langle N,j \rangle}$ is the tax-mismatch at node $\langle N, j \rangle$. So another way of satisfying the after-tax hedging condition is to require that the tax-mismatch be zero across the final nodes, i.e., that

the following equation is true for $j = 0, 1, ..., 2^N - 1$:

$$\mathfrak{M}_{\langle N,j \rangle} = 0. \tag{2.7}$$

The concept of the tax-mismatch and the requirement that it be zero across all the final nodes on the tree is fundamental to the operation of the tax-adjusted hedging algorithm.

Evolution of the stock price

It is worth pointing out at this early stage that we evolve the stock price in exactly the same manner as in CRR. The usual formula for an "up" and "down" move over one period are:

$$u = \exp\left[\sigma\sqrt{\frac{T}{N}}\right], \tag{2.8}$$

$$d = \exp\left[-\sigma\sqrt{\frac{T}{N}}\right], \tag{2.9}$$

where σ is the annual volatility, T is time to maturity of the option in years, and N is the number of periods on the tree.

Therefore, the evolved stock price forms a recombining tree. The recombining stock price tree has to be superimposed on the non-recombining tree that we use to value the option. As an example, let us consider a two-period non-recombining tree. Over the first period, the stock price moves to $S_{\langle 1,0 \rangle} = S_{\langle 0,0 \rangle}u$ or $S_{\langle 1,1 \rangle} = S_{\langle 0,0 \rangle}d$. Over the second period we have $S_{\langle 2,0 \rangle} = S_{\langle 0,0 \rangle}u^2$, $S_{\langle 2,1 \rangle} = S_{\langle 0,0 \rangle}ud$, $S_{\langle 2,2 \rangle} = S_{\langle 0,0 \rangle}du$, and $S_{\langle 2,3 \rangle} = S_{\langle 0,0 \rangle}d^2$. Of course, $S_{\langle 2,1 \rangle} = S_{\langle 2,2 \rangle} = S_{\langle 0,0 \rangle}$, so although we have four distinct nodes after two periods, the stock price is the same at two of them.

2.3.1 The after-tax payoffs from the stock, bond and call at the final nodes

The tax liability on the stock, bond and option transactions at the final nodes on the tree

1. The tax on the stock transactions. We need to know the net profit (or loss) along each path on the tree due to the stock transactions and this is given by the realised net gain

(RNG). The tax liability on the stock transactions is:

$$-\tau_s(RNG_{\langle N,j\rangle}).$$

The RNG is explained more fully below.

2. The tax on the bond transactions. We need to know the net profit (or loss) along each path on the tree due to the bond transactions. This is simply the sum of the interest paid or received on the bond holdings along a particular path on the tree. The tax liability on the bond transactions is:

$$-\tau_b\left((R-1)\sum_{k=1}^{N}B_{\langle N,j,-k\rangle}\right).$$

3. The tax on the option transactions. We need to know the net profit (or loss) on the option transactions. For a long option position, the net profit is given by the payoff minus the initial cost of the option. The tax liability on the long option position is:

$$-\tau_o(X_{\langle N,j\rangle} - X_{\langle 0,0\rangle}).$$

The reason why the tax liability for the option has been given on a long option position, whereas we need it in terms of a short position, is because of the way Equation (2.6) is written; we define $ATO_{\langle N,j\rangle}$ in terms of a long option position, so $-ATO_{\langle N,j\rangle}$, as it appears in (2.6), represents the after-tax short option position.

We will now formally define the after-tax positions in the stock, bond and option at the final nodes on the tree.

Definition 2.1: The After-Tax Stock payoffs at the final nodes, $ATS_{\langle N,j\rangle}$

$$ATS_{\langle N,j\rangle} = \Delta_{\langle N,j,-1\rangle}S_{\langle N,j\rangle} - \tau_s(RNG_{\langle N,j\rangle}). \tag{2.10}$$

Definition 2.2: The After-Tax Bond payoffs at the final nodes, $ATB_{\langle N,j \rangle}$

$$ATB_{\langle N,j \rangle} = RB_{\langle N,j,-1 \rangle} - \tau_b \left((R-1) \left(\sum_{k=1}^{N} B_{\langle N,j,-k \rangle} \right) \right). \tag{2.11}$$

Definition 2.3: The After-Tax Option payoffs at the final nodes, $ATO_{<N,j>}$

$$ATO_{\langle N,j \rangle} = (1 - \tau_o) X_{\langle N,j \rangle} + \tau_o X_{\langle 0,0 \rangle}, \tag{2.12}$$

where X is the generic notation for an option, either a European put or call.

It is important to remember that we are defining the tax-adjusted hedging price of the option. Definitions 2.1, 2.2 and 2.3 all relate to the writer of the option holding the hedging portfolio of stock and bonds, and a short position in a call option written on the stock. Therefore, the tax rates and assumptions about the deductibility of losses, all relate to the writer of the option. The after-tax payoff from the option given in (2.12) is for a long position in the option, and looks as though it relates to the buyer of the option, since the buyer would receive the payoff multiplied by one minus the tax rate, plus a rebate given by the premium paid multiplied by the tax rate. However, as we explained above when discussing the tax liability on the option, when Equation (2.12) is substituted into the tax-mismatch equation, (2.6), the negative sign correctly specifies the short option position that the writer holds.

2.3.2 The realised net gain

The RNG is the key quantity that incorporates the tax treatment into the model, since this determines the tax on the stock transactions. Here we use a RNG that is intended to capture the marked-to-market method of recording the profit or loss, which, certainly in the UK, seems to be the most reasonable way to calculate the profit or loss on the stock transactions while delta hedging an equity option. With this method, a profit or loss is generated on each movement of the stock price.

Definition 2.4: The marked-to-market realised net gain, $RNG_{\langle i,j\rangle}^{mkt}$

$$RNG_{\langle N,j\rangle}^{mkt} = \sum_{k=1}^{N} \Delta_{\langle N,j,-k\rangle} \left(S_{\langle N,j,-k+1\rangle} - S_{\langle N,j,-k\rangle} \right) \qquad (2.13)$$

From now, when we refer to RNG in the text or $RNG_{\langle i,j\rangle}$ in an equation, we will mean this marked-to-market version.

2.3.3 The after-tax hedging condition for the tax-adjusted deltas and bonds

Let us denote the unique[1] tax-adjusted deltas and bonds as $\Delta_{\langle i,j\rangle}^{*}$ and $B_{\langle i,j\rangle}^{*}$, where $i = 0, 1, ..., N-1$ and $j = 0, 1, ..., 2^N - 1$, such that the after-tax hedging condition, as specified by Equation (2.5), is satisfied . Substituting for $ATS_{\langle N,j\rangle}$ (2.10), $RNG_{\langle N,j\rangle}$ (2.13), $ATB_{\langle N,j\rangle}$ (2.11) and $ATO_{\langle N,j\rangle}$ (2.12) in (2.5), we get:

$$
\begin{aligned}
0 &= \Delta_{\langle N,j,-1\rangle}^{*} S_{\langle N,j\rangle} - \tau_s \sum_{k=1}^{N} \Delta_{\langle N,j,-k\rangle}^{*} \left(S_{\langle N,j,-k+1\rangle} - S_{\langle N,j,-k\rangle} \right) \qquad (2.14) \\
&\quad + RB_{\langle N,j,-1\rangle}^{*} - \tau_b \left((R-1) \left(\sum_{k=1}^{N} B_{\langle N,j,-k\rangle}^{*} \right) \right) \\
&\quad - (1 - \tau_o) X_{\langle N,j\rangle} - \tau_o X_{\langle 0,0\rangle}^{*} \\
&= \mathfrak{M}_{\langle N,j\rangle}^{*}.
\end{aligned}
$$

As with the no-tax CRR case, the self-financing property of the hedging portfolio means that we have the following relationship between the portfolios at connected nodes $\langle i,j,-1\rangle$ and $\langle i,j\rangle$:

$$\Delta_{\langle i,j,-1\rangle}^{*} S_{\langle i,j\rangle} + RB_{\langle N,j,-1\rangle}^{*} = \Delta_{\langle i,j\rangle}^{*} S_{\langle i,j\rangle} + B_{\langle i,j\rangle}^{*}. \qquad (2.15)$$

Of course, the initial tax-adjusted hedging price of the option is given by:

$$X_{\langle 0,0\rangle}^{*} = \Delta_{\langle 0,0\rangle}^{*} S_{\langle 0,0\rangle} + B_{\langle 0,0\rangle}^{*}. \qquad (2.16)$$

---

[1]Chapter 3 deals with the question of uniqueness. For now we will assume that we have unique tax-adjusted deltas and bonds.

2.3.4 The after-tax hedging condition for arbitrary deltas and bonds

In general, for deltas and bonds other than $\Delta^*_{\langle i,j\rangle}$ and $B^*_{\langle i,j\rangle}$, the after-tax hedging condition will not be satisfied across all the final nodes on the tree. Let us denote these deltas and bonds by $\Delta^0_{\langle i,j\rangle}$ and $B^0_{\langle i,j\rangle}$, where Equations (2.14), (2.15) and (2.16) become:

$$
\begin{aligned}
\mathfrak{M}^0_{\langle N,j\rangle} \;=\;& \Delta^0_{\langle N,j,-1\rangle}S_{\langle N,j\rangle} - \tau_s \sum_{k=1}^{N} \Delta^0_{\langle N,j,-k\rangle}\left(S_{\langle N,j,-k+1\rangle} - S_{\langle N,j,-k\rangle}\right) \qquad (2.17)\\
&+RB^0_{\langle N,j,-1\rangle} - \tau_b\left((R-1)\left(\sum_{k=1}^{N} B^0_{\langle N,j,-k\rangle}\right)\right)\\
&-(1-\tau_o)X_{\langle N,j\rangle} - \tau_c X^0_{\langle 0,0\rangle}\\
\neq\;& 0,
\end{aligned}
$$

$$
\Delta^0_{\langle i,j,-1\rangle}S_{\langle i,j\rangle} + RB^0_{\langle N,j,-1\rangle} = \Delta^0_{\langle i,j\rangle}S_{\langle i,j\rangle} + B^0_{\langle i,j\rangle}, \qquad (2.18)
$$

and

$$
X^0_{\langle 0,0\rangle} = \Delta^0_{\langle 0,0\rangle}S_{\langle 0,0\rangle} + B^0_{\langle 0,0\rangle}. \qquad (2.19)
$$

A two-period example

Figure 2.1 shows a two-period tree and the tax-mismatch across each of the final four nodes. We can immediately see why we need a non-recombining tree since the tax-mismatches at nodes $\langle 2,1\rangle$ and $\langle 2,2\rangle$, equivalent in the no-tax CRR tree in the sense that they are both zero, are clearly distinct in our taxable environment.

$$\langle 2,0\rangle \quad \mathfrak{M}^0_{<2,0>} = \Delta^0_{\langle 1,0\rangle}S_{\langle 2,0\rangle} - \tau_s\left[\Delta^0_{\langle 0,0\rangle}\left(S_{\langle 1,0\rangle} - S_{\langle 0,0\rangle}\right) + \Delta^0_{\langle 1,0\rangle}\left(S_{\langle 2,0\rangle} - S_{\langle 1,0\rangle}\right)\right]$$
$$+ RB^0_{\langle 1,0\rangle} - \tau_b(R-1)\left(B^0_{\langle 0,0\rangle} + B^0_{\langle 1,0\rangle}\right) - (1-\tau_o)X_{\langle 2,0\rangle} - \tau_c X^0_{\langle 0,0\rangle}$$

$$\langle 2,1\rangle \quad \mathfrak{M}^0_{<2,1>} = \Delta^0_{\langle 1,0\rangle}S_{\langle 2,1\rangle} - \tau_s\left[\Delta^0_{\langle 0,0\rangle}\left(S_{\langle 1,0\rangle} - S_{\langle 0,0\rangle}\right) + \Delta^0_{\langle 1,0\rangle}\left(S_{\langle 2,1\rangle} - S_{\langle 1,0\rangle}\right)\right]$$
$$+ RB^0_{\langle 1,0\rangle} - \tau_b(R-1)\left(B^0_{\langle 0,0\rangle} + B^0_{\langle 1,0\rangle}\right) - (1-\tau_o)X_{\langle 2,1\rangle} - \tau_c X^0_{\langle 0,0\rangle}$$

$$\langle 2,2\rangle \quad \mathfrak{M}^0_{<2,2>} = \Delta^0_{\langle 1,1\rangle}S_{\langle 2,2\rangle} - \tau_s\left[\Delta^0_{\langle 0,0\rangle}\left(S_{\langle 1,1\rangle} - S_{\langle 0,0\rangle}\right) + \Delta^0_{\langle 1,1\rangle}\left(S_{\langle 2,2\rangle} - S_{\langle 1,1\rangle}\right)\right]$$
$$+ RB^0_{\langle 1,1\rangle} - \tau_b(R-1)\left(B^0_{\langle 0,0\rangle} + B^0_{\langle 1,1\rangle}\right) - (1-\tau_o)X_{\langle 2,2\rangle} - \tau_c X^0_{\langle 0,0\rangle}$$

$$\langle 2,3\rangle \quad \mathfrak{M}^0_{<2,3>} = \Delta^0_{\langle 1,1\rangle}S_{\langle 2,3\rangle} - \tau_s\left[\Delta^0_{\langle 0,0\rangle}\left(S_{\langle 1,1\rangle} - S_{\langle 0,0\rangle}\right) + \Delta^0_{\langle 1,1\rangle}\left(S_{\langle 2,3\rangle} - S_{\langle 1,1\rangle}\right)\right]$$
$$+ RB^0_{\langle 1,1\rangle} - \tau_b(R-1)\left(B^0_{\langle 0,0\rangle} + B^0_{\langle 1,1\rangle}\right) - (1-\tau_o)X_{\langle 2,3\rangle} - \tau_c X^0_{\langle 0,0\rangle}$$

Figure 2.1: A two-period tree with the tax-mismatch at the final nodes

2.3.5 The tax-adjusted hedging algorithm

In the previous two sections we have looked at the unique tax-adjusted deltas and bonds, $\Delta^*_{\langle i,j\rangle}$ and $B^*_{\langle i,j\rangle}$, that satisfy the after-tax hedging condition at the final nodes so the tax-mismatch is zero across all of these nodes, and arbitrary deltas and bonds, $\Delta^0_{\langle i,j\rangle}$ and $B^0_{\langle i,j\rangle}$, where the after-tax hedging condition is not satisfied at the final nodes so the tax-mismatch is not equal to zero across all these nodes. How do we get from our arbitrary deltas and bonds, $\Delta^0_{\langle i,j\rangle}$ and $B^0_{\langle i,j\rangle}$, to our unique tax-adjusted delta and bonds, $\Delta^*_{\langle i,j\rangle}$ and $B^*_{\langle i,j\rangle}$, and hence $X^*_{\langle 0,0\rangle}$, the tax-adjusted hedging price of the option? One method is to use the algorithm that is given below.

The basic idea behind the algorithm is to form adjusted payoffs for the option which are used to compute new bonds and deltas in the tree using the usual CRR recursive procedure. The tax-mismatch is used to provide feedback and to adjust the payoffs.

41

The algorithm

1. Set $g = 0$.

2. Form the synthetic payoff $\mathfrak{X}^g_{\langle N,j\rangle}$ for the gth iteration from the previous synthetic payoff minus the previous tax-mismatch.

   (a) If $g > 0$, $\mathfrak{X}^g_{\langle N,j\rangle} = \mathfrak{X}^{g-1}_{\langle N,j\rangle} - \mathfrak{M}^{g-1}_{\langle N,j\rangle}$, where $j = 0, 1, ..., 2^N - 1$.

   (b) If $g = 0$, $\mathfrak{X}^g_{\langle N,j\rangle} = \mathfrak{X}^0_{\langle N,j\rangle} = X_{\langle N,j\rangle}$, where $j = 0, 1, ..., 2^N - 1$.

3. Using the normal recursive procedure as in CRR, work back through the tree computing the delta and bonds at each node.

   (a) $\Delta^g_{\langle N,j,-1\rangle} S_{\langle N,j\rangle} + RB^g_{\langle N,j,-1\rangle} = \mathfrak{X}^g_{\langle N,j\rangle}$, and $\Delta^g_{\langle i,j,-1\rangle} S_{\langle i,j\rangle} + RB^g_{\langle i,j,-1\rangle} = \Delta^g_{\langle i,j\rangle} S_{\langle i,j\rangle} + B^g_{\langle i,j\rangle}$ where $i = 1, 2, ..., N - 1$ and $j = 0, 1, ..., 2^i - 1$.

4. Form the tax-adjusted hedging price of the option at the initial node for the gth iteration.

   (a) $X^g_{\langle 0,0\rangle} = \Delta^g_{\langle 0,0\rangle} S_{\langle 0,0\rangle} + B^g_{\langle 0,0\rangle}$.

5. Form the tax-mismatch using the deltas and bonds computed in Step 3.

   (a) $\mathfrak{M}^g_{\langle N,j\rangle} = ATS^g_{\langle N,j\rangle} + ATB^g_{\langle N,j\rangle} - ATO^g_{\langle N,j\rangle}$.

6. Test whether the tax-mismatch has been reduced to zero across all the final nodes. If it has, we have found the unique tax-adjusted delta and bonds and hence the tax-adjusted hedging price of the option.

   (a) If $\mathfrak{M}^g_{\langle N,j\rangle} = 0 \ \forall j$, then $\Delta^g_{\langle i,j\rangle} = \Delta^*_{\langle i,j\rangle}$ and $B^g_{\langle i,j\rangle} = B^*_{\langle i,j\rangle}$, for $i = 0, 1, ..., N - 1$ and $j = 0, 1, ..., 2^i - 1$, and $X^g_{\langle 0,0\rangle} = X^*_{\langle 0,0\rangle}$. End.

   (b) Else $g = g + 1$. Return to Step 2.

Notes to the algorithm

1. For $g = 0$ the synthetic option payoffs are simply the normal CRR option payoffs. Consequently, $\Delta^0_{\langle i,j\rangle} = \Delta^{CRR}_{\langle i,j\rangle}$ and $B^0_{\langle i,j\rangle} = B^{CRR}_{\langle i,j\rangle}$ where $i = 0, 1, ..., N - 1$ and $j = 0, 1, ..., 2^i - 1$, and $X^0_{\langle 0,0\rangle} = X^{CRR}_{\langle 0,0\rangle}$. The superscript CRR indicates no-tax CRR quantities.

2. When computing the tax-mismatch in Step 5, the option payoffs that are used are the classical CRR payoffs, $X_{\langle N,j \rangle}$, as shown, for example, in Equation (2.14), and not the current synthetic payoffs, $\mathcal{X}^g_{\langle N,j \rangle}$.

3. For a European call option, $X_{\langle N,j \rangle} = \max\left[S_{\langle N,j \rangle} - K, 0\right]$, and for a European put, $X_{\langle N,j \rangle} = \max\left[K - S_{\langle N,j \rangle}, 0\right]$.

Tax-adjusted option prices at the intermediate nodes

In the classical CRR case the no-arbitrage price of the option at an intermediate node $\langle i,j \rangle$ (where $i = 1, 2, ..., N-1$) is given by the value of the hedging portfolio at this node. In the tax-adjusted model this is not the case; the hedging portfolio at an intermediate node is specified on a pre-tax basis and so does not represent the tax-adjusted option price at that node.

The tax-adjusted hedging price at an intermediate node on the tree is equal to the price obtained by applying the tax-adjusted hedging algorithm to that node, i.e., by treating the intermediate node as the initial node. Therefore, at node $\langle i,j \rangle$ where $i = 1, 2, ..., N-1$, the tax-adjusted hedging price is given by:

$$X^*_{\langle i,j \rangle} = \bar{X}^*_{\langle 0,0 \rangle}, \tag{2.20}$$

where $X^*_{\langle i,j \rangle}$ is the tax-adjusted hedging price, $\bar{X}^*_{\langle 0,0 \rangle}$ is the tax-adjusted initial price of an option obtained by applying the tax-adjusted hedging algorithm with the following parameter adjustments:

$$
\begin{aligned}
\bar{N} &= N - i \\
\bar{T} &= T - i\frac{T}{N} \\
\bar{S}_{\langle 0,0 \rangle} &= S_{\langle i,j \rangle}.
\end{aligned}
$$

It is as if we sell the option at the intermediate node and hedge it until maturity in the usual (tax-adjusted) way.

2.4 The results obtained from a C++ implementation of the tax-adjusted hedging algorithm

The tax-adjusted hedging algorithm has been implemented in C++ using a 633Mhz, 512MB Pentium III PC.

Memory requirements and speed of computation

The algorithm is very demanding of memory since we have to store all the values in the tree for the stock, delta, option and bonds, and store the RNG at the final nodes in the tree. The size of each of the arrays for these variables therefore increases exponentially with the number of periods used, because the tree is non-recombining.

The speed of computation slows with an increase in the number of periods because the time taken to perform one iteration to adjust the payoff and compute the bonds and deltas is increased. The total number of iterations required to achieve convergence stays more or less the same as the number of periods increases, assuming all other parameters are unchanged.

Table 2.1 shows the number of iterations required for the algorithm to converge and the CPU time taken as the number of periods increases. The parameters used were: $S = K = 100, r = 0.05, \sigma = 0.25, T = 1, \tau_b = 0.4, \tau_s = 0.3, \tau_o = 0.2$.

Periods, N	Iterations required to converge, g	CPU time (s)
$1 - 14$	< 9	< 1
15	8	1
16	8	2
17	8	5
18	8	10
19	8	20
20	8	40
21	8	80
22	9	180
≥ 23	Memory requirements too great	NA

Table 2.1: Number of iterations and time taken for algorithm to converge

2.4.1 Results: an overview

The results in this section are presented in Tables 2.2 and 2.3 and Figures 2.2 to 2.7. The tax-adjusted hedging algorithm has been used to examine the following three cases:

Case 2.1: The bond is taxed as income and the stock and option are taxed as capital gains

$\tau_b = \tau_i$ and $\tau_s = \tau_o = \tau_{cg}$, where $\tau_i \geq \tau_{cg}$. If the option is written and hedged in the same tax-jurisdiction, this is the most likely situation in which the stock, bond and option would be taxed at different rates.

Case 2.2: The option and the hedge are taxed at different rates

$\tau_s = \tau_b \neq \tau_o$. This could occur if the option is written in one tax-jurisdiction and hedged in another.

45

Case 2.3: The bond and option are taxed as income and the stock is taxed as capital gains

$\tau_b = \tau_o = \tau_i$ and $\tau_s = \tau_{cg}$, where $\tau_i \geq \tau_{cg}$. This situation is not very likely since the option is likely to be taxed as capital gains. However, this scenario is included because in Scholes (1976) it is assumed that the option is taxed as income when deriving the taBS equation.

Tax rates			ITM call (K=80)		ATM call (K=100)		OTM call (K=120)	
τ_b	τ_s	τ_o	Price	% diff CRR	Price	% diff CRR	Price	% diff CRR
0	0	0	25.4191*	0.0	12.4876*	0.0	4.92731*	0.0
0.25	0	0	24.642	-3.1	11.882	-4.8	4.5861	-7.0
0.4	0	0	24.1722	-4.9	11.5221	-7.7	4.38745	-11.0
0.25	0.25	0.25	25.4191	0.0	12.4876	0.0	4.92731	0.0
0.4	0.25	0.25	24.7981	-2.4	12.0025	-3.9	4.65334	-5.6
0.4	0.4	0.4	25.4191	0.0	12.4876	0.0	4.92731	0.0
0	0	0.25	25.0124	-1.6	12.2878	-1.6	4.84849	-1.6
0	0	0.4	24.6186	-3.1	12.0943	-3.1	4.77215	-3.1
0.25	0.25	0	25.7328	+1.2	12.6417	+1.2	4.98813	+1.2
0.25	0.25	0.4	25.1129	-1.2	12.3371	-1.2	4.86796	-1.2
0.4	0.4	0	25.9248	+2.0	12.736	+2.0	5.02535	+2.0
0.4	0.4	0.25	25.6695	+1.0	12.6106	+1.0	4.97585	+1.0
0.25	0	0.25	24.3416	-4.2	11.7371	-6.0	4.53019	-8.1
0.4	0	0.4	23.7007	-6.8	11.2974	-9.5	4.30186	-12.7
0.4	0.25	0.4	24.5562	-3.4	11.8855	-4.8	4.60795	-6.5

Table 2.2: Tax-adjusted call option prices obtained from the tax-adjusted hedging algorithm
*Indicates CRR price

Tax rates			ITM put (K=120)		ATM put (K=100)		OTM put (K=80)	
τ_b	τ_s	τ_o	Price	% diff CRR	Price	% diff CRR	Price	% diff CRR
0	0	0	19.0748*	0.0	7.6105*	0.0	1.51743*	0.0
0.25	0	0	20.1426	+5.6	8.17904	+7.5	1.67969	+10.7
0.4	0	0	20.8061	+9.1	8.53768	+12.2	1.78464	+17.6
0.25	0.25	0.25	19.0748	0.0	7.6105	0.0	1.51743	0.0
0.4	0.25	0.25	19.9252	+4.5	8.06245	+5.9	1.646	+8.5
0.4	0.4	0.4	19.0748	0.0	7.6105	0.0	1.51743	0.0
0	0	0.25	18.7697	-1.6	7.48876	-1.6	1.49315	-1.6
0	0	0.4	18.4742	-3.1	7.37085	-3.1	1.46964	-3.1
0.25	0.25	0	19.3103	+1.2	7.70444	+1.2	1.53615	+1.2
0.25	0.25	0.4	18.8451	-1.2	7.51882	-1.2	1.49915	-1.2
0.4	0.4	0	19.4544	+2.0	7.76192	+2.0	1.54762	+2.0
0.4	0.4	0.25	19.2627	+1.0	7.68546	+1.0	1.53237	+1.0
0.25	0	0.25	19.897	+4.3	8.07931	+6.2	1.65921	+9.3
0.4	0	0.4	20.4002	+6.9	8.37112	+10.0	1.74983	+15.3
0.4	0.25	0.4	19.7309	+3.4	7.98381	+4.9	1.62995	+7.4

Table 2.3: Tax-adjusted put option prices obtained from the tax-adjusted hedging algorithm
*Indicates CRR price

Notes to Tables 2.2 and 2.3

1. The following parameters were used: $N = 15, S = 100, r = 0.05, \sigma = 0.25, T = 1$.

2. The top six rows contain the results relating to Case 2.1.

3. The following six rows contain the results relating to Case 2.2.

4. The bottom three rows contain the results relating to Case 2.3.

Notes to Figures 2.2 to 2.7:

1. Figures 2.2 and 2.5 contain results relating to Case 2.1 for a call and put respectively.

2. Figures 2.3 and 2.6 contain results relating to Case 2.2.

3. Figures 2.4 and 2.7 contain results relating to Case 2.3.

4. In Cases 2.1 and 2.3 there is the following restriction on the tax rate: $\tau_i \geq \tau_{cg}$. Figure 2.2, 2.4, 2.5 and 2.7 do not include this restriction and show the whole option price surface.

5. The diagonal line along the surface that corresponds to the tax rates all being equal, in each of the figures, indicates the CRR price of the option.

6. ITM stands for "in-the-money"; OTM stands for "out-of-the-money"; ATM stands for "at-the-money".

Key results contained in Tables 2.2 and 2.3 and Figures 2.2 to 2.7

Option prices produced by the tax-adjusted hedging algorithm show the following:

1. When $\tau_b = \tau_s = \tau_o$, $X^*_{<0,0>} = X^{CRR}_{<0,0>}$.

2. Case 2.1 and Case 2.3

    (a) When $\tau_i > \tau_{cg}$, $c^*_{<0,0>} < c^{CRR}_{<0,0>}$ and $p^*_{<0,0>} > p^{CRR}_{<0,0>}$.

    (b) For both puts and calls the relative price difference (cf CRR) increases as we move from ITM through ATM to OTM options.

3. Case 2.2

    (a) When $\tau_o > \tau_b = \tau_s$, $X^*_{<0,0>} < X^{CRR}_{<0,0>}$.

    (b) When $\tau_o < \tau_b = \tau_s$, $X^*_{<0,0>} > X^{CRR}_{<0,0>}$.

    (c) In both 3a and 3b the relative price differences (cf CRR) stay constant whether the option is a call or a put or whether it is ITM, ATM or OTM.

Figure 2.2: ATM call option price as a function of the bond tax rate and the stock/option tax rate
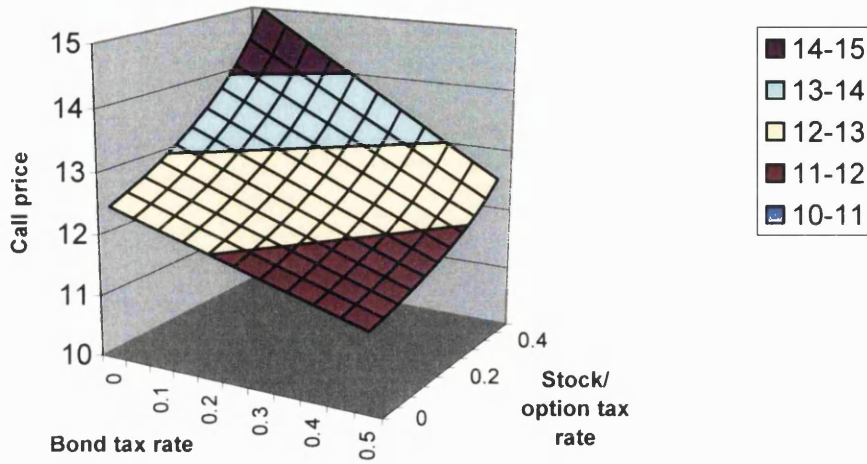
Figure 2.3: ATM call option price as a function of the option tax rate and the stock/bond tax rate
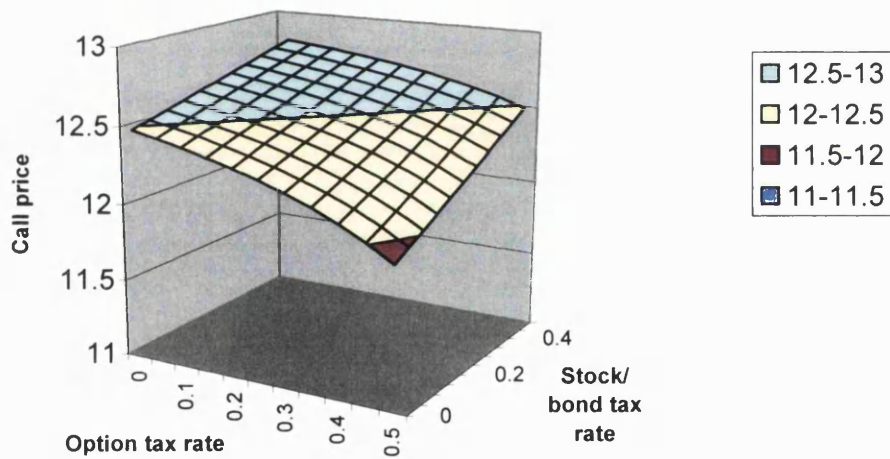
Figure 2.4: ATM call option price as a function of the stock tax rate and the option/bond tax rate
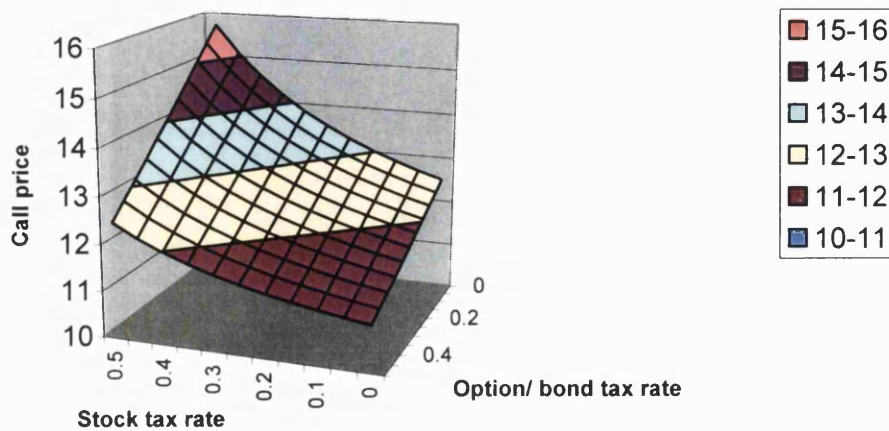
Figure 2.5: ATM put option price as a function of the bond tax rate and the stock/option tax rate
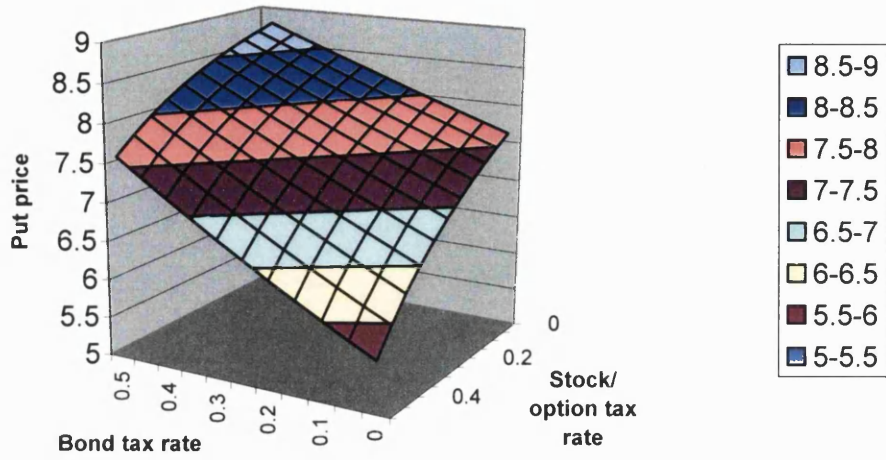
Figure 2.6: ATM put option price as a function of the option tax rate and the stock/bond tax rate
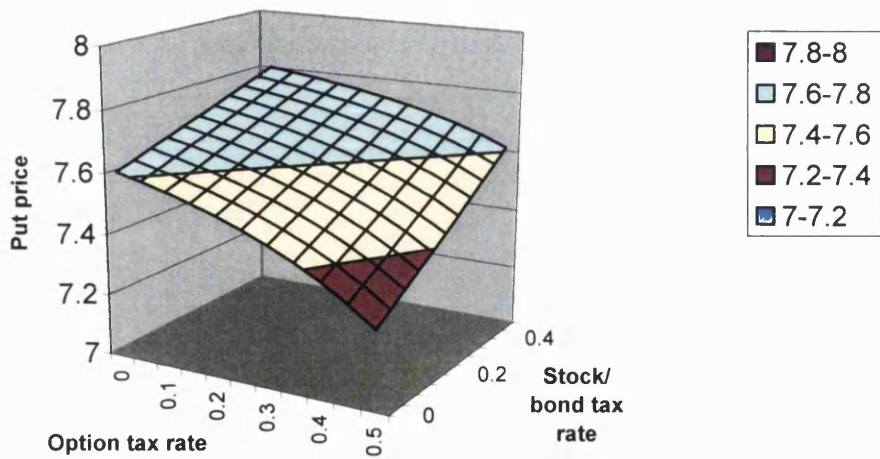
Figure 2.7: ATM put option price as a function of the stock tax rate and the option/bond tax rate
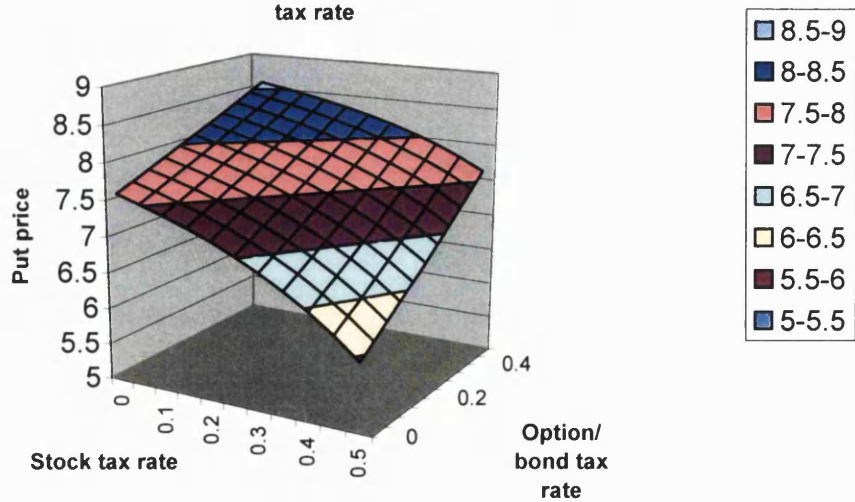
2.5 Numerical examples

It is useful to look at some numerical examples to explicitly show the tax-adjusted deltas and bonds in the tree, and to confirm that the after-tax hedging condition is satisfied.

2.5.1 A comparison of the tax-adjusted tree with the CRR tree

If we compare the values for the option delta and bond, and the option prices in a tax-adjusted tree with a CRR tree, we can gain some insight into how the tax-adjusted hedging algorithm described in Section 2.3.5 modifies these values throughout the tree to produce a tax-adjusted hedging price for the option. Figure 2.8 shows the stock, delta, bonds and option price in a two period tree for a call and a put for the tax-adjusted and CRR cases.

Notes to Figure 2.8

1. The following parameters were used: $N = 2, S = K = 100, r = 0.05, \sigma = 0.25, T = 1, \tau_b = 0.4, \tau_s = \tau_o = 0.25$.

a. No-tax call

```
                                         142.412
                                         0
                                         0
                        119.336          42.119
                        1
                        -97.531          100
                        21.8055          0
                                         0
          100                            0
          0.613551
          -50.1442                       100
          11.211                         0
                        83.7967          0
                        0                0
                        0
                        0                70.2189
                                         0
                                         0
                                         0
```

b. With-tax call

```
                                         142.412
                                         0
                                         0
                        119.336          41.665
                        1
                        -98.2595         100
                        21.077           0
                                         0
          100                            -0.746943
          0.599933
          -49.2696                       100
          10.7237                        0
                        83.7967          0
                        0                -0.250691
                        -0.244502
                        -0.244502        70.2189
                                         0
                                         0
                                         -0.250691
```

STOCK
DELTA
BONDS
OPTION

c. No-tax put

```
                                         142.412
                                         0
                                         0
                        119.336          0
                        0
                        0                100
                        0                0
                                         0
          100                            0
          -0.386449
          44.9788                        100
          6.33391                        0
                        83.7967          0
                        -1               0
                        97.531
                        13.7343          70.2189
                                         0
                                         0
                                         29.7811
```

d. With-tax put

```
                                         142.412
                                         0
                                         0
                        119.336          0.238076
                        0
                        0.232198         100
                        0.232198         0
                                         0
          100                            0.238076
          -0.400067
          46.7903                        100
          6.78358                        0
                        83.7967          0
                        -1               0.734328
                        98.2472
                        14.4505          70.2189
                                         0
                                         0
                                         30.5155
```

Figure 2.8: Tax-adjusted delta, bonds, and option values in a two period tree compared with CRR

We can see from Figure 2.8 that the synthetic tax-adjusted payoff[2] from the with-tax call is slightly lower when compared to the no-tax call (Figures 2.8a and 2.8b), and the synthetic tax-adjusted payoff from the with-tax put is slightly higher when compared to the no-tax put (Figures 2.8c and 2.8d). Moreover, when the call (put) finishes out-of-the-money the with-tax tree shows a slight negative (positive) payoff. The decreased (increased) payoff for the call (put) explains why its price on the initial node is lower (higher) than in the CRR case.

---

[2] The term "synthetic tax-adjusted payoff" refers to the adjusted payoff that is produced by the algorithm such that the hedging portfolios satisfy the after-tax hedging condition across the final nodes on the tree.

It is also interesting to note that the delta is unchanged as a result of the modified payoffs when the option finishes out-of-the-money. At the initial node on the tree, the long (short) delta decreases (increases) for a call (put).

Figures 2.8b and 2.8d also show explicitly how taxes introduce path-dependency to the tree. For both these two trees, the middle two states in the final period, show different payoffs for the option. In the CRR case, these two states are identical given that there is no path dependency.

Risk-neutral valuation

In the CRR setting it is possible to calculate the option price by using risk-neutral valuation. In this case, the initial price of the option is found by discounting its expected future payoff, where the expectation is taken under the risk-neutral measure. Given the tax-adjusted hedging algorithm operates in a CRR environment by using synthetic tax-adjusted payoffs to calculate the tax-adjusted hedging price, applying the CRR risk-neutral probabilities to the adjusted payoffs will give us the tax-adjusted option price at the initial node:

$$X^*_{\langle 0,0 \rangle} = \frac{1}{R^N} E^Q \left[X^*_{\langle N,j \rangle} \right], \qquad (2.21)$$

where $X^*_{\langle 0,0 \rangle}$ is the tax-adjusted price of the option (a European put or call), $1/R^N$ is the discount rate over the life of the option, $X^*_{\langle N,j \rangle}$ represents the synthetic tax-adjusted payoffs from the option at the final nodes, and $E^Q[\cdot]$ is the expectation operator where the expectation is taken under the same risk neutral measure as in the CRR case.

Example Using the parameters used to produce the trees given in Figure 2.8 we get:

$$
\begin{aligned}
u &= \exp(\sigma\sqrt{T/N}) = 1.193365 \\
d &= \exp(-\sigma\sqrt{T/N}) = 0.837967 \\
R &= \exp(r\frac{T}{N}) = 1.025315
\end{aligned}
$$

The risk neutral probability of an "up" move is given by Cox (1979):

$$\pi = \frac{R-d}{u-d} = 0.527151 \qquad (2.22)$$

The tax-adjusted value of the call shown in Figure 2.8b is given by (2.21):

$$
\begin{aligned}
c^*_{<0,0>} &= \frac{1}{R^2}\left(41.665\pi^2 - (0.746943 + 0.250691)(1-\pi)\pi - 0.250691(1-\pi)^2\right) \\
&= 10.723675
\end{aligned}
$$

We see from the Figure 2.8b that the value of the call at time zero is 10.7237 currency units.

This analysis confirms that the tax -adjusted hedging algorithm described in Section 2.3.5 does exactly what it is intended to do: it adjusts the payoffs from the option so that the hedging condition holds on an after-tax basis. In other words, the tax-mismatch given by Equation (2.6) is zero across all the final nodes on the tree. The risk-neutral probabilities are exactly the same as those on the corresponding CRR tree, and to value the option using risk-neutrality we can discount the expectation of the adjusted payoffs under this measure. The problem is how to determine these adjusted payoffs, and this is achieved by using the iterative procedure given in Section 2.3.5.

2.5.2 The after-tax hedging condition and the hedging portfolio

A numerical example of the after-tax hedging condition

We can work through the calculations to show explicitly that the model produces a hedging portfolio that is equivalent to the option payoff at the final nodes, on an after-tax basis, for Figure 2.8b.

Let us calculate the after-tax position on the stock transactions. We first need to calculate the RNG at each of the final nodes in the tree, and to do this we use the following equation:

$$RNG_{\langle N,j\rangle} = \sum_{k=1}^{N} \Delta_{\langle N,j,-k\rangle}\left(S_{\langle N,j,-k+1\rangle} - S_{\langle N,j,-k\rangle}\right) \qquad (2.23)$$

The RNGs at the final nodes are:

$$RNG_{\langle 2,0 \rangle} = 0.600(119.336 - 100) + 1.000(142.412 - 119.336) = 34.678;$$

$$RNG_{\langle 2,1 \rangle} = 0.600(119.336 - 100) + 1.000(100 - 119.336) = -7.734;$$

$$RNG_{\langle 2,2 \rangle} = 0.600(83.797 - 100) + 0 = -9.722;$$

$$RNG_{\langle 2,3 \rangle} = 0.600(83.797 - 100) + 0 = -9.722.$$

The after-tax stock (ATS) positions are given by the following equation:

$$ATS_{\langle N,j \rangle} = \Delta^*_{\langle N,j,-1 \rangle} S_{\langle N,j \rangle} - \tau_s(RNG_{\langle N,,j \rangle}) \tag{2.24}$$

At the final nodes the ATS positions are:

$$ATS_{\langle 2,0 \rangle} = 1.00 \times 142.412 - 0.25 \times 34.678 = 133.743;$$

$$ATS_{\langle 2,1 \rangle} = 1.00 \times 100 + 0.25 \times 7.734 = 101.934;$$

$$ATS_{\langle 2,2 \rangle} = 0.000 \times 100 + 0.25 \times 9.722 = 2.431;$$

$$ATS_{\langle 2,3 \rangle} = 0.000 \times 70.219 + 0.25 \times 9.722 = 2.431.$$

Now let us calculate the after-tax bond (ATB) positions at the final node. The first step is to calculate the total bond interest (BI) for each path. The equation for bond interest is:

$$BI_{\langle N,j \rangle} = (R - 1)\left(\sum_{k=1}^{N} B^*_{\langle N,j,-k \rangle}\right) \tag{2.25}$$

The values calculated for tree 2.8b are:

$$BI_{\langle 2,0 \rangle} = (\exp(0.5 \times 0.05) - 1)(-49.270 - 98.260) = -3.735;$$

$$BI_{\langle 2,1 \rangle} = (\exp(0.5 \times 0.05) - 1)(-49.270 - 98.260) = -3.735;$$

$$BI_{\langle 2,2 \rangle} = (\exp(0.5 \times 0.05) - 1)(-49.270 - 0.244) = -1.253;$$

$$BI_{\langle 2,3 \rangle} = (\exp(0.5 \times 0.05) - 1)(-49.270 - 0.244) = -1.253;$$

The ATB positions at the final nodes are given by the following equation:

$$ATB_{\langle N,j \rangle} = RB^*_{\langle N,j,-1 \rangle} - \tau_b BI_{\langle N,j \rangle} \qquad (2.26)$$

For tree 2.8b we find,

$$ATB_{\langle 2,0 \rangle} = -98.260 \times \exp\left(0.5 \times 0.05\right) + 0.4 \times 3.735 = -99.253;$$

$$ATB_{\langle 2,1 \rangle} = -98.260 \times \exp\left(0.5 \times 0.05\right) + 0.4 \times 3.735 = -99.253;$$

$$ATB_{\langle 2,2 \rangle} = -0.244 \times \exp\left(0.5 \times 0.05\right) + 0.4 \times 1.253 = +0.250;$$

$$ATB_{\langle 2,3 \rangle} = -0.244 \times \exp\left(0.5 \times 0.05\right) + 0.4 \times 1.253 = +0.250.$$

Now we need to find the after-tax call (ATC) positions at the final nodes. These are given by the following equation:

$$ATC_{\langle N,j \rangle} = (1 - \tau_c)c_{\langle N,j \rangle} + \tau_c c^*_{\langle 0,0 \rangle}$$

For tree 2.8b we find,

$$ATC_{\langle 2,0 \rangle} = 0.75 \times 42.412 + 0.25 \times 10.724 = \mathbf{34.490};$$

$$ATC_{\langle 2,1 \rangle} = 0 + 0.25 \times 10.724 = \mathbf{2.681};$$

$$ATC_{\langle 2,2 \rangle} = 0 + 0.25 \times 10.724 = \mathbf{2.681};$$

$$ATC_{\langle 2,3 \rangle} = 0 + 0.25 \times 10.724 = \mathbf{2.681}.$$

For the hedging condition to be satisfied on an after-tax basis, we want the following equation to hold across the final nodes:

$$ATS_{\langle N,j \rangle} + ATB_{\langle N,j \rangle} = ATC_{\langle N,j \rangle}$$

Substituting the values into the above equation we get:

$$133.743 - 99.253 \quad = \quad \mathbf{34.490} = ATC_{\langle 2,0 \rangle};$$

$$101.934 - 99.253 \quad = \quad \mathbf{2.681} = ATC_{\langle 2,1 \rangle};$$

$$2.431 + 0.250 \quad = \quad \mathbf{2.681} = ATC_{\langle 2,2 \rangle};$$

$$2.431 + 0.250 \quad = \quad \mathbf{2.681} = ATC_{\langle 2,3 \rangle}.$$

We have demonstrated explicitly that the call price given by the algorithm is consistent with the after-tax hedging condition.

An explicit example showing the operation of the hedging portfolio

It is useful to explicitly go through the numbers at the final nodes to show that the writer of the option is fully hedged on an after-tax basis. We again use the tree given in Figure 2.8b, and consider each of the final nodes in turn. The payments due from the option writer at maturity are listed on the left side, and the receipts due to the writer on the right side. Examples of the calculations can be found in the previous section.

Node <2,0>

DESCRIPTION OF PAYMENT	VALUE	DESCRIPTION OF RECEIPT	VALUE
Payoff from option	42.41	Tax rebate from option	7.93
Repayment of bonds	100.75	Tax rebate on bonds	1.49
Tax due on stock transactions	8.67	Liquidation of stock	142.41
Total	**151.83**	Total	**151.83**

Node <2,1>

Payoff from option	0.00	Tax rebate from bonds	1.49
Repayment of bonds	100.75	Tax rebate from stock transactions	1.94
Tax charge on option premium	2.68	Liquidation of stock	100.00
Total	**103.43**	Total	**103.43**

Node <2,2>

Payoff from option	0.00	Tax rebate from bonds	0.50
Repayment of bonds	0.25	Tax rebate from stock transactions	2.43
Tax charge on option premium	2.68	Liquidation of stock	0.00
Total	**2.93**	Total	**2.93**

Node <2,3>

Payoff from option	0.00	Tax rebate from bonds	0.50
Repayment of bonds	0.25	Tax rebate from stock transactions	2.43
Tax charge on option premium	2.68	Liquidation of stock	0.00
Total	**2.93**	Total	**2.93**

We can see from the above analysis that the option writer is perfectly hedged (i.e., the payments made at maturity match the receipts exactly), on an after-tax basis, if they follow the trading strategy advocated in Figure 2.8b.

We can also see from this analysis that the values at the final nodes for the call option, the synthetic tax-adjusted payoffs, are purely there to establish the delta and bond amounts in the tree and do not represent an actual after-tax payoff from the option. This is the reason why they are called the *synthetic tax-adjusted payoffs*.

2.6 A numerical approximation to the continuous-time price of an option and tax-adjusted put-call parity

It is possible to obtain an approximation to the continuous-time price of an option from the discrete prices calculated using binomial trees. With an approximation to the continuous-time option price, we can examine tax-adjusted put-call parity.

2.6.1 A continuous-time price approximation from discrete time prices

Using CRR, if we plot a graph of option price as a function of even periods and option price as a function of odd periods on the same axes, we see that each graph appears to converge to some value as the number of periods increases: odd periods converge from above and even periods converge from below. We know in the CRR case that the limiting price as the number of periods goes to infinity is the BS price, since CRR is the discrete-time precursor to the continuous-time BS model. Figures 2.9 and 2.11 show this for a CRR call and put respectively. If we take the two prices we have at any one period, one on the even period plot and the other on the odd period plot, and average them, we should obtain a price that is close to the BS price, assuming that the two plots converge at the same rate. Thus, we can obtain an approximation to the continuous-time price from discrete-time prices. We will explain this method more formally below.

Figures 2.10 (call) and 2.12 (put) show the variation in option price with odd and even periods for the tax-adjusted case. We can see that these graphs are very similar to the CRR cases, and the plots appear to converge to some value as the number of periods increases. Therefore, we should be able to apply the same method, as advocated with CRR, to obtain continuous-time price approximations in the tax-adjusted case.

Notes to Figures 2.9 to 2.12

1. The following parameters were used: $S = K = 100, r = 0.05, \sigma = 0.25, T = 1$, and additionally for Figures 2.10 and 2.12, $\tau_b = 0.4, \tau_s = \tau_o = 0.25$.

2. To obtain the odd-period data points on the even-period plot, and vice-versa, the two adjacent option prices were averaged. For example, the option price for period nine on the even-period plot is given by the sum of option price at period eight and the option price at period ten, divided by two.

3. Figures 2.9 and 2.11 show the BS price of the option as a straight line, whereas Figures 2.10 and 2.12 show the taBS price as a straight line.

A method to find an approximation to the continuous-time option price from a binomial tree

1. Set n equal to the period at which we wish to approximate the continuous-time option price.

2. Compute $_nX_{<0,0>}$, $_{(n+1)}X_{<0,0>}$ and $_{(n-1)}X_{<0,0>}$, using a binomial tree method (either CRR or using the tax-adjusted hedging algorithm), where $_nX_{<0,0>}$ represents the initial value of an option using a tree with $N = n$.

3.

$$_nX^{cnts} = \frac{1}{2}\left(_nX_{<0,0>} + \frac{1}{2}\left(_{(n+1)}X_{<0,0>} +_{(n-1)} X_{<0,0>}\right)\right), \qquad (2.27)$$

where $_nX^{cnts}$ is the approximation to the continuous-time option price. The leading subscript n can been retained to indicate the approximation is derived from a binomial tree with n periods. In future we will use X^{cnts*} to denote the tax-adjusted case, dropping the leading subscript to ease the notation.

Example of a calculation

Setting $n = 15$ and using CRR to value a call option with the same parameters as used for Figures 2.9-2.12, we get $_{14}X_{<0,0>} = 12.1611$, $_{15}X_{<0,0>} = 12.4876$, and $_{16}X_{<0,0>} = 12.1828$. Using (2.27) we find:

$$
\begin{aligned}
_{15}X^{cnts} &= \frac{1}{2}(12.4876 + \frac{1}{2}(12.1828 + 12.1611)) \\
&= 12.3298
\end{aligned}
$$

The BS value is found to be 12.3347, and so $_{15}X^{cnts}(= 12.3298)$ is valued 0.04% lower compared to BS.

Results

Tax rates			ITM call (K=80)	ATM call (K=100)	OTM call (K=120)
τ_b	τ_s	τ_o	c^{cnts*} (BS=25.4131)	c^{cnts*} (BS=12.3347)	c^{cnts*} (BS=5.02526)
0	0	0	25.4363 $^{+}$(+0.09%)	12.3298 $^{+}$(-0.04%)	5.02074 $^{+}$(-0.09%)
0.25	0	0	24.6598	11.7218	4.67691
0.4	0	0	24.1903	11.3606	4.47659
0.4	0.25	0.25	24.8157	11.8428	4.74468

Table 2.4: Approximations to continuous-time tax-adjusted call prices

Tax rates			ITM put (K=120)	ATM put (K=100)	OTM put (K=80)
τ_b	τ_s	τ_o	p^{cnts*} (BS=19.1728)	p^{cnts*} (BS=7.45761)	p^{cnts*} (1.51148)
0	0	0	19.1682 $^{+}$(-0.02%)	7.45270 $^{+}$(-0.07%)	1.53465 $^{+}$(-1.53%)
0.25	0	0	20.2334	8.01885	1.69745
0.4	0	0	20.8952	8.37622	1.80271
0.4	0.25	0.25	20.0166	7.90272	1.66366

Table 2.5: Approximations to continuous-time tax-adjusted put prices

$^{+}$indicates approximation to BS value.

Notes to Tables 2.4 and 2.5

1. The following parameters were used: $n = 15, S = 100, r = 0.05, \sigma = 0.25, T = 1$.

2. The BS value for each strike used is given in brackets.

3. The first row is equivalent to using CRR to approximate BS and the difference in price, compared to BS, is given in brackets.

Figure 2.9: Variation of CRR call price with odd and even periods

T=1; vol=0.25; r=0.05; S=K=100; tr-option/stock=0.25; tr-bond=0.4

Legend: CRR call odd periods — CRR call even periods — BS
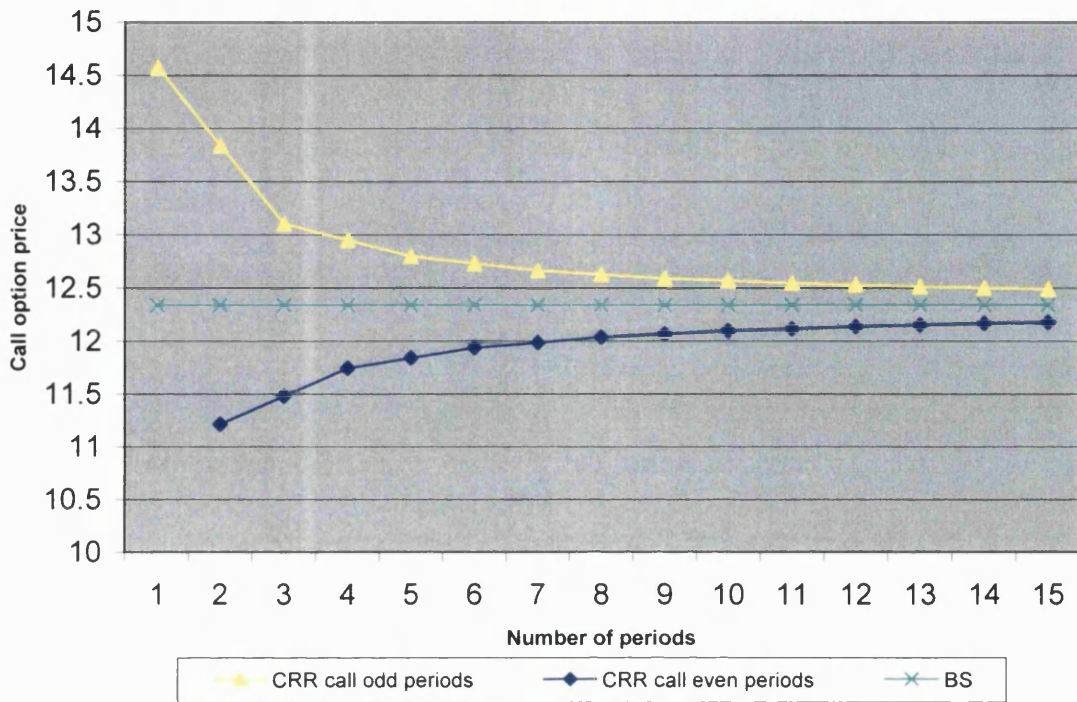
Figure 2.10: Variation of tax-adjusted call price with even and odd periods

T=1; vol=0.25; r=0.05; S=K=100; tr-option/stock=0.25; tr-bond=0.4

Legend: tax-adjusted call odd periods — tax-adjusted call even periods — taBS call
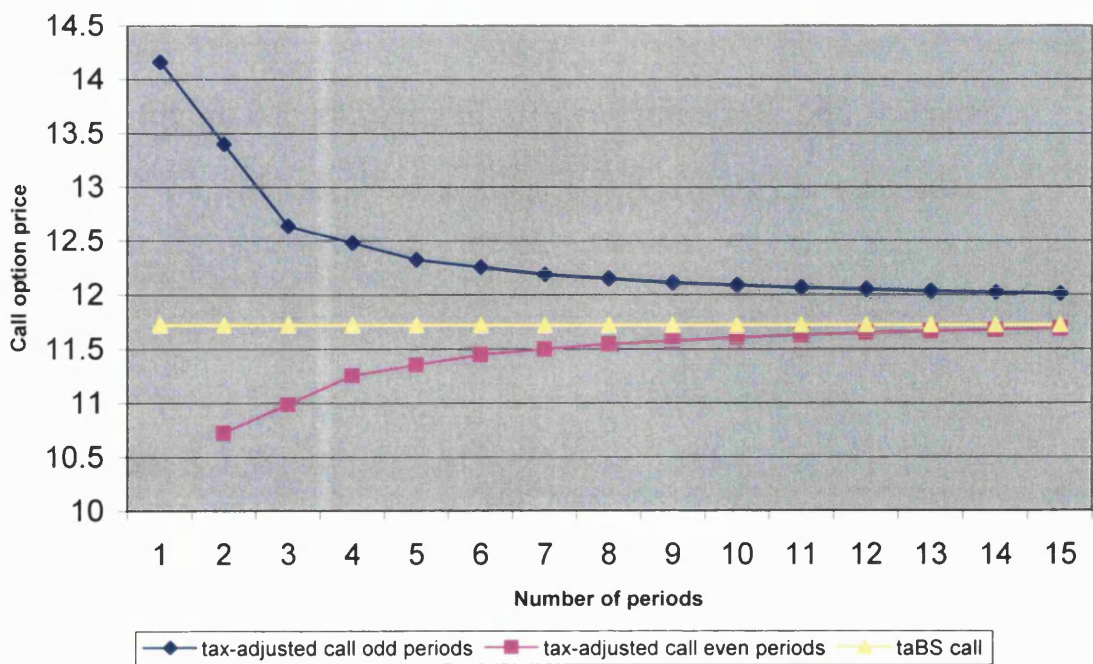
Figure 2.11: Variation of CRR put price with odd and even periods

T=1; vol=0.25; r=0.05; S=K=100; tr-option/stock=0.25; tr-bond=0.4
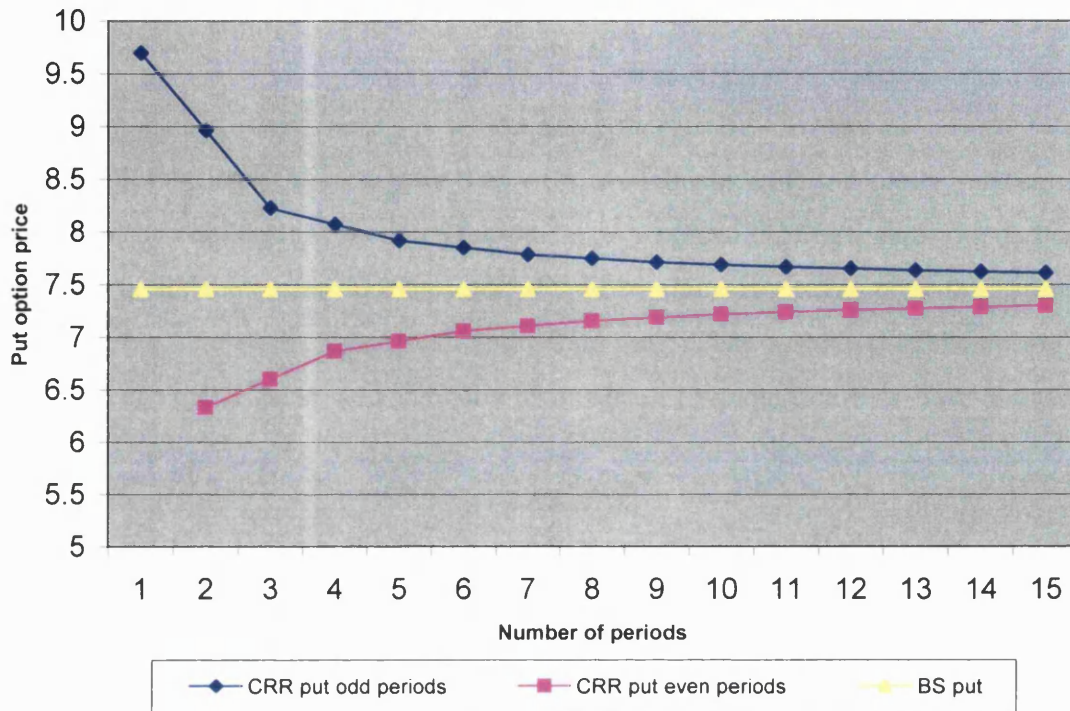
Figure 2.12: Variation of tax-adjusted put price with odd and even periods

T=1; vol=0.25; r=0.05; S=K=100; tr-option/stock=0.25; tr-bond=0.4
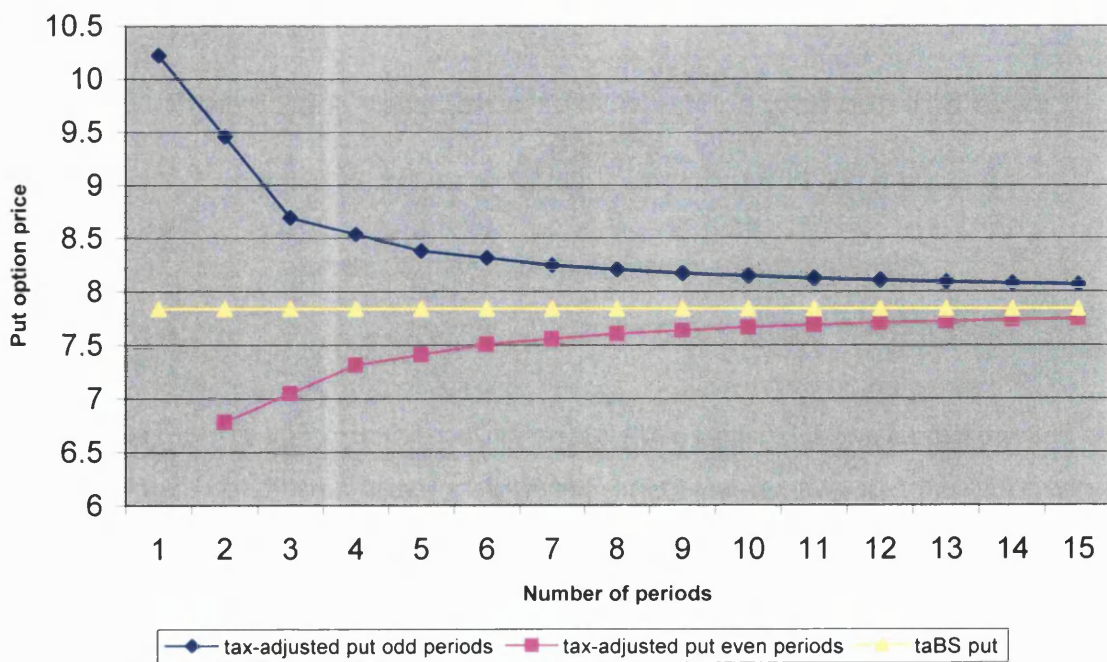
2.6.2 Tax-adjusted put-call parity

For a European put and call with no dividends put-call parity is given by the following relationship:

$$c + Ke^{-rt} = S + p. \tag{2.28}$$

Milevsky and Prisman have derived a tax-adjusted put-call parity relationship (Milevsky 1997a) by arguing that the after-tax rate of return from holding a zero-coupon bond with a face value of K, the options' strike price, should be the same as the after-tax rate of return from holding a long stock, a short call and a long put. They assume that the option and stock are taxed as capital gains, and the bond as income. The equation they have derived is as follows:

$$S + p^* - c^* = aKe^{-rt}, \tag{2.29}$$

where,

$$a = \frac{(1 - \tau_{cg})}{(1 - \tau_i) + (\tau_i - \tau_{cg})e^{-rt}}. \tag{2.30}$$

Example

Using c^{cnts*} and p^{cnts*} for the ATM strike from Tables 2.4 and 2.5, where $\tau_s = \tau_o = \tau_{cg} = 0.25$ and $\tau_b = \tau_i = 0.4$ we find:

$$a = 1.00985,$$

and,

$$
\begin{aligned}
c^{cnts*} - p^{cnts*} &= 11.8428 - 7.90272 = 3.94008 \\
S - aKe^{-rt} &= 100 - 1.00985 \times 100e^{-0.05} = 3.94009
\end{aligned}
$$

We see that the approximate continuous-time tax-adjusted prices for the put and call calculated using the tax-adjusted hedging algorithm satisfy the tax-adjusted put-call parity relationship

derived by Milevsky and Prisman.

Results

Tax rates				$K = 80$		$K = 100$	
τ_b	τ_s	τ_o	a	$S - aKe^{-rt}$	$c^{cnts*} - p^{cnts*}$	$S - aKe^{-rt}$	$c^{cnts*} - p^{cnts*}$
0	0	0	1.00000	23.90165*	23.90165*	4.87706*	4.87707*
0.25	0	0	1.01234	22.96235	22.96232	3.70294	3.70294
0.4	0	0	1.01990	22.38756	22.38756	2.98446	2.98442
0.4	0.25	0.25	1.00985	23.15206	23.15209	3.94009	3.94008

Tax rates				$K = 120$	
τ_b	τ_s	τ_o	a	$S - aKe^{-rt}$	$c^{cnts*} - p^{cnts*}$
0	0	0	1.00000	-14.14753	-14.14751
0.25	0	0	1.01234	-15.55647	-15.55649
0.4	0	0	1.01990	-16.41865	-16.41863
0.4	0.25	0.25	1.00985	-15.27791	-15.27189

Table 2.6: Tax-adjusted put-call parity using option prices from Tables 2.4 and 2.5

Notes to Table 2.6

1. The following parameters were used: $n = 15, S = 100, r = 0.05, \sigma = 0.25, T = 1$.

2.7 Summary

The following summarises the main results of this chapter:

1. A basic guide to taxation is given, which is intended to capture some of the generic basic elements of tax systems.

2. We argue that the "tax-adjusted hedging price" is equivalent to "the tax-adjusted no-arbitrage price" if and only if all market participants have the same tax position.

3. A tax-adjusted hedging algorithm is developed that finds the tax-adjusted hedging price of European equity call options. At the heart of the numerical method is an algorithm

65

that forces the tax-mismatch to zero via an iterative procedure.

4. Some option prices obtained using the tax-adjusted hedging algorithm are examined:

   (a) If the tax rates on the stock, bond and option are all equal, the tax-adjusted hedging price of both puts and calls are seen to be equal to their CRR counterparts.

   (b) The algorithm prices call (put) options below (above) their CRR counterparts if the tax rate on the stock and option is less than the tax rate on the bond.

   (c) The algorithm prices call and put options above (below) their CRR counterparts if the tax rate on the stock and bond is greater (less) than the tax rate on the option.

5. A method to find an approximation to the continuous-time price of an option from discrete-time prices is presented. This allows us to show that the tax-adjusted hedging algorithm produces tax-adjusted option prices (which are used to find "continuous-time" tax-adjusted option prices) that satisfy the tax-adjusted put-call parity relationship.

Chapter 3

The Tax-Adjusted Hedging Price of Options: A Simultaneous Equation Approach and One-Period Analysis

3.1 Introduction

In the previous chapter we looked at how to find the tax-adjusted hedging price of an option using the tax-adjusted hedging algorithm. That approach owed much to the classical CRR method in that we were using backward recursion in a binomial tree environment, albeit iteratively. In this chapter we will see that the problem of finding the tax-adjusted hedging price of an option, using exactly the same framework as in Chapter 2, is the same as solving a system of simultaneous equations (Section 3.2). In Section 3.2 we will also derive the general form for the delta at any node for N-periods, and the general form for the bond at the initial node. These equations form the basis for the simultaneous equation algorithm, which is computationally faster and less demanding of memory, than the tax-adjusted hedging algorithm of Chapter 2.

In Section 3.3 we will look at some analysis of the one-period case, given that we can find an analytic formula for the one-period tax-adjusted option price using the simultaneous equation approach. Specifically, we will look at how the tax-adjusted option price is related to the CRR price and examine tax-adjusted put-call parity. The last part of Section 3.3 demonstrates, in

the one-period case, that the tax-adjusted hedging algorithm converges to the analytic formula for the option price as the number of iterations goes to infinity.

The assumptions and notation are the same as in Chapter 2. Any new notation will be explained as it is used in the text.

3.2 A simultaneous equation approach

We can consider the problem of finding the tax-adjusted hedging price of an option, $X^*_{\langle 0,0 \rangle}$, as being the same as solving a system of simultaneous equations.

3.2.1 Formulating the system of equations

Let us assume that the tax-mismatch is zero across all the final nodes at maturity. Equation (2.14) can be written as follows:

$$
\underbrace{\Delta^*_{\langle N,j,-1 \rangle} S_{\langle N,j \rangle} - \tau_s \sum_{k=1}^{N} \Delta^*_{\langle N,j,-k \rangle} \left(S_{\langle N,j,-k+1 \rangle} - S_{\langle N,j,-k \rangle} \right)}_{ATS_{<N,j>}} \tag{3.1}
$$

$$
\underbrace{+ RB^*_{\langle N,j,-1 \rangle} - \tau_b \left(R - 1 \right) \left(\sum_{k=1}^{N} B^*_{\langle N,j,-k \rangle} \right)}_{ATB_{<N,j>}}
$$

$$
\underbrace{- \left((1 - \tau_o) X_{\langle N,j \rangle} + \tau_o \left(\Delta^*_{\langle 0,0 \rangle} S_{\langle 0,0 \rangle} + B^*_{\langle 0,0 \rangle} \right) \right)}_{ATO_{<N,j>}} = 0,
$$

where we have substituted for $X^*_{\langle 0,0 \rangle} = \Delta^*_{\langle 0,0 \rangle} S_{\langle 0,0 \rangle} + B^*_{\langle 0,0 \rangle}$. We have 2^N equations of the form of (3.1), one for each of the final nodes on the tree.

For each of the $\langle i,j \rangle$ intermediate nodes, where $i = 1, 2, ...N - 1$, we have equations of the form:

$$
\Delta^*_{\langle i,j \rangle} S_{\langle i,j \rangle} + B^*_{\langle i,j \rangle} = \Delta^*_{\langle i,j,-1 \rangle} S_{\langle i,j \rangle} + RB^*_{\langle i,j,-1 \rangle}, \tag{3.2}
$$

being the self-financing property of the hedging portfolio. We have $2^N - 2$ intermediate nodes and consequently $2^N - 2$ equations of the form of (3.2).

In total we have $2^N + 2^N - 2 = 2^{N+1} - 2$ equations.

Determining the number of unknowns

The unknowns consist of $\Delta^*_{\langle i,j \rangle}$ and $B^*_{\langle i,j \rangle}$ at each of the $\langle i,j \rangle$ nodes prior to maturity, where $i = 0, 1, ...N - 1$. We have $2^N - 1$ nodes prior to maturity and consequently $2\left(2^N - 1\right) = 2^{N+1} - 2$ unknowns.

Uniqueness of the $\Delta^*_{\langle i,j \rangle}s$ and $B^*_{\langle i,j \rangle}s$

We have $2^{N+1} - 2$ unknowns and $2^{N+1} - 2$ linearly independent equations. Because we have the same number of unknowns as we have *linearly independent* equations, we can solve them simultaneously to find unique $\Delta^*_{\langle i,j \rangle}s$ and $B^*_{\langle i,j \rangle}s$ and consequently a unique $X^*_{\langle 0,0 \rangle}$, the tax-adjusted hedging price of the option.

3.2.2 The single-period case

We have two unknowns, $_1\Delta^*_{\langle 0,0 \rangle}$ and $_1B^*_{\langle 0,0 \rangle}$, and two equations, being the two mismatch equations at nodes $\langle 1,0 \rangle$ and $\langle 1,1 \rangle$. The leading subscript denotes the number of periods.

$$
\langle 1,0 \rangle \quad
\begin{aligned}
0 = {}& _1\Delta^*_{\langle 0,0 \rangle}S_{\langle 1,0 \rangle} - \tau_s \, _1\Delta^*_{\langle 0,0 \rangle}\left(S_{\langle 1,0 \rangle} - S_{\langle 0,0 \rangle}\right) + R \, _1B^*_{\langle 0,0 \rangle} - \tau_b(R - 1) \, _1B^*_{\langle 0,0 \rangle} \\
& -(1 - \tau_o)X_{\langle 1,0 \rangle} - \tau_o\left(_1\Delta^*_{\langle 0,0 \rangle}S_{\langle 0,0 \rangle} + \, _1B^*_{\langle 0,0 \rangle}\right)
\end{aligned}
$$

$\langle 0,0 \rangle$

$$
\langle 1,1 \rangle \quad
\begin{aligned}
0 = {}& _1\Delta^*_{\langle 0,0 \rangle}S_{\langle 1,1 \rangle} - \tau_s \, _1\Delta^*_{\langle 0,0 \rangle}\left(S_{\langle 1,1 \rangle} - S_{\langle 0,0 \rangle}\right) + R \, _1B^*_{\langle 0,0 \rangle} - \tau_b(R - 1) \, _1B^*_{\langle 0,0 \rangle} \\
& -(1 - \tau_o)X_{\langle 1,1 \rangle} - \tau_o\left(_1\Delta^*_{\langle 0,0 \rangle}S_{\langle 0,0 \rangle} + \, _1B^*_{\langle 0,0 \rangle}\right)
\end{aligned}
$$

Figure 3.1: The tax-mismatch equations for a one-period tree

Solving the above two equations simultaneously we get:

$$_1\Delta^*_{\langle 0,0\rangle} = \frac{(1-\tau_o)\left(X_{\langle 1,0\rangle} - X_{\langle 1,1\rangle}\right)}{(1-\tau_s)\left(S_{\langle 1,0\rangle} - S_{\langle 1,1\rangle}\right)}, \tag{3.3}$$

and,

$$_1B^*_{\langle 0,0\rangle} = \frac{1}{[R - \tau_b(R-1) - \tau_o]} \times \tag{3.4}$$
$$\left[_1\Delta^*_{\langle 0,0\rangle}\left\{\tau_s\left(S_{\langle 1,0\rangle} - S_{\langle 0,0\rangle}\right) - S_{\langle 1,0\rangle} + \tau_o S_{\langle 0,0\rangle}\right\} + (1-\tau_o)X_{\langle 1,0\rangle}\right].$$

Note, if $\tau_o = \tau_s$ the delta in the single-period case collapses to its CRR equivalent. As we will see, this only occurs in the single-period case.

3.2.3 Multiple periods

We can work through the algebra for two, three and four periods to find a general N-period expression for $_N\Delta^*_{\langle i,j\rangle}$, representing all the deltas in the N-period environment where $i = 0,1,...,N-1$ and $j = 0,1,...,2^i - 1$, and an expression for $_NB^*_{\langle 0,0\rangle}$. The algebra quickly becomes extremely involved; going to four periods means we have 30 equations to deal with.

The expressions for $_N\Delta^*_{\langle i,j\rangle}$ and $_NB^*_{\langle i,j\rangle}$ are given below. Appendix C contains the derivations.

$$_N\Delta^*_{\langle i,j\rangle} = \frac{1}{\left(S_{\langle i+1,2j\rangle} - S_{\langle i+1,2j+1\rangle}\right)\left(-\tau_s + R^{N-1-i} - \tau_b(R-1)\sum_{a=0}^{N-2-i} R^a\right)} \times \tag{3}$$

$$\left[\begin{array}{l} + (1-\tau_o)\left(X_{\langle N,j2^{N-i}\rangle} - X_{\langle N,(2j+1)2^{N-i-1}\rangle}\right) \\[2mm] - \sum_{a=i+1}^{N-1} {}_N\Delta^*_{\langle a,j2^{a-i}\rangle}\left[\begin{array}{l} + S_{\langle a+1,j2^{a+1-i}\rangle}\left(R^{N-a-1} - \tau_b(R-1)\sum_{b=0}^{N-2-a} R^b\right) \\[1mm] - S_{\langle a,j2^{a-i}\rangle}\left(R^{N-a} - \tau_b(R-1)\sum_{b=0}^{N-1-a} R^b\right) \\[1mm] - \tau_s\left(S_{\langle a+1,j2^{a+1-i}\rangle} - S_{\langle a,j2^{a-i}\rangle}\right) \end{array}\right] \\[6mm] + \sum_{a=i+1}^{N-1} {}_N\Delta^*_{\langle a,(2j+1)2^{a-i-1}\rangle}\left[\begin{array}{l} + S_{\langle a+1,(2j+1)2^{a-i}\rangle}\left(R^{N-a-1} - \tau_b(R-1)\sum_{b=0}^{N-2-a} R^b\right) \\[1mm] - S_{\langle a,(2j+1)2^{a-i-1}\rangle}\left(R^{N-a} - \tau_b(R-1)\sum_{b=0}^{N-1-a} R^b\right) \\[1mm] - \tau_s\left(S_{\langle a+1,(2j+1)2^{a-i}\rangle} - S_{\langle a,(2j+1)2^{a-i-1}\rangle}\right) \end{array}\right] \end{array}\right]$$

$$NB^*_{\langle 0,0\rangle} = \frac{1}{\left(R^N - \tau_o - \tau_b(R-1)\sum_{a=0}^{N-1} R^a\right)} \times \qquad (3$$

$$\begin{bmatrix} + (1-\tau_o)\,X_{\langle N,0\rangle} \\[4pt] -N\Delta^*_{\langle 0,0\rangle}\left[-\tau_o S_{\langle 0,0\rangle} - \tau_s\left(S_{\langle 1,0\rangle} - S_{\langle 0,0\rangle}\right) + S_{\langle 1,0\rangle}\left(R^{N-1} - \tau_b(R-1)\sum_{a=0}^{N-2} R^a\right)\right] \\[4pt] -\sum_{a=1}^{N-1} {}_N\Delta^*_{\langle a,0\rangle}\begin{bmatrix} -\tau_s\left(S_{\langle a+1,0\rangle} - S_{\langle a,0\rangle}\right) - S_{\langle a,0\rangle}\left(R^{N-a} - \tau_b(R-1)\sum_{b=0}^{N-1-a} R^b\right) \\[4pt] +S_{\langle a+1,0\rangle}\left(R^{N-a-1} - \tau_b(R-1)\sum_{b=0}^{N-2-a} R^b\right) \end{bmatrix} \end{bmatrix}$$

The simultaneous equation algorithm

We can use (3.5) and (3.6) to form an algorithm that finds the tax-adjusted hedging price of an option. The overall idea being the algorithm is to start at the $N-1$ nodes and move back through the tree (if we still want to think of the environment as that of a non-recombining tree) calculating the deltas as we go. Once we have the deltas we can find $_NB^*_{\langle 0,0\rangle}$ and hence $_NX^*_{\langle 0,0\rangle}$.

1. Set $i = N - 1$.

2. Set $j = 0$.

3. Using Equation (3.5), compute $_N\Delta^*_{\langle i,j\rangle}$ and store in an array. This requires a sub-routine:

   (a) Set $a = i + 1$.

   (b) Set *sum_upper_delta* = *sum_lower_delta* = 0.

   (c) Form the sums $\sum_{b=0}^{N-2-a} R^b$ and $\sum_{b=0}^{N-1-a} R^b$.

   (d) Form the two coefficients, *upper_coefficient* and *lower_coefficient*, to $_N\Delta^*_{\langle a,j2^{a-i}\rangle}$ and $_N\Delta^*_{\langle a,(2j+1)2^{a-i-1}\rangle}$ respectively (with the help of the sum formed in Step 3c).

   (e) Set:

      i. *sum_upper_delta* = *sum_upper_delta* $-$ $_N\Delta^*_{\langle a,j2^{a-i}\rangle}$ $\times$ *upper_coefficient*;

      ii. *sum_lower_delta* = *sum_lower_delta* $+$ $_N\Delta^*_{\langle a,(2j+1)2^{a-i-1}\rangle}$ $\times$ *lower_coefficient*.

   (f) If $a < N - 1$,

i. then, $a = a + 1$; return to Step 3c;

ii. else, go to Step 3g.

(g) Compute $_N\Delta^*_{\langle i,j \rangle}$ and store in an array. We can calculate $_N\Delta^*_{\langle i,j \rangle}$ here because we have found *sum_upper_delta* (which equals the $-\sum_{a=i+1}^{N-1} {_N}\Delta^*_{\langle a,j2^{a-i} \rangle}$ term in (3.5)) and *sum_lower_delta* (which equals the $+\sum_{a=i+1}^{N-1} {_N}\Delta^*_{\langle a,(2j+1)2^{a-i-1} \rangle}$ term in (3.5)), and all the other terms are known or straightforward to calculate.

4. If $j < 2^i - 1$,

(a) then, $j = j + 1$; return to Step 3;

(b) else, go to Step 5.

5. If $i > 0$,

(a) then, $i = i - 1$; return to Step 2.

(b) else, go to Step 6.

6. Using Equation (3.6), compute $_NB^*_{\langle 0,0 \rangle}$. This requires a sub-routine:

(a) Set $a = 1$.

(b) Set *sum_delta* $= 0$.

(c) Form the sums $\sum_{b=0}^{N-2-a} R^b$ and $\sum_{b=0}^{N-1-a} R^b$.

(d) Form the coefficient (*coefficient*) to $_N\Delta^*_{\langle a,0 \rangle}$ (with the help of the sums formed in Step 6c).

(e) Set *sum_delta* $=$ *sum_delta* $- {_N}\Delta^*_{\langle a,j2^{a-i} \rangle} \times coefficient$.

(f) If $a < N - 1$,

i. then, $a = a + 1$; return to Step 6c;

ii. else, go to Step 6g.

(g) Compute $_NB^*_{\langle 0,0 \rangle}$, given that we have found *sum_delta* which equals the $-\sum_{a=1}^{N-1} {_N}\Delta^*_{\langle a,0 \rangle}$ term in (3.6).

7. Compute $_NX^*_{\langle 0,0 \rangle} = {_N}\Delta^*_{\langle 0,0 \rangle} S_{\langle 0,0 \rangle} + {_N}B^*_{\langle 0,0 \rangle}$.

Results

The simultaneous equation algorithm has been implemented in C++ using a 633Mhz, 512MB Pentium III PC. The results obtained from this algorithm agree with the results obtained from the tax-adjusted hedging algorithm given in Chapter 2, as we would expect.

Memory requirements and speed of computation

The algorithm is demanding on memory since we have to store all the values in the tree for the stock and delta and the size of each of these arrays increases exponentially with the number of periods. However, the simultaneous equation algorithm only requires two arrays (of approximate size 2^N) as opposed to the five arrays (again, of approximate size 2^N) that were used in the tax-adjusted hedging algorithm of Chapter 2. Consequently, the simultaneous equation algorithm is less demanding on memory than the tax-adjusted hedging algorithm.

The speed of computation slows with an increase in the number of periods because the number of deltas to be computed increases as 2^N. The simultaneous equation algorithm is faster than the tax-adjusted hedging algorithm because the deltas only have to be computed once, and only one bond term is found. This is born out by Table 3.1, which shows a comparison of the CPU time taken for the simultaneous equation and tax-adjusted hedging[1] algorithms to reach the tax-adjusted hedging price of the option, as the number of periods increases. We can also see that the reduction in memory requirements for the simultaneous equation algorithm allows us to use three more periods, compared with the tax-adjusted hedging algorithm, before we run out of memory.

The parameters used to compile Table 3.1 were: $S = K = 100, r = 0.05, \sigma = 0.25, T = 1, \tau_b = 0.4, \tau_s = 0.3, \tau_o = 0.2$.

---

[1]From Table 2.1.

73

Periods, N	Simultaneous equation alg. CPU time (s)	Tax-adjusted hedging alg. CPU time (s)
$1 - 14$	< 1	< 1
15	< 1	1
16	< 1	2
17	1	5
18	3	10
19	5	20
20	10	40
21	18	80
22	35	180
23	70	Memory requirements too great
24	175	Memory requirements too great
25	25 mins 10s	Memory requirements too great
≥ 26	Memory requirements too great	Memory requirements too great

Table 3.1: Time taken for algorithms to find tax-adjusted price

3.3 Single-period analysis

3.3.1 Tax-adjusted hedging prices: relationship to CRR and tax-adjusted put-call parity

Relationship to CRR

The well known CRR delta and bond formula (Cox (1979)) in the single-period case are:

$$_1\Delta_{\langle 0,0 \rangle}^{CRR} = \frac{X_{\langle 1,0 \rangle} - X_{\langle 1,1 \rangle}}{S_{\langle 1,0 \rangle} - S_{\langle 1,1 \rangle}}, \tag{3.7}$$

and,

$$_1B_{\langle 0,0 \rangle}^{CRR} = \frac{1}{R} \left[X_{\langle 1,0 \rangle} - \frac{\left(X_{\langle 1,0 \rangle} - X_{\langle 1,1 \rangle} \right)}{\left(S_{\langle 1,0 \rangle} - S_{\langle 1,1 \rangle} \right)} S_{\langle 1,0 \rangle} \right]. \tag{3.8}$$

We can see immediately from Equation (3.3) that the tax-adjusted delta and the CRR delta are related by:

$$_1\Delta^*_{\langle 0,0 \rangle} = {}_1\Delta^{CRR}_{\langle 0,0 \rangle} \frac{(1 - \tau_o)}{(1 - \tau_s)}. \tag{3.9}$$

We need to write Equation (3.4) in a different way to see how the tax-adjusted bond and the CRR bond are related. If we substitute for $_1\Delta^*_{\langle 0,0 \rangle}$ and bring $(1 - \tau_o)$ outside the main bracket we get the following:

$$_1B^*_{\langle 0,0 \rangle} = \frac{(1 - \tau_o)}{[R - \tau_b(R - 1) - \tau_o]} \left[\begin{array}{l} +X_{\langle 1,0 \rangle} - \frac{\left(X_{\langle 1,0 \rangle} - X_{\langle 1,1 \rangle}\right)}{\left(S_{\langle 1,0 \rangle} - S_{\langle 1,1 \rangle}\right)} S_{\langle 1,0 \rangle} \\ -S_{\langle 0,0 \rangle} \frac{\left(X_{\langle 1,0 \rangle} - X_{\langle 1,1 \rangle}\right)}{\left(S_{\langle 1,0 \rangle} - S_{\langle 1,1 \rangle}\right)} \frac{(\tau_s - \tau_o)}{(1 - \tau_s)} \end{array} \right]. \tag{3.10}$$

We can recognise the top line in the square bracket as being equal to $_1B^{CRR}_{\langle 0,0 \rangle} R$. We find that the tax-adjusted bond and CRR bond are related by:

$$_1B^*_{\langle 0,0 \rangle} = \frac{(1 - \tau_o)}{[R - \tau_b(R - 1) - \tau_o]} \left[\begin{array}{l} + {}_1B^{CRR}_{\langle 0,0 \rangle} R \\ -S_{\langle 0,0 \rangle} \frac{\left(X_{\langle 1,0 \rangle} - X_{\langle 1,1 \rangle}\right)}{\left(S_{\langle 1,0 \rangle} - S_{\langle 1,1 \rangle}\right)} \frac{(\tau_s - \tau_o)}{(1 - \tau_s)} \end{array} \right]. \tag{3.11}$$

With Equations (3.9) and (3.11) established, we can look at tax-adjusted option prices compared to CRR option prices in the single-period environment. We will look at Case 2.1 $(\tau_b = \tau_i$ and $\tau_s = \tau_o = \tau_{cg}$, where $\tau_i > \tau_{cg})$ which was given in Section 2.4.1.

Theorem 3.1

If the tax rates are those given in Case 2.1 then,

$$_1c^*_{\langle 0,0 \rangle} < {}_1c^{CRR}_{\langle 0,0 \rangle}, \tag{3.12}$$

and,

$$_1p^*_{\langle 0,0 \rangle} > {}_1p^{CRR}_{\langle 0,0 \rangle}. \tag{3.13}$$

Note, the relationships in Theorem 3.1 are consistent with the results found in Section 2.4.1.

Proof

Substituting $\tau_b = \tau_i$ and $\tau_s = \tau_o = \tau_{cg}$ into (3.9) and (3.11) we get:

$$_1\Delta^*_{\langle 0,0\rangle} = \, _1\Delta^{CRR}_{\langle 0,0\rangle}, \tag{3.14}$$

$$_1B^*_{\langle 0,0\rangle} = \alpha \, _1B^{CRR}_{\langle 0,0\rangle}, \tag{3.15}$$

where,

$$\alpha = \frac{R\left(1 - \tau_{cg}\right)}{[R - \tau_i(R-1) - \tau_{cg}]} \tag{3.16}$$

We need to look at α in the above equation:

$$
\begin{aligned}
\tau_i &> \tau_{cg} \\
\tau_i(R-1) &> (R-1)\tau_{cg} \\
\tau_i R - \tau_i - R &> \tau_{cg}R - \tau_{cg} - R \\
R(1-\tau_i) + \tau_i &< R(1-\tau_{cg}) + \tau_{cg} \\
R(1-\tau_i) + \tau_i - \tau_{cg} &< R(1-\tau_{cg}) \\
1 &< \frac{R(1-\tau_{cg})}{R - \tau_i(R-1) - \tau_{cg}} = \alpha.
\end{aligned}
$$

The fact that $R > 1$ has been used above.

So we find that,

$$\alpha > 1. \tag{3.17}$$

The price of the option is given by:

$$_1X^*_{\langle 0,0\rangle} = S_{\langle 0,0\rangle} \, _1\Delta^{CRR}_{\langle 0,0\rangle} + \alpha \, _1B^{CRR}_{\langle 0,0\rangle}. \tag{3.18}$$

For a call, $_1B^{CRR}_{\langle 0,0 \rangle} < 0$. Because $\alpha > 1$,

$$_1c^{*}_{\langle 0,0 \rangle} = S_{\langle 0,0 \rangle} \; _1\Delta^{CRR}_{\langle 0,0 \rangle} + \alpha \; _1B^{CRR}_{\langle 0,0 \rangle} < \; _1c^{CRR}_{\langle 0,0 \rangle}. \tag{3.19}$$

For a put, $_1B^{CRR}_{\langle 0,0 \rangle} > 0$. Because $\alpha > 1$,

$$_1p^{*}_{\langle 0,0 \rangle} = S_{\langle 0,0 \rangle} \; _1\Delta^{CRR}_{\langle 0,0 \rangle} + \alpha \; _1B^{CRR}_{\langle 0,0 \rangle} > \; _1p^{CRR}_{\langle 0,0 \rangle}. \tag{3.20}$$

Theorem 3.2: Tax-adjusted put-call parity

If the tax rates are those given in Case 2.1 then,

$$S_{\langle 0,0 \rangle} + \; _1p^{*}_{\langle 0,0 \rangle} - \; _1c^{*}_{\langle 0,0 \rangle} = aKR^{-1}, \tag{3.21}$$

where,

$$a = \frac{(1 - \tau_{cg})}{(1 - \tau_i) + (\tau_i - \tau_{cg})R^{-1}}. \tag{3.22}$$

This is the tax-adjusted put-call parity relationship given by Equations (2.29) and (2.30) in Section 2.6.2, where $R^{-1} = e^{-rt}$.

Proof

Using Equations (3.19) and (3.20) we can write,

$$\begin{aligned}
S_{\langle 0,0 \rangle} + \; _1p^{*}_{\langle 0,0 \rangle} - \; _1c^{*}_{\langle 0,0 \rangle} &= S_{\langle 0,0 \rangle} \left(1 + \; _1\Delta^{put}_{\langle 0,0 \rangle} - \; _1\Delta^{call}_{\langle 0,0 \rangle} \right) \\
&+ \alpha \left(_1B^{put}_{\langle 0,0 \rangle} - \; _1B^{call}_{\langle 0,0 \rangle} \right),
\end{aligned} \tag{3.23}$$

where the superscripts *put* and *call* denote the CRR quantity for a put and call respectively.

It is easy to show that:

$$_1\Delta^{put}_{\langle 0,0 \rangle} - \; _1\Delta^{call}_{\langle 0,0 \rangle} = -1, \tag{3.24}$$

77

and consequently (from classical put-call parity) that:

$$_1B^{put}_{\langle 0,0 \rangle} - {}_1B^{call}_{\langle 0,0 \rangle} = KR^{-1}. \tag{3.25}$$

Therefore, (3.23) becomes:

$$S_{\langle 0,0 \rangle} + {}_1p^*_{\langle 0,0 \rangle} - {}_1c^*_{\langle 0,0 \rangle} = \alpha KR^{-1}. \tag{3.26}$$

Now we know that:

$$\begin{aligned}
\alpha &= \frac{R(1 - \tau_{cg})}{R - \tau_i(R - 1) - \tau_{cg}} \\
&= \frac{(1 - \tau_{cg})}{1 - \tau_i(1 - \frac{1}{R}) - \frac{\tau_{cg}}{R}} \\
&= \frac{(1 - \tau_{cg})}{(1 - \tau_i) + (\tau_i - \tau_{cg})R^{-1}}.
\end{aligned}$$

So,

$$\alpha = a.$$

This gives (3.26) as:

$$S_{\langle 0,0 \rangle} + {}_1p^*_{\langle 0,0 \rangle} - {}_1c^*_{\langle 0,0 \rangle} = aKR^{-1}.$$

3.3.2 The convergence of the tax-adjusted hedging algorithm of Chapter 2 to the analytic tax-adjusted hedging price (one period case)

It was noted in the results part of Section 3.2.3 that empirically the tax-adjusted option prices found from the simultaneous equation approach agree with prices obtained from the tax-adjusted hedging algorithm of Chapter 2. In this section we will show, in a more rigorous manner, that the tax-adjusted hedging algorithm given in Section 2.3.5 converges to the analytic formula for the option price found from the simultaneous equation approach of this chapter, as the number of iterations goes to infinity. We will only do this in the single-period case because even here the algebra is very time-consuming.

We will start by finding a general equation for the delta for the gth iteration, in terms of the delta at all previous iterations, and the same for the bond (in terms of the previous bonds and deltas). Once we have these expressions we will be able to look at how $_1\Delta^g_{\langle 0,0 \rangle} \rightarrow {}_1\Delta^*_{\langle 0,0 \rangle}$, and $_1B^g_{\langle 0,0 \rangle} \rightarrow {}_1B^*_{\langle 0,0 \rangle}$ as $g \rightarrow \infty$. As a reminder, the superscript, g, represents the iteration that the tax-adjusted hedging algorithm has reached.

A general expression for $_1\Delta^g_{\langle 0,0 \rangle}$ and $_1B^g_{\langle 0,0 \rangle}$

The equations that are used in the tax-adjusted hedging algorithm are:

$$\mathfrak{M}^g_{\langle N,j \rangle} = ATS^g_{\langle N,j \rangle} + ATB^g_{\langle N,j \rangle} - ATO^g_{\langle N,j \rangle}; \tag{3.27}$$

$$\mathfrak{X}^g_{\langle N,j \rangle} = \mathfrak{X}^{g-1}_{\langle N,j \rangle} - \mathfrak{M}^{g-1}_{\langle N,j \rangle}, \text{ where } \mathfrak{X}^0_{\langle N,j \rangle} = X_{\langle N,j \rangle}; \tag{3.28}$$

and,

$$\Delta^g_{\langle N,j,-1 \rangle} S_{\langle N,j \rangle} + RB^g_{\langle N,j,-1 \rangle} = \mathfrak{X}^g_{\langle N,j \rangle}. \tag{3.29}$$

In the single-period environment we have two equations of the form of (3.27):

$$\begin{aligned}
\mathfrak{M}^g_{\langle 1,0 \rangle} = {}&{}_1\Delta^g_{\langle 0,0 \rangle} S_{\langle 1,0 \rangle} - \tau_s {}_1\Delta^g_{\langle 0,0 \rangle} \left(S_{\langle 1,0 \rangle} - S_{\langle 0,0 \rangle} \right) \\
&+ R {}_1B^g_{\langle 0,0 \rangle} - \tau_b(R-1) {}_1B^g_{\langle 0,0 \rangle} - (1-\tau_o)X_{\langle 1,0 \rangle} - \tau_o X^g_{\langle 0,0 \rangle},
\end{aligned} \tag{3.30}$$

and,

$$\begin{aligned}
\mathfrak{M}^g_{\langle 1,1 \rangle} = {}&{}_1\Delta^g_{\langle 0,0 \rangle} S_{\langle 1,1 \rangle} - \tau_s {}_1\Delta^g_{\langle 0,0 \rangle} \left(S_{\langle 1,1 \rangle} - S_{\langle 0,0 \rangle} \right) \\
&+ R {}_1B^g_{\langle 0,0 \rangle} - \tau_b(R-1) {}_1B^g_{\langle 0,0 \rangle} - (1-\tau_o)X_{\langle 1,1 \rangle} - \tau_o X^g_{\langle 0,0 \rangle}.
\end{aligned} \tag{3.31}$$

We have two equations of the form of (3.28):

$$\mathfrak{X}^g_{\langle 1,0 \rangle} = \mathfrak{X}^{g-1}_{\langle 1,0 \rangle} - \mathfrak{M}^{g-1}_{\langle 1,0 \rangle}, \text{ where } \mathfrak{X}^0_{\langle 1,0 \rangle} = X_{\langle 1,0 \rangle}, \tag{3.32}$$

and,

$$\mathfrak{X}_{\langle 1,1 \rangle}^{g} = \mathfrak{X}_{\langle 1,1 \rangle}^{g-1} - \mathfrak{M}_{\langle 1,1 \rangle}^{g-1}, \text{ where } \mathfrak{X}_{\langle 1,1 \rangle}^{0} = X_{\langle 1,1 \rangle}. \tag{3.33}$$

We have two equations of the form of (3.29):

$$_1\Delta_{\langle 0,0 \rangle}^{g} S_{\langle 1,0 \rangle} + R \ _1B_{\langle 0,0 \rangle}^{g} = \mathfrak{X}_{\langle 1,0 \rangle}^{g}, \tag{3.34}$$

and,

$$_1\Delta_{\langle 0,0 \rangle}^{g} S_{\langle 1,1 \rangle} + R \ _1B_{\langle 0,0 \rangle}^{g} = \mathfrak{X}_{\langle 1,1 \rangle}^{g}. \tag{3.35}$$

And of course,

$$X_{\langle 0,0 \rangle}^{g} = \ _1\Delta_{\langle 0,0 \rangle}^{g} S_{\langle 0,0 \rangle} + \ _1B_{\langle 0,0 \rangle}^{g}. \tag{3.36}$$

If we start at $g = 0$ and solve for $\Delta_{\langle 0,0 \rangle}^{0}$ and $B_{\langle 0,0 \rangle}^{0}$ we get:

$$_1\Delta_{\langle 0,0 \rangle}^{0} = \frac{(X_{\langle 1,0 \rangle} - X_{\langle 1,1 \rangle})}{(S_{\langle 1,0 \rangle} - S_{\langle 1,1 \rangle})} = \ _1\Delta_{\langle 0,0 \rangle}^{CRR},$$

and,

$$_1B_{\langle 0,0 \rangle}^{0} = \frac{1}{R} \left[\frac{(X_{\langle 1,1 \rangle}S_{\langle 1,0 \rangle} - X_{\langle 1,0 \rangle}S_{\langle 1,1 \rangle})}{(S_{\langle 1,0 \rangle} - S_{\langle 1,1 \rangle})} \right] = \ _1B_{\langle 0,0 \rangle}^{CRR},$$

as we expect.

The delta   Now let's look at what happens to the delta for $g = 1, 2,$ and 3:

$$_1\Delta_{\langle 0,0 \rangle}^{1} = \frac{1}{(S_{\langle 1,0 \rangle} - S_{\langle 1,1 \rangle})} \times \tag{3.37}$$
$$\left[\begin{array}{l} +(X_{\langle 1,0 \rangle} - X_{\langle 1,1 \rangle}) + (X_{\langle 1,0 \rangle} - X_{\langle 1,1 \rangle})(1 - \tau_o) \\ - \ _1\Delta_{\langle 0,0 \rangle}^{0}(S_{\langle 1,0 \rangle} - S_{\langle 1,1 \rangle})(1 - \tau_s) \end{array} \right];$$

80

$$_1\Delta^2_{\langle 0,0\rangle} \;=\; \frac{1}{(S_{\langle 1,0\rangle} - S_{\langle 1,1\rangle})} \times \tag{3.38}$$

$$\left[\begin{array}{l} +(X_{\langle 1,0\rangle} - X_{\langle 1,1\rangle}) + 2(X_{\langle 1,0\rangle} - X_{\langle 1,1\rangle})\,(1-\tau_o) \\ -\left[{}_1\Delta^0_{\langle 0,0\rangle} + {}_1\Delta^1_{\langle 0,0\rangle}\right](S_{\langle 1,0\rangle} - S_{\langle 1,1\rangle})\,(1-\tau_s) \end{array}\right] \; ;$$

$$_1\Delta^3_{\langle 0,0\rangle} \;=\; \frac{1}{(S_{\langle 1,0\rangle} - S_{\langle 1,1\rangle})} \times \tag{3.39}$$

$$\left[\begin{array}{l} +(X_{\langle 1,0\rangle} - X_{\langle 1,1\rangle}) + 3(X_{\langle 1,0\rangle} - X_{\langle 1,1\rangle})\,(1-\tau_o) \\ -\left[{}_1\Delta^0_{\langle 0,0\rangle} + {}_1\Delta^1_{\langle 0,0\rangle} + {}_1\Delta^2_{\langle 0,0\rangle}\right](S_{\langle 1,0\rangle} - S_{\langle 1,1\rangle})\,(1-\tau_s) \end{array}\right] \; .$$

We can write down the general case for g-iterations as:

$$_1\Delta^g_{\langle 0,0\rangle} \;=\; \frac{1}{(S_{\langle 1,0\rangle} - S_{\langle 1,1\rangle})} \times \tag{3.40}$$

$$\left[\begin{array}{l} +(X_{\langle 1,0\rangle} - X_{\langle 1,1\rangle}) + g(X_{\langle 1,0\rangle} - X_{\langle 1,1\rangle})\,(1-\tau_o) \\ -\left[\sum_{i=0}^{g-1} {}_1\Delta^i_{\langle 0,0\rangle}\right](S_{\langle 1,0\rangle} - S_{\langle 1,1\rangle})\,(1-\tau_s) \end{array}\right] \; .$$

The bond Now for the bond for $g = 1, 2$, and 3:

$$_1B^1_{\langle 0,0\rangle} = \frac{1}{R}\left[\begin{array}{l} \dfrac{\left(X_{\langle 1,1\rangle}S_{\langle 1,0\rangle} - X_{\langle 1,0\rangle}S_{\langle 1,1\rangle}\right) + \left(X_{\langle 1,1\rangle}S_{\langle 1,0\rangle} - X_{\langle 1,0\rangle}S_{\langle 1,1\rangle}\right)(1-\tau_o)}{\left(S_{\langle 1,0\rangle} - S_{\langle 1,1\rangle}\right)} \\[2mm] -\,{}_1\Delta^0_{\langle 0,0\rangle}\,(\tau_s - \tau_o)\,S_{\langle 0,0\rangle} - {}_1B^0_{\langle 0,0\rangle}\,(R - \tau_b(R-1) - \tau_o) \end{array}\right] \; ; \tag{3.41}$$

$$_1B^2_{\langle 0,0\rangle} = \frac{1}{R}\left[\begin{array}{l} \dfrac{\left(X_{\langle 1,1\rangle}S_{\langle 1,0\rangle} - X_{\langle 1,0\rangle}S_{\langle 1,1\rangle}\right) + 2\left(X_{\langle 1,1\rangle}S_{\langle 1,0\rangle} - X_{\langle 1,0\rangle}S_{\langle 1,1\rangle}\right)(1-\tau_o)}{\left(S_{\langle 1,0\rangle} - S_{\langle 1,1\rangle}\right)} \\[2mm] -\left[{}_1\Delta^0_{\langle 0,0\rangle} + {}_1\Delta^1_{\langle 0,0\rangle}\right](\tau_s - \tau_o)\,S_{\langle 0,0\rangle} \\[2mm] -\left[{}_1B^0_{\langle 0,0\rangle} + {}_1B^1_{\langle 0,0\rangle}\right](R - \tau_b(R-1) - \tau_o) \end{array}\right] \; ; \tag{3.42}$$

$$_1B^3_{\langle 0,0\rangle} = \frac{1}{R}\left[\begin{array}{l} \dfrac{\left(X_{\langle 1,1\rangle}S_{\langle 1,0\rangle} - X_{\langle 1,0\rangle}S_{\langle 1,1\rangle}\right) + 3\left(X_{\langle 1,1\rangle}S_{\langle 1,0\rangle} - X_{\langle 1,0\rangle}S_{\langle 1,1\rangle}\right)(1-\tau_o)}{\left(S_{\langle 1,0\rangle} - S_{\langle 1,1\rangle}\right)} \\[2mm] -\left[{}_1\Delta^0_{\langle 0,0\rangle} + {}_1\Delta^1_{\langle 0,0\rangle} + {}_1\Delta^2_{\langle 0,0\rangle}\right](\tau_s - \tau_o)\,S_{\langle 0,0\rangle} \\[2mm] -\left[{}_1B^0_{\langle 0,0\rangle} + {}_1B^1_{\langle 0,0\rangle} + {}_1B^2_{\langle 0,0\rangle}\right](R - \tau_b(R-1) - \tau_o) \end{array}\right] \; ; \tag{3.43}$$

81

We can write down the general case for g-iterations as:

$$_1B^g_{\langle 0,0 \rangle} = \frac{1}{R} \left[\begin{array}{c} \frac{\left(X_{\langle 1,1 \rangle}S_{\langle 1,0 \rangle}-X_{\langle 1,0 \rangle}S_{\langle 1,1 \rangle}\right)+g\left(X_{\langle 1,1 \rangle}S_{\langle 1,0 \rangle}-X_{\langle 1,0 \rangle}S_{\langle 1,1 \rangle}\right)(1-\tau_o)}{\left(S_{\langle 1,0 \rangle}-S_{\langle 1,1 \rangle}\right)} \\ -\left[\sum_{i=0}^{g-1} {}_1\Delta^i_{\langle 0,0 \rangle}\right](\tau_s-\tau_o)S_{\langle 0,0 \rangle} \\ -\left[\sum_{i=0}^{g-1} {}_1B^i_{\langle 0,0 \rangle}\right](R-\tau_b(R-1)-\tau_o) \end{array} \right]. \tag{3.44}$$

With the expressions for $_1\Delta^g_{\langle 0,0 \rangle}$ and $_1B^g_{\langle 0,0 \rangle}$ given by (3.40) and (3.44), we can look to finding an expression for each in terms of only the stock, tax rates, option payoffs (pre-tax) and R, from which the behaviour as g goes to infinity can be examined. We will look at the delta and bond separately, starting with the delta.

Convergence of $_1\Delta^g_{\langle 0,0 \rangle}$ to $_1\Delta^*_{\langle 0,0 \rangle}$

We want to show that:

$$\lim_{g\to\infty} {}_1\Delta^g_{\langle 0,0 \rangle} = {}_1\Delta^*_{\langle 0,0 \rangle} = \frac{\left(X_{\langle 1,0 \rangle}-X_{\langle 1,1 \rangle}\right)(1-\tau_o)}{\left(S_{\langle 1,0 \rangle}-S_{\langle 1,1 \rangle}\right)(1-\tau_s)}. \tag{3.45}$$

Using (3.40) we can eliminate the previous deltas by repeated substitution. The algebra is quite involved and time-consuming. Eventually we find:

$$_1\Delta^g_{\langle 0,0 \rangle} = \frac{\left(X_{\langle 1,0 \rangle}-X_{\langle 1,1 \rangle}\right)}{\left(S_{\langle 1,0 \rangle}-S_{\langle 1,1 \rangle}\right)}\left[1+\sum_{i=1}^{g}\tau_s^i-\tau_o\sum_{i=0}^{g-1}\tau_s^i\right] \tag{3.46}$$

We can recognise the square bracket above as being the sum of two geometric series:

$$1+\sum_{i=1}^{g}\tau_s^i = 1+\tau_s+\tau_s^2+...+\tau_s^g, \tag{3.47}$$

and,

$$-\tau_o\sum_{i=0}^{g-1}\tau_s^i = -\tau_o\left[1+\tau_s+\tau_s^2+...+\tau_s^{g-1}\right]. \tag{3.48}$$

The geometric series is convergent iff $|\tau_s|<1$ as $g\to\infty$, and has the sum $1/(1-\tau_s)$ (see pp786-787 Kreyszig (1993)).

Therefore,

$$
\begin{aligned}
{}_1\Delta_{\langle 0,0\rangle}^{\infty} &= \frac{\left(X_{\langle 1,0\rangle} - X_{\langle 1,1\rangle}\right)}{\left(S_{\langle 1,0\rangle} - S_{\langle 1,1\rangle}\right)} \left[\frac{1}{(1-\tau_s)} - \frac{\tau_o}{(1-\tau_s)}\right] \\
&= \frac{\left(X_{\langle 1,0\rangle} - X_{\langle 1,1\rangle}\right)}{\left(S_{\langle 1,0\rangle} - S_{\langle 1,1\rangle}\right)} \left[\frac{(1-\tau_o)}{(1-\tau_s)}\right],
\end{aligned}
\tag{3.49}
$$

since $0 \le \tau_s < 1$.

Thus, we have shown that ${}_1\Delta_{\langle 0,0\rangle}^{g} \to {}_1\Delta_{\langle 0,0\rangle}^{*}$ as $g \to \infty$.

Convergence of ${}_1B_{\langle 0,0\rangle}^{g}$ to ${}_1B_{\langle 0,0\rangle}^{*}$

We want to show that:

$$
{}_1B_{\langle 0,0\rangle}^{g} \overset{g\to\infty}{=} {}_1B_{\langle 0,0\rangle}^{*}
\tag{3.50}
$$

$$
= \frac{1}{[R - \tau_b(R-1) - \tau_o]} \times
\tag{3.51}
$$

$$
\left[\frac{\left(X_{\langle 1,1\rangle}S_{\langle 1,0\rangle} - X_{\langle 1,0\rangle}S_{\langle 1,1\rangle}\right)}{\left(S_{\langle 1,0\rangle} - S_{\langle 1,1\rangle}\right)} (1-\tau_o) + \frac{\left(X_{\langle 1,1\rangle} - X_{\langle 1,0\rangle}\right)}{\left(S_{\langle 1,0\rangle} - S_{\langle 1,1\rangle}\right)} \frac{(1-\tau_o)}{(1-\tau_s)} (\tau_o - \tau_s) S_{\langle 0,0\rangle}\right],
\tag{3.52}
$$

where we have written ${}_1B_{\langle 0,0\rangle}^{*}$ in a different way to that given in (3.4).

Using (3.44) and (3.40) we can eliminate the previous bonds and deltas by repeated substitution. The algebra is even more demanding than in the case of the delta above. Eventually we find:

$$
\begin{aligned}
{}_1B_{\langle 0,0\rangle}^{g} &= \frac{\left(X_{\langle 1,1\rangle}S_{\langle 1,0\rangle} - X_{\langle 1,0\rangle}S_{\langle 1,1\rangle}\right)}{R\left(S_{\langle 1,0\rangle} - S_{\langle 1,1\rangle}\right)} \left[\sum_{i=0}^{g}(-Y)^i \binom{g}{i} + \sum_{i=0}^{g-1}(-Y)^i \binom{g}{i+1}(1-\tau_o)\right] \\
&+ \frac{\left(X_{\langle 1,0\rangle} - X_{\langle 1,1\rangle}\right)}{R\left(S_{\langle 1,0\rangle} - S_{\langle 1,1\rangle}\right)} (\tau_o - \tau_s) S_{\langle 0,0\rangle} \sum_{i=0}^{g-1}(-Y)^i \left[\binom{g}{i+1} + \sum_{j=1}^{g-1-i}\binom{g-j}{i+1}(\tau_s - \tau_o)\tau_s^{j-1}\right],
\end{aligned}
\tag{3.53}
$$

where,

$$
Y = \frac{[R - \tau_b(R-1) - \tau_o]}{R}.
\tag{3.54}
$$

83

So we actually want to show that for the $\frac{\left(X_{\langle 1,1\rangle}S_{\langle 1,0\rangle} - X_{\langle 1,0\rangle}S_{\langle 1,1\rangle}\right)}{R\left(S_{\langle 1,0\rangle} - S_{\langle 1,1\rangle}\right)}$ term,

$$\lim_{g\to\infty}\left[\sum_{i=0}^{g}(-Y)^i\binom{g}{i} + \sum_{i=0}^{g-1}(-Y)^i\binom{g}{i+1}(1-\tau_o)\right] = Y^{-1}(1-\tau_o), \tag{3.55}$$

and for the $\frac{\left(X_{\langle 1,0\rangle} - X_{\langle 1,1\rangle}\right)}{R\left(S_{\langle 1,0\rangle} - S_{\langle 1,1\rangle}\right)}$ term,

$$\lim_{g\to\infty}\sum_{i=0}^{g-1}(-Y)^i\left[\binom{g}{i+1} + \sum_{j=1}^{g-1-i}\binom{g-j}{i+1}(\tau_s - \tau_o)\tau_s^{j-1}\right] = Y^{-1}\frac{(1-\tau_o)}{(1-\tau_s)}. \tag{3.56}$$

The $\frac{\left(X_{\langle 1,1\rangle}S_{\langle 1,0\rangle} - X_{\langle 1,0\rangle}S_{\langle 1,1\rangle}\right)}{R\left(S_{\langle 1,0\rangle} - S_{\langle 1,1\rangle}\right)}$ term The first term in the square bracket of (3.55) is straightforward as this is a binomial series:

$$\sum_{i=0}^{g}(-Y)^i\binom{g}{i} = (1-Y)^g. \tag{3.57}$$

The second term in the square bracket of (3.55) is almost a binomial series, but we need to play around with it a little:

$$(1-\tau_o)\sum_{i=0}^{g-1}(-Y)^i\binom{g}{i+1} = (1-\tau_o)S_g,$$

where,

$$S_g = \left[\binom{g}{1} - Y\binom{g}{2} + Y^2\binom{g}{3} - Y^3\binom{g}{4} + \dots + (-Y)^{g-1}\binom{g}{g}\right]. \tag{3.58}$$

We want to find S_g. If we start with a binomial series multiplied by $(-Y)^{-1}$ we get:

$$\begin{aligned}-Y^{-1}(1-Y)^g &= -Y^{-1} + \binom{g}{1} - Y\binom{g}{2} + Y^2\binom{g}{3} + \dots + (-Y)^{g-1}\binom{g}{g} \tag{3.59}\\ &= -Y^{-1} + S_g.\end{aligned}$$

Therefore:

$$S_g = Y^{-1} - Y^{-1}(1-Y)^g \tag{3.60}$$

Bringing it all together we find that the $\frac{\left(X_{\langle 1,1\rangle}S_{\langle 1,0\rangle}-X_{\langle 1,0\rangle}S_{\langle 1,1\rangle}\right)}{R\left(S_{\langle 1,0\rangle}-S_{\langle 1,1\rangle}\right)}$ term is:

$$\sum_{i=0}^{g}(-Y)^i\left[\binom{g}{i}+\binom{g}{i+1}(1-\tau_o)\right]=(1-Y)^g+(1-\tau_o)Y^{-1}[1-(1-Y)^g]$$

Because $0<Y\leq 1$ (as we will show below),

$$\lim_{g\to\infty}(1-Y)^g+(1-\tau_o)Y^{-1}[1-(1-Y)^g]=(1-\tau_o)Y^{-1}. \tag{3.61}$$

Therefore,

$$\lim_{g\to\infty}\sum_{i=0}^{g}(-Y)^i\left[\binom{g}{i}+\binom{g}{i+1}(1-\tau_o)\right]=(1-\tau_o)Y^{-1}, \tag{3.62}$$

which is as we require from (3.55).

Subject to proving that $0<Y\leq 1$, we have shown that the $\frac{\left(X_{\langle 1,1\rangle}S_{\langle 1,0\rangle}-X_{\langle 1,0\rangle}S_{\langle 1,1\rangle}\right)}{R\left(S_{\langle 1,0\rangle}-S_{\langle 1,1\rangle}\right)}$ term in (3.53) converges to the correct quantity as $g\to\infty$.

The $\frac{\left(X_{\langle 1,0\rangle}-X_{\langle 1,1\rangle}\right)}{R\left(S_{\langle 1,0\rangle}-S_{\langle 1,1\rangle}\right)}$ **term** We can immediately recognise the first term in the square brackets of (3.56) as S_g. Therefore:

$$\sum_{i=0}^{g-1}(-Y)^i\binom{g}{i+1}=Y^{-1}-Y^{-1}(1-Y)^g \tag{3.63}$$

from (3.60).

After further time-consuming algebra we find that the second term in the square brackets

85

of (3.56) is:

$$\sum_{i=0}^{g-1}(-Y)^i\sum_{j=1}^{g-1-i}\binom{g-j}{i+1}(\tau_s-\tau_o)\tau_s^{j-1} \tag{3.64}$$

$$= (\tau_s-\tau_o)\begin{bmatrix} \binom{g-1}{1}-Y\binom{g-1}{2}+Y^2\binom{g-1}{3}-Y^3\binom{g-1}{4}+\ldots+(-Y)^{g-2}\binom{g-1}{g-1} \\ +\tau_s\left\{\binom{g-2}{1}-Y\binom{g-2}{2}+Y^2\binom{g-2}{3}+\ldots+(-Y)^{g-3}\binom{g-2}{g-2}\right\} \\ +\tau_s^2\left\{\binom{g-3}{1}-Y\binom{g-3}{2}+Y^2\binom{g-3}{3}+\ldots+(-Y)^{g-4}\binom{g-3}{g-3}\right\} \\ +\ldots \\ +\tau_s^{g-3}\left\{\binom{2}{1}-Y\binom{2}{2}\right\} \\ +\tau_s^{g-2}\left\{\binom{1}{1}\right\} \end{bmatrix}.$$

Applying the same method that we used to find (3.60) to each of the series inside the square brackets of (3.64), we find:

$$\sum_{i=0}^{g-1}(-Y)^i\sum_{j=1}^{g-1-i}\binom{g-j}{i+1}(\tau_s-\tau_o)\tau_s^{j-1} \tag{3.65}$$

$$= (\tau_s-\tau_o)\begin{bmatrix} Y^{-1}-Y^{-1}(1-Y)^{g-1} \\ +\tau_s\left\{Y^{-1}-Y^{-1}(1-Y)^{g-2}\right\} \\ +\tau_s^2\left\{Y^{-1}-Y^{-1}(1-Y)^{g-3}\right\} \\ +\ldots \\ +\tau_s^{g-3}\left\{Y^{-1}-Y^{-1}(1-Y)^2\right\} \\ +\tau_s^{g-2}\left\{Y^{-1}-Y^{-1}(1-Y)\right\} \end{bmatrix}.$$

We can re-write (3.65) as two series in the following manner:

$$\sum_{i=0}^{g-1}(-Y)^i\sum_{j=1}^{g-1-i}\binom{g-j}{i+1}(\tau_s-\tau_o)\tau_s^{j-1} \tag{3.66}$$

$$= (\tau_s-\tau_o)Y^{-1}\times$$
$$\begin{bmatrix} 1+\tau_s+\tau_s^2+\ldots+\tau_s^{g-3}+\tau_s^{g-2} \\ -\left\{(1-Y)^{g-1}+\tau_s(1-Y)^{g-2}+\tau_s^2(1-Y)^{g-3}+\ldots+\tau_s^{g-3}(1-Y)^2+\tau_s^{g-2}(1-Y)\right\} \end{bmatrix}.$$

The first series in (3.66) is a geometric one and it is straightforward to show that:

$$1 + \tau_s + \tau_s^2 + ... + \tau_s^{g-3} + \tau_s^{g-2} = \frac{\left(1 - \tau_s^{g-1}\right)}{(1 - \tau_s)}. \tag{3.67}$$

The second series in (3.66) needs a little more work:

$$Q_{g-2} = (1 - Y)^{g-1} + \tau_s (1 - Y)^{g-2} + \tau_s^2 (1 - Y)^{g-3} + ... + \tau_s^{g-3} (1 - Y)^2 + \tau_s^{g-2} (1 - Y). \tag{3.68}$$

So:

$$\frac{\tau_s}{(1 - Y)} Q_{g-2} = \tau_s (1 - Y)^{g-2} + \tau_s^2 (1 - Y)^{g-3} + \tau_s^3 (1 - Y)^{g-4} + ... + \tau_s^{g-1}. \tag{3.69}$$

Subtracting (3.69) from (3.68) gives:

$$Q_{g-2} - \frac{\tau_s}{(1 - Y)} Q_{n-2} = (1 - Y)^{g-1} - \tau_s^{g-1}$$

$$Q_{g-2} \left(1 - \frac{\tau_s}{(1 - Y)}\right) = (1 - Y)^{g-1} - \tau_s^{g-1}.$$

Therefore,

$$Q_{g-2} = \left(\frac{(1 - Y)}{(1 - Y) - \tau_s}\right) \left((1 - Y)^{g-1} - \tau_s^{g-1}\right) \tag{3.70}$$

Bringing it all together we find that the $\frac{(X_{\langle 1,0 \rangle} - X_{\langle 1,1 \rangle})}{R(S_{\langle 1,0 \rangle} - S_{\langle 1,1 \rangle})}$ term is:

$$\sum_{i=0}^{g-1} (-Y)^i \left[\binom{g}{i+1} + \sum_{j=1}^{g-1-i} \binom{g-j}{i+1} (\tau_s - \tau_o) \tau_s^{j-1} \right] \tag{3.71}$$

$$= Y^{-1} \left[\begin{array}{l} 1 - (1 - Y)^g \\ + (\tau_s - \tau_o) \left[\frac{\left(1 - \tau_s^{g-1}\right)}{(1 - \tau_s)} - \left(\frac{(1 - Y)}{(1 - Y) - \tau_s}\right) \left((1 - Y)^{g-1} - \tau_s^{g-1}\right) \right] \end{array} \right]$$

Because $0 \leq \tau_s < 1$, and as we will show below, $0 < Y \leq 1$,

$$\lim_{g \to \infty} Y^{-1} \left[\begin{array}{l} 1 - (1-Y)^g \\ + (\tau_s - \tau_o) \left[\frac{(1-\tau_s^{g-1})}{(1-\tau_s)} - \left(\frac{(1-Y)}{(1-Y)-\tau_s} \right) \left((1-Y)^{g-1} - \tau_s^{g-1} \right) \right] \end{array} \right]$$
$$= Y^{-1} \left[1 + \frac{(\tau_s - \tau_o)}{(1-\tau_s)} \right]$$
$$= Y^{-1} \left[\frac{(1-\tau_o)}{(1-\tau_s)} \right] .$$

Therefore,

$$\lim_{g \to \infty} \sum_{i=0}^{g-1} (-Y)^i \left[\binom{g}{i+1} + \sum_{j=1}^{g-1-i} \binom{g-j}{i+1} (\tau_s - \tau_o) \tau_s^{j-1} \right] = Y^{-1} \left[\frac{(1-\tau_o)}{(1-\tau_s)} \right] ,$$

which is as we require from (3.56).

Subject to showing that $0 < Y \leq 1$, we have shown that $_1 B_{\langle 0,0 \rangle}^g \to {}_1 B_{\langle 0,0 \rangle}^*$ as $g \to \infty$.

Proof that $0 < Y \leq 1$

We know that $R = \exp \left[r \frac{T}{N} \right]$, where r is the annual risk-free rate. Because $r \geq 0$, $R \geq 1$.

$$1 \leq R,$$
$$0 \leq R - 1,$$
$$-(R-1) \leq 0.$$

Since $0 \leq \tau_b < 1$:

$$-(R-1) < -\tau_b (R-1) \leq 0.$$

And since $0 \leq \tau_o < 1$:

$$
\begin{aligned}
-(R-1)-1 &< -\tau_b(R-1)-\tau_o \leq 0, \\
-R &< -\tau_b(R-1)-\tau_o \leq 0, \\
-1 &< \frac{-\tau_b(R-1)-\tau_o}{R} \leq 0, \\
0 &< 1+\left(\frac{-\tau_b(R-1)-\tau_o}{R}\right) \leq 1, \\
0 &< \frac{R-\tau_b(R-1)-\tau_o}{R} \leq 1.
\end{aligned}
$$

Therefore,

$$
0 < Y \leq 1.
$$

With the following conditions satisfied we know that the one-period algorithm will converge to the correct $X^*_{\langle 0,0 \rangle}$:

$$
\begin{aligned}
0 &\leq \tau_b < 1, \\
0 &\leq \tau_s < 1, \\
0 &\leq \tau_o < 1, \\
R &\geq 1.
\end{aligned}
$$

3.4 Summary

The following summarises the main results of this chapter:

1. In the non-recombining tree environment we have $2^{N+1} - 2$ unknowns (being the deltas and bonds at each node) and $2^{N+1} - 2$ linearly independent equations. Therefore we can find $_N\Delta^*_{\langle 0,0 \rangle}$ and $_NB^*_{\langle 0,0 \rangle}$ (and hence $_NX^*_{\langle 0,0 \rangle}$) by solving the equations simultaneously.

2. Using the simultaneous equation approach we derive a general form for $_N\Delta^*_{\langle i,j \rangle}$ (the deltas at every node) and $_NB^*_{\langle 0,0 \rangle}$. These equations are used to form the basis for the simultaneous equation algorithm, which finds the tax-adjusted hedging price of the option. This

algorithm is computationally faster and less demanding on memory, than the tax-adjusted hedging algorithm of Chapter 2. Empirically we note that the two algorithms find exactly the same price, as we would expect since $_N\Delta^*_{\langle 0,0\rangle}$ and $_NB^*_{\langle 0,0\rangle}$ are unique.

3. The simultaneous equation approach allows us to find an analytic formula for the tax-adjusted hedging price of the option. For the one-period case, two theorems are proven for the situation where $\tau_b = \tau_i$, $\tau_s = \tau_o = \tau_{cg}$, and $\tau_i > \tau_{cg}$.

4. For the one-period case we demonstrate that the tax-adjusted hedging algorithm of Chapter 2 converges to the analytic formula for the option price as the number of iterations goes to infinity:

$$_1\Delta^g_{\langle 0,0\rangle} \quad \rightarrow \quad _1\Delta^*_{\langle 0,0\rangle} \text{ as } g \rightarrow \infty,$$

$$_1B^g_{\langle 0,0\rangle} \quad \rightarrow \quad _1B^*_{\langle 0,0\rangle} \text{ as } g \rightarrow \infty,$$

and therefore,

$$_1X^g_{\langle 0,0\rangle} \quad \rightarrow \quad _1X^*_{\langle 0,0\rangle} \text{ as } g \rightarrow \infty.$$

Chapter 4

The Tax-Adjusted Option Price when the Tax Year-End Occurs During the Option's Life

4.1 Introduction

In this chapter we allow a tax-year end to occur during the life of the option. This relaxes the assumption given in Section 1.4.2, that the tax falls due at the maturity date of the option, at the final nodes on the tree. The approach we take is the same as that of Chapter 3 as we consider the problem to be one of formulating and solving a system of simultaneous equations.

In Section 4.2 we set up the framework that we are working in and formulate the system of equations that we need to work with. In Section 4.2 we also derive the general form for the delta at any node for N-periods, and the general form for the bond at the initial node. These general forms constitute the basis for the tax year-adjusted simultaneous equation algorithm of Section 4.3. In Section 4.4 we look at some option prices obtained from the algorithm and compare them to equivalent prices obtained when the tax year-end coincides with the option's maturity.

4.2 The tax year-end occurs during the life of the option

Here we allow the tax year-end to fall at any time during the life of the option. Figure 4.1 illustrates the situation in which the option's life spans a tax year-end. We have the following additional assumptions:

1. To simplify the analysis, we impose the restriction that the option has a maturity of one year or less.

2. The option is tax marked-to-market at the first tax year-end. This is explained in Section 4.2.1.

$$T = \text{maturity of option}$$
$$N = \text{number of periods until maturity}$$
$$m = \text{period when current tax year ends}$$
$$h = T/N$$

1 year to next tax year end

h

T-mh

1-(T-mh)

Current tax year end
Period m
Time mh

Option maturity
Period N
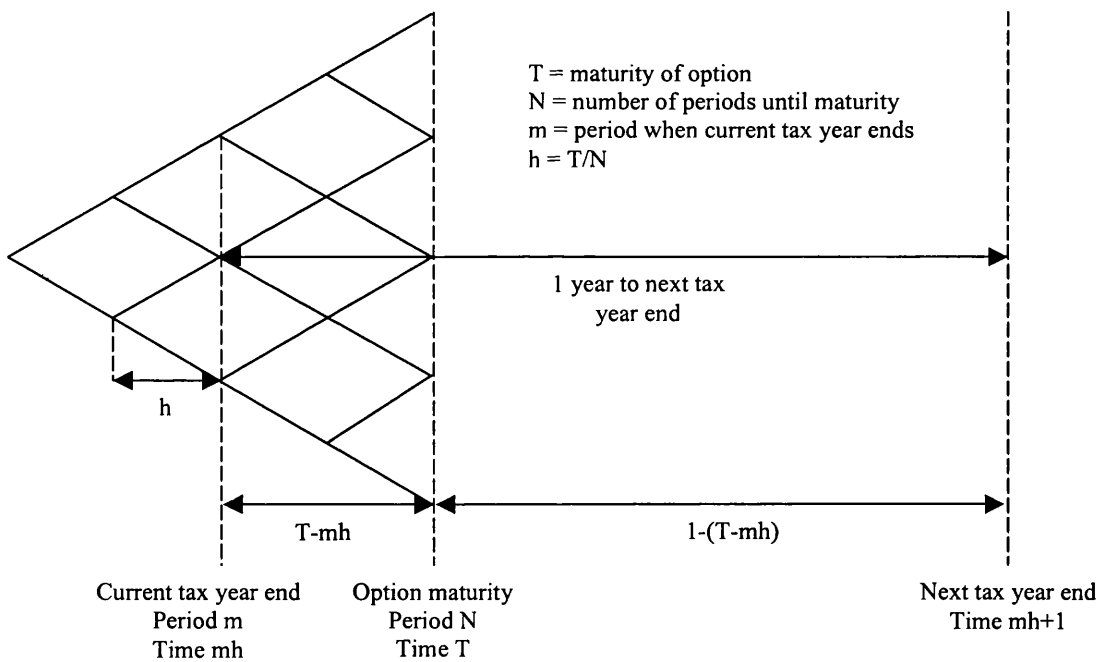Time T

Next tax year end
Time mh+1

Figure 4.1: The tax year-end occurs during the life of the option

Notes to Figure 4.1

1. Figure 4.1 shows a four-period binomial tree (a recombining tree is shown for simplicity, although we still require a non-recombining tree) with the tax year-end at period $m(=$

2), which occurs at time mh (since the first node on the tree is at time zero, and the discretisation interval is h).

2. The next tax year-end will occur one year later, at time $(mh + 1)$, and this is marked towards the right side of the figure.

3. The length of time between the maturity of the option, at time T, and the next tax year-end is given by $(1 - (T - mh))$, or equivalently by, $(1 - (N - m)h)$.

Approach to the problem

We will use the simultaneous equation approach, advocated in Chapter 3. This seems the best way to tackle the problem since we now have two tax charges to calculate, one for each tax year. Consequently we need to alter the equations, compared to the situation in Chapter 3, to take account of the tax charge that is paid (or received, if a loss) during the life of the option, at period m. Let us now formulate these equations.

4.2.1 Formulating the system of equations

We have two tax years to deal with. We will look at them separately.

The first tax year: periods 0 to m

For each of the $\langle i, j \rangle$ intermediate nodes, where $i = 1, 2, ..., m - 1$, we have equations of the form:

$$\substack{m \\ N}\Delta^*_{\langle i,j,-1 \rangle} S_{\langle i,j \rangle} + R \substack{m \\ N}B^*_{\langle i,j,-1 \rangle} = \substack{m \\ N}\Delta^*_{\langle i,j \rangle} S_{\langle i,j \rangle} + \substack{m \\ N}B^*_{\langle i,j \rangle}, \tag{4.1}$$

being the self-financing property of the hedging portfolio. The leading subscript denotes the maturity period, and the leading superscript the tax year-end period.

When we get to period m the above equation does not apply because we have to take into account the tax charge on the transactions that take place during the first tax year. We have

the following equation:

$$
\underbrace{{}^m_N\Delta^*_{\langle m,j,-1\rangle}S_{\langle m,j\rangle} - \tau_s \sum_{k=1}^m {}^m_N\Delta^*_{\langle m,j,-k\rangle}\left(S_{\langle m,j,-k+1\rangle} - S_{\langle m,j,-k\rangle}\right)}_{ATS_{<m,j>}} \qquad (4.2)
$$

$$
\underbrace{+R\,{}^m_N B^*_{\langle m,j,-1\rangle} - \tau_b\,(R-1)\left(\sum_{k=1}^m {}^m_N B^*_{\langle m,j,-k\rangle}\right)}_{ATB_{<m,j>}}
$$

$$
\underbrace{-\tau_o\left(\left({}^m_N\Delta^*_{\langle 0,0\rangle}S_{\langle 0,0\rangle} + {}^m_N B^*_{\langle 0,0\rangle}\right) - {}^{mkt}X_{\langle m,j\rangle}\right)}_{ATO_{<m,j>}}
$$

$$
= \;\; {}^m_N\Delta^*_{\langle m,j\rangle}S_{\langle m,j\rangle} + {}^m_N B^*_{\langle m,j\rangle}.
$$

The above equation is of a similar form to (3.1). Let us look at each of the elements of (4.2) in turn:

$ATS_{<m,j>}$ This is the usual after-tax stock position, this time relating to the stock transactions that take place in the first tax year.

$ATB_{<m,j>}$ This is the usual after-tax bond position, this time relating to the bond transactions that take place in the first tax year.

$ATO_{<m,j>}$ This is the tax marked-to-market option position at the end of the first tax year. The writer has a gain (equal to $(\Delta^*_{\langle 0,0\rangle}S_{\langle 0,0\rangle} + B^*_{\langle 0,0\rangle})$) from the sale of the option at time zero, and is assumed to have bought it back (nominally, for tax purposes) for ${}^{mkt}X_{\langle m,j\rangle}$, which represents the market price of the option at the tax year-end. The net position of the two, multiplied by the tax rate that applies to the option, gives the marked-to-market tax charge relating to the option.

We will discuss how we determine ${}^{mkt}X_{\langle m,j\rangle}$ in Section 4.3.1.

$\Delta^*_{\langle m,j\rangle}S_{\langle m,j\rangle} + B^*_{\langle m,j\rangle}$ This term is the value of the hedging portfolio at the start of the new tax year, and would, of course, be equal to zero if the option had expired at the end of period m.

The second tax year: periods $m + 1$ to N

For each of the $\langle i, j \rangle$ intermediate nodes, where $i = m + 1, m + 2, ..., N - 1$, we have equations of the form:

$$_N^m \Delta_{\langle i,j,-1 \rangle}^* S_{\langle i,j \rangle} + R \, _N^m B_{\langle i,j,-1 \rangle}^* = \, _N^m \Delta_{\langle i,j \rangle}^* S_{\langle i,j \rangle} + \, _N^m B_{\langle i,j \rangle}^*, \tag{4.3}$$

being the self-financing property of the hedging portfolio.

At the final nodes, at maturity, we have:

$$\underbrace{_N^m \Delta_{\langle N,j,-1 \rangle}^* S_{\langle N,j \rangle} - \lambda \tau_s \sum_{k=1}^{N-m} \, _N^m \Delta_{\langle N,j,-k \rangle}^* \left(S_{\langle N,j,-k+1 \rangle} - S_{\langle N,j,-k \rangle} \right)}_{ATS_{\langle N-m,j \rangle}} \tag{4.4}$$

$$\underbrace{+ R \, _N^m B_{\langle N,j,-1 \rangle}^* - \lambda \tau_b (R - 1) \left(\sum_{k=1}^{N-m} \, _N^m B_{\langle N,j,-k \rangle}^* \right)}_{ATB_{\langle N-m,j \rangle}}$$

$$\underbrace{- \left((1 - \lambda \tau_o) X_{\langle N,j \rangle} + \lambda \tau_o \, ^{mkt} X_{\langle N,j,-N+m \rangle} \right)}_{ATO_{\langle N-m,j \rangle}} = 0,$$

where $\lambda = \exp[-(1 - (N - m)h)r]$ and is a discount factor used to recognise the fact that the tax charges relating to the second tax year are not payable until the end of the second tax year, which occurs after the option has expired at time $(1 - (N - m)h)$ (see Figure 4.1).

In (4.4) $ATO_{\langle N-m,j \rangle}$ includes the market price of the option at period m on the path that leads to node $\langle N, j \rangle$. This time $^{mkt} X_{\langle N,j,-N+m \rangle}$ appears as a gain, whereas in (4.2) it appeared as a loss, as we require when marking-to-market.

Solving the equations

We obviously still have the same number of unknowns $(2^{N+1} - 2)$ as in Chapter 3. We also have the same number of linearly independent equations $(2^{N+1} - 2)$, although some are now of the form of (4.2). Therefore, we can solve them simultaneously to find unique $\Delta_{\langle i,j \rangle}^* s$ and $B_{\langle i,j \rangle}^* s$ and consequently a unique $X_{\langle 0,0 \rangle}^*$, the tax-adjusted hedging price of the option.

Let us now look at the two-period case, being the simplest given we want a tax year-end to

occur during the option's life.

4.2.2 The two-period case

We have six unknowns and six linearly independent equations.

Given $m = 1$, we have two equations, relating to the first tax year, of the form:

$$
\begin{aligned}
& \tfrac{1}{2}\Delta^*_{\langle 0,0\rangle} S_{\langle 1,j\rangle} - \tau_s \, \tfrac{1}{2}\Delta^*_{\langle 0,0\rangle} \left(S_{\langle 1,j\rangle} - S_{\langle 0,0\rangle} \right) + R \, \tfrac{1}{2}B^*_{\langle 0,0\rangle} - \tau_b \left(R - 1 \right) \tfrac{1}{2}B^*_{\langle 0,0\rangle} \\
& \quad - \tau_o \left(\left(\tfrac{1}{2}\Delta^*_{\langle 0,0\rangle} S_{\langle 0,0\rangle} + \tfrac{1}{2}B^*_{\langle 0,0\rangle} \right) - {}^{mkt}X_{\langle 1,j\rangle} \right) \\
& = \quad \tfrac{1}{2}\Delta^*_{\langle 1,j\rangle} S_{\langle 1,j\rangle} + \tfrac{1}{2}B^*_{\langle 1,j\rangle},
\end{aligned}
\tag{4.5}
$$

where $j = 0, 1$.

We have four equations, relating to the second tax year, of the form:

$$
\begin{aligned}
& \tfrac{1}{2}\Delta^*_{\langle 2,j,-1\rangle} S_{\langle 2,j\rangle} - \lambda \tau_s \, \tfrac{1}{2}\Delta^*_{\langle 2,j,-1\rangle} \left(S_{\langle 2,j\rangle} - S_{\langle 2,j,-1\rangle} \right) \\
& \quad + R \, \tfrac{1}{2}B^*_{\langle 2,j,-1\rangle} - \lambda \tau_b \left(R - 1 \right) \tfrac{1}{2}B^*_{\langle 2,j,-1\rangle} - \left(\left(1 - \lambda \tau_o \right) X_{\langle 2,j\rangle} - \lambda \tau_o \, {}^{mkt}X_{\langle 2,j,-1\rangle} \right) \\
& = \quad 0,
\end{aligned}
\tag{4.6}
$$

where $j = 0, 1, 2, 3$.

Solving the system of six equations simultaneously, we get:

$$
\tfrac{1}{2}\Delta^*_{\langle 1,j\rangle} = \frac{\left(1 - \lambda \tau_o \right) \left(X^*_{\langle 2,2j\rangle} - X^*_{\langle 2,2j+1\rangle} \right)}{\left(1 - \lambda \tau_s \right) \left(S_{\langle 2,2j\rangle} - S_{\langle 2,2j+1\rangle} \right)},
\tag{4.7}
$$

where $j = 0, 1$;

$$\tfrac{1}{2}\Delta^*_{\langle 0,0 \rangle} = \frac{1}{\left(S_{\langle 1,0 \rangle} - S_{\langle 1,1 \rangle}\right)\left(1 - \tau_s\right)\left[R - \lambda\tau_b(R - 1)\right]} \times \tag{4.8}$$

$$\left[
\begin{array}{l}
+ \left(1 - \lambda\tau_o\right)\left(X_{\langle 2,0 \rangle} - X_{\langle 2,2 \rangle}\right) \\[4pt]
- \left(^{mkt}X_{\langle 1,0 \rangle} - {^{mkt}}X_{\langle 1,1 \rangle}\right)\tau_o\left[R - \lambda\tau_b(R - 1) - \lambda\right] \\[4pt]
- \tfrac{1}{2}\Delta^*_{\langle 1,0 \rangle}\left[
\begin{array}{l}
S_{\langle 2,0 \rangle} - S_{\langle 1,0 \rangle}\left(R - \lambda\tau_b(R - 1)\right) \\[4pt]
-\lambda\tau_s\left(S_{\langle 2,0 \rangle} - S_{\langle 1,0 \rangle}\right)
\end{array}
\right] \\[12pt]
+ \tfrac{1}{2}\Delta^*_{\langle 1,1 \rangle}\left[
\begin{array}{l}
S_{\langle 2,2 \rangle} - S_{\langle 1,1 \rangle}\left(R - \lambda\tau_b(R - 1)\right) \\[4pt]
-\lambda\tau_s\left(S_{\langle 2,2 \rangle} - S_{\langle 1,1 \rangle}\right)
\end{array}
\right]
\end{array}
\right],$$

and,

$$\tfrac{1}{2}B^*_{\langle 0,0 \rangle} = \frac{1}{\left[R - \tau_b(R - 1) - \tau_o\right]\left[R - \lambda\tau_b(R - 1)\right]} \times \tag{4.9}$$

$$\left[
\begin{array}{l}
+ \left(1 - \lambda\tau_o\right)X_{\langle 2,0 \rangle} \\[4pt]
-\tau_o\,^{mkt}X_{\langle 1,0 \rangle}\left[R - \lambda\tau_b(R - 1) - \lambda\right] \\[4pt]
- \tfrac{1}{2}\Delta^*_{\langle 0,0 \rangle}\left[S_{\langle 1,0 \rangle} - \tau_s\left(S_{\langle 1,0 \rangle} - S_{\langle 0,0 \rangle}\right) - \tau_o S_{\langle 0,0 \rangle}\right]\left[R - \lambda\tau_b(R - 1)\right] \\[4pt]
- \tfrac{1}{2}\Delta^*_{\langle 1,0 \rangle}\left[
\begin{array}{l}
+S_{\langle 2,0 \rangle} - S_{\langle 1,0 \rangle}\left(R - \lambda\tau_b(R - 1)\right) \\[4pt]
-\lambda\tau_s\left(S_{\langle 2,0 \rangle} - S_{\langle 1,0 \rangle}\right)
\end{array}
\right]
\end{array}
\right].$$

4.2.3 Multiple periods

We can work through the algebra for two, three and four periods in an attempt to find a general N-period expression for $^m_N\Delta^*_{\langle i,j \rangle}$, representing all the deltas in the N-period environment where $i = 0, 1, ..., N - 1$, $j = 0, 1, ..., 2^i - 1$, and $0 < m < N$, and an expression for $^m_N B^*_{\langle 0,0 \rangle}$. As with Chapter 3, the algebra quickly becomes extremely involved; going to four periods means we have 30 equations to deal with.

The expressions for $^m_N B^*_{\langle i,j \rangle}$ and $^m_N\Delta^*_{\langle i,j \rangle}$ are given below. Appendix D contains the derivations.

$$
{}^{m}_{N}B^{*}_{\langle 0,0\rangle} \tag{4.10}
$$

$$
= \frac{1}{\left[R^{m} - \tau_{o} - \tau_{b}(R-1)\sum_{a=0}^{m-1}R^{a}\right]\left[R^{N-m} - \lambda\tau_{b}(R-1)\sum_{a=0}^{N-m-1}R^{a}\right]} \times
$$

$$
\left[
\begin{array}{l}
+ (1 - \lambda\tau_{o})\,X_{\langle N,0\rangle} \\[2mm]
-\tau_{o}\,{}^{mkt}X_{\langle m,0\rangle}\left[R^{N-m} - \lambda - \lambda\tau_{b}(R-1)\sum\limits_{a=0}^{N-m-1}R^{a}\right] \\[4mm]
-\,{}^{m}_{N}\Delta^{*}_{\langle 0,0\rangle}
\left[
\begin{array}{l}
-\tau_{o}S_{\langle 0,0\rangle} - \tau_{s}\left(S_{\langle 1,0\rangle} - S_{\langle 0,0\rangle}\right) \\[2mm]
+S_{\langle 1,0\rangle}\left(R^{m-1} - \tau_{b}(R-1)\sum\limits_{a=0}^{m-2}R^{a}\right)
\end{array}
\right]
\left[
\begin{array}{l}
R^{N-m} \\[2mm]
-\lambda\tau_{b}(R-1)\sum\limits_{a=0}^{N-m-1}R^{a}
\end{array}
\right] \\[10mm]
-\sum\limits_{a=1}^{m-1}\,{}^{m}_{N}\Delta^{*}_{\langle a,0\rangle}
\left[
\begin{array}{l}
-\tau_{s}\left(S_{\langle a+1,0\rangle} - S_{\langle a,0\rangle}\right) \\[2mm]
-S_{\langle a,0\rangle}\left(
\begin{array}{l}
R^{m-a} \\[2mm]
-\tau_{b}(R-1)\sum\limits_{b=0}^{m-1-a}R^{b}
\end{array}
\right) \\[6mm]
+S_{\langle a+1,0\rangle}\left(
\begin{array}{l}
R^{m-a-1} \\[2mm]
-\tau_{b}(R-1)\sum\limits_{b=0}^{m-2-a}R^{b}
\end{array}
\right)
\end{array}
\right]
\left[
\begin{array}{l}
R^{N-m} \\[2mm]
-\lambda\tau_{b}(R-1)\sum\limits_{b=0}^{N-m-1}R^{b}
\end{array}
\right] \\[16mm]
-\sum\limits_{a=m}^{N-1}\,{}^{m}_{N}\Delta^{*}_{\langle a,0\rangle}
\left[
\begin{array}{l}
-\lambda\tau_{s}\left(S_{\langle a+1,0\rangle} - S_{\langle a,0\rangle}\right) - S_{\langle a,0\rangle}\left(R^{N-a} - \lambda\tau_{b}(R-1)\sum\limits_{b=0}^{N-1-a}R^{b}\right) \\[4mm]
+S_{\langle a+1,0\rangle}\left(R^{N-a-1} - \lambda\tau_{b}(R-1)\sum\limits_{b=0}^{N-2-a}R^{b}\right)
\end{array}
\right]
\end{array}
\right]
$$

$$_N^m \Delta^*_{\langle i,j \rangle} \tag{4.11}$$

$$
= \frac{1}{\beta \left(S_{\langle i+1,2j \rangle} - S_{\langle i+1,2j+1 \rangle} \right)} \times
$$

$$
\left[
\begin{array}{l}
+(1 - \lambda \tau_o) \left(X_{\langle N, j2^{N-i} \rangle} - X_{\langle N, (2j+1)2^{N-i-1} \rangle} \right) \\[2mm]
-I_{i<m} \left({}^{mkt}X_{\langle m, j2^{m-i} \rangle} - {}^{mkt}X_{\langle m, (2j+1)2^{m-i-1} \rangle} \right) \tau_o \left[
\begin{array}{l}
R^{N-m} - \lambda \\[1mm]
-\lambda \tau_b (R-1) \sum\limits_{a=0}^{N-m-1} R^a
\end{array}
\right] \\[5mm]
-\sum\limits_{a=i+1}^{N-1} {}_N^m\Delta^*_{\langle a, j2^{a-i} \rangle} \times
\left[
\begin{array}{l}
I_{a<m} \left[
\begin{array}{l}
+S_{\langle a+1, j2^{a+1-i} \rangle} \left(R^{m-a-1} - \tau_b(R-1) \sum\limits_{b=0}^{m-2-a} R^b \right) \\[2mm]
-S_{\langle a, j2^{a-i} \rangle} \left(R^{m-a} - \tau_b(R-1) \sum\limits_{b=0}^{m-1-a} R^b \right) \\[2mm]
-\tau_s \left(S_{\langle a+1, j2^{a+1-i} \rangle} - S_{\langle a, j2^{a-i} \rangle} \right)
\end{array}
\right] \\[8mm]
\left[
\begin{array}{l}
R^{N-m} \\[1mm]
-\lambda \tau_b(R-1) \sum_{b=0}^{N-m-1} R^b
\end{array}
\right] \\[4mm]
I_{a \geq m} \left[
\begin{array}{l}
+S_{\langle a+1, j2^{a+1-i} \rangle} \left(R^{N-a-1} - \lambda \tau_b(R-1) \sum\limits_{b=0}^{N-2-a} R^b \right) \\[2mm]
-S_{\langle a, j2^{a-i} \rangle} \left(R^{N-a} - \lambda \tau_b(R-1) \sum\limits_{b=0}^{N-1-a} R^b \right) \\[2mm]
-\lambda \tau_s \left(S_{\langle a+1, j2^{a+1-i} \rangle} - S_{\langle a, j2^{a-i} \rangle} \right)
\end{array}
\right]
\end{array}
\right] \\[10mm]
+\sum\limits_{a=i+1}^{N-1} {}_N^m\Delta^*_{\langle a, (2j+1)2^{a-i-1} \rangle} \times
\left[
\begin{array}{l}
I_{a<m} \left[
\begin{array}{l}
+S_{\langle a+1, (2j+1)2^{a-i} \rangle} \left(
\begin{array}{l}
R^{m-a-1} \\
-\tau_b(R-1) \sum\limits_{b=0}^{m-2-a} R^b
\end{array}
\right) \\[3mm]
-S_{\langle a, (2j+1)2^{a-i-1} \rangle} \left(
\begin{array}{l}
R^{m-a} \\
-\tau_b(R-1) \sum\limits_{b=0}^{m-1-a} R^b
\end{array}
\right) \\[3mm]
-\tau_s \left(S_{\langle a+1, (2j+1)2^{a-i} \rangle} - S_{\langle a, (2j+1)2^{a-i-1} \rangle} \right)
\end{array}
\right] \\[8mm]
\left[
\begin{array}{l}
R^{N-m} \\[1mm]
-\lambda \tau_b(R-1) \sum\limits_{b=0}^{N-m-1} R^b
\end{array}
\right] \\[4mm]
I_{a \geq m} \left[
\begin{array}{l}
+S_{\langle a+1, (2j+1)2^{a-i} \rangle} \left(
\begin{array}{l}
R^{N-a-1} \\
-\lambda \tau_b(R-1) \sum\limits_{b=0}^{N-2-a} R^b
\end{array}
\right) \\[3mm]
-S_{\langle a, (2j+1)2^{a-i-1} \rangle} \left(
\begin{array}{l}
R^{N-a} \\
-\lambda \tau_b(R-1) \sum\limits_{b=0}^{N-1-a} R^b
\end{array}
\right) \\[3mm]
-\lambda \tau_s \left(S_{\langle a+1, (2j+1)2^{a-i} \rangle} - S_{\langle a, (2j+1)2^{a-i-1} \rangle} \right)
\end{array}
\right]
\end{array}
\right]
\end{array}
\right]
$$

where:

if $i < m$,

$$\beta = \left[\begin{array}{c} R^{N-m} \\ -\lambda \tau_b (R-1) \sum_{a=0}^{N-m-1} R^a \end{array} \right] \left[\begin{array}{c} R^{m-1-i} - \tau_s \\ -\tau_b (R-1) \sum_{a=0}^{m-2-i} R^a \end{array} \right]; \qquad (4.12)$$

if $i \geq m$,

$$\beta = \left[R^{N-i-1} - \lambda \tau_s - \lambda \tau_b (R-1) \sum_{a=0}^{N-i-2} R^a \right]; \qquad (4.13)$$

and,

$$I_{a<m} = \left\{ \begin{array}{l} 1 \text{ if } a < m \\ 0 \text{ otherwise} \end{array} \right\}$$

etc.

We can see that (4.10) and (4.11) are similar to (3.6) and (3.5). The main driving factor behind the additional terms we have in (4.10) and (4.11) is the relationship between the quantities we are calculating and the tax year-end. For example, if we are calculating the deltas at period-i we need to know whether i is greater than or equal to, or less than, m because this determines the delta coefficient, β, as given by (4.12) and (4.13).

4.3 The tax year-adjusted simultaneous equation algorithm

We can use (4.10) and (4.11) to form an algorithm that finds the tax-adjusted price of an option, when a tax year-end occurs during the life of the option. Before we look at the algorithm we need to think about what to use as a proxy for the market prices of the option at the tax year-end, since we obviously do not know these when the option is written.

4.3.1 Determining the market prices of the option

In Section 2.2 we discussed what we meant by the tax-adjusted hedging price and the tax-adjusted no-arbitrage price of the option. If we are finding the tax-adjusted hedging price of

the option then we are not arguing that the market prices of options follow their tax-adjusted prices. In other words, the market prices of options are constrained by no-arbitrage arguments that ignore taxation, rather than those that are tax-adjusted. In this case we will assume that the market prices of options are equal to their CRR prices, since these are defined using no-arbitrage arguments that ignore taxation.

If we are finding the tax-adjusted no-arbitrage price of the option, the market prices of the option at the tax year-end must, by tax-adjusted no-arbitrage arguments, be equal to their tax-adjusted no-arbitrage values.

Sub-algorithm: setting the market prices of the option at the tax year-end to the CRR prices

1. Set $j = 0$.

2. Set $^{mkt}X^{CRR}_{\langle m,j \rangle}$ equal to the CRR price with the following parameters:

   (a) Number of periods, $\bar{N} = N - m$.

   (b) Time to maturity, $\bar{T} = T - m\frac{T}{N}$.

   (c) Initial stock price, $\bar{S}_{\langle 0,0 \rangle} = S_{\langle m,j \rangle}$.

   (d) All other parameters are the same as for the tax-adjusted option we are pricing.

3. If $j < 2^m - 1$ then $j = j + 1$. Go back to Step 2.

4. End.

Sub-algorithm: setting the market prices of the option at the tax year-end to the tax-adjusted prices

Note, in this algorithm the tax rates are adjusted by the discount factor, λ. This is because the maturity date of the option in the second tax year does not coincide with the end of that tax year. Consequently, we need to discount the tax charge to recognise the fact that the tax is not payable until some time after the option matures (see Figure 4.1).

1. Set $j = 0$.

2. Set $^{mkt}X^{tax}_{\langle m,j\rangle}$ equal to the tax-adjusted option price (using the simultaneous equation algorithm of Chapter 3) with the following parameters:

   (a) Number of periods, $\bar{N} = N - m$.

   (b) Time to maturity, $\bar{T} = T - m\frac{T}{N}$.

   (c) Initial stock price, $\bar{S}_{\langle 0,0\rangle} = S_{\langle m,j\rangle}$.

   (d) The tax rates, $\bar{\tau}_b = \lambda\tau_b$, $\bar{\tau}_s = \lambda\tau_s$ and $\bar{\tau}_o = \lambda\tau_o$.

   (e) All other parameters are the same as for the tax-adjusted option we are pricing.

3. If $j < 2^m - 1$ then $j = j + 1$. Go back to Step 2.

4. End.

The tax year-adjusted simultaneous equation algorithm

As with the simultaneous equation algorithm of Chapter 3, the overall idea is to start at the $N - 1$ nodes and move back through the tree calculating the deltas as we go. Once we have the deltas we can find $^m_N B^*_{\langle 0,0\rangle}$, and hence $^m_N X^*_{\langle 0,0\rangle}$.

1. Set $^{mkt}X_{\langle m,j\rangle}$ equal to either $^{mkt}X^{tax}_{\langle m,j\rangle}$ or $^{mkt}X^{CRR}_{\langle m,j\rangle}$ depending on whether we are finding the tax-adjusted no-arbitrage price, or the tax-adjusted hedging price of the option. We use the relevant sub-routine given above to achieve this.

2. Set $i = N - 1$.

3. If $i < m$, $I_{i<m} = 1$ otherwise $I_{i<m} = 0$.

4. Set $j = 0$.

5. Using Equation (4.11), compute $^m_N\Delta^*_{\langle i,j\rangle}$ and store in an array. This requires a sub-routine:

   (a) Set $a = i + 1$.

   (b) If $a < m$, $I_{a<m} = 1$ and $I_{a\geq m} = 0$; otherwise $I_{i<m} = 0$ and $I_{a\geq m} = 1$.

   (c) Set $sum\_upper\_delta = sum\_lower\_delta = 0$.

(d) Form the two coefficients, *upper_coefficient* and *lower_coefficient*, to ${}^m_N\Delta^*_{\langle a,j2^{a-i}\rangle}$ and ${}^m_N\Delta^*_{\langle a,(2j+1)2^{a-i-1}\rangle}$ respectively (given we know $I_{a<m}$ and $I_{a\geq m}$).

(e) Set:

  i. $sum\_upper\_delta = sum\_upper\_delta - {}^m_N\Delta^*_{\langle a,j2^{a-i}\rangle} \times upper\_coefficient$;

  ii. $sum\_lower\_delta = sum\_lower\_delta + {}^m_N\Delta^*_{\langle a,(2j+1)2^{a-i-1}\rangle} \times lower\_coefficient$.

(f) If $a < N - 1$,

  i. then, $a = a + 1$; return to Step 5b;

  ii. else, go to Step 5g.

(g) Compute ${}^m_N\Delta^*_{\langle i,j\rangle}$ and store in an array. We can calculate ${}^m_N\Delta^*_{\langle i,j\rangle}$ here because we have found sum_upper_delta (which equals the $-\sum_{a=i+1}^{N-1} {}^m_N\Delta^*_{\langle a,j2^{a-i}\rangle}$ term in (4.11)) and sum_lower_delta (which equals the $+\sum_{a=i+1}^{N-1} {}^m_N\Delta^*_{\langle a,(2j+1)2^{a-i-1}\rangle}$ term in (4.11)), and all the other terms are known or are straightforward to calculate. (We know $I_{i<m}$ and so we know which version of β to use - either (4.12) or (4.13).)

6. If $j < 2^i - 1$,

  (a) then, $j = j + 1$; return to Step 5;

  (b) else, go to Step 7.

7. If $i > 0$,

  (a) then, $i = i - 1$; return to Step 3.

  (b) else, go to Step 8.

8. Using Equation (4.10), compute ${}^m_N B^*_{\langle 0,0\rangle}$. This requires a sub-routine:

  (a) Set $a = 1$.

  (b) Set $sum\_delta = 0$.

  (c) Form the coefficient (*coefficient*) to ${}^m_N\Delta^*_{\langle a,0\rangle}$:

    i. If $a < m$, then use the $-\sum_{a=1}^{m-1} {}^m_N\Delta^*_{\langle a,0\rangle}$ sum in (4.10) to form the coefficient;

    ii. else, use the $-\sum_{a=m}^{N-1} {}^m_N\Delta^*_{\langle a,0\rangle}$ sum in (4.10) to form the coefficient.

(d) Set $sum\_delta = sum\_delta - {}_N^m\Delta^*_{\langle a,j2^{a-i}\rangle} \times coefficient.$

(e) If $a < N - 1$,

    i. then, $a = a + 1$; return to Step 8c;

    ii. else, go to Step 8f.

(f) Compute ${}_N^m B^*_{\langle 0,0\rangle}$, given that we have found sum_delta which equals the ${}_N^m\Delta^*_{\langle a,0\rangle}$ term in (4.10), and all the other terms are known or are straightforward to calculate.

9. Compute ${}_N^m X^*_{\langle 0,0\rangle} = {}_N^m\Delta^*_{\langle 0,0\rangle}S_{\langle 0,0\rangle} + {}_N^m B^*_{\langle 0,0\rangle}.$

Memory requirements and speed of computation

The tax year-adjusted simultaneous equation algorithm has been implemented in C++ using a 633Mhz, 512MB Pentium III PC. The algorithm is more demanding on memory than the simultaneous equation algorithm of Chapter 3, since we have to store the values for the market price of the option at the tax year end m, and the size of this array increases approximately as 2^m. In fact, from Table 4.1 we can see that the memory requirements vary depending on whether we are using ${}^{mkt}X^{tax}_{\langle m,j\rangle}$ or ${}^{mkt}X^{CRR}_{\langle m,j\rangle}$ for the market prices of the option. The tax-adjusted market prices require more memory because we have to use the simultaneous equation algorithm of Chapter 3 and this algorithm is more demanding on memory than a CRR procedure (whose memory requirements grow linearly with the number of periods).

The speed of computation slows with an increase in the number of periods because the number of deltas to be computed increases as 2^N. Table 4.1 shows that if we are using tax-adjusted market prices for the option, the speed of computation is slower than for CRR market prices. Again, this is due to the fact that we have to use the simultaneous equation algorithm of Chapter 3 to calculate the tax-adjusted market prices, and this algorithm is much slower than a CRR procedure.

The parameters used to compile Table 4.1 are: $S = K = 100, r = 0.05, \sigma = 0.25, T = 1, \tau_b = 0.4, \tau_s = 0.3, \tau_o = 0.2.$

Periods, N	Option marked to CRR price (m=N-2)	Option marked to tax-adjusted price (m=N-2)
$3 - 14$	< 1	< 1
15	< 1	1
16	1	2
17	2	4
18	4	6
19	6	10
20	15	21
21	30	45
22	60	135
23	120	Memory requirements too great
24	300	Memory requirements too great
≥ 25	Memory requirements too great	Memory requirements too great

Table 4.1: CPU time (s) taken for the algorithm to find the tax-adjusted price

4.4 Results

The results are presented in Tables 4.2 and 4.3. The three cases examined in Section 2.4.1 are looked at again, this time using the tax year-adjusted simultaneous equation algorithm. The percentage differences between the prices obtained here and those obtained in Chapter 2 (where the tax year-end coincides with the option's maturity) are given for comparative purposes.

Tax rates			Call mkd-to-mkt with CRR price		Call mkd-to-mkt with tax-adjusted price	
τ_b	τ_s	τ_o	Price	% diff Table2.2	Price	% diff Table2.2
0	0	0	12.4876*	0.00	12.4876*	0.00
0.25	0	0	11.8834	+0.01	11.8834	+0.01
0.4	0	0	11.5258	+0.03	11.5258	+0.03
0.25	0.25	0.25	12.4876	0.00	12.4876	0.00
0.4	0.25	0.25	12.0009	-0.01	12.005	+0.02
0.4	0.4	0.4	12.4876	0.00	12.4876	0.00
0	0	0.25	12.2854	-0.02	12.2878	0.00
0	0	0.4	12.0851	-0.08	12.0943	0.00
0.25	0.25	0	12.6413	-0.003	12.6413	-0.003
0.25	0.25	0.4	12.3348	-0.02	12.3377	+0.005
0.4	0.4	0	12.735	-0.01	12.735	-0.01
0.4	0.4	0.25	12.611	+0.003	12.6099	-0.006
0.25	0	0.25	11.7316	-0.05	11.739	+0.02
0.4	0	0.4	11.2818	-0.14	11.3018	+0.04
0.4	0.25	0.4	11.8784	-0.06	11.8885	+0.03

Table 4.2: Tax-adjusted call option prices where the option's life spans a tax year-end
*Indicates CRR price

Tax rates			Put mkd-to-mkt with CRR price		Put mkd-to-mkt with tax-adjusted price	
τ_b	τ_s	τ_o	Price	% diff Table2.3	Price	% diff Table2.3
0	0	0	7.6105*	0.00	7.6105*	0.00
0.25	0	0	8.17762	-0.02	8.17762	-0.02
0.4	0	0	8.53392	-0.04	8.53392	-0.04
0.25	0.25	0.25	7.6105	0.0	7.6105	0.0
0.4	0.25	0.25	8.06393	+0.02	8.06008	-0.03
0.4	0.4	0.4	7.6105	0.0	7.6105	0.0
0	0	0.25	7.48732	-0.02	7.48876	0.00
0	0	0.4	7.36522	-0.08	7.37085	0.00
0.25	0.25	0	7.70421	-0.003	7.70421	-0.003
0.25	0.25	0.4	7.51741	-0.02	7.51919	+0.006
0.4	0.4	0	7.76132	-0.01	7.76132	-0.01
0.4	0.4	0.25	7.68571	+0.003	7.68508	-0.005
0.25	0	0.25	8.08275	+0.04	8.07816	-0.01
0.4	0	0.4	8.38109	+0.12	8.36808	-0.04
0.4	0.25	0.4	7.98821	+0.06	7.98186	-0.02

Table 4.3: Tax-adjusted put option prices where the option's life spans a tax year-end
*Indicates CRR price

Notes to Tables 4.2 and 4.3

1. The following parameters were used: $N = 15, m = 4, S = 100, K = 100, r = 0.05, \sigma = 0.25, T = 1$.

2. The top six rows contain the results relating to Case 2.1.

3. The following six rows contain the results relating to Case 2.2.

4. The bottom three rows contain the results relating to Case 2.3.

5. Where duplication of results would occur in two or more of the three cases, only the first

instance is included.

Key results contained in Tables 4.2 and 4.3

Option prices produced using the tax year-adjusted simultaneous equation algorithm show the following:

1. The price differences with respect to option prices in Tables 2.2 and 2.3 (obtained from the tax-adjusted hedging algorithm where the tax year-end coincides with the option's maturity) are very small, being of the order of 0.01 percent.

2. When $\tau_b = \tau_s = \tau_o$, $\frac{m}{N}X^*_{\langle 0,0 \rangle} = X^{CRR}_{\langle 0,0 \rangle}$.

3. Put and call prices, when tax rates are not all equal, move in opposite directions when compared to option prices in Tables 2.2 and 2.3, except for those in Case 2.2 where the price differences are the same whether the option is a put or a call. For example, if the call price is below its value in Table 2.2 then the corresponding put price is above its value in 2.3.

4.5 Summary

The following summarises the main results of this chapter:

1. We relax the assumption that the tax year-end coincides with the maturity of the option and allow a tax year end to occur at period m, during the life of the option (we impose the restriction that the option's life is less than or equal to one year).

2. We have to consider two tax years: the first, which ends at the end of period m during the life of the option, and the second, which ends one year later, after the option has expired. Therefore we have two dates when the tax is due and we alter the equations accordingly:

   (a) At first tax year-end when we move from the $\langle m, j, -1 \rangle$ nodes to the $\langle m, j \rangle$ nodes we pay the tax liability due on the transactions that took place in the first tax year and form the hedging portfolio at the start of the second tax year;

(b) At the second tax year-end we pay the tax liability due on the transactions that took place in the second tax year. Given the second tax year-end occurs some time after the option has expired, we discount these cash flows back to the expiry date of the option.

3. We assume that the option is tax marked-to-marked at the first tax year-end. This requires us to have a proxy for the market prices of the option at this tax year-end. There are two choices:

   (a) The CRR prices, if options are valued using no-arbitrage arguments that ignore taxation. In this case we are finding the tax-adjusted hedging price of the option.

   (b) The tax-adjusted hedging prices, if options are valued using tax-adjusted no-arbitrage arguments. In this case we are finding the tax-adjusted no-arbitrage price of the option.

4. We have the same number of unknowns as linearly independent equations $(2^{N+1} - 2)$ as in Chapter 3, and so we can find ${}_N^m\Delta^*_{\langle 0,0\rangle}$ and ${}_N^m B^*_{\langle 0,0\rangle}$ (and hence ${}_N^m X^*_{\langle 0,0\rangle}$) by solving the equations simultaneously.

5. Using the simultaneous equation approach we can derive a general form for ${}_N^m\Delta^*_{\langle i,j\rangle}$ (the deltas at every node) and ${}_N B^*_{\langle 0,0\rangle}$. These equations are used to form the basis for the tax year-adjusted simultaneous equation algorithm, which finds the tax-adjusted hedging price of the option when a tax year-end occurs at period m. This algorithm is computationally slower and more demanding on memory than the simultaneous equation algorithm of Chapter 3, because we have to find the market prices of the option at the tax year-end.

6. We see that the option prices we obtained using the tax year-adjusted simultaneous equation algorithm show a difference, of the order of 0.01 percent, with respect to corresponding prices obtained using the tax-adjusted hedging algorithm of Chapter 2 (or equivalently, the simultaneous equation algorithm of Chapter 3).

Chapter 5

The Tax-Adjusted Binomial Option Pricing Model and Convergence to the Tax-Adjusted Black-Scholes Equation

5.1 Introduction

In Chapter 1 we looked at an extension to the Black-Scholes PDE, derived in Scholes (1976), which incorporates tax into the model. The resulting PDE was termed the tax-adjusted Black-Scholes (taBS) equation, and its derivation is contained in Appendix A. We know that the original binomial option pricing model converges to the Black-Scholes model as a special limiting case (Cox (1979)). In this chapter we will look at an analogous discrete-time option pricing model with taxes (Section 5.2), which converges to the taBS model (Section 5.3), again as a special limiting case. This tax-adjusted model we will call the "tax-adjusted binomial option pricing model" (taBOPM).

Section 5.4 presents an alternative to taBS, the "generalised tax-adjusted Black-Scholes" (gtaBS) equation, which does not assume the derivative is taxed as income, as is the case in taBS.

Throughout this chapter we are looking at the "tax-adjusted no-arbitrage price" of an option, and so we make the assumption that all market participants have the same tax position (i.e., are subject to the same tax rates, have the same ability to use losses to offset other gains, and their tax is due for payment on the same date).

5.2 The tax-adjusted binomial option pricing formula

This section uses the same framework as CRR to develop a tax-adjusted binomial option pricing formula for a European call or put. This means we are working in a recombining binomial tree environment, where there is a linear relationship between the number of periods and the memory requirements.

5.2.1 The single-period setting

Assumptions

We assume that the stock price follows a binomial process over discrete periods. The rate of return of the stock over each period can have two possible values: $u - 1$ with probability q, or $d - 1$ with probability $1 - q$. Thus, if the current stock price is S, the stock price at the end of the next period will be either uS or dS. We evolve the stock price on a pre-tax basis since the option payoff is determined pre-tax.

The other assumptions are listed below:

- The interest rate is constant;

- Individuals may borrow or lend as much as they wish at the interest rate;

- There are no transaction costs, or margin requirements;

- The stock pays no dividends;

- The option writer has other gains that can be offset with losses;

- The tax is due at the end of the period.

We require that there be no arbitrage opportunities involving the stock and riskless borrowing or lending, on an after-tax basis. Therefore, we need to define the tax-adjusted total returns.

The tax-adjusted total returns

The total return for an up or down movement in the stock price can be adjusted to take into account the tax on the transaction. Since stock transactions are associated with capital gains, the relevant tax rate is τ_{cg}, the capital gains tax rate.

Definition 5.1: The tax-adjusted total return for an 'up' movement in the stock price

$$u^* = u - (u - 1)\tau_{cg}.$$ (5.1)

Definition 5.2: The tax-adjusted total return for a 'down' movement in the stock price

$$d^* = d - (d - 1)\tau_{cg}.$$ (5.2)

The total gain (loss) from investing (borrowing) at the risk-free rate will be taxed as income, and so the relevant tax rate is τ_i, the income tax rate.

Definition 5.3: The tax-adjusted total risk-free return

$$R^* = R - (R - 1)\tau_i,$$ (5.3)

where R is one plus the risk-free interest rate over one period.

The no-arbitrage condition

We can now apply the no-arbitrage condition that must exist at every period on the tree on an after-tax basis, since we wish to value options on an after-tax basis. This is similar to the

classical CRR case, but now we are using tax-adjusted total returns:

$$u^* > R^* > d^*. \tag{5.4}$$

The option

If the stock price moves up, the option price at the end of the period is given by X_u ($X_u = \max[Su - K, 0]$ for a call and $X_u = \max[K - Su, 0]$ for a put), and if the stock moves down the option price at the end of the period is given by X_d (where d replaces u in the payoffs for the up state). We need to work out the after-tax position.

Definition 5.4: The tax-adjusted option position for an 'up' move in the stock

$$X_u^* = X_u - \tau_o \left(X_u - X^* \right), \tag{5.5}$$

since the profit is $X_u - X$ (the buyer has received X_u after paying X^*, the initial price of the option). The tax rate that applies to the option is τ_o.

Definition 5.5: The tax-adjusted option position for an 'down' move in the stock

$$X_d^* = X_d - \tau_o \left(X_d - X^* \right), \tag{5.6}$$

since the profit is $X_d - X$ (the buyer has received X_d after paying X^*).

The hedging portfolio

We form a hedging portfolio at beginning of the period given by Δ^* of stock and B^* of bonds such that,

$$\Delta^* u^* S + R^* B^* = X_u^*, \tag{5.7}$$

113

and,

$$\Delta^* d^* S + R^* B^* = X_d^*. \tag{5.8}$$

Solving for Δ^* and B^* in (5.7) and (5.8), we find:

$$\Delta^* = \frac{X_u^* - X_d^*}{(u^* - d^*)S} \tag{5.9}$$

and,

$$B^* = \frac{u^* X_d^* - d^* X_u^*}{(u^* - d^*)R^*} \tag{5.10}$$

If there are to be no arbitrage opportunities, it must be true that,

$$
\begin{aligned}
X^* &= \Delta^* S + B^* \\
&= \frac{X_u^* - X_d^*}{(u^* - d^*)} + \frac{u^* X_d^* - d^* X_u^*}{(u^* - d^*)R^*},
\end{aligned}
\tag{5.11}
$$

where X^* is the current tax-adjusted price of the option.

The tax-adjusted risk-neutral probabilities

We can re-write (5.11) in the following way:

$$X^* = \frac{1}{R^*} \left[\pi^* X_u^* + (1 - \pi^*) X_d^* \right], \tag{5.12}$$

where,

$$\pi^* = \frac{(R^* - d^*)}{(u^* - d^*)}. \tag{5.13}$$

Now π^* is always greater than zero and less than one, so it has the properties of a probability. It is the value that q, the real probability of an up move, would have if investors were risk-neutral on an after-tax basis. In an after-tax risk-neutral world the after-tax expected rate of

return on the stock would be the after-tax riskless interest rate. Thus:

$$q(u^* S) + (1 - q)(d^* S) = R^* S, \tag{5.14}$$

and,

$$q = \frac{(R^* - d^*)}{(u^* - d^*)} = \pi^*.$$

Therefore π^* is the tax-adjusted risk-neutral probability.

We can derive π^* using an alternative method. Given the no-arbitrage condition in (5.4), there exists a strictly positive number, π^*, that takes a value between zero and one such that,

$$\pi^* u^* + (1 - \pi^*) d^* = R^*. \tag{5.15}$$

Solving for π^* we get (5.13)

Substituting (5.1), (5.2) and (5.3) into (5.13) we obtain the tax-adjusted risk-neutral probability for an up move in the stock price:

$$\pi^* = \frac{R - d - (R - 1)\tau_i + (d - 1)\tau_{cg}}{u - d - (u - 1)\tau_{cg} + (d - 1)\tau_{cg}}, \tag{5.16}$$

and for a down move:

$$1 - \pi^* = \frac{u - R + (R - 1)\tau_i - (u - 1)\tau_{cg}}{u - d - (u - 1)\tau_{cg} + (d - 1)\tau_{cg}}. \tag{5.17}$$

We see that (5.16) and (5.17) collapse to their no-tax counterparts when the tax rates are set to zero. We can also see that if τ_{cg} is set equal to τ_i, (5.16) and (5.17) again collapse to their no-tax counterparts.

Risk-neutral valuation when $\tau_o = \tau_i$

If we substitute (5.5) and (5.6) into (5.12) we get:

$$
\begin{aligned}
X^* &= \frac{1}{R^*} \left[\pi^* \left(X_u - \tau_o \left(X_u - X^* \right) \right) + (1 - \pi^*) \left(X_d - \tau_o \left(X_d - X^* \right) \right) \right] \qquad (5.18) \\
&= \frac{1}{R^*} \left[\pi^* \left(1 - \tau_o \right) X_u + (1 - \pi^*) \left(1 - \tau_o \right) X_d + \tau_o X^* \right] .
\end{aligned}
$$

Rearranging we find:

$$
X^* \left(\frac{R^* - \tau_o}{R^*} \right) = \frac{(1 - \tau_o)}{R^*} \left[\pi^* X_u + (1 - \pi^*) X_d \right] . \qquad (5.19)
$$

Therefore:

$$
X^* = \left(\frac{(1 - \tau_o)}{R^* - \tau_o} \right) \left[\pi^* X_u + (1 - \pi^*) X_d \right] . \qquad (5.20)
$$

If we substitute for R^* using (5.3) and assume that the option is taxed as income (which is the assumption made in Scholes (1976)) so $\tau_o = \tau_i$, we get:

$$
X^* = \frac{1}{R} \left[\pi^* X_u + (1 - \pi^*) X_d \right] . \qquad (5.21)
$$

The above equation uses the pre-tax discount factor, $1/R$, and the pre-tax option values, X_u and X_d.

5.2.2 The multiperiod setting

The single-period analysis can easily be extended to a multiperiod setting, as is done in Cox (1979). We assume that the tax is paid at the end of every period, and the option is taxed as income.

The tax-adjusted binomial option pricing formula

The tax-adjusted binomial option pricing formula for a call option is given by:

$$
c^{taBOPM} = S \Phi[a; n, \pi^{*\prime}] - K R^{-n} \Phi[a; n, \pi^*] \qquad (5.22)
$$

where,

$$\pi^* = \frac{R - d - (R - 1)\tau_i + (d - 1)\tau_{cg}}{u - d - (u - 1)\tau_{cg} + (d - 1)\tau_{cg}}$$

$$\pi^{*\prime} = \pi^*(u/R)$$

a = the smallest non-negative integer greater than $\ln\left(\dfrac{K}{Sd^n}\right) / \ln\left(\dfrac{u}{d}\right)$

c = 0 if $a > n$

n = the number of periods on the tree

$\Phi[a; n, \pi^*]$ = the complementary binomial distribution function.

See Cox (1979) for the derivation of the classical precursor to Equation (5.22), which is the same as (5.22) except that π^* is replaced with π, the classical risk-neutral probability of an up move in the stock price.

5.2.3 The effect of taxes on option prices

Theorem 5.1

Calls (puts) are valued by taBOPM at a level less (greater) than their CRR counterparts if the income tax rate is greater than the capital gains tax rate.

More formally,

$$c^{taBOPM} < c^{CRR},$$

$$p^{taBOPM} > p^{CRR},$$

$$\text{if } \tau_i > \tau_{cg}.$$

Proof

Let us look at the effect of taxes on the tax-adjusted risk-neutral probability.

We can write (5.16) in the following way:

$$
\begin{aligned}
\pi^* &= \frac{R - d - (R-1)\tau_i + (d-1)\tau_{cg}}{u - d - (u-1)\tau_{cg} + (d-1)\tau_{cg}} \\
&= \frac{R - d - (R-1)\tau_i + (d-1)\tau_{cg}}{(u-d)\left(1 - \tau_{cg}\right)}.
\end{aligned}
$$

Dividing by $(R-d)$ gives:

$$
\frac{\pi^*\left(u-d\right)}{\left(R-d\right)} = \frac{R - d - (R-1)\tau_i + (d-1)\tau_{cg}}{\left(R-d\right)\left(1 - \tau_{cg}\right)}.
$$

Substituting in π and adding zero to the right side gives:

$$
\begin{aligned}
\frac{\pi^*}{\pi} &= \frac{R\left(1-\tau_i\right) - d\left(1-\tau_{cg}\right) + \left(\tau_i - \tau_{cg}\right)}{\left(R-d\right)\left(1-\tau_{cg}\right)} + \frac{R\left(1-\tau_{cg}\right) - R\left(1-\tau_{cg}\right)}{\left(R-d\right)\left(1-\tau_{cg}\right)} \\
&= 1 + \frac{R\left(1-\tau_i\right) + \left(\tau_i - \tau_{cg}\right) - R\left(1-\tau_{cg}\right)}{\left(R-d\right)\left(1-\tau_{cg}\right)} \\
&= 1 + \frac{\left(1-R\right)\left(\tau_i - \tau_{cg}\right)}{\left(R-d\right)\left(1-\tau_{cg}\right)}
\end{aligned}
$$

$\Rightarrow$

$$
\frac{\pi^*}{\pi} - 1 = \frac{\left(1-R\right)\left(\tau_i - \tau_{cg}\right)}{\left(R-d\right)\left(1-\tau_{cg}\right)} \tag{5.23}
$$

Let us look at the terms on the right side of the above equation.

Numerator

$$
\begin{aligned}
\left(1-R\right) &< 0, \\
\left(\tau_i - \tau_{cg}\right) &> 0, \text{ if } \tau_i > \tau_{cg}.
\end{aligned}
$$

$\Rightarrow$

$$
\left(1-R\right)\left(\tau_i - \tau_{cg}\right) < 0. \tag{5.24}
$$

Denominator

$$(R - d) > 0,$$

$$(1 - \tau_{cg}) > 0.$$

$$\Rightarrow$$

$$(R - d)(1 - \tau_{cg}) > 0. \tag{5.25}$$

Overall Relationships (5.24) and (5.25) imply:

$$\frac{\pi^*}{\pi} - 1 < 0,$$

$$\frac{\pi^*}{\pi} < 1.$$

Therefore:

$$\pi^* < \pi.$$

So the tax-adjusted risk-neutral probability of an up move is less than the CRR risk-neutral probability of an up move, and as a result the tax-adjusted probability of a down move is greater than the classical probability of a down move. Calls (puts) finish in the money when the stock moves up (down), above (below) the strike price. Consequently, if the tax-adjusted risk-neutral probability of an up move is lower than the CRR probability, calls (puts) will be priced below (above) their CRR counterparts.

A numerical example

It is useful to consider a numerical example to see the effect that taxes have on the risk-neutral probability. The parameters used are: $S = K = 100, r = 0.05, \sigma = 0.25, T = 1, \tau_b = 0.4, \tau_s = \tau_o = 0.25$.

The risk-neutral probabilities for the CRR tree are found from $\pi = \frac{R-d}{u-d}$, and we find using

the parameters above:

$$\pi = 0.51693,$$

and,

$$1 - \pi = 0.48307.$$

The tax-adjusted risk-neutral probabilities are found from Equation (5.16), and we find for the parameters above:

$$\pi^* = 0.50796,$$

and,

$$1 - \pi^* = 0.49204$$

Immediately, we can see that if the income tax rate is greater than the capital gains tax rate, as is the case here, then the risk-neutral probability of an up (down) move is lower (higher) with taxes than without taxes.

Applying the CRR risk-neutral probability and the tax-adjusted risk neutral probability to a five period tree, with parameters given above, we find the prices of the options given in Table 5.1.

	Option value	% difference CRR
CRR call	12.7946	0.0
taBOPM call	12.1888	-4.7
CRR put	7.91759	0.0
taBOPM put	8.30276	+4.9

Table 5.1: CRR vs tax-adjusted binomial model option prices

With reference to Table 5.1 we can see that a call priced under the tax-adjusted probability

is less expensive than the CRR call, and a put priced under the tax-adjusted probability is more expensive than the CRR put, as we expect from Theorem 5.1.

It is worth reiterating that the payoffs used in the tax-adjusted binomial model are exactly the same as those used in the CRR model; the only change is the risk-neutral probability.

5.3   Convergence to the tax-adjusted Black-Scholes model

We can apply the method advocated in Section 2.6, which approximates the continuous-time option price from discrete-time prices, to taBOPM. Table 5.2 shows the approximate continuous-time prices of put and call options obtained using discrete-time prices from the taBOPM and CRR models. The parameters used to compile Table 5.2 were: $N = 1000, S = K = 100, r = 0.05, \sigma = 0.25, T = 1, \tau_b = 0.4, \tau_s = 0.25$.

	Option value	% diff taBS		Option value	% diff BS
taBS call	11.7186	0.000	Black-Scholes call	12.3347	0.000
taBOPM call	11.7192	0.005	CRR call	12.3359	0.009
taBS put	7.8365	0.000	Black-Scholes put	7.4576	0.000
taBOPM put	7.8371	0.008	CRR put	7.4588	0.016

Table 5.2: The convergence of the taBOPM to the taBS model

We can see from Table 5.2 that the prices obtained from the taBOPM for both calls and puts are very close to the values calculated using the taBS model. In fact, the differences are slightly smaller than those found when comparing CRR prices with Black-Scholes prices.

Theorem 5.2

The discrete time precursor to the taBS model is the taBOPM, in the same way that CRR is the discrete time precursor to Black-Scholes.

Proof to Theorem 5.2

See the next section for the derivation of the implied continuous-time trading equation.

5.3.1 The implied continuous-time trading equation

For convergence of the CRR model to the Black-Scholes model, the following three conditions are required (Cox (1979)):

1. $u = e^{\sigma\sqrt{h}}$

2. $d = e^{-\sigma\sqrt{h}}$

3. $R = r^h$

Where t is the time to expiration, n is the number of periods in the tree, h is the discretisation interval (also equal to t/n), r is the risk-free rate over fixed length of calendar time, and σ is the volatility.

The no-arbitrage equation in the tax-adjusted binomial model is given by:

$$\pi^* X_u + [1 - \pi^*] X_d - Rc = 0, \tag{5.26}$$

where X_u is the option price on an up move in the stock, X_d is the option price on a down move, and X is the current call option price.

Equation (5.26), after substitution for the tax-adjusted risk-neutral probabilities derived above in (5.16) and (5.17), becomes:

$$\left[\frac{R - d - (R-1)\tau_i + (d-1)\tau_{cg}}{u - d - (u-1)\tau_{cg} + (d-1)\tau_{cg}}\right] X_u + \left[\frac{u - R + (R-1)\tau_i - (u-1)\tau_{cg}}{u - d - (u-1)\tau_{cg} + (d-1)\tau_{cg}}\right] X_d - RX = 0. \tag{5.27}$$

We can also include the functional dependencies:

$$\left[\frac{R - d - (R-1)\tau_i + (d-1)\tau_{cg}}{u - d - (u-1)\tau_{cg} + (d-1)\tau_{cg}}\right] X(uS, t - h) \tag{5.28}$$

$$+ \left[\frac{u - R + (R-1)\tau_i - (u-1)\tau_{cg}}{u - d - (u-1)\tau_{cg} + (d-1)\tau_{cg}}\right] X(dS, t - h) - RX$$

$$= 0.$$

After substitution for u, d, and R given by three conditions above, we get:

$$\left[\frac{r^h - e^{-\sigma\sqrt{h}} - (r^h - 1)\tau_i + (e^{-\sigma\sqrt{h}} - 1)\tau_{cg}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}} - (e^{\sigma\sqrt{h}} - 1)\tau_{cg} + (e^{-\sigma\sqrt{h}} - 1)\tau_{cg}} \right] X(e^{\sigma\sqrt{h}}S, t - h) \qquad (5.29)$$

$$+ \left[\frac{e^{\sigma\sqrt{h}} - r^h + (r^h - 1)\tau_i - (e^{\sigma\sqrt{h}} - 1)\tau_{cg}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}} - (e^{\sigma\sqrt{h}} - 1)\tau_{cg} + (e^{-\sigma\sqrt{h}} - 1)\tau_{cg}} \right] X(e^{-\sigma\sqrt{h}}S, t - h) - r^h X$$

$$= 0.$$

We can expand the "up" value as a Taylor series around $X(S,t)$, keeping only terms multiplied by $\sqrt{h}$ or h, since the remaining terms become negligible as h becomes small. This gives:

$$X(e^{\sigma\sqrt{h}}S, t - h) = X(S,t) + (e^{\sigma\sqrt{h}} - 1)S\frac{\partial X}{\partial S} + \frac{1}{2}(e^{\sigma\sqrt{h}} - 1)S^2\frac{\partial^2 X}{\partial S^2} - h\frac{\partial X}{\partial t}. \qquad (5.30)$$

The expansion for the "down" value is the same, except $-\sigma\sqrt{h}$ replaces $\sigma\sqrt{h}$.

We can also expand the exponential functions and r^h as Taylor series:

$$e^{\sigma\sqrt{h}} = 1 + \sigma\sqrt{h} + \frac{1}{2}\sigma^2 h + ... \qquad (5.31)$$

$$e^{-\sigma\sqrt{h}} = 1 - \sigma\sqrt{h} + \frac{1}{2}\sigma^2 h - ... \qquad (5.32)$$

$$r^h = 1 + h\ln r + ... \qquad (5.33)$$

Substituting (5.30), (5.31), (5.32), and (5.33) into (5.29), and retaining only terms up to order h, we obtain:

$$\frac{1}{2}h\sigma^2 S^2\frac{\partial^2 X}{\partial S^2} + \frac{(1 - \tau_i)}{(1 - \tau_{cg})}(\ln r)\, hS\frac{\partial X}{\partial S} - h\frac{\partial X}{\partial t} - (\ln r)hX(S,t) + \Re = 0, \qquad (5.34)$$

where $\Re$ is the remainder term.

If we divide through by h, the $\Re/h$ term goes to zero, and we obtain the tax-adjusted Black-Scholes PDE:

$$\frac{1}{2}\sigma^2 S^2\frac{\partial^2 X}{\partial S^2} + \frac{(1 - \tau_i)}{(1 - \tau_{cg})}(\ln r)\, S\frac{\partial X}{\partial S} - \frac{\partial X}{\partial t} - (\ln r)X(S,t) = 0. \qquad (5.35)$$

123

5.4 An alternative tax-adjusted Black-Scholes equation

We have shown in Section 5.3 that the taBOPM converges to the taBS model in the same way that the BOPM converges to the Black-Scholes model. We saw that when formulating the taBOPM, the tax rate arising from the option transactions does not feature in the model if we assume that the option is taxed as income (see (5.20) and (5.21)). The same occurs in the derivation of taBS, shown in Appendix A: the income tax term in Equation (A.16) relating to the bond and derivative cancels to give the taBS equation (Equation (A.17)).

However, what if we can't assume that the derivative is taxed as income (it is more likely to be taxed as capital gains)? In this case we have the following equation, which is given as (A.16) in Appendix A where it is derived:

$$\frac{\partial V_t}{\partial t} + rS_t\frac{(1-\tau_i)}{(1-\tau_{cg})}\frac{\partial V_t}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V_t}{\partial S_t^2} - r\frac{(1-\tau_i)}{(1-\tau_o)}V_t = 0, \tag{5.36}$$

where τ_o is the tax rate that applies to the option. We can write this equation in the following way:

$$\frac{\partial V_t}{\partial t} + (\hat{r} - r^*)S_t\frac{\partial V_t}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V_t}{\partial S_t^2} - \hat{r}V_t = 0, \tag{5.37}$$

where,

$$\hat{r} = \frac{(1-\tau_i)}{(1-\tau_o)}, \tag{5.38}$$

and,

$$r^* = r\frac{(1-\tau_i)(\tau_{cg}-\tau_o)}{(1-\tau_{cg})(1-\tau_o)} \tag{5.39}$$

Equation (5.37) is termed the "generalised tax-adjusted Black-Scholes" equation (gtaBS). The solution for a the price of a European call, c, or put, p, is given by the following:

$$c = e^{-r^*(T-t)}SN(d_1) - Ke^{-\hat{r}(T-t)}N(d_2), \tag{5.40}$$

$$p = Ke^{-\hat{r}(T-t)}N(-d_2) - e^{-r^*(T-t)}SN(-d_1). \tag{5.41}$$

where d_1 and d_2 are given by:

$$d_1 = \frac{\ln(S/K) + (r^* - \hat{r} + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \tag{5.42}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}. \tag{5.43}$$

5.5 Summary

The following summarises the main results of this chapter:

1. The tax-adjusted total returns are defined for:

    (a) an up move in the stock price, $u^* = u - (u-1)\tau_{cg}$;

    (b) a down move in the stock price, $d^* = d - (d-1)\tau_{cg}$; and,

    (c) the risk free rate, $R^* = R - (R-1)\tau_i$.

2. The after-tax positions in the option are defined as:

    (a) $X_u^* = X_u - \tau_i(X_u - X^*)$, for an up move; and,

    (b) $X_d^* = X_d - \tau_i(X_d - X^*)$, for a down move.

3. The taBOPM uses the tax-adjusted total returns and after-tax option positions to form the tax-adjusted hedging portfolio and the tax-adjusted risk-neutral probability of an up move in the stock price. Option valuation can be achieved either by recursively forming the tax-adjusted hedging portfolio at each node, or by risk-neutral valuation using the tax-adjusted risk-neutral probability.

4. Two theorems, relating to the taBOPM, are proven:

    (a) Theorem 5.1 states that tax-adjusted calls (puts) are priced below (above) their CRR counterparts if the income tax rate is greater than the capital gains tax rate ($\tau_i > \tau_{cg}$), because the tax-adjusted risk-neutral probability of an up move is less than the corresponding CRR probability; and,

(b) Theorem 5.2 states that the taBOPM is the discrete time precursor to the taBS model.

5. An alternative to the taBS equation, the gtaBS equation, is proposed that does not assume that the derivative is taxed as income. The solutions to gtaBS for a put and call is given.

Chapter 6

Conclusions

In Chapter 2 we take M&P's work (in particular, Milevsky (1997a)) as a starting point and develop a tax-adjusted hedging algorithm that finds the tax-adjusted hedging price for a put or call option. We argue that the tax-adjusted hedging price can only be thought of as a tax-adjusted no-arbitrage price if, and only if, all market participants have the same tax position. The tax-adjusted hedging price is the price that the option writer must charge to be perfectly hedged on an after-tax basis. If the tax-adjusted option price is greater than the CRR price, then an option writer performing a classical CRR hedge will not be fully hedged on an after-tax basis at the expiry of the option.

The tax-adjusted hedging algorithm uses a non-recombining binomial tree framework. We calculate the after-tax positions for the three elements that are involved in delta hedging an option at the final nodes on the tree: the after-tax stock position, which is given by the usual CRR stock calculation at the final nodes, minus the tax liability on all the stock transactions made during that path on the tree; the after-tax bond position, which is given by the usual CRR bond calculation at the final nodes, minus the tax liability on the bond transactions made during that path on the tree; and, the after-tax option position, which is given by the usual CRR option payoff, minus the tax liability on the option for that path on the tree.

We see that the tax liability (or rebate, if a loss is made) on the stock transactions is path dependent, which is why we need to use a non-recombining binomial tree. The tax liability on the stock transactions is calculated using a marked-to-market method, whereby a tax charge is generated on each movement of the stock price. The tax charge on the option includes the

initial premium for the option, which is, of course, not known at the final nodes.

In general at a specific final node on the tree, the after-tax stock position plus the after-tax bond position does not equal the after-tax option position. This gives rise to the concept of the tax-mismatch. This forms the heart of the tax-adjusted hedging algorithm, which forces the tax-mismatch to zero via an iterative procedure. Iterations are required because of the fact that the initial option premium is included in the after-tax option calculation at the final nodes on the tree.

Option prices obtained using the tax-adjusted hedging algorithm show the following:

1. If tax rates on the stock, bond and option are all equal, tax-adjusted option prices are equal to their CRR counterparts.

2. Call (put) options are priced below (above) their CRR counterparts if the tax rate on the stock and option is less than the tax rate on the bond.

3. Call and put options are priced above (below) their CRR counterparts if the tax rate on the stock and bond is greater (less) than the tax rate on the option.

4. Approximate continuous-time tax-adjusted option prices satisfy the tax-adjusted put-call parity relationship.

In Chapter 3 we recognise that we can find the tax-adjusted hedging price of the option by solving a system of simultaneous equations. This is because we have the same number of unknowns (being the deltas and bond amounts at every node on the tree prior to maturity) as linearly independent equations. Taking this approach we can derive a general form for the delta at any node and a general form for the bond amount at the initial node. These general forms are used to form the simultaneous equation algorithm, which finds the same tax-adjusted hedging price for the option as the tax-adjusted hedging algorithm. The simultaneous equation algorithm is less demanding on memory, and computationally faster than the tax-adjusted hedging algorithm.

Using the simultaneous equation approach, we can find an analytic formula for the tax-adjusted option price in the single-period case. This analytic formula allows us to prove in the single-period case that:

128

1. Call (put) options are priced below (above) their CRR counterparts if the tax rate on the stock and option is less than the tax rate on the bond.

2. Tax-adjusted put and call prices satisfy the tax-adjusted put-call parity relationship.

3. The tax-adjusted simultaneous hedging algorithm converges to the tax-adjusted option price formula as the number of iterations it performs goes to infinity.

In Chapter 4 we relax the assumption that the tax year-end coincides with the maturity of the option at the final nodes on the tree. We allow the tax year-end to occur during the option's life. Using the simultaneous equation approach, we again derive a general form for the delta at any node and a general form for the bond amount at the initial node. These general forms are used to form the tax year-adjusted simultaneous equation algorithm. This algorithm is more demanding on memory, and computationally slower than the simultaneous equation algorithm. This is because we assume that the option is tax-marked-to-market at the tax year-end and this requires us to compute and store values for the option's market price.

We argue that if we are finding the tax-adjusted no-arbitrage price for the option, then it should be marked-to-market with its tax-adjusted value at the tax year-end; otherwise, we assume option prices are constrained by pre-tax no-arbitrage arguments, and we mark-to-market with the CRR price at the tax year-end.

Option prices obtained using the tax year-adjusted simultaneous equation algorithm show very small differences (of the order of 0.01%) compared with their counterparts valued using the simultaneous equation algorithm.

In Chapter 5 we look at Scholes (1976) and his extension to the Black-Scholes equation that includes tax - the tax-adjusted Black-Scholes equation.

The tax-adjusted total returns for an up movement in the stock price, a down movement in the stock price, and the risk free rate are defined. This allows us to derive a tax-adjusted risk-neutral probability and form the tax-adjusted binomial option pricing model. Under the assumptions given in Scholes (1976), the derivative is taxed as income and the stock and option as capital gains, we can prove that the tax-adjusted binomial option pricing model is the discrete-time precursor to the taBS equation, as CRR is the discrete-time precursor to the BS equation.

The assumption that the derivative is taxed as income can be relaxed and this allows us to derive a generalised taBS equation.

In this thesis we have assumed that the option writer has losses which he or she can use to offset other gains. Thus, if a loss is made the tax charge becomes, in effect, a tax rebate and the writer receives a nominal cashflow from the tax authorities. However, it is not certain that the writer will have other gains available, or will be able to use them in this way. Future research could look at the question of restricting the ability of the writer to offset gains with losses. Preliminary thoughts would indicate that this would cause the hedging-price of the option to increase since the writer would no longer receive rebates if he or she makes a loss.

Throughout, we have been considering equity put and call options. It would be interesting to look at the post-tax position of exotic options, and at the tax implications for products with different underlyings, most obviously interest rate derivatives.

Bibliography

[Black 1973]   BLACK, F. & M. SCHOLES (1973) The pricing of options and corporate liabilities. Journal of Political Economy. 81:637-659.

[Baxter 1996]   BAXTER, M.W., & A.J.O.RENNIE (1996) Financial calculus (Cambridge University Press).

[Cox 1979]   COX, J.C., S.A. ROSS & M. RUBINSTEIN (1979) Option pricing: a simplified approach. Journal of Financial Economics. 7(3):229-263.

[Dewynne 1994]   DEWYNNE, J., S.HOWISON & P.WILMOTT (1994) Option pricing: mathematical models and computation (Oxford Financial Press).

[Dermody 1991]   DERMODY, J.C. (1991) Cash stream valuation in the face of transaction costs and taxes. Mathematical Finance. 1(1):31-54.

[Dermody 1995]   DERMODY, J.C. (1991) Are transaction cost and tax effects worth the trouble in valuation theory? Working Paper, New York University Stern School of Business, New York, USA.

[Dothan 1990]   DOTHAN, M.U. (1990) Prices in financial markets (Oxford University Press).

[Duffie 1988]   DUFFIE, D. (1988) Security markets stochastic models (Academic Press).

[Kreyszig (1993)] KREYSZIG, E. (1993) Advanced engineering mathematics, 7th edn (John Wiley & Sons, Inc.)

131

[Merton 1973]   MERTON, R. (1973) Theory of rational option pricing. Bell Journal of Economics and Management Science. 4:141-183.

[Milevsky 1996] MILEVSKY, M. (1996) Hedging and pricing derivative securities with taxes. Unpublished PhD dissertation, Schulich School of Business, York University, Ontario, Canada.

[Milevsky 1996a] MILEVSKY, M. & E PRISMAN (1996) Tax effects in Canadian equity option markets. Working Paper, Schulich School of Business, York University, Ontario, Canada.

[Milevsky 1996b] MILEVSKY, M. & E PRISMAN (1996) Hedging and pricing with tax law uncertainty. Working Paper, Schulich School of Business, York University, Ontario, Canada.

[Milevsky 1996c] MILEVSKY, M. & E PRISMAN (1996) Derivative securities and taxation. Working Paper, Schulich School of Business, York University, Ontario, Canada.

[Milevsky 1997] MILEVSKY, M. & E PRISMAN (1997) A tax-adjusted algorithm for pricing derivative securities using the symbolic computational language MAPLE. IEEE/IAFE 1997 Conference on Computational Intelligence for Financial Engineering, New York.

[Milevsky 1997a] MILEVSKY, M. & E.PRISMAN (1997) Optional taxes. Risk. 10(9):133-137.

[Neftci 1996]   NEFTCI, SALIH N. (1996) An introduction to the mathematics of financial derivatives (Academic Press).

[Scholes 1976]   SCHOLES, M.S (1976) Taxes and the pricing of options. Journal of Finance. 31(2):319-332.

[Tolley]   Butterworths Yellow Tax Handbook (Butterworths).

Appendix A

Derivation of the tax-adjusted

Black-Scholes equation

This appendix is included because the method used gives us a better insight into the tax-adjusted Black-Scholes equation than the original PDE derivation (Scholes (1976)), and it contains the derivation of Equation (5.36) given in Chapter 5

The assumptions are the same as for the original BS equation (Black (1973)), except that the transactions in the bond and the underlying are taxed at rates of τ_i and τ_c[1], respectively, on a continuous basis. This assumes that the bond and option are taxed as ordinary income and the underlying is taxed as capital gains. We assume that tax liabilities (or rebates) are not only incurred, but also paid on a continuous basis.

The asset price follows geometric Brownian motion, where:

$$dS_t = \mu S_t dt + \sigma S_t dW_t^P. \tag{A.1}$$

The analysis here is complicated because we have to develop tradable stock and tradable bond processes to take into account the continuous tax liability (or rebate) that is generated over the life of the option. These are denoted $\tilde{S_t}$ and $\tilde{B_t}$ respectively.

---

[1]Note that $0 \leq \tau < 1$, for a generic tax rate τ.

Form the tradable bond process B_t^-

A tax liability (rebate) is generated when there is a gain (loss) on the bond. The notation dB_t^- represents an infinitesimal change in B_t^-. Because B_t^- changes, a tax liability (or rebate) is generated equal to $\tau_i r B_t^- dt$. Therefore we can write:

$$
\begin{aligned}
dB_t^- &= r B_t^- dt - \tau_i r B_t^- dt \\
&= r(1 - \tau_i) B_t^- dt.
\end{aligned}
\tag{A.2}
$$

The solution for B_t^- is:

$$
B_t^- = \exp\left[r(1 - \tau_i)t\right].
\tag{A.3}
$$

Form the tradable stock process S_t^-

A tax liability (rebate) is generated when there is a gain (loss) on the stock. The infinitesimal change in S_t^- is given by dS_t^- and so the tax liability (or loss) is $\tau_c dS_t^-$. Therefore we can write:

$$
\begin{aligned}
dS_t^- &= \mu S_t^- dt + \sigma S_t^- dW_t^P - \tau_{cg}(\mu S_t^- dt + \sigma S_t^- dW_t^P) \\
&= (1 - \tau_{cg})\mu S_t^- dt + (1 - \tau_{cg})\sigma S_t^- dW_t^P.
\end{aligned}
\tag{A.4}
$$

Ito's lemma verifies that the solution for S_t^- is:

$$
S_t^- = S_0 \exp\left[(1 - \tau_{cg})\left[\left(\mu - \frac{1}{2}\sigma^2(1 - \tau_{cg})\right)t + \sigma W_t^P\right]\right].
\tag{A.5}
$$

Convert $\left(B_t^-\right)^{-1} S_t^-$ into a martingale

Find the total differential of $\left(B_t^-\right)^{-1} S_t^-$:

$$
d\left[\left(B_t^-\right)^{-1} S_t^-\right] = \left(B_t^-\right)^{-1} dS_t^- + S_t^- d\left[\left(B_t^-\right)^{-1}\right].
\tag{A.6}
$$

Substitute for $d\left[\left(B_t^-\right)^{-1}\right]$ from Equation (A.2), and dS_t^- from Equation (A.4):

$$d\left[\left(B_t^-\right)^{-1}S_t^-\right] = \left(B_t^-\right)^{-1}S_t^-\left[(1-\tau_{cg})\mu - r(1-\tau_i)\right]dt + \left(B_t^-\right)^{-1}S_t^-(1-\tau_{cg})\sigma dW_t^P. \quad \text{(A.7)}$$

We require the drift term in (A.7) to be zero for $\left(B_t^-\right)^{-1}S_t^-$ to be a martingale. We use the Girsanov theorem to convert $\left(B_t^-\right)^{-1}S_t^-$ into a martingale:

$$dW_t^Q = dW_t^P + dX_t. \quad \text{(A.8)}$$

With

$$dX_t = \frac{1}{\sigma}\left[\mu - \frac{(1-\tau_i)}{(1-\tau_{cg})}r\right]dt, \quad \text{(A.9)}$$

we can see that substituting for dW_t^P in (A.7), using (A.8), gives us:

$$d\left[\left(B_t^-\right)^{-1}S_t^-\right] = \left(B_t^-\right)^{-1}S_t^-\sigma(1-\tau_{cg})dW_t^Q. \quad \text{(A.10)}$$

The above equation is now driftless, and so we have converted $\left(B_t^-\right)^{-1}S_t^-$ into a martingale by changing the probability measure from P to Q.

Form a new equation for dS_t under the risk-neutral measure

We use the Girsanov theorem to change the measure in the original equation for dS_t (that is Equation (A.1), and not the tradable stock equation). This is achieved by substituting (A.8) and (A.9) into (A.1). The result is:

$$dS_t = rS_t\frac{(1-\tau_i)}{(1-\tau_{cg})}dt + \sigma S_t dW_t^Q. \quad \text{(A.11)}$$

Convert $\left(B_t^-\right)^{-1}V^-(S_t,t)$ into a martingale

Find the total differential of $\left(B_t^-\right)^{-1}V^-(S_t,t)$:

$$d\left[\left(B_t^-\right)^{-1}V^-(S_t,t)\right] = \left(B_t^-\right)^{-1}dV_t^- + V_t d\left[\left(B_t^-\right)^{-1}\right], \quad \text{(A.12)}$$

where $V_t^{\tilde{}}$ is used to denote $V^{\tilde{}}(S_t, t)$. Note, we are using the tradable bond equation.

Using Ito's Lemma we can write the tradable equation for the derivative:

$$dV_t^{\tilde{}} = \left[\frac{\partial V_t^{\tilde{}}}{\partial t} dt + \frac{\partial V_t^{\tilde{}}}{\partial S_t} dS_t + \frac{1}{2} \frac{\partial^2 V_t^{\tilde{}}}{\partial S_t^2} \sigma^2 S_t^2 dt \right] (1 - \tau_o). \qquad (A.13)$$

We can now write:

$$d\left[(B_t^{\tilde{}})^{-1} V^{\tilde{}}(S_t, t) \right] = (B_t^{\tilde{}})^{-1} \left[\frac{\partial V_t^{\tilde{}}}{\partial t} dt + \frac{\partial V_t^{\tilde{}}}{\partial S_t} dS_t + \frac{1}{2} \frac{\partial^2 V_t^{\tilde{}}}{\partial S_t^2} \sigma^2 S_t^2 dt \right] (1 - \tau_o) - r (B_t^{\tilde{}})^{-1} V_t^{\tilde{}} (1 - \tau_i) d \qquad (A.14)$$

Substituting for dS_t as given in (A.11) gives:

$$d\left[(B_t^{\tilde{}})^{-1} V^{\tilde{}}(S_t, t) \right] = \sigma S_t \frac{\partial V_t^{\tilde{}}}{\partial S_t} (1 - \tau_o) dW_t^Q \qquad (A.15)$$

$$+ (B_t^{\tilde{}})^{-1} \left[\left(\begin{array}{c} \frac{\partial V_t^{\tilde{}}}{\partial t} + r S_t \frac{(1-\tau_i)}{(1-\tau_{cg})} \frac{\partial V_t^{\tilde{}}}{\partial S_t} \\ + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V_t^{\tilde{}}}{\partial S_t^2} \end{array} \right) (1 - \tau_o) - r(1 - \tau_i) V_t^{\tilde{}} \right] dt.$$

For $(B_t^{\tilde{}})^{-1} V^{\tilde{}}(S_t, t)$ to be a martingale, Equation (A.15) has to be driftless. This will be the case when:

$$\frac{\partial V_t^{\tilde{}}}{\partial t} + r S_t \frac{(1 - \tau_i)}{(1 - \tau_{cg})} \frac{\partial V_t^{\tilde{}}}{\partial S_t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V_t^{\tilde{}}}{\partial S_t^2} - r \frac{(1 - \tau_i)}{(1 - \tau_o)} V_t^{\tilde{}} = 0. \qquad (A.16)$$

In the original derivation of taBS, it is assumed that the tax rate that applies to the derivative is τ_i. Setting $\tau_o = \tau_i$ results in:

$$\frac{\partial V_t}{\partial t} + (r - \bar{r}) S_t \frac{\partial V_t}{\partial S_t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V_t}{\partial S_t^2} - r V_t = 0, \qquad (A.17)$$

which is the tax-adjusted Black-Scholes PDE, where,

$$\bar{r} = \frac{r(\tau_i - \tau_{cg})}{(1 - \tau_{cg})}. \qquad (A.18)$$

The solution for a the price of a European call, c, or put, p, is given by the following:

$$c = e^{-\bar{r}(T-t)}SN(d_1) - Ke^{-r(T-t)}N(d_2), \tag{A.19}$$

$$p = Ke^{-r(T-t)}N(-d_2) - e^{-\bar{r}(T-t)}SN(-d_1). \tag{A.20}$$

where d_1 and d_2 are given by:

$$d_1 = \frac{\ln(S/K) + (r - \bar{r} + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \tag{A.21}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}. \tag{A.22}$$

Appendix B

Derivation of the tax-adjusted Black-Scholes equation with dividends

This appendix is included because the method used gives us a better insight into the derivation of the tax-adjusted Black-Scholes equation with dividends than the original PDE derivation (Milevsky (1997a)).

Here we assume that the underlying pays a continuous dividend and all the transactions are taxed (see Appendix A).

The analysis here is complicated because we have to develop a tradable stock process to take into account the continuous tax liability (or rebate) that is generated from the stock and the reinvested dividends. We also need a tradable bond process to take into account the continuous tax liability from the bond. These tradeable processes are denoted S_t^- and B_t^- respectively.

Form the tradable bond process B_t^-

A tax liability (rebate) is generated when there is a gain (loss) on the bond. The notation dB_t^- represents an infinitesimal change in B_t^-. Because B_t^- changes, a tax liability (or rebate)

is generated equal to $\tau_i r B_t^- dt$. Therefore we can write:

$$
\begin{aligned}
dB_t^- &= r B_t^- dt - \tau_i r B_t^- dt \\
&= r(1 - \tau_i) B_t^- dt.
\end{aligned}
\tag{B.1}
$$

The solution for B_t^- is:

$$
B_t^- = \exp\left[r(1 - \tau_i)t\right].
\tag{B.2}
$$

Form the tradable stock process S_t^-

The tradable stock process that includes tax is (see Appendix B):

$$
dS_t^* = (1 - \tau_c)\mu S_t^* dt + (1 - \tau_c)\sigma S_t^* dW_t^P
\tag{B.3}
$$

We need to include dividends aswell. The dividend payment is $q S_t^- dt$, of which, $\tau_q q S_t^- dt$ is paid as tax. Therefore, the total dividend received that is reinvested in the stock is $(1 - \tau_q) q S_t^- dt$. Therefore our overall tradable stock is written as:

$$
dS_t^- = [(1 - \tau_c)\mu + (1 - \tau_q)q]\, S_t^- dt + (1 - \tau_c)\sigma S_t^- dW_t^P.
\tag{B.4}
$$

Ito's lemma verifies that the solution for S_t^- is:

$$
S_t^- = S_0 \exp\left[(1 - \tau_c)\left[\left(\mu + \frac{q(1 - \tau_q)}{(1 - \tau_c)} - \frac{1}{2}\sigma^2(1 - \tau_c)\right)t + \sigma W_t^P\right]\right].
\tag{B.5}
$$

Convert $\left(B_t^-\right)^{-1} S_t^-$ into a martingale

Find the total differential of $\left(B_t^-\right)^{-1} S_t^-$:

$$
d\left[\left(B_t^-\right)^{-1} S_t^-\right] = \left(B_t^-\right)^{-1} dS_t^- + S_t^- d\left[\left(B_t^-\right)^{-1}\right].
\tag{B.6}
$$

Substitute for $d\left[\left(B_t^-\right)^{-1}\right]$ from Equation (B.1), and dS_t^- from Equation (B.4):

$$d\left[\left(B_t^-\right)^{-1} S_t^-\right] = \left(B_t^-\right)^{-1} S_t^- \left[\mu(1-\tau_c) + q(1-\tau_q) - r(1-\tau_i)\right] dt + \left(B_t^-\right)^{-1} S_t^-(1-\tau_c)\sigma dW_t^P.$$

$$(B.7)$$

We require the drift term in (B.7) to be zero for $\left(B_t^-\right)^{-1} S_t^-$ to be a martingale. We use the Girsanov theorem to convert $\left(B_t^-\right)^{-1} S_t^-$ into a martingale:

$$dW_t^Q = dW_t^P + dX_t.$$

$$(B.8)$$

With

$$dX_t = \frac{1}{\sigma}\left[\mu - r\frac{(1-\tau_i)}{(1-\tau_c)} + q\frac{(1-\tau_q)}{(1-\tau_c)}\right] dt,$$

$$(B.9)$$

we can see that substituting for dW_t^P in (B.7), using (B.8), gives us:

$$d\left[\left(B_t^-\right)^{-1} S_t^-\right] = \left(B_t^-\right)^{-1} S_t^- \sigma(1-\tau_c)dW_t^Q.$$

$$(B.10)$$

The above equation is now driftless, and so we have converted $\left(B_t^-\right)^{-1} S_t^-$ into a martingale by changing the probability measure from P to Q.

Form a new equation for dS_t under the risk-neutral measure

We use the Girsanov theorem to change the measure in the original equation for dS_t (that is Equation (A.1), and not the tradable stock equation). This is achieved by substituting (B.8) and (B.9) into (A.1). The result is:

$$dS_t = S_t\left[r\frac{(1-\tau_i)}{(1-\tau_{cg})} - q\frac{(1-\tau_q)}{(1-\tau_{cg})}\right] dt + \sigma S_t dW_t^Q.$$

$$(B.11)$$

Convert $\left(B_t^-\right)^{-1} V^-(S_t, t)$ into a martingale

Find the total differential of $\left(B_t^-\right)^{-1} V^-(S_t, t)$:

$$d\left[\left(B_t^-\right)^{-1} V^-(S_t, t)\right] = \left(B_t^-\right)^{-1} dV_t^- + V_t d\left[\left(B_t^-\right)^{-1}\right], \tag{B.12}$$

where V_t^- is used to denote $V^-(S_t, t)$. Note, we are using the tradable bond equation.

Using Ito's Lemma we can write the tradable equation for the derivative:

$$dV_t^- = \left[\frac{\partial V_t^-}{\partial t} dt + \frac{\partial V_t^-}{\partial S_t} dS_t + \frac{1}{2}\frac{\partial^2 V_t^-}{\partial S_t^2}\sigma^2 S_t^2 dt\right](1 - \tau_o). \tag{B.13}$$

We can now write:

$$d\left[\left(B_t^-\right)^{-1} V^-(S_t, t)\right] = \left(B_t^-\right)^{-1}\left[\frac{\partial V_t^-}{\partial t} dt + \frac{\partial V_t^-}{\partial S_t} dS_t + \frac{1}{2}\frac{\partial^2 V_t^-}{\partial S_t^2}\sigma^2 S_t^2 dt\right](1 - \tau_o) - r\left(B_t^-\right)^{-1} V_t^-(1 - \tau_i)d$$

$$\tag{B.14}$$

Substituting for dS_t as given in (B.11) gives:

$$d\left[\left(B_t^-\right)^{-1} V^-(S_t, t)\right] = \sigma S_t \frac{\partial V_t^-}{\partial S_t}(1 - \tau_o)dW_t^Q \tag{B.1}$$

$$+ \left(B_t^-\right)^{-1}\left[\begin{array}{c}\frac{\partial V_t^-}{\partial t} + S_t\left[r\frac{(1-\tau_i)}{(1-\tau_c)} - q\frac{(1-\tau_q)}{(1-\tau_c)}\right]\frac{\partial V_t^-}{\partial S_t} \\ +\frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V_t^-}{\partial S_t^2}\end{array}\right](1 - \tau_o)dt - r(1 - \tau_i)V_t^- d$$

For $\left(B_t^-\right)^{-1} V^-(S_t, t)$ to be a martingale, Equation (B.15) has to be driftless. This will be the case when:

$$\frac{\partial V_t}{\partial t} + (r^* - q^*) S_t \frac{\partial V_t}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V_t}{\partial S_t^2} - r^* V_t = 0, \tag{B.16}$$

which is the tax-adjusted Black-Scholes PDE with dividends as derived by Milevsky and Prisman (Milevsky 1997a), where,

$$r^* = r\frac{(1 - \tau_i)}{(1 - \tau_o)}, \tag{B.17}$$

and,

$$q^* = r\frac{(\tau_o - \tau_{cg})(1 - \tau_i)}{(1 - \tau_{cg})(1 - \tau_o)} + q\frac{(1 - \tau_q)}{(1 - \tau_{cg})}. \tag{B.18}$$

The solution for the price of a European call, c or put, p, is given by the following:

$$c = e^{-q^*(T-t)}SN(d_1) - Ke^{-r^*(T-t)}N(d_2), \tag{B.19}$$

$$p = Ke^{-r^*(T-t)}N(-d_2) - e^{-q^*(T-t)}SN(-d_1), \tag{B.20}$$

where d_1 and d_2 are given by:

$$d_1 = \frac{\ln(S/K) + (r^* - q^* + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \tag{B.21}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}. \tag{B.22}$$

Appendix C

Derivation of the N-Period form for the Deltas and Bonds

If we solve the system of simultaneous equations for $N = 1, 2, 3$ and 4, and find formulas for the bonds and deltas in each case, then we may be able to see a pattern and derive a formula for each in the general case.

C.0.1 The deltas

$N = 1$

$$_1\Delta^*_{\langle 0,0 \rangle} = \frac{(1 - \tau_o)\left(X_{\langle 1,0 \rangle} - X_{\langle 1,1 \rangle}\right)}{(1 - \tau_s)\left(S_{\langle 1,0 \rangle} - S_{\langle 1,1 \rangle}\right)}. \tag{C.1}$$

$N = 2$

$$_2\Delta^*_{\langle 0,0 \rangle} = \frac{1}{\left(S_{\langle 1,0 \rangle} - S_{\langle 1,1 \rangle}\right)\left(-\tau_s + R - \tau_b(R - 1)\right)} \times \tag{C.2}$$

$$\begin{bmatrix} +(1 - \tau_o)\left(X_{\langle 2,0 \rangle} - X_{\langle 2,2 \rangle}\right) \\ -\,{}_1\Delta^*_{\langle 1,0 \rangle} \begin{bmatrix} S_{\langle 2,0 \rangle} - S_{\langle 1,0 \rangle}\left(R - \tau_b(R - 1)\right) \\ -\tau_s\left(S_{\langle 2,0 \rangle} - S_{\langle 1,0 \rangle}\right) \end{bmatrix} \\ +\,{}_1\Delta^*_{\langle 1,1 \rangle} \begin{bmatrix} S_{\langle 2,2 \rangle} - S_{\langle 1,1 \rangle}\left(R - \tau_b(R - 1)\right) \\ -\tau_s\left(S_{\langle 2,2 \rangle} - S_{\langle 1,1 \rangle}\right) \end{bmatrix} \end{bmatrix}$$

143

At period one, where $j = 0, 1$, we have:

$$_2\Delta^*_{\langle 1,j \rangle} = \frac{(1 - \tau_o)\left(X_{\langle 2,2j \rangle} - X_{\langle 2,2j+1 \rangle}\right)}{(1 - \tau_s)\left(S_{\langle 2,2j \rangle} - S_{\langle 2,2j+1 \rangle}\right)}.$$

(C.3)

$N = 3$

$$_3\Delta^*_{\langle 0,0 \rangle} = \frac{1}{\left(S_{\langle 1,0 \rangle} - S_{\langle 1,1 \rangle}\right)\left[-\tau_s - \tau_b(R-1) + R\left(R - \tau_b\left(R-1\right)\right)\right]} \times$$

(C.4)

$$\begin{bmatrix} + (1 - \tau_o)\left(X_{\langle 3,0 \rangle} - X_{\langle 3,4 \rangle}\right) \\ - {}_3\Delta^*_{\langle 1,0 \rangle} \begin{bmatrix} +S_{\langle 2,0 \rangle}\left(R - \tau_b\left(R-1\right)\right) \\ -S_{\langle 1,0 \rangle}\left[R\left(R - \tau_b(R-1)\right) - \tau_b(R-1)\right] \\ -\tau_s\left(S_{\langle 2,0 \rangle} - S_{\langle 1,0 \rangle}\right) \end{bmatrix} \\ - {}_3\Delta^*_{\langle 2,0 \rangle} \begin{bmatrix} S_{\langle 3,0 \rangle} - S_{\langle 2,0 \rangle}\left(R - \tau_b(R-1)\right) \\ -\tau_s\left(S_{\langle 3,0 \rangle} - S_{\langle 2,0 \rangle}\right) \end{bmatrix} \\ + {}_3\Delta^*_{\langle 1,1 \rangle} \begin{bmatrix} +S_{\langle 2,2 \rangle}\left(R - \tau_b\left(R-1\right)\right) \\ -S_{\langle 1,1 \rangle}\left[R\left(R - \tau_b(R-1)\right) - \tau_b(R-1)\right] \\ -\tau_s\left(S_{\langle 2,2 \rangle} - S_{\langle 1,1 \rangle}\right) \end{bmatrix} \\ + {}_3\Delta^*_{\langle 2,2 \rangle} \begin{bmatrix} S_{\langle 3,4 \rangle} - S_{\langle 2,2 \rangle}\left(R - \tau_b(R-1)\right) \\ -\tau_s\left(S_{\langle 3,4 \rangle} - S_{\langle 2,2 \rangle}\right) \end{bmatrix} \end{bmatrix}$$

At period one, where $j = 0, 1$, we have:

$$_3\Delta^*_{\langle 1,j \rangle} = \frac{1}{\left(S_{\langle 2,2j \rangle} - S_{\langle 2,2j+1 \rangle}\right)\left(-\tau_s + R - \tau_b(R-1)\right)} \times$$

(C.5)

$$\begin{bmatrix} + (1 - \tau_o)\left(X_{\langle 3,j2^2 \rangle} - X_{\langle 3,2(2j+1) \rangle}\right) \\ - {}_3\Delta^*_{\langle 2,2j \rangle} \begin{bmatrix} S_{\langle 3,j2^2 \rangle} - S_{\langle 2,2j \rangle}\left(R - \tau_b(R-1)\right) \\ -\tau_s\left(S_{\langle 3,j2^2 \rangle} - S_{\langle 2,2j \rangle}\right) \end{bmatrix} \\ + {}_3\Delta^*_{\langle 2,2j+1 \rangle} \begin{bmatrix} S_{\langle 3,2(2j+1) \rangle} - S_{\langle 2,2j+1 \rangle}\left(R - \tau_b(R-1)\right) \\ -\tau_s\left(S_{\langle 3,2(2j+1) \rangle} - S_{\langle 2,2j+1 \rangle}\right) \end{bmatrix} \end{bmatrix}$$

At period two, where $j = 0, 1, 2, 3$, we have:

$$3\Delta^*_{\langle 2,j \rangle} = \frac{(1 - \tau_o) \left(X^*_{\langle 3,2j \rangle} - X^*_{\langle 3,2j+1 \rangle} \right)}{(1 - \tau_s) \left(S_{\langle 3,2j \rangle} - S_{\langle 3,2j+1 \rangle} \right)}. \tag{C.6}$$

We can see already that the form for the delta at period i is a function of the number of periods from maturity, $N - i$: $1\Delta^*_{\langle 0,0 \rangle}$ given by (C.1) is of the same form as $2\Delta^*_{\langle 1,j \rangle}$ (C.3) and $3\Delta^*_{\langle 2,j \rangle}$ (C.6); $2\Delta^*_{\langle 0,0 \rangle}$ (C.2) is of the same form as $3\Delta^*_{\langle 1,j \rangle}$ (C.5).

$N = 4$

$$4\Delta^*_{\langle 0,0 \rangle} = \frac{1}{\left(S_{\langle 1,0 \rangle} - S_{\langle 1,1 \rangle} \right) \left[-\tau_s - \tau_b(R-1) + R\left\{ R\left(R - \tau_b(R-1)\right) - \tau_b(R-1) \right\} \right]} \times \tag{C.7}$$

$$
\begin{bmatrix}
+ (1 - \tau_o) \left(X_{\langle 4,0 \rangle} - X_{\langle 4,8 \rangle} \right) \\[2mm]
- 4\Delta^*_{\langle 1,0 \rangle} \begin{bmatrix} +S_{\langle 2,0 \rangle} \{R(R - \tau_b(R-1)) - \tau_b(R-1)\} \\ -S_{\langle 1,0 \rangle} \{R[R(R - \tau_b(R-1)) - \tau_b(R-1)] - \tau_b(R-1)\} \\ -\tau_s \left(S_{\langle 2,0 \rangle} - S_{\langle 1,0 \rangle} \right) \end{bmatrix} \\[6mm]
- 4\Delta^*_{\langle 2,0 \rangle} \begin{bmatrix} +S_{\langle 3,0 \rangle} (R - \tau_b(R-1)) \\ -S_{\langle 2,0 \rangle} \{R(R - \tau_b(R-1)) - \tau_b(R-1)\} \\ -\tau_s \left(S_{\langle 3,0 \rangle} - S_{\langle 2,0 \rangle} \right) \end{bmatrix} \\[6mm]
- 4\Delta^*_{\langle 3,0 \rangle} \begin{bmatrix} +S_{\langle 4,0 \rangle} - S_{\langle 3,0 \rangle} (R - \tau_b(R-1)) \\ -\tau_s \left(S_{\langle 4,0 \rangle} - S_{\langle 3,0 \rangle} \right) \end{bmatrix} \\[6mm]
+ 4\Delta^*_{\langle 1,1 \rangle} \begin{bmatrix} +S_{\langle 2,2 \rangle} \{R(R - \tau_b(R-1)) - \tau_b(R-1)\} \\ -S_{\langle 1,1 \rangle} \{R[R(R - \tau_b(R-1)) - \tau_b(R-1)] - \tau_b(R-1)\} \\ -\tau_s \left(S_{\langle 2,2 \rangle} - S_{\langle 1,1 \rangle} \right) \end{bmatrix} \\[6mm]
+ 4\Delta^*_{\langle 2,2 \rangle} \begin{bmatrix} +S_{\langle 3,4 \rangle} (R - \tau_b(R-1)) \\ -S_{\langle 2,2 \rangle} \{R(R - \tau_b(R-1)) - \tau_b(R-1)\} \\ -\tau_s \left(S_{\langle 3,4 \rangle} - S_{\langle 2,2 \rangle} \right) \end{bmatrix} \\[6mm]
+ 4\Delta^*_{\langle 3,4 \rangle} \begin{bmatrix} +S_{\langle 4,8 \rangle} - S_{\langle 3,4 \rangle} (R - \tau_b(R-1)) \\ -\tau_s \left(S_{\langle 4,8 \rangle} - S_{\langle 3,4 \rangle} \right) \end{bmatrix}
\end{bmatrix}
$$

The form of the deltas at periods one, two and three is given by Equations (C.4), (C.5) and (C.6), respectively.

The general case

Now we will show how we can use the deltas for $N = 1, 2, 3$ and 4, to find the general case, $_N\Delta^*_{\langle i,j \rangle}$ where $i = 0, 1, ..., N - 1$ and $j = 0, 1, ..., 2^i - 1$.

The overall coefficient The stock terms are simply the stock at the two connected nodes, one period later. The form for the other bracket, containing R and the tax rates, can be seen by more clearly from the following table.

$N - i$	Bracket containing R and the tax rates
1	$-\tau_s + 1$
2	$-\tau_s + R - \tau_b(R - 1)(1)$
3	$-\tau_s + R^2 - \tau_b(R - 1)(1 + R)$
4	$-\tau_s + R^3 - \tau_b(R - 1)\left(1 + R + R^2\right)$
$N - i$	$-\tau_s + R^{N-1-i} - \tau_b(R - 1)\sum_{a=0}^{N-2-i} R^a$

Table C.1: Development of the overall coefficient

So the general form for the overall coefficient is:

$$\frac{1}{\left(S_{\langle i+1,2j \rangle} - S_{\langle i+1,2j+1 \rangle}\right)\left(-\tau_s + R^{N-1-i} - \tau_b(R - 1)\sum_{a=0}^{N-2-i} R^a\right)}. \tag{C.8}$$

The bracket containing the option payoffs We start at node $\langle i, j \rangle$: the first payoff is given by following the path u^{N-i}, which gives $X_{\langle N,j2^{N-i} \rangle}$; the second payoff is given by following the path du^{N-i-1}, which gives $X^*_{\langle N,(2j+1)2^{N-i-1} \rangle}$. The usual notation to represent an "up" move (u) and a "down" move (d) has been used.

The general form for the bracket containing the option payoffs is:

$$(1 - \tau_o)\left(X_{\langle N,j2^{N-i} \rangle} - X_{\langle N,(2j+1)2^{N-i-1} \rangle}\right). \tag{C.9}$$

The deltas inside the main square bracket We have two sets of deltas. The first set, whose elements are all negative, is given by all the deltas along path u^{N-i}, starting at node $\langle i,j \rangle$ but not including this node. The second set, whose elements are all positive, is given by all the deltas along path du^{N-i-1}, starting at node $\langle i,j \rangle$ but not including this node. Therefore, the first set is:

$$-\sum_{a=i+1}^{N-1} {}_N\Delta^*_{\langle a,j2^{a-i} \rangle},$$
(C.10)

and the second set is:

$$+\sum_{a=i+1}^{N-1} {}_N\Delta^*_{\langle a,(2j+1)2^{a-i-1} \rangle}.$$
(C.11)

The coefficients to the deltas inside the main square bracket The coefficients are of the same form for the two sets of deltas at each period. For example, the coefficients to $(-{}_4\Delta^*_{\langle 1,0 \rangle})$ and $(+{}_4\Delta^*_{\langle 1,1 \rangle})$ in (C.7) are of the same form. We will only look at the coefficients to the negative set of deltas, since the coefficients to the positive set will follow immediately.

Let us consider the coefficient to the following delta, $(-{}_N\Delta^*_{\langle a,j2^{a-i} \rangle})$. There are three terms to the coefficient: a positive stock term, where the stock is that at the "up" node from the delta, $(S_{\langle a+1,j2^{a+1-i} \rangle})$; a negative stock term, where the stock is that at the same node as the delta, $(S_{\langle a,j2^{a-i} \rangle})$; and a term involving τ_s and the two stock prices, given simply by $-\tau_s \left(S_{\langle a+1,j2^{a+1-i} \rangle} - S_{\langle a,j2^{a-i} \rangle} \right)$.

$N-a$	$S_{\langle a+1,j2^{a+1-i} \rangle}$ term	$-S_{\langle a,j2^{a-i} \rangle}$ term
1	1	$R - \tau_b(R-1)\,(1)$
2	$R - \tau_b(R-1)\,(1)$	$R^2 - \tau_b(R-1)\,(1+R)$
3	$R^2 - \tau_b(R-1)\,(1+R)$	$R^3 - \tau_b(R-1)\left(1+R+R^2\right)$
$N-a$	$R^{N-1-a} - \tau_b(R-1)\sum_{b=0}^{N-2-a} R^b$	$R^{N-a} - \tau_b(R-1)\sum_{b=0}^{N-1-a} R^b$

Table C.2: Development of the stock terms for the delta coefficients

Table C.2 gives the stock terms in the delta coefficients.

The general form for the coefficients to the deltas is:

$$
{}_N\Delta^*_{\langle a,j2^{a-i}\rangle}
\begin{bmatrix}
+S_{\langle a+1,j2^{a+1-i}\rangle}\left(R^{N-a-1}-\tau_b(R-1)\sum_{b=0}^{N-2-a}R^b\right) \\[2mm]
-S_{\langle a,j2^{a-i}\rangle}\left(R^{N-a}-\tau_b(R-1)\sum_{b=0}^{N-1-a}R^b\right) \\[2mm]
-\tau_s\left(S_{\langle a+1,j2^{a+1-i}\rangle}-S_{\langle a,j2^{a-i}\rangle}\right)
\end{bmatrix}.
$$

Combining the elements When we combine all the elements together we get the general form for ${}_N\Delta^*_{\langle i,j\rangle}$:

$$
{}_N\Delta^*_{\langle i,j\rangle} = \frac{1}{\left(S_{\langle i+1,2j\rangle}-S_{\langle i+1,2j+1\rangle}\right)\left(-\tau_s+R^{N-1-i}-\tau_b(R-1)\sum_{a=0}^{N-2-i}R^a\right)} \times \tag{C.}
$$

$$
\begin{bmatrix}
+(1-\tau_o)\left(X_{\langle N,j2^{N-i}\rangle}-X_{\langle N,(2j+1)2^{N-i-1}\rangle}\right) \\[2mm]
-\displaystyle\sum_{a=i+1}^{N-1}{}_N\Delta^*_{\langle a,j2^{a-i}\rangle}
\begin{bmatrix}
+S_{\langle a+1,j2^{a+1-i}\rangle}\left(R^{N-a-1}-\tau_b(R-1)\sum_{b=0}^{N-2-a}R^b\right) \\[2mm]
-S_{\langle a,j2^{a-i}\rangle}\left(R^{N-a}-\tau_b(R-1)\sum_{b=0}^{N-1-a}R^b\right) \\[2mm]
-\tau_s\left(S_{\langle a+1,j2^{a+1-i}\rangle}-S_{\langle a,j2^{a-i}\rangle}\right)
\end{bmatrix} \\[4mm]
+\displaystyle\sum_{a=i+1}^{N-1}{}_N\Delta^*_{\langle a,(2j+1)2^{a-i-1}\rangle}
\begin{bmatrix}
+S_{\langle a+1,(2j+1)2^{a-i}\rangle}\left(R^{N-a-1}-\tau_b(R-1)\sum_{b=0}^{N-2-a}R^b\right) \\[2mm]
-S_{\langle a,(2j+1)2^{a-i-1}\rangle}\left(R^{N-a}-\tau_b(R-1)\sum_{b=0}^{N-1-a}R^b\right) \\[2mm]
-\tau_s\left(S_{\langle a+1,(2j+1)2^{a-i}\rangle}-S_{\langle a,(2j+1)2^{a-i-1}\rangle}\right)
\end{bmatrix}
\end{bmatrix}
$$

C.0.2 The bonds

We can write the initial bond amount in terms of the deltas for $N=1,2,3$ and 4. The deltas are given by (C.12).

$N=1$

$$
{}_1B^*_{\langle 0,0\rangle} = \frac{1}{[R-\tau_b(R-1)-\tau_o]}
\begin{bmatrix}
+(1-\tau_o)X_{\langle 1,0\rangle} \\[2mm]
-{}_1\Delta^*_{\langle 0,0\rangle}\left[-\tau_o S_{\langle 0,0\rangle}-\tau_s\left(S_{\langle 1,0\rangle}-S_{\langle 0,0\rangle}\right)+S_{\langle 1,0\rangle}\right]
\end{bmatrix}.
$$

$$\tag{C.13}$$

$N = 2$

$$
{}_2B^*_{\langle 0,0\rangle} = \frac{1}{[-\tau_o - \tau_b(R-1) + R\{R - \tau_b(R-1)\}]} \times \tag{C.14}
$$

$$
\begin{bmatrix}
+ (1 - \tau_o)\, X_{\langle 2,0\rangle} \\[4pt]
- {}_2\Delta^*_{\langle 0,0\rangle}\left[-\tau_o S_{\langle 0,0\rangle} - \tau_s\left(S_{\langle 1,0\rangle} - S_{\langle 0,0\rangle}\right) + S_{\langle 1,0\rangle}\left(R - \tau_b(R-1)\right) \right] \\[4pt]
- {}_2\Delta^*_{\langle 1,0\rangle}\begin{bmatrix} + S_{\langle 2,0\rangle} - S_{\langle 1,0\rangle}\left(R - \tau_b(R-1)\right) \\[4pt] -\tau_s\left(S_{\langle 2,0\rangle} - S_{\langle 1,0\rangle}\right) \end{bmatrix}
\end{bmatrix} .
$$

$N = 3$

$$
{}_3B^*_{\langle 0,0\rangle} = \frac{1}{[-\tau_o - \tau_b(R-1) + R\left[R\left(R - \tau_b(R-1)\right) - \tau_b(R-1)\right]]} \times \tag{C.15}
$$

$$
\begin{bmatrix}
+ (1 - \tau_o)\, X_{\langle 3,0\rangle} \\[4pt]
- {}_3\Delta^*_{\langle 0,0\rangle}\begin{bmatrix} -\tau_o S_{\langle 0,0\rangle} - \tau_s\left(S_{\langle 1,0\rangle} - S_{\langle 0,0\rangle}\right) \\[4pt] + S_{\langle 1,0\rangle}\left\{R\left(R - \tau_b(R-1)\right) - \tau_b(R-1)\right\} \end{bmatrix} \\[4pt]
- {}_3\Delta^*_{\langle 1,0\rangle}\begin{bmatrix} + S_{\langle 2,0\rangle}\left(R - \tau_b(R-1)\right) - S_{\langle 1,0\rangle}\left\{R\left(R - \tau_b(R-1)\right) - \tau_b(R-1)\right\} \\[4pt] -\tau_s\left(S_{\langle 2,0\rangle} - S_{\langle 1,0\rangle}\right) \end{bmatrix} \\[4pt]
- {}_3\Delta^*_{\langle 2,0\rangle}\begin{bmatrix} + S_{\langle 3,0\rangle} - S_{\langle 2,0\rangle}\left(R - \tau_b(R-1)\right) \\[4pt] -\tau_s\left(S_{\langle 3,0\rangle} - S_{\langle 2,0\rangle}\right) \end{bmatrix}
\end{bmatrix} .
$$

$$_4B^*_{\langle 0,0\rangle} \;=\; \cfrac{1}{\left[-\tau_o - \tau_b(R-1) + R\left\{\begin{array}{l} R\left[R\left(R-\tau_b(R-1)\right)-\tau_b(R-1)\right] \\ -\tau_b(R-1)\end{array}\right\}\right]} \;\times \qquad \text{(C.16)}$$

$$\left[\begin{array}{l} +\left(1-\tau_o\right)X_{\langle 4,0\rangle} \\[2mm] -\,_4\Delta^*_{\langle 0,0\rangle}\left[\begin{array}{l} -\tau_o S_{\langle 0,0\rangle} - \tau_s\left(S_{\langle 1,0\rangle} - S_{\langle 0,0\rangle}\right) \\[1mm] +S_{\langle 1,0\rangle}\left\{R\left[R\left(R-\tau_b(R-1)\right)-\tau_b(R-1)\right]-\tau_b(R-1)\right\} \end{array}\right] \\[4mm] -\,_4\Delta^*_{\langle 1,0\rangle}\left[\begin{array}{l} +S_{\langle 2,0\rangle}\left\{R\left(R-\tau_b(R-1)\right)-\tau_b(R-1)\right\} \\[1mm] -S_{\langle 1,0\rangle}\left\{R\left[R\left(R-\tau_b(R-1)\right)-\tau_b(R-1)\right]-\tau_b(R-1)\right\} \\[1mm] -\tau_s\left(S_{\langle 2,0\rangle}-S_{\langle 1,0\rangle}\right) \end{array}\right] \\[4mm] -\,_4\Delta^*_{\langle 2,0\rangle}\left[\begin{array}{l} +S_{\langle 3,0\rangle}\left(R-\tau_b(R-1)\right) \\[1mm] -S_{\langle 2,0\rangle}\left\{R\left(R-\tau_b(R-1)\right)-\tau_b(R-1)\right\} \\[1mm] -\tau_s\left(S_{\langle 3,0\rangle}-S_{\langle 2,0\rangle}\right) \end{array}\right] \\[4mm] -\,_4\Delta^*_{\langle 3,0\rangle}\left[\begin{array}{l} +S_{\langle 4,0\rangle} - S_{\langle 3,0\rangle}\left(R-\tau_b(R-1)\right) \\[1mm] -\tau_s\left(S_{\langle 4,0\rangle}-S_{\langle 3,0\rangle}\right) \end{array}\right] \end{array}\right].$$

The general case

The overall coefficient Table C.3 shows the pattern for the overall coefficient.

N	1/(Overall Coefficient)
1	$-\tau_o + R - \tau_b(R-1)(1)$
2	$-\tau_o + R^2 - \tau_b(R-1)\left(1+R\right)$
3	$-\tau_o + R^3 - \tau_b(R-1)\left(1+R+R^2\right)$
4	$-\tau_o + R^4 - \tau_b(R-1)\left(1+R+R^2+R^3\right)$
N	$-\tau_o + R^N - \tau_b(R-1)\sum_{a=0}^{N-1} R^a$

Table C.3: Development of the overall coefficient for the initial bond term

From Table F.3 we can see that the overall coefficient for the general case is:

$$\frac{1}{\left(R^N - \tau_o - \tau_b(R-1)\sum_{a=0}^{N-1} R^a\right)}.$$ (C.17)

The option payoff term   This is straightforward, and the general case is:

$$(1 - \tau_o)\, X_{\langle N,0\rangle}.$$ (C.18)

The deltas   The set of deltas is that given by following the path u^{N-1}, starting at node $\langle 0,0\rangle$. The general case is:

$$-\sum_{a=0}^{N-1} {}_N\Delta^*_{\langle a,0\rangle}.$$

The coefficients to the deltas   For ${}_N\Delta^*_{\langle 0,0\rangle}$, the τ_o and τ_s terms are straightforward. The coefficients of $S_{\langle 1,0\rangle}$ are the same as those given in Table C.3, with $N = N - 1$ and no τ_o term. The general case is:

$$S_{\langle 1,0\rangle}\left(R^{N-1} - \tau_b(R-1)\sum_{a=0}^{N-2} R^a\right).$$ (C.19)

Now for the rest of the deltas.   Let us consider the coefficient to the following delta, $\left(-{}_N\Delta^*_{\langle a,0\rangle}\right)$. There are three terms to the coefficient: a positive stock term, where the stock is that at the "up" node from the delta, $(S_{\langle a+1,0\rangle})$; a negative stock term, where the stock is that at the same node as the delta, $(-S_{\langle a,0\rangle})$; and a term involving τ_s and the two stock prices, given simply by $-\tau_s\left(S_{\langle a+1,0\rangle} - S_{\langle a,0\rangle}\right)$.

$N-a$	$S_{\langle a+1,0\rangle}$ term	$-S_{\langle a,0\rangle}$ term
1	1	$R - \tau_b(R-1)(1)$
2	$R - \tau_b(R-1)(1)$	$R^2 - \tau_b(R-1)(1+R)$
3	$R^2 - \tau_b(R-1)(1+R)$	$R^3 - \tau_b(R-1)\left(1+R+R^2\right)$
$N-a$	$R^{N-1-a} - \tau_b(R-1)\sum_{b=0}^{N-2-a} R^b$	$R^{N-a} - \tau_b(R-1)\sum_{b=0}^{N-1-a} R^b$

Table C.4: Development of the stock terms for the delta coefficients

Table C.4 gives the stock terms in the delta coefficients. Note, Table C.4 is the same as Table C.2.

The general form for the coefficients to the deltas is:

$$
{}_N\Delta^*_{\langle a,0\rangle}
\begin{bmatrix}
-\tau_s \left(S_{\langle a+1,0\rangle} - S_{\langle a,0\rangle}\right) - S_{\langle a,0\rangle}\left(R^{N-a} - \tau_b(R-1)\sum_{b=0}^{N-1-a} R^b\right) \\
+S_{\langle a+1,0\rangle}\left(R^{N-a-1} - \tau_b(R-1)\sum_{b=0}^{N-2-a} R^b\right)
\end{bmatrix}.
\tag{C.20}
$$

Combining the elements When we combine all the elements together we get the general form for ${}_N B^*_{\langle 0,0\rangle}$:

$$
{}_N B^*_{\langle 0,0\rangle} = \frac{1}{\left(R^N - \tau_o - \tau_b(R-1)\sum_{a=0}^{N-1} R^a\right)} \times \tag{C}
$$

$$
\begin{bmatrix}
+ (1-\tau_o) X_{\langle N,0\rangle} \\
-{}_N\Delta^*_{\langle 0,0\rangle}\left[-\tau_o S_{\langle 0,0\rangle} - \tau_s\left(S_{\langle 1,0\rangle} - S_{\langle 0,0\rangle}\right) + S_{\langle 1,0\rangle}\left(R^{N-1} - \tau_b(R-1)\sum_{a=0}^{N-2} R^a\right)\right] \\
-\sum_{a=1}^{N-1} {}_N\Delta^*_{\langle a,0\rangle}
\begin{bmatrix}
-\tau_s\left(S_{\langle a+1,0\rangle} - S_{\langle a,0\rangle}\right) - S_{\langle a,0\rangle}\left(R^{N-a} - \tau_b(R-1)\sum_{b=0}^{N-1-a} R^b\right) \\
+S_{\langle a+1,0\rangle}\left(R^{N-a-1} - \tau_b(R-1)\sum_{b=0}^{N-2-a} R^b\right)
\end{bmatrix}
\end{bmatrix}
$$

Appendix D

Derivation of the N-Period form for the Deltas and Bonds when the Tax Year Occurs During the Life of the Option

As with Appendix C, if we solve the system of simultaneous equations for $N = 2, 3$ and 4, and find formulas for the bonds and deltas in each case, then we may be able to see a pattern and derive a formula for each in the general case. The situation is more complicated here because the tax year-end can occur at any intermediate period on the tree: for these values of N we have six different versions of the deltas and bonds - $N = 2$, $m = 1$; $N = 3$, $m = 1, 2$; $N = 4$, $m = 1, 2, 3$ - as opposed to three versions in Appendix C.

The basic form of the expressions follows those in Appendix C. The additional terms arise due to the tax year-end at m, and in particular the relationship between the tax year-end and the quantity we are looking at.

153

D.0.3 The deltas

$N = 2$, $m = 1$

$$\tfrac{1}{2}\Delta^*_{\langle 0,0 \rangle} = \frac{1}{\left(S_{\langle 1,0 \rangle} - S_{\langle 1,1 \rangle}\right)\left(1 - \tau_s\right)\left[R - \lambda \tau_b(R - 1)\right]} \times$$

$$\begin{bmatrix} + \left(1 - \lambda \tau_o\right)\left(X_{\langle 2,0 \rangle} - X_{\langle 2,2 \rangle}\right) \\ - \left(^{mkt}X_{\langle 1,0 \rangle} - {}^{mkt}X_{\langle 1,1 \rangle}\right)\tau_o\left[R - \lambda \tau_b(R - 1) - \lambda\right] \\ - \tfrac{1}{2}\Delta^*_{\langle 1,0 \rangle}\begin{bmatrix} S_{\langle 2,0 \rangle} - S_{\langle 1,0 \rangle}\left(R - \lambda \tau_b(R - 1)\right) \\ -\lambda \tau_s\left(S_{\langle 2,0 \rangle} - S_{\langle 1,0 \rangle}\right) \end{bmatrix} \\ + \tfrac{1}{2}\Delta^*_{\langle 1,1 \rangle}\begin{bmatrix} S_{\langle 2,2 \rangle} - S_{\langle 1,1 \rangle}\left(R - \lambda \tau_b(R - 1)\right) \\ -\lambda \tau_s\left(S_{\langle 2,2 \rangle} - S_{\langle 1,1 \rangle}\right) \end{bmatrix} \end{bmatrix}. \quad (D.1)$$

At period one, where $j = 0, 1$, we have:

$$\tfrac{1}{2}\Delta^*_{\langle 1,j \rangle} = \frac{\left(1 - \lambda \tau_o\right)\left(X^*_{\langle 2,2j \rangle} - X^*_{\langle 2,2j+1 \rangle}\right)}{\left(1 - \lambda \tau_s\right)\left(S_{\langle 2,2j \rangle} - S_{\langle 2,2j+1 \rangle}\right)}. \quad (D.2)$$

$N = 3$, $m = 2$

$$\tfrac{2}{3}\Delta^*_{\langle 0,0\rangle} = \frac{1}{\left(S_{\langle 1,0\rangle} - S_{\langle 1,1\rangle}\right)\left[R - \tau_b\left(R-1\right) - \tau_s\right]\left[R - \lambda\tau_b(R-1)\right]} \times \qquad \text{(D.3)}$$

$$
\begin{bmatrix}
+ \left(1 - \lambda\tau_o\right)\left(X_{\langle 3,0\rangle} - X_{\langle 3,4\rangle}\right) \\[2pt]
- \left({}^{mkt}X_{\langle 2,0\rangle} - {}^{mkt}X_{\langle 2,2\rangle}\right)\tau_o\left[R - \lambda - \lambda\tau_b(R-1)\right] \\[4pt]
- \tfrac{2}{3}\Delta^*_{\langle 1,0\rangle}
\begin{bmatrix}
+S_{\langle 2,0\rangle} \\
-S_{\langle 1,0\rangle}\left[R - \tau_b(R-1)\right] \\
-\tau_s\left(S_{\langle 2,0\rangle} - S_{\langle 1,0\rangle}\right)
\end{bmatrix}
\left[R - \lambda\tau_b\left(R-1\right)\right] \\[4pt]
- \tfrac{2}{3}\Delta^*_{\langle 2,0\rangle}
\begin{bmatrix}
S_{\langle 3,0\rangle} - S_{\langle 2,0\rangle}\left(R - \lambda\tau_b(R-1)\right) \\
-\lambda\tau_s\left(S_{\langle 3,0\rangle} - S_{\langle 2,0\rangle}\right)
\end{bmatrix} \\[4pt]
+ \tfrac{2}{3}\Delta^*_{\langle 1,1\rangle}
\begin{bmatrix}
+S_{\langle 2,2\rangle} \\
-S_{\langle 1,1\rangle}\left[R - \tau_b(R-1)\right] \\
-\tau_s\left(S_{\langle 2,2\rangle} - S_{\langle 1,1\rangle}\right)
\end{bmatrix}
\left[R - \lambda\tau_b\left(R-1\right)\right] \\[4pt]
+ \tfrac{2}{3}\Delta^*_{\langle 2,2\rangle}
\begin{bmatrix}
S_{\langle 3,4\rangle} - S_{\langle 2,2\rangle}\left(R - \lambda\tau_b(R-1)\right) \\
-\lambda\tau_s\left(S_{\langle 3,4\rangle} - S_{\langle 2,2\rangle}\right)
\end{bmatrix}
\end{bmatrix}
$$

At period one, where $j = 0, 1$, we have:

$$\tfrac{2}{3}\Delta^*_{\langle 1,j\rangle} = \frac{1}{\left(S_{\langle 2,2j\rangle} - S_{\langle 2,2j+1\rangle}\right)\left[1 - \tau_s\right]\left[R - \lambda\tau_b(R-1)\right]} \times \qquad \text{(D.4)}$$

$$
\begin{bmatrix}
+ \left(1 - \lambda\tau_o\right)\left(X_{\langle 3,j2^2\rangle} - X_{\langle 3,2(2j+1)\rangle}\right) \\[2pt]
- \left({}^{mkt}X_{\langle 2,2j\rangle} - {}^{mkt}X_{\langle 2,2j+1\rangle}\right)\tau_o\left[R - \lambda - \lambda\tau_b(R-1)\right] \\[4pt]
- \tfrac{2}{3}\Delta^*_{\langle 2,2j\rangle}
\begin{bmatrix}
S_{\langle 3,j2^2\rangle} - S_{\langle 2,2j\rangle}\left(R - \lambda\tau_b(R-1)\right) \\
-\lambda\tau_s\left(S_{\langle 3,j2^2\rangle} - S_{\langle 2,2j\rangle}\right)
\end{bmatrix} \\[4pt]
+ \tfrac{2}{3}\Delta^*_{\langle 2,2j+1\rangle}
\begin{bmatrix}
S_{\langle 3,2(2j+1)\rangle} - S_{\langle 2,2j+1\rangle}\left(R - \lambda\tau_b(R-1)\right) \\
-\lambda\tau_s\left(S_{\langle 3,2(2j+1)\rangle} - S_{\langle 2,2j+1\rangle}\right)
\end{bmatrix}
\end{bmatrix}
$$

At period two, where $j = 0, 1, 2, 3$, we have:

$$\tfrac{2}{3}\Delta^*_{\langle 2,j\rangle} = \frac{(1 - \lambda\tau_o)\left(X^*_{\langle 3,2j\rangle} - X^*_{\langle 3,2j+1\rangle}\right)}{(1 - \lambda\tau_s)\left(S_{\langle 3,2j\rangle} - S_{\langle 3,2j+1\rangle}\right)}. \tag{D.5}$$

$N = 3,\ m = 1$

$$\tfrac{1}{3}\Delta^*_{\langle 0,0\rangle} = \frac{1}{\left(S_{\langle 1,0\rangle} - S_{\langle 1,1\rangle}\right)\left[1 - \tau_s\right]\left[R^2 - \lambda\tau_b(R-1)(R+1)\right]} \times \tag{D.6}$$

$$\left[\begin{array}{l} + (1 - \lambda\tau_o)\left(X_{\langle 3,0\rangle} - X_{\langle 3,4\rangle}\right) \\[4pt] - \left(^{mkt}X_{\langle 1,0\rangle} - {}^{mkt}X_{\langle 1,1\rangle}\right)\tau_o\left[R^2 - \lambda\tau_b(R-1)(R+1) - \lambda\right] \\[6pt] - \tfrac{1}{3}\Delta^*_{\langle 1,0\rangle}\left[\begin{array}{l} +S_{\langle 2,0\rangle}\left[R - \lambda\tau_b(R-1)\right] \\[3pt] -S_{\langle 1,0\rangle}\left[R^2 - \lambda\tau_b(R-1)(R+1)\right] \\[3pt] -\lambda\tau_s\left(S_{\langle 2,0\rangle} - S_{\langle 1,0\rangle}\right) \end{array}\right] \\[20pt] - \tfrac{1}{3}\Delta^*_{\langle 2,0\rangle}\left[\begin{array}{l} S_{\langle 3,0\rangle} - S_{\langle 2,0\rangle}\left[R - \lambda\tau_b(R-1)\right] \\[3pt] -\lambda\tau_s\left(S_{\langle 3,0\rangle} - S_{\langle 2,0\rangle}\right) \end{array}\right] \\[16pt] + \tfrac{1}{3}\Delta^*_{\langle 1,1\rangle}\left[\begin{array}{l} +S_{\langle 2,2\rangle}\left[R - \lambda\tau_b(R-1)\right] \\[3pt] -S_{\langle 1,1\rangle}\left[R^2 - \lambda\tau_b(R-1)(R+1)\right] \\[3pt] -\lambda\tau_s\left(S_{\langle 2,2\rangle} - S_{\langle 1,1\rangle}\right) \end{array}\right] \\[20pt] + \tfrac{1}{3}\Delta^*_{\langle 2,2\rangle}\left[\begin{array}{l} S_{\langle 3,4\rangle} - S_{\langle 2,2\rangle}\left[R - \lambda\tau_b(R-1)\right] \\[3pt] -\lambda\tau_s\left(S_{\langle 3,4\rangle} - S_{\langle 2,2\rangle}\right) \end{array}\right] \end{array}\right]$$

At period one, where $j = 0, 1$, we have:

$$\tfrac{1}{3}\Delta^*_{\langle 1,j\rangle} = \frac{1}{\left(S_{\langle 2,2j\rangle} - S_{\langle 2,2j+1\rangle}\right)\left[R - \lambda\tau_b(R-1) - \lambda\tau_s\right]} \times \tag{D.7}$$

$$\left[\begin{array}{l} + (1 - \lambda\tau_o)\left(X_{\langle 3,j2^2\rangle} - X_{\langle 3,2(2j+1)\rangle}\right) \\[6pt] - \tfrac{1}{3}\Delta^*_{\langle 2,2j\rangle}\left[\begin{array}{l} S_{\langle 3,j2^2\rangle} - S_{\langle 2,2j\rangle}\left[R - \lambda\tau_b(R-1)\right] \\[3pt] -\lambda\tau_s\left(S_{\langle 3,j2^2\rangle} - S_{\langle 2,2j\rangle}\right) \end{array}\right] \\[18pt] + \tfrac{1}{3}\Delta^*_{\langle 2,2j+1\rangle}\left[\begin{array}{l} S_{\langle 3,2(2j+1)\rangle} - S_{\langle 2,2j+1\rangle}\left[R - \lambda\tau_b(R-1)\right] \\[3pt] -\lambda\tau_s\left(S_{\langle 3,2(2j+1)\rangle} - S_{\langle 2,2j+1\rangle}\right) \end{array}\right] \end{array}\right]$$

At period two, where $j = 0, 1, 2, 3$, we have:

$$\tfrac{1}{3}\Delta^*_{\langle 2,j\rangle} = \frac{(1 - \lambda\tau_o)\left(X^*_{\langle 3,2j\rangle} - X^*_{\langle 3,2j+1\rangle}\right)}{(1 - \lambda\tau_s)\left(S_{\langle 3,2j\rangle} - S_{\langle 3,2j+1\rangle}\right)}. \tag{D.8}$$

$N = 4$, $m = 3$

$$\tfrac{3}{4}\Delta^*_{\langle 0,0\rangle} = \frac{1}{\left(S_{\langle 1,0\rangle} - S_{\langle 1,1\rangle}\right)\left[R^2 - \tau_b(R-1)(R+1) - \tau_s\right]\left[R - \lambda\tau_b(R-1)\right]} \times \tag{D.9}$$

$$\left[\begin{array}{l} + (1 - \lambda\tau_o)\left(X_{\langle 4,0\rangle} - X_{\langle 4,8\rangle}\right) \\[4pt] - \left(^{mkt}X_{\langle 3,0\rangle} - {}^{mkt}X_{\langle 3,4\rangle}\right)\tau_o\left[R - \lambda - \lambda\tau_b(R-1)\right] \\[4pt] - \tfrac{3}{4}\Delta^*_{\langle 1,0\rangle}\left[\begin{array}{l} +S_{\langle 2,0\rangle}\left[R - \tau_b(R-1)\right] \\ -S_{\langle 1,0\rangle}\left[R^2 - \tau_b(R-1)(R+1)\right] \\ -\tau_s\left(S_{\langle 2,0\rangle} - S_{\langle 1,0\rangle}\right) \end{array} \right]\left[R - \lambda\tau_b(R-1)\right] \\[20pt] - \tfrac{3}{4}\Delta^*_{\langle 2,0\rangle}\left[\begin{array}{l} +S_{\langle 3,0\rangle} \\ -S_{\langle 2,0\rangle}\left[R - \tau_b(R-1)\right] \\ -\tau_s\left(S_{\langle 3,0\rangle} - S_{\langle 2,0\rangle}\right) \end{array} \right]\left[R - \lambda\tau_b(R-1)\right] \\[20pt] - \tfrac{3}{4}\Delta^*_{\langle 3,0\rangle}\left[\begin{array}{l} +S_{\langle 4,0\rangle} - S_{\langle 3,0\rangle}\left[R - \lambda\tau_b(R-1)\right] \\ -\lambda\tau_s\left(S_{\langle 4,0\rangle} - S_{\langle 3,0\rangle}\right) \end{array} \right] \\[16pt] + \tfrac{3}{4}\Delta^*_{\langle 1,1\rangle}\left[\begin{array}{l} +S_{\langle 2,2\rangle}\left[R - \tau_b(R-1)\right] \\ -S_{\langle 1,1\rangle}\left[R^2 - \tau_b(R-1)(R+1)\right] \\ -\tau_s\left(S_{\langle 2,2\rangle} - S_{\langle 1,1\rangle}\right) \end{array} \right]\left[R - \lambda\tau_b(R-1)\right] \\[20pt] + \tfrac{3}{4}\Delta^*_{\langle 2,2\rangle}\left[\begin{array}{l} +S_{\langle 3,4\rangle} \\ -S_{\langle 2,2\rangle}\left[R - \tau_b(R-1)\right] \\ -\tau_s\left(S_{\langle 3,4\rangle} - S_{\langle 2,2\rangle}\right) \end{array} \right]\left[R - \lambda\tau_b(R-1)\right] \\[20pt] + \tfrac{3}{4}\Delta^*_{\langle 3,4\rangle}\left[\begin{array}{l} +S_{\langle 4,8\rangle} - S_{\langle 3,4\rangle}\left[R - \lambda\tau_b(R-1)\right] \\ -\lambda\tau_s\left(S_{\langle 4,8\rangle} - S_{\langle 3,4\rangle}\right) \end{array} \right] \end{array} \right]$$

The form of the deltas at periods one, two and three is given by Equations (D.3), (D.4) and (D.5), respectively.

The general case

As mentioned above, the basic form for the delta equations is derived in Appendix C. To derive the general form for the delta, $_N^m\Delta^*_{\langle i,j\rangle}$, as given in Equation (4.11) we start with the basic framework given in Appendix C and look for a pattern in the terms considering the number of periods, N, the tax year-end, m, and the period we are evaluating the delta for, i. To go through the entire procedure would be extremely involved. Instead let us consider an example and show how the overall coefficient, given by β in (4.11), is arrived at.

Finding β in (4.11) Table D.1 shows β for all the deltas given above where $i \geq m$.

Delta	$N - i$	β
$_2^1\Delta^*_{\langle 1,j\rangle}$	1	$[1 - \lambda\tau_s]$
$_3^2\Delta^*_{\langle 2,j\rangle}$	1	$[1 - \lambda\tau_s]$
$_3^1\Delta^*_{\langle 2,j\rangle}$	1	$[1 - \lambda\tau_s]$
$_3^1\Delta^*_{\langle 1,j\rangle}$	2	$[R - \lambda\tau_s - \lambda\tau_b(R - 1)]$
$_4^3\Delta^*_{\langle 3,0\rangle}$	1	$[1 - \lambda\tau_s]$

Table D.1: Coefficient β when $i \geq m$

From Table D.1 we can write down the general form as:

$$\beta = \left[R^{N-i-1} - \lambda\tau_s - \lambda\tau_b(R - 1)\sum_{a=0}^{N-i-2} R^a\right], \tag{D.10}$$

if $i \geq m$.

Table D.2 shows β for all the deltas given above where $i < m$.

Delta	$N-m$	$m-i-1$	β
$\frac{1}{2}\Delta^*_{\langle 0,0\rangle}$	1	0	$[R - \lambda\tau_b(R-1)]\,[1 - \tau_s]$
$\frac{2}{3}\Delta^*_{\langle 1,j\rangle}$	1	0	$[R - \lambda\tau_b(R-1)]\,[1 - \tau_s]$
$\frac{2}{3}\Delta^*_{\langle 0,0\rangle}$	1	1	$[R - \lambda\tau_b(R-1)]\,[R - \tau_b(R-1) - \tau_s]$
$\frac{1}{3}\Delta^*_{\langle 0,0\rangle}$	2	0	$[R^2 - \lambda\tau_b(R-1)(1+R)]\,[1 - \tau_s]$
$\frac{3}{4}\Delta^*_{\langle 2,j\rangle}$	1	0	$[R - \lambda\tau_b(R-1)]\,[1 - \tau_s]$
$\frac{3}{4}\Delta^*_{\langle 1,j\rangle}$	1	1	$[R - \lambda\tau_b(R-1)]\,[R - \tau_b(R-1) - \tau_s]$
$\frac{3}{4}\Delta^*_{\langle 0,0\rangle}$	1	2	$[R - \lambda\tau_b(R-1)]\,[R^2 - \tau_b(R-1)(1+R) - \tau_s]$

Table D.2: Coefficient β when $i<m$

From Table D.2 we can write down the general form as:

$$\beta = \begin{bmatrix} R^{N-m} \\ -\lambda\tau_b(R-1)\sum_{a=0}^{N-m-1} R^a \end{bmatrix} \begin{bmatrix} R^{m-1-i} - \tau_s \\ -\tau_b(R-1)\sum_{a=0}^{m-2-i} R^a \end{bmatrix} ; \qquad \text{(D.11)}$$

if $i < m$.

We can repeat this procedure for all the terms until we arrive at the general form for $\frac{m}{N}\Delta^*_{\langle i,j\rangle}$, given by Equation (4.11).

D.0.4 The bonds

$N = 2$, $m = 1$

$$\frac{1}{2}B^*_{\langle 0,0\rangle} = \frac{1}{[R - \tau_b(R-1) - \tau_o]\,[R - \lambda\tau_b(R-1)]} \times \qquad \text{(D.12)}$$

$$\begin{bmatrix} +(1 - \lambda\tau_o)X_{\langle 2,0\rangle} \\ -\tau_o{}^{mkt}X_{\langle 1,0\rangle}[R - \lambda\tau_b(R-1) - \lambda] \\ -\frac{1}{2}\Delta^*_{\langle 0,0\rangle}\big[S_{\langle 1,0\rangle} - \tau_s\big(S_{\langle 1,0\rangle} - S_{\langle 0,0\rangle}\big) - \tau_o S_{\langle 0,0\rangle}\big][R - \lambda\tau_b(R-1)] \\ -\frac{1}{2}\Delta^*_{\langle 1,0\rangle}\begin{bmatrix} +S_{\langle 2,0\rangle} - S_{\langle 1,0\rangle}[R - \lambda\tau_b(R-1)] \\ -\lambda\tau_s\big(S_{\langle 2,0\rangle} - S_{\langle 1,0\rangle}\big) \end{bmatrix} \end{bmatrix} .$$

$N = 3$, $m = 2$

$$\tfrac{2}{3}B^*_{\langle 0,0\rangle} = \frac{1}{[R^2 - \tau_b(R-1)(R+1) - \tau_o][R - \lambda\tau_b(R-1)]} \times \tag{D.13}$$

$$\left[\begin{array}{l} + (1 - \lambda\tau_o)\, X_{\langle 3,0\rangle} \\[4pt] -\tau_o\,{}^{mkt}X_{\langle 2,0\rangle}\,[R - \lambda - \lambda\tau_b(R-1)] \\[4pt] -\tfrac{2}{3}\Delta^*_{\langle 0,0\rangle} \left[\begin{array}{l} -\tau_o S_{\langle 0,0\rangle} - \tau_s\left(S_{\langle 1,0\rangle} - S_{\langle 0,0\rangle}\right) \\[4pt] + S_{\langle 1,0\rangle}\,[R - \tau_b(R-1)] \end{array} \right] [R - \lambda\tau_b(R-1)] \\[18pt] -\tfrac{2}{3}\Delta^*_{\langle 1,0\rangle} \left[\begin{array}{l} + S_{\langle 2,0\rangle} - S_{\langle 1,0\rangle}\,[R - \tau_b(R-1)] \\[4pt] -\tau_s\left(S_{\langle 2,0\rangle} - S_{\langle 1,0\rangle}\right) \end{array} \right] [R - \lambda\tau_b(R-1)] \\[18pt] -\tfrac{2}{3}\Delta^*_{\langle 2,0\rangle} \left[\begin{array}{l} + S_{\langle 3,0\rangle} - S_{\langle 2,0\rangle}\,[R - \lambda\tau_b(R-1)] \\[4pt] -\lambda\tau_s\left(S_{\langle 3,0\rangle} - S_{\langle 2,0\rangle}\right) \end{array} \right] \end{array} \right].$$

$N = 3$, $m = 1$

$$\tfrac{1}{3}B^*_{\langle 0,0\rangle} = \frac{1}{[R^2 - \lambda\tau_b(R-1)(R+1)][R - \tau_b(R-1) - \tau_o]} \times \tag{D.14}$$

$$\left[\begin{array}{l} + (1 - \lambda\tau_o)\, X_{\langle 3,0\rangle} \\[4pt] -\tau_o\,{}^{mkt}X_{\langle 1,0\rangle}\,[R^2 - \lambda - \lambda\tau_b(R-1)(R+1)] \\[4pt] -\tfrac{1}{3}\Delta^*_{\langle 0,0\rangle} \left[\begin{array}{l} -\tau_o S_{\langle 0,0\rangle} - \tau_s\left(S_{\langle 1,0\rangle} - S_{\langle 0,0\rangle}\right) \\[4pt] + S_{\langle 1,0\rangle} \end{array} \right] [R^2 - \lambda\tau_b(R-1)(R+1)] \\[18pt] -\tfrac{1}{3}\Delta^*_{\langle 1,0\rangle} \left[\begin{array}{l} + S_{\langle 2,0\rangle}\,[R - \lambda\tau_b(R-1)] \\[4pt] - S_{\langle 1,0\rangle}\,[R^2 - \lambda\tau_b(R-1)(R+1)] \\[4pt] -\lambda\tau_s\left(S_{\langle 2,0\rangle} - S_{\langle 1,0\rangle}\right) \end{array} \right] \\[24pt] -\tfrac{1}{3}\Delta^*_{\langle 2,0\rangle} \left[\begin{array}{l} + S_{\langle 3,0\rangle} - S_{\langle 2,0\rangle}\,[R - \lambda\tau_b(R-1)] \\[4pt] -\lambda\tau_s\left(S_{\langle 3,0\rangle} - S_{\langle 2,0\rangle}\right) \end{array} \right] \end{array} \right].$$

$$N = 4, \; m = 3$$

$$
{}^{3}_{4}B^*_{\langle 0,0\rangle} = \frac{1}{[R^3 - \tau_b(R-1)(R^2+R+1) - \tau_o][R - \lambda\tau_b(R-1)]} \times
$$
(D.15)

$$
\left[
\begin{array}{l}
+\left(1 - \lambda\tau_o\right) X_{\langle 4,0\rangle} \\[4pt]
-\tau_o \,{}^{mkt}X_{\langle 3,0\rangle}\left[R - \lambda\tau_b(R-1) - \lambda\right] \\[4pt]
- {}^{3}_{4}\Delta^*_{\langle 0,0\rangle}
\left[
\begin{array}{l}
-\tau_o S_{\langle 0,0\rangle} - \tau_s\left(S_{\langle 1,0\rangle} - S_{\langle 0,0\rangle}\right) \\[4pt]
+S_{\langle 1,0\rangle}\left[R^2 - \tau_b(R-1)(R+1)\right]
\end{array}
\right]\left[R - \lambda\tau_b(R-1)\right] \\[14pt]
- {}^{3}_{4}\Delta^*_{\langle 1,0\rangle}
\left[
\begin{array}{l}
+S_{\langle 2,0\rangle}\left[R - \tau_b(R-1)\right] \\[4pt]
-S_{\langle 1,0\rangle}\left[R^2 - \tau_b(R-1)(R+1)\right] \\[4pt]
-\tau_s\left(S_{\langle 2,0\rangle} - S_{\langle 1,0\rangle}\right)
\end{array}
\right]\left[R - \lambda\tau_b(R-1)\right] \\[20pt]
- {}^{3}_{4}\Delta^*_{\langle 2,0\rangle}
\left[
\begin{array}{l}
+S_{\langle 3,0\rangle} \\[4pt]
-S_{\langle 2,0\rangle}\left[R - \tau_b(R-1)\right] \\[4pt]
-\tau_s\left(S_{\langle 3,0\rangle} - S_{\langle 2,0\rangle}\right)
\end{array}
\right]\left[R - \lambda\tau_b(R-1)\right] \\[20pt]
- {}^{3}_{4}\Delta^*_{\langle 3,0\rangle}
\left[
\begin{array}{l}
+S_{\langle 4,0\rangle} - S_{\langle 3,0\rangle}\left[R - \lambda\tau_b(R-1)\right] \\[4pt]
-\lambda\tau_s\left(S_{\langle 4,0\rangle} - S_{\langle 3,0\rangle}\right)
\end{array}
\right]
\end{array}
\right].
$$

The general case

We use a similar procedure to that used to find β for the deltas, for all the terms in ${}^{m}_{N}B^*_{\langle 0,0\rangle}$ until we arrive at the general form given by Equation (4.10).

Appendix E

List of Figures and Tables

Figures

Tables