



Wavenumber-Explicit Regularity Estimates on the Acoustic Single- and Double-Layer Operators

Jeffrey Galkowski and Euan A. Spence 

Abstract. We prove new, sharp, wavenumber-explicit bounds on the norms of the Helmholtz single- and double-layer boundary-integral operators as mappings from $L^2(\partial\Omega) \rightarrow H^1(\partial\Omega)$ (where $\partial\Omega$ is the boundary of the obstacle). The new bounds are obtained using estimates on the restriction to the boundary of quasimodes of the Laplacian, building on recent work by the first author and collaborators. Our main motivation for considering these operators is that they appear in the standard second-kind boundary-integral formulations, posed in $L^2(\partial\Omega)$, of the exterior Dirichlet problem for the Helmholtz equation. Our new wavenumber-explicit $L^2(\partial\Omega) \rightarrow H^1(\partial\Omega)$ bounds can then be used in a wavenumber-explicit version of the classic compact-perturbation analysis of Galerkin discretisations of these second-kind equations; this is done in the companion paper (Galkowski, Müller, and Spence in Wavenumber-explicit analysis for the Helmholtz h-BEM: error estimates and iteration counts for the Dirichlet problem, 2017. [arXiv:1608.01035](https://arxiv.org/abs/1608.01035)).

Mathematics Subject Classification. 31B10, 31B25, 35J05, 35J25, 65R20.

Keywords. Helmholtz equation, Layer-potential operators, High frequency, Semiclassical, Boundary integral equation.

1. Introduction

1.1. Statement of the Main Results

Let $\Phi_k(x, y)$ be the fundamental solution of the Helmholtz equation ($\Delta u + k^2 u = 0$) given by

$$\Phi_k(x, y) := \frac{i}{4} H_0^{(1)}(k|x-y|), \quad d = 2, \quad \Phi_k(x, y) := \frac{e^{ik|x-y|}}{4\pi|x-y|}, \quad d = 3, \quad (1.1)$$

where d is the spatial dimension. Let Ω be a bounded Lipschitz open set such that the open complement $\Omega_+ := \mathbb{R}^d \setminus \bar{\Omega}$ is connected (so that the scattering problem with obstacle Ω is well-defined). Recall that, for almost every $x \in \partial\Omega$,

there exists a unique outward-pointing unit normal vector, which we denote by $n(x)$. For $\phi \in L^2(\partial\Omega)$ and $x \in \partial\Omega$, the single- and double-layer potential operators are defined by

$$S_k\phi(x) := \int_{\partial\Omega} \Phi_k(x, y)\phi(y) \, ds(y), \quad D_k\phi(x) := \int_{\partial\Omega} \frac{\partial\Phi_k(x, y)}{\partial n(y)}\phi(y) \, ds(y), \tag{1.2}$$

and the adjoint-double-layer operator is defined by

$$D'_k\phi(x) := \int_{\partial\Omega} \frac{\partial\Phi_k(x, y)}{\partial n(y)}\phi(y) \, ds(y) \tag{1.3}$$

(recall that D'_k is the adjoint of D_k with respect to the real-valued $L^2(\partial\Omega)$ inner product; see, e.g., [7, p. 120]).

Before stating our main results, we need to make the following definitions.

Definition 1.1. (*Smooth hypersurface*). We say that $\Gamma \subset \mathbb{R}^d$ is a *smooth hypersurface* if there exists $\tilde{\Gamma}$ a compact embedded smooth $d - 1$ dimensional submanifold of \mathbb{R}^d , possibly with boundary, such that Γ is an open subset of $\tilde{\Gamma}$, with Γ strictly away from $\partial\tilde{\Gamma}$, and the boundary of Γ can be written as a disjoint union

$$\partial\Gamma = \left(\bigcup_{\ell=1}^n Y_\ell \right) \cup \Sigma,$$

where each Y_ℓ is an open, relatively compact, smooth embedded manifold of dimension $d - 2$ in $\tilde{\Gamma}$, Γ lies locally on one side of Y_ℓ , and Σ is closed set with $d - 2$ measure 0 and $\Sigma \subset \overline{\bigcup_{\ell=1}^n Y_\ell}$. We then refer to the manifold $\tilde{\Gamma}$ as an extension of Γ .

For example, when $d = 3$, the interior of a 2-d polygon is a smooth hypersurface, with Y_i the edges and Σ the set of corner points.

Definition 1.2. (*Curved*). We say a smooth hypersurface is *curved* if there is a choice of normal so that the second fundamental form of the hypersurface is everywhere positive definite.

Recall that the principal curvatures are the eigenvalues of the matrix of the second fundamental form in an orthonormal basis of the tangent space, and thus “curved” is equivalent to the principal curvatures being everywhere strictly positive (or everywhere strictly negative, depending on the choice of the normal).

Definition 1.3. (*Piecewise smooth*). We say that a hypersurface Γ is *piecewise smooth* if $\Gamma = \bigcup_{i=1}^N \bar{\Gamma}_i$ where Γ_i are smooth hypersurfaces and $\Gamma_i \cap \Gamma_j = \emptyset$.

Definition 1.4. (*Piecewise curved*). We say that a piecewise smooth hypersurface Γ is *piecewise curved* if Γ is as in Definition 1.3 and each Γ_j is curved.

The main results of this paper are contained in the following theorem. We use the notation that $a \lesssim b$ if there exists a $C > 0$, independent of k , such that $a \leq Cb$.

Theorem 1.5. (Bounds on $\|S_k\|_{L^2(\partial\Omega)\rightarrow H^1(\partial\Omega)}$, $\|D_k\|_{L^2(\partial\Omega)\rightarrow H^1(\partial\Omega)}$, $\|D'_k\|_{L^2(\partial\Omega)\rightarrow H^1(\partial\Omega)}$).

Let Ω be a bounded Lipschitz open set such that the open complement $\Omega_+ := \mathbb{R}^d \setminus \bar{\Omega}$ is connected.

- (a) If $\partial\Omega$ is a piecewise smooth hypersurface (in the sense of Definition 1.3), then, given $k_0 > 1$,

$$\|S_k\|_{L^2(\partial\Omega)\rightarrow H^1(\partial\Omega)} \lesssim k^{1/2} \log k, \quad (1.4)$$

for all $k \geq k_0$. Moreover, if $\partial\Omega$ is piecewise curved (in the sense of Definition 1.4), then, given $k_0 > 1$, the following stronger estimate holds for all $k \geq k_0$

$$\|S_k\|_{L^2(\partial\Omega)\rightarrow H^1(\partial\Omega)} \lesssim k^{1/3} \log k. \quad (1.5)$$

- (b) If $\partial\Omega$ is a piecewise smooth, $C^{2,\alpha}$ hypersurface, for some $\alpha > 0$, then, given $k_0 > 1$,

$$\|D_k\|_{L^2(\partial\Omega)\rightarrow H^1(\partial\Omega)} + \|D'_k\|_{L^2(\partial\Omega)\rightarrow H^1(\partial\Omega)} \lesssim k^{5/4} \log k$$

for all $k \geq k_0$. Moreover, if $\partial\Omega$ is piecewise curved, then, given $k_0 > 1$, the following stronger estimates hold for all $k \geq k_0$

$$\|D_k\|_{L^2(\partial\Omega)\rightarrow H^1(\partial\Omega)} + \|D'_k\|_{L^2(\partial\Omega)\rightarrow H^1(\partial\Omega)} \lesssim k^{7/6} \log k.$$

- (c) If Ω is convex and $\partial\Omega$ is C^∞ and curved (in the sense of Definition 1.2) then, given $k_0 > 0$,

$$\|S_k\|_{L^2(\partial\Omega)\rightarrow H^1(\partial\Omega)} \lesssim k^{1/3},$$

$$\|D_k\|_{L^2(\partial\Omega)\rightarrow H^1(\partial\Omega)} + \|D'_k\|_{L^2(\partial\Omega)\rightarrow H^1(\partial\Omega)} \lesssim k \quad (1.6)$$

for all $k \geq k_0$.

Note that the requirement in Part (b) of Theorem 1.5 that $\partial\Omega$ is $C^{2,\alpha}$ arises since this is the regularity required of $\partial\Omega$ for D_k and D'_k to map $L^2(\partial\Omega)$ to $H^1(\partial\Omega)$; see [38, Theorem 4.2], [15, Theorem 3.6].

Remark 1.6. (Sharpness of the bounds in Theorem 1.5). In Sect. 3 we show that, modulo the factor $\log k$, all of the bounds in Theorem 1.5 are sharp (i.e. the powers of k in the bounds are optimal). The sharpness (modulo the factor $\log k$) of the $L^2(\partial\Omega) \rightarrow L^2(\partial\Omega)$ bounds in Theorem 2.10 was proved in [31, §A.2–A.3]. Earlier work in [6, §4] proved the sharpness of some of the $L^2(\partial\Omega) \rightarrow L^2(\partial\Omega)$ bounds in 2-d; we highlight that Sect. 3 and [31, §A.2–A.3] contain the appropriate generalisations to multidimensions of some of the arguments of [6, §4] (in particular [6, Theorems 4.2 and 4.4]).

Remark 1.7. (Comparison to previous results). The only previously-existing bounds on the $L^2(\partial\Omega) \rightarrow H^1(\partial\Omega)$ -norms of S_k , D_k , and D'_k are the following:

$$\|S_k\|_{L^2(\partial\Omega)\rightarrow H^1(\partial\Omega)} \lesssim k^{(d-1)/2} \quad (1.7)$$

when $\partial\Omega$ is Lipschitz [29, Theorem 1.6 (i)], and

$$\|D_k\|_{L^2(\partial\Omega)\rightarrow H^1(\partial\Omega)} + \|D'_k\|_{L^2(\partial\Omega)\rightarrow H^1(\partial\Omega)} \lesssim k^{(d+1)/2} \quad (1.8)$$

when $\partial\Omega$ is $C^{2,\alpha}$ [29, Theorem 1.6 (ii)].

We see that (1.7) is a factor of $\log k$ sharper than the bound (1.4) when $d = 2$, but otherwise all the bounds in Theorem 1.5 are sharper than (1.7) and (1.8).

Remark 1.8. (Bounds for general dimension and $k \in \mathbb{R}$). We have restricted attention to 2- and 3-dimensions because these are the most practically-interesting ones. From a semiclassical point of view, it is natural work in $d \geq 1$, and the results of Theorem 1.5 apply for any $d \geq 1$ (although when $d = 1$ it is straightforward to get sharper bounds; see [26, §1]). We have also restricted attention to the case when k is positive and bounded away from 0. Nevertheless, the methods used to prove the bounds in Theorem 1.5 show that if one replaces $\log k$ by $\log\langle k \rangle$ [where $\langle \cdot \rangle = (2 + |\cdot|^2)^{1/2}$] and includes an extra factor of $\log\langle k^{-1} \rangle$ when $d = 2$, then the resulting bounds hold for all $k \in \mathbb{R}$.

As explained in Sect. 1.2 below, the motivation for proving the $L^2(\partial\Omega) \rightarrow H^1(\partial\Omega)$ bounds of Theorem 1.5 comes from interest in second-kind Helmholtz boundary integral equations (BIEs) posed in $L^2(\partial\Omega)$. However, there is also a large interest in both first- and second-kind Helmholtz BIEs posed in the trace spaces $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$ (see, e.g., [47, §3.9], [51, §7.6]). The k -explicit theory of Helmholtz BIEs in the trace spaces is much less developed than the theory in $L^2(\partial\Omega)$, so we therefore highlight that the $L^2(\partial\Omega) \rightarrow H^1(\partial\Omega)$ bounds in Theorem 1.5 can be converted to $H^{s-1/2}(\partial\Omega) \rightarrow H^{s+1/2}(\partial\Omega)$ bounds for $|s| \leq 1/2$.

Corollary 1.9. (Bounds from $H^{s-1/2}(\partial\Omega) \rightarrow H^{s+1/2}(\partial\Omega)$ for $|s| \leq 1/2$). *Theorem 1.5 is valid with all the norms from $L^2(\partial\Omega) \rightarrow H^1(\partial\Omega)$ replaced by norms from $H^{s-1/2}(\partial\Omega) \rightarrow H^{s+1/2}(\partial\Omega)$ for $|s| \leq 1/2$.*

Remark 1.10. (The idea behind Theorem 1.5). The bounds of Theorem 1.5 are proved using estimates on the restriction of quasimodes of the Laplacian to hypersurfaces from [5, 14, 32, 52, 54], and [53] (and recapped in Sect. 2.3 below). The reason why these restriction estimates can be used to prove bounds on boundary-integral operators is explained in Sect. 2.4.2 below; this idea was first introduced in [26], [31, Appendix A] and [24], where $L^2(\partial\Omega) \rightarrow L^2(\partial\Omega)$ bounds were proved on S_k, D_k, D'_k .

1.2. Motivation for Proving Theorem 1.5

Our motivation for proving Theorem 1.5 has four parts.

1. The integral operators S_k, D_k , and D'_k appear in the standard second-kind BIE formulations of the exterior Dirichlet problem for the Helmholtz equation.
2. The standard analysis of the Galerkin method applied to these second-kind BIEs is based on the fact that, when $\partial\Omega$ is C^1 , the operators S_k, D_k , and D'_k are all compact, and thus $A'_{k,\eta}$ and $A_{k,\eta}$ are compact perturbations of $\frac{1}{2}I$.
3. To perform a k -explicit analysis of the Galerkin method applied to $A'_{k,\eta}$ or $A_{k,\eta}$ via these compact-perturbation arguments, we need to have

k -explicit information about the smoothing properties of S_k , D_k , and D'_k .

4. When $\partial\Omega$ is $C^{2,\alpha}$, the operators S_k , D_k , and D'_k all map $L^2(\partial\Omega)$ to $H^1(\partial\Omega)$, and k -explicit bounds on these norms therefore give the required k -explicit smoothing information.

Regarding Point 1: if u is the solution of the exterior Dirichlet problem for the Helmholtz equation

$$\Delta u(x) + k^2 u(x) = 0, \quad x \in \Omega_+,$$

satisfying the Sommerfeld radiation condition

$$\frac{\partial u}{\partial r}(x) - ik u(x) = o\left(\frac{1}{r^{(d-1)/2}}\right)$$

as $r := |x| \rightarrow \infty$, uniformly in x/r , then Green's integral representation theorem implies that

$$u(x) = - \int_{\partial\Omega} \Phi_k(x, y) \partial_n^+ u(y) ds(y) + \int_{\partial\Omega} \frac{\partial \Phi_k(x, y)}{\partial n(x)} \gamma^+ u(y) ds(y), \quad x \in \Omega_+, \quad (1.9)$$

where $\partial_n^+ u$ is the (unknown) Neumann trace on $\partial\Omega$ and $\gamma^+ u$ is the (known) Dirichlet trace. Taking the Dirichlet and Neumann traces of (1.9), using the jump relations for the single- and double-layer potentials (see, e.g. [7, Equations 2.41–2.43]), and then taking a linear combination of the resulting equations, we obtain the so-called “direct” BIE

$$A'_{k,\eta} \partial_n^+ u = f_{k,\eta} \quad (1.10)$$

where

$$A'_{k,\eta} := \frac{1}{2}I + D'_k - i\eta S_k, \quad (1.11)$$

$\eta \in \mathbb{R} \setminus \{0\}$, and $f_{k,\eta}$ is given in terms of the known Dirichlet trace; see, e.g., [7, Equation 2.68] (the exact form of $f_{k,\eta}$ is not important for us here). Alternatively, one can pose the ansatz

$$u(x) = \int_{\partial\Omega} \frac{\partial \Phi_k(x, y)}{\partial n(y)} \phi(y) ds(y) - i\eta \int_{\partial\Omega} \Phi_k(x, y) \phi(y) ds(y), \quad (1.12)$$

for $x \in \Omega_+$, $\phi \in L^2(\partial\Omega)$, and $\eta \in \mathbb{R} \setminus \{0\}$. Taking the Dirichlet trace of (1.12), we obtain the so-called “indirect” BIE

$$A_{k,\eta} \phi = \gamma^+ u, \quad (1.13)$$

where

$$A_{k,\eta} := \frac{1}{2}I + D_k - i\eta S_k. \quad (1.14)$$

The motivation for considering these “combined BIEs” (i.e. BIEs involving a linear combination of S_k , D_k , and D'_k) is that, when $\eta \in \mathbb{R} \setminus \{0\}$, the operators $A'_{k,\eta}$ and $A_{k,\eta}$ are bounded, invertible operators on $L^2(\partial\Omega)$ for all $k > 0$ (see, e.g., [7, Theorem 2.27]). In contrast, the integral operators S_k , $(\frac{1}{2}I + D'_k)$, and $(\frac{1}{2}I + D_k)$ are *not* invertible for all $k > 0$ (see, e.g., [7, §2.5]).

Regarding Point 2: S_k is compact when $\partial\Omega$ is Lipschitz (since $S_k : L^2(\partial\Omega) \rightarrow H^1(\partial\Omega)$ in this case [56, Theorem 1.6]), and D_k and D'_k are compact when $\partial\Omega$ is C^1 [23, Theorem 1.2(c)].

Regarding Points 3 and 4: [29] performed a k -explicit version of the classic compact-perturbation argument appearing in, e.g., [3, Chapter 3]. The two k -explicit ingredients were the $L^2(\partial\Omega) \rightarrow H^1(\partial\Omega)$ bounds on S_k, D_k , and D'_k discussed in Remark 1.7 (and proved in [29, Theorem 1.6]) and the sharp $L^2(\partial\Omega) \rightarrow L^2(\partial\Omega)$ bound on $(A'_{k,\eta})^{-1}$ and $A_{k,\eta}^{-1}$ when Ω is star-shaped with respect to a ball from [12, Theorem 4.3]. The paper [25] shows how the results of [29] are improved by using the new, sharp $L^2(\partial\Omega) \rightarrow H^1(\partial\Omega)$ bounds on S_k, D_k , and D'_k from Theorem 1.5, along with the sharp $L^2(\partial\Omega) \rightarrow L^2(\partial\Omega)$ bounds on $(A'_{k,\eta})^{-1}$ and $A_{k,\eta}^{-1}$ for nontrapping Ω from [4, Theorem 1.13].

1.3. Discussion of the Results of Theorem 1.5 in the Context of Using Semiclassical Analysis in the Numerical Analysis of the Helmholtz Equation

In the last 10 years, there has been growing interest in using results about the k -explicit analysis of the Helmholtz equation from semiclassical analysis to design and analyse numerical methods for the Helmholtz equation.¹ The activity has occurred in, broadly speaking, four different directions:

1. The use of the results of Melrose and Taylor [41]—on the rigorous $k \rightarrow \infty$ asymptotics of the solution of the Helmholtz equation in the exterior of a smooth convex obstacle with strictly positive curvature—to design and analyse k -dependent approximation spaces for integral-equation formulations [2, 17, 19–21, 28].
2. The use of the results of Melrose and Taylor [41], along with the work of Ikawa [37] on scattering from several convex obstacles, to analyse algorithms for multiple scattering problems [1, 22].
3. The use of bounds on the Helmholtz solution operator (also known as *resolvent estimates*) due to Vainberg [55] (using the propagation of singularities results of Melrose and Sjöstrand [40]) and Morawetz [45] to prove bounds on both $\|(A'_{k,\eta})^{-1}\|_{L^2(\partial\Omega) \rightarrow L^2(\partial\Omega)}$ and the inf-sup constant of the domain-based variational formulation [4, 12, 13, 48], and also to analyse preconditioning strategies [27].
4. The use of identities originally due to Morawetz [45] to prove coercivity of $A'_{k,\eta}$ [50] and to introduce new coercive formulations of Helmholtz problems [44, 49].

This paper concerns a fifth direction, namely proving sharp k -explicit bounds on S_k, D_k and D'_k using estimates on the restriction of quasimodes of the Laplacian to hypersurfaces from [5, 14, 32, 52, 54], and [53] (and recapped in Sect. 2.3 below). This direction was initiated in [26], [31, Appendix A], and [24], where sharp, k -explicit $L^2(\partial\Omega) \rightarrow L^2(\partial\Omega)$ bounds on S_k, D_k and D'_k were proved using this idea. The present paper extends this method to obtain sharp $L^2(\partial\Omega) \rightarrow H^1(\partial\Omega)$ bounds. The companion paper [25] then explores

¹A closely-related activity is the design and analysis of numerical methods for the Helmholtz equation based on proving *new* results about the $k \rightarrow \infty$ asymptotics of Helmholtz solutions for polygonal obstacles; see [9, 11, 34, 35], and [33].

the implications of both the $L^2(\partial\Omega) \rightarrow L^2(\partial\Omega)$ and $L^2(\partial\Omega) \rightarrow H^1(\partial\Omega)$ bounds (used in conjunction with the results in Points 3 and 4 above) on the k -explicit numerical analysis of the Galerkin method applied to the second-kind Eqs. (1.10) and (1.13).

1.4. Outline of the Paper

In Sect. 2 we prove Theorem 1.5 [the $L^2(\partial\Omega) \rightarrow H^1(\partial\Omega)$ bounds] and Corollary 1.9, and in Sect. 3 we show that the bounds in Theorem 1.5 are sharp in their k -dependence.

2. Proof of Theorem 1.5 and Corollary 1.9

In this section we prove Theorem 1.5 and Corollary 1.9. The vast majority of the work will be in proving Parts (a) and (b) of Theorem 1.5, with Part (c) of Theorem 1.5 following from the results in [24, Chapter 4], and Corollary 1.9 following from the results of [29].

The outline of this section is as follows: in Sect. 2.1 we discuss some preliminaries from the theory of semiclassical pseudodifferential operators, with our default references the texts [57] and [18]. In Sect. 2.2 we recap facts about function spaces on piecewise smooth hypersurfaces. In Sect. 2.3 we recap restriction bounds on quasimodes—these results are central to our proof of Theorem 1.5. In Sect. 2.4 we prove of Parts (a) and (b) of Theorem 1.5, in Sect. 2.5 we prove Part (c) of Theorem 1.5 Sect. 2.5, and in Sect. 2.6 we prove Corollary 1.9.

We drop the \lesssim notation in this section and state every bound with a constant C (independent of k); we do this because later in the proof it will be useful to be able to indicate whether or not the constant in our estimates depends on the order s of the Sobolev space, or on a particular hypersurface Γ (we do this via the subscript s and Γ —see, e.g., (2.20) below).

2.1. Semiclassical Preliminaries

2.1.1. Symbols and Quantization. Following [57, §3.3], for $k > 0$ and $u \in \mathcal{S}(\mathbb{R}^d)$, we define the semiclassical Fourier transform $\mathcal{F}_k(u)$ by

$$\mathcal{F}_k(u)(\xi) := \int_{\mathbb{R}^d} \exp(-ik\langle y, \xi \rangle) u(y) dy, \quad (2.1)$$

where $\langle x, \xi \rangle := \sum_{j=1}^d x_j \xi_j$. We recall the inversion formula

$$u(x) := \frac{k^d}{(2\pi)^d} \int_{\mathbb{R}^d} \exp(ik\langle x, \xi \rangle) \mathcal{F}_k(u)(\xi) d\xi.$$

We use the standard notation that $D := -i\partial$, so that $\mathcal{F}_k(k^{-1}D_j u)(\xi) = \xi_j \mathcal{F}_k(u)(\xi)$. We let $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$ and, following [18, §E.1.2], we say that $a(x, \xi; k) \in C^\infty(\mathbb{R}_{x,\xi}^{2d})$ lies in $S^m(\mathbb{R}_{x,\xi}^{2d})$ if for all $\alpha, \beta \in \mathbb{N}^d$ and $K \Subset \mathbb{R}^d$, there exists $C_{\alpha,\beta,K} > 0$ so that

$$\sup_{x \in K, \xi \in \mathbb{R}^d} \langle \xi \rangle^{-m+|\beta|} |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha,\beta,K}.$$

From here on, we follow the usual convention of suppressing the dependence of $a(x, \xi; k)$ on k , writing instead $a(x, \xi)$ (see, e.g., [57, Remark on p. 72]), and also writing $S^m(\mathbb{R}^{2d})$ instead of $S^m(\mathbb{R}_{x, \xi}^{2d})$. We write $S^{-\infty}(\mathbb{R}^{2d}) = \bigcap_{m \in \mathbb{R}} S^m(\mathbb{R}^{2d})$. We say that $a \in S^{\text{comp}}(\mathbb{R}^{2d})$ if $a \in S^{-\infty}(\mathbb{R}^{2d})$ with $\text{supp } a \subset K$ for some compact set $K \subset \mathbb{R}^{2d}$ independent of k .

For an element $a \in S^m$, we define its quantization to be the operator

$$u \mapsto a(x, k^{-1}D)u := \frac{k^d}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \exp(ik\langle x - y, \xi \rangle) a(x, \xi) u(y) dy d\xi \tag{2.2}$$

for $u \in \mathcal{S}(\mathbb{R}^d)$. These operators can be defined by duality on $u \in \mathcal{S}'(\mathbb{R}^d)$. We say that an operator $A(k) : C_c^\infty(\mathbb{R}^d) \rightarrow \mathcal{D}'(\mathbb{R}^d)$ is $\mathcal{O}_{\Psi^{-\infty}}(k^{-\infty})$ if it is smoothing (i.e. its Schwartz kernel \mathcal{K} is smooth) and each seminorm of \mathcal{K} on $C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ is $\mathcal{O}(k^{-\infty})$. Note that, by introducing an operator $R = \mathcal{O}_{\Psi^{-\infty}}(k^{-\infty})$ as an error, we can make the operator $a(x, k^{-1}D)$ properly supported (i.e. so that for any $K \Subset \mathbb{R}^d$, the kernel \mathcal{K} of $a(x, k^{-1}D) + R$ has the property that both $\pi_R^{-1}(K) \cap \text{supp } \mathcal{K}$ and $\pi_L^{-1}(K) \cap \text{supp } \mathcal{K}$ are compact where $\pi_R, \pi_L : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ are projection onto the right and left factors respectively).

Now, we say that $A(k)$ is a pseudodifferential operator of order m and write $A(k) \in \Psi^m(\mathbb{R}^d)$ if $A(k)$ is properly supported and for some $a \in S^m(\mathbb{R}^{2d})$,

$$A(k) := a(x, k^{-1}D) + \mathcal{O}_{\Psi^{-\infty}}(k^{-\infty}).$$

We say that $A(k) \in \Psi^{\text{comp}}(\mathbb{R}^d)$ if

$$A(k) = a(x, k^{-1}D) + \mathcal{O}_{\Psi^{-\infty}}(k^{-\infty})$$

for some $a \in S^{\text{comp}}(\mathbb{R}^{2d})$.

Suppose that $A(k) \in \Psi^m(\mathbb{R}^d)$ has $A(k) = a(x, k^{-1}D) + \mathcal{O}_{\Psi^{-\infty}}(k^{-\infty})$. Then we call a the *full symbol* of A . The *principal symbol* of $A \in \Psi^m(\mathbb{R}^d)$, denoted by $\sigma(A)$, is defined by

$$\sigma(A) := a \pmod{k^{-1}S^{m-1}(\mathbb{R}^{2d})}.$$

Lemma 2.1. [18, Proposition E.16]. *Let $a \in S^{m_1}(\mathbb{R}^{2d})$ and $b \in S^{m_2}(\mathbb{R}^{2d})$. Then we have*

$$\begin{aligned} a(x, k^{-1}D)b(x, k^{-1}D) &= (ab)(x, k^{-1}D) + k^{-1}r_1(x, k^{-1}D) + \mathcal{O}_{\Psi^{-\infty}}(k^{-\infty}) \\ [a(x, k^{-1}D), b(x, k^{-1}D)] &:= a(x, k^{-1}D)b(x, k^{-1}D) - b(x, k^{-1}D)a(x, k^{-1}D) \\ &= \frac{1}{ik} \{a, b\}(x, k^{-1}D) + k^{-2}r_2(x, k^{-1}D) + \mathcal{O}_{\Psi^{-\infty}}(k^{-\infty}) \end{aligned}$$

where $r_1 \in S^{m_1+m_2-1}(\mathbb{R}^{2d})$, $r_2 \in S^{m_1+m_2-2}(\mathbb{R}^{2d})$, $\text{supp } r_i \subset \text{supp } a \cap \text{supp } b$, and the Poisson bracket $\{a, b\}$ is defined by

$$\{a, b\} := \sum_{j=1}^d (\partial_{\xi_j} a)(\partial_{x_j} b) - (\partial_{\xi_j} b)(\partial_{x_j} a).$$

2.1.2. Action on Semiclassical Sobolev Spaces. We define the Semiclassical Sobolev spaces $H_k^s(\mathbb{R}^d)$ to be the space $H^s(\mathbb{R}^d)$ equipped with the norm

$$\|u\|_{H_k^s(\mathbb{R}^d)}^2 := \|\langle k^{-1}D \rangle^s u\|_{L^2(\mathbb{R}^d)}^2.$$

Note that for s an integer, this norm is equivalent to

$$\|u\|_{H_k^s(\mathbb{R}^d)}^2 = \sum_{|\alpha| \leq s} \|(k^{-1}\partial)^\alpha u\|_{L^2(\mathbb{R}^d)}^2.$$

The definition of the semiclassical Sobolev spaces on a smooth compact manifold Γ of dimension $d-1$, i.e. $H_k^s(\Gamma)$ for $|s| \leq 1$, follows from the definition of $H_k^s(\mathbb{R}^{d-1})$ (see, e.g., [39, p. 98]). Because solutions of the Helmholtz equation $(-k^{-2}\Delta - 1)u = 0$ oscillate at frequency k , scaling derivatives by k^{-1} makes the k -dependence of these norms uniform in the number of derivatives.

With these definitions in hand, we have the following lemma on boundedness of pseudodifferential operators.

Lemma 2.2. [18, Proposition E.22]. *Let $A \in \Psi^m(\mathbb{R}^d)$. Then for $\chi_1, \chi_2 \in C_c^\infty(\mathbb{R}^d)$, $\|\chi_2 A \chi_1\|_{H_k^s(\mathbb{R}^d) \rightarrow H_k^{s-m}(\mathbb{R}^d)} \leq C$.*

2.1.3. Ellipticity. For $A \in \Psi^m(\mathbb{R}^d)$, we say that $(x, \xi) \in \mathbb{R}^{2d}$ is in the *elliptic set* of A , denoted $\text{ell}(A)$, if there exists U a neighborhood of (x, ξ) such that for some $\delta > 0$,

$$\inf_U |\sigma(A)(x, \xi)| \geq \delta.$$

We then have the following lemma

Lemma 2.3. [18, Proposition E.31]. *Suppose that $A \in \Psi^{m_1}(\mathbb{R}^d)$, $b \in S^{\text{comp}}(\mathbb{R}^{2d})$ with $\text{supp } b \subset \text{ell}(A)$. Then there exists $R_1, R_2 \in \Psi^{\text{comp}}(\mathbb{R}^d)$ with*

$$R_1 A = b(x, k^{-1}D) + \mathcal{O}_{\Psi^{-\infty}}(k^{-\infty}), \quad AR_2 = b(x, k^{-1}D) + \mathcal{O}_{\Psi^{-\infty}}(k^{-\infty}).$$

Moreover, if $b \in S^{m_2}(\mathbb{R}^{2d})$ and there exists $M > 0$, $\delta > 0$

$$\inf_{\text{supp } b} |\sigma(A)| \langle \xi \rangle^{-m_1} > \delta,$$

then the same conclusions hold with $R_i \in \Psi^{m_2 - m_1}(\mathbb{R}^d)$.

2.1.4. Pseudodifferential Operators on Manifolds. Since we only use the notion of a pseudodifferential operator on a manifold in passing (in Lemma 2.15 and Sect. 2.5 below), we simply note that it is possible to define pseudodifferential operators on manifolds (see, e.g., [57, Chapter 14]). The analogues of Lemmas 2.1, 2.2, and 2.3 all hold in this setting. Moreover, the principal symbol map can still be defined although its definition is somewhat more involved.

2.2. Function Spaces on Piecewise Smooth Hypersurfaces

We now define the spaces $\overline{H}^s(\Gamma)$ and $\dot{H}^s(\Gamma)$ (with the notation for these spaces taken from [36, §B.2]).

Definition 2.4. (*Extendable Sobolev space $\overline{H^s}(\Gamma)$ on a smooth hypersurface*). Let Γ be a smooth hypersurface of \mathbb{R}^d (in the sense of Definition 1.1) and let $\tilde{\Gamma}$ be an extension of Γ . Given $s \in \mathbb{R}$, we say that $u \in \overline{H^s}(\Gamma)$ if there exists $\underline{u} \in H^s_{\text{comp}}(\tilde{\Gamma})$ such that $\underline{u}|_{\Gamma} = u$.

Let $(U_j, \psi_j)_{j \in J}$ be an atlas of $\tilde{\Gamma}$ such that $U_j \cap \partial\Gamma \cap \partial\tilde{\Gamma} = \emptyset$ for all $j \in J$, and let

$$J_{\Gamma} := \{j \in J, U_j \cap \Gamma \neq \emptyset\} \quad \text{and} \quad J_{\partial} := \{j \in J, U_j \cap \partial\Gamma \neq \emptyset\}$$

(observe that if $\partial\Gamma = \emptyset$ then $J_{\partial} = \emptyset$). Let $(\chi_j)_{j \in J}$ be a partition of unity of $\tilde{\Gamma}$ subordinated to $(U_j)_{j \in J}$. Given $\chi \in C^{\infty}_c(\text{Int}(\tilde{\Gamma}))$ such that $\chi = 1$ in a neighborhood of Γ , we define

$$\begin{aligned} \|u\|_{\overline{H^s}(\Gamma)} &= \sum_{j \in J_{\Gamma} \setminus J_{\partial}} \|(\chi_j u) \circ \psi_j^{-1}\|_{H^s(\mathbb{R}^{d-1})} \\ &+ \inf_{\underline{u} \in H^s_{\text{comp}}(\tilde{\Gamma}), \underline{u}|_{\Gamma} = u} \sum_{j \in J_{\partial}} \|(\chi_j \chi \underline{u}) \circ \psi_j^{-1}\|_{H^s(\mathbb{R}^{d-1})}. \end{aligned} \tag{2.3}$$

We make two remarks:

1. The definition of the norm $\overline{H^s}(\Gamma)$ depends on $\tilde{\Gamma}$, χ , and the choice of charts (U_j, ψ_j) and partition of unity (χ_j) . One can however prove that two different choices of charts (U_j, ψ_j) and partition of unity (χ_j) lead to equivalent norms $\overline{H^s}(\Gamma)$. In what follows, (U_j, ψ_j, χ_j) will be traces on $\tilde{\Gamma}$ of charts and partition of unity on \mathbb{R}^d .
2. This definition is the same as, e.g., the definition of $H^s(\Gamma)$ for $\Gamma \subset \mathbb{R}^d$ any non-empty open set in [39, p. 77]. However, we use the specific notation $\overline{H^s}(\Gamma)$ for the following two reasons: (i) parallelism with the space $\overline{H^s}(\partial\Omega)$ in Definition 2.6 below, and (ii) the fact that, without using the overline, $H^s(\cdot)$ would be defined differently depending on whether the \cdot is a smooth hypersurface or the boundary of a Lipschitz domain.

Definition 2.5. (*Sobolev space $\dot{H}^s(\Gamma)$ on a smooth hypersurface*). Let Γ be a smooth hypersurface of \mathbb{R}^d (in the sense of Definition 1.1) and let $\tilde{\Gamma}$ be an extension of Γ . Given $s \in \mathbb{R}$, We say that $u \in \dot{H}^s(\Gamma)$ if $u \in H^s_{\text{comp}}(\tilde{\Gamma})$ and $\text{supp } u \subset \bar{\Gamma}$. Then,

$$\|u\|_{\dot{H}^s(\Gamma)} := \|u\|_{H^s(\tilde{\Gamma})}.$$

Since Γ has C^0 boundary, one can show [10, Theorem 3.3, Lemma 3.15] that the dual of $\overline{H^s}(\Gamma)$ is given by $\dot{H}^{-s}(\Gamma)$ with the dual pairing inherited from that of $H^s_{\text{comp}}(\tilde{\Gamma})$ and $H^{-s}_{\text{comp}}(\tilde{\Gamma})$.

For piecewise smooth $\partial\Omega$, it is useful to consider the following ‘‘piecewise- H^s ’’ spaces.

Definition 2.6. (*Sobolev space $\overline{H^s}(\partial\Omega)$*). Let Ω be a bounded Lipschitz open set such that its open complement is connected and $\partial\Omega$ is a piecewise smooth hypersurface (in the sense of Definition 1.3); i.e., $\partial\Omega = \bigcup_{i=1}^N \bar{\Gamma}_i$ where Γ_i are smooth hypersurfaces. With $|s| \leq 1$, we say that $u \in \overline{H^s}(\partial\Omega)$ if

$$u = \sum_{i=1}^N u_i, \quad \text{for } u_i \in \overline{H^s}(\Gamma_i), \quad \text{and we let } \|u\|_{\overline{H^s}(\partial\Omega)} := \sqrt{\sum_{i=1}^N \|u_i\|_{\overline{H^s}(\Gamma_i)}^2}.$$

We similarly define the norms $\overline{H}_k^s(\Gamma)$ and $\dot{H}_k^s(\Gamma)$ replacing $\|\cdot\|_{H^s(\mathbb{R}^{d-1})}$ in (2.3) with the weighted-norm $\|\cdot\|_{H_k^s(\mathbb{R}^{d-1})}$ (see, e.g., [18, Definition E.21]).

The following lemma implies that, when $S_k, D_k,$ and D'_k map $L^2(\partial\Omega)$ to $H^1(\partial\Omega)$, to bound the $H^1(\partial\Omega)$ norms of $S_k\phi, D_k\phi,$ and $D'_k\phi$, it is sufficient to bound their $\overline{H^1}(\partial\Omega)$ norms.

Lemma 2.7. *Let Ω be a bounded Lipschitz open set such that its open complement is connected and $\partial\Omega$ is a piecewise smooth hypersurface (in the sense of Definition 1.3). If $u \in H^1(\partial\Omega)$ then*

$$\|u\|_{H^1(\partial\Omega)} \leq \|u\|_{\overline{H^1}(\partial\Omega)} \tag{2.4}$$

Proof. Recall that $H^1(\partial\Omega)$ can be defined as the completion of $C_{\text{comp}}^\infty(\partial\Omega) := \{u|_{\partial\Omega} : u \in C_0^\infty(\mathbb{R}^d)\}$ with respect to the norm

$$\int_{\partial\Omega} (|\nabla_{\partial\Omega} u|^2 + |u|^2) ds \tag{2.5}$$

[7, pp. 275–276] where $\nabla_{\partial\Omega}$ is the *surface gradient*, defined in terms of a parametrisation of the boundary by, e.g., [7, Equations (A.13) and (A.14)]. By the definition of the $\overline{H^1}(\Gamma_i)$ norm from Definition 2.4, u restricted to Γ_i satisfies

$$\begin{aligned} \|u\|_{\overline{H^1}(\Gamma_i)}^2 &= \int_{\Gamma_i} (|\nabla_{\Gamma_i} u|^2 + |u|^2) ds(\Gamma_i) + \inf_{\underline{u}|_{\Gamma=U}} \int_{\tilde{\Gamma}_i \setminus \Gamma_i} (|\nabla_{\tilde{\Gamma}_i} \underline{u}|^2 + |\underline{u}|^2) ds(\tilde{\Gamma}_i), \\ &\geq \int_{\Gamma_i} (|\nabla_{\Gamma_i} u|^2 + |u|^2) ds(\Gamma_i). \end{aligned}$$

Then,

$$\begin{aligned} \|u\|_{\overline{H^1}(\partial\Omega)}^2 &= \int_{\partial\Omega} (|\nabla_{\partial\Omega} u|^2 + |u|^2) ds = \sum_{i=1}^N \int_{\Gamma_i} (|\nabla_{\Gamma_i} u|^2 + |u|^2) ds(\Gamma_i) \\ &\leq \sum_{i=1}^N \|u\|_{\overline{H^1}(\Gamma_i)}^2 = \|u\|_{\overline{H^1}(\partial\Omega)}^2 \end{aligned}$$

and the proof is complete. □

Observe that Lemma 2.7 also holds when $H^1(\partial\Omega)$ and $\overline{H^1}(\partial\Omega)$ are replaced by $H_k^1(\partial\Omega)$ and $\overline{H}_k^1(\partial\Omega)$ respectively.

2.3. Recap of Restriction Estimates for Quasimodes

Theorem 2.8. *Let $U \subset \mathbb{R}^d$ be open and precompact with Γ a smooth hypersurface (in the sense of Definition 1.1) satisfying $\overline{\Gamma} \subset U$. Given $k_0 > 0$, there exists $C > 0$ (independent of k) so that if $\|u\|_{L^2(U)} = 1$ with*

$$(-k^{-2}\Delta - 1)u = \mathcal{O}_{L^2(U)}(k^{-1}), \tag{2.6}$$

(i.e. $\|k^{-2}\Delta u + u\|_{L^2(U)} = \mathcal{O}(k^{-1})$) then, for all $k \geq k_0$,

$$\|u\|_{L^2(\Gamma)} \leq \begin{cases} Ck^{1/4}, \\ Ck^{1/6}, & \Gamma \text{ curved,} \end{cases} \tag{2.7}$$

and

$$\|\partial_\nu u\|_{L^2(\Gamma)} \leq Ck \tag{2.8}$$

where ∂_ν is a choice of normal derivative to Γ .

References for the Proof of Theorem 2.8. The bound (2.7) for general Γ is proved in [52, Theorem 1.7] and [5, Theorem 1] and for curved Γ in [32, Theorem 1.3]. The bound (2.8) is proved in [53, Theorem 0.2] (with the analogous estimate for proper eigenfunctions appearing in [14, Theorem 1.1]).

We highlight that the analogues of the estimates (2.7) and (2.8) in the context of the wave equation on smooth Riemannian manifolds appear in [54, Theorem 1] (along with their L^p generalizations in [54, Theorem 8]), with [54, pp. 187 and 188] noting that the L^2 bounds are a corollary of an estimate in [30]. \square

Remark 2.9. (Smoothness of Γ required for the quasimode estimates). The $k^{1/4}$ -bound in (2.7) is valid when Γ is only $C^{1,1}$, and the $k^{1/6}$ -bound is valid when Γ is $C^{2,1}$ and curved. Therefore, with some extra work it should be possible to prove that the bounds on S_k in Theorem 1.5 hold with the assumption ‘‘piecewise smooth’’ replaced by ‘‘piecewise $C^{1,1}$ ’’ and ‘‘piecewise $C^{2,1}$ and curved’’ respectively. On the other hand, the bound (2.8) is not known in the literature for lower regularity Γ .

2.4. Proof of Parts (a) and (b) of Theorem 1.5

When proving these results, it is more convenient to work in semiclassical Sobolev spaces, i.e. to prove the bounds from $L^2(\partial\Omega)$ to $H_k^1(\partial\Omega)$, where (following Sect. 2.1.2),

$$\|w\|_{H_k^1(\partial\Omega)}^2 := k^{-2} \|\nabla_{\partial\Omega} w\|_{L^2(\partial\Omega)}^2 + \|w\|_{L^2(\partial\Omega)}^2, \tag{2.9}$$

where $\nabla_{\partial\Omega}$ is the surface gradient on $\partial\Omega$ (defined by, e.g., [7, Equations (A.13) and (A.14)]). We therefore now restate Theorem 1.5 as Theorem 2.10 below, working in these spaces.

Theorem 2.10. (Restatement of Theorem 1.5 as bounds from $L^2(\partial\Omega) \rightarrow H_k^1(\partial\Omega)$).

Let Ω be a bounded Lipschitz open set such that the open complement $\Omega_+ := \mathbb{R}^d \setminus \overline{\Omega}$ is connected.

- (a) If $\partial\Omega$ is a piecewise smooth hypersurface (in the sense of Definition 1.3), then, given $k_0 > 1$, there exists $C > 0$ (independent of k) such that

$$\|S_k\|_{L^2(\partial\Omega) \rightarrow H_k^1(\partial\Omega)} \leq C k^{-1/2} \log k. \tag{2.10}$$

for all $k \geq k_0$. Moreover, if $\partial\Omega$ is piecewise curved (in the sense of Definition 1.4), then, given $k_0 > 1$, the following stronger estimate holds for all $k \geq k_0$

$$\|S_k\|_{L^2(\partial\Omega) \rightarrow H_k^1(\partial\Omega)} \leq C k^{-2/3} \log k. \tag{2.11}$$

- (b) If $\partial\Omega$ is a piecewise smooth, $C^{2,\alpha}$ hypersurface, for some $\alpha > 0$, then, given $k_0 > 1$, there exists $C > 0$ (independent of k) such that

$$\|D_k\|_{L^2(\partial\Omega) \rightarrow H_k^1(\partial\Omega)} + \|D'_k\|_{L^2(\partial\Omega) \rightarrow H_k^1(\partial\Omega)} \leq Ck^{1/4} \log k. \quad (2.12)$$

Moreover, if $\partial\Omega$ is piecewise curved, then, given $k_0 > 1$, there exists $C > 0$ (independent of k) such that the following stronger estimates hold for all $k \geq k_0$

$$\|D_k\|_{L^2(\partial\Omega) \rightarrow H_k^1(\partial\Omega)} + \|D'_k\|_{L^2(\partial\Omega) \rightarrow H_k^1(\partial\Omega)} \lesssim k^{1/6} \log k.$$

- (c) If Ω is convex and $\partial\Omega$ is C^∞ and curved (in the sense of Definition 1.2) then, given $k_0 > 1$, there exists C such that, for $k \geq k_0$,

$$\begin{aligned} \|S_k\|_{L^2(\partial\Omega) \rightarrow H_k^1(\partial\Omega)} &\leq Ck^{-2/3}, \\ \|D_k\|_{L^2(\partial\Omega) \rightarrow H_k^1(\partial\Omega)} + \|D'_k\|_{L^2(\partial\Omega) \rightarrow H_k^1(\partial\Omega)} &\leq C. \end{aligned}$$

Because Theorem 2.10 works in the weighted space $H_k^1(\partial\Omega)$, the $L^2(\partial\Omega) \rightarrow L^2(\partial\Omega)$ bounds contained in Theorem 2.10 are one power of k stronger than those contained in Theorem 1.5. The $L^2(\partial\Omega) \rightarrow L^2(\partial\Omega)$ bounds contained in Theorem 2.10 were originally proved in [26, Theorem 1.2], [31, Appendix A], and [24, Theorems 4.29, 4.32].

In Sect. 2.4.2 below, we give an outline of the proof of Parts (a) and (b). This outline, however, requires the definitions of S_k , D_k , and D'_k in terms of the free resolvent (a.k.a. the Newtonian, or volume, potential), given in the next subsection.

2.4.1. S_k , D_k , and D'_k Written in Terms of the Free Resolvent $R_0(k)$. We now recall the definitions of S_k , D_k , and D'_k in terms of the free resolvent $R_0(k)$, these expressions are well-known in the theory of BIEs on Lipschitz domains [16], [39, Chapters 6 and 7]. We then specialise these to the case when $\partial\Omega$ is a piecewise smooth hypersurface (in the sense of Definition 1.3)

Let $R_0(k)$ be the free (outgoing) resolvent at k ; i.e. for $\psi \in C_{\text{comp}}^\infty(\mathbb{R}^d)$ we have

$$(R_0(k)\psi)(x) := \int_{\mathbb{R}^d} \Phi_k(x, y)\psi(y) dy,$$

where $\Phi_k(x, y)$ is the (outgoing) fundamental solution defined by (1.1) for $d = 2, 3$. Recall that $R_0(k) : H_{\text{comp}}^s(\mathbb{R}^d) \rightarrow H_{\text{loc}}^{s+2}(\mathbb{R}^d)$; see, e.g., [39, Equation 6.10].

With Ω a bounded Lipschitz open set with boundary $\partial\Omega$ and $1/2 < s < 3/2$, let $\gamma^+ : H_{\text{loc}}^s(\Omega_+) \rightarrow H^{s-1/2}(\partial\Omega)$ and $\gamma^- : H^s(\Omega) \rightarrow H^{s-1/2}(\partial\Omega)$, be the trace maps [16, Lemma 3.6], [39, Theorem 3.38]. When $\gamma^+u = \gamma^-u$ we write both as γu (so that $\gamma : H_{\text{loc}}^s(\mathbb{R}^d) \rightarrow H^{s-1/2}(\partial\Omega)$), and we then let $\gamma^* : H^{-s+1/2}(\partial\Omega) \rightarrow H_{\text{comp}}^{-s}(\mathbb{R}^d)$ be the adjoint of γ [39, Equation 6.14]. Then S_k can be written as

$$S_k = \gamma R_0(k) \gamma^* \quad (2.13)$$

[39, p. 202 and Equation 7.5], [16, Proof of Theorem 1]. With ∂_n^* denoting the adjoint of the normal derivative trace (see, e.g., [39, Equation 6.14]), we have that the double-layer potential, \mathcal{D}_k , is defined by

$$D_k := R_0(k)\partial_n^*$$

[39, p. 202]. Recalling that the normal vector n points out of Ω and into Ω_+ , we have that the traces of D_k from Ω_{\pm} to Γ are given by

$$\gamma^{\pm}D_k = \pm\frac{1}{2}I + D_k$$

[39, Equation 7.5 and Theorem 7.3] and thus

$$D_k = \mp\frac{1}{2}I + \gamma^{\pm}R_0(k)\partial_n^*. \tag{2.14}$$

Similarly, results about the normal-derivative traces of the single-layer potential S_k imply that

$$\partial_n^{\pm}S_k = \mp\frac{1}{2}I + D'_k$$

so

$$D'_k = \pm\frac{1}{2}I + \partial_n^{\pm}R_0(k)\gamma^*. \tag{2.15}$$

When $\partial\Omega$ is Lipschitz, $S_k : L^2(\partial\Omega) \rightarrow H^1(\partial\Omega)$ by [56, Theorem 1.6] (see also, e.g., [42, Chapter 15, Theorem 5], [43, Proposition 3.8]), and when $\partial\Omega$ is $C^{2,\alpha}$ for some $\alpha > 0$, then $D_k, D'_k : L^2(\partial\Omega) \rightarrow H^1(\partial\Omega)$ by [38, Theorem 4.2] (see also [15, Theorem 3.6]).

We now consider the case when $\partial\Omega$ is a piecewise smooth hypersurface (in the sense of Definition 1.3) and use the notation that $\tilde{\Gamma}_i$ are the compact embedded smooth manifolds of \mathbb{R}^d such that, for each i , Γ_i is an open subset of $\tilde{\Gamma}_i$. Let L_i be a vector field whose restriction to $\tilde{\Gamma}_i$ is equal to ∂_{ν_i} , the unit normal to $\tilde{\Gamma}_i$ that is outward pointing with respect to $\partial\Omega$. Let $\gamma_i : H^s_{\text{loc}}(\mathbb{R}^d) \rightarrow H^{s-1/2}(\Gamma_i)$ denote restriction to Γ_i . We note that γ_i^* is the inclusion map $f \mapsto f\delta_{\Gamma_i}$ where δ_{Γ_i} is $d - 1$ dimensional Hausdorff measure on Γ . Finally, we let γ_i^{\pm} denote restrictions from the interior and exterior respectively, where ‘interior’ and ‘exterior’ are defined via considering Γ_i as a subset of $\partial\Omega$. With these notations, we have that

$$D_k = \mp\frac{1}{2}I + \sum_{i,j} \gamma_i^{\pm}R_0(k)L_j^*\gamma_j^* \tag{2.16}$$

and

$$D'_k = \pm\frac{1}{2}I + \sum_{i,j} \gamma_i^{\pm}L_iR_0(k)\gamma_j^*; \tag{2.17}$$

the advantage of these last two expressions over (2.14) and (2.15) is that they involve γ_i and L_i instead of ∂_n^* and ∂_n^{\pm} .

In the rest of this section, we use the formulae (2.13), (2.16), and (2.17) as the definitions of S_k , D_k , and D'_k . Note that we slightly abuse notation by omitting the sums in (2.16) and (2.17) and instead writing

$$D_k = \pm\frac{1}{2}I + \gamma^{\pm}R_0(k)L\gamma^*, \quad D'_k = \mp\frac{1}{2}I + \gamma^{\pm}LR_0(k)\gamma^*. \tag{2.18}$$

2.4.2. Outline of the Proof of Parts (a) and (b) of Theorem 2.10. The proof of Parts (a) and (b) of Theorem 2.10 will follow in two steps. In Lemma 2.11, we obtain estimates on frequencies $\leq Mk$ and in Lemma 2.20 we complete the proof by estimating the high frequencies ($\geq Mk$).

To estimate the low frequency components, we spectrally decompose the resolvent using the Fourier transform. We are then able to reduce the proof of the low-frequency estimates to the estimates on the restriction of eigenfunctions (or more precisely quasimodes) to $\partial\Omega$ that we recalled in Sect. 2.3. To understand this reduction, we proceed formally. From the description of S_k in terms of the free resolvent, (2.13), the spectral decomposition of S_k via the Fourier transform is formally

$$S_k f = \int_0^\infty \frac{1}{r^2 - (k + i0)^2} \langle f, \gamma u(r) \rangle_{L^2(\partial\Omega)} \gamma u(r) dr \quad (2.19)$$

where $u(r)$ is a generalized eigenfunction of $-\Delta$ with eigenvalue r^2 , and $k + i0$ denotes the limit of $k + i\varepsilon$ as $\varepsilon \rightarrow 0^+$. Observe that the integral in (2.19) is not well-defined (hence why this calculation is only formal), but (2.19) nevertheless indicates that estimating S_k amounts to estimating the restriction of the generalized eigenfunction $u(r)$ to $\partial\Omega$.

At very high frequency, we compare the operators S_k , D_k , and D'_k with the corresponding operators when $k = 1$ (recall that the mapping properties of boundary integral operators with $k = 1$ have been extensively studied on rough domains; see, e.g. [42, Chapter 15], [39, 43]). By using a description of the resolvent at very high frequency as a pseudodifferential operator, we are able to see that these differences gain additional regularity and hence obtain estimates on them easily.

The new ingredients in our proof compared to [26] and [31] are that we have H^s norms in Lemma 2.11 and Lemma 2.20 rather than the L^2 norms appearing in the previous work.

2.4.3. Proof of Parts (a) and (b) of Theorem 2.10.

Low-Frequency Estimates. Following the outline in Sect. 2.4.2, our first task is to estimate frequencies $\leq kM$. We start by proving a conditional result that assumes a certain estimate on restriction of the Fourier transform of surface measures to the sphere of radius r (Lemma 2.11). In Lemma 2.13 we then show that the hypotheses in Lemma 2.11 are a consequence of restriction estimates for quasimodes. In Lemma 2.17 we show how the low-frequency estimates on S_k , D_k , and D'_k follow from Lemma 2.11.

In this section we denote the sphere of radius r by S_r^{d-1} and we denote the surface measure on S_r^{d-1} by $d\sigma$. We also use $\widehat{\cdot}$ to denote the *non-semiclassical* Fourier transform, i.e. $\widehat{u}(\xi)$ is defined by the right-hand side of (2.1) with $k = 1$.

Lemma 2.11. *Suppose that for $\Gamma \subset \mathbb{R}^d$ any precompact smooth hypersurface, $s \geq 0$, $f \in \dot{H}^{-s}(\Gamma)$, and some $\alpha, \beta > 0$,*

$$\int_{S_r^{d-1}} |\widehat{L^* f \delta_\Gamma}|^2(\xi) d\sigma(\xi) \leq C_\Gamma \langle r \rangle^{2\alpha+2s} \|f\|_{\dot{H}^{-s}(\Gamma)}^2, \tag{2.20}$$

$$\int_{S_r^{d-1}} |\widehat{f \delta_\Gamma}|^2(\xi) d\sigma(\xi) \leq C_\Gamma \langle r \rangle^{2\beta+2s} \|f\|_{\dot{H}^{-s}(\Gamma)}^2. \tag{2.21}$$

Let $\Gamma_1, \Gamma_2 \subset \mathbb{R}^d$ be compact embedded smooth hypersurfaces. Recall that L_i is a vector field with $L_i = \partial_\nu$ on Γ_i for some choice of unit normal ν on Γ_i and $\psi \in C_c^\infty(\mathbb{R})$ with $\psi \equiv 1$ in neighborhood of 0. With the frequency cutoff $\psi(k^{-1}D)$ defined as in (2.2), we then define for $f \in \dot{H}^{-s_1}(\Gamma_1)$, $g \in \dot{H}^{-s_2}(\Gamma_2)$, $s_i \geq 0$,

$$\begin{aligned} Q_S(f, g) &:= \int_{\mathbb{R}^d} R_0(k) (\psi(k^{-1}D) f \delta_{\Gamma_1}) \bar{g} \delta_{\Gamma_2} dx, \\ Q_D(f, g) &:= \int_{\mathbb{R}^d} R_0(k) (\psi(k^{-1}D) L_1^*(f \delta_{\Gamma_1})) \bar{g} \delta_{\Gamma_2} dx, \\ Q_{D'}(f, g) &:= \int_{\mathbb{R}^d} R_0(k) (\psi(k^{-1}D) f \delta_{\Gamma_1}) \overline{L_2^*(g \delta_{\Gamma_2})} dx. \end{aligned}$$

Then there exists $C_{\Gamma_1, \Gamma_2, \psi}$ so that for $k > 1$,

$$|Q_S(f, g)| \leq C_{\Gamma_1, \Gamma_2, \psi} \langle k \rangle^{2\beta-1+s_1+s_2} \log \langle k \rangle \|f\|_{\dot{H}^{-s_1}(\Gamma_1)} \|g\|_{\dot{H}^{-s_2}(\Gamma_2)}, \tag{2.22}$$

$$\begin{aligned} &|Q_D(f, g)| + |Q_{D'}(f, g)| \\ &\leq C_{\Gamma_1, \Gamma_2, \psi} \langle k \rangle^{\alpha+\beta-1+s_1+s_2} \log \langle k \rangle \|f\|_{\dot{H}^{-s_1}(\Gamma_1)} \|g\|_{\dot{H}^{-s_2}(\Gamma_2)}. \end{aligned} \tag{2.23}$$

The key point is that, modulo the frequency cutoff $\psi(k^{-1}D)$, $Q_S(f, g)$, $Q_D(f, g)$, and $Q_{D'}(f, g)$ are given respectively by $\langle S_k f, g \rangle_\Gamma$, $\langle D_k f, g \rangle_\Gamma$, and $\langle D'_k f, g \rangle_\Gamma$, where f is supported on Γ_1 and g on Γ_2 .

Proof of Lemma 2.11. We follow [26, 31] to prove the lemma. First, observe that due to the compact support of $f \delta_{\Gamma_i}$, (2.20) and (2.21) imply that for $\Gamma \subset \mathbb{R}^d$ precompact,

$$\int_{S_r^{d-1}} \left| \nabla_\xi \widehat{L^* f \delta_\Gamma}(\xi) \right|^2 d\sigma(\xi) \leq C_\Gamma \langle r \rangle^{2\alpha+2s} \|f\|_{\dot{H}^{-s}(\Gamma)}^2, \tag{2.24}$$

$$\int_{S_r^{d-1}} \left| \nabla_\xi \widehat{f \delta_\Gamma}(\xi) \right|^2 d\sigma(\xi) \leq C_\Gamma \langle r \rangle^{2\beta+2s} \|f\|_{\dot{H}^{-s}(\Gamma)}^2. \tag{2.25}$$

Indeed, $\nabla_\xi \widehat{f \delta_\Gamma} = \widehat{x f \delta_\Gamma}$ and since Γ is compact,

$$\|x f\|_{\dot{H}^{-s}(\Gamma)} \leq C_\Gamma \|f\|_{\dot{H}^{-s}(\Gamma)}.$$

Also, $\nabla_\xi \widehat{L^*(f \delta_\Gamma)} = \mathcal{F}(x L^*(f \delta_\Gamma))$. Thus,

$$x L^*(f \delta_\Gamma) = L^*(x f \delta_\Gamma) + [x, L^*] f \delta_\Gamma$$

and $[x, L^*] \in C^\infty$. Therefore, using compactness of Γ ,

$$\|x f\|_{\dot{H}^{-s}(\Gamma)} + \|[x, L^*] f\|_{\dot{H}^{-s}(\Gamma)} \leq C_\Gamma \|f\|_{\dot{H}^{-s}(\Gamma)}.$$

Now, $g\delta_{\Gamma_2} \in H^{\min(-s, -1/2-\epsilon)}(\mathbb{R}^d)$, $L_2^*(g\delta_{\Gamma_2}) \in H^{\min(-s-1, -3/2-\epsilon)}(\mathbb{R}^d)$ and since $\psi \in C_c^\infty(\mathbb{R})$,

$$R_0(k)(\psi(k^{-1}|D|)L^*(f\delta_{\Gamma_1})) \in C^\infty(\mathbb{R}^d), \quad R_0(k)(\psi(k^{-1}|D|))f\delta_{\Gamma_1} \in C^\infty(\mathbb{R}^d).$$

By Plancherel's theorem,

$$\begin{aligned} Q_S(f, g) &= \int_{\mathbb{R}^d} \psi(k^{-1}|\xi|) \frac{\widehat{f\delta_{\Gamma_1}}(\xi) \overline{\widehat{g\delta_{\Gamma_2}}(\xi)}}{|\xi|^2 - (k + i0)^2} d\xi, \\ Q_D(f, g) &= \int_{\mathbb{R}^d} \psi(k^{-1}|\xi|) \frac{\widehat{L_1^* f \delta_{\Gamma_1}}(\xi) \overline{\widehat{g\delta_{\Gamma_2}}(\xi)}}{|\xi|^2 - (k + i0)^2} d\xi, \\ \text{and } Q_{D'}(f, g) &= \int_{\mathbb{R}^d} \psi(k^{-1}|\xi|) \frac{\widehat{f\delta_{\Gamma_1}}(\xi) \overline{\widehat{L_2^* g \delta_{\Gamma_2}}(\xi)}}{|\xi|^2 - (k + i0)^2} d\xi, \end{aligned}$$

where $k + i0$ is understood as the limit of $k + i\epsilon$ as $\epsilon \rightarrow 0^+$.

Therefore, to prove the lemma, we only need to estimate

$$\int_{\mathbb{R}^d} \psi(k^{-1}|\xi|) \frac{F(\xi) G(\xi)}{|\xi|^2 - (k + i0)^2} d\xi \quad (2.26)$$

where, by (2.20), (2.21), (2.24), and (2.25),

$$\|F\|_{L^2(S_r^{d-1})} + \|\nabla_\xi F\|_{L^2(S_r^{d-1})} \leq C_{\Gamma_1} \langle r \rangle^{\delta_1 + s_1} \|f\|_{\dot{H}^{-s_1}(\Gamma_1)}, \quad \text{and} \quad (2.27)$$

$$\|G\|_{L^2(S_r^{d-1})} + \|\nabla_\xi G\|_{L^2(S_r^{d-1})} \leq C_{\Gamma_2} \langle r \rangle^{\delta_2 + s_2} \|g\|_{\dot{H}^{-s_2}(\Gamma_2)}. \quad (2.28)$$

Consider first the integral in (2.26) over $||\xi| - |k|| \geq 1$. Since $||\xi|^2 - k^2| \geq ||\xi|^2 - |k|^2|$, by the Schwartz inequality, (2.20), and (2.21), this piece of the integral is bounded by

$$\begin{aligned} & \int_{||\xi| - |k|| \geq 1} \left| \psi(k^{-1}|\xi|) \frac{F(\xi) G(\xi)}{|\xi|^2 - k^2} \right| d\xi \\ & \leq \int_{Mk \geq |r - |k|| \geq 1} \frac{1}{r^2 - |k|^2} \int_{S_r^{d-1}} F(r\theta) G(r\theta) d\sigma(\theta) dr \\ & \leq C_{\Gamma_1, \Gamma_2} \|f\|_{\dot{H}^{-s_1}(\Gamma_1)} \|g\|_{\dot{H}^{-s_2}(\Gamma_2)} \int_{M|k| \geq |r - |k|| \geq 1} \langle r \rangle^{\delta_1 + \delta_2 + s_1 + s_2} |r^2 - |k|^2|^{-1} dr \\ & \leq C_{\Gamma_1, \Gamma_2, \psi} \|f\|_{\dot{H}^{-s_1}(\Gamma_1)} \|g\|_{\dot{H}^{-s_2}(\Gamma_2)} |k|^{\delta_1 + \delta_2 - 1 + s_1 + s_2} \int_{M|k| \geq |r - |k|| \geq 1} |r - |k||^{-1} dr \\ & \leq C_{\Gamma_1, \Gamma_2, \psi} |k|^{\delta_1 + \delta_2 - 1 + s_1 + s_2} \log |k| \|f\|_{\dot{H}^{-s_1}(\Gamma_1)} \|g\|_{\dot{H}^{-s_2}(\Gamma_2)}, \quad (2.29) \end{aligned}$$

where the constant M in the intermediate steps depends on the support of ψ . Since $k > 1$, we write

$$\frac{1}{|\xi|^2 - (k + i0)^2} = \frac{1}{|\xi| + (k + i0)} \frac{\xi}{|\xi|} \cdot \nabla_\xi \log(|\xi| - (k + i0)),$$

where the logarithm is well defined since $\text{Im}(|\xi| - (k + i0)) < 0$. In particular, we may take the branch cut of the logarithm that has $\log(x) \in \mathbb{R}$ for $x \in (0, \infty)$ and has the branch cut on $i[0, \infty)$. Let $\chi(r) = 1$ for $|r| \leq 1$ and vanish

for $|r| \geq 3/2$. We then use integration by parts, together with (2.27) and (2.28)

$$\left| \int_{\mathbb{R}^d} \chi(|\xi| - |k|) \psi(k^{-1}|\xi|) \frac{1}{|\xi| + k + i0} F(\xi) G(\xi) \frac{\xi}{|\xi|} \cdot \nabla_\xi \log(|\xi| - (k + i0)) \, d\xi \right| \leq C_{\Gamma_1, \Gamma_2, \psi} |k|^{\delta_1 + \delta_2 - 1 + s_1 + s_2} \|f\|_{\dot{H}^{-s_1}(\Gamma_1)} \|g\|_{\dot{H}^{-s_2}(\Gamma_2)}.$$

Now, taking $\delta_1 = \delta_2 = \beta$ gives (2.22), and taking $\delta_1 = \alpha$ and $\delta_2 = \beta$ gives (2.23). \square

Remark 2.12. The estimate (2.29) is the only term where the $\log |k|$ appears, which leads to the $\log k$ factors in the bounds of Theorem 1.5 (without which these bounds would be sharp).

The proofs of the estimates (2.20) and (2.21) are contained in the following lemma.

Lemma 2.13. *Let $\Gamma \subset \mathbb{R}^d$ be a precompact smooth hypersurface. Then estimate (2.21) holds with $\beta = 1/4$. For $L = \partial_\nu$ on Γ , estimate (2.20) holds with $\alpha = 1$. Moreover, if Γ is curved then (2.21) holds with $\beta = 1/6$.*

To prove this lemma, we need to understand certain properties of the operator T_r defined by

$$T_r \phi(x) := \int_{S_r^{d-1}} \phi(\xi) e^{i\langle x, \xi \rangle} \, d\sigma(\xi). \tag{2.30}$$

Indeed, with $A : H^s(\mathbb{R}^d) \rightarrow H^{s-1}(\mathbb{R}^d)$, to estimate

$$\int_{S_r^{d-1}} |A^*(\widehat{f\delta_\Gamma})(\xi)|^2 \, d\sigma(\xi),$$

we write

$$\begin{aligned} \langle A^*(\widehat{f\delta_\Gamma})(\xi), \phi(\xi) \rangle_{S_r^{d-1}} &= \int_{S_r^{d-1}} \int_{\mathbb{R}^d} A^*(f(x)\delta_\Gamma) \overline{\phi(\xi) e^{i\langle x, \xi \rangle}} \, dx \, d\sigma(\xi) \\ &= \int_\Gamma \overline{f AT_r \phi} \, dx = \langle f, AT_r \phi \rangle_\Gamma, \end{aligned} \tag{2.31}$$

with T_r defined by (2.30).

Before proving Lemma 2.13 we prove two lemmas (Lemma 2.14 and 2.15) collecting properties of T_r .

Lemma 2.14. *Let T_r be defined by (2.30) and $\chi \in C_c^\infty(\mathbb{R}^d)$. Then,*

$$\|\chi T_r \phi\|_{L^2(\mathbb{R}^d)} \leq C \|\phi\|_{L^2(S_r^{d-1})}.$$

Proof of Lemma 2.14. We estimate $B := (\chi T_r)^* \chi T_r : L^2(S_r^{d-1}) \rightarrow L^2(S_r^{d-1})$. This operator has kernel

$$B(\xi, \eta) = \int_{\mathbb{R}^d} \chi^2(y) \exp(i\langle y, \xi - \eta \rangle) \, dy = \widehat{\chi^2}(\eta - \xi).$$

Now, for $\eta \in S_r^{d-1}$, and any $N > 0$,

$$\int_{S_r^{d-1}} |\widehat{\chi^2}(\eta - \xi)| \, d\sigma(\xi) \leq \int_{B(0, r/2)} \langle \xi' \rangle^{-N} \left[1 - \frac{|\xi'|^2}{r^2} \right]^{-1/2} \, d\xi' + C \langle r \rangle^{-N} \leq C.$$

Thus, by Schur's inequality, B is bounded on $L^2(S_r^{d-1})$ uniformly in r . Therefore,

$$\|\chi T_r \phi\|_{L^2(\mathbb{R}^d)}^2 \leq C \|\phi\|_{L^2(S_r^{d-1})}^2. \quad \square$$

In the next lemma, we use r (the radius of S_r^{d-1}) as a semiclassical parameter, with the space $\overline{H}_r^s(\Gamma)$ defined in exactly the same way as $\overline{H}_k^s(\Gamma)$ is defined in Sect. 2.2.

Lemma 2.15. *With T_r be defined by (2.30), let $\tilde{\Gamma}$ denote an extension of Γ , $\chi \in C_c^\infty(\mathbb{R}^d)$ and $A \in \Psi^1(\mathbb{R}^d)$ with $\chi \equiv 1$ in a neighborhood of $\tilde{\Gamma}$. Then for $s \in \mathbb{R}$,*

$$\|\chi A T_r \phi\|_{\overline{H}_r^s(\Gamma)} \leq C_s \|\chi A T_r \phi\|_{L^2(\tilde{\Gamma})}.$$

Proof of Lemma 2.15. Since $\widehat{T_r \phi}$ is supported on $|\xi| \leq 2r$, $\chi T_r \phi$ is compactly microlocalized in the sense that for $\psi \in C_c^\infty(\mathbb{R})$ with $\psi \equiv 1$ on $[-2, 2]$ with support in $[-3, 3]$,

$$\psi(r^{-1}|D|)\chi A T_r \phi = \chi A T_r \phi + \mathcal{O}_{\Psi^{-\infty}}(r^{-\infty})\chi T_r \phi.$$

(Note that $\psi(r^{-1}|D|)$ can be defined using (2.2) since $\psi(t)$ is constant near $t = 0$.)

Let $\gamma_{\tilde{\Gamma}}$ denote restriction to $\tilde{\Gamma}$, and γ_{Γ} restriction to Γ . Let $\chi_{\Gamma} \in C_c^\infty(\tilde{\Gamma})$ with $\chi_{\Gamma} \equiv 1$ on Γ . Then for $\psi_1 \in C_c^\infty(\mathbb{R})$ with $\psi_1 \equiv 1$ on $[-4, 4]$,

$$\chi_{\Gamma} \psi_1(r^{-1}|D'|_g)\chi_{\Gamma} \gamma_{\tilde{\Gamma}} \chi A T_r \phi = \chi_{\Gamma}^2 \gamma_{\tilde{\Gamma}} \chi A T_r \phi + \mathcal{O}_{\Psi^{-\infty}}(r^{-\infty})\gamma_{\tilde{\Gamma}} \chi T_r \phi$$

where $\psi_1(r^{-1}|D'|_g)$ is a pseudodifferential operator on $\tilde{\Gamma}$ with symbol $\psi_1(|\xi'|_g)$ and $|\cdot|_g$ denotes the metric induced on $T^*\tilde{\Gamma}$ from \mathbb{R}^d (see Remark 2.16 below).

Hence, for $r > 1$,

$$\|\gamma_{\Gamma} \chi A T_r \phi\|_{\overline{H}_r^s(\Gamma)} \leq C_s \|\chi A T_r \phi\|_{L^2(\tilde{\Gamma})}. \quad \square$$

Remark 2.16. (The definition of $|\cdot|_g$ used in the proof of Lemma 2.15). We now briefly review the definition of $|\cdot|_g$ from Riemannian geometry. Observe that the metric on \mathbb{R}^d is given by $g_e = \sum_{i=1}^d (dy^i)^2$ where y^i are standard coordinates on \mathbb{R}^d . To induce a metric on $\tilde{\Gamma}$, at a point x_0 we identify $V, W \in T_{x_0}\tilde{\Gamma}$ with $V', W' \in T_{x_0}\mathbb{R}^d$ and define $g(V, W) = g_e(V', W')$. That is, if $V = \sum_i V^i \partial_{y_i}$, $W = \sum_i W^i \partial_{y_i}$, then $g(V, W) = \sum_i V^i W^i$. By doing this at each point $x_0 \in \tilde{\Gamma}$, we obtain a metric on $\tilde{\Gamma}$. Next, choose coordinates x^i on $\tilde{\Gamma}$ and write the metric g as $g(\sum a^i \partial_{x^i}, \sum b^j \partial_{x^j}) = \sum_{i,j} g_{ij}(x) a^i b^j$. Then, for the corresponding dual coordinates ξ on $T^*\tilde{\Gamma}$, we have $|\xi|_g = \sqrt{\sum_{i,j} g^{ij} \xi_i \xi_j}$ where g^{ij} denotes the inverse matrix of g_{ij} . Note that this definition is independent of all of the choices of coordinates.

We are now in a position to prove Lemma 2.13.

Proof of Lemma 2.13. The key observation for the proof of Lemma 2.13 is that for $\chi \in C_c^\infty(\mathbb{R}^d)$, with $\chi \equiv 1$ in a neighborhood of Γ , $\chi T_r \phi$ is a quasimode of the Laplacian with $k = r$ in the sense of (2.6) in Theorem 2.8. To see this, observe first that $-\Delta T_r \phi = r^2 T_r \phi$ by the definition (2.30). Therefore,

$$-\Delta \chi T_r \phi = r^2 \chi T_r \phi + [-\Delta, \chi] T_r \phi.$$

Now, observe that for $\tilde{\chi} \in C_c^\infty(\mathbb{R}^d)$ with $\text{supp } \tilde{\chi} \subset \{\chi \equiv 1\}$, $\tilde{\chi}[-\Delta, \chi] = 0$. Therefore, taking such a $\tilde{\chi}$ with $\tilde{\chi} \equiv 1$ in a neighborhood, U of Γ shows that $\chi T_r \phi$ is a quasimode.

To prove (2.21), we let $A = I$. Then, by the bounds (2.7) in Theorem 2.8 together with Lemmas 2.14 and 2.15, for $s \geq 0$,

$$\begin{aligned} \|\chi T_r \phi\|_{\overline{H^s(\Gamma)}} &\leq C_s \langle r \rangle^s \|\chi T_r \phi\|_{L^2(\tilde{\Gamma})} \leq C_s \langle r \rangle^{\frac{1}{4}+s} \|\chi T_r \phi\|_{L^2(\mathbb{R}^d)} \\ &\leq C_s \langle r \rangle^{\frac{1}{4}+s} \|\phi\|_{L^2(S_r^{d-1})}, \end{aligned} \tag{2.32}$$

and if Γ is curved then

$$\|\chi T_r \phi\|_{\overline{H^s(\Gamma)}} \leq C \langle r \rangle^{\frac{1}{6}+s} \|\phi\|_{L^2(S_r^{d-1})}. \tag{2.33}$$

To prove (2.20), we take $A = L$. Observe that

$$\gamma_{\tilde{\Gamma}} \chi L T_r \phi = \gamma_{\tilde{\Gamma}} L \chi T_r \phi.$$

Hence, using the fact that $L = \partial_\nu$ on Γ together with the bound (2.8) in Theorem 2.8, we can estimate $L T_r \phi$.

$$\|\chi L T_r \phi\|_{L^2(\tilde{\Gamma})} = \|L \chi T_r \phi\|_{L^2(\tilde{\Gamma})} \leq C \langle r \rangle \|\chi T_r \phi\|_{L^2(\mathbb{R}^d)}. \tag{2.34}$$

In particular, for $s \geq 0$,

$$\|\chi L T_r \phi\|_{\overline{H^s(\Gamma)}} \leq C_s \langle r \rangle^{s+1} \|\phi\|_{L^2(S_r^{d-1})}.$$

Applying the Cauchy–Schwarz inequality together with (2.31), (2.32), (2.33) and (2.34) completes the proof of Lemma 2.13, since we have shown that

$$\begin{aligned} |\langle \widehat{f \delta_\Gamma}(\xi), \phi(\xi) \rangle_{L^2(S_r^{d-1})}| &\leq C_s \langle r \rangle^{\frac{1}{4}+s} \|f\|_{\dot{H}^{-s}(\Gamma)} \|\phi\|_{L^2(S_r^{d-1})}, \\ |\langle \widehat{L^*(f \delta_\Gamma)}(\xi), \phi(\xi) \rangle_{L^2(S_r^{d-1})}| &\leq C_s \langle r \rangle^{1+s} \|f\|_{\dot{H}^{-s}(\Gamma)} \|\phi\|_{L^2(S_r^{d-1})}, \end{aligned}$$

and if Γ is curved,

$$|\langle \widehat{f \delta_\Gamma}(\xi), \phi(\xi) \rangle_{L^2(S_r^{d-1})}| \leq C_s \langle r \rangle^{\frac{1}{6}+s} \|f\|_{\dot{H}^{-s}(\Gamma)} \|\phi\|_{L^2(S_r^{d-1})}. \quad \square$$

Lemma 2.17. (Low-frequency estimates). *Let s_2 be either 0 or 1. If $\partial\Omega$ is both Lipschitz and piecewise smooth (in the sense of Definition 1.3), then*

$$\|\gamma^\pm R_0(k) \psi(k^{-1} D) \gamma^* f\|_{H^{s_2}(\partial\Omega)} \leq C_{\partial\Omega, \psi} \langle k \rangle^{2\beta-1+s_2} \log \langle k \rangle \|f\|_{L^2(\partial\Omega)} \tag{2.35}$$

$$\|\gamma R_0(k) \psi(k^{-1} D) L_1^* \gamma^* f\|_{H^{s_2}(\partial\Omega)} \leq C_{\partial\Omega, \psi} \langle k \rangle^{\beta+s_2} \log \langle k \rangle \|f\|_{L^2(\partial\Omega)} \tag{2.36}$$

$$\|\gamma^\pm L_2 R_0(k) \psi(k^{-1} D) \gamma^* f\|_{H^{s_2}(\partial\Omega)} \leq C_{\partial\Omega, \psi} \langle k \rangle^{\beta+s_2} \log \langle k \rangle \|f\|_{L^2(\partial\Omega)}. \tag{2.37}$$

with $\beta = 1/4$. If $\partial\Omega$ is piecewise curved and Lipschitz then (2.35)–(2.37) hold with $\beta = 1/6$.

Proof of Lemma 2.17. By the duality property of $\overline{H^s}(\Gamma)$ and $\dot{H}^{-s}(\Gamma)$ (discussed after Definition 2.5), Lemma 2.13 and the estimates (2.22) and (2.23) imply for $s_1, s_2 \geq 0$ that there exists $C > 0$ independent of $k > 1$ so that

$$\|\gamma_{\Gamma_2} R_0(k) \psi(k^{-1}D) \gamma_{\Gamma_1}^* f\|_{\overline{H^{s_2}}(\Gamma_2)} \leq C_{\Gamma_1, \Gamma_2, \psi} \langle k \rangle^{2\beta-1+s_1+s_2} \log \langle k \rangle \|f\|_{\dot{H}^{-s_1}(\Gamma_1)}, \quad (2.38)$$

$$\|\gamma_{\Gamma_2} R_0(k) \psi(k^{-1}D) L_1^* \gamma_{\Gamma_1}^* f\|_{\overline{H^{s_2}}(\Gamma_2)} \leq C_{\Gamma_1, \Gamma_2, \psi} \langle k \rangle^{\beta+s_1+s_2} \log \langle k \rangle \|f\|_{\dot{H}^{-s_1}(\Gamma_1)}, \quad (2.39)$$

$$\|\gamma_{\Gamma_2} L_2 R_0(k) \psi(k^{-1}D) \gamma_{\Gamma_1}^* f\|_{\overline{H^{s_2}}(\Gamma_2)} \leq C_{\Gamma_1, \Gamma_2, \psi} \langle k \rangle^{\beta+s_1+s_2} \log \langle k \rangle \|f\|_{\dot{H}^{-s_1}(\Gamma_1)}. \quad (2.40)$$

Since $\partial\Omega$ is piecewise smooth, $\partial\Omega = \sum_{i=1}^N \Gamma_i$ with Γ_i smooth hypersurfaces. Since $\psi(k^{-1}D)$ is a smoothing operator on \mathcal{S}' , by elliptic regularity $R_0(k) \psi(k^{-1}D)$ is smoothing and hence its restriction to $\partial\Omega$ maps compactly supported distributions into $H^1(\partial\Omega)$. Applying (2.38)–(2.40) with $s_1 = 0$, $\Gamma = \Gamma_i$, summing over i , and using Definition 2.6, we find that, for $0 \leq s_2 \leq 1$,

$$\|\gamma R_0(k) \psi(k^{-1}D) \gamma^* f\|_{\overline{H^{s_2}}(\partial\Omega)} \leq C_{\partial\Omega, \psi} \langle k \rangle^{2\beta-1+s_2} \log \langle k \rangle \|f\|_{L^2(\partial\Omega)} \quad (2.41)$$

$$\|\gamma^\pm R_0(k) \psi(k^{-1}D) L_1^* \gamma^* f\|_{\overline{H^{s_2}}(\partial\Omega)} \leq C_{\partial\Omega, \psi} \langle k \rangle^{\beta+s_2} \log \langle k \rangle \|f\|_{L^2(\partial\Omega)} \quad (2.42)$$

$$\|\gamma^\pm L_2 R_0(k) \psi(k^{-1}D) \gamma^* f\|_{\overline{H^{s_2}}(\partial\Omega)} \leq C_{\partial\Omega, \psi} \langle k \rangle^{\beta+s_2} \log \langle k \rangle \|f\|_{L^2(\partial\Omega)}. \quad (2.43)$$

Applying (2.41)–(2.43) with $s_2 = 1$ (using the norm bound (2.4)) and $s_2 = 0$, we obtain the estimates (2.35)–(2.37). \square

High Frequency Estimates. Next, we obtain an estimate on the high frequency ($\geq kM$) components of S_k , D_k , and D'_k . We start by analyzing the high frequency components of the free resolvent, proving two lemmata on the structure of the free resolvent there.

Lemma 2.18. *Suppose that $z \in [-E, E]$. Let $\psi \in C_c^\infty(\mathbb{R})$ with $\psi \equiv 1$ on $[-2E^2, 2E^2]$. Then for $\chi \in C_c^\infty(\mathbb{R}^d)$.*

$$\chi R_0(zk) \chi (1 - \psi(|k^{-1}D|)) = B_1, \quad (1 - \psi(|k^{-1}D|)) \chi R_0(zk) \chi = B_2$$

where $B_i \in k^{-2} \Psi^{-2}(\mathbb{R}^d)$ with

$$\sigma(B_i) = \frac{\chi^2 k^{-2} (1 - \psi(|\xi|))}{|\xi|^2 - z^2}.$$

Proof of Lemma 2.18. Let $\chi_0 = \chi \in C_c^\infty(\mathbb{R}^d)$ and $\chi_n \in C_c^\infty(\mathbb{R}^d)$ have $\chi_n \equiv 1$ on $\text{supp } \chi_{n-1}$ for $n \geq 1$. Let $\psi_0 = \psi \in C_c^\infty(\mathbb{R})$ have $\psi \equiv 1$ on $[-2E^2, 2E^2]$, let $\psi_n \in C_c^\infty(\mathbb{R})$ have $\psi_n \equiv 1$ on $[-3E^2/2, 3E^2/2]$ and $\text{supp } \psi_n \subset \{\psi_{n-1} \equiv 1\}$ for $n \geq 1$. Finally, let $\varphi_n = (1 - \psi_n)$ and $\varphi = \varphi_0 = (1 - \psi)$. Then,

$$k^2 \chi R_0(zk) \chi (-k^{-2} \Delta - z^2) = (\chi^2 - \chi k^2 \chi_1 R_0(zk) \chi_1 [\chi, k^{-2} \Delta]). \quad (2.44)$$

Now, by Lemma 2.3 there exists $A_0 \in k^{-2} \Psi^{-2}(\mathbb{R}^d)$ with

$$A_0 = k^{-2} a_0(x, k^{-1}D) + \mathcal{O}_{\Psi^{-\infty}}(k^{-\infty}), \quad \text{supp } a_0 \subset \{\text{supp } \varphi_0\} \quad (2.45)$$

such that

$$k^2(-k^{-2}\Delta - z^2)A_0 = \varphi(|k^{-1}D|) + \mathcal{O}_{\Psi^{-\infty}}(k^{-\infty}) \tag{2.46}$$

and A_0 has

$$\sigma(A_0) = \frac{k^{-2}\varphi(|\xi|)}{|\xi|^2 - z^2}. \tag{2.47}$$

(Indeed, since we are working on \mathbb{R}^d ,

$$k^2(-k^{-2}\Delta - z^2)\frac{k^{-2}\varphi(|k^{-1}D|)}{|k^{-1}D|^2 - z^2} = \varphi(|k^{-1}D|)$$

with no remainder.)

Composing (2.44) on the right with A_0 , we have

$$\begin{aligned} \chi R_0 \chi \varphi(|k^{-1}D|) &= \chi^2 A_0 - k^2 \chi \chi_1 R_0 \chi_1 \varphi_1(|k^{-1}D|) [\chi, k^{-2}\Delta] A_0 + \mathcal{O}_{\Psi^{-\infty}}(k^{-\infty}), \\ &= \chi^2 A_0 + \chi \chi_1 R_0 \chi_1 \varphi_1(|k^{-1}D|) k^{-1} E_1 + \mathcal{O}_{\Psi^{-\infty}}(k^{-\infty}), \end{aligned}$$

where $E_1 \in \Psi^{-1}(\mathbb{R}^d)$ and we have used that $\varphi_1 \equiv 1$ on $\text{supp } \varphi_0$ and hence

$$\varphi_1(|k^{-1}D|) [\chi, k^{-2}\Delta] A_0 = [\chi, k^{-2}\Delta] A_0 + \mathcal{O}_{\Psi^{-\infty}}(k^{-\infty}).$$

Now, applying the same arguments, but with A_n such that

$$k^2(-k^{-2}\Delta - z^2)A_n = \varphi_n(|k^{-1}D|)E_n + \mathcal{O}_{\Psi^{-\infty}}(k^{-\infty})$$

there exists $A_n \in k^{-2}\Psi^{-2-n}(\mathbb{R}^d)$ such that

$$\begin{aligned} \chi_n R_0 \chi_n \varphi_n(|k^{-1}D|)E_n &= \chi_n^2 A_n + \chi_n \chi_{n+1} R_0 \chi_{n+1} \varphi_{n+1}(|k^{-1}D|)k^{-1}E_{n+1} \\ &\quad + \mathcal{O}_{\Psi^{-\infty}}(k^{-\infty}) \end{aligned}$$

with $E_{n+1} \in \Psi^{-1-n}(\mathbb{R}^d)$. Let

$$B_1^N := \chi^2 A_0 + \sum_{j=1}^{N-1} \chi k^{-j} A_j.$$

and assume that

$$\chi R_0 \chi \varphi(|k^{-1}D|) = B_1^N + \chi \chi_N R_0 \chi_N \varphi_N(|k^{-1}D|)k^{-N}E_N + \mathcal{O}_{\Psi^{-\infty}}(k^{-\infty}).$$

for some N . Then,

$$\begin{aligned} \chi \chi_N R_0 \chi_N \varphi_N(|k^{-1}D|)k^{-N}E_N &= \chi(\chi_N^2 A_N + \chi_N \chi_{N+1} R_0 \chi_{N+1} \varphi_{N+1}(|k^{-1}D|)k^{-1}E_{N+1} + \mathcal{O}_{\Psi^{-\infty}}(k^{-\infty})) \\ &= \chi k^{-N} A_N + \chi \chi_{N+1} R_0 \chi_{N+1} \varphi_{N+1}(|k^{-1}D|)k^{-(N+1)}E_{N+1} + \mathcal{O}_{\Psi^{-\infty}}(k^{-\infty}) \end{aligned}$$

and thus by induction, for all $N \geq 1$,

$$\begin{aligned} \chi R_0 \chi \varphi(|k^{-1}D|) &= B_1^{N+1} + \chi \chi_{N+1} R_0 \chi_{N+1} \varphi_{N+1}(|k^{-1}D|)k^{-(N+1)}E_{N+1} + \mathcal{O}_{\Psi^{-\infty}}(k^{-\infty}). \end{aligned}$$

Since $A_k \in \Psi^{-2-k}(\mathbb{R}^d)$, we may let

$$B_1 \sim \chi^2 A_0 + \sum_{j=1}^{\infty} \chi k^{-j} A_j$$

to obtain

$$\chi R_0 \chi \varphi(|k^{-1}D|) = B_1 \in k^{-2} \Psi^{-2}(\mathbb{R}^d)$$

with

$$\sigma(B_1) = \frac{k^{-2} \chi^2 (1 - \psi(|\xi|))}{|\xi|^2 - z^2}.$$

The proof of the statement for B_2 is identical. \square

Next, we prove an estimate on the difference between the resolvent at high energy and that at fixed energy.

Lemma 2.19. *Suppose that $z \in [0, E]$. Let $\psi \in C_c^\infty(\mathbb{R})$ with $\psi \equiv 1$ on $[-2E^2, 2E^2]$. Then for $\chi \in C_c^\infty(\mathbb{R}^d)$,*

$$\chi(R_0(zk) - R_0(1))\chi(1 - \psi(|k^{-1}D|)) \in k^{-2} \Psi^{-4}(\mathbb{R}^d).$$

Proof of Lemma 2.19. We proceed as in the proof of Lemma 2.18. Let $\chi_0 = \chi \in C_c^\infty(\mathbb{R}^d)$ and $\chi_n \in C_c^\infty(\mathbb{R}^d)$ have $\chi_n \equiv 1$ on $\text{supp } \chi_{n-1}$ for $n \geq 1$. Let $\psi_0 = \psi \in C_c^\infty(\mathbb{R})$ have $\psi \equiv 1$ on $[-2E^2, 2E^2]$, let $\psi_n \in C_c^\infty(\mathbb{R})$ have $\psi_n \equiv 1$ on $[-3E^2/2, 3E^2/2]$ and $\text{supp } \psi_n \subset \{\psi_{n-1} \equiv 1\}$ for $n \geq 1$. Finally, let $\varphi_n = (1 - \psi_n)$. Then,

$$\begin{aligned} & k^2 \chi(R_0(zk) - R_0(1))\chi(-k^{-2}\Delta - z^2) \\ &= \chi R_0(1) (z^2 k^2 - 1) \chi - \chi k^2 \chi_1(R_0(zk) - R_0(1))\chi_1[\chi, k^{-2}\Delta]. \end{aligned} \quad (2.48)$$

Now, by Lemma 2.3 there exists $A_0 \in k^{-2} \Psi^{-2}(\mathbb{R}^d)$ such that (2.45), (2.46), and (2.47) hold. Composing (2.48) on the right with $k^{-2}A_0$, we have

$$\begin{aligned} & \chi(R_0(zk) - R_0(1))\chi\varphi(|k^{-1}D|) \\ &= k^2 \chi R_0(1)\chi(z^2 - k^{-2})A_0 \\ & \quad - k^2 \chi \chi_1(R_0(zk) - R_0(1))\chi_1\varphi_1(|k^{-1}D|)[\chi, k^{-2}\Delta]A_0 + \mathcal{O}_{\Psi^{-\infty}}(k^{-\infty}). \end{aligned} \quad (2.49)$$

In particular, iterating using the same argument to write

$$\begin{aligned} & \chi_1(R_0(zk) - R_0(1))\chi_1\varphi_1(|k^{-1}D|) \\ &= k^2 \chi_1 R_0(1)\chi_1(z^2 - k^{-2})A_1 \\ & \quad - k^2 \chi_1 \chi_2(R_0(zk) - R_0(1))\chi_2\varphi_2(|k^{-1}D|)[\chi_1, k^{-2}\Delta]A_1 + \mathcal{O}_{\Psi^{-\infty}}(k^{-\infty}), \end{aligned}$$

we see that the right hand side of (2.49) is in $k^{-2} \Psi^{-4}(\mathbb{R}^d)$. \square

With Lemmas 2.18 and 2.19 in hand, we obtain the high-frequency estimates of the boundary-integral operators by comparing them to those at fixed frequency.

Lemma 2.20. (High-frequency estimates). *Let $M > 1$ and $\psi \in C_c^\infty(\mathbb{R})$ with $\psi \equiv 1$ for $|\xi| < M$. Suppose that $\partial\Omega$ is both Lipschitz and piecewise smooth (in the sense of Definition 1.3). Then for $k > 1$ and $\chi \in C_c^\infty(\mathbb{R}^d)$*

$$\gamma R_0(k)\chi(1 - \psi(|k^{-1}D|))\gamma^* = \mathcal{O}_{L^2(\partial\Omega) \rightarrow H_k^1(\partial\Omega)}(k^{-1}(\log k)^{1/2}). \quad (2.50)$$

If, in addition, $\partial\Omega$ is $C^{2,\alpha}$ for some $\alpha > 0$, then

$$\mp \frac{1}{2}I + \gamma^\pm R_0(k)\chi(1 - \psi(|k^{-1}D|))L^*\gamma^* = \mathcal{O}_{L^2(\partial\Omega) \rightarrow H_k^1(\partial\Omega)}(\log k) \quad (2.51)$$

$$\pm \frac{1}{2}I + \gamma^\pm LR_0(k)\chi(1 - \psi(|k^{-1}D|))\gamma^* = \mathcal{O}_{L^2(\partial\Omega) \rightarrow H_k^1(\partial\Omega)}(\log k). \quad (2.52)$$

Remark 2.21. The factors of $\log k$ in the bounds of Lemma 2.20 are likely artifacts of our proof, but since they do not affect our final results, we do not attempt to remove them here. In fact, if $\partial\Omega$ is smooth (rather than piecewise smooth), then one can show that the logarithmic factors can be removed from the bounds in Lemma 2.20 using the analysis in [24, Section 4.4].

Proof of Lemma 2.20. By Lemma 2.19,

$$A_k := \chi(R_0(k) - R_0(1))\chi(1 - \psi(k^{-1}D)) \in k^{-2}\Psi^{-4}.$$

Note that for $s > 1/2$,

$$\gamma = \mathcal{O}_{H_k^s(\mathbb{R}^d) \rightarrow H_k^{s-1/2}(\partial\Omega)}(k^{1/2}); \quad (2.53)$$

this bound follows from repeating the proof of the trace estimate in [39, Lemma 3.35] but working in semiclassically rescaled spaces.

Let $B_k := \gamma A_k \gamma^*$, $C_k := \gamma A_k L^* \gamma^*$, $C'_k := \gamma L A_k \gamma^*$. Then, using (2.53) and the fact that $L, L^* = \mathcal{O}_{H_k^s \rightarrow H_k^{s-1}}(k)$, we have that $B_k = \mathcal{O}_{L^2 \rightarrow H_k^1}(k^{-1})$ and $C_k, C'_k = \mathcal{O}_{L^2 \rightarrow H_k^1}(1)$.

Recalling the notation for S_k (2.13), D_k , and D'_k (2.18), and the mapping properties recapped in Sect. 2.4.1, we have

$$\gamma R_0(1)\chi\gamma^* : L^2(\partial\Omega) \rightarrow H^1(\partial\Omega)$$

when $\partial\Omega$ is Lipschitz, and

$$\begin{aligned} &\pm \frac{1}{2}I + \gamma^\pm R_0(1)\chi L^* \gamma^* : L^2(\partial\Omega) \rightarrow H^1(\partial\Omega) \\ &\mp \frac{1}{2}I + \gamma^\pm LR_0(1)\chi\gamma^* : L^2(\partial\Omega) \rightarrow H^1(\partial\Omega) \end{aligned}$$

when $\partial\Omega$ is $C^{2,\alpha}$.

Now, note that for $\tilde{\Gamma}$ a precompact smooth hypersurface, and $\psi \in C_c^\infty(\mathbb{R})$,

$$\begin{aligned} &\|\psi(|k^{-1}D|)\gamma_{\tilde{\Gamma}}^*\|_{L^2(\tilde{\Gamma}) \rightarrow H^s(\mathbb{R}^d)} + \|\gamma_{\tilde{\Gamma}}\psi(|k^{-1}D|)\|_{H^{-s}(\mathbb{R}^d) \rightarrow \overline{H^{-s-1/2}}(\tilde{\Gamma})} \\ &\leq C \begin{cases} 1 & s < -1/2 \\ (\log k)^{1/2} & s = -1/2 \\ k^{(s+1/2)} & s > -1/2. \end{cases} \end{aligned}$$

Thus, since $\psi(|k^{-1}D|) : \mathcal{S}'(\mathbb{R}^d) \rightarrow C^\infty(\mathbb{R}^d)$ and in particular, $\gamma\psi(|k^{-1}D|) : \mathcal{S}'(\mathbb{R}^d) \rightarrow \overline{H^1}(\Gamma)$,

$$\begin{aligned} &\|\psi(|k^{-1}D|)\gamma^*\|_{L^2(\Gamma) \rightarrow H^s(\mathbb{R}^d)} + \|\gamma\psi(|k^{-1}D|)\|_{H^{-s}(\mathbb{R}^d) \rightarrow \overline{H^{-s-1/2}}(\Gamma)} \\ &\leq C \begin{cases} 1 & s < -1/2 \\ (\log k)^{1/2} & s = -1/2 \\ k^{(s+1/2)} & s > -1/2. \end{cases} \quad (2.54) \end{aligned}$$

Furthermore, notice that by Lemma 2.18, if $\psi_1 \in C_c^\infty(\mathbb{R})$ has $\psi_1 \equiv 1$ on $\text{supp } \psi$, then

$$\chi R_0(1)\chi\psi(|k^{-1}D|) = \psi_1(|k^{-1}D|)\chi R_0(1)\chi\psi(|k^{-1}D|) + \mathcal{O}_{\Psi^{-\infty}}(k^{-\infty}).$$

In particular, using this estimate together with (2.54) and that $\chi R_0(1)\chi : H^s(\mathbb{R}^d) \rightarrow H^{s+2}(\mathbb{R}^d)$,

$$\begin{aligned} \gamma^\pm R_0(1)\chi\psi(|k^{-1}D|)\gamma^* &= \begin{cases} \mathcal{O}_{L^2(\Gamma) \rightarrow \overline{H^1}(\Gamma)}((\log k)^{1/2}), \\ \mathcal{O}_{L^2(\Gamma) \rightarrow L^2(\Gamma)}(1), \end{cases} \\ \gamma^\pm R_0(1)\chi\psi(|k^{-1}D|)L^*\gamma^* &= \begin{cases} \mathcal{O}_{L^2(\Gamma) \rightarrow \overline{H^1}(\Gamma)}(k), \\ \mathcal{O}_{L^2(\Gamma) \rightarrow L^2(\Gamma)}(\log k), \end{cases} \\ \gamma^\pm LR_0(1)\chi\psi(|k^{-1}D|)\gamma^* &= \begin{cases} \mathcal{O}_{L^2(\Gamma) \rightarrow \overline{H^1}(\Gamma)}(k), \\ \mathcal{O}_{L^2(\Gamma) \rightarrow L^2(\Gamma)}(\log k). \end{cases} \end{aligned}$$

Hence,

$$\begin{aligned} \gamma R_0(k)\chi(1 - \psi(|k^{-1}D|))\gamma^* \\ = \gamma R_0(1)\chi(1 - \psi(|k^{-1}D|))\gamma^* + B_k = \mathcal{O}_{L^2 \rightarrow \overline{H^1}}((\log k)^{1/2}). \end{aligned}$$

Furthermore, since $R_0(k)\chi(1 - \psi(|k^{-1}D|)) \in k^{-2}\Psi^{-2}(\mathbb{R}^d)$, and we have (2.53),

$$\gamma R_0(k)\chi(1 - \psi(|k^{-1}D|))\gamma^* = \mathcal{O}_{L^2 \rightarrow L^2}(k^{-1}). \quad (2.55)$$

Next, observe that

$$\begin{aligned} \mp \frac{1}{2} + \gamma^\pm R_0(k)\chi(1 - \psi(|k^{-1}D|))L^*\gamma^* \\ = \mp \frac{1}{2} + \gamma^\pm R_0(1)\chi(1 - \psi(|k^{-1}D|))L^*\gamma^* + C_k \\ = \begin{cases} \mathcal{O}_{L^2 \rightarrow \overline{H^1}}(k) \\ \mathcal{O}_{L^2 \rightarrow L^2}(\log k), \end{cases} \end{aligned} \quad (2.56)$$

$$\begin{aligned} \pm \frac{1}{2} + \gamma^\pm LR_0(k)\chi(1 - \psi(|k^{-1}D|))\gamma^* \\ = \pm \frac{1}{2} + \gamma^\pm LR_0(1)\chi(1 - \psi(|k^{-1}D|))\gamma^* + C'_k \\ = \begin{cases} \mathcal{O}_{L^2 \rightarrow \overline{H^1}}(k) \\ \mathcal{O}_{L^2 \rightarrow L^2}(\log k). \end{cases} \end{aligned} \quad (2.57)$$

Since $\partial\Omega$ is piecewise smooth, $\partial\Omega = \sum_{i=1}^N \Gamma_i$. Applying (2.55)–(2.57) with $\Gamma = \Gamma_i$, summing over i , and then using the result (2.4) we obtain (2.50)–(2.52). \square

Proof of Parts (a) and (b) of Theorem 2.10. This follows from combining the low-frequency estimates (2.35)–(2.37) in Lemma 2.17 with the high-frequency estimates (2.50)–(2.52) in Lemma 2.20, recalling the decompositions (2.13) and (2.18). \square

2.5. Proof of Part (c) of Theorem 2.10

Proof of Part (c) of Theorem 2.10. Observe that [24, Theorems 4.29, 4.32] imply that for $\psi \in C_c^\infty(\mathbb{R})$ with $\psi \equiv 1$ on $[-2, 2]$,

$$\psi(|k^{-1}D'|)S_k = \mathcal{O}_{L^2 \rightarrow H_k^1}(k^{-2/3}), \quad \psi(|k^{-1}D'|)D_k = \mathcal{O}_{L^2 \rightarrow H_k^1}(1).$$

Then [24, Lemma 4.25] shows that $(1 - \psi(|k^{-1}D'|))S_k \in k^{-1}\Psi^{-1}(\partial\Omega)$ and $(1 - \psi(|k^{-1}D'|))D_k \in k^{-1}\Psi^{-1}(\partial\Omega)$ and hence

$$(1 - \psi(|k^{-1}D'|))S_k = \mathcal{O}_{L^2 \rightarrow H_k^1}(k^{-1}), \quad (1 - \psi(|k^{-1}D'|))D_k = \mathcal{O}_{L^2 \rightarrow H_k^1}(k^{-1}).$$

An identical analysis shows that

$$D'_k = \mathcal{O}_{L^2 \rightarrow H_k^1}(1). \quad \square$$

2.6. Proof of Corollary 1.9

This follows in exactly same way as [29, Proof of Corollary 1.2, p. 193]. The two ideas are that (i) the relationships

$$\int_{\Gamma} \phi S_k \psi \, ds = \int_{\Gamma} \psi S_k \phi \, ds, \quad \text{and} \quad \int_{\Gamma} \phi D_k \psi \, ds = \int_{\Gamma} \psi D'_k \phi \, ds,$$

for $\phi, \psi \in L^2(\partial\Omega)$ (see, e.g., [7, Equation 2.37]), and the duality of $H^1(\partial\Omega)$ and $H^{-1}(\partial\Omega)$ (see, e.g., [39, p. 98]) allow us to convert bounds on $S_k, D_k,$ and D'_k as mappings from $L^2(\partial\Omega) \rightarrow H^1(\partial\Omega)$ into bounds on these operators as mappings from $H^{-1}(\partial\Omega) \rightarrow L^2(\partial\Omega)$; and

(ii) interpolation then allows us to obtain bounds from $H^{s-1/2}(\partial\Omega) \rightarrow H^{s+1/2}(\partial\Omega)$ for $|s| \leq 1/2$.

Remark 2.22. (Using the triangle inequality on $\|D'_k - i\eta S_k\|_{L^2(\partial\Omega) \rightarrow H^1(\partial\Omega)}$). As explained in Sect. 1.2, the motivation for proving the $L^2(\partial\Omega) \rightarrow H^1(\partial\Omega)$ bounds on $S_k, D_k,$ and D'_k is so that they can be used to estimate (in a k -explicit way) the smoothing power of $D'_k - i\eta S_k$ in the analysis of the Galerkin method via the classic compact-perturbation argument (see [25, Proof of Theorem 1.10]). We now show that we do not lose anything, from the point of view of k -dependence, by using the triangle inequality $\|D'_k - i\eta S_k\|_{L^2(\partial\Omega) \rightarrow H^1(\partial\Omega)} \leq \|D'_k\|_{L^2(\partial\Omega) \rightarrow H^1(\partial\Omega)} + |\eta| \|S_k\|_{L^2(\partial\Omega) \rightarrow H^1(\partial\Omega)}$. As a consequence, therefore, the bounds on $\|D'_k - i\eta S_k\|_{L^2(\partial\Omega) \rightarrow H^1(\partial\Omega)}$ obtained from using the bounds on $\|D_k\|_{L^2(\partial\Omega) \rightarrow H^1(\partial\Omega)}$ and $\|S_k\|_{L^2(\partial\Omega) \rightarrow H^1(\partial\Omega)}$ in Theorem 1.5 are sharp.

First, recall that D'_k and S_k have wavefront set relation given by the billiard ball relation (see for example [24, Chapter 4]). Let $B^*\partial\Omega$ and $S^*\partial\Omega$ denote respectively the unit coball and cosphere bundles in $\partial\Omega$. That is,

$$B^*\partial\Omega := \{(x, \xi) \in T^*\partial\Omega : |\xi|_{g(x)} < 1\}, \quad S^*\partial\Omega := \{(x, \xi) \in T^*\partial\Omega : |\xi|_{g(x)} = 1\}.$$

Denote the relation by $C_\beta \subset \overline{B^*\partial\Omega} \times \overline{S^*\partial\Omega}$ i.e.

$$C_\beta = \{(x, \xi, y, \eta) : (x, \xi) = \beta(y, \eta)\}$$

where β is the billiard ball map (see Fig. 1). To see that the optimal bound in terms of powers of k for $\|D'_k - i\eta S_k\|_{L^2(\partial\Omega) \rightarrow H^1(\partial\Omega)}$ is equal to that for $\|D'_k\|_{L^2(\partial\Omega) \rightarrow H^1(\partial\Omega)} + |\eta| \|S_k\|_{L^2(\partial\Omega) \rightarrow H^1(\partial\Omega)}$, observe that the largest norm

for S_k corresponds microlocally to points $(q_1, q_2) \in C_\beta \cap (S^* \partial \Omega \times S^* \partial \Omega)$ (i.e. “glancing” to “glancing”). On the other hand, these points are damped (relative to the worst bounds) for D'_k . In particular, microlocally near such points, one expects that

$$\|D'_k f_{q_2}\|_{H^1(\partial \Omega)} \leq Ck, \quad \|S_k f_{q_2}\|_{H^1(\partial \Omega)} \geq \begin{cases} Ck^{1/2}, & \partial \Omega \text{ flat,} \\ Ck^{1/3}, & \partial \Omega \text{ curved,} \end{cases}$$

where $\|f_{q_2}\|_{L^2(\partial \Omega)} = 1$ and f_{q_2} is microlocalized near q_2 .

The norm for D'_k is maximized microlocally near $(p_1, p_2) \in C_\beta \cap (S^* \partial \Omega \times B^* \partial \Omega)$ (i.e. “transversal” to “glancing”), but near these points, the norm of S_k is damped relative to its worst bound. In particular, microlocally near (p_1, p_2) , one expects

$$\|D'_k f_{p_2}\|_{H^1(\partial \Omega)} \geq \begin{cases} Ck^{5/4}, & \partial \Omega \text{ flat,} \\ Ck^{7/6}, & \partial \Omega \text{ curved,} \end{cases}$$

$$\|S_k f_{p_2}\|_{H^1(\partial \Omega)} \leq \begin{cases} Ck^{1/4}, & \partial \Omega \text{ flat,} \\ Ck^{1/6}, & \partial \Omega \text{ curved,} \end{cases}$$

where $\|f_{p_2}\|_{L^2(\partial \Omega)} = 1$ and f_{p_2} is microlocalized near p_2 . Therefore, even if $|\eta|$ is chosen so that $\|D'_k\|_{L^2(\partial \Omega) \rightarrow H^1(\partial \Omega)} \sim |\eta| \|S_k\|_{L^2(\partial \Omega) \rightarrow H^1(\partial \Omega)}$, this analysis shows that there cannot be cancellation since the worst norms occur at different points of phase space.

3. Sharpness of the Bounds in Theorem 1.5

We now prove that the powers of k in the $\|S_k\|_{L^2(\partial \Omega) \rightarrow H^1(\partial \Omega)}$ bounds in Theorem 2.10 are optimal. The analysis in [31, §A.3] proves that the powers of k in the $\|D_k\|_{L^2(\partial \Omega) \rightarrow L^2(\partial \Omega)}$ bounds are optimal, but can be adapted in a similar way to below to prove the sharpness of the $\|D_k\|_{L^2(\partial \Omega) \rightarrow H^1(\partial \Omega)}$ bounds.

In this section we write $x \in \mathbb{R}^d$ as $x = (x', x_d)$ for $x' \in \mathbb{R}^{d-1}$, and $x' = (x_1, x'')$ (in the case $d = 2$, the x'' variable is superfluous).

Lemma 3.1. *(Lower bound on $\|S_k\|_{L^2(\partial \Omega) \rightarrow H^1(\partial \Omega)}$ when $\partial \Omega$ contains a line segment). If $\partial \Omega$ contains the set*

$$\{(x_1, 0) : |x_1| < \delta\}$$

for some $\delta > 0$ and is C^2 in a neighborhood thereof (i.e. $\partial \Omega$ contains a line segment), then there exists $k_0 > 0$ and $C > 0$ independent of k , such that, for all $k \geq k_0$,

$$\|S_k\|_{L^2(\partial \Omega) \rightarrow L^2(\partial \Omega)} \geq Ck^{-1/2} \quad \text{and} \quad \|S_k\|_{L^2(\partial \Omega) \rightarrow H^1(\partial \Omega)} \geq Ck^{1/2}.$$

Lemma 3.1 shows that the bound (1.4), when $\partial \Omega$ is piecewise smooth, is sharp up to a factor of $\log k$.

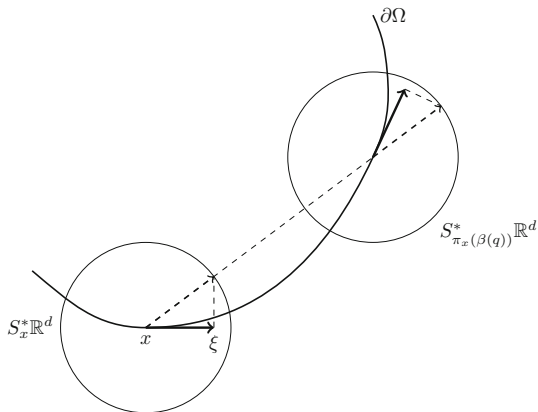


FIGURE 1. A recap of the billiard ball map. Let $q = (x, \xi) \in B^* \partial \Omega$ (the unit ball in the cotangent bundle of $\partial \Omega$). The solid black arrow on the left denotes the covector $\xi \in B_x^* \partial \Omega$, with the dashed arrow denoting the unique inward-pointing unit vector whose tangential component is ξ . The dashed arrow on the right is the continuation of the dashed arrow on the left, and the solid black arrow on the right is $\xi(\beta(q)) \in B_{\pi_x(\beta(q))}^* \partial \Omega$. The center of the left circle is x and that of the right is $\pi_x(\beta(q))$. If this process is repeated, then the dashed arrow on the right is reflected in the tangent plane at $\pi_x(\beta(q))$: the standard “angle of incidence equals angle of reflection” rule

Lemma 3.2. (General lower bound on $\|S_k\|_{L^2(\partial \Omega) \rightarrow H^1(\partial \Omega)}$) *If $\partial \Omega$ is C^2 in a neighborhood of a point then there exists $k_0 > 0$ and $C > 0$ (independent of k), such that, for all $k \geq k_0$,*

$$\|S_k\|_{L^2(\partial \Omega) \rightarrow L^2(\partial \Omega)} \geq Ck^{-2/3} \quad \text{and} \quad \|S_k\|_{L^2(\partial \Omega) \rightarrow H^1(\partial \Omega)} \geq Ck^{1/3}.$$

Lemma 3.2 shows that the bound (1.5), when $\partial \Omega$ is piecewise curved, is sharp up to a factor of $\log k$ and that the bound (1.6), when $\partial \Omega$ is smooth and curved, is sharp.

Remark 3.3. The lower bound $\|S_k\|_{\dot{H}^s(\Gamma) \rightarrow H^{s+1}(\Gamma)} \geq Ck^{1/2}$ when Γ is a flat screen (i.e. a bounded and relatively open subset of $\{x \in \mathbb{R}^d : x_d = 0\}$) and $s \in \mathbb{R}$ is proved in [8, Remark 4.2] (recall that $\dot{H}^s(\Gamma)$ is defined in Definition 2.5).

Proof of Lemma 3.1. By assumption $\Gamma \subset \partial \Omega$, where

$$\Gamma := \left\{ (x_1, x'', \gamma(x')) : |x'| < \delta \right\}$$

for some $\gamma(x')$ with $\gamma(x_1, 0) = 0$ for $|x_1| < \delta$ (since the line segment $\{(x_1, 0) : |x_1| < \delta\} \subset \Gamma$).

By the definition of the operator norm, it is sufficient to prove that there exists $u_k \in L^2(\partial \Omega)$ with $\text{supp } u_k \subset \Gamma$, $k_0 > 0$, and $C > 0$ (independent of k),

such that, for all $k \geq k_0$,

$$\|S_k u_k\|_{L^2(\Gamma)} \geq Ck^{-1/2}\|u\|_{L^2(\Gamma)} \quad \text{and} \quad \|\partial_{x_1} S_k u_k\|_{L^2(\Gamma)} \geq Ck^{1/2}\|u\|_{L^2(\Gamma)}. \quad (3.1)$$

We begin by observing that the definition of $\Phi_k(x, y)$ (1.1) and the asymptotics of Hankel functions for large argument and fixed order (see, e.g., [46, §10.17]) imply that

$$\begin{aligned} \Phi_k(x, y) &= C_d k^{d-2} e^{ik|x-y|} \left((k|x-y|)^{-(d-1)/2} + \mathcal{O}\left((k|x-y|)^{-(d+1)/2}\right) \right), \quad (3.2) \\ \langle V, \partial_x \rangle \Phi_k(x, y) &= C'_d k^{d-1} \frac{\langle V, x-y \rangle}{|x-y|} e^{ik|x-y|} \left((k|x-y|)^{-(d-1)/2} \right. \\ &\quad \left. + \mathcal{O}\left((k|x-y|)^{-(d+1)/2}\right) \right). \quad (3.3) \end{aligned}$$

Let $\chi \in C_c^\infty(\mathbb{R})$ with $\text{supp } \chi \subset [-2, 2]$, $\chi(0) \equiv 1$ on $[-1, 1]$ and define

$$\chi_{\epsilon, \gamma_1, \gamma_2}(x') = \chi(\epsilon^{-1} k^{\gamma_1} x_1) \chi(\epsilon^{-1} k^{\gamma_2} |x''|), \quad (3.4)$$

In what follows, we suppress the dependence of u on k for convenience. Let $u(x', \gamma(x')) := e^{ikx_1} \chi_{\epsilon, 0, 1/2}(x')$. The definition of χ implies that

$$\text{supp } u = \left\{ (x', \gamma(x')) : |x_1| \leq 2\epsilon, |x''| \leq 2\epsilon k^{-1/2} \right\},$$

and thus $\text{supp } u \subset \Gamma$ for ϵ sufficiently small and k sufficiently large (say $\epsilon < (2\sqrt{2})^{-1}\delta$ and $k > 1$); for the rest of the proof we assume that ϵ and k are such that this is the case. Observe also that

$$\|u\|_{L^2(\Gamma)} \sim C_\epsilon k^{-(d-2)/4}. \quad (3.5)$$

Let

$$U := \left\{ (x', \gamma(x')) : M\epsilon \leq x_1 \leq 2M\epsilon, |x''| \leq \epsilon k^{-1/2}, \quad M \gg 1 \right\};$$

the motivation for this choice comes from the analysis in Remark 2.22 below. Indeed, we know that S_k is largest microlocally near points that are glancing in both the incoming and outgoing variables. Since u concentrates microlocally at $x = 0$, $\xi = (1, 0)$ up to scale $k^{-1/2}$, the billiard trajectory emanating from this point is $\{t(1, 0) : t > 0\}$. This ray is always glancing since Γ is flat. Therefore, we choose U to contain this ray up to scale $k^{-1/2}$.

Then for $x \in U$, $y \in \text{supp } u$,

$$|(x', \gamma(x')) - (y', \gamma(y'))|^2 = (x_1 - y_1)^2 + |x'' - y''|^2 + |\gamma(x') - \gamma(y')|^2,$$

Then, observe that by Taylor's formula

$$\begin{aligned} \gamma(x') - \gamma(y') &= \gamma(x_1, 0) - \gamma(y_1, 0) + \partial_{x''} \gamma(x_1, 0)(x'' - y'') \\ &\quad + y'' (\partial_{x''} \gamma(x_1, 0) - \partial_{x''} \gamma(y_1, 0)) + \mathcal{O}(|x''|^2 + |y''|^2). \end{aligned}$$

Since $\gamma(x_1, 0) = 0$ for $|x_1| < \delta$,

$$|\gamma(x') - \gamma(y')|^2 = \mathcal{O}(|x'' - y''|^2) + \mathcal{O}(|x''|^2 + |y''|^2).$$

In particular,

$$\begin{aligned} |(x', \gamma(x')) - (y', \gamma(y'))| &= (x_1 - y_1) \\ &\quad + \mathcal{O}\left(\left(|x'' - y''|^2 + |x''|^2 + |y''|^2\right)|x_1 - y_1|^{-1}\right) \\ &= x_1 - y_1 + \mathcal{O}(k^{-1}M^{-1}\epsilon), \end{aligned} \tag{3.6}$$

$$= x_1 \left(1 + \mathcal{O}(M^{-1}) + \mathcal{O}(k^{-1}M^{-2})\right). \tag{3.7}$$

We have from the Hankel-function asymptotics (3.2) and the definition of u that, for $x \in U$,

$$\begin{aligned} S_k u(x) &= C_d k^{d-2} \int_{\Gamma} e^{ik|x-y|+iky_1} \left(k^{-(d-1)/2}|x-y|^{-(d-1)/2}\right. \\ &\quad \left.+ \mathcal{O}\left((k|x-y|)^{-(d+1)/2}\right)\right) \chi_{\epsilon,0,1,2}(y') ds(y), \end{aligned}$$

and then using the asymptotics (3.6) in the exponent of the integrand and the asymptotics (3.7) in the rest of the integrand, we have, for $x \in U$,

$$\begin{aligned} S_k u(x) &= C_d k^{d-2} \frac{e^{ikx_1}}{k^{(d-1)/2}|x_1|^{(d-1)/2}} \\ &\quad \times \int_{\Gamma} \left(1 + \mathcal{O}(M^{-1}\epsilon)\right) \left(1 + \mathcal{O}(M^{-1}) + \mathcal{O}_{\epsilon,M}(k^{-1})\right) \chi_{\epsilon,0,1/2}(y') ds(y). \end{aligned}$$

Therefore, with M large enough, ϵ small enough, and then k_0 large enough, the contribution from the integral over Γ is determined by the cutoff $\chi_{\epsilon,0,1/2}$, yielding $k^{-(d-2)/2}$, and thus

$$|S_k u(x')| \geq C k^{(d-2)/2} \frac{1}{k^{(d-1)/2}|x_1|^{(d-1)/2}}, \quad x' \in U, k \geq k_0. \tag{3.8}$$

In the step of taking ϵ sufficiently small, we can also take ϵ small enough to ensure that $U \subset \Gamma$ for all $k \geq 1$. Using (3.8), along with the fact that the measure of $U \sim k^{-(d-2)/2}$, we have that

$$\|S_k u\|_{L^2(U)} \geq C k^{-1/2-(d-2)/4}. \tag{3.9}$$

Since we have ensured that $U \subset \Gamma$, (3.9) and (3.5) imply that the first bound in (3.1) holds. It easy to see that if we repeat the argument above but with (3.3) instead of (3.2), then we obtain the second bound in (3.1). \square

Proof of Lemma 3.2. Let $x_0 \in \partial\Omega$ be a point so that $\partial\Omega$ is C^2 in a neighborhood of x_0 and let x' be coordinates near x_0 so that

$$\Gamma := \left\{ (x', \gamma(x')) : |x'| < \delta \right\} \subset \partial\Omega, \quad \text{with } \gamma \in C^2, \gamma(0) = \partial\gamma(0) = 0.$$

Similar to the proof of Lemma 3.1, it is sufficient to prove that there exists $u_k \in L^2(\partial\Omega)$ with $\text{supp } u_k \subset \Gamma$, $k_0 > 0$, and $C > 0$ (independent of k), such that

$$\|S_k u_k\|_{L^2(\Gamma)} \geq C k^{-1/2} \|u_k\|_{L^2(\Gamma)} \quad \text{and} \quad \|\partial_{x_1} S_k u\|_{L^2(\Gamma)} \geq C k^{1/2} \|u\|_{L^2(\Gamma)} \tag{3.10}$$

for all $k \geq k_0$.

The idea in the curved case is the same as in the flat case: choose u concentrating as close as possible to a glancing point and measure near the point given by the billiard map. More practically, this amounts to ensuring that $|x - y|$ looks like $x_1 - y_1$ modulo terms that are much smaller than k^{-1} . The fact that Γ may be curved will force us to choose u differently and cause our estimates to be worse than in the flat case (leading to the weaker but still sharp lower bound).

With $\chi_{\epsilon, \gamma_1, \gamma_2}$ defined by (3.4), let $u(x', \gamma(x')) := e^{ikx_1} \chi_{\epsilon, 1/3, 2/3}(x')$ where, as in the proof of Lemma 3.1, we have $x' = (x_1, x'')$ and as in Lemma 3.1, $\text{supp } u \subset \Gamma$ for ϵ sufficiently small and k sufficiently large, and for the rest of the proof we assume that this is the case. Then

$$\|u\|_{L^2(\Gamma)} \leq C_\epsilon k^{-1/6} k^{-(d-2)/3}. \quad (3.11)$$

Define

$$U := \left\{ (x', \gamma(x')) : M\epsilon k^{-1/3} \leq x_1 \leq 2M\epsilon k^{-1/3}, |x''| \leq \epsilon k^{-2/3}, M \gg 1 \right\}.$$

Then, for $y \in \text{supp } u$ and $x \in U$,

$$\begin{aligned} & |(x', \gamma(x')) - (y', \gamma(y'))| \\ &= (x_1 - y_1) + \mathcal{O}(|x'|^2 + |y'|^2 |x_1 - y_1|^{-1}) + \mathcal{O}(|x'' - y''|^2 |x_1 - y_1|^{-1}) \\ &= x_1 - y_1 + \mathcal{O}(k^{-1} M^3 \epsilon^3) + \mathcal{O}(\epsilon k^{-1} M^{-1}) \end{aligned} \quad (3.12)$$

$$= x_1 \left(1 + \mathcal{O}(M^{-1}) + \mathcal{O}(k^{-2/3} M^2 \epsilon^2) + \mathcal{O}(k^{-2/3} M^{-2}) \right). \quad (3.13)$$

From (3.2) and the definition of u , we have for $x' \in U$,

$$\begin{aligned} S_k u(x) &= C_d k^{d-2} \int_{\Gamma} e^{ik|x-y|+iky_1} \left(k^{-(d-1)/2} |x-y|^{-(d-1)/2} \right. \\ &\quad \left. + \mathcal{O}\left((k|x-y|)^{-(d+1)/2} \right) \right) \chi_{\epsilon, 1/3, 2/3}(y') ds(y), \end{aligned}$$

and then, using (3.12) in the exponent of the integrand and (3.13) in the rest, we have, for $x' \in U$,

$$\begin{aligned} S_k u(x) &= C_d k^{d-2} \frac{e^{ikx_1}}{k^{(d-1)/2} |x_1|^{(d-1)/2}} \\ &\quad \times \int_{\Gamma} \left(1 + \mathcal{O}(M^3 \epsilon^3) + \mathcal{O}(M^{-1} \epsilon) \right) \\ &\quad \times \left(1 + \mathcal{O}(M^{-1}) + \mathcal{O}_{\epsilon, M}(k^{-2/3}) \right) \chi_{\epsilon, 1/3, 2/3}(y') ds(y). \end{aligned}$$

Thus, fixing M large enough, then ϵ small enough, then k_0 large enough, we have

$$|S_k u(x')| \geq C k^{(d-2)/3} \frac{1}{k^{(d-1)/2} |x_1|^{(d-1)/2}} k^{-1/3}, \quad x' \in U, k \geq k_0. \quad (3.14)$$

In the step of taking ϵ sufficiently small, we can also take ϵ small enough so that when $x' \in U$, $|x'| < \delta$, and thus $x' \in \Gamma$. Using the lower bound (3.14), and the fact that the measure of $U \sim k^{-1/3} k^{-2(d-2)/3}$, we have that

$$\|S_k u\|_{L^2(\Gamma)} \geq \|S_k u\|_{L^2(U)} \geq C k^{-2/3-1/6-(d-2)/3},$$

and so using (3.11) we obtain the first bound in (3.10). Similar to before, if we repeat this argument with (3.3) instead of (3.2), we find the second bound in (3.10). \square

Acknowledgements

JG thanks the US National Science Foundation for support under the Mathematical Sciences Postdoctoral Research Fellowship DMS-1502661. EAS thanks the UK Engineering and Physical Sciences Research Council for support under Grant EP/R005591/1.

Open Access. This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

- [1] Anand, A., Boubendir, Y., Ecevit, F., Reitich, F.: Analysis of multiple scattering iterations for high-frequency scattering problems. II: the three-dimensional scalar case. *Numerische Mathematik* **114**(3), 373–427 (2010)
- [2] Asheim, A., Huybrechs, D.: Extraction of uniformly accurate phase functions across smooth shadow boundaries in high frequency scattering problems. *SIAM J. Appl. Math.* **74**(2), 454–476 (2014)
- [3] Atkinson, K.E.: *The Numerical Solution of Integral Equations of the Second Kind*. Cambridge Monographs on Applied and Computational Mathematics (1997)
- [4] Baskin, D., Spence, E.A., Wunsch, J.: Sharp high-frequency estimates for the Helmholtz equation and applications to boundary integral equations. *SIAM J. Math. Anal.* **48**(1), 229–267 (2016)
- [5] Burq, N., Gérard, P., Tzvetkov, N.: Restrictions of the Laplace–Beltrami eigenfunctions to submanifolds. *Duke Math. J.* **138**(3), 445–486 (2007)
- [6] Chandler-Wilde, S.N., Graham, I.G., Langdon, S., Lindner, M.: Condition number estimates for combined potential boundary integral operators in acoustic scattering. *J. Integral Equ. Appl.* **21**(2), 229–279 (2009)
- [7] Chandler-Wilde, S.N., Graham, I.G., Langdon, S., Spence, E.A.: Numerical-asymptotic boundary integral methods in high-frequency acoustic scattering. *Acta Numerica* **21**(1), 89–305 (2012)
- [8] Chandler-Wilde, S.N., Hewett, D.P.: Wavenumber-explicit continuity and coercivity estimates in acoustic scattering by planar screens. *Integral Equ. Oper. Theory* **82**(3), 423–449 (2015)
- [9] Chandler-Wilde, S.N., Hewett, D.P., Langdon, S., Twigger, A.: A high frequency boundary element method for scattering by a class of nonconvex obstacles. *Numerische Mathematik* **129**(4), 647–689 (2015)

- [10] Chandler-Wilde, S.N., Hewett, D.P., Moiola, A.: Sobolev spaces on non-Lipschitz subsets of \mathbb{R}^n with application to boundary integral equations on fractal screens. *Integral Equ. Oper. Theory* **87**(2), 179–224 (2017)
- [11] Chandler-Wilde, S.N., Langdon, S.: A Galerkin boundary element method for high frequency scattering by convex polygons. *SIAM J. Numer. Anal.* **45**(2), 610–640 (2007)
- [12] Chandler-Wilde, S.N., Monk, P.: Wave-number-explicit bounds in time-harmonic scattering. *SIAM J. Math. Anal.* **39**(5), 1428–1455 (2008)
- [13] Chandler-Wilde, S.N., Spence, E.A., Gibbs, A., Smyshlyaev, V.P.: High-frequency bounds for the Helmholtz equation under parabolic trapping and applications in numerical analysis. (2017) arXiv preprint. [arXiv:1708.08415](https://arxiv.org/abs/1708.08415)
- [14] Christianson, H., Hassell, A., Toth, J.A.: Exterior Mass estimates and L^2 -restriction bounds for Neumann data along hypersurfaces. *Int. Math. Res. Notices* **6**, 1638–1665 (2015)
- [15] Colton, D., Kress, R.: *Inverse Acoustic and Electromagnetic Scattering Theory*. Springer (1998)
- [16] Costabel, M.: Boundary integral operators on Lipschitz domains: elementary results. *SIAM J. Math. Anal.* **19**, 613–626 (1988)
- [17] Domínguez, V., Graham, I.G., Smyshlyaev, V.P.: A hybrid numerical-asymptotic boundary integral method for high-frequency acoustic scattering. *Numerische Mathematik* **106**(3), 471–510 (2007)
- [18] Dyatlov, S., Zworski, M.: *Mathematical theory of scattering resonances*. Book in progress (2018). <http://math.mit.edu/~dyatlov/res/>
- [19] Ecevit, F.: Frequency independent solvability of surface scattering problems. *Turkish J. Math.* **42**(2), 407–417 (2018)
- [20] Ecevit, F., Eruslu, H.H.: A Galerkin BEM for high-frequency scattering problems based on frequency-dependent changes of variables. *IMA J. Numer. Anal.* to appear (2018)
- [21] Ecevit, F., Özen, H.Ç.: Frequency-adapted galerkin boundary element methods for convex scattering problems. *Numerische Mathematik* **135**(1), 27–71 (2017)
- [22] Ecevit, F., Reitich, F.: Analysis of multiple scattering iterations for high-frequency scattering problems. Part I: the two-dimensional case. *Numerische Mathematik* **114**, 271–354 (2009)
- [23] Fabes, E.B., Jodeit, M., Riviere, N.M.: Potential techniques for boundary value problems on C^1 domains. *Acta Mathematica* **141**(1), 165–186 (1978)
- [24] Galkowski, J.: Distribution of resonances in scattering by thin barriers. arXiv preprint. [arXiv:1404.3709](https://arxiv.org/abs/1404.3709) (to appear in *Memoirs of the AMS*) (2014)
- [25] Galkowski, J., Müller, E.H., Spence, E.A.: Wavenumber-explicit analysis for the Helmholtz h -BEM: error estimates and iteration counts for the Dirichlet problem (2017). [arXiv:1608.01035](https://arxiv.org/abs/1608.01035)
- [26] Galkowski, J., Smith, H.F.: Restriction bounds for the free resolvent and resonances in lossy scattering. *Int. Math. Res. Notices* **16**, 7473–7509 (2015)
- [27] Gander, M.J., Graham, I.G., Spence, E.A.: Applying GMRES to the Helmholtz equation with shifted Laplacian preconditioning: What is the largest shift for which wavenumber-independent convergence is guaranteed? *Numerische Mathematik* **131**(3), 567–614 (2015)

- [28] Ganesh, M., Hawkins, S.: A fully discrete Galerkin method for high frequency exterior acoustic scattering in three dimensions. *J. Comput. Phys.* **230**, 104–125 (2011)
- [29] Graham, I.G., Löhndorf, M., Melenk, J.M., Spence, E.A.: When is the error in the h -BEM for solving the Helmholtz equation bounded independently of k ? *BIT Numer. Math.* **55**(1), 171–214 (2015)
- [30] Greenleaf, A., Seeger, A.: Fourier integral operators with fold singularities. *J. Reine Angew. Math.* **455**, 35–56 (1994)
- [31] Han, X., Tacy, M.: Sharp norm estimates of layer potentials and operators at high frequency. *J. Funct. Anal.* **269**(9), 2890–2926 (2015). with an appendix by J. Galkowski
- [32] Hassell, A., Tacy, M.: Semiclassical L^p estimates of quasimodes on curved hypersurfaces. *J. Geom. Anal.* **22**(1), 74–89 (2012)
- [33] Hewett, D.P.: Shadow boundary effects in hybrid numerical-asymptotic methods for high-frequency scattering. *Eur. J. Appl. Math.* **26**(05), 773–793 (2015)
- [34] Hewett, D.P., Langdon, S., Chandler-Wilde, S.N.: A frequency-independent boundary element method for scattering by two-dimensional screens and apertures. *IMA J. Numer. Anal.* **35**(4), 1698–1728 (2014)
- [35] Hewett, D.P., Langdon, S., Melenk, J.M.: A high frequency hp boundary element method for scattering by convex polygons. *SIAM J. Numer. Anal.* **51**(1), 629–653 (2013)
- [36] Hörmander, L.: *The Analysis of Linear Partial Differential Operators III: Pseudo-Differential Operators*. Springer (1985)
- [37] Ikawa, M.: Decay of solutions of the wave equation in the exterior of several convex bodies. *Ann. Inst. Fourier* **38**(2), 113–146 (1988)
- [38] Kirsch, A.: Surface gradients and continuity properties for some integral operators in classical scattering theory. *Math. Methods Appl. Sci.* **11**(6), 789–804 (1989)
- [39] McLean, W.C.H.: *Strongly Elliptic Systems and Boundary Integral Equations*. Cambridge University Press, Cambridge (2000)
- [40] Melrose, R.B., Sjöstrand, J.: Singularities of boundary value problems. II. *Commun. Pure Appl. Math.* **35**(2), 129–168 (1982)
- [41] Melrose, R.B., Taylor, M.E.: Near peak scattering and the corrected Kirchhoff approximation for a convex obstacle. *Adv. Math.* **55**(3), 242–315 (1985)
- [42] Meyer, Y., Coifman, R.: *Wavelets: Calderón-Zygmund and Multilinear Operators*. Cambridge University Press, Cambridge (2000)
- [43] Mitrea, M., Taylor, M.: Boundary layer methods for Lipschitz domains in Riemannian manifolds. *J. Funct. Anal.* **163**(2), 181–251 (1999)
- [44] Moiola, A., Spence, E.A.: Is the Helmholtz equation really sign-indefinite? *SIAM Rev.* **56**(2), 274–312 (2014)
- [45] Morawetz, C.S.: Decay for solutions of the exterior problem for the wave equation. *Commun. Pure Appl. Math.* **28**(2), 229–264 (1975)
- [46] NIST. Digital Library of Mathematical Functions (2018). <http://dlmf.nist.gov/>
- [47] Sauter, S.A., Schwab, C.: *Boundary Element Methods*. Springer, Berlin (2011)
- [48] Spence, E.A.: Wavenumber-explicit bounds in time-harmonic acoustic scattering. *SIAM J. Math. Anal.* **46**(4), 2987–3024 (2014)

- [49] Spence, E.A., Chandler-Wilde, S.N., Graham, I.G., Smyshlyaev, V.P.: A new frequency-uniform coercive boundary integral equation for acoustic scattering. *Commun. Pure Appl. Math.* **64**(10), 1384–1415 (2011)
- [50] Spence, E.A., Kamotski, I.V., Smyshlyaev, V.P.: Coercivity of combined boundary integral equations in high-frequency scattering. *Commun. Pure Appl. Math.* **68**(9), 1587–1639 (2015)
- [51] Steinbach, O.: *Numerical Approximation Methods for Elliptic Boundary Value Problems: Finite and Boundary Elements*. Springer, New York (2008)
- [52] Tacy, M.: Semiclassical L^p estimates of quasimodes on submanifolds. *Commun. Partial Differ. Equ.* **35**(8), 1538–1562 (2010)
- [53] Tacy, M.: Semiclassical L^2 estimates for restrictions of the quantisation of normal velocity to interior hypersurfaces. arXiv preprint (2014). [arxiv:1403.6575](https://arxiv.org/abs/1403.6575)
- [54] Tataru, D.: On the regularity of boundary traces for the wave equation. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **26**(1), 185–206 (1998)
- [55] Vainberg, B.R.: On the short wave asymptotic behaviour of solutions of stationary problems and the asymptotic behaviour as $t \rightarrow \infty$ of solutions of non-stationary problems. *Russian Mathematical Surveys* **30**(2), 1–58 (1975)
- [56] Verchota, G.: Layer potentials and regularity for the Dirichlet problem for Laplace’s equation in Lipschitz domains. *Journal of Functional Analysis* **59**(3), 572–611 (1984)
- [57] Zworski, M.: *Semiclassical analysis*, volume 138 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI (2012)

Jeffrey Galkowski
Department of Mathematics
Northeastern University
567 Lake Hall
Boston MA02115
USA
e-mail: jeffrey.galkowski@northeastern.edu

Euan A. Spence (✉)
Department of Mathematical Sciences
University of Bath
Bath BA2 7AY
UK
e-mail: E.A.Spence@bath.ac.uk

Received: July 26, 2018.

Revised: January 2, 2019.