MULTIPLE EQUILIBRIA IN THEORY AND PRACTICE

Ahmed Waqar Anwar

University College London

Department of Economics

Thesis submitted for the degree of

Doctor of Philosophy

University of London

September 1998

To my Parents

Acknowledgements

I would like to thank Ken Binmore for his careful supervision and Tilman Börgers for his help during the course of the thesis. I am also thankful to Sandra Semple for her help and support especially over the weeks leading up to submission. Finally, I would like to thank all my colleagues who have made my time at UCL so enjoyable. Funding from the ESRC was greatly appreciated.

ABSTRACT

The first part of the thesis studies equilibrium selection. We use a stochastic evolutionary model characterised by small probability shocks or mutations which perturb the system away from its deterministic evolution, allowing it to move between equilibria over a long period of time. Much of the literature has concentrated on the result that, in the limit as the mutation rate approaches zero, the stationary distribution becomes concentrated on the risk-dominant equilibrium because it is easier to flow into. However, it has been shown that in models of local interaction, allowing player movement eases the flow into the efficient equilibrium. We look at the consequences of such player movement when there are capacity constraints which limit the number of agents who can reside at each location. The limit distribution may then become concentrated on a mixed state in which different locations coordinate on different equilibria.

The second part looks at the problem of characterising equilibria in multi-unit auctions. Surprisingly little is known about optimal mechanism design for multi-unit auctions relative to the single-unit auctions. This is highlighted by the continuing debate on whether the US Treasury should use a discriminatory or uniform pricing rule. These questions have become of wider practical interest as a result of the innovative use of auction theory in the England and Wales Electricity Pool. We compare the two pricing rules in a common-value model with capacity constraints and uncertain demand and show that the discriminatory pricing rule performs better. We also present a model of the Electricity Pool and show that a discriminatory pricing rule would lead to more competitive prices than the current uniform pricing rule. The ranking holds even in the repeated game case, despite the problem of multiple equilibria.

Contents

List of figures 8

List of tables 8

1 Introduction 9

1.1 The Equilibrium Selection Problem 11

1.1.1 Equilibrium Refinements 11

1.1.2 The Evolutionary Approach 15

1.2 Equilibria in Multi-Unit Auctions 23

1.3 Summary of Results 32

2 Equilibrium Selection in Games 35

2.1 Stochastic Techniques 40

2.1.1 Friedlin and Wentzell 40

2.1.2 Kandori, Mailath and Rob 42

2.1.3 Young 46

2.2 When Does Immigration Facilitate Efficiency 47

2.2.1 Two Islands 47

2.2.2 Three Islands 57

2.2.3 Inertia in Strategy Revision 62

2.2.4 No Capacity Constraints 65

2.3 Conclusions 67

2.4 Appendix 68

3 Multi-Unit, Common-Value Auctions 73

3.1 Multiple Equilibria in Uniform-price, Multi-Unit Auctions 76

3.2 Discrete Units 79

3.3 Capacity Constraints and Uncertain Demand 83

3.3.1 Discriminatory Auction 84

3.3.2 Uniform-price Auction 94

3.3.3 Ranking 100

3.4 Conclusions 103

3.5 Appendix 105

4 Modelling the Electricity Pool 107

4.1 Price Determination Process in the England And Wales Pool 111

4.2 Electricity Pool Literature 114

4.2.1 Green and Newbery 114

4.2.2 von der Fehr and Harbord 119

4.2.3 Wolak and Patrick 123

4.3 The Case for a Discriminatory Auction in the England and Wales Pool 126

4.3.1 A model of the England and Wales Pool 127

4.3.2 Alternative Pricing Rules 141

4.3.3 Ranking the Pricing Rules in the England and Wales Pool 144

4.3.4 Repeated Game Analysis 146

4.4 Conclusions 148

4.5 Appendix 150

References 153

List of figures.

Figure 1.1	12
Figure 1.2	14
Figure 2.1	42
Figure 2.2	45
Figure 2.3	47
Figure 2.4	53
Figure 2.5	53
Figure 2.6	56
Figure 2.7	60
Figure 2.8	60
Figure 2.9	61
Figure 2.10	64
Figure 2.11	70
Figure 3.1	76
Figure 3.2	78
Figure 3.4	80
Figure 4.1	116
Figure 4.2	117
Figure 4.3	130
Figure 4.4	133
Figure 4.5	134
Figure 4.6	135
Figure 4.7	150
Figure 4.8	152
Figure 4.9	152

List of tables.

Table 2.1	54
Table 4.1	151
Table 4.2	152

Chapter 1
Introduction

The first part of the thesis looks at the question of equilibrium selection. The limitations of the early approach of equilibrium refinements is illustrated by the failure of this literature to predict equilibrium play correctly in numerous laboratory experiments. Most of the literature also ignores the question of selecting between strict Nash equilibria. Evolutionary game theory addresses the question of equilibrium selection by modelling the way agents adjust their strategies out of equilibrium. The evolutionary approach has been successful both in explaining some of the experimental evidence and addressing the question of selecting between strict Nash equilibria. These ideas are discussed in more detail in section 1.1. In chapter 2, we address the question of selecting between strict Nash equilibria. We present a stochastic evolutionary model with player movement and capacity constraints limiting the number of agents who can reside at each location.

The second part of the thesis looks at equilibria in multi-unit auctions. Since the seminal work of Vickrey (1961), there has been an explosion of literature on single-unit auctions. However, surprisingly little is known about optimal mechanism design for multi-unit auctions relative to the single-unit auctions. This is highlighted by the continuing debate on whether the US Treasury should use a discriminatory or uniform pricing rule. These questions have become of wider practical interest as a result of the innovative use of auction theory in the

England and Wales Electricity Pool. Section 1.2 looks at the main results on single-unit auctions and at what the multi-unit auction literature has to say about the discriminatory auction *vs* uniform-price auction debate. In chapters 3 and 4, we compare the auctions in specific multi-unit models. Chapter 3 looks at a common-value auction with capacity constraints where the quantity for auction is uncertain. In chapter 4, we present an explicit model of the England and Wales Electricity Pool.

1.1 The Equilibrium Selection Problem.

A Nash equilibrium is a set of strategies such that each player is optimising given the strategies of the other players. Nash (1950) showed that every finite game has at least one Nash equilibrium. A question that has been the focus of much research since is which equilibrium will be played when a game has multiple equilibria. The early approach was to refine the set of equilibria by eliminating equilibria that are not plausible when the game is played by rational agents. This approach is briefly discussed in the next section. However, experimental evidence has shown that the predictions of the equilibrium refinement literature are not always correct. Game theorists have turned instead to the evolutionary approach. By modelling how agents adjust their strategies out of equilibrium we can analyse how a population settles on one of the equilibria. This approach is discussed in section 1.1.2.

1.1.1 Equilibrium Refinements.

Most of the equilibrium refinements are based on eliminating weakly dominated strategies. The first prominent refinement was Selten's subgame perfection. To illustrate the idea, consider the Chain-Store Game of figure 1.1. Player 1, the entrant, moves first by deciding whether to enter the market and compete with the incumbent monopolist. If he stays out, his payoff is zero and the incumbent firm earns the monopoly rent. If player 1 enters, his payoff depends on whether the incumbent fights, F, or acquiesces, A. The game has two pure-strategy equilibria, {O,F} and {I,A}. However, the first equilibrium involves a threat by player 2 to fight in the event that player 1 enters, even though he would be better off acquiescing.

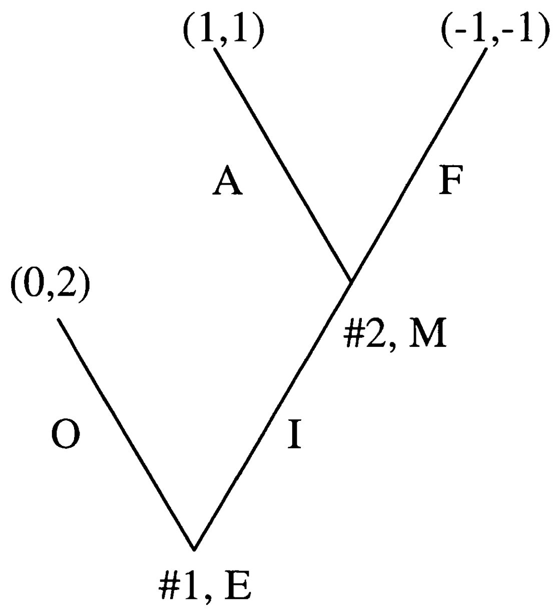


Figure 1.1
The Chain-Store Game.

Selten (1965) argued that for an equilibrium to be plausible it must consist of an equilibrium in every subgame, including those that are not on the equilibrium path. Such an equilibrium is referred to as subgame perfect. Hence the equilibrium {O, F} is not subgame perfect as the play in the subgame where the monopolist makes a move is not an equilibrium. Subgame perfection therefore involves eliminating some of the weakly dominated strategies. A refinement that takes this elimination a step further is Selten's *trembling hand* perfection (1975). The idea underlying perfection is that players sometimes make mistakes with the consequence that a strategy is selected at random. If there is a positive probability of selecting every strategy then every information set is reached with a positive probability and an equilibrium would therefore involve maximising behaviour at every information set. Selten defines a perfect equilibrium as one that arises when the mistake probability goes to zero. Myerson (1978) shows that adding strictly dominated strategies may change the set of perfect equilibria and he introduces a further refinement. He postulates that rational players are more likely to make mistakes that are less costly. One of many other refinements to the perfectness concept is strict perfectness (Okada 1981). A strictly perfect equilibrium is stable against arbitrary slight perturbations.

Kreps and Wilson (1982) take an alternative approach to refining subgame perfection. They assume that agents will maximise utility in the face of uncertainty using subjective probabilities. Hence, when an information set that is not on an equilibrium path is reached, the agent will make a best reply according to his beliefs about the state of the game. If there exists a set of beliefs such that each player optimises by continuing to play according to the equilibrium, the equilibrium is sequential. For a comprehensive guide to the equilibrium refinement literature the reader is referred to van Damme (1991). Although the refinement literature has had some success in dealing with the question of equilibrium selection it has two major problems: some of the predictions of the refinement literature have been refuted by experimental evidence; equilibrium refinements have nothing to say about the choice between strict equilibria.

Game theorists have been critical of many of the experiments that refute game-theoretic predictions. Binmore (1992) asserts that if one is to have any faith in the experimental results then the following conditions should be satisfied: the game must be reasonably simple; the incentives should be adequate; there should be sufficient time for trial and error learning to take place. In practice the game-theoretic predictions have performed much better when these conditions are satisfied (Ledyard 1995). However, there are some exceptions, the most notable of which is the Ultimatum Game. Binmore, Gale and Samuelson (1995) argue that it was wrong for game theorists to assume that players would always play according to a subgame-perfect equilibrium, which is based on eliminating weakly dominated strategies. They use an alternative evolutionary approach based on interactive learning to explain the experimental outcomes. This is discussed in the next section.

The second difficulty with the equilibrium refinement approach is that it has nothing to say about the choice between strict equilibria. Consider the Coordination Game of figure 1.2.

	s_1	s_2
s_1	5,5	0,3
s_2	3,0	4,4

Figure 1.2

The game has two strict pure-strategy equilibria, $\{s_1, s_1\}$ and $\{s_2, s_2\}$, which survive all the equilibrium refinements based on eliminating weakly dominated strategies. Hence the scope of the equilibrium-refinement literature is limited. To address the question of choosing between strict equilibria in a Coordination Game, Schelling (1960) appeals to the prominence of efficiency. Agents will play for the prominent payoff-dominant equilibrium, $\{s_1, s_1\}$, in the expectation that other agents will be similarly attracted by its focal status. But Harsanyi and Selten (1988) have emphasised that such an expectation may not be well-founded. If each player optimises on the assumption that the opponent is equally likely to play either strategy, the outcome will be $\{s_2, s_2\}$, which therefore also has a focal status that may outweigh that of the payoff-dominant equilibrium. Evolutionary game theory has given the argument another perspective. By modelling the process by which agents adjust their strategies out of equilibrium we can analyse how it is that one equilibrium is selected rather than another. In chapter 2, we use an evolutionary model to address the question of equilibrium selection in 2x2 Coordination Games.

1.1.2 The Evolutionary Approach

Evolutionary game theory was developed by biologists to model situations in which the fitness (or reproductive success) of a gene depends on the current mix of genes in the population. One can think of the evolutionary game as being played between genes which are programmed to give their host certain characteristics. Together with the current mix of genes in the population, the fitness of a gene is determined by these characteristics as the population of hosts are competing for scarce resources. Genes that engender successful behaviour will therefore gain in frequency relative to those that result in lower reproductive success. A dynamic system known as the replicator dynamics (Taylor and Jonker 1978, Zeeman 1981) based on this type of selection is derived below.

The equilibrium notion in evolutionary game theory is an Evolutionary Stable Strategy (ESS) which is a refinement of Nash equilibrium. In a symmetric game an ESS is a strategy that is a best reply to itself and a better reply to any alternative best reply than the alternative is to itself (Smith and Price 1973). The basic idea is that an equilibrium in which an ESS is being used by the whole population is stable against invasion by a small number of mutants.

To look at these ideas more formally we need to introduce some notation. Consider a population of agents who are randomly paired to play a symmetric two-player game, G. Let S=(s_1, s_2,.....s_n) be the set of pure strategies of G. Denote the set of mixed strategies by ΔS. The payoff to an agent using strategy x against one using strategy y is given by P[x, y] where x,y ∈ ΔS. Then Smith and Price define an ESS as a strategy x that satisfies the following two conditions

$$
\begin{aligned}
&P[x,x] \geq P[y,x] \qquad\qquad \forall y,\\
&P[x,x] = P[y,x] \Rightarrow P[x,y] > P[y,y].
\end{aligned}
$$

To get a more intuitive understanding of an ESS consider the following alternative definition used by Taylor and Jonker (1978). A state x is an ESS if for every y$\neq$x and for a sufficiently small ε>0

$$P[x,\varepsilon y+(1-\varepsilon)x] > P[y,\varepsilon y+(1-\varepsilon)x].$$

In a biological context, if all the population is using an ESS then the equilibrium is stable against mutations in genes. If a small number of mutations occur, the reproductive rate of the mutants will be less than the average rate and as a result the proportion using the mutant strategy will diminish. In an economic context, one can think of an ESS as a convention. Given that everybody else in the population is using the strategy, the optimal thing to do is conform to the convention. Furthermore, the strategy is stable against experimenters as the strategy does better against a population mix made up partly of experimenters than experimenters do, as long as the number of experimenters is sufficiently small. Hence the experimenters will soon switch back to the convention. The ESS concept says nothing about how an equilibrium is reached. It simply gives a stability property that an equilibrium 'should' have.

The most commonly used dynamic system in the biological literature is the replicator dynamics of Taylor and Jonker (1978). We now think of players as being programmed with pure strategies. This differs from the ESS analysis which allows for mixed strategies. Let the number of players programmed to use strategy s_i at time t be $p_i(t)$ and $x_i(t)$ be the corresponding proportion of the population using the strategy. Hence the expected payoff to a player using strategy s_i when randomly matched with someone in the population is

$$\sum_{j=1}^{n} x_j(t)P[s_i, s_j].$$

This is formally equivalent to playing against an opponent using the mixed strategy $x(t)=(x_1(t), x_2(t), \ldots x_n(t))$ with a payoff of $P[s_i,x]$. Similarly the average payoff in the population is $P[x,x]$. These proportions change over time as some of the hosts die and new hosts programmed to use the strategy of their single parent are born. If reproduction takes place continuously over time then $P[s_i, x(t)]$ represents the incremental effect on the birth rate from playing the game. If there is a background birth rate β that is independent of the game and the death rate is α, then the population dynamics is given by

$$\dot{p}_i(t) = [\beta + P[s_i, x(t)] - \alpha]p_i(t).$$

To derive the dynamics for the population proportions, $x_i(t)$, take the time derivative of both sides of the identity $p(t)x_i(t)=p_i(t)$ where $p(t)$ is the size of the population at time t. This gives $p(t)\dot{x}_i(t) = \dot{p}_i(t) - \dot{p}(t)x_i$. Substituting for $\dot{p}_i(t)$ and $\dot{p}(t)$ gives

$$\dot{x}_i(t) = [P[s_i, x(t)] - P[x(t), x(t)]x_i(t).$$

This derivation of the standard continuous time replicator equation is given in Weibull (1995). The growth rate of a strategy at any time is simply the difference between the payoff from using the strategy and the average payoff in the population.

The application of evolutionary game theory to an economic setting involves a re-labelling. The characteristics that genes are programmed to give their host are replaced by strategies that people use in interaction with each other. Fitness is replaced by the utility of agents and selection is driven by an increase in the proportion of the population adopting strategies that result in a higher than average utility. This allows us to address the question of equilibrium selection by modelling the actual behaviour of the population out of equilibrium. The assumption that agents are mathematical machines programmed to always make optimal choices can be put aside. Agents are assumed to maximise utility but are only assumed to be *boundedly rational* and at any point in time there may be agents who are making sub-optimal choices. Over a period of time, however, agents will adjust these choices as they learn about the game and imitate those who are using strategies that yield relatively high payoffs. Hence selection in a biological context is replaced by learning in an economic context. Eventually the population will settle on an equilibrium of the game which is represented by a rest point of the dynamics. Crucially, the equilibria that are selected in this way do not always correspond to those selected in the equilibrium-refinement literature. In particular weakly dominated equilibria are not necessarily eliminated.

Binmore, Gale and Samuelson (1995) demonstrate this in the Ultimatum Game. The Ultimatum Game is played between two players, a proposer and a responder. The proposer makes an offer to divide a cake which the responder can either accept or reject. If the responder accepts then the cake is divided according to the offer. If he rejects then both players get nothing. The unique subgame-perfect equilibrium of this involves the responder accepting any offer and the proposer offering nothing. This prediction has been consistently refuted by experimental evidence. Binmore, Gale and Samuelson show that an evolutionary analysis can lead to equilibria that involve the proposer offering a substantial amount to the responder thus going some way towards explaining the experimental results. They assume that agents are randomly matched in pairs from a population of proposers and responders to play a version of the Ultimatum Game where the proposer makes an offer from the set $\{1,2,3,....,40\}$ and a strategy for the responder is a minimum acceptable offer. Using the simple replicator dynamics with uniform initial conditions, they find that the system converges to an equilibrium where the modal offer of the proposer is 9. They also consider a *noisy* version of the replicator dynamics where each period a small proportion of each population chooses a strategy at random. The subgame-perfect equilibrium only appears from the uniform initial conditions if the responder population is sufficiently less noisy than the proposer population.

The key lesson from this is that we cannot eliminate weakly dominated equilibria if the process by which an equilibrium is reached is *evolutive* in nature. The question of equilibrium selection should then be addressed by modelling the dynamic process by which an equilibrium is reached. The replicator dynamics is used quite generally in a biological context but there is no reason to assume that a model of strategy adjustment through interactive learning will lead to these dynamics. In fact, the dynamics will vary according to the assumptions made about the strategy adjustment process.

However, a surprising amount can be said about equilibrium selection with minimal assumptions on the dynamic process. If the dynamics are such that strategies that currently yield a higher than average payoff are used by a greater proportion of the population in future periods, then the equilibrium the system converges to will simply depend on the point where the process began. In a 2x2 Coordination Game such as the one given in figure 1.2, if the process starts at a point where a significant number of agents are playing strategy s_1 then the optimum response is to play s_1. Agents who are using the other strategy will adjust their strategy when they learn that it is better to switch. The process will then converge on the equilibrium where everyone plays s_1.

However, this is not a complete description of the system if agents make mistakes by occasionally choosing a strategy at random. If each agent makes a mistake with some probability ε by simply choosing a strategy at random and each strategy is then selected with a positive probability, the system can be described by an aperiodic and irreducible Markov process which has a unique stationary distribution. If the mutation rate is very small then from any initial condition the process is likely to follow the 'learning' dynamics and converge to one of the equilibria of the game. However, there will eventually be enough simultaneous mutations to move the system into the vicinity of another equilibrium. The system is then likely to converge to this equilibrium and will remain there until it is once again moved by a large number of simultaneous mutations. Over a long period of time one will find that the relative time spent in or close to some equilibria will be greater than others. Kandori *et al* (1993) and Young (1993) consider the limiting distribution, as the mutation rate is allowed to go to zero. They show the stationary distribution then becomes concentrated on a subset of the equilibria and very often on a unique equilibrium. An equilibrium selected in this way is referred to as a long-run equilibrium. This is a very strong result when one considers the minimal assumptions made on the strategy adjustment process[1].

The theoretical literature has concentrated attention on the 2x2 Coordination Game of the type given in figure 1.2, where one equilibrium is risk-dominant and the other one is payoff-dominant. Does an evolutionary analysis pick Schelling's payoff-dominant equilibrium or Harsanyi and Selten's risk-dominant equilibrium? We focus on this question in chapter 2. In the models presented by Kandori *et al* and Young the unique long-run equilibrium of the game is the risk-dominant equilibrium.

We explain the techniques used to characterise the limit of the stationary distribution in section 2.1. We present a model of local interaction where agents are only paired with players from the same location in section 2.2. We consider

[1] However, it is necessary to assume that the probability with which each strategy is selected when a mutation occurs is independent of ε. If the ratio of probabilities with which strategies are selected when a mutation occurs are allowed to approach zero as ε approaches zero then Bergin and Lipman (1996) show that any outcome can be obtained in the limit.

the effects of allowing agents to move, with a capacity constraint limiting the number of agents who can reside at each location. In the case where there are two locations, we find that in the limit it is possible to have equilibria where one location coordinates on the risk-dominant equilibrium and the other one on the efficient one. We show that the situation where only the risk-dominant equilibrium is played in the limit requires that the capacity constraint, limiting the number of agents that can reside at each location, to be sufficiently tight.

1.2 Equilibria in Multi-Unit Auctions.

In theory one can design the optimal mechanism for the sale of multiple objects. By the Revelation Principle the designer can restrict attention to direct, incentive-compatible mechanisms. The problem of multiple equilibria would not arise as the mechanism would be designed to elicit truthful revelation. In practice, relatively little is known about optimal mechanisms for the sale of multiple objects. A first step to obtaining a greater understanding is to compare mechanisms that are currently used. Even this has proved difficult however. In the sale of Treasury bonds, for example, the US Treasury have switched between using a discriminatory auction and a uniform-price auction. In both cases the participants submit demand schedules reflecting the maximum price they are willing to pay for various quantities. These bids are used to construct the aggregate demand schedule and if the number of bonds for sale is n then the n highest bids win. Under a discriminatory auction the bid price is paid for units won. With a uniform pricing rule, all winning bids pay the bid price of the lowest winning bid. The question of which one results in the higher revenue is still open.

The theory of single-unit auctions is relatively well developed. The three auction formats that are commonly used are, the English auction where the price is raised until only one bidder remains, the Dutch auction where the price is lowered from an initial high level until a bidder accepts the current price and the first-price sealed-bid auction where the highest bidder wins and pays his bid price. Another possibility that has been considered in the literature but is seldom used in reality is the second-price or Vickrey sealed-bid auction, where the highest bidder wins and pays the second highest bid price.

It is well understood that the first-price auction and the Dutch auction are strategically equivalent as the bidders in a Dutch auction must simply decide the price at which they will stop the auction. In a model of independent private values[2], the second-price auction and the English auction are also equivalent. In the second-price auction it is a dominant strategy to submit a bid equal to your valuation, while in an English auction it is a dominant strategy to bid until the auction price reaches your valuation. Both auctions therefore result in an outcome that is efficient, as the object is sold to the bidder that values it most highly. The Dutch and first-price auctions also result in an efficient outcome as in equilibrium, all bidders shade their bids symmetrically. Moreover, in such equilibria the bidder who values the object most highly will optimise by bidding at the expected value of the second highest bid. This gives the famous revenue equivalence result: the expected revenue to the seller is the same under all four auctions (Vickrey (1961), Myerson (1981)). In fact, with an optimally determined reserve price, the four auctions are also optimal mechanisms (Myerson (1981)). This result is one of the major achievements of mechanism design theory as it shows that there is no elaborate mechanism that will result in a higher expected revenue than the four simple auctions.

The prevalence of the English auction in practice can be explained by relaxing the assumptions on which the Revenue Equivalence Theorem is based. For example, when bidders' valuations are affiliated, the English auction yields a higher expected revenue than the other three auction formats (Milgrom and Weber 1982). The reason is that the English auction process conveys information

[2] Where each bidder has a private value that is independent of the valuations of the other bidders.

to the bidders on the valuations of other bidders. Under the other three auction formats, no such information is conveyed and bidders therefore reduce their bids to account for the fact that they have the highest valuation if they win[3]. The first-price auction is also used widely in practice, and this can be explained by the fact that it performs better than the English auction if the bidders are risk-averse (Holt 1980). This is because under a first-price auction, bidders will increase their bids towards their valuation if they are risk-averse, to increase the probability that they win.

However, the English auction is not the optimal mechanism with affiliated values and the first-price auction is not the optimal mechanism when bidders are risk-averse. More elaborate mechanisms can be designed that increase the expected revenue to the seller. For example, when bidders are risk-averse the optimal auction involves subsidising high losing bidders and penalising low bidders (Maskin and Riley 1984). Such mechanisms are rarely observed in practice. One reason for this is that the optimal mechanism is difficult to characterise when a single assumption is relaxed. Hence most of the literature concentrates on optimal mechanisms when just one or two assumptions are relaxed. The problem of designing optimal mechanisms for complex economic environments is considered to be intractable. A second reason that elaborate mechanisms are rarely used in practice is that they are complicated relative to the simple auctions. Real economic agents are at best boundedly rational, unlike the idealised agents of orthodox implementation theory. The rules of the mechanism therefore need to be sufficiently simple for all to understand.

[3] A bidder that does not take this into account will find that their valuation of the object is reduced in the event that they win the object. This is known as the "winner's curse".

A great deal of work has been done in extending the single-unit results to the case of multiple units. The Revenue Equivalence Theorem extends to the case where there are multiple units and each bidder demands one unit (Harris and Raviv 1981, Maskin and Riley 1989). Maskin and Riley also investigate optimal auctions in the case that bidders demand multiple units and show that neither the discriminatory auction nor the uniform-price auction is optimal. However, as I pointed out earlier, elaborate mechanisms are not used in practice. The optimal auction can be complicated under simple assumptions even in the single-unit case. It is correspondingly more complicated in the multi-unit case. The problem of designing the optimal auction for the sale of Treasury bonds is therefore very difficult and this is generally true of complex economic environments.

The public debate on the mechanism used for the sale of Treasury bonds has therefore concentrated on the choice between the discriminatory auction and the uniform-price auction. Even this question, however, remains unresolved. The conceptual difficulty of multi-unit auctions is highlighted by a false analogy that is made between the second-price, sealed-bid auction for a single unit and a uniform-price auction for multiple units. For example McAfee and McMillan (1987) state: "Both the discriminatory auction and the uniform-price auction have been used to sell Treasury Bills. Because this is a common-value setting, theory predicts the uniform-price auction, which is similar to the second-price auction, yields more revenue than the discriminatory auction, which corresponds to the first-price auction". The theory they are referring to is the affiliated valuations model of Milgrom and Weber (1982).

Milton Friedman and Merton Miller seem to take the analogy even further by suggesting that under a uniform-price auction it is a dominant strategy to bid one's true demand curve. In the *Wall Street Journal* (August 28, 1991) Friedman states: "A [uniform-price] auction proceeds precisely as a discriminatory auction with one crucial exception: All successful bidders pay the same price, the cut-off price. An apparently minor change, yet it has the major consequence that no one is deterred from bidding by fear of being stuck with an excessively high price. You do not have to be a specialist. You need only to know the maximum price you are willing to pay for different quantities." Merton Miller in an interview with the New York Times explains why the bidders have an incentive to shade their bids under a discriminatory auction. He then says of a uniform-price auction: "You just bid what you think it's worth." This argument was part of the reason the US Treasury experimented with the uniform-price auction in the early 90's. A report by the Treasury Department, the Securities and Exchange Commission and the Federal Reserve Board[4] concluded that: "Moving to a uniform-price award method permits bidding at the auction to reflect the true nature of investor preferences... . In the case envisioned by Friedman, uniform-price awards would make the auction demand curve identical to the secondary market demand curve."

Recently Back and Zender (1993), Wang and Zender (1995), Ausubel and Crampton (1995) and Binmore and Swierzbinski (1997), have all illustrated that the bidders do have an incentive to shade their bids under a uniform pricing rule. The reason is there is a chance that one of the bids of a bidder will be the marginal bid that determines the uniform price. Bidders can reduce the price they pay for all the units they win in this event by shading bids. In fact, this was first noted by

[4] Joint Report on the Government Securities Market (1992, p. B-21).

Vickrey (1961), who also gives the correct multi-unit extension of the second-price auction[5]. Back and Zender and Wang and Zender use the share auction framework of Wilson (1979), where the good is perfectly divisible and has a common value. They find that any price between the reservation price and the lower bound of the common-value distribution can be supported as a symmetric Nash equilibrium. There is therefore a multiple-equilibrium problem. Binmore and Swierzbinski show that the multiple-equilibrium result holds when the bidders have private values.

Ausubel and Crampton (1995) concentrate on the relative efficiency of the auctions. They prove an inefficiency theorem for the uniform-price auction which applies when there is a private-values component and bidders demand more than one unit. The inefficiency arises from the fact that large bidders will shade more than small bidders and sometimes lose units to small bidders who actually value them less. They propose an ascending-bid auction based on the multi-unit Vickrey (sealed-bid) auction. The chief advantage of a Vickrey auction is that it is a dominant strategy to bid true valuations, with the result that the outcome is efficient. They also show that the revenue ranking is ambiguous and give examples where the Vickrey auction revenue-dominates both the uniform-price auction and the discriminatory auction.

The revenue-ranking debate has resulted in empirical research using natural experiments. Simon (1994) estimates that the switch from a discriminatory pricing rule to a uniform one in the 1970's cost the US Treasury \$7 thousand to \$8 thousand for every \$1 million of bonds sold. However, Umlauf (1993)

[5] The discrete multi-unit Vickrey auction is presented in the appendix of chapter 4.

estimates that the Mexican Treasury gained by switching to a uniform-price auction for the sale of 30-day bills although the gains were relatively small. Tenorio (1993) looks at the Zambian government's sale of US dollars to importers who switched from a uniform-price auction to a discriminatory one. The conclusion after controlling for factors such as an increase in the number of dollars auctioned was that the switch resulted in a loss to the government even though the average price received per dollar was substantially increased. The evidence from the natural experiments is therefore inconclusive.

It is clear that unlike in the single-unit case, relatively little can be said in general about the revenue ranking of the auctions in the multi-unit case. The difficulty of modelling auctions with multiple units has led to misguided analogies with single-unit auctions. In practice this has resulted in institutions experimenting with both the uniform-price and discriminatory auctions with inconclusive results.

The practical significance of multi-unit auctions has widened as a result of their use in cases where a single buyer wishes to purchase multiple units of a good from a number of sellers. We refer to such auctions as 'reverse' auctions. The methodology is exactly the same. Suppliers submit supply schedules detailing the minimum price at which they are willing to supply various quantities. The bids are ranked and the lowest bids win. Such a system has been used in the supply of electricity since 1990 in England and Wales. As part of the privatisation process, the generators of the state owned monopoly were split between three companies. These companies compete to supply electricity to the *Electricity Pool*, through a uniform-price, multi-unit, reverse auction. Similar systems have been adopted elsewhere.

A constraint that arises naturally in reverse auctions is a limit on the number of units a firm can bid. The constraint simply reflects the output capacity of the suppliers. For example, in the case of the Electricity Pool, the constraint simply reflects the generating capacity. The models of Ausubel and Crampton and Back and Zender account for the case where each firm has a maximum amount they can bid for but only to the extent that there is always competition for every unit. In the case of the Electricity Pool, the larger firms will have some residual monopoly in periods of high demand as the total capacity of the other firms will be insufficient to meet demand.

In chapter 3, we look at common-value, multi-unit auctions. We show that the multiple equilibria that Back and Zender find in the case when the good is perfectly divisible do not carry over to the case where units are discrete. In section 3.2, we present a discrete multi-unit, common-value auction model with capacity constraints, where the quantity for auction is uncertain and compare the equilibria under the uniform and discriminatory pricing rules. We show that the discriminatory auction results in a lower expected cost to the buyer (higher expected revenue for the seller). Although the assumptions are motivated by reverse auctions, the results can be applied to conventional auctions.

Theoretical papers on the Electricity Pool have concentrated on the performance of the current uniform pricing rule and the consequences of increasing the number of bidders, (Green and Newbery 1992, von der Fehr and Harbord 1993). Wolak and Patrick (1996) characterise the actual bidding behaviour using data from the Pool. They explain why the present structure gives the generators an incentive to withhold capacity and present evidence of this strategy being used. In chapter 4, we present a model of the Pool which captures the incentive to withhold capacity under a uniform pricing rule. We then look at equilibria under two alternative pricing rules, discriminatory and Vickrey.

1.3 Summary of Results.

In chapter 2, we use a stochastic evolutionary model to address the question of equilibrium selection in 2x2 Coordination Games. Much of the literature has concentrated on the result that, in the limit as the mutation rate approaches zero, the stationary distribution becomes concentrated on the risk-dominant equilibrium because it is easier to flow into. However, it has been shown that in models of local interaction, allowing player movement eases the flow into the efficient equilibrium. We look at the consequences of such player movement when there are capacity constraints which limit the number of agents who can reside at each location.

In the case of two locations, we show that, when the capacity constraints are sufficiently tight, the risk-dominant equilibrium continues to be selected. However, as the capacity constraint is relaxed, the equilibrium switches from the risk-dominant equilibrium to states in which one location coordinates on the efficient equilibrium and the other on the risk-dominant equilibrium. We extend the analysis to the case of three locations and also to the case where there is inertia in strategy revision. In the three location case, the equilibrium switches from the risk-dominant equilibrium to states in which two locations coordinate on the efficient equilibrium and the other on the risk-dominant equilibrium. We show that the results are the same when we model inertia in the strategy-adjustment process, although the equilibria are slightly more difficult to characterise. When the capacity constraint is relaxed, the equilibrium selected involves everyone playing the efficient strategy.

In chapter 3, we model common-value, multi-unit auctions. We show that the multiple-equilibrium problem in uniform-price auctions that has been identified in the literature disappears with discrete units as long as bids are allowed in sufficiently small increments. We then present a discrete model with capacity constraints and uncertain demand. When there is no binding capacity constraint there is a unique type of pure-strategy equilibrium under both auction formats in which the marginal price is equal to marginal cost. Both auction formats therefore result in a competitive equilibrium. When it is certain that each firm will have some residual market share, however, the expected cost is greater under a uniform-price auction. Under the uniform pricing rule there is a unique type of equilibrium in which the marginal price (and therefore the price paid for all units) is equal to the maximum permissible price. Under a discriminatory auction there is no pure-strategy equilibrium. We characterise a mixed-strategy equilibrium that holds for any distribution of the quantity up for auction.

In chapter 4, we present a model of the Electricity Pool. Under a uniform pricing rule, the firms maximise profits by withholding base-load capacity to increase the probability that the marginal price is set by peak-load units which can be bid at much higher prices. This results in prices substantially above marginal cost.

Such an incentive does not exist under a discriminatory pricing rule as the price paid for each unit is simply the bid price. The average prices in the discriminatory equilibrium are therefore much lower than under a uniform-price auction. The case for a discriminatory auction is even stronger in the repeated game for two reasons: 1) collusive behaviour would be easy to detect as it would involve bidding high prices for all units and not just manipulating the marginal price; 2) even if the firms could collude they are limited in the profits they can make. In fact, in our model, we show that in the monopoly outcome under a discriminatory pricing rule results in a lower cost than the stage-game, capacity-withholding equilibrium of the uniform-price auction.

A third alternative that has been suggested in the literature is the Vickrey pricing rule. This is advocated on the grounds of efficiency as it is a weakly dominant strategy to bid all units at marginal cost. However, we show that the Vickrey rule can result in a high cost as peak-load prices are paid to some units when no peak-load capacity is required. Also, in a repeated game setting, the firms can collude on weakly dominated equilibria of the stage game which substantially increase profits.

Chapter 2
Equilibrium Selection in Games.

How do players know which equilibrium to play when a game has multiple equilibria? This question has been at the heart of much research in game theory. The focus of attention has been the 2x2 Coordination Game such as the one given in figure 1.2 that has two Nash equilibria in pure strategies, one of which is Pareto-efficient but is riskier to play than the other. Harsanyi and Selten (1988) call the former equilibrium payoff-dominant and the latter risk-dominant. Schelling (1960) appeals to the prominence of efficiency to suggest that agents will play for the payoff-dominant equilibrium in the expectation that other agents will be similarly attracted by its focal status. But Harsanyi and Selten have emphasised that such an expectation may not be well-founded. If each player optimises on the assumption that the opponent is equally likely to play either strategy, the outcome will be the risk-dominant equilibrium of the game, which therefore also has a focal status that may outweigh that of the payoff-dominant equilibrium.

Evolutionary game theory has given the argument another perspective. By modelling the process by which agents adjust their strategies out of equilibrium we can analyse how it is that one equilibrium strategy rather than another may be selected. The principle underlying the dynamic systems studied in evolutionary game theory is that successful strategies will be used by a greater proportion of the population in future periods.

To illustrate the idea consider again the Coordination Game of figure 1.2. The game has two pure-strategy equilibria, $e_1 = (s_1, s_1)$ and $e_2 = (s_2, s_2)$. Notice that e_1 is payoff-dominant while e_2 is risk-dominant. There is also a mixed-strategy equilibrium where s_1 is played with probability 2/3. When expressed in terms of the fraction q of the population using strategy s_1, these Nash equilibria correspond respectively to q=1, q=0 and q=2/3. We begin by studying a specific dynamic system for which the population states q=1 and q=0 are stable stationary points. Denote these stationary states by E_1 and E_2 respectively.

Assume that members of the population are randomly matched each period to play this game. They adjust their choice by playing the strategy that yielded the highest expected payoff in the previous period when they are given the chance to do so. Now consider the case where q>2/3. If a revision opportunity arises, then the optimal response against the current state is to play s_1. The proportion playing s_1 will therefore grow over time until the state where everyone plays s_1 is reached. The basin of attraction of E_1 is therefore (2/3, 1], since it will be selected from any state where q>2/3. Similarly the basin of attraction of E_2, where everyone plays s_2, is [0, 2/3). A third possible stationary state (provided the population size N is infinite) is q = 2/3. At this point, no agent has an incentive to change his strategy. However, only E_1 and E_2 are locally stable.

Kandori *et al* (1993) and Young (1993) added to this analysis by assuming that agents sometimes "mutate" by changing their strategies at random. Each agent has a positive probability of mutating each period. There is therefore a small but positive probability that there will be a large number of simultaneous mutations. Once in an equilibrium, it is therefore no longer the case that the system will stay there forever because enough simultaneous mutations will eventually occur to move the system into the other basin of attraction. The system therefore needs to be described in terms of a probability distribution over the states, with much of the time spent in or close to the two stable states when the mutation rate is small.

Kandori *et al* show that, when the probability of mutation goes to zero, the distribution becomes concentrated entirely on the risk-dominant equilibrium, E_2. The reason for this is that more mutations are required to move from E_2 to E_1 than from E_1 to E_2. As the mutation rate goes to zero the probability of the first transition becomes negligible compared with the second. The time-limit of the distribution over population states therefore puts all its mass on E_2 when the mutation rate becomes vanishingly small. Following Kandori *et al*, equilibria that have a positive probability as the mutation rate goes to zero will be called long-run equilibria.

A criticism of this model is the huge expected waiting time to move from E_1 to the long-run equilibrium E_2 when the population size is large. If the system is in E_1 and the mutation rate is very small, then although it is true that the stationary distribution will be concentrated on E_2, it is likely to be a very long time before there are enough simultaneous mutations to move the system out of the basin of attraction of E_1. Ellison (1993) introduces a local interaction structure which dramatically reduces waiting times whilst maintaining the result that the state where everyone plays the risk-dominant strategy is the unique long-run equilibrium. In his model, players are located around a circle and interact only with a subset of the population who are close to them. He shows that a small number of mutations concentrated together may be enough to upset the payoff-dominant equilibrium[1].

Although the overwhelming consensus of this literature is that the risk-dominant equilibrium E_2 will be selected, this is not always the case when local interaction is modelled. In Kandori *et al*, the location structure does not matter since each agent is equally likely to be matched with every other agent in the population. In models of local interaction, agents are more likely to be matched with neighbouring players. An agent's choice of location is therefore important, since this will determine his or her expected payoff. Thus, if agents are given the chance, they will move to a location where they get a higher expected payoff. In Ellison's model, however, this phenomenon is absent, since agents are located at *fixed* positions around a circle and remain there. If this assumption is relaxed, a few mutations need no longer be enough to upset the payoff-dominant equilibrium because agents may move away from a locality in which deviant mutations have occurred in search of a higher payoff. Similarly, the risk-dominant equilibrium may now be easier to upset since a few localised mutations may entice movement towards this locality. Ely (1995) presents a model based

[1] Suppose for example that each agent only interacts with the four closest players or neighbours either side of him and they are randomly matched with these neighbours to play the game in figure 1.2. Then each player has 8 neighbours and if at least 3 of them play s_2 then the best response is to play s_2. Now consider the state in which everyone plays s_1. If there are 4 neighbours who simultaneously mutate then they each have 3 neighbours who are playing s_2 and they will therefore continue to play s_2. There are now another 4 players who are currently playing s_1 but have at least 3 neighbours playing s_2. They will revise their strategies when given the chance and in this way the strategy s_2 will spread throughout the population. Four well placed mutations are enough to move the system into the basin of attraction of the risk-dominant equilibrium and so the expected waiting time is much smaller and independent of the population size.

on this idea in which such movement makes the long-run equilibrium E_1 rather than E_2.

In this chapter, we consider the consequences of movement with the restriction that there is a capacity constraint limiting the number of agents who can reside at each location. We begin by analysing the case where strategy[2] revision is instantaneous, i.e. everybody revises their strategy each period, but the chance to move to another location only arises with some positive probability. This model is analysed with two and three locations or islands. In the two location case, it is shown that there is a range of parameter values for which the long-run equilibria involve one island playing the efficient equilibrium and the other playing the risk-dominant one with the first island full to capacity. This extends to the three-location case, where two islands play the efficient equilibrium.

In section 2.2.3, we show that the results hold when there is inertia in strategy revision. Section 2.2.4 looks at the consequences of relaxing the capacity constraint altogether. As in Ely (1995), the efficient equilibria are then favoured.

---

[2] A strategy is simply a choice between s_1 and s_2 and does not include location.

2.1 Stochastic Techniques.

Kandori *et al* (1993) & Young (1993) use a result due to Friedlin & Wentzell (1984) to characterise the stationary distribution of a Markov chain. This enables them to analyse the behaviour of the distribution as the mutation rate becomes vanishingly small. Using this characterisation, Kandori *et al* show that in the limit the stationary distribution becomes concentrated on a set of states which they call long-run equilibria, and that these states have the property that they require the lowest number of mutations to move to from all other states taken together. Young shows that to find the long-run equilibria, it is sufficient to look at the number of mutations required to move between the set of equilibria rather than the set of states. In this section, we give a brief review of the stochastic techniques developed in these papers.

2.1.1 Friedlin & Wentzell

Consider a finite Markov chain, P, with state space S=(1,2,...,N). A stationary distribution of a Markov chain satisfies $\mu=\mu P$. It is well known that an irreducible and aperiodic Markov chain has a unique stationary distribution. For large N the problem of solving for μ becomes intractable. However, there is a useful way of characterising the unique stationary distribution which is sufficient for our purposes.

A z-tree, h, defined on state space S, is a set of ordered pairs, $(i \to j)$ $i, j \in S$, such that each state $i \neq z$ is the initial point of one arrow and from every state there is a path which leads to z. Denote the set of all z-trees by H_z.

Then define the number u_z,

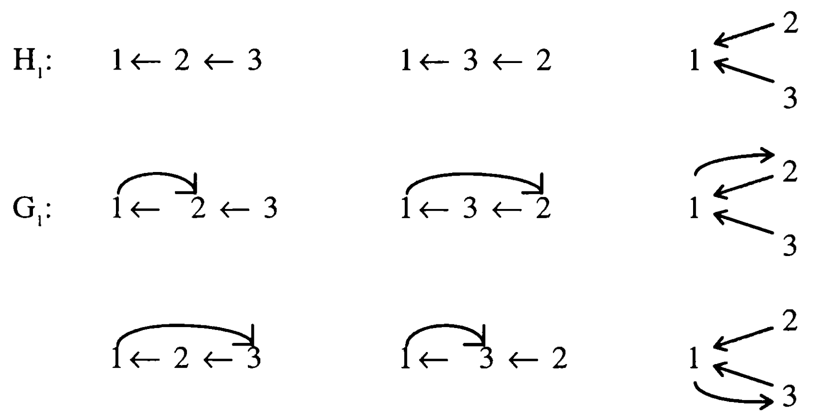
$$u_z = \sum_{h \in H_z} \prod_{(i \to j) \in h} P_{ij} .$$

Now consider the directed graph, g, where each state $i \in S$ is the initial point of one arrow and there is a unique loop which contains z. The set of all possible graphs for state z is denoted G_z .

Define the number,

$$x_z = \sum_{g \in G_z} \prod_{(i \to j) \in g} P_{ij} .$$

The sets H_1 and G_1 are illustrated for the case S=(1,2,3).



x_z can be written in terms of u_i as follows,

$$x_z = \sum_{i \neq z} u_i P_{iz} = \sum_{i \neq z} u_z P_{zi} .$$

That is we can either take each i-tree, $i \neq z$, and add the transition $i \to z$ or take each z-tree and add the transition $z \to i$ for all $i \neq z$.

Hence $\sum_{i \neq z} u_z P_{zi} = u_z \sum_{i \neq z} P_{zi} = u_z (1 - P_{zz})$,

$$\sum_{i \neq z} u_i P_{iz} + u_z P_{zz} = u_z ,$$

$$\sum_i u_i P_{iz} = u_z \quad \Rightarrow \quad u=Pu \quad \text{where} \quad u=(u_1, \ldots, u_N),$$

$$\therefore \quad \mu_z = \frac{u_z}{\sum_j u_j}.$$

So normalising the vector u gives us the unique stationary distribution.

2.1.2 Kandori Mailath and Rob.

This paper looks at the consequences of introducing ongoing mutations into an evolutionary model. The main result is that the set of equilibria is drastically reduced in the limit as the mutation rate goes to zero. Consider a finite population that is randomly matched each period to play the 2 x 2 symmetric game of figure 2.1.

	s_1	s_2
s_1	A, A	B, C
s_2	C, B	D, D

Figure 2.1

At the beginning of every period each agent chooses a strategy that he uses for that period. The average payoff to a player using strategy s_i, π_i, is equal to the expected payoff this strategy yields against a mixed strategy where s_1 is played with a probability equal to the proportion of the remainder of the population using s_1. This average payoff is consistent with an infinite number of random matches each period or with each player being matched exactly once with every other player in each period. Denote the state of the system by the number of agents using s_1, z_t. When agents adjust their strategy they adopt the strategy that yielded the highest expected payoff in the previous period. In a 2 x 2

Coordination Game this will result in convergence to either the state 0 or N depending on the initial point. Without mutations the system will then remain there forever. If we now allow agents to change their strategy with some positive probability ε independently of each other, then the system will no longer get stuck in one of the equilibrium states. In fact, there will be a positive probability of going from any state to any other, as any number of simultaneous mutations can occur. We therefore have an irreducible and aperiodic Markov process, P, on state space S=(0,1,....,N). Each transition probability, P_{ij}, is a polynomial in ε. We now make use of the characterisation of the unique stationary distribution given in section 2.1.1.

The value u_z is constructed by taking the product of transition probabilities along each z-tree and summing this over all z-trees. Hence u_z is also a polynomial in ε. The stationary distribution is just a normalisation of the vector u and is given by $\mu\ (\varepsilon) = (\mu_1(\varepsilon),..........., \mu_N(\varepsilon))$ where $\mu_z(\varepsilon) = \dfrac{u_z(\varepsilon)}{\sum_j u_j(\varepsilon)}$.

We are interested in $\lim_{\varepsilon \to 0} \mu(\varepsilon)$. Let the lowest power of ε in u_z be L_z and define $L^* = \min_{z \in S} L_z$.

If $L_z > L^*$ then $\dfrac{u_z(\varepsilon)}{\sum_j u_j(\varepsilon)} \to 0$ as $\varepsilon \to 0$.

If $L_z = L^*$ then $\dfrac{u_z(\varepsilon)}{\sum_j u_j(\varepsilon)} \to f$ as $\varepsilon \to 0$ where $0 < f \leq 1$.

Hence the limit distribution $\overline{\mu} = \lim_{\varepsilon \to 0} \mu(\varepsilon)$ will put a positive probability on state z only if the lowest power of ε in $u_z(\varepsilon)$ is L^*. Now consider the determinant of the lowest power. Let c_{ij} be the lowest power of ε in P_{ij}. We call this the cost of the transition $i \rightarrow j$, since it is the minimum number of mutations required for the transition. The cost of a z-tree, h, is the minimum number of mutations required to move along it. This is given by $c_h = \sum_{i \rightarrow j \in h} c_{ij}$. The lowest power of ε in u_z will be determined by the z-tree which has the lowest cost. Thus $L_z = \min_{h \in H_z} c_h$.

Therefore L^* will be determined by the state that has the lowest cost z-tree of all states. So all we need to do to characterise the limit distribution is to find the state which has the lowest cost z-tree.

To illustrate the idea, consider the following example. A population of 10 individuals are randomly matched to play the Coordination Game given in figure 1.2. The state of the system at time t is given by the number of agents playing s_1, q_t. Assume the dynamics are such that each period one agent revises his strategy: if seven or more agents play s_1, the deterministic dynamics will move one place towards the state 10; if six or less play s_1, the dynamics will move one place towards the state 0.

From any initial position the system will move to state 10 or 0 and then stay there. The introduction of mutations allows the system to move between equilibria. Each individual changes his strategy independently and with probability ε. Hence we have an irreducible and aperiodic Markov chain, P. The transition probability P_{ij} encompasses all the possible combinations of mutations. For example, consider the transition $7 \rightarrow 6$. With no mutations the deterministic dynamics will take the system to state $q_{t+1}=8$, where eight agents play s_1 and two play s_2. If two of the s_1 players mutate and none of the s_2 players, then the system will move to $q_{t+1} = 6$. There are other ways of moving to state 6. For example, we could have four of the s_1 players and both of the s_2 players mutating. This requires six mutations. Note that in this case the smallest power of ε in P_{76} is 2.

State 9 cannot have the minimum cost z-tree because it includes an arrow out of state 10 which must have a positive cost. A 10-tree will not have this cost and there is a transition from 9 to 10 at zero cost. Therefore for every 9-tree there is a lower cost 10-tree. This is true for all states in the basin of attraction of state 10 and also for all states in the basin of attraction of state 0. So that leaves us with two candidates for minimum cost z-tree, state 0 and state 10. The minimum cost 0-tree is achieved by just enough mutations to get into its basin of attraction.

$$\begin{tikzcd}
0 & 1 \arrow[l, "0"'] & 2 \arrow[l, "0"'] & 3 \arrow[l, "0"'] & 4 \arrow[l, "0"'] & 5 \arrow[l, "0"'] & 6 \arrow[l, "0"'] & 7 \arrow[r, "0"] & 8 \arrow[r, "0"] & 9 \arrow[r, "0"] & 10 \arrow[llll, bend right, "4"']
\end{tikzcd}$$

$$\begin{tikzcd}
0 & 1 \arrow[l, "0"'] & 2 \arrow[l, "0"'] & 3 \arrow[l, "0"'] & 4 \arrow[l, "0"'] & 5 \arrow[l, "0"'] & 6 \arrow[l, "0"'] & 7 \arrow[r, "0"] & 8 \arrow[r, "0"] \arrow[ll, bend right, "3"'] & 9 \arrow[r, "0"] & 10 \arrow[ll, bend right, "2"']
\end{tikzcd}$$

Figure 2.2

Figure 2.2 illustrates that two jumps to get out of the basin of attraction will require more mutations than one jump since the dynamics will push back towards state 10. Similarly, the minimum cost 10-tree has a cost of 7. Therefore $L^* = 4$ and the limiting distribution puts a probability of 1 on the state where everybody plays s_2, the risk-dominant equilibrium.

2.1.3 Young

Peyton Young goes one step further to show that we only need to consider the cost of moving between the recurrent communication classes of the unperturbed process. In the above example the unperturbed process is deterministic. In general the unperturbed process may follow a Markov chain, with recurrent communication classes which have the following properties: from any state $x \in S$ the process will move to a state which is in one of these classes; once there the process will move between states within the class. If the perturbed process is irreducible, then the system can move between classes but this will involve a cost since mutations are required. Denote the classes by X_1 .....X_J. Let r_{ij} be the lowest cost of moving from class i to j. Now define a j-tree exactly as a z-tree but where the vertices are the indices (1,2,..., J). Let γ_j be the cost of the least cost j-tree and $\gamma(z)$ the cost of the least cost z-tree. Then he shows $\gamma(x)=\gamma_j$ for all $x \in X_j$. The intuition is easy to see. If we want the minimum cost z-tree for a state in class i then it must involve a transition out of the other classes at some cost. But apart from this no cost is needed since we allow all other transitions to be zero cost. This gives us a simple method for finding the long-run equilibria. Intuitively the equilibria which are easiest to flow into from all the other equilibria are selected as $\varepsilon \to 0$.

2.2 When Does Immigration Facilitate Efficiency.

In this section, we present a model of local interaction with movement between locations. We assume that agents are randomly matched with someone at the same location to play the game of figure 2.3 in which A>C, D>B, A>D and A+B<C+D. Hence $e_1=(s_1, s_1)$ is the payoff-dominant equilibrium while $e_2=(s_2, s_2)$ is risk-dominant. The probability with which s_1 is played in the mixed-strategy equilibrium is $q^* = (D-B)/(A-C+D-B) > 1/2$.

	s_1	s_2
s_1	A, A	B, C
s_2	C, B	D, D

Figure 2.3

Each period some agents are given the chance to move locations. We begin by looking at the case where there are two locations and strategy revision at each location is instantaneous. The analysis is then extended to the case of three locations and to the case where there is inertia in strategy revision.

2.2.1 Two Islands

Players are randomly matched on each of two isolated islands to play the game of figure 2.3. The global population is 2N and the capacity of each island is (1+d)N. Strategy revision is instantaneous, that is everybody chooses a strategy that is a best response to the state in the previous period. The chance to change islands arises with a positive probability each period. When such an opportunity arises, the agent will choose the location and strategy that would have maximised their expected payoff in the previous period. If the agent is indifferent between two choices then we assume they choose either with a positive probability.

However, an agent cannot move to an island that is full to capacity. If the number of agents who wish to move to island i is greater than (N(1+d)-n_i), where n_i is the current number on the island, then only (N(1+d)-n_i) of them will be allowed to move. N is sufficiently large that the following set of numbers are all integers, {N(1-d), N(1+d), q*N, (1+d)q*N, (1-d)q*N, (1-d)(1-q*)N, (1+d)(1-q*)N}. One can think of the following story underlying these dynamics. At the end of each period players gather information on the proportion of the population using each strategy on their island. With some positive probability, they also learn the proportions on the other island. At the beginning of the next period they choose a location and strategy to use for that period. If they have no information about the other island then they stay where they are and choose the strategy that is a best reply to the proportions in the previous period. If they do learn the proportions on the other island then they will want to move if a best reply on the other island yields a higher expected payoff. If the island has spare capacity they will move and play the best reply. If it is full then they play a best reply on their current island.

The state space is

$$S = \{(\frac{n_1^1}{n_1}, \frac{n_2^1}{2N - n_1}, n_1) : n_1^1 \in (0,1,\ldots,n_1), n_2^1 \in (0,1,\ldots,2N - n_1), N(1-d) \le n_1 \le N(1+d)\},$$

where n_i^1 is the number playing strategy s_1 on island i and n_1 is the number of agents on island 1. Denote a state of the system by s=(q_1 ,q_2 ,n_1)$\in$S, where q_i is the proportion of the population playing s_1 on islands i.

The dynamics give rise to a Markov process, P, on state space S. From any initial condition, the system will move to a state or set of states where it remains. Following Young (1993), such a set will be called a recurrent communication class. The recurrent communication classes are characterised latter.

Without mutations, the system will move to one of these classes and remain there. Now assume that each agent mutates independently, with probability ε, with the consequence that a strategy[3] is re-selected at random on their current island. This allows the system to move between classes and gives rise to the perturbed transition matrix P^{ε} where,

$$P_{ij}^{\varepsilon} = P_{ij}(1-\varepsilon)^{2N} + \sum_{k=1}^{2N} c_{ijk}\varepsilon^{k}(1-\varepsilon)^{2N-k}. \tag{2.1}$$

P_{ij} is the ijth element of P, the unperturbed transition Matrix.

Proposition 2.1: *P^{ε} has a unique stationary distribution $\mu(\varepsilon)$ and $\lim_{\varepsilon \to o} \mu(\varepsilon)$ exists and is equal to one of the stationary distributions of P.*

Proof: Young (1993) shows that this is true if P^{ε} is a 'regular perturbation' of P. If P^{ε} is a regular perturbation of P then the following conditions must hold,

i) P^{ε} is aperiodic and irreducible

ii) $\lim_{\varepsilon \to o} P^{\varepsilon}_{ij} = P_{ij}$

iii) $P^{\varepsilon}_{ij} > 0$ for some ε implies $\exists r \geq 0 \;\; s.t. \;\; 0 \prec \lim_{\varepsilon \to 0} \varepsilon^{-r} P_{ij}^{\varepsilon} \prec \infty$.

From (2.1) conditions (ii) and (iii) are clearly satisfied. If $P^{\varepsilon}_{ij} > 0$ then r is 0 if $P_{ij} > 0$ or equal to the lowest value of k such that $c_{ijk} > 0$. We now show that P^{ε}

[3] All the results go through if we assume a strategy and location is re-selected at random with the restriction that the capacity constraint cannot be broken. If the number who re-select a location at random would take that location over its capacity then that island becomes full to capacity and some agents select a strategy at random on their current island instead.

is aperiodic and irreducible. The diagonal elements of P^{ε} are all positive. This is because in any state, there is a positive probability that nobody moves and that there are mutations that keep the same numbers playing each strategy on both islands. Hence P^{ε} is aperiodic. There is a positive probability of going from any state to the class in which both islands coordinate on the same equilibrium. This simply requires a certain number of mutations on each island. We can then have any number of agents on each island up to N(1+d) and for a given number of agents on each island, we can have any number playing each strategy, as there is a positive probability that nobody moves while a certain number mutate. It is therefore possible to go from any state to any other and the process is irreducible. QED.

Definition 2.1: *The set of states in the support of* $\lim_{\varepsilon \to o} \mu(\varepsilon)$ *will be called the long-run equilibria.*

Definition 2.2: *A k-tree, h, defined on state space S (the set of recurrent communication classes), is a set of ordered pairs, $(i \to j)$ $i, j \in S$, such that each state $x \neq k$ is the initial point of one arrow and from every state there is a path which leads to k.*

Let r_{ij} be the minimum number of mutations required to go from class i to j. We know that such a number exists because P^{ε} is irreducible. The cost of a k-tree is $\sum_{(i \to j) \in h} r_{ij}$.

Proposition 2.2: *The long-run equilibria are the set of states in the recurrent communication class which has the lowest cost k-tree.*

For the proof the reader is referred to Young (1993). The intuition is clear. The long-run equilibria are the set of states in the recurrent communication class that is easiest to flow into from all other recurrent communication classes. Hence to find the long-run equilibria we need to characterise the recurrent communication classes and the costs r_{ij} of moving between them and then find the class that has the lowest cost k-tree.

Recurrent communication classes.

One recurrent communication class is the set of all states where $q_1 = q_2 = 0$. The basin of attraction of this class is $\{(q_1, q_2): q_1 \leq q^*, q_2 \leq q^*\}$, since best replies will[4] lead both islands to the risk-dominant equilibrium. In this class the system will move between states where $q_1 = q_2 = 0$ and $n_1 \in (N(1-d), N(1+d))$, since agents move with a positive probability when they are indifferent and n_1 must lie in this range due to the capacity constraint.

Now consider any initial condition with $q_1 \geq q^*$ and $q_2 \leq q^*$. Best replies will move the system towards $q_1 = 1$ and $q_2 = 0$. This will result in movement into island 1, since the higher payoff equilibrium is being played there. The system will eventually move to the equilibrium state (1,0,N(1+d)). Similarly the set of states with $q_1 \leq q^*$ and $q_2 \geq q^*$ form the basin of attraction of the equilibrium (0,1,N(1-d)). The final possibility is for both populations to coordinate on the

[4]When $q_1 = q^*$ or $q_2 = q^*$ the dynamics can go either way.

payoff-dominant equilibrium. The basin of attraction for this class is $\{(q_1, q_2): q_1 \geq q^*, q_2 \geq q^*\}$, and the recurrent communication class is the set of all states with $q_1=q_2=1$ and $n_1 \in (N(1-d), N(1+d))$. The four recurrent communication classes are illustrated in figure 2.4.

Minimum costs of moving between recurrent communication classes.

Consider the transition from class 1 to 2^A. We want the minimum number of mutations required to get into the basin of attraction of class two, $\{(q_1, q_2): q_1 \geq q^*, q_2 \leq q^*\}$, from a state in class one, $(0,0,n_1)$. Hence we require a proportion q^* of island 1 to mutate. Now the less populated island 1 is, the lower the number of mutations required to achieve this. The minimum value of n_1 is $N(1-d)$ so the minimum number of mutations required is $N(1-d)\ q^*$. The dynamics will then move the system to the state $(1,0,N(1+d))$. The cost of moving back is $N(1+d)\ (1-q^*)$ since we require the system to move back to a state where $q_1 \leq q^*$ and island 1 is full to capacity.

A direct jump will not necessarily yield the minimum number of mutations. For example consider the transition from class 1 to 3. A direct jump from class 1 to 3 requires $2Nq^*$ simultaneous mutations. However, it is easier to go from class 1 to 2 and then from 2 to 3 since this only requires $2(1-d)Nq^*$ mutations. Hence the minimum number of mutations required to go from class 1 to 3 is $2(1-d)Nq^*$. All the minimum costs are given in figure 2.5.

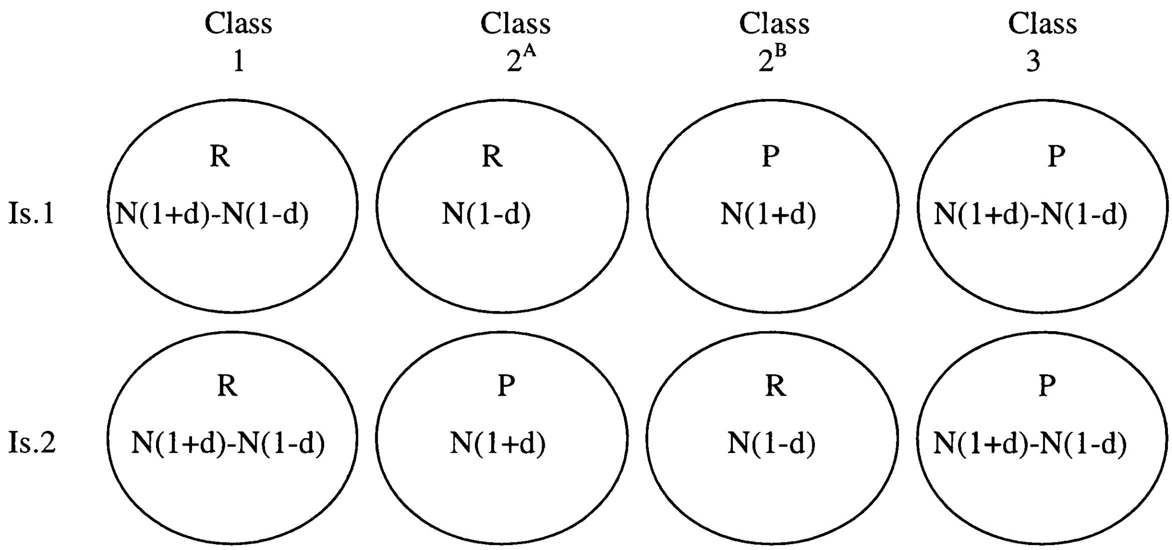


Figure 2.4. Recurrent communication classes:
Row i of circles illustrate the equilibrium played on island i
in each of the classes (risk-dominant, R or payoff-dominant, P),
plus the range of values of n_i that are consistent with the class.

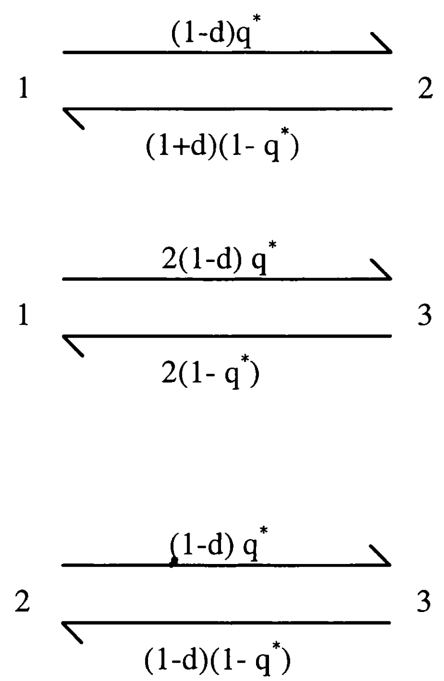


Figure 2.5
Minimum costs of moving between recurrent
communication classes.

Lemma 1: *The two location model has 4 recurrent communication classes. To find the class which has the minimum cost k-tree it is sufficient to find the minimum cost trees between just three classes, ruling out either 2^A or 2^B.*

Proof: Let r_{ij} denote the minimum cost of the transition $i \to j$.

$$\text{Then} \quad r_{i2^A} = r_{i2^B} \qquad r_{2^A i} = r_{2^B i} \qquad i \in (1,2,3) .$$

Let h be a minimum cost k-tree. Adjust the tree so that at least one of 2^A or 2^B have no predecessors without changing the cost. This is easy to do since $i \to 2^A$ can be transferred to $i \to 2^B$ (or vice versa) leaving 2^A with no predecessors. We can split the adjusted k-tree into two parts, a minimum cost k'-tree defined on the vertices (1,2,3) and 2^A added at minimum cost. It must be a minimum cost k'-tree because any adjustments which reduce the cost would also reduce the cost of the k-tree but we started with a minimum cost k-tree. Hence we can find the minimum cost k-tree by first finding the minimum cost k'-tree and then adding a 2-state at minimum cost. This cost will be common to all k-trees and so does not need to be considered. QED

This leaves nine trees that we need to compare (3 for each communication class). These trees are illustrated in table 2.1.

1-trees	2-trees	3-trees
A^1: $1 \leftarrow 2 \leftarrow 3$	A^2: $2 \leftarrow 1 \leftarrow 3$	A^3: $3 \leftarrow 2 \leftarrow 1$
B^1: $1 \leftarrow 3 \leftarrow 2$	B^2: $2 \leftarrow 3 \leftarrow 1$	B^3: $3 \leftarrow 1 \leftarrow 2$
C^1: $2 \rightarrow 1 \leftarrow 3$	C^2: $1 \rightarrow 2 \leftarrow 3$	C^3: $2 \rightarrow 3 \leftarrow 1$

Table 2.1.
k-trees

Proposition 2.3: *The long-run equilibria are:*

the set of states in class 1 if $d<2q^*-1$,

and states 2^A and 2^B if $d>2q^*-1$.

Proof: From proposition 2.2, the long-run equilibria are the set of states in the recurrent communication class which has the lowest cost k-tree. It is a simple exercise to see that the lowest cost 1-tree is A^1: $1\leftarrow 2\leftarrow 3$. The other two 1-trees include the transition $1\leftarrow 3$, which has the same cost as A^1 but also include a transition from class 2 at some cost. Similarly, the lowest cost 3-tree is A^3: $3\leftarrow 2\leftarrow 1$ as the other two 3-trees include the transition $3\leftarrow 1$, which has the same cost as A^3. Finally, the lowest cost 2-tree is C^2: $1\rightarrow 2\leftarrow 3$. The other two 2-trees include the transitions $3\rightarrow 1$ and $1\rightarrow 3$. In each case the cost is reduced by going directly to class 2.

The only difference between the cost of C^2 and A^3 is in the transition between classes 2 and 3. Since $r_{23}>r_{32}$ (as $q^*>\frac{1}{2}$), C^2 always has a lower cost. This leaves two candidates for minimum cost k-tree, A^1 and C^2. The cost of A^1 is less than the cost of C^2 if $r_{21}<r_{12}$. Hence class 1 has the lowest cost k-tree if

$$(1+d)(1-q^*) < (1-d)q^* \Rightarrow d<2q^*-1.$$

If the inequality is reversed then class 2 has the minimum cost k-tree. QED.

The long-run equilibria are illustrated in figure 2.6. Hence the long-run equilibria are the set of states where everyone plays s_2, the risk-dominant strategy if $d<2q^*-1$. The critical value of d where class 1 becomes the long-run equilibrium

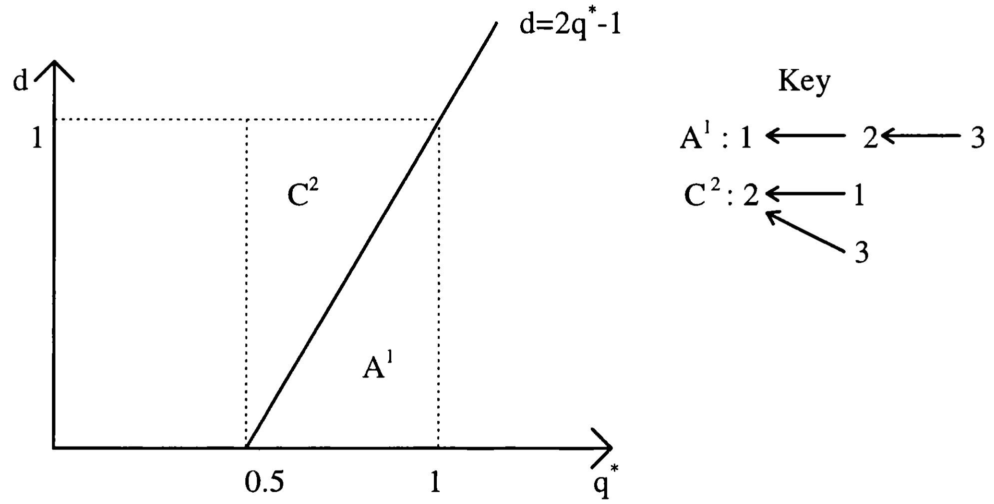


Figure 2.6
Minimum cost j-trees.

increases with the degree of risk-dominance. If d is above this critical value then class 2 has the minimum cost k-tree. The long-run equilibria are the two states where the two islands play different equilibria. In fact, from the symmetry of the cost structure, states 2^A and 2^B will each have a probability of one half in the limit-distribution. It is easy to see why higher values of d upset the risk-dominant equilibrium. In class 2, the island playing the payoff-dominant equilibrium becomes more populated as d increases because in equilibrium it is full to capacity. The transition to class 1 therefore becomes more difficult.

By the same token the island playing the risk-dominant equilibrium in class 2 becomes smaller as d increases and so easier to convert. Hence the transition from class 2 to class 3 where both islands play the payoff-dominant equilibrium becomes easier. However, we never observe class 3 as the long-run equilibrium. The reason for this is that although the cost of class 3 is decreasing, the cost of class 2 is also decreasing and is always less. Consider the minimum costs of the transitions $1 \rightarrow 2$ and $3 \rightarrow 2$. As d increases the smallest possible size of an island, (1-d)N, falls. In classes 1 and 3 the transition to a state where one island has a population size of ((1-d)N) has zero cost. The transition to class 2 then only requires enough mutations on the small island. The minimum cost therefore falls as d increases.

2.2.2 Three islands.

We now extend the model to the case of 3 islands. Assume that the global population is now 3N. We only consider the cases where there is a positive population on each island. Hence the capacity constraint is (1+d)N where $0<d<\frac{1}{2}$. All other details of the model are the same. The state space is now

$$S=\{(\frac{n_1^1}{n_1},\frac{n_2^1}{n_2},\frac{n_3^1}{3N-n_1-n_2},n_1,n_2):n_1^1\in(0,1,\dots,n_1),n_2^1\in(0,1,\dots,n_2),$$
$$n_3^1\in(0,1,\dots,3N-n_1-n_2),N(1-2d)\le n_1,n_2\le N(1+d)\}.$$

where n_i^1 is the number playing strategy s_1 on island i and n_i is the number of agents on island i. Denote the state of the system by $s=(q_1,q_2,q_3,n_1,n_2)\in S$ where q_i is the proportion of island i playing s_1.

As before the dynamics give rise to a Markov process, P', on state space S. From any initial condition the system will move to one of the recurrent communication classes which are characterised latter. An element of the perturbed Markov process is now

$$P_{ij}^{/\varepsilon}=P_{ij}^{/}(1-\varepsilon)^{3N}+\sum_{k=1}^{3N}c_{ijk}\varepsilon^k(1-\varepsilon)^{3N-k}.$$

From propositions 2.1 and 2.2 we know that to find the long-run equilibria we need to find the recurrent communication class that has the lowest cost k-tree. To do this we need to characterise the recurrent communication classes and the costs of moving between them.

Recurrent communication classes.

One recurrent communication class is the set of all states where $q_1=q_2=q_3=0$. The basin of attraction is $\{(q_1,q_2,q_3): q_1 \leq q^*, q_2 \leq q^*, q_3 \leq q^*\}$ since best replies will lead both islands to the risk-dominant equilibrium. In this class the system will move between states where $q_1=q_2=q_3=0$ and $n_1,n_2 \in (N(1-2d), N(1+d))$ since agents move with a positive probability when they are indifferent and n_i must lie in this range due to the capacity constraint.

Now consider any initial condition with $q_1 \geq q^*$, $q_2 \leq q^*$ and $q_3 \leq q^*$. Best replies will move the system towards $q_1 =1$, $q_2=0$, $q_3=0$. This will result in movement into island 1, since the higher payoff equilibrium is being played there. The system will eventually move to the class $(1,0,0,N(1+d),n_2)$ where $n_2 \in (N(1-2d), N(1+d))$. Similarly the set of states with $q_1 \leq q^*$, $q_2 \geq q^*$ and $q_3 \leq q^*$ form the basin of attraction of the class $(0,1,0,n_1,N(1+d))$ where $n_1 \in (N(1-2d), N(1+d))$ and the set of states with $q_1 \leq q^*$, $q_2 \leq q^*$ and $q_3 \geq q^*$ form the basin of attraction of the class $(0,0,1,n_1,n_2)$ where $n_1 \in (N(1-2d), N(1+d))$ and $n_1+n_2=N(2-d)$. Hence there are three classes where one island plays the efficient equilibrium and the other two play the risk-dominant one.

There are also three classes where two islands play the efficient equilibrium and one plays the risk-dominant one. The basin of attraction of the class where the first two islands play the efficient equilibrium and the third island plays the risk-dominant one is, with $q_1 \geq q^*$, $q_2 \geq q^*$ and $q_3 \leq q^*$. Similarly the set of states with $q_1 \leq q^*$, $q_2 \geq q^*$ and $q_3 \geq q^*$ form the basin of attraction of the class (0,1,1,N(1-2d),N(1+d)) and the set of states with $q_1 \geq q^*$, $q_2 \leq q^*$ and $q_3 \geq q^*$ form the basin of attraction of the class (1,0,1,N(1+d),N(1-2d)).

The final possibility is for all three populations to play the payoff-dominant equilibrium. The basin of attraction is $\{(q_1, q_2, q_3): q_1 \geq q^*, q_2 \geq q^*, q_3 \geq q^*\}$, and the recurrent communication class is the set of all states with $q_1 = q_2 = q_3 = 1$ and $n_1, n_2 \in (N(1-d), N(1+d))$. The eight recurrent communication classes are illustrated in figure 2.7.

Lemma 2.2: *In the three islands model there are 8 recurrent communication classes. The 8 classes can be split into 4 similar groups and to find the minimum cost k-tree it is sufficient to find the minimum cost tree spanning this group.*

This is shown in the appendix. The minimum costs of moving between these groups are given in figure 2.8.

Proposition 2.4: *The long-run equilibria are:*

the set of states in class one if $d < \dfrac{2q^*-1}{1+q^*}$,

the states 3^A, 3^B *and* 3^C *if* $d > \dfrac{2q^*-1}{1+q^*}$.

This is shown in the appendix. The results are illustrated in figure 2.9.

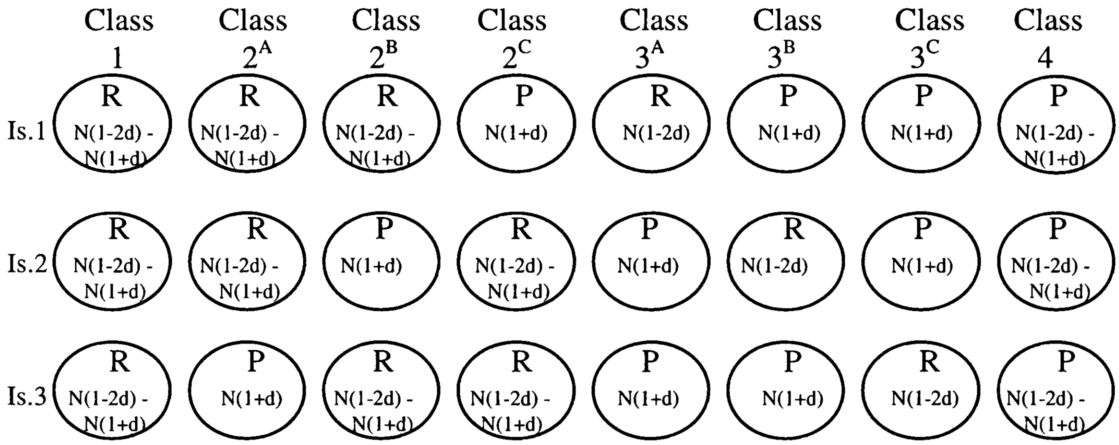


Figure 2.7. Recurrent communication classes:
Row i of circles illustrate the equilibrium played on island i
in each of the classes (risk-dominant, R or payoff-dominant, P),
plus the range of values of n_i that are consistent with the class.

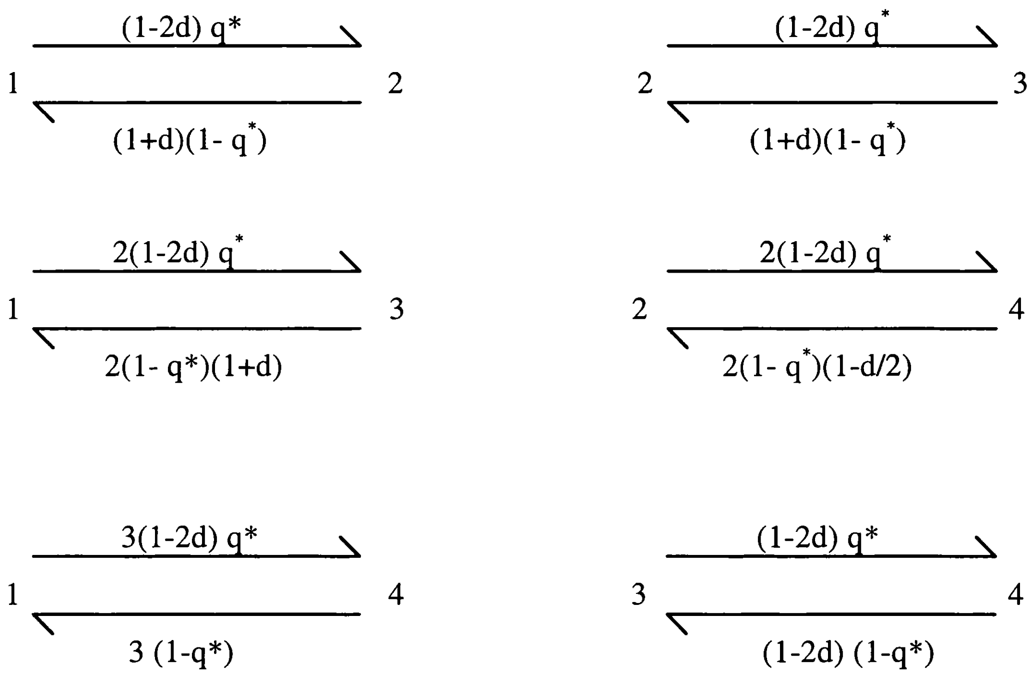
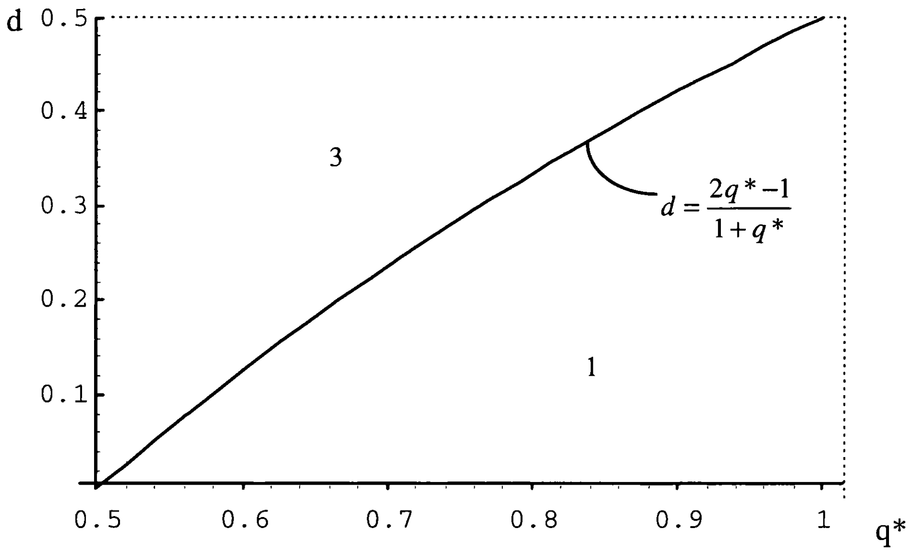


Figure 2.8

Minimum costs of moving between recurrent
communication classes.



Key

1 : $1 \leftarrow 2 \leftarrow 3 \leftarrow 4$
3 : $4 \rightarrow 3 \leftarrow 2 \leftarrow 1$

Figure 2.9
Minimum cost j-trees.

Much like the two island case the graph is split into two sections. When the capacity constraint is below some critical value, the long-run equilibria are the set of states where everyone plays the risk-dominant strategy. If it is above this critical value, then the long-run equilibria are the set of states in group 3 where two islands play the payoff-dominant equilibrium and the other island plays the risk-dominant one. It is interesting that we never observe the case where two islands play the risk-dominant equilibrium and the other island plays the payoff-dominant one. This is because the cost of class 3 becomes less than the cost of class 2 just as the cost of class 2 becomes less than the cost of class 1. The intuition behind not observing the set of states where everyone plays payoff-dominant strategy is the same as in the 2 island case.

2.2.3 Inertia in strategy revision.

The results of sections 2 and 3 rely on the assumption that each agent plays a best reply at their location. We now extend the 2 island model to the case where there is a positive probability that agents simply continue to use the strategy they used in the previous period. However, if they do revise their strategy they do so by playing a best reply. As before, there is a positive probability that they are given the chance to move islands and agents will then choose the location and strategy that would have maximised their expected payoff in the previous period, as long as this does not involve moving to an island that is full to capacity.

One can now think of the following story underlying these dynamics. At the end of each period the following events occur with a positive probability for each agent: 1) the agent observes nothing about the proportions using each strategy, 2) the agent only observes the proportions on his current island and 3) the agent observes the proportions on both islands. In the first case he simply continues to use the same strategy in the next period. In the second case he chooses a best reply on his current island. Finally, in the third case, he will want to move if a best reply on the other island yields a higher expected payoff than a best reply on his current island. If the island has spare capacity he will move and play the best reply. If it is full then he plays a best reply on his current island. The previous models look at the extreme case where the probability of the first event is zero. In this model the probability of each event is positive.

The state space is S, is the same as in the case with no inertia. The above dynamics, however, give rise to a different transition matrix, $P^{//}$. All other aspects of the model are the same. The perturbed transition matrix is given by

$$P_{ij}^{//\varepsilon} = P_{ij}^{//}(1-\varepsilon)^{2N} + \sum_{k=1}^{2N} c_{ijk}\varepsilon^k(1-\varepsilon)^{2N-k}.$$

The same reasoning as before can be used to show that $P^{//\varepsilon}$ is aperiodic and irreducible. Hence we can apply propositions 2.1 and 2.2 and find the long-run equilibria by finding the recurrent communication class that has the lowest cost k-tree. The recurrent communication classes are the same as in the model with no inertia and are illustrated in figure 2.3. However, the basins of attraction of the recurrent communication classes are now different. This changes the cost of moving between classes. In the model without inertia, the minimum cost of the transition $2 \rightarrow 1$ is (1+d)(1-q*). We can now achieve this transition with fewer mutations because after a certain number of mutations on the efficient island, it will be optimal for agents to move and get a payoff of D. If there are (1-d)(1-q*) mutations followed by movement, then there is a positive probability that 2Nd agents move and that all the agents that move were playing s_1, while nobody revises their strategy on the efficient island. Hence the proportion playing s_1 will be (N(1+d)-2Nd-(1-d)(1-q*)N)/(N(1+d)-2Nd)=q*. However, we must ensure that it is optimal to move and this requires a proportion $(1-q')$ of the efficient island to mutate, where q' satisfies $Aq'+B(1-q') = D$ or $q' = (D-B)/(A-B)$. Hence $(1- q')(1+d)N$ mutations are required before anyone will move. Since the number of mutations must satisfy both of the above conditions, the minimum number of mutations required will be max[(1- q')(1+d),(1-q*(1-d)]. The minimum costs of moving between recurrent communication classes are given in figure 2.10.

$$1 \underset{\max[(1-d)(1-q^*),\,(1-q')(1+d)]}{\overset{(1-d)q^*}{\rightleftarrows}} 2$$

$$1 \underset{(1-d)(1-q^*)+\max[(1-d)(1-q^*),\,(1-q')(1+d)]}{\overset{2(1-d)q^*}{\rightleftarrows}} 3$$

$$2 \underset{(1-d)(1-q^*)}{\overset{(1-d)q^*}{\rightleftarrows}} 3$$

Figure 2.10

Minimum costs of moving between recurrent communication classes.

Proposition 2.5: *The long-run equilibria are:*

all states in class 1 if $d< f(q^*, q')$

and states 2^A *and* 2^B *if* $d> f(q^*, q')$,

where $f(q^*, q') = \dfrac{q^*+q'-1}{q^*+1-q'}$.

Proof. From proposition 2.2 we know that we need to find the class with the lowest cost k-tree. Also to find the class which has the minimum cost k-tree it is sufficient to find the minimum cost trees between just three classes, ruling out either 2^A or 2^B (lemma 1). Of the nine trees (table 2.1), it is a simple exercise to see that the minimum cost tree is either $1 \rightarrow 2 \leftarrow 3$ or $1 \leftarrow 2 \leftarrow 3$[5]. Hence the set of states in class 1 will be the long-run equilibria when

$$\max[(1\text{-}d)(1\text{-}q^*),(1\text{-}q')(1+d)]<(1\text{-}d)q^*.$$

This condition reduces to

$$(1\text{-}q')(1+d)<(1\text{-}d)q^* \Rightarrow d<\frac{q^*+q'-1}{q^*+1-q'}{}^{6}.$$

If the inequality is reversed then class 2 has the minimum cost k-tree. Q.E.D.

The function $f(q^*, q')$ is increasing in q^*. Apart from the transitions $2 \rightarrow 1$ and $3 \rightarrow 1$ the minimum costs of moving between recurrent communication classes are the same as the model with no inertia. Since c_{21} is smaller than in the earlier case, class 1 has a slightly larger range over which it is the long-run equilibrium. Otherwise the long-run equilibria are similar - class 1 if d is below some critical value and class 2 if it is above this value, where the critical value is increasing with q^*.

2.2.4 No capacity constraints.

We now consider the consequences of removing the capacity constraints altogether. To do this it is necessary to make some assumptions on what happens when an island becomes empty. If we assume that the payoff of being alone at a location is less than D then there are 4 equilibrium states, all of which involve an

[5] Any tree that includes the transition $1 \leftarrow 3$ (or $3 \leftarrow 1$) cannot be the minimum cost tree because $1 \leftarrow 2 \leftarrow 3$ (or $3 \leftarrow 2 \leftarrow 1$) has the same cost but includes no more transitions. Also $3 \leftarrow 2 \leftarrow 1$ always has a higher cost than $3 \rightarrow 2 \leftarrow 1$ because $r_{32}<r_{23}$ for all values of d.

[6] $(1\text{-}d)(1\text{-}q^*)< (1\text{-}d)q^*$ when $q^*>1/2$ which is always true.

empty island[7]. To ensure the perturbed process is irreducible, we only consider the case where agents choose a strategy and location at random when they mutate[8]. Only one mutation is required to move from an equilibrium where everyone plays s_2 to one where everyone plays s_1 but to move in the reverse direction requires $2N(1-q^*)$ mutations. Clearly the long-run equilibria are the two states where one island is empty and the other one plays the efficient equilibrium. The empty location plays a coordinating role.

[7]When the islands are playing different equilibria, then everyone will move to the efficient one. When they are playing the same equilibrium, then one island will eventually become empty without mutations as agents move with a positive probability when they are indifferent.
[8] Otherwise once an island becomes empty it remains empty.

2.3 Conclusions

Kandori *et al* and Young have developed techniques for characterising the limit of the stationary distribution when the mutation rate goes to zero. This allows us to address the question of equilibrium selection in the long-run when the mutation rate is very small. Using these techniques they show that the equilibrium selected is the risk-dominant one rather than the efficient one.

The results of the local interaction model essentially show that the introduction of movement may upset the long-run equilibrium where everyone plays the risk-dominant strategy, but will not necessarily lead to a long-run equilibrium where everyone plays the efficient one. The alternatives are states in which some islands coordinate on the efficient equilibrium and others on the risk-dominant one. These equilibria are observed when there is a binding capacity constraint. However, risk-dominance still has a role to play in the determination of the long-run equilibria. The important feature is the degree of risk-dominance which will determine how lax the capacity constraint needs to be before the long-run equilibria switch from purely risk-dominant to the mixture of equilibria. When the capacity constraint is relaxed altogether the long-run equilibria involve everyone playing the efficient strategy.

Hence when agents are able to move between locations, the state in which everyone plays the risk-dominant strategy is no longer the unique long-run equilibrium. However, results that show that movement leads to a unique long-run equilibrium where everyone plays the efficient strategy rely on spare capacity. In fact, with a binding capacity constraint, the long-run equilibria will depend on the degree of risk-dominance and the strictness of the capacity constraint.

2.4 Appendix: Proofs.

Proof of lemma 2.2.

Lemma 2: *In the three islands model, there are 8 recurrent communication classes. To find the class which has the minimum cost k-tree it is sufficient to find the minimum cost trees between the 4 groups (1,2,3,4).*

Proof: Let r_{ij} denote the minimum cost of the transition $i \rightarrow j$. Then,

$$r_{ix^A} = r_{ix^B} = r_{ix^C}, \quad r_{x^A i} = r_{x^B i} = r_{x^C i}.$$

where, $i \in (1,4)$, $x \in (2,3)$.

However, the minimum cost of moving between groups 2 and 3 are not the same. For example, $r_{2^A 3^C} \succ r_{2^A 3^B}$. This is because class 2^A is exactly the opposite of class 3^C since all three islands are playing a different equilibrium. The cost of moving between these states is greater than the minimum cost given for transitions between classes 2 and 3 since this requires only one island to convert (for example $r_{2^A 3^B}$). The same applies for (2^B and 3^B) and (2^C and 3^A). However the minimum cost tree will never include transitions between these states since if such a transition is present then there will always be a tree which has a lower cost.

Let $3 \rightarrow 2$ be a high cost transition in a k-tree, h. Then follow this process of adjustment. Transfer $3 \rightarrow 2'$ if this is possible. This will not be possible if both the other 2-classes are predecessors of 3. In this case swap 3 with $3'$. $3' \rightarrow 2$ is no longer high cost but the change may result in a high cost $2 \rightarrow 3$ transition elsewhere. If this is the case, repeat the process with $2 \rightarrow 3$. Eventually the first part of the process will be possible since we are moving backward along the tree. The end result will be a tree with the same structure but no high cost transitions between classes 2 and 3.

Let h be a minimum cost k-tree. Consider the path from $i \rightarrow j$ where $i \in (1, 4)$. If the successor of i is a 3-class, adjust the tree so that there is only one 3 state along this path. This will not change the cost since $r_{i3} = r_{i3'}$. If there is more than one 2-class along this path in the adjusted tree, then make the following adjustment. Let 2 be the first 2-class in the path and $2'$ be the last. Transfer $n \rightarrow 2$ to $n \rightarrow 2'$. If this results in a high cost transition $3 \rightarrow 2'$ then swap 2 and $2'$. If the successor of i is a 2-class then do the same but with the roles of 2 and 3 reversed.

The resulting tree will still be minimum cost and can be split into 2-parts, a minimum cost k'-tree defined over the vertices (1,2,3,4) and the remaining 2-classes and 3-classes added on at minimum cost. It must be a minimum cost k'-tree because any adjustments which reduce the cost would also reduce the cost of the j-tree but we started with a minimum cost k-tree. Hence we can find the minimum cost k-tree by first finding the minimum cost k'-tree and then adding the remaining 2-classes and 3-classes at minimum cost. This cost will be common to all k-trees and so does not need to be considered. QED.

Proof of proposition 2.4.

Proposition 2.4: *The long-run equilibria are:*

$$\textit{the set of states in class one if } d < \frac{2q^*-1}{1+q^*},$$
$$\textit{the states } 3^A,\ 3^B \textit{ and } 3^C \textit{ if } d > \frac{2q^*-1}{1+q^*}.$$

From proposition 2.2 we know that to find the long-run equilibrium we need to find the recurrent communication class with the lowest cost k-tree. We only need to consider the cost of moving between the four groups, (lemma 2). Hence for each group there are 16 k-trees. All 16 1-trees are illustrated in figure 2.11.

1^1	$1 \leftarrow 2 \leftarrow 3 \leftarrow 4$	2^1	$1 \leftarrow 2 \leftarrow 4 \leftarrow 3$
3^1	$1 \leftarrow 3 \leftarrow 2 \leftarrow 4$	4^1	$1 \leftarrow 3 \leftarrow 4 \leftarrow 2$
5^1	$1 \leftarrow 4 \leftarrow 3 \leftarrow 2$	6^1	$1 \leftarrow 4 \leftarrow 2 \leftarrow 3$
7^1	$2 \rightarrow 1 \leftarrow 3 \leftarrow 4$	8^1	$2 \rightarrow 1 \leftarrow 4 \leftarrow 3$
9^1	$3 \rightarrow 1 \leftarrow 2 \leftarrow 4$	10^1	$3 \rightarrow 1 \leftarrow 4 \leftarrow 2$
11^1	$4 \rightarrow 1 \leftarrow 2 \leftarrow 3$	12^1	$4 \rightarrow 1 \leftarrow 3 \leftarrow 2$
13^1	$1 \leftarrow 2 \begin{array}{l} \swarrow 3 \\ \nwarrow 4 \end{array}$	14^1	$1 \leftarrow 3 \begin{array}{l} \swarrow 2 \\ \nwarrow 4 \end{array}$
15^1	$1 \leftarrow 4 \begin{array}{l} \swarrow 2 \\ \nwarrow 3 \end{array}$	16^1	$1 \begin{array}{l} \swarrow 2 \\ \leftarrow 3 \\ \nwarrow 4 \end{array}$

Figure 2.11
1-trees.

Claim 1: The lowest cost 1-tree is 1^1: $1 \leftarrow 2 \leftarrow 3 \leftarrow 4$.

Claim 2: The lowest cost 2-tree is 1^2: $1 \rightarrow 2 \leftarrow 3 \leftarrow 4$.

Claim 3: The lowest cost 3-tree is 1^3: $1 \rightarrow 2 \rightarrow 3 \leftarrow 4$.

Claim 4: The lowest cost 4-tree is 1^4: $1 \rightarrow 2 \rightarrow 3 \rightarrow 4$.

1^4 has a higher cost than 1^3 as $r_{43}<r_{34}$. Hence class 4 is never the long-run equilibrium. 1^1 has a lower cost than 1^3 if

$$r_{21}<r_{12} \Rightarrow (1+d)(1-q^*)<(1-2d)q^* \Rightarrow d < \frac{2q^*-1}{1+q^*}.$$

The condition for 1^1 having a lower cost than 1^3 is the same as $r_{32}=r_{21}$ and $r_{12}=r_{23}$. If the condition holds then the long-run equilibria are the set of states in class 1. Now compare the cost of 1^2 and 1^3. The condition for 1^3 to have a lower cost is,

$$r_{32}>r_{23} \Rightarrow d > \frac{2q^*-1}{1+q^*}.$$

The final possibility is $d = \frac{2q^*-1}{1+q^*}$. In this case, states in classes 1, 2 and 3 all form the set of long-run equilibria. QED.

Proof of claim 1: Consider a tree which includes the transition $1 \leftarrow 3$ and has a minimum cost of c. This tree must also include a transition $i \leftarrow 2$ where i is 1, 3 or 4. Such a tree cannot be the lowest cost tree as $r_{31}=r_{32}+r_{21}$. Hence there is another tree which has cost $c-r_{2i}$. Similarly a tree with the transition $1 \leftarrow 4$ cannot be the lowest cost tree as $r_{41}=r_{43}+r_{32}+r_{21}$. The minimum cost tree must therefore include the transition $1 \leftarrow 2$. Now repeating the argument for any tree that includes the transition $2 \leftarrow 4$ shows that the minimum cost tree must have $2 \leftarrow 3$. Finally the lowest cost path from 4 is $3 \leftarrow 4$. QED.

Proof of claim 2: Consider a tree which has a transition $1 \rightarrow i$ where $i \neq 2$. This cannot be a lowest cost 2-tree. If we change the transition $1 \rightarrow i$ to $1 \rightarrow 2$ then the resulting 2-tree will have a lower cost. Hence the lowest cost 2-tree must include $1 \rightarrow 2$. A similar argument shows that the minimum cost tree must include $3 \rightarrow 2$. Finally the lowest cost way of adding 4 is $4 \rightarrow 3$. QED.

Proof of claim 3: The same reasoning in the previous proof can be used to show that the minimum cost 3-tree must include the transitions $3 \leftarrow 4$ and $3 \leftarrow 2$. The lowest cost way of adding 1 is $2 \leftarrow 1$. QED.

Proof of claim 4: Consider a tree which includes the transition $4 \leftarrow 2$ and has a minimum cost of c. This tree must also include a transition $i \leftarrow 3$ where i is 1, 2 or 3. Such a tree cannot be the lowest cost tree as $r_{24}=r_{23}+r_{34}$. Hence there is another tree which has cost $c-r_{3i}$. Similarly a tree with the transition $4 \leftarrow 1$ cannot be the lowest cost tree as $r_{14}=r_{12}+r_{23}+r_{34}$. The minimum cost tree must therefore include the transition $4 \leftarrow 3$. Now repeating the argument for any tree that includes the transition $3 \leftarrow 1$ shows that the minimum cost tree must have $3 \leftarrow 2$. Finally, the lowest cost path from 1 is $2 \leftarrow 1$. QED.

Chapter 3

Multi-Unit, Common-Value Auctions.

The introduction highlighted the difficulties associated with the theory of multi-unit auctions. This chapter addresses some of these difficulties in the case of multi-unit, common-value auctions. In common-value auctions, the value of all units is the same for all bidders. There may, however, be incomplete information about this common value. Perhaps the best example of the use of multi-unit auctions where the units have a common value is the sale of index-linked bonds by the US Treasury. The common value is the price the bonds will fetch in the secondary market. In the case of reverse auctions, where suppliers submit supply schedules to the auctioneer, the common-value assumption simply implies a constant marginal cost.

One of the problems associated with multi-unit auctions is the existence of multiple equilibria. Back and Zender (1993) show that in uniform-price, common-value, multi-unit auctions any price between the reservation price and the lower bound of the common value can be supported as a symmetric equilibrium when the good is perfectly divisible. We illustrate this result when there is complete information in section 3.1, and show that the multiple equilibria disappear when units are discrete. If quantities are discrete and price bids are continuous then, in the complete information case with common values, the unique pure-strategy equilibrium results in a competitive market clearing price equal to the common value.

A second problem is characterising the equilibria. Section 3.2 investigates equilibria in common-value auctions with complete information and capacity constraints, where the quantity for auction is uncertain. Capacity constraints arise naturally in reverse auctions. For example, in the case of the Electricity Pool, the constraints simply reflect the generating capacity of each firm. In the case of Treasury auctions there may be a limit on the number of units for which each firm can bid. The capacity constraints are significant when units are required from all bidders to meet demand. Each firm will then know that they have some 'residual market' share irrespective of their bids. The uncertainty in the quantity up for auction is also motivated by the application to the Electricity Pool, as the residual demand faced by the generators is uncertain. In Treasury auctions the number of bonds up for sale is known.

Section 3.2.1 looks at the case of a discriminatory auction. Without capacity constraints, we show there is a unique pure-strategy equilibrium, where the market clearing price is equal to the marginal cost (or the common value, in the case of a conventional auction). If, however, there is a positive probability that each firm will have a residual market then there is no pure-strategy equilibrium. We characterise a mixed-strategy equilibrium for the duopoly case that holds for any distribution of the quantity up for auction, when this probability is one. If the probability is less than one, then there is a similar mixed-strategy equilibrium that holds for a large class of distributions.

Section 3.2.2 studies the uniform case. Most of the results of this section are due to von der Fehr and Harbord (1993). They use a uniform-price, multi-unit auction to model the Electricity Pool. In their model each firm has a constant marginal cost but this may vary between firms. We only consider the common-value case where all firms have the same constant marginal cost. As with the discriminatory case, without capacity constraints there is a pure-strategy equilibrium where the market clearing price is equal to the marginal cost. If it is certain that each firm will have a residual market, then there is a unique set of pure-strategy equilibria where the market clearing price is equal to the maximum permissible price (or reserve price in the case of a conventional auction). If it is not certain but possible that each firm will have a residual market then for most distributions there is no pure-strategy equilibrium. We show that the mixed-strategy equilibrium which von der Fehr and Harbord derive for this case does not extend to the case where each firm has multiple units.

All the illustrations of section 3.1 and results of section 3.2 are presented for reverse auctions, where suppliers compete to sell goods. The results apply equally to conventional auctions, where the uncertainty in demand is replaced by uncertainty in supply, the constant marginal cost is replaced by a common value and the maximum permissible price by a reservation price. We present the results for reverse auctions because some of the results are used in chapter 4.

3.1 Multiple Equilibria in Uniform-Price Multi-Unit Auctions.

To illustrate the multiple equilibria, consider the following simple model. There are two firms that produce a perfectly divisible good at a constant marginal cost, c. The auctioneer announces a fixed quantity, Q, that is required (hence demand is perfectly inelastic) and a maximum price, p^u, the firms can bid. The two firms must submit non-decreasing supply schedules $s:[0,p^u] \rightarrow [0,Q]$. Let $s_i(p)$ denote the supply schedule of firm i. The market clearing price, p*, is the minimum price at which $s_1(p)+s_2(p)\geq Q$. Each firm sells the quantity bid at a price less than or equal to p*[1] and is paid p* for the quantity sold. Call the above game G. Let q_i be the quantity firm i sells in equilibrium. The market clearing price and quantities are illustrated in figure 3.1.

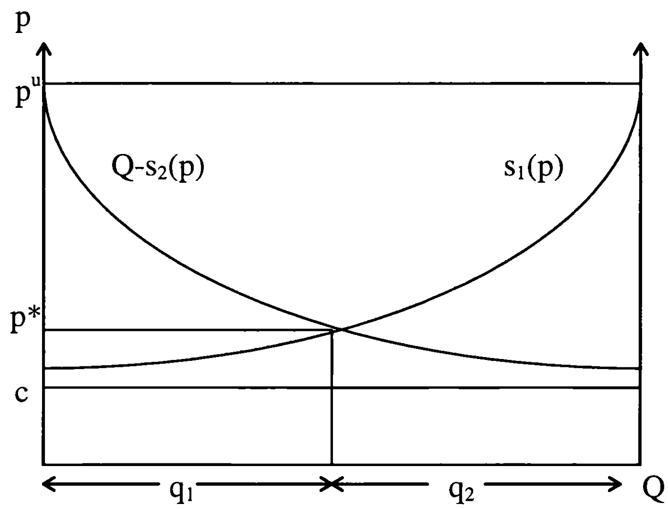


Figure 3.1
Market clearing price.

The figure plots firm 1's supply curve against the residual demand he faces

[1] If at p* the aggregate supply curve is flat, then all units bid at this price are rationed. The rationing rule is not important for our purposes.

once we take out the supply of firm 2. The intersection of these curves therefore determines the market clearing price p* and the quantity each firm sells, q_1 and q_2. The profit function of firm i is π_i=qi(p*-c). An iso-profit curve is therefore given by $I=\frac{\pi}{q}+c$. We can draw iso-profit curves of both firms on the same diagram as $q_2=Q-q_1$. This is illustrated in figure 3.2.

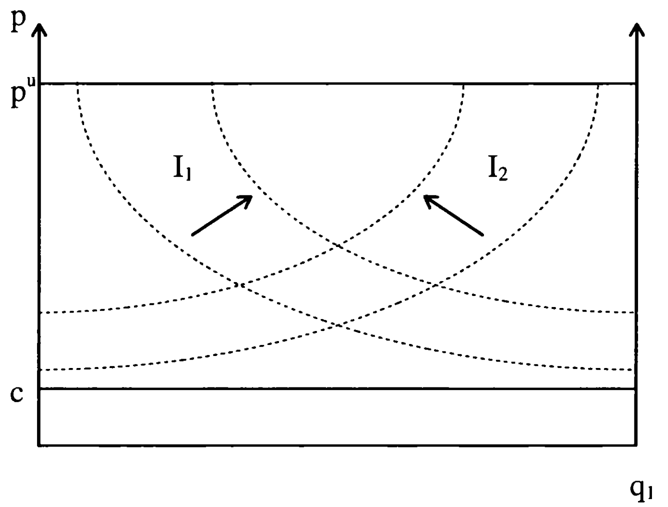


Figure 3.2
Iso-profit curves.

Proposition 3.1: *Any price, $\bar{p}\in[c,p^u]$, can be sustained as an equilibrium of G. Moreover, for any price, $\bar{p}\in[c,p^u]$, there is an equilibrium where firm 1 gets $\bar{q}_1\in(0,Q)$ and firm 2 gets $\bar{q}_2=Q-\bar{q}_1$.*

Proof: The optimal price and quantity combination for firm 1 is $\bar{p}$ and $\bar{q}_1$ if firm 2 submits a supply curve such that the residual supply is tangent to an iso-profit curve at this point and lies below the iso-profit curve at all other points. Any supply curve that passes through this point will then be optimal for firm 1. If firm 1 submits a supply curve which is tangent to firm 2's iso-profit curve that passes through this point, then any supply curve that passes through this point is also optimal for firm 2. Hence any pair of supply curves that pass through this point, that are tangent to the other firm's iso-profit curve at this point and that do not cross the iso-profit curve at any other point form an equilibrium. QED.

The diagram illustrates such an equilibrium with linear supply curves.

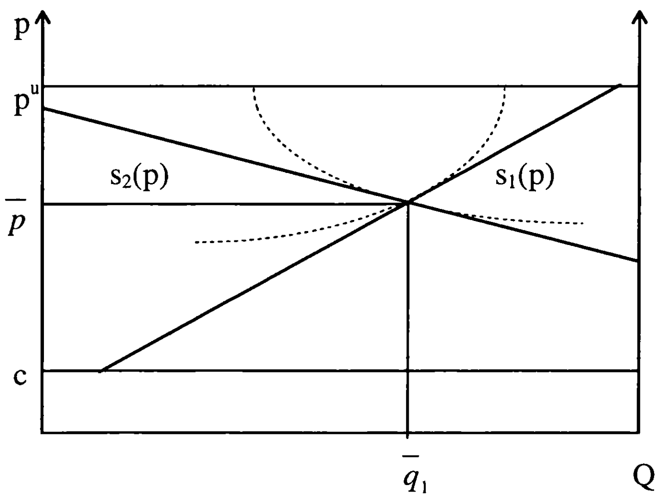


Figure 3.3
Multiple equilibria.

Klemperer and Meyer (1989) obtain a similar result with downward sloping demand, although they only consider twice differentiable continuous supply functions (section 4.1). Back and Zender (1993) show that this is true when there is incomplete information about the true common value. They show that any price between the reserve price and the lower bound of the common-value distribution can be a market clearing price in a symmetric equilibrium. Finally, Binmore and Swierzbinski (1997) use the same framework to illustrate multiple equilibria in the case of private values. All these papers assume the good is perfectly divisible. In the next section, we show that the multiple equilibria in pure strategies disappear when bids are for discrete units.

3.2 Discrete Units.

Now consider a model where the bids must be for discrete units. Call this game G_d. As in G, the auctioneer announces a quantity Q that is required, where Q is an integer, and a maximum permissible price, p^u. Each firm submits a vector of Q prices, s_i, where the n^{th} element of the vector represents the minimum price at which the firm is willing to supply n units. One can think of each element of the vector as a bid for a particular unit. The vector is then a list of these bids in increasing order. Denote by s_i/p the vector that only includes prices up to p and by $N(s_i/p)$ the number of elements in s_i/p. Hence $N(s_i/p)$ is the maximum number of units firm i is willing to supply at a price, p. The market clearing price is the lowest price such that $N(s_1/p)+N(s_2/p) \geq Q$. If $N(s_1/p^*) + N(s_2/p^*)=Q$ then each firm sells $N(s_i/p^*)$ units at p^*. If $N(s_1/p^*) + N(s_2/p^*)>Q$ then all the units bid at p^* are rationed. Denote by $s_i//p$ the vector that only includes prices less than p and by $N(s_i//p)$ the number of elements in $s_i//p$. Then the number of units rationed if $N(s_1/p^*)+N(s_2/p^*)>Q$ is $Q-N(s_1//p^*)-N(s_2//p^*)$. Assume there is a proportional rationing rule. Hence each of the rationed units are assigned to firm i with probability,

$$r_i = \frac{N(s_i / p^*) - N(s_i // p^*)}{N(s_i / p^*) - N(s_i // p^*) + N(s_j / p^*) - N(s_j // p^*)} . \tag{3.1}$$

The expected number firm i sells is therefore,

$$q_i = r_i (Q - N(s_1 // p^*) - N(s_2 // p^*)) + N(s_i // p^*) . \tag{3.2}$$

Assume the firms are risk-neutral and maximise expected profits. The market clearing price is illustrated in figure 3.4. As before we draw firm 2's supply backward. This represents the residual demand faced by firm 1. The expected profit of each firm is $\pi_i(s_1,s_2)=q_i(s_1,s_2)\ (p^*(s_1,s_2)-c)$.

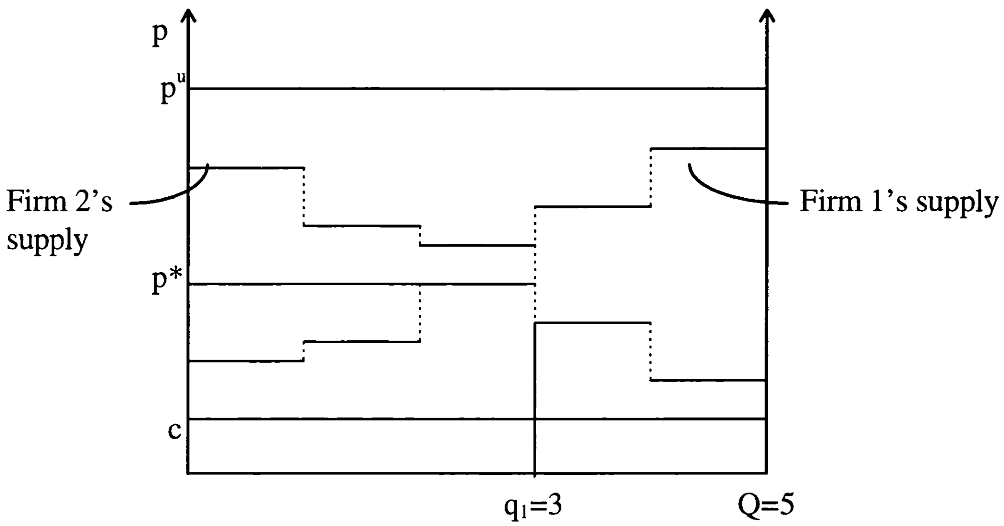


Figure 3.4
Market clearing price with discrete units.

Proposition 3.2: *G_d has a unique pure-strategy equilibrium where each firm bids c for all Q units.*

Proof: Consider a situation where the bids (s_1,s_2) are such that $p^*>c$. Then the quantity each firm sells is $q_i= x_i +N(s_i//p^*)$. If both firms have bid units at p^* then $x_1,x_2>0$. Each firm can then gain by submitting the $Q-N(s_i//p^*)$ units that are currently at a price $p\geq p^*$ at a price slightly below p^*. This will result in them getting the other firm's rationed quantity without significantly affecting price. If only firm j has bid for units at p^* then $x_i=0$. Firm i can then gain by undercutting in the same way. If all bids are at c then no firm can gain by cutting bids as this would result in a negative profit, and no firm can gain by raising bids as these units would not be sold. QED.

The proof relies on being able to undercut a price by some small amount. In reality prices are not continuous. Price bids must be a multiple of some finite amount, f. Hence undercutting the market clearing price will reduce the price received for all units by f. Call the game where price bids must be some multiple of f G'_d. Assume c and p^u are also multiples of f.

Proposition 3.3: *If the minimum price increment, f, is sufficiently large then any price,* $\bar{p} \in \{c, c+f, c+2f, ..., p^u\}$, *can be sustained as an equilibrium of* G'_d.

Proof: Consider the following supply schedules. Firm i submits d_i units at some price $p<\bar{p}$ where $d_i>0$ and $d_1+d_2=Q-1$, and all other units at $\bar{p}$. Each firm then sells the units bid at $p<\bar{p}$ and one more unit with a probability r_i. Increasing the price of some of the units submit at $\bar{p}$ will reduce this probability and therefore expected profits. Reducing the price of one of these units by f will increase this probability to 1 as there will be no rationing. This, however, will reduce the profits of this firm if,

$$(\bar{p}-c)(d_i+r_i) > (\bar{p}-c-f)(d_i+1) \Rightarrow f > (\bar{p}-c)(1-\frac{d_i+r_i}{d_i+1}).$$

If the above inequality holds for i=1,2 then neither firm can gain by reducing the price. Finally, the firms will have no incentive to increase the price of the d_i units bid at $p<\bar{p}$ to a price $p \geq \bar{p}$, as this would reduce the expected quantity sold without affecting the price. QED.

Hence the multiple equilibria problem returns if the increments in which the firms are allowed to bid are sufficiently large. A simple solution for an auctioneer wishing to avoid this problem is to make the increments very small. We then get the competitive equilibrium result in proposition 3.2. This perhaps explains why bids in Treasury auctions can be any multiple of a fraction of one basis point which translates to very small increments in prices. The most suitable model in applications therefore seems to be one with discrete units where price bids are allowed in very small increments. In the next section, we investigate equilibra in common-value auctions with capacity constraints, where units are discrete and price bids continuous.

3.3 Capacity Constraints and Uncertain Demand.

This section studies equilibria in multi-unit, common-value auctions, where there is a limit on the number of units each firm can bid for and the quantity for auction is uncertain. As before, the reverse auction case is studied. The limit then represents the maximum number of units the firms have a capacity to supply. In a conventional auction a limit may be imposed by the auctioneer.

There are m firms who each have enough capacity to supply k units. Normalise the total capacity of each firm to be 1 so the size of each unit is 1/k. All firms produce at a constant marginal cost, c, up to capacity. The firms submit a vector s_i, of k prices where the n^{th} element of the vector is the minimum price at which the firm is willing to supply n units. As before, one can think of each element of the vector as a bid for a particular unit. The vector is then a list of these bids in increasing order. After the firms submit their supply schedules, nature chooses the level of demand, d. Let $(\underline{d},\overline{d})$ be the minimum and maximum values that demand can take. The level of demand need not be an integer as in the previous section. The firms know the demand distribution.

The market clearing price is the lowest price, p(d), such that $\sum_{i=1}^{m} N(s_i / p) \geq d$. If $\sum_{i=1}^{m} N(s_i / \mathrm{p(d)}) = d$ then each firm sells $N(s_i/\ \mathrm{p(d)})$ units at p_d. If $\sum_{i=1}^{m} N(s_i / \mathrm{p(d)}) > d$ then $d - \sum_{i=1}^{m} N(s_i // \mathrm{p(d)})$ units are rationed. Once again assume there is a proportional rationing rule. Since d may not be a multiple of 1/k one of the firms may be assigned a fraction of a unit. Each of the rationed units plus this fraction is assigned to firm i with probability,

$$r_i = \frac{N(s_i\,/\,\mathrm{p(d)}) - N(s_i\,//\,\mathrm{p(d)})}{\sum_{j=1}^{m} N(s_j\,/\,\mathrm{p(d)}) - \sum_{j=1}^{m} N(s_j\,//\,\mathrm{p(d)})}. \tag{3.3}$$

The expected number firm i sells is therefore,

$$q_i = r_i (d - \sum_{j=1}^{m} N(s_j\,//\,\mathrm{p(d)})) + N(s_i\,//\,\mathrm{p(d)}). \tag{3.4}$$

The firms are risk-neutral and maximise expected profits. Let (p^1, p^2,..., p^{mk}) be the vector of m.k bids in increasing order. The next two sections study equilibria of this model under a discriminatory and uniform-price auction format. In section 3.3.4, we compare the two auction formats in terms of the expected cost to the buyer (in a conventional auction this amounts to a comparison of revenue to the seller).

3.3.1 Discriminatory auction

Under a discriminatory auction the sellers are paid the bid prices for the units they are assigned. This section studies the equilibria under the discriminatory pricing rule. We begin by presenting a general result on the type of pure-strategy equilibria that can exist.

Proposition 3.4: *In any pure strategy equilibrium the marginal price is c.*

Proof: If $p(\bar{d})=p^1>c$ then any firm can gain by reducing their bids slightly below $p(\bar{d})$. This increases the expected number of units dispatched (as it avoids rationing) without significantly affecting the price. Now assume the aggregate supply is increasing. Let i be the largest number such that $p^i<p(\bar{d})$. If only one

firm has bid at this price, then that firm can gain by increasing the price towards $p(\bar{d})$. If two or more firms have bid a unit at this price and these units are rationed with a positive probability, then these firms can gain by reducing their bids slightly and thereby increasing the expected quantity dispatched. If the units are not rationed then the firms can gain by raising these bids towards $p(\bar{d})$. Hence there is no equilibrium in which $p(\bar{d})>c$. QED.

Proposition 3.5: *If $\bar{d}<m-1$ then there is a pure-strategy equilibrium in which at least $\bar{d}+1$ units are bid at c.*

Proof: This is the standard Bertrand type result. Reducing the price below marginal cost will result in negative profits. Raising price above marginal cost will result in a zero probability of being dispatched. QED.

Proposition 3.6: *If $\bar{d}>m-1$, then there is no pure-strategy equilibrium.*

Proof: From proposition 3.4 there is no pure-strategy equilibrium where the marginal price is greater than c. Hence in any pure-strategy equilibrium profits must be equal to 0. However, each firm can make a positive profit by setting a positive price for all units, as they will be dispatched in the event that demand is greater than m-1. QED.

For the remainder of this section, we concentrate on the duopoly case, m=2. Where there are no pure-strategy equilibria we look for mixed-strategy equilibria. For simplicity assume the marginal cost is zero.

Proposition 3.7: *If Pr(d>1)=1 then there is a mixed-strategy equilibrium in which each firm submits a price* $p \in (p_v, p^u)$ *with probability* $f(p) = \frac{E(d)-1}{2-E(d)} \frac{(p^u - c)}{(p-c)^2}$ *for all k units, where* $p_v = (E(d)-1)p^u + (2-E(d))c$ *and E(d) is the expected value of demand.*

Proof: Suppose firm 2 is submitting a price $p \in (p_v, p^u)$ for all units, according to the distribution function F(p). Let f(p) be the corresponding density function. Then player 1's expected payoff from submitting a price p_1 for all k units is

$$\Pi(p_1) = \int_{p_1}^{p^u} (p_1 - c) f(p) dp + \int_{p_v}^{p_1} (p_1 - c)(E(d) - 1) f(p) dp .$$

In equilibrium $\pi'(p_1)=0$ for all $p_1 \in (p_v, p^u)$. This gives

$$pf(p) + F(p) = \frac{1}{(2 - E(d))} .$$

The unique solution of this differential equation with boundary condition $F(p^u)=1$[2] is

$$f(p) = \frac{E(d)-1}{2-E(d)} \frac{(p^u - c)}{(p-c)^2}, \tag{3.5}$$

$$F(p) = \frac{p}{(2-E(d))(p-c)} - \frac{(E(d)-1)p^u + (2-E(d))c}{(2-E(d))(p-c)} . \tag{3.6}$$

Solving $F(p_v)=0$ gives $p_v = (E(d)-1)p^u + (2-E(d))c$. Hence given firm 2 is using the mixed strategy, firm 1 is indifferent as regards submission of any price in the

[2] It is not possible to have a mixed-strategy equilibrium with an upper bound less than p^u as the same quantity will be assigned by setting a price p^u as this upper bound but the price received is greater.

interval $[p_v, p^u]$ for all k units. We now show that, given player 2 is using this mixed strategy, player 1 cannot gain by submitting an increasing supply function. To do this we need to introduce some further notation.

Let P be a vector of n prices, $\{p_1,\ldots p_n\}$ where $n \le k$, $p_1 > p_2 > \ldots > p_n$, $p_1 \le p^u$, $p_n \ge p_v$[3]. Let α_i be the number of units bid at p_i multiplied by 1/k. Hence $\sum_{i=1}^{n} \alpha_i = 1$. Let pr^i be the probability that $2 - \sum_{j=o}^{i-1} \alpha_j > d > 2 - \sum_{j=o}^{i} \alpha_j$ and d^i be, $E(d/(2 - \sum_{j=o}^{i-1} \alpha_j > d > 2 - \sum_{j=o}^{i} \alpha_j))$, where $\alpha_0 = 0$. The expected profit of firm 1 as a function of the price vector is,

[3] Any bid below p_v cannot be optimal as firm 1 will be assigned such units with probability 1 but can increase the payment received for these units by increasing the bid to p_v.

$$
\begin{aligned}
\pi(P) = {} & pr^1\Big(\int_{p_1}^{p^u}\sum_{i=1}^{n}\alpha_i(p_i-c)f(p)dp \\
& \quad + \int_{p_v}^{p_1}\sum_{i=2}^{n}\alpha_i(p_i-c)+(d^1-(2-\alpha_1))(p_1-c)f(p)dp\Big) \\
& + pr^2\Big(\int_{p_1}^{p^u}\sum_{i=1}^{n}\alpha_i(p_i-c)f(p)dp+\int_{p_2}^{p_1}\sum_{i=2}^{n}\alpha_i(p_i-c)f(p)dp \\
& \quad + \int_{p_v}^{p_2}\sum_{i=3}^{n}\alpha_i(p_i-c)+(d^2-(2-\alpha_1-\alpha_2))(p_2-c)f(p)dp\Big) \\
& \vdots \\
& + pr^l\Big(\int_{p_1}^{p^u}\sum_{i=1}^{n}\alpha_i(p_i-c)f(p)dp+\int_{p_2}^{p_1}\sum_{i=2}^{n}\alpha_i(p_i-c)f(p)dp+ \\
& \quad + \sum_{m=3}^{l}\Big(\int_{p_m}^{p_{m-1}}\sum_{i=m}^{n}\alpha_i(p_i-c)f(p)dp\Big) \\
& \qquad + \int_{p_v}^{p_l}\sum_{i=l+1}^{n}\alpha_i(p_i-c)+(d^l-(2-\sum_{j=1}^{l}\alpha_j))(p_l-c)f(p)dp\Big) \\
& \vdots \\
& + pr^n\Big(\int_{p_1}^{p^u}\sum_{i=1}^{n}\alpha_i(p_i-c)f(p)dp+\int_{p_2}^{p_1}\sum_{i=2}^{n}\alpha_i(p_i-c)f(p)dp \\
& \quad + \sum_{m=3}^{n}\Big(\int_{p_m}^{p_{m-1}}\sum_{i=m}^{n}\alpha_i(p_i-c)f(p)dp\Big) \\
& \qquad + \int_{p_v}^{p_n}(d^n-1)(p_n-c)\,f(p)dp\Big).
\end{aligned}
$$

The first line is the expected profit conditional on $2>d>2-\alpha_1$ multiplied by the probability of this event. If the other firm's price is greater than firm 1's highest price then firm 1 simply sells all units at the bid prices. If the price is less than firm 1's highest price then firm 1 sells all units priced below p_1 at the bid prices (as $d>2-\alpha_1$) and some of the units priced at p_1. The expected quantity he sells at p_1 is $d^1-(2-\alpha_1)$. The second line is the expected profits conditional on

$2-\alpha_1 > d > 2-\alpha_1-\alpha_2$ multiplied by the probability of this event. As before if the other firm's price is greater than firm 1's highest price then firm 1 simply sells all units at the bid prices. If the other firm's price is between p_l and p_2 then firm 1 sells all units priced below p_l at bid prices and none of the units priced at p_l, as $d<2-\alpha_1$ and firm 2 will therefore supply the remainder of the units. If firm 2's price is less than p_2 then firm 1 sells all units priced below p_2 at the bid prices (as $d>2-\alpha_1-\alpha_2$) and some of the units priced at p_2. The expected quantity he sells at p_2 is $d^2-(2-\alpha_1-\alpha_2)$. The third line gives the general term for the expected profits conditional on $2-\sum_{j=o}^{l-1}\alpha_j > d > 2-\sum_{j=o}^{l}\alpha_j$ multiplied by the probability of this event. Now consider the terms in the profit function that include p_1.

$$
\begin{aligned}
&pr^1\Big(\int_{p_1}^{p^u}\sum_{i=1}^{n}\alpha_i(p_i-c)f(p)dp \\
&\qquad+\int_{p_V}^{p_1}\sum_{i=2}^{n}\alpha_i(p_i-c)+(d^1-(2-\alpha_1))(p_1-c)f(p)dp\Big) \\
&+pr^2\Big(\int_{p_1}^{p^u}\sum_{i=1}^{n}\alpha_i(p_i-c)f(p)dp+\int_{p_2}^{p_1}\sum_{i=2}^{n}\alpha_i(p_i-c)f(p)dp\Big) \\
&\ \vdots \\
&+pr^l\Big(\int_{p_1}^{p^u}\sum_{i=1}^{n}\alpha_i(p_i-c)f(p)dp+\int_{p_2}^{p_1}\sum_{i=2}^{n}\alpha_i(p_i-c)f(p)dp\Big) \\
&\ \vdots \\
&+pr^n\Big(\int_{p_1}^{p^u}\sum_{i=1}^{n}\alpha_i(p_i-c)f(p)dp+\int_{p_2}^{p_1}\sum_{i=2}^{n}\alpha_i(p_i-c)f(p)dp\Big).
\end{aligned}
$$

But $pr^1 + pr^2 + \ldots + pr^n = 1$. The expression then simplifies to,

$$
\begin{aligned}
&(1-F(p_1))(\alpha_1(p_1-c)+\alpha_2(p_2-c)+\dots\ \alpha_n(p_n-c)) \\
&\quad +F(p_1)(\alpha_2(p_2-c)+\dots\ \alpha_n(p_n-c)) \\
&\qquad + pr^1F(p_1)(d^1-(2-\alpha_1))(p_1-c) \\
&=(1-F(p_1))\alpha_1(p_1-c)+pr^1F(p_1)(d^1-(2-\alpha_1))(p_1-c).
\end{aligned}
$$

Substituting for F(p) using 3.6 gives,

$$(-\frac{E(d)-1}{2-E(d)}\alpha_1+pr^1(d^1-(2-\alpha_1))\frac{1}{2-E(d)})p_1.$$

Let d^{-1} be $E(d\,/\,2-\alpha_1>d)$. Then $E(d)=(1-p^1)\,d^{-1}+p^1\,d^1$. Substituting in for E(d) in the numerator and simplifying gives,

$$\frac{(-p^1(2-d^1)(1-\alpha_1)-(1-p)(d^{-1}-1)\alpha_1)p_1}{2-E(d)}.$$

Hence

$$\frac{\partial\pi(P)}{\partial p_1}=\frac{-p^1(2-d^1)(1-\alpha_1)-(1-p^1)(d^{-1}-1)\alpha_1}{2-E(d)}<0. \tag{3.7}$$

Firm 1 can therefore increase profits by reducing the highest price towards the second highest price. If he sets $p_1=p_2$ then we have a new price vector with n-1 prices. Firm 1 can then gain by reducing the new highest price towards the second highest price. Repeating the argument n-1 times, firm 1 maximises profits by reducing all bids to p_n. Hence, given firm 2 is using the mixed strategy, firm 1 will optimise by submitting a single price between p_v and p_u for all units. From the symmetry of the game the same applies for firm 2 if firm 1 is using the mixed strategy and we therefore have a mixed-strategy equilibrium. QED.

The expected cost to the buyer under the mixed-strategy equilibrium is

$$C=\int_{p_v}^{p^u}\left[\int_{p_1}^{p^u}(p_1+(E(d)-1)p_2)f(p_2)dp_2\right.$$

$$\left.+\int_{p_v}^{p_1}(p_1(E(d)-1)+p_2)f(p_2)dp_2\right]f(p_1)dp_1 \qquad (3.8)$$

$$=2(E(d)-1)p^u+(2-E(d))c.$$

In the appendix, we characterise the corresponding distribution for the case of a conventional auction. A similar result holds for the case where $\underline{d}<1<\overline{d}$. However, it does not hold for any demand distribution. To see this we repeat the previous analysis for this case. Suppose firm 2 is submitting a price $p \in (p_v, p^u)$ for all units, according to the distribution function G(p). Let g(p) be the corresponding density function. Then player 1's expected payoff from submitting a price p_1 for all units is

$$\Pi(p_1)=p^+\left(\int_{p_1}^{p^u}(p_1-c)g(p)dp+\int_{p_V}^{p_1}(p_1-c)(d^+-1)g(p)dp\right)$$
$$+p^-\int_{p_1}^{p^u}(p_1-c)d^-g(p)dp$$

where $p^+=\Pr(d>1)$, $p^-=\Pr(d<1)$, $d^+=E(d/d>1)$, $d^-=E(d/d<1)$.

In equilibrium $\pi'(p)=0$ for all $p \in (p_v, p^u)$. This gives

$$(p-c)g(p)+G(p)=\frac{p^++p^-d^-}{p^+(2-d^+)+p^-d^-}.$$

The unique solution of this differential equation with boundary condition $G(p^u)=1$ is

$$g(p)=\frac{(d^+-1)p^+}{(p^+(2-d^+)+p^-d^-)}\frac{(p^u-c)}{(p-c)^2}, \qquad (3.9)$$

$$G(p)=\frac{p^{+}+p^{-}d^{-}}{(p^{+}(2-d^{+})+p^{-}d^{-})(p-c)} \\ -\frac{(d^{+}-1)p^{+}p^{u}+(p^{+}(2-d^{+})+p^{-}d^{-})c}{(p^{+}(2-d^{+})+p^{-}d^{-})(p-c)}. \tag{3.10}$$

Solving $G(p_v)=0$ gives $p_v=(p^+(d^+-1)p^u+(p^+(2-d^+)+p^-d^-)c)/(p^++p^-d^-)$. For simplicity we only consider whether firm 1 can gain by submitting two prices, $\{p_1, p_2\}$where $p_1>p_2$. Let α be the quantity bid at p_1. Then firm 1's profit given firm 2 is using the mixed strategy is

$$\begin{aligned}\pi(p_1,p_2)=\ & p^a\Big(\int_{p_1}^{p^u}\alpha(p_1-c)+(1-\alpha)(p_2-c)g(p)dp \\ & \quad+\int_{p_V}^{p_1}(1-\alpha)(p_2-c)+(d^a-(2-\alpha))(p_1-c)g(p)dp\Big) \\ & +p^b\Big(\int_{p_1}^{p^u}\alpha(p_1-c)+(1-\alpha)(p_2-c)g(p)dp \\ & \quad+\int_{p_2}^{p_1}(1-\alpha)(p_2-c)g(p)dp)+\int_{p_V}^{p_2}(d^b-1))(p_2-c)g(p)dp\Big) \\ & +p^c\Big(\int_{p_1}^{p^u}(d^c-(1-\alpha))(p_1-c)+(1-\alpha)(p_2-c)g(p)dp \\ & \quad+\int_{p_2}^{p_1}(1-\alpha)(p_2-c)g(p)dp\Big) \\ & +p^d\int_{p_2}^{p^u}d^d(p_2-c)g(p)dp\end{aligned}$$

where $p^a=\Pr(2<d<2-\alpha)$, $p^b=\Pr(2-\alpha<d<1)$, $p^c=\Pr(1<d<1-\alpha)$, $p^d=\Pr(1-\alpha<d<0)$, $d^a=E(d/2<d<2-\alpha)$, $d^b=E(d/2-\alpha<d<1)$, $d^c=E(d/1<d<1-\alpha)$, $d^d=E(d/1-\alpha<d<0)$.

Taking out the terms that involve p_1, substituting for $G(p_1)$ using (3.10) and simplifying as before, the first derivative of the profit function with respect to p_1 is,

$$\frac{\partial\pi(p_1,p_2,\alpha)}{\partial p_1}=(-(p^a+p^b)(p^a(2-d^a)(1-\alpha)+p^b(d^b-1)\alpha)$$
$$-(d^c-(1-\alpha))p^c(p^a(d^a-1) \tag{3.11}$$
$$+p^b(d^b-1))+(d^a-(2-\alpha))p^a(p^c d^c+p^d d^d))/$$
$$(2(p^+(d^+-1)+p^-d^-)).$$

The case where demand is always greater than 1 is the special case where $p^a+p^b=1$. The derivative is then given by (3.7). In that case, the derivative is always negative regardless of the demand distribution. The denominator of (3.11) is always positive. All three terms in the numerator of (3.11) are positive and the overall sign is therefore ambiguous and depends on the demand distribution. The mixed-strategy equilibrium holds for any distribution where the derivative is negative for all $\alpha\in[0,1]$. If, however, the derivative is positive for some $\alpha\in[0,1]$, then each firm can gain by submitting two prices when the other uses the mixed strategy. We now show that the mixed-strategy equilibrium holds when demand is distributed uniformly.

Assume $d\sim U[0,2]$. Then $p^a=\alpha/2$, $p^b=(1-\alpha)/2$, $p^c=\alpha/2$, $p^d=(1-\alpha)/2$, $d^a=(4-\alpha)/2$, $d^b=(3-\alpha)/2$, $d^c=(1-\alpha)/2$, $d^d=(1-\alpha)/2$. Substituting these values into the numerator of (3.11) gives,

$$-\frac{1}{8}(\alpha^2(1-\alpha)+\alpha(1-\alpha^2)<0.$$

This is less than zero for any value of α and each firm can increase profits by reducing the higher price to the lower one. We now give an example of a distribution where the equilibrium does not hold. Assume $\alpha=.5$, $d^a=1.75$, $d^b=1.25$,

d^c=.75, d^d=.25. Substituting these values into the numerator of (3.11) gives,

$$p^a(p^d - 4p^b - 2p^a) - 2p^{b^2} - p^c p^b. \tag{3.12}$$

This expression is greater than zero when,

$$p^d > \frac{2p^{b^2} + p^c p^b}{p^a} + 4p^b + 2p^a. \tag{3.13}$$

For example, if p^a =.05, p^b=.05, p^c=.05 and p^d=.85 then the inequality holds. Each firm can then increase profits by increasing the distance between the prices. The mixed-strategy equilibrium, where they set one price for all units, no longer holds. If, however, we substitute p^a =.1, p^b=.1, p^c=.1 and p^d=.7 into (3.12) then the term is negative and the equilibrium continues to hold. It is clear from (3.11) and (3.13) that, for the equilibrium not to hold, the demand distribution must be heavily skewed towards low levels of demand. In particular, the value of p^d needs to be very high. For any distribution that is not skewed in this way the equilibrium holds.

The expected cost to the buyer under the mixed-strategy equilibrium is,

$$\begin{aligned} C = \int_{p_v}^{p^u}\Bigg[& \int_{p_1}^{p^u} (p^+(p_1 + (d^+ - 1)p_2) + p^- d^- p_1) g(p_2) dp_2 \\ & + \int_{p_v}^{p_1} (p^+(p_1(d^+ - 1) + p_2) + p^- d^- p_2) g(p_2) dp_2 \Bigg] g(p_1) dp_1 \\ = {} & 2p^+(d^+ - 1)p^u + (p^+(2 - d^+) + p^- d^-)c. \end{aligned} \tag{3.14}$$

3.3.2 Uniform-price auction

Under a uniform-price auction the sellers are paid the market clearing price for the units that are assigned to them. This case is analysed by von der Fehr and Harbord (1993) when each firm has a different constant marginal cost. We reproduce and extend their results for the common-value case where all the firms have the same constant marginal cost.

Proposition 3.8: *If $Pr(d \leq m\text{-}1)=1$ then there is a unique type of pure-strategy equilibrium where a quantity no less than $\bar{d}$ is bid at marginal cost by the m-1 firms excluding firm i, for all i.*

Proof: It is simple to see that this is an equilibrium as, if any firm raises their bids above marginal cost these units will have a zero probability of being sold. We now show that there is no other type of pure-strategy equilibrium. If $p^{mk}>p(\bar{d})>c$ then units bid at p^{mk} have a zero probability of being sold. Firms submitting at p^{mk} can undercut $p(\bar{d})$ slightly, which will have a negligible effect on the marginal price but will strictly increase the expected quantity sold. If $p^{mk}=p(\bar{d})$ then there must be more than one firm submitting at this price, as the capacity of each firm is 1. These firms can gain by undercutting this price, thereby avoiding rationing. Hence there is no equilibrium where $p(\bar{d})>c$. If $p(\bar{d})=c$ then the only way a firm can gain is by raising $p(\bar{d})$, but they cannot do this if at least $\bar{d}$ is submitted at c by all the other firms. QED.

Proposition 3.9*: If $Pr(d>m\text{-}1)=1$ then there is a unique type of pure-strategy equilibrium where one firm sets p^u for a quantity greater than m- $\underline{d}$ and all other firms set a price sufficiently low that this firm cannot gain by lowering the price of these units.*

Remark: The simplest form of such an equilibrium is where one firm bids p^u for all k units and all other firms bid c for all k units. The high price firm only sells a quantity equal to d-(m-1) while all other firms sell k units.

Proof: There is no equilibrium where $p(\bar{d})=c$ as all firms can gain by increasing bids. It is also not possible to have an equilibrium with bids greater than $p(\bar{d})$ since these units will not be assigned. Let i be the lowest number such that $p^i>c$. If $p(\underline{d})=c$, the firms only make a positive profit when demand is sufficiently high for p^i to be the marginal price. However, each firm can increase the marginal price in the event demand is less than this, by increasing the bids currently at c. As d>m-1 this will increase the marginal price in the event demand is equal to $\underline{d}$. Hence, if $p(\underline{d})=c$, then each firm can gain by increasing bids currently at c towards p^i, as this will only affect the ranking in the event where they were previously making no profit and will result in a positive profit in this event. There is therefore no equilibrium where $p(\underline{d})=c$.

Suppose there are two firms with bids in the interval $[p(\underline{d}), p(\bar{d})]$. If only one firm has units bid at $p(\underline{d})$ then this firm can gain by increasing these bids towards the next highest bid of another firm. If more than one firm has bids equal to $\underline{d}$ then each firm can gain by reducing the bids slightly to avoid rationing. Hence there is no equilibrium where more than one firm has bids greater than or equal to $p(\underline{d})$, and in equilibrium only one firm sets the marginal price.

Since the payment of this firm is increasing in the marginal price, they will set the highest permissible price. All the other firms are indifferent between setting prices in the interval $[c, p^u)$ as they sell k units for p^u. However, they must set prices sufficiently low that the high price firm cannot gain by lowering prices and increasing the quantity sold. QED.

Let P(d) be any demand distribution on the interval $[\underline{d}, \overline{d}]$ where $P(d_1)-P(d_2)>0$ for all $[d_2,d_1] \subseteq [\underline{d}, \overline{d}]$.

Proposition 3.10: *If $\underline{d}<m-1<\overline{d}$ and demand is distributed according to some function P(d) that satisfies the above conditions then there is no pure-strategy equilibrium.*

Proof: Much of the proof is the same as for proposition 3.9. It differs in the proof that firms can gain by increasing bids currently at c towards the next highest bid when $p(\underline{d})=c$. As before, there is no equilibrium where $p(\overline{d})=c$ as all firms can gain by increasing bids. If $p(\underline{d})=c$ then firms who have bid units at c can gain by increasing these bids towards the next highest bid because such units will become marginal with probability one as every interval of demand has positive weight. Applying other parts of the proof of proposition 3.9, there is no equilibrium where more than one firm has bids greater than or equal to $p(\underline{d})$. However, it is not possible for only one firm to bid units above $p(\underline{d})$ as $\underline{d}<m-1$ and the capacity of each firm is only 1. QED.

To see why it was necessary to assume the demand distribution P(d), consider the following discrete example in the duopoly case. Demand takes two values with positive probability, 0.8 and 1.95. Assume also that k=0.1. Then there is a pure-strategy equilibrium where one firm bids one unit at p^u and all others at c and the other firm bids c for all units. The part of the proof of proposition 3.10 that breaks down is that firms cannot gain by raising bids above marginal cost when $p(\underline{d})=c$ and this is because there is a hole in the distribution. If, however, $\overline{d}-\underline{d}<1$, then there is no pure-strategy equilibrium whatever the distribution.

Proposition 3.13: *If $\underline{d} \leq m-1<\overline{d}$ and $\overline{d}-\underline{d} \leq 1$, then there is no pure-strategy equilibrium.*

Proof: As with proposition 3.10, most of the proof is the same as for proposition 3.9. It differs in the proof that firms can gain by increasing bids currently at c towards the next highest bid when $p(\underline{d})=c$. There is no equilibrium where $p(\overline{d})=c$ as all firms can gain by increasing bids. We now show that there is no equilibrium where $p(\underline{d})=c$. First consider the extreme case where $\underline{d}=m-1$ and $\overline{d}=m$. If only one firm has bid units above c, then every other firm can increase $p(\underline{d})$ by increasing all bids. (The total number of units bid above c will then be the firm's capacity of one plus the units the high bidding firm is submitting above c). If two or more firms have bid units above c and $p(\underline{d})=c$ then each firm can increase $p(\underline{d})$ by increasing bids currently at c. (The total number of units bid above c will then be the firm's capacity of one plus the units the other firms have bid above c). Increasing bids in this way will increase the marginal price in the event demand is equal to $\underline{d}$ and the firms will therefore increase profits by increasing the bids towards the next highest bid.

Now consider the case where $\overline{d}$<m. Recall that in any equilibrium p($\overline{d}$)>c and there can be no bids above p($\overline{d}$). If more than one firm has units bid at $\overline{d}$ then they can gain by undercutting the others as this avoids rationing (which arises if there are no bids above p($\overline{d}$)). Hence in equilibrium only one firm can bid units at p($\overline{d}$). If p($\underline{d}$)=c then the firms that have no bids at p($\overline{d}$) can increase p($\underline{d}$) by increasing bids currently at c. (The firm with units at p($\overline{d}$) must have more than 2-$\overline{d}$ units at this price. Increasing bids on a quantity of one will ensure p($\underline{d}$) increases as $\overline{d}$-$\underline{d}$ $\leq$1.) Hence there is no pure-strategy equilibrium where p($\underline{d}$)=c.

Applying other parts of the proof of proposition 3.9, there is no equilibrium where more than one firm has bids greater than or equal to p($\underline{d}$). However, it is not possible for only one firm to bid units above p($\underline{d}$) as $\underline{d}$ $\leq$m-1 and the capacity of each firm is only 1. QED.

It is clear that for a large class of distributions there is no pure-strategy equilibrium when $\underline{d}$ $\leq$m-1<$\overline{d}$. We are unable to find any mixed-strategy equilibria for these cases. If the firms are only allowed to set one price for all units then it is a simple exercise to find a mixed-strategy equilibrium of the type given in proposition 3.8. An example of such an equilibrium is given by von der Fehr and Harbord (1993). They assume that demand is discrete and takes two values, 1 and 2, with probabilities r and 1-r. From proposition 3.13 there is no pure-strategy equilibrium in this case. They derive a mixed-strategy equilibrium by finding the distribution function the other firm needs to use for each firm to be indifferent between bidding a price in the interval $[p_v, p^u]$. Let the corresponding density function for firm 2 be t(p). We now show that this equilibrium does not hold if firm 1 can split his unit in two and set two prices. The profit of firm 1 if he sets two prices and firm 2 is using the mixed strategy is,

$$\pi_1(p_1, p_2) = (1-r)\left(\int_{p_1}^{p^u}(p-c)t(p)dp + \int_{p_v}^{p_1}(p_1-c)t(p)dp\right)$$
$$+r\left(\int_{p_1}^{p^u}(p-c)t(p)dp + \int_{p_2}^{p_1}\frac{1}{2}(p_1-c)t(p)dp\right)$$
$$=\pi_1(p_1) + r\int_{p_2}^{p_1}\frac{1}{2}(p_1-c)t(p)dp.$$

Hence the profit is equal to the profit the firm would get by setting one price plus some positive amount.

3.2.4 Ranking

We now rank the auctions in terms of the cost to the buyer (or in the conventional auction case, in terms of revenue for the seller). For simplicity we restrict attention to the duopoly case. The case where Pr(d<1) is straightforward as there is a unique type of pure-strategy equilibrium where the marginal price is always c with both auction formats. Hence in each case the buyer simply pays marginal cost for all units. In a conventional auction this translates to the seller receiving the common value. It is not surprising that we get this competitive result as each firm can supply the entire demand. The case where $\underline{d} < 1 < \overline{d}$ is problematic as we were unable to find any equilibrium for the uniform-price auction case. Hence we cannot rank the auctions in this case.

The most interesting case is when Pr(d>1)=1. Each firm is then sure to sell some units as the capacity of the other firm is 1. In the uniform-price auction case there is a unique set of pure-strategy equilibria where the marginal price is always p^u (from proposition 3.11). The expected cost for the buyer is then p^u E(d). Under a discriminatory auction there is no pure-strategy equilibrium for this case (proposition 3.7). In the mixed-strategy equilibrium given in proposition 3.8 the expected cost for the buyer is $2p^u(E(d)-1)+(2-E(d))c$. This is less than the cost under the pure-strategy equilibrium in the uniform case as $c<p^u$. In fact we can show there is no equilibrium in the discriminatory case that results in an expected cost of p^u E(d). This would require both firms to bid p^u for all units (as they get paid their bids) but each firm can then gain by undercutting the other (proposition 3.7). Hence we have a clear ranking when each firm is certain to have some residual demand which makes the discriminatory auction less costly than the uniform one.

We can translate these results to apply to a conventional auction. The application to treasury auctions is particularly convenient as supply is not uncertain. We therefore only have two cases to consider, $s<1$ and $s>1$ where s is the quantity of bonds for sale, and we can say something about the ranking in each case. When $s<1$ we get the competitive outcome with both auction formats. When $s>1$ then under a uniform-price auction there is an equilibrium of the type given in proposition 3.11 where one firm sets the reservation price , r, for enough units to ensure the marginal price is r and the other firm bids sufficiently high prices. This type of equilibrium would be particularly bad for the seller as they only get the reservation price for all units. The mixed-strategy equilibrium given in proposition 3.8 for the discriminatory case can also be translated into an equilibrium in a conventional auction. The corresponding equilibrium is given in the appendix. The expected revenue of the seller is then $v(2-s)+2r(s-1)$ where v is the common value. This results in a higher revenue for the seller as $r<v$.

3.4 Conclusions.

One of the arguments used against the use of a uniform-price, multi-unit auction is that there are multiple equilibria in pure–strategies, some of which are very bad for the buyer (or seller in a conventional auction). We show that this result depends on the good being perfectly divisible. If quantities are discrete and price bids are continuous then all pure-strategy equilibria result in a competitive market clearing price. The multiple equilibria reappear if the increments in which price bids are allowed is sufficiently large. To ensure that these equilibria do not exist, the auctioneer should allow for very small increments in price bids. The set of pure-strategy equilibria are the same if a discriminatory pricing rule is used and there is nothing to tell between the two pricing rules in terms of cost/revenue to the buyer/seller.

However, if capacity constraints are present we find that the discriminatory pricing rule performs better. We show that if each firm is certain to have some residual demand/supply then the discriminatory auction results in a lower expected cost to the buyer/a higher expected revenue for the seller than the uniform-price auction. The results show that it is a bad idea for a seller such as the Treasury to impose restrictions on quantity bids to such an extent that the bidders are certain to get some units whatever they bid, as this effectively gives them some market power and decreases the expected revenue whatever the pricing rule.

In the case of reverse auctions these constraints arise naturally as they represent the capacity of the firms. In periods when demand is certain to be sufficiently high that capacity is required from both firms, each firm will have some market power. We show that under these circumstances a discriminatory pricing rule results in a lower expected cost to the buyer than a uniform pricing rule. We therefore give a case for adopting a discriminatory pricing rule.

In the next section, we apply these results to the Electricity Pool where generators compete to supply electricity from their generating plants. The above results show that, in a very simple model of the pool, a discriminatory pricing rule performs better than the current uniform pricing rule in periods when demand for electricity is high. In a more detailed model where generators withhold capacity, we show the uniform pricing rule performs even worse while the capacity withholding does not effect the equilibria in the discriminatory auction.

3.5 Appendix

Mixed-strategy equilibrium in a conventional auction.

We now characterise the mixed-strategy equilibrium for a conventional auction, corresponding to the reverse auction equilibrium given in section 3.3.1 (proposition 3.8).

The model is the same except we now have bidders submitting demand functions for k units of a good, we replace the constant marginal cost with a common valuation, v, for each unit of the good and the maximum permissible bid is replaced by a minimum permissible bid or reservation price, r. Suppose firm 2 is submitting a price $p \in (r, p_m)$ for all units, according to the distribution function H(p). Let h(p) be the corresponding density function and E(s) the expected value of supply. Then player 1's expected payoff from submitting a price p_1 for all k units is

$$\Pi(p_1) = \int_{p_1}^{p_m} (E(s)-1)(v-p_1)h(p)dp + \int_{r}^{p_1} (v-p_1)h(p)dp .$$

In equilibrium $\pi'(p_1)=0$ for all $p_1 \in (r, p_m)$. This gives

$$(v-p)h(p) - H(p) = \frac{E(s)-1}{2-E(s)} .$$

The unique solution of this differential equation with boundary condition F(r)=0[4] is

$$h(p) = \frac{(E(s)-1)(v-r)}{(2-E(s))(v-p)^2} ,$$

[4] It is not possible to have a mixed-strategy equilibrium with a lower bound greater than r as the same quantity will be assigned by setting a price r as this lower bound but the price paid is less.

$$H(p)=\frac{E(s)-1}{(2-E(s))(v-p)}p-\frac{E(s)-1}{(2-E(s))(v-p)}r.$$

Solving $H(p_m)=1$ gives $p_m=(2-E(s))v+(E(s)-1)r$. Hence given firm 2 is using the mixed strategy, firm 1 is indifferent between submitting any price in the interval $[r,p_m]$ for all k units. The proof that player 1 cannot gain by submitting an increasing supply function when player 2 is using this mixed strategy is the same as in the text. The expected revenue for the seller from the mixed-strategy equilibrium is,

$$\begin{aligned}R&=\int_r^{p_m}\left[\int_{p_1}^{p_m}(p_1(E(s)-1)+p_2)h(p_2)dp_2\right.\\&\qquad\left.+\int_r^{p_1}(p_1+(E(s)-1)p_2)h(p_2)dp_2\right]h(p_1)dp_1\\&=2(E(s)-1)r+(2-E(s))v.\end{aligned}\tag{3.15}$$

Chapter 4
Modelling the Electricity Pool

The electricity industry in England and Wales has gone through a radical transformation over the last 8 years. The 1990 reform brought to an end the nationalised system which was set up in 1947. Under this system the Central Electricity Generating Board (CEGB) had a monopoly on the supply and high voltage transmission of electricity, while twelve regional electricity boards were responsible for the distribution and sale of electricity. The regional electricity boards were privatised as they were, and became known as the Regional Electricity Companies (RECs). The supply side, however, was restructured. The generating plants of the CEGB were split between 3 companies, National Power, PowerGen and Nuclear Electric. The high voltage transmission network was separated from generation and put in the hands of the National Grid Company, who were also given the role of central dispatcher.

At the heart of the reform was the innovative way in which competition in generation was introduced. The Electricity Pool is a spot market for the sale of electricity. Generators compete to supply electricity from their plants by submitting bids for the minimum price at which they are willing to supply electricity from each of their plants. The central dispatcher then constructs the least cost rank order of plants for each half-hour. All units dispatched are paid the price of the marginal unit. The rationale behind a competitive pool is that competition in generation will result in a more productive and efficient generating system. Since 1990 similar systems have been adopted elsewhere.

One of the key features of the restructuring was that the thermal plants of the CEGB were divided between only two companies, National Power and PowerGen. Their plants accounted for 48% and 30% of the total generation capacity available to the Pool. Nuclear Electric took control of the nuclear plants which accounted for 14% of capacity. Electricite de France and producers in Scotland also supply electricity to the grid although this is limited by transmission capacity constraints and in 1990 accounted for only 5% of the total capacity. The hope that this market structure would lead to competition in generation has not been realised with pool prices above competitive levels. The main problem has been the market power of the two large generators, National Power and PowerGen.

The Office of Electricity Regulation (OFFER) was set up to regulate the newly privatised industry. The main role of OFFER was to regulate the transmission network and the RECs who had regional monopolies on distribution. There were to be no explicit controls over the Electricity Pool, as it was thought that competition between the generators would render any regulation unnecessary. Nevertheless, the Electricity Pool became the subject of a number of OFFER enquiries as it became apparent that the generators were exercising market power. One of the roles of the regulator was to promote competition but the only significant weapon the regulator had to achieve this was to refer the companies to the Monopolies and Mergers Commission. With this threat the regulator was successful in getting National Power and PowerGen to agree on price caps between 1994 -1996 and on the divestiture of plant.

Since 1990 there has been a significant amount of entry which, together with plant divestiture, has reduced the market share of National Power and PowerGen which stood at 34% and 28% of the total capacity available to the Pool in 1996. Independent Power Producers have entered with combined cycle gas turbines in the so called 'dash for gas'. By 1996 they accounted for 10% of the market. They have been encouraged to enter by high pool prices and the security of gas contracts and long-term contracts with the RECs. There is a widely held view that pool prices above competitive levels have resulted in excessive investment in new generating plants. Despite this, the market power of the two dominant firms is still a concern. A complete review of the Pool is currently taking place and among other things the government is looking at ways in which the Pool can be made more competitive.

In section 4.2, we look at two theoretical papers which show why we should not expect competitive prices under this framework. The first is Green and Newbery (1992) who use the supply function framework of Klemperer and Meyer (1989) and the second is von der Fehr and Harbord (1993) who use a discrete framework. We also look at an empirical paper by Wolak and Patrick (1996) who show that the current structure gives the generators an incentive to withhold base-load capacity and present evidence of this strategy being used.

An important aspect of the Pool is the pricing rule. We explain the price determination process in detail in section 4.1. In simple terms one can think of the current set-up as a multi-unit uniform-price auction. In section 4.3, we present a model of the Pool where the firms have an incentive to withhold capacity under a uniform pricing rule. We show that the discriminatory auction equilibria result in a much lower cost. A third possibility is a multi-unit Vickrey auction. The main advantage of the Vickrey auction is that it is a weakly dominant strategy for each firm to bid at marginal cost. We show that this equilibrium is worse for the buyer than the discriminatory auction.

We also consider the repeated game and show that under a discriminatory auction the worst possible equilibrium for the buyer results in a lower cost than the one-shot pure-strategy equilibrium of the uniform-price auction. With a Vickrey auction the firms can collude on weakly dominated equilibria of the one-shot game and thereby achieve high prices in the Pool. We therefore put a strong case for the use of a discriminatory auction format rather than a uniform one.

4.1 Price Determination Process in the England and Wales Electricity Pool.

Since April 1 1990, all licensed generators and suppliers have had to buy and sell electricity through the Electricity Pool. The Pool operates a day ahead spot market to determine the generating units that are dispatched from each generator and the price at which trade takes place for each half-hour of the following day. The generators are required to submit bids for the price at which they are willing to supply electricity from each of their generating units[1]. These bids are fixed for the 48 half-hour periods of the following day. For each half-hour they also declare the units they wish to make available. The price bids and availability declarations are used to construct a supply curve for each half-hour of the following day.

Suppliers do not make bids for the price at which they are willing to purchase various quantities of electricity, they simply pay the price set by the Pool for whatever quantity is demanded in each half-hour. Hence suppliers play no role in the price determination process. In place of demand side bids, the Pool uses forecasted demand. The intersection of the vertical forecasted demand and the step aggregate supply curve determines the system marginal price (SMP)[2] for each half-hour and the generating sets that are scheduled to be dispatched[3]. All units dispatched are paid at SMP. This part of the price determination process can therefore be thought of as a daily multi-unit reverse auction with a uniform pricing rule.

---

[1] In fact, for each generating unit they submit 3 incremental price bids plus bids for start up costs and no load rate to reflect fixed costs.
[2] i.e. the price of the marginal unit that is required to meet forecasted demand.
[3] This is subject to transmission constraints.

However, in addition to the marginal price the generators are paid a capacity charge CC which is given by

$$CC=LOLP(VOLL-SMP) \tag{4.1}$$

where LOLP is the loss of load probability and VOLL is the value of lost load. This element is designed to account for the stochastic element of demand. The loss of load probability is the probability that there will be insufficient capacity to meet demand. This probability is inversely related to the reserve margin which is the capacity made available net of the forecasted demand. The LOLP is negligible for reasonable levels of the reserve margin but rises rapidly as the reserve margin becomes very small. The relationship is therefore extremely convex with a significant LOLP only occurring when the total capacity made available is very close to forecasted demand. The VOLL reflects the marginal value of electricity to the consumer in the event that there is insufficient capacity to meet their demand. This was set at £2000/MWH for 1990/91 and has increased annually at the rate of inflation. The total price paid to the generators is therefore SMP+CC and is referred to as the Pool Purchase Price (PPP). The formula for PPP can be written as,

$$PPP=(1-LOLP)SMP+LOLP.VOLL.$$

If the generators bid at marginal cost then PPP is a weighted average of the marginal price of producing an extra unit of electricity and the marginal value to the consumer in the event demand is rationed. It is therefore intended to signal optimally to the generators on investment decisions. However, this formula gives the generators an incentive to withhold capacity, as this reduces the reserve margin and allows them to get prices substantially above SMP. In section 4.2.3, we look at a paper by Wolak and Patrick who present evidence of this strategy being used.

The aggregate supply curve used to set SMP is known as the unconstrained merit order. The units scheduled to be dispatched may differ from this merit order due to transmission constraints. The problem is the uneven distribution of generating plants across the country with a concentration in the North close to the generating inputs. To minimise cost subject to transmission constraints, the grid operator revises the dispatch schedule. Generating sets that are 'constrained off' are paid (PPP-bid) while 'constrained on' unit are paid (bid+CC). This rule has given generators with plants that are likely to be constrained on because of their location in the network, an incentive to bid these units at very high prices.

The price paid by the suppliers is PPP+UPLIFT and is referred to as the Pool Selling Price (PSP). UPLIFT includes the cost incurred when units are constrained off and on. It also covers availability payments which are made to plants that are made available but not scheduled to run. The availability payment is LOLP(VOLL-max{SMP,bid price}). Costs associated with demand forecasting errors and ancillary services are also covered by UPLIFT.

4.2 Electricity Pool Literature.

In this section, we present three papers that look at the England and Wales Electricity Pool. The first two papers take different theoretical approaches to model the Electricity Pool. Green and Newbery (1992) use the supply function framework of Klemperer and Meyer (1989) while von der Fehr and Harbord (1993) use a discrete framework. Green and Newbery use their model to simulate the Electricity Pool by calibrating it to the electricity supply industry. The third paper is an empirical paper by Wolak and Patrick (1996). They look at the time series properties of the 48 half-hourly prices using data from the Pool between 1991-95. They argue that the empirical evidence is consistent with the generators withholding capacity to achieve occasionally high pool prices which result in yearly revenues significantly above production costs.

4.2.1 Green and Newbery.

Green and Newbery use the supply function model of Klemperer and Meyer to model the England and Wales Pool. The 'supply function equilibria' approach looks at equilibria in supply functions in oligopolistic competition rather than Cournot equilibria where firms choose quantities and Bertrand equilibria where firms choose prices. Klemperer and Meyer show that when demand is uncertain, it is optimal for each firm to commit to a supply curve rather than simply choosing one price or quantity. This is because there is a different price and quantity combination that is optimal for the firm for each realisation of demand. They show that without demand uncertainty, there is nothing to be gained by committing to a supply function because, whatever the other firms do, each firm faces a residual demand and optimises by choosing a point on this residual demand that maximises profits. When demand is uncertain, however, the residual demand is also uncertain and it is therefore optimal to commit to a supply curve such that for each realisation of demand the firm commits to a point on the residual demand curve that is profit maximising. We now illustrate their results in the duopoly case.

Assume there are two firms with identical marginal cost functions, C(q) where $C'(q) \geq 0$ and $C''(q) \geq 0$ for all $q \geq 0$. Demand is subject to an exogenous shock, ε, which has positive density everywhere on $[\underline{\varepsilon}, \overline{\varepsilon}]$. The demand is equal to $D(p,\varepsilon)$ where $-\infty < D_p < 0$, $D_{pp} \leq 0$, $D_\varepsilon > 0$ and $D_{p\varepsilon} = 0$. A strategy for firm k=1,2 is a function $S^k(p):[0, \infty) \rightarrow (-\infty, \infty)$. Attention is restricted to twice differentiable supply functions. Let $R^i(p,\varepsilon) = D(p,\varepsilon) - S^j(p)$ be the residual demand faced by firm i. First consider the case where there is no demand uncertainty. The profit maximisation problem of firm i is

$$\text{Max}_p\text{: } pR^i(p) - C(R^i(p)).$$

The first order condition is

$$R^{i'}(p) = -\frac{R^i(p)}{p - C'(R^i(p))}. \tag{4.2}$$

We now show that any output pair $(\overline{q}_i, \overline{q}_j)$ can be supported as an equilibrium outcome at a market price $\overline{p}$, where $\overline{q}_i + \overline{q}_j = D(\overline{p})$, if both firms cover marginal cost. To see this, substitute these values into (4.1). This gives,

$$R^{i'}(\overline{p}) = -\frac{\overline{q}_i}{\overline{p} - C'(\overline{q}_i)}. \tag{4.3}$$

Hence the price and quantity pair ($\bar{p}$, $\bar{q}_i$) is optimal for firm i if his residual demand curve has a slope given by (4.3) at this point. This is illustrated in figure 4.1.

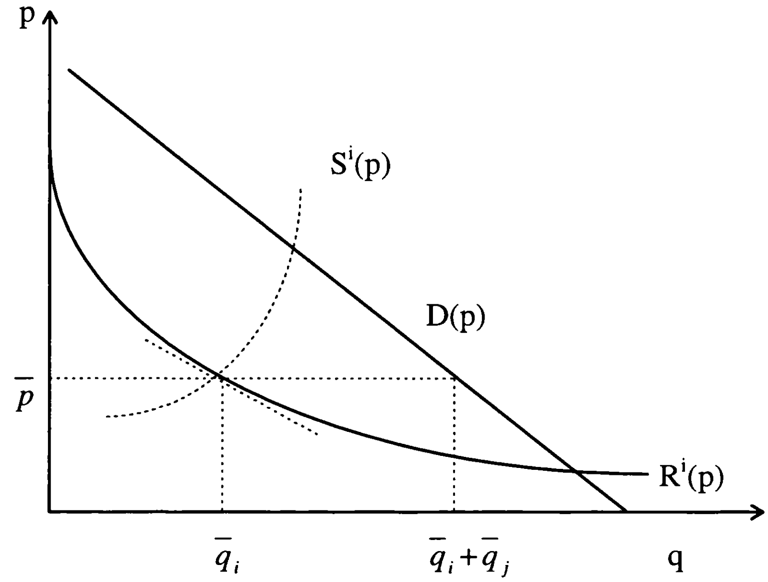


Figure 4.1
Equilibrium with no uncertainty

Any supply curve that passes through this point is optimal for firm i. Repeating the argument for firm j shows that an equilibrium simply requires both firms to submit a supply curve with the correct slope at the equilibrium point. This slope is given by (substituting for $R^{i\prime}(\bar{p})$ in (4.3)),

$$S^{j\prime}(\bar{p}) = \frac{\bar{q}_i}{\bar{p} - C'(\bar{q}_i)} + D'(\bar{p}) . \tag{4.4}$$

Hence any outcome where each firm makes a positive profit can be supported by a multiplicity of equilibria. The number of equilibria can be dramatically reduced when demand is uncertain as there is a unique optimal point for each realisation of demand and an equilibrium therefore specifies the slope at each point along the supply schedule. In the symmetric case where each firm supplies half of the demand in equilibrium, the first order condition in the uncertain demand case is,

$$S'(p) = \frac{S(p)}{\overline{p} - C'(S)} + D'(p). \tag{4.5}$$

The difference between this and (4.4) is that it specifies a locus of points such that each firm's first order condition is satisfied for every value of the demand shock, ε. This is illustrated in figure 4.4.

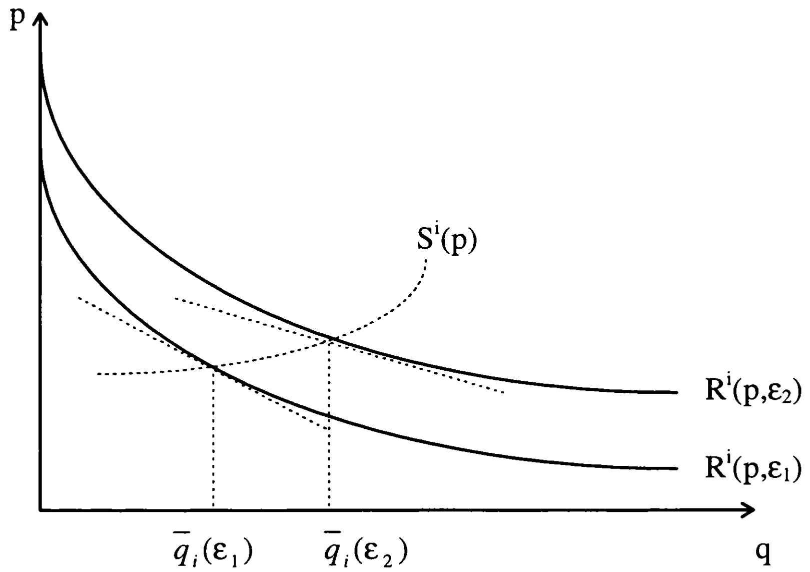


Figure 4.2
Equilibrium with uncertainty

Klemperer and Meyer go on to show that the equilibrium supply functions lie between the Bertrand and Cournot supply functions. If the demand shock is bounded then there is a connected set of equilibria. They give an example where there is a unique equilibrium but this requires unbounded support.

To apply this model to the electricity industry Green and Newbery replace the demand uncertainty with demand variation over time. The demand curve is then given by the load duration curve D(p,t) where t is the number of hours that demand is above D during the day. The idea is that the generators have to commit to one supply curve over a period of one day and choose a supply curve to maximise profits given varying levels of demand over the day. This is equivalent to committing to a supply curve when demand is uncertain and choosing a supply curve to maximise expected profits given the demand distribution. They use the load duration curve so that demand is monatomic over time.

Using this framework, Green and Newbery model a duopoly competing to supply to the England and Wales Pool. This is an assumption made more generally and the justification is the following. The nuclear plants have a very high fixed cost but once up and running the marginal cost of producing up to capacity is relatively small. Nuclear Electric have therefore tended to bid their units at very low prices to ensure dispatch. This is also true of the Scottish producers and Electricite de France who bid low prices to ensure they sell their excess energy. This creates a virtual duopoly in the Electricity Pool, with National Power and PowerGen competing to supply the residual demand once the supply of these other generators is taken out[4].

Using this theoretical framework Green and Newbery simulate the England and Wales spot market to measure the extent and cost of market power. They fit a simple cost function to data from the *CEGB Statistical Yearbook*, and use demand and output data from 1988/89. Using a linear demand with elasticity -.25 they find that in the lowest-supply, highest price equilibrium, the price to

[4] Since privatisation a significant amount of entry has taken place by Independent Power Producers. However, they have entered on the back of contracts with the Regional Electricity Companies which effectively fix the price they get for electricity. They therefore also bid at low prices to ensure dispatch. The assumption of a duopoly is therefore still valid.

suppliers is nearly double the competitive level and output is 10% less. They estimate that this results in a deadweight loss of £340 million a year. Using different values for the elasticity they find the losses are much greater, the more inelastic the demand. They repeat the analysis for the case where there are five identical firms (with National Power divided into three and PowerGen into two). They estimate that the equilibrium price in the highest price equilibrium would be much closer to the competitive level with deadweight losses down to only £20 million. They also go on to show that the present structure induces too much entry leading to additional welfare losses. Green and Newbery are therefore able to show that splitting the thermal plants of the CEGB between only two companies was a costly mistake and the assumption that Bertrand type competition in the Pool would lead to competitive prices was ill-founded.

A major difficulty with the supply function approach is the assumption that the generators submit continuously differentiable supply curves. In reality the generators submit step functions with price bids for discrete units. In section 3.1, we showed that the type of equilibria in models with perfectly divisible units do not carry through to the case where units are discrete. The next paper we look at models the Electricity Pool with discrete units.

4.2.2 von der Fehr and Harbord.

von der Fehr and Harbord model the Pool as a uniform-price, sealed-bid, multi-unit, private-value, reverse auction. It is uniform-price because all units dispatched are paid the bid price of the marginal unit, sealed-bid because each firm bids without knowledge of the bids of other firms, and private-value because the firms have different marginal costs. In section 3.2.3, we presented the special case where all firms have the same constant marginal cost. The results for the case where firms have different marginal costs are very similar in nature. For simplicity we will concentrate on the duopoly case although many of the results extend to the oligopoly case.

Two generators compete to supply electricity from their plants to the Pool. Let the marginal cost of firm n be c_n where $c_2>c_1>0$. Each firm has a capacity of k_n consisting of a number of discrete sized plants. Each firm submits bids for the minimum price at which they are willing to supply electricity from each of their plants and these bids are used to construct the aggregate supply curve The highest permissible bid price is $\overline{p}$. If there is a price at which more than one unit has been bid, then these units are equally likely to be called into operation. The level of demand is random and is not known at the time the bids are made. However, the generators do know the probability distribution, G(d). The support of the demand distribution is $[\underline{d},\overline{d}] \subseteq [0, k_1 + k_2]$. The Pool operator equates the aggregate supply with the realised level of demand to determine the units that are dispatched. The firms are paid the bid price of the marginal unit that is required to meet the realised level of demand, for all units dispatched.

They begin by showing that in any pure-strategy equilibrium only one firm can set the marginal price with positive probability. However, this result depends on demand having full support on the interval $[\underline{d},\overline{d}]$[5]. The equilibria that they characterise using this proposition, do not apply to discrete distributions. The

[5] On page 98 we give a discrete example where this proposition does not hold.

support of the demand distribution plays an important role in the characterisation of the equilibria. They therefore split all possible types of support into three categories: 'low demand periods' when the support is such that each firm can satisfy demand; 'high demand periods' when neither firm has enough capacity to satisfy demand; and 'variable demand periods' when there is a positive probability that demand will be low enough for both firms to satisfy demand and high enough that neither firm can satisfy demand.

Low demand periods

If $\Pr(d<\min\{k_1,k_2\})=1$ then there is a unique type of pure-strategy equilibrium, where the marginal price is equal to c_2 for any level of demand. This is basically an extension of proposition 3.8 to the case where firms have different marginal costs. In this case only the most efficient generator produces.

High demand periods

If $\Pr(d>\max\{k_1,k_2\})=1$ then there is a unique type of pure-strategy equilibrium, where one firm bids the maximum permissible price and the other sufficiently low prices[6]. This corresponds to proposition 3.9 and is the case where both firms know that they will face some residual demand whatever they bid. An important additional feature when firms have different marginal costs is that there are inefficient equilibria, as the low-cost firm can be the high pricing generator.

---

[6] In fact, these equilibria hold when $\Pr(d>\min\{k_1,k_2\})=1$. If $\min\{k_1,k_2\}<\underline{d}<\max\{k_1,k_2\}$ then the firm with the larger capacity must be the high price firm.

Variable demand periods

The intermediate cases that are not covered by the low demand period and the high demand period are where $\underline{d}<\min\{k_1,k_2\}<\bar{d}$. However, von der Fehr and Harbord do not cover all these cases. They only consider the cases where $\bar{d}-\underline{d}>\max\{k_1,k_2\}$ and show that there is no pure-strategy equilibrium when this condition holds. This follows from the proposition that shows that in any pure-strategy equilibrium only one firm can set the marginal price with positive probability. As the support of the demand distribution is greater than the capacity of either firm it is not possible for only one firm to set the marginal price with a positive probability. They go on to characterise a mixed-strategy equilibrium for the case where each firm has one unit. They also assume demand is discrete[7]. They show that there is a potentially significant probability that the high-cost generator submits the lower price. Hence, as in the high demand period case the equilibrium is not efficient. They also find that, in the oligopoly case, the expected pool price is lower in a more fragmented industry.

Both discrete and continuous frameworks therefore lead to the conclusion that splitting the thermal generators of the CEGB between more than two companies would have led to lower pool prices. Both papers also show that the present rules lead to equilibria that are not efficient. To eradicate inefficient dispatching, von der Fehr and Harbord suggest the use of the Vickrey auction but make a slight mistake in extending it to the case of multiple units. The correct

[7] In section 3.3.2, we show that this equilibrium does not extend to the case where firms have multiple units.

version is given in the appendix and compared with the uniform and discriminatory pricing rules in the model presented in section 4.3.

4.2.3 Wolak and Patrick.

The previous two papers assume that the price paid to the generators for units that are dispatched is the bid price of the marginal unit. In the England and Wales Pool an additional element known as the capacity charge is also paid. The formula for the capacity charge is given by CC= LOLP(VOLL-SMP) and is explained in section 4.1. LOLP is the loss of load probability, the probability that there will be insufficient capacity to meet demand. This probability is inversely related to the reserve margin which is the capacity made available net of the forecasted demand. Wolak and Patrick show that the two large firms can manipulate this probability by withholding capacity and present evidence that this strategy has been used.

They argue that the following features allow the two firms to exercise market power. The two firms know the maximum amount of capacity that can be made available by all the other generators and therefore know the periods when they are likely to face a large residual demand. They know the forecasted demand that will be used to determine SMP and CC, and the only uncertainty they face is the capacity made available by the other firms as the forecasted demand is independent of price. Finally, the nature of the technology gives rise to a step marginal cost function that rises rapidly towards the end.

The market structure gives the two firms two strategic weapons that they can use to exercise market power, the price bids and the quantity choice (the capacity they make available to the Pool). The price bids determine SMP while the availability declarations are important in determining CC. Wolak and Patrick argue that the use of the first strategy is easier to regulate. It would be difficult for the firms to justify bids above £100 per MWH for anything but peak-load units and bids of this nature would eventually lead to intervention as it would be obvious that the firms were exercising market power. A strategy that is more difficult to regulate is bidding at close to marginal cost and withholding base-load capacity. In high-demand periods this results in a high SMP as the higher cost plants are marginal and a high CC as the reserve margin is low. Given the large revenues obtained, it is only necessary to do this in a relatively small number of periods.

Evidence that this strategy has been used comes from data on half-hourly pool prices from 1 April 1991 to 31 March 1995. During the sample period they find that CC is extremely volatile. Most of the revenue from CC is acquired in a relatively small number of periods when the reserve-margin is very small and CC is very large. During these periods the ratio CC/SMP which is normally small is also very large. This evidence is consistent with their hypothesis that the firms are withholding capacity for a relatively small number of periods to get extremely high pool prices. They estimate the upper bound of the marginal cost of National Power and PowerGen by taking the minimum bid for each generating unit. They show that the actual bids indicate that a large amount of base-load capacity is not made available in the summer. Some of this can be explained by scheduled maintenance but there is also a strategic incentive to obtain a significant payment for CC in the summer when demand is lower. It is difficult for the regulator to check whether the maintenance was necessary and as unscheduled maintenance is a random event it is difficult to prove that capacity has been withheld for strategic reasons.

The profitability of the capacity withholding strategy can be dramatically reduced by simply removing CC from the pool price calculation. CC is designed to signal optimally to the generators on investment decisions, but given that the firms have some control over the reserve margin and therefore over CC, this has not worked out in practice. However, removing CC would not eradicate the incentive to withhold capacity as withholding base-load capacity makes it more likely that SMP will be set by units with a higher marginal cost. As it was argued above, the maximum price bids are a function of the marginal cost, since the firms would find it difficult to justify bids that are significantly above marginal cost. A strategy that is more difficult to regulate is that of submitting bids at close to marginal cost and withholding base-load units to ensure the marginal price is set by a high-cost unit. We make this argument more formal in the next section. We present a model where the firms choose prices and quantity optimally given the regulatory constraint on bids. We show that under a discriminatory pricing rule there is no incentive to withhold capacity.

4.3 The Case for a Discriminatory Auction in the England and Wales Electricity Pool.

In this section, we look at how a change in the pricing rule would affect the extent of market power of PowerGen and National Power. We model a duopoly competing to supply to the Pool.

We present a model where the maximum price the firms can submit for units is a mark-up on marginal cost. The bid price of high-cost, peak-load units can therefore be much higher. Under a uniform pricing rule this gives the firms an incentive to withhold low-cost base-load units. We show that these results extend to the case where the firms have entered into financial contracts with the distributors.

The equilibria under a discriminatory pricing rule do not involve withholding capacity as there is nothing to be gained by having a high marginal price, and the equilibrium average pool price is therefore much lower. We also look at the multiple unit extension of the Vickrey auction. Under a Vickrey pricing rule it is a weakly dominant strategy to bid all units at marginal cost. This rule has been advocated on the basis of efficiency. However, we show that it results in a higher cost to the buyer than the discriminatory pricing rule.

Finally, we look at the repeated game. In practice the game is played every day and a repeated game analysis is therefore very important. A major problem with studying the repeated game is that the set of equilibria for each auction format is huge. However, we are able to show that the highest cost monopoly outcome under a discriminatory auction results in a cost to the buyer that is less than the cost in the stage-game equilibrium of the uniform-price auction. Also under a Vickrey auction there are equilibria in the repeated game where the firms withhold capacity.

4.3.1 A model of the England and Wales Pool.

Two large firms compete to supply the residual demand to the Pool[8]. Each firm owns a number of generating plants. The firms submit bids for the minimum price at which they are willing to supply electricity from each of their plants. In addition they also submit bids for the generating capacity they wish to make available to the Pool. This may involve taking out an entire plant or only making part of a plant available. The bids are used to construct the merit order[9]. The marginal price is the bid price of the marginal unit that is required to meet the realised level of demand[10]. Each generator dispatches the units that were bid below SMP plus some or all that were bid at SMP[11]. Demand is distributed according to P(d), $d \in (\underline{d}, \overline{d})$, and this is common knowledge.

The regulator knows the marginal cost of each generating unit and will observe that the firms are exercising market power if these bids are significantly above marginal cost. We therefore assume that price bids cannot be more than m units above marginal cost. There are two types of generating plant, low-cost base-load and high-cost peak-load, which have marginal cost c_b and c_p respectively.

---

[8] This assumption is justified on page 118.
[9] The step aggregate supply curve.
[10] In the England and Wales Electricity Pool forecasted demand is used to set System Marginal Price. There are two reasons for modelling uncertain demand. First, the two large firms compete to supply the residual demand which is uncertain due to the variable amount that will be made available by the small firms. Second, if the firms can only submit one supply function for the entire day then maximising with uncertain demand is equivalent to choosing a supply function that maximises profits over the day.
[11] In the event that two or more units have been bid at the marginal price, these units are equally likely to be dispatched.

Each firm owns k units of base-load plants. We assume that only firm 2 owns peak-load plants[12] and that $c_p>m+c_b$. Hence the marginal cost of the peak-load units is greater than the maximum permissible base-load price. If insufficient base-load capacity is made available then the marginal price is set by high-cost, peak-load units. We begin by characterising the price equilibria given the capacity choices. Let (y_1,y_2) be the base-load capacity made available by the two firms. Then we have the following equilibria in prices.

Equilibria in prices.

In equilibrium, firm 2 will never set a price below c_p for peak-load units as they will make a loss in the event these units are dispatched. Also, since $c_p>m+c_b$, firm 2 cannot undercut a base-load unit with a peak-load unit. The strategic use of peak-load units comes from the fact that the marginal price is high when it is set by peak-load units. Hence firm 2 will set the maximum price c_p+m for all peak-load units. We now consider equilibria in base-load prices.

Proposition 4.1: *If $Pr(d<min\{y_1,y_2\})=1$ then there is a unique type of pure-strategy equilibrium where the marginal price is always c_b.*

This follows from proposition 3.8. When firms have different capacities, it is necessary for demand to be less than the capacity of each firm.

[12] We make this assumption to avoid complications from mixed strategy peak-load prices The focus of this model is on capacity choices and competition to dispatch base-load units.

Proposition 4.2: *If $y_1+y_2>\underline{d}>min\{y_1,y_2)$ then there is a unique type of pure-strategy equilibrium, where one firm sets the maximum permissible price for a quantity greater than $y_1+y_2-\underline{d}$ and the other firm sets a price sufficiently low so that this firm cannot gain by lowering the price of these units.*

This follows from proposition 3.9. If $\min\{y_1,y_2\}<\underline{d}<\max\{y_1,y_2\}$ then the firm with the larger capacity must be the high price firm.

Quantity choices.

For a given demand distribution it is clear why a capacity withholding strategy would be profitable. Restricting the amount of capacity made available will increase the chance that marginal price is set at c_p+m. This is illustrated in the figure 4.3. However, the desire to withhold capacity must be balanced against the profitability of making a surplus on units not made available. The equilibrium will involve a balance between these opposing forces. We assume that the capacity choices must lie in the interval $[\underline{q},\overline{q}]$. The upper limit simply reflects the total capacity of each firm. The lower limit is a regulatory constraint which imposes a limit on the amount of capacity the firms can make unavailable without intervention from the regulator.

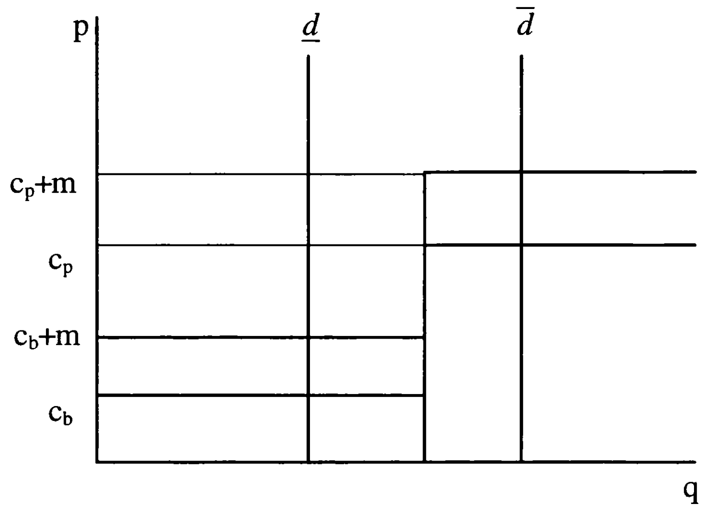


Figure 4.3
Quantity choices

Equilibria in prices and quantity.

To investigate equilibria in prices and quantity we analyse the model with a uniform demand distribution. From proposition 4.2 when $\min(y_1,y_2)<\underline{d}$ the equilibrium in prices will involve one firm setting a low price and the other setting the maximum c_b+m. In this type of equilibrium let firm 1 be the low price firm submitting all units at marginal cost and firm 2 the high price firm submitting all units at c_b+m. The condition for this equilibrium then becomes $y_1<\underline{d}$. For every equilibrium we characterise where firm 1 is the low price firm there is a similar[13] equilibrium where firm 2 is the low price firm. We begin by characterising the equilibria when there are no binding quantity constraints. Let $\overline{d}-\underline{d}=\Delta_d$ and $c_p-c_b=\Delta_c$.

[13] But not symmetric, as firm 2 owns peak-load units.

The profit functions conditional on $\underline{d} < y_1 + y_2$ are

$$\pi_1 = \frac{\bar{d} - y_1 - y_2}{\bar{d} - \underline{d}}(\Delta_c + m)y_1 + \frac{y_1 + y_2 - \underline{d}}{\bar{d} - \underline{d}} m y_1,$$

$$\pi_2 = \int_{y_1+y_2}^{\bar{d}} \frac{(d - y_1 - y_2)m}{\bar{d} - \underline{d}} dd + \frac{\bar{d} - y_1 - y_2}{\bar{d} - \underline{d}}(\Delta_c + m)y_2 + \int_{\underline{d}}^{y_1+y_2} \frac{(d - y_1)m}{\bar{d} - \underline{d}} dd \tag{4.6}$$

$$= \frac{m}{\bar{d} - \underline{d}}\left(\frac{\bar{d}^2}{2} - (y_1 + y_2)\bar{d} + \frac{(y_1 + y_2)^2}{2}\right)$$
$$+ \frac{\bar{d} - y_1 - y_2}{\bar{d} - \underline{d}}(\Delta_c + m)y_2 + \frac{m}{\bar{d} - \underline{d}}\left(\frac{(y_1 + y_2)^2}{2} - (y_1 + y_2)y_1 - \frac{\underline{d}}{2} + \underline{d}y_1\right).$$

Both profit functions are concave given $c_p > 0$. From the first order condition we get the following reaction functions

$$\overline{R_1}(y_2) = \frac{\bar{d}\Delta_c - \Delta_d m}{2\Delta_c} - \frac{1}{2}y_2,$$
$$\overline{R_2}(y_1) = \frac{\bar{d}}{2} - \frac{1}{2}y_1. \tag{4.7}$$

The solution is

$$y_1{*} = \frac{\bar{d}\Delta_c + 2\Delta_d m}{3\Delta_c},$$
$$y_2{*} = \frac{\bar{d}\Delta_c - \Delta_d m}{3\Delta_c}. \tag{4.8}$$

The profit functions conditional on $\underline{d} > y_1 + y_2$ are

$$\pi_1^b = (c_p + m)y_1,$$
$$\pi_2^b = (c_p + m)y_2 + (E(d) - y_1 - y_2)m. \tag{4.9}$$

These are strictly increasing in quantity. Hence, if $\underline{d} < y_1 + y_2$, firm i will want to set quantity to at least $\underline{d} - y_j$. It will set quantity to $\overline{R_i}(y_j)$ if $\overline{R_1}(y_2) > \underline{d} - y_j$. The reaction functions are therefore

$$R_1(y_2) = \max[\overline{R}_1(y_2), (\underline{d} - y_2)],$$
$$R_2(y_1) = \max[\overline{R}_2(y_1), (\underline{d} - y_1)]. \tag{4.10}$$

Proposition 4.3: *There is a set of equilibria where each firm bids at least $\overline{d}$ at marginal cost.*

Proof: If both firms make enough capacity available to cover demand then from proposition 4.1, the unique equilibrium in prices involves bidding at least $\overline{d}$ at marginal cost. Neither firm can gain by withholding capacity if the other firm has bid $\overline{d}$ at marginal cost. QED.

Proposition 4.4: *If $\underline{d}$ satisfies (4.11) then there is a continuum of equilibria where $y_1 + y_2 = \underline{d}$ irrespective of the base-load price bids.*

$$\underline{d} > \frac{\Delta_d (2\Delta_c + m)}{\Delta_c}. \tag{4.11}$$

Proof: Assume firm 2 is bidding at $m+c_b$. If $y_1^* + y_2^* > \underline{d}$ then the reaction functions (4.10) cross along a continuum of points where $y_1 + y_2 = \underline{d}$. Substituting for y_1^* and y_2^* from (4.8) and solving for $\underline{d}$ yields the inequality. If it is an equilibrium to set the total capacity at $\underline{d}$ when an increase in quantity yields a chance that the marginal price will be reduced to $m+c_b$, then it is also an equilibrium if the base-load price bids are less than $m+c_b$. QED.

This equilibrium is illustrated in figure 4.4. Hence, if the level of demand is high enough holding Δ_d fixed, then there is a capacity withholding equilibrium where the marginal price is always set by peak-load units. The critical value of $\underline{d}$ depends on the difference between the marginal cost of the peak-load and base-load units, Δ_c. The greater this difference the lower the critical value.

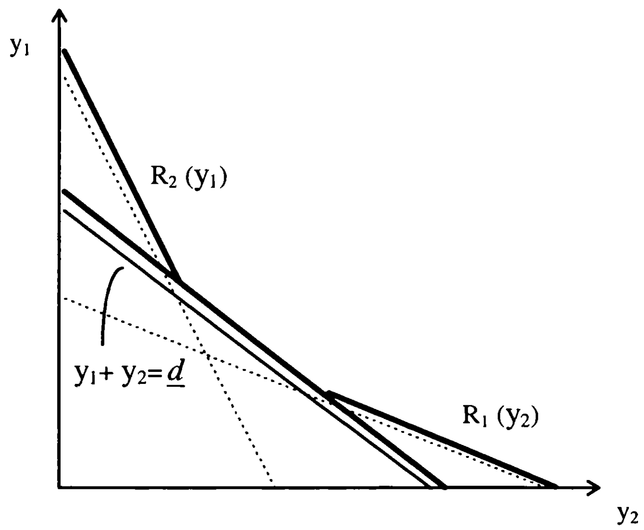


Figure 4.4.
Continuum of equilibria.

Proposition 4.5: *If $\underline{d}$ satisfies (4.12) then there is an equilibrium in prices and quantities where $y_1=y_1^*$, $y_2=y_2^*$ and firm one submits base-load units at marginal cost and firm two sets the maximum price, $m+c_b$.*

$$\frac{\Delta_d(\frac{1}{2}\Delta_c+m)}{\Delta_c}<\underline{d}<\frac{\Delta_d(2\Delta_c+m)}{\Delta_c}. \tag{4.12}$$

Proof: If $y_1^*+y_2^*>\underline{d}$, then the reaction functions are given by (4.7). They cross at (y_1^*,y_2^*). If $y_1^*<\underline{d}$ then the price equilibria given by proposition 4.2 hold (where firm 1 is the low price firm). Substituting for y_1* using (4.8) gives,

$$y_1{}^* < \underline{d} \Rightarrow \frac{\Delta_d(\frac{1}{2}\Delta_c + m)}{\Delta_c} < \underline{d}\text{. QED.}$$

This equilibrium is illustrated in figure 4.5. The expected pool price in this equilibrium is $\frac{\overline{d} - y_1{}^* - y_2{}^*}{\Delta_d}(c_p + m) + \frac{y_1{}^* + y_2{}^* - \underline{d}}{\Delta_d}(c_b + m)$. The total base-load capacity made available under this equilibrium, y_1*+y_2*, is decreasing in Δ_c. The expected pool price is therefore increasing in Δ_c.

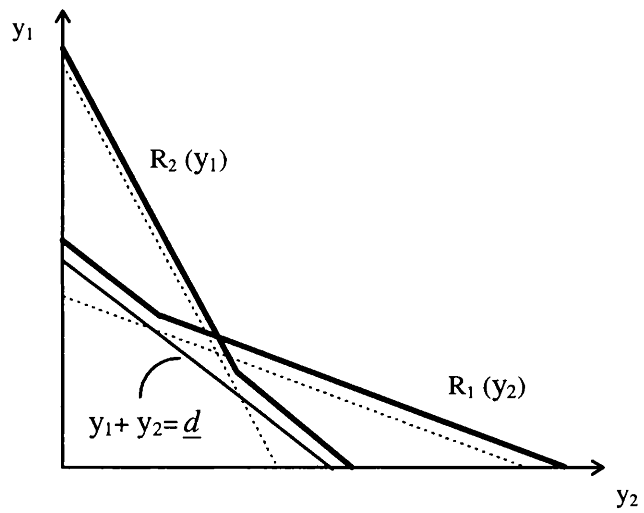


Figure 4.5
Unique equilibrium

If $\underline{d} < \frac{\Delta_d(c_p + m)^2}{\Delta_c(\Delta_c + c_p + m)}$ then the base-load price equilibrium breaks down as y_1*>$\underline{d}$. This is illustrated in figure 4.6. Although there is an incentive to withhold capacity, there is no pure-strategy equilibrium in base-load prices. In this case there may be mixed-strategy equilibria in base-load prices.

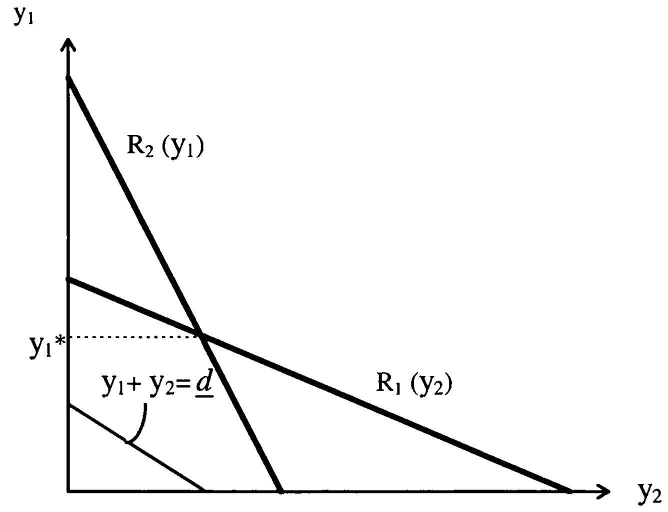


Figure 4.6.
No base-load price equilibrium.

Capacity constraints.

Now consider the model with the capacity constraints. Assume each firm has a total capacity of 1. Then the equilibria given in proposition 4.3 only hold when each firm can make enough capacity available to satisfy the highest possible level of demand. Hence the equilibria only exist when $\bar{d}<1$. Now consider the capacity withholding equilibria. If demand is so high that y_1*>1 or y_2*>1 then the constrained firm will set $y_i=\bar{q}$ (since the profit function is concave and therefore increasing in quantity when $y_i<y_i$*). The other firm will then set $\min(R_j^*(\bar{q}),\bar{q})$. This will result in a greater probability of marginal price being set by the peak-load units. If demand is so low that $y_1^*<\underline{q}$ or $y_1^*<\underline{q}$ then the constrained firm will set $y_i=\underline{q}$ and the other firm will set $\max((R_j^*(\underline{q}),\underline{q})$. This will result in a smaller probability of marginal price being set by the peak-load units.

The contract market.

Although suppliers must purchase all electricity through the Pool they have managed to effectively fix the price of some of their purchases through contracts for differences (CDFs). The contracts are primarily between the generators and distributors. An initial portfolio of contracts was drawn up by the government to ensure a smooth transition into the new mechanism. Most of these contracts have now expired and new contracts have been negotiated with no regulatory oversight. It has been argued that the price at which electricity is traded is determined outside the Pool through these financial contracts, and that the generators therefore have no incentive to manipulate the pool price. We extend our model to include contracts and show that this is not the case.

Most of the contracts have been between RECs and generators. A two-way CFD fixes a strike price at which the contracted quantity is traded. If the pool price is above the strike price then a payment equal to this difference times the contracted quantity is made by the generator to the REC. If the pool price is below the strike price then a payment equal to the difference times the contracted quantity is made by the REC to the generator. These contracts are purely financial contracts and are not physical contracts to deliver electricity at a stipulated price. They do, however, effectively fix the price the generators receive and the RECs pay for the contracted quantities. Another form of contract that insures the RECs against volatile pool prices is a one-way CFD, where a payment by the generator to the REC is made when the pool price is above the strike price.

A number of issues concerning contracts have been identified in the literature. There is a general consensus that the existence of contracts reduces the average pool price as generators effectively compete to supply a smaller residual demand. Incumbent generators may then wish to supply contracts to drive down the pool price (Newbery 1996). It is also possible that competition in the supply of contracts will lead to much lower pool prices (Green 1992). Von der Fehr and Harbord (1994) extend their framework to show that if contracts are held in large enough quantities they reduce the spot price to the contract strike price. They also identify a strategic incentive to sell large quantities of contracts to commit to a low-pricing strategy in the Pool. However, these papers do not model the demand side of the contract market, concentrating on the strategic incentives of the generators.

Powell (1993) presents a more complete model of the contract market. He assumes the distributors are risk-averse and are willing to pay a premium to insure against volatile pool prices. He looks at the polar cases where generators compete in the contract market and drive down the strike price to the pool price and where they act as monopolist in the contract market in which case they set a price above the expected pool price and the distributors only partially cover. The overall effect of the contract market is to increase the average cost to the buyer, as the distributors are paying a premium above the average pool price for the contracted quantity.

We extend the capacity withholding model to show that the firms still have an incentive to withhold capacity to obtain a high pool price on capacity that is not covered. We do not model the contract market but argue that the equilibria in the Pool are important in determining the contract prices and therefore the cost to the buyer.

Assume that the generators have entered into two-way difference contracts with the distributors. If the pool price is higher than the strike price, f, of the contract then a payment of (pool price - f) x_i is made by generator i to the distributors, where x_i is the contracted quantity. If the pool price is less than the strike price then a payment of (f - pool price) x_i is made by the distributors to the generator. The quantity covered by the each firm is less than the base-load generating capacity, $x_i<1$.

The profit function of generator i conditional on $\underline{d}<y_1+y_2$ is

$$\pi'_i = \pi_i + (f - E(p))x_{Gi},$$
$$E(p) = ((\overline{d} - y_1 - y_2)(c_p + m) + (y_1 + y_2 - \underline{d})(c_b + m)) / (\overline{d} - \underline{d}),$$

where π_i is given in (4.6).

The reaction functions are

$$\overline{R}'_1 = \overline{R}_1 + \frac{1}{2}x_1,$$
$$\overline{R}'_2 = \overline{R}_2 + \frac{1}{2}x_2.$$

where $\overline{R}_i$ is given in (4.7).

Hence the existence of contracts will shift each generator's reaction function out, which will result in each firm submitting a greater quantity. This will result in a lower expected pool price.

The profit functions conditional on $\underline{d}>y_1+y_2$ are now

$$\pi_1^b = (c_p + m - c_b)y_1 + (f - (c_p + m - c_b))x_1,$$

$$\pi_2^b = (c_p + m - c_b)y_2 + (E(d) - y_1 - y_2)m + (f - (c_p + m - c_b))x_2.$$

As before, these are strictly increasing in quantity. Hence, if $\underline{d} < y_1 + y_2$, firm i will want to set quantity to at least $\underline{d}$ - y_j. It will set quantity to $\overline{R_i}(y_j)$ if $\overline{R_1}(y_2) > \underline{d}$ - y_j. The reaction function of firm i is therefore

$$R'_i(y_j) = \max[\overline{R}'_i(y_j), (\underline{d} - y_j)]$$

If we repeat the comparative static analysis with contracts then we can simply replace the conditions (4.11) and (4.12) with (4.13) and (4.14), respectively, and apply propositions 4.3-4.5.

$$\underline{d} > \frac{\Delta_d(2\Delta_c + m)}{\Delta_c} + x_1 + x_2. \tag{4.13}$$

$$\frac{\Delta_d(\frac{1}{2}\Delta_c + m)}{\Delta_c} + x_1 - \frac{1}{2}x_2 < \underline{d} < \frac{\Delta_d(2\Delta_c + m)}{\Delta_c} + x_1 + x_2. \tag{4.14}$$

The critical level of demand where the marginal price is always c_p+m (figure 4.1) is now increased by the quantity of the contract coverage. Hence a higher level of demand is needed to be in this regime. The capacity withholding equilibria where there is a unique equilibrium in quantities (figure 4.2) now involves a lower expected pool price than before, as the reaction functions have shifted out. In all the capacity withholding equilibria, the quantity made available is greater than the contracted quantity. However, there is now a possibility that a capacity withholding equilibrium does not exist.

Proposition 4.6: *If* $\underline{d}$ *satisfies (4.15) then there is no incentive to withhold capacity.*

$$\underline{d} < \frac{\Delta_d(m-\Delta_c)}{\Delta_c} + x_1 + x_2 \tag{4.15}$$

Proof: If $y_1^*+y_2^*>\bar{d}$ then there is no gain to be made by ensuring the marginal price is set by peak-load units. Substituting for y_1^* and y_2^* using (4.8) yields the inequality. QED.

Hence when the level of demand is sufficiently low there is no incentive to withhold capacity. This does not occur without contracts as $c_p>m+c_b$ and $\underline{d}$ would have to be negative. The capacity withholding strategy will only be used in the higher demand periods.

This simple analysis bypasses the problem of modelling the bargaining process that takes place in negotiating the contracts and looks only at the resultant effect on the pool price. We show that there is still an incentive to withhold capacity, although the pool prices are lower. However, this does not mean the cost to the buyer is less. As we discussed earlier, one of the reasons the distributors enter into contracts is to insure against volatile pool prices for which they pay a premium. This would be reflected in a contract strike price that is above the average pool price without contracts. The overall cost to the buyer with a contract market would then be even greater than the cost given in the model without contracts. However one models the contract market, it is clear that the contract market and the Pool are interdependent. If rules can be introduced that make one of the markets competitive then the other market would fall in line. For example, if the pool mechanism guaranteed competitive prices then the contract strike price would mirror this. Since it would be more difficult to govern the contract market the most important question to address should be how one can change the pool rules in such a way as to make generation more competitive.

4.3.2 Alternative pricing rules.

In the previous section, we modelled the England and Wales Pool as a multi-unit, uniform-price auction with constraints on bids that are related to the marginal cost. We now consider the effects of changing the pricing rule. Two alternatives to the uniform pricing rule are the discriminatory rule, where the generators are simply paid the bid price for units they are assigned, and the Vickrey rule which is explained below.

Discriminatory pricing rule.

Under a discriminatory pricing rule there is no incentive to manipulate the marginal price as the generators are simply paid their bid prices for the units that are dispatched. Also, firm 2 does not gain by submitting peak-load units rather than base-load units, as the maximum profit they can make on any unit is m, irrespective of the marginal cost. Withholding capacity will only decrease the expected quantity dispatched. The model for base-load units is then formally equivalent to the common value model of section 3.3.1.

Price equilibria.

Peak-load units will never be bid below c_p as firm 2 would make a loss on these units. Therefore in equilibrium, peak-load units are only dispatched when the demand is greater than the total base-load capacity. As in the previous model, firm 2 will therefore set the maximum price, c_p+m for all peak-load units.

The base-load price equilibria correspond to those given in the simple model of section 4.3.1. If $\bar{d}<1$ then there is a pure-strategy equilibrium where firms submit base-load units at marginal cost. If $\bar{d}>1$ then there is no pure-strategy equilibrium in base-load prices. We characterise a mixed-strategy equilibrium where each firm submits one price for all units according to (3.9)[14]. We need to make one change to the notation in (3.9) to account for the case where $\bar{d}\geq 2$

$$d^+ = \frac{\Pr(1<d<2)(E(d/1<d<2)+2\Pr(d\geq 2)}{\Pr(d>1)}. \tag{4.16}$$

The expected cost of base-load units under the mixed-strategy equilibrium is, $2Pr(d>1)(d^+-1)(m+c_b)+(Pr(d>1)(2-d^+)+Pr(d<1)E(d/d<1))c_b$ (from 3.14). The cost of any peak-load units used will always be m+c_p. Hence the overall cost of the mixed-strategy equilibrium is

$$2Pr(d>1)(d^+-1)(m+c_b)+(Pr(d>1)(2-d^+)+Pr(d<1)E(d/d<1))c_b$$

$$+Pr(d>2)(E(d/d>2)-2)(m+c_p).$$

[14] If $\underline{d}<1$ then this equilibrium holds as long as the demand distribution is not heavily skewed to the left. See section 3.3.1 for details.

Vickrey auction.

Under a Vickrey auction the pricing rule is as follows. For the lowest priced unit dispatched, each firm is paid the bid price of the unit that would be required to meet demand if the capacity of this firm were not available. Let the capacity of the first unit be k. For the second unit dispatched the firm is paid the price of the marginal unit that would be required to meet (demand-k), if the capacity of this firm were not available and so on[15]. Hence the price paid to a firm is independent of that firm's bids.

Proposition 4.7: *Under the Vickrey pricing rule it is a weakly dominant strategy for each firm to bid at marginal cost.*

This is shown in the appendix. If the generators use this strategy then the following results hold.

If Pr(d<1)=1 then each firm is paid at marginal cost for all units dispatched. This is because both firms can satisfy demand with their capacity and if they bid all units at marginal cost then the other firm will be paid at marginal cost. If $1<\bar{d}<2$, then in equilibrium each firm will be paid c_p for the first d-1 units dispatched and at marginal cost for the rest. Hence 2(d-1) units will be paid at c_p and the other (2-d) at c_b. When $\bar{d}>2$, all units dispatched are paid at c_p. The overall expected cost is therefore

$Pr(d<1)E(d/d<1)c_b$

[15] In the event that there is insufficient capacity to meet demand when the capacity of a firm is taken out we assume that the firm will be paid at the marginal cost of the peak-load units.

$+Pr(1<d<2)(2(E(d/1<d<2)-1)c_p+(2-E(d/1<d<2))c_b$

$+Pr(d>2)(2c_p+(E(d/d>2)-2))c_p).$

This can be rewritten as

$2Pr(d>1)(d^+-1)c_p+(Pr(d>1)(2-d^+)+Pr(d<1)E(d/d<1))c_b$

$+Pr(d>2)(E(d/d>2)-2)c_p.$

where d^+ is given by (4.16).

4.3.3 Ranking the pricing rules in the England and Wales Pool.

A summary of the equilibria and the overall expected cost to the Pool under the three auction formats is given in tables 4.1 and 4.2. Table 4.1 summarises the case where $\bar{d}<1$. There is a competitive equilibrium under all three auction formats where the buyers pay at marginal cost for all electricity purchased. However, under the uniform-price auction there are also capacity withholding equilibria.

Now consider the case where $\bar{d}>1$. It is simple to see that the expected cost in the capacity withholding equilibrium is always greater than under the mixed-strategy equilibrium of the discriminatory auction. All peak-load units are sold at $m+c_p$ in both cases. However, under the capacity withholding equilibrium, all of the base-load units are sold at a cost of at least $m+c_b$, the maximum permissible base-load price, and some are sold at $m+c_p$, the maximum permissible peak-load price, whereas under a discriminatory auction, all the base-load units are sold at a price less than or equal to $m+c_b$ [16].

If Pr(d>2)=0 (when the probability that there is enough base-load capacity

[16] The price will be equal to $m+c_b$ if $\underline{d}>2$.

to meet demand is one), the Vickrey auction results in a greater cost than the mixed-strategy equilibrium of the discriminatory auction as $c_p > m$. If $\Pr(d>2)>0$ then the ranking is ambiguous. The peak-load units always go for (c_p+m) under the discriminatory equilibrium because we have assumed that only one firm has peak-load capacity. If both firms had peak-load capacity, the equilibrium peak-load price would be lower. Also, we have assumed that under a Vickrey auction, in the event that there is insufficient capacity to meet demand when the capacity of a firm is taken out, the firm will only be paid at the marginal cost of the peak-load units, c_p. Despite these assumptions, the overall cost will be less under the discriminatory auction if the peak-load capacity used is small relative to base-load capacity and/or the marginal cost of the peak-load units is large relative to the marginal cost of base-load units. The problem with the Vickrey auction is that the payment rule allows for peak-load prices to be paid to some units when no peak-load capacity is required which results in a high average cost.

Pricing Rule	Equilibrium	Cost
Uniform	Competitive Capacity withholding	$E(d)c_b$ $Pr(d>y_1^*+y_2^*)E(d/d>y_1^*+y_2^*)(c_p+m)$ $+Pr(d<y_1^*+y_2^*)E(d/d<y_1^*+y_2^*)(m+c_b)$ or $E(d)p^p$[17]
Discriminatory	Competitive	$E(d)c_b$
Vickrey	Competitive	$E(d)c_b$

TABLE 4.1
Cost of auctions when $\bar{d}<1$.

Pricing Rule	Equilibrium	Cost
Uniform	Capacity withholding	$Pr(d>y_1^*+y_2^*)E(d/d>y_1^*+y_2^*)(c_p+m)$ $+Pr(d<y_1^*+y_2^*)E(d/d<y_1^*+y_2^*)(m+c_b)$ or $E(d)\ (c_p+m)$
Discriminatory	Mixed strategy	$(Pr(d>1)(2-d^+)+Pr(d<1)E(d/d<1))c_b$ $+2Pr(d/d>1)(d^+-1)(m+c_b)$ $+Pr(d>2)(E(d/d>2)-2)(m+c_p)$
Vickrey	Competitive	$(Pr(d>1)(2-d^+)+Pr(d<1)E(d/d<1))c_b$ $+2Pr(d/d>1)(d^+-1)c_p$ $+Pr(d>2)(E(d/d>2)-2)c_p$

TABLE 4.2
Cost of auctions when $\bar{d}>1$.

4.3.4 Repeated game analysis.

The above analysis looks at the stage game or one-shot equilibria under the various pricing rules. In reality the game is played repeatedly every day. This gives the generators an opportunity to collude on repeated game equilibria that result in high levels of profits. Armstrong, Cowans and Vickers (1994) argue that

[17] In the case where there is a continuum of equilibria (proposition 4.4)

the market structure is particularly conducive to tacit collusion as all information on capacity and price bids is publicly available and the players are matched frequently to play the game.

Under a discriminatory pricing rule, the best outcome the firms can coordinate on is the monopoly outcome, where all units are bid at the maximum permissible price. This would be a strategy that is easy for the regulator to detect. It would also be relatively easy to prove that the generators are colluding and to take action. Even if the generators were able to get away with colluding in this fashion the expected cost to the buyer is still less than under the capacity withholding equilibria of the one shot game under a uniform pricing rule (table 4.2). One would expect that in a repeated game setting the firms would make even greater profits.

Under a Vickrey pricing rule it is a weakly dominant strategy to bid all units at marginal cost. Withholding capacity does not increase profits since the price that a firm is paid is independent of his bids. However, withholding capacity and/or increasing bids does increase the profits of the other firm. In a repeated game setting, the firms can collude on a weakly dominated equilibrium. This would be a self-enforcing equilibrium, as neither firm would gain by deviating in the one-shot game, but would be punished in future periods. Since withholding capacity is more difficult to monitor than simply looking at price bids, the firms can collude on a strategy where they take out more capacity for maintenance than they need to. Hence the case for using a discriminatory pricing rule is even stronger in the repeated game case.

4.4 Conclusions.

The England and Wales Electricity Pool was set up to introduce competition in the generation part of the electricity supply industry. However, competitive pool prices have not been realised and attention has turned to ways in which the market can be re-structured to increase competition. In section 4.2, we look at two theoretical models of the Pool which show that the notion that a duopoly would lead to Bertrand type equilibria was ill-founded. There is also some evidence that the current structure has resulted in generators withholding capacity to obtain high pool prices.

We present a model of the Pool that captures the capacity withholding incentive. We show that under a uniform pricing rule the firms maximise profits by withholding base-load capacity to increase the probability that the marginal price is set by peak-load units, which can be bid at much higher prices. Such an incentive does not exist under a discriminatory pricing rule as the price paid for each unit is simply the bid price.

The case for a discriminatory auction is even stronger in the repeated game for two reasons: 1) collusive behaviour would be easy to detect as it would involve bidding high prices for all units and not just manipulating the marginal price; 2) even if the firms could collude they are limited in the profits they can make. In fact, in our model, we show that the monopoly outcome under a discriminatory pricing rule results in a lower cost than the stage-game capacity-withholding equilibrium of the uniform-price auction.

A third alternative that has been suggested in the literature is the Vickrey pricing rule. This is advocated on the grounds of efficiency as it is a weakly dominant strategy to bid all units at marginal cost. However, it can result in a high cost, as peak-load prices are paid to some units when no peak-load capacity is required. Also, in a repeated game setting, the firms can collude on weakly dominated equilibria of the stage-game which substantially increase profits.

We therefore put a strong case for the use of a discriminatory pricing rule in the England and Wales Pool, on the grounds that it minimises the market power of the generators and results in more competitive pool prices. One question we do not address, however, is the relative efficiency of the auction formats and this is clearly an area for future research.

4.5 Appendix

Vickrey pricing rule.

Under a Vickrey auction the pricing rule is as follows. For the lowest priced unit dispatched the firm is paid the price of the unit that would be required to meet demand if the capacity of this firm were not available. Let the capacity of this unit be k. For the second unit dispatched the firm is paid the price of the marginal unit that would be required to meet (demand-k) if the capacity of this firm were not available and so on. This is illustrated in figure 4.7 for the case where there are two firms.

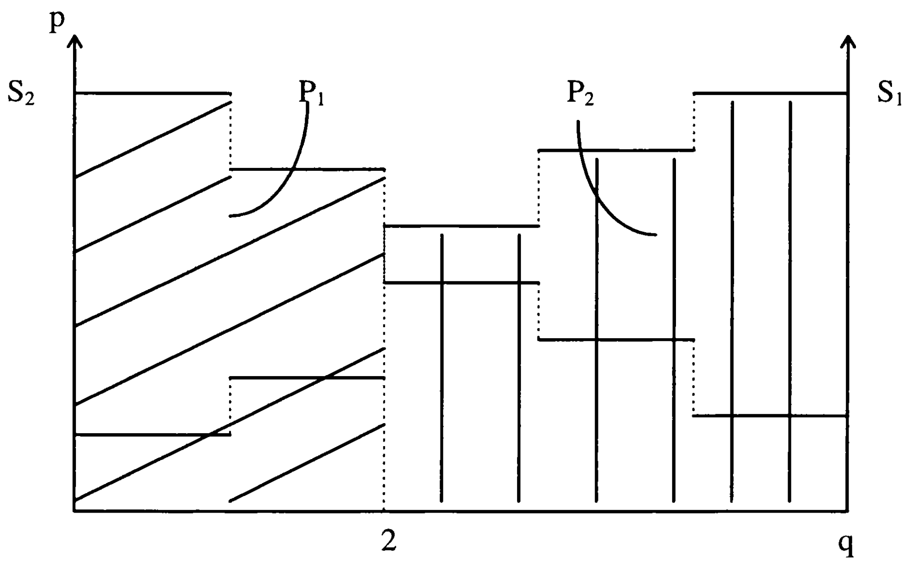


Figure 4.7
Vickrey payment rule

The figure plots firm 1's bids (S_1) for 5 units in the conventional manner and firm 2's bids (S_2) in reverse. In equilibrium firm 1 gets two units and firm 2 gets 3 units. The payment firm 1 receives for the first unit is the price of the unit that would be required to meet demand if the capacity of this firm 1 were not available. As there are only two firms this is the bid of firm 2's fifth unit. The overall payment for the two units is therefore the area under the last two bids of firm 2, P_1. Similarly, the payment to firm 2 for the 3 units is the area under firm 1's last three bids, P_2. We now show that under this pricing rule it is a weakly dominant strategy for each firm to bid at marginal cost.

Proof of Proposition 4.7: The firms can do nothing to affect the price they are paid for units dispatched through their bids as this depends on the residual demand curve. They can only use their bids to determine the units that are dispatched. Assume firm 1 has bid a unit at b, below its marginal cost, c. There is a possibility that this unit will be the marginal unit dispatched and that it will be paid a price below marginal cost. This is illustrated in figure 4.8. The firm will then make a loss on this unit, s. The firm can then increase profits by increasing this bid to be above the next highest bid so that this unit is not dispatched. Bidding below marginal cost is therefore weakly dominated by bidding at marginal cost. Now assume firm 1 has bid a unit at b, above marginal cost, c. Then there is a possibility this unit is the lowest rejected bid and the marginal price is above its marginal cost. This is illustrated in figure 4.9. The firm can make a profit, p, on this unit by reducing this bid to just below the next lowest bid. Bidding below marginal cost is therefore weakly dominated by bidding at marginal cost. QED.

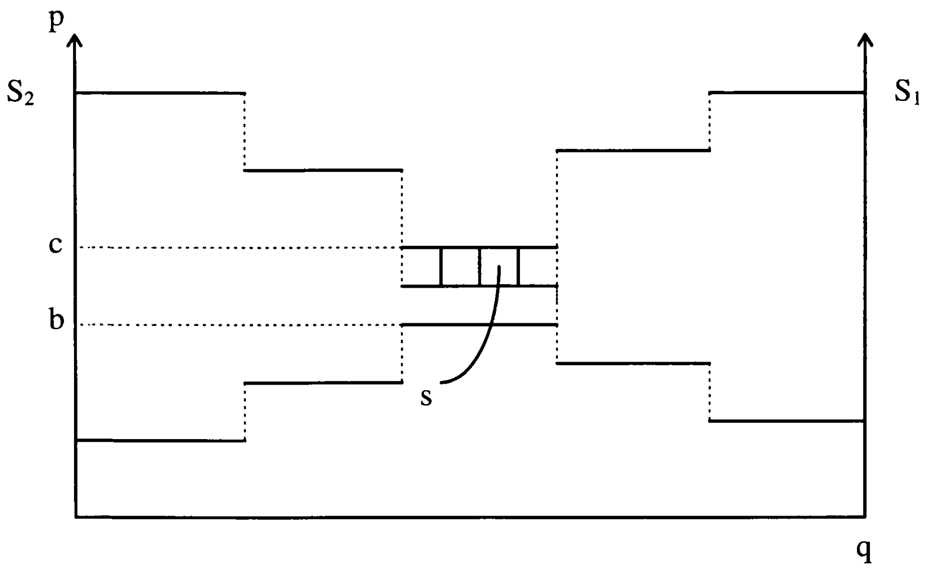


Figure 4.8
Bidding below marginal cost is weakly
dominated by bidding at marginal cost.

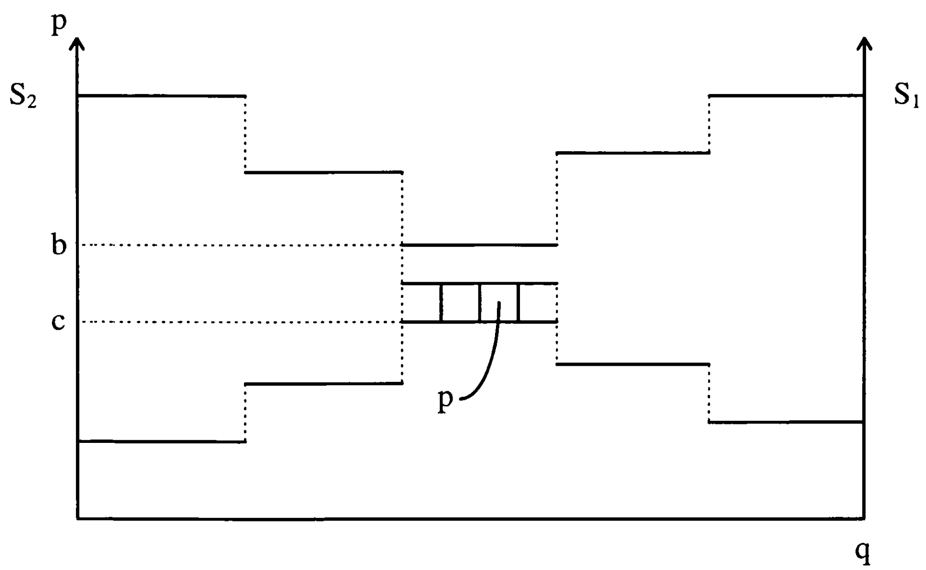


Figure 4.9
Bidding above marginal cost is weakly
dominated by bidding at marginal cost.

References

Armstrong, M., S. Cowans and Vickers. 1994. *Regulatory Reform-Economic Analysis and British Experience.* Oxford University Press: Oxford.

Ausubel, L.M. and P.Crampton. 1995. Demand Reduction and Inefficiency in Multi-Unit Auctions. Working Paper 96-07. University of Maryland.

Ausubel, L.M. 1997. An Efficient Ascending-Bid Auction for Multiple Objects. Working Paper 96-06. University of Maryland.

Back, K. and J. Zender. 1993. Auctions of Divisible Goods: On the rationale for the Treasury Experiments. *Review of Financial Studies* 6, 733-764

Bergmin, J. and B. Lipman. 1996. Evolution with State-Dependent Mutations. *Econometrica* 64, 943-956.

Binmore, K. 1992. Fun and Games. D.C.Heath, Lexington, MA.

Binmore, K., J. Gale and L. Samuelson. 1995. Learning to be Imperfect: The Ultimatum Game. *Games and Economic Behaviour* 8, 56-90.

Binmore, K. and J. Swierzbinski. 1997. Uniform or Discriminatory? ELSE advisory paper on Treasury Auctions. Department of Economics, University College London.

Blume, L. 1993. The Statistical Mechanics of Strategic Interaction. *Games and Economic Behaviour* 4, 387-424.

Ellison, G. 1993. Learning, Local Interaction, and Coordination," *Econometrica* 61, 1047-1071.

Ely, J. 1995. Local Conventions. Working Paper, Economics Department, University of California at Berkeley.

Fehr, N-H von der, and D. Harbord. 1992. Spot Market Competition in the UK Electricity Industry. University of Oslo, Department of Economics. Memorandum 9.

Fehr, N-H von der, and D. Harbord. 1993. Spot Market Competition in the UK Electricity Industry. *Economic Journal* 103, 531-546.

Fehr, N-H. von der, and D. Harbord. 1994. Long-Term Contracts and Imperfectly Competitive Spot Markets: A study of the UK Electricity Industry. University of Oslo, Department of Economics, Memorandum 14.

Foster, D. and H.P. Young 1990. Stochastic Evolutionary Game Dynamics. *Journal of Theoretical Population Biology* 38, 219-230.

Friedlin, M.I. and A.D. Wentzell. 1984. *Random Perturbations of Dynamical Systems.* New York: Springer Verlag.

Green, R.J. 1992. Contracts and the Pool: The British Electricity Market. Department of Applied Economics, Cambridge.

Green, R.J. and D.M. Newbery. 1992. Competition in the British Electricity Spot Market. *Journal of Political Economy* 100, 929-953.

Harris, M. and A. Raviv. 1981. Allocation Mechanisms and the Design of Auctions. *Econometrica* 49, 1477-1499.

Harsanyi, J.C. and R.Selten. 1988. *A General Theory of Equilibrium Selection in Games*. MIT Press, Cambridge, MA.

Holt, C.A. 1980. Competitive Bidding for Contracts Under Alternative Auction Procedures. *Journal of Political Economy* 88, 433-445.

Kandori, M., G. Mailath and R. Rob. 1993. Learning, Mutation, and Long Run Equilibria in Games," *Econometrica* 61, 29-56.

Kandori, M., and R. Rob. 1995. Evolution of Equilibria in the Long Run: A General Theory and Applications." *Journal of Economic Theory*, 65, 383-414.

Klemperer, P.D. and M.A. Meyer. 1989. Supply Function Equilibria in Oligopoly Under Uncertainty. *Econometrica* 57,1243-1277.

Kreps, D.M. and R. Wilson. 1982. Sequential Equilibria. *Econometrica* 50, 863-894.

Ledyard, J.O. 1995. Public Goods: A Survey of Experimental Research. In J. Kagel and A.E. Roth,eds., *Handbook of Experimental Economics*, pp.111-194. Princeton University Press, Princeton, NJ.

Mailath, G., L. Samuelson, and A. Shaked. 1994. Evolution and Endogenous Interactions. Economics Working Paper Archive, Washington University at St. Louis.

Maskin, E.S. and J.G. Riley. 1984. Optimal Auctions with Risk-Averse Buyers. *Econometrica* 52, 1473-1518.

Maskin, E.S. and J.G. Riley. 1989. Optimal Multi-Unit Auctions. In F. Hahn (ed), *The Economics of Missing Markets, Information, and Games*. Oxford University Press, 312-335.

Maynard Smith, J. and G.R. Price. 1973. The Logic of Animal Conflict. *Nature* 246, 15-18.

McAfee, R.P. and J. MaMillan. 1987. Auctions and Bidding. *Journal of Economic Literature* 25, 699-738.

Milgrom, P.R. and R.J. Weber. 1982. A Theory of Auctions and Competitive Bidding. *Econometrica* 50, 1089-1122.

Myerson, R.B. 1978. Refinements of the Nash Equilibrium Concept. *International Journal of Game Theory* 7, 73-80.

Myerson, R.B. 1981. Optimal Auction Design. *Mathematics of Operational Research* 6, 58-72.

Nash, J. 1950. Equilibrium Points in n-person Games. *Proceedings of the National Academy of Sciences* 36, 48-49.

Newbery, D.M. 1996. Competition, Contracts and Entry in the Electricity Spot Market. Department of Applied Economics, Cambridge.

Okada, A. 1981. On the Stability of Perfect Equilibrium Points. *International Journal of Game Theory* 7, 67-73.

Powell, A. 1993. Trading Forward in an Imperfect Market: The case of Electricity in Britain. *Economic Journal* 103, 444-453.

Tenorio, R. 1993. Revenue Equivalence and Bidding Behaviour in a Multi-Unit Auction Market: An Empirical Analysis. Review of Economics and Statistics 75, 302-314.

Schelling, T. 1960. *The Strategy of Conflict.* Harved University Press.

Selten, R. 1965. Spieltheoretische Behandlung eines Oligopolmodells mit Nachfragetragheit. *Zeitscrift für die gesgamte Staatswissenschaft* 12, 301-324.

Selten, R. 1975. Re-examination of the Perfectness Concept for Equilibrium Points in Extensive Games. *International Journal of Game Theory* 4, 25-55.

Simon, D.P. 1994. The Treasury's Experiment with Single-Price Auctions in the Mid-1970s: Winner's Curse or Taxpayer's curse? *Review of Economics and Statistics* 76, 754-760.

Taylor, P.D. and L.B. Jonker. 1978. Evolutionary Stable Strategies and Game Dynamics. *Mathematical Biosciences* 40, 145-156.

Umlauf, S. 1993. An Empirical Study of the Mexican Treasury Bill Auction. *Journal of Financial Economics* 33, 313-340.

Van Damme, E.E.C. 1991. *Stability and Perfection of Nash Equilibria.* Springer, Berlin, 1991.

Vickrey, W. 1961. Counterspeculation, Auctions, and Competitive Sealed Tenders. *Journal of Finance* 16, 8-37.

Wang, J.J.D. and J.F. Zender. 1995. Auctioning Divisible Goods. Working Paper, University of Utah.

Weibull, J.1995. *Evolutionary Game Theory.* MIT Press, Cambridge, MA.

Wilson, R. 1979. Auctions of Shares. *Quarterly Journal of Economics* 93, 675-698.

Wolak, F. and R. Patrick. 1996. The Impact of Market Rules and Market Structure on the Price Determination Process in the England and Wales Electricity Market. *CEPR Discussion Paper* 463, Stanford University.

Wolfram, C.D. 1997. Strategic Bidding in a Multi-Unit Auction: An Empirical Analysis of Bids to Supply Electricity in England and Wales. Working Paper 6269. National Bureau of Economic Research.

Young, H.P. 1993. The Evolution of Conventions. *Econometrica* 61, 57-84.

Zeeman, E.C. 1981. Dynamics of Evolution of Animal Conflicts. *Journal of Theoretical Biology* 89, 249-270.