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Thermal Velocity Effects in Axially Symmetric Solid Beams

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Electron trajectory analysis was added to the test program from which the optical properties of lenses may be calculated including focal points and lengths. In another test case the properties of a particular microscope lens were computed and compared with experimental results. These are shown in Fig. (1). The solid lines were determined by computing the field for the given configuration and then computing electron trajectories for a number of voltage parameter values. The experimental points (and the definition of the voltage parameter) were taken from Heise.[13] The small but consistant differences between the theoretical and experimental values are believed to be due to experimental error in the determination of the voltage parameter.

COMPUTING NOTES

Typical three-electrode lenses were easily defined by a total of 70 to 75 points along the three interfaces, requiring the inversion of a matrix of that order in the induction analysis. The complete program including the inversion, computation of a detailed voltage map, and the optical analysis required 0.155 h on the IBM 7090 computer. Program time varies between the square and the cube of the number of defining points.

The matrix is well conditioned; all elements are positive and the largest element in any column lies on the main diagonal. The largest loss of precision in the inversion due to differencing error was four bits in the cases tested.

Numerical integration was by third-order Gaussian coefficients[14] except where a pole occurs in the integration interval; in that case it was assumed that the integrand went to infinity as the log of distance. An eight parameter formula, taken from Hastings,[15] estimates the elliptic integral to 7 decimal places.

The program is written in the FORTRAN language and is available from the author.

[13] Von F. Heise, "Bestimmung von Verzeichnung und Öffnungsfehler elektrostatischer Linsen aus Hauptflächen und Brennpunkt-Hilfsflächen." Optik 5, (1949).

[14] W. E. Milne, *Numerical Calculus* (Princeton University Press, Princeton, New Jersey, 1949), p. 285ff.

[15] C. Hastings, *Approximations for Digital Computers* (Princeton University Press, Princeton, New Jersey, 1955), p. 171.

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Thermal Velocity Effects in Axially Symmetric Solid Beams

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The effect of a Maxwellian distribution of emission velocities on the longitudinal current density variation in axially symmetric solid beams is investigated to the paraxial approximation. The effects of both longitudinal and transverse initial emission velocities are included. The treatment permits variable magnetic fields and apertures which intercept some of the beam. Some particles may be turned back because all their energy is forced into the transverse motion; the reduction of the transmitted current density is evaluated. A simple illustrative example is given.

INTRODUCTION

SEVERAL authors have estimated the effects of thermal velocities in electron beams. In some cases the treatment has been very general[1,2] and only some special cases have been treated in detail. Others have considered electrostatic[3–5] or electromagnetic[6–8] paraxial beams without allowing for the effects of longitudinal velocity spread. In this paper the work of Refs. 3–7 is extended by including the effects of longitudinal velocity spread, nonuniform cathode emission, and nonuniform magnetic fields, in one treatment. In practical cases, the transverse dimension of the beam is limited; therefore, the treatment allows the beam to be restricted by one or more round apertures. The approach of this paper relies heavily on the work in Refs. 5 and 7. The introductions of these two references could also well serve as introduction to this paper.

In all the references on paraxial beams quoted, it is assumed that the electromagnetic fields in the finite temperature beam are the same, to first order, as those in the zero temperature case. If this is not so, higher-order corrections are required. The higher-order corrections are discussed in Refs. 4 and 5 for electrostatic beams, and can be applied in much the same way for

[1] P. A. Lindsay, Advan. Electron. 13, 181 (1961).

[2] J. R. Pierce, *Theory and Design of Electron Beams* (D. Van Nostrand Company, Inc., New York, 1954).

[3] C. C. Cutler and M. E. Hines, Proc. I.R.E. 43, 307 (1955).

[4] W. E. Danielson, J. L. Rosenfeld, and J. A. Saloom, Bell System Tech. J. 35, 375 (1956).

[5] P. T. Kirstein, IRE Trans. Electron. Devices 10, 69 (1963).

[6] E. Ash, *Proceedings 4th International Conference on Microwave Tubes 1963* (to be published).

[7] G. Herman, J. Appl. Phys. 29, 127 (1958).

[8] Y. V. Troitski, Zh. Techn. Fiz. 30, 25 (1960) [English transl.: Soviet Phys.—Tech. Phys. 5, 22 (1960)].

the beams of this paper. A more thorough and general treatment of higher-order corrections is given in Ref. 9.

In a paraxial theory it is assumed that the forces due to space charge vary linearly with distance from the axis. This assumption is equivalent to one that the effect of transverse variation in charge density on the electric field are neglected. Since the space-charge term is itself usually a first-order term, variation in it may be considered second order. Thus it is not inconsistent to consider thermal effects in beams with nonuniform cathode emission to the paraxial approximation.

The paraxial theory relates the transverse position and velocity, and hence transverse energy, of a particle at one plane z in terms of their initial values at the cathode; for solid axially symmetric beams, this reaction does not, to first order, depend on the initial longitudinal velocity. Moreover, the total kinetic energy of a particle at z is related to its energy at the cathode by the conservation of energy. From the difference of these two relations, a negative longitudinal energy may be required by a particle to reach a given plane with given transverse energy. This is physically impossible, and means the particle would have been turned back and cannot reach the plane z of interest. An aperture in the system would also intercept particles, and so impose additional restrictions on the range of permissible transverse velocities of particles which reach the plane z at a specified distance from the axis.

In Sec. 2, the transverse position, velocity, and total energy of a particle at a plane z is related to its values at the cathode in general terms. An axially symmetric current-density distribution is assumed at the cathode, having a half-Maxwellian velocity distribution. Based on Liouville's theorem, the longitudinal current density at z is related to that at the cathode by an integral over-all permissible velocities. The range of permissible velocities is discussed. In Sec. 3, the results of Sec. 2 are used to find the longitudinal current density on the axis for uniform cathode current density; while in Sec. 4 the methods of Sec. 3 are applied to points off the axis. In Sec. 5, the extension to nonuniform cathode emission and to sheet and hollow beams with curvilinear ray axes is discussed. Finally, in Sec. 6, a simple illustrative example is given. In this example, a beam comes from an electrostatic cylindrical Pierce gun, passes through a defining aperture, and then drifts in a uniform magnetic field. For algebraic simplicity the uniform field is such that its cyclotron frequency is equal to the plasma frequency of the beam.

2. THE GENERAL FORMALISM

In this section general expressions are derived for the axial current density in solid, axially symmetric paraxial beams assuming a Maxwellian distribution of emission, and the regions over which the appropriate integrals must be evaluated are discussed.

The Transformation Laws for a Single Particle

Let $(x, y, \dot{x}, \dot{y}, \dot{z})$ represent the transverse position and the velocity components of an electron at a plane z, and $(X,Y,\dot{X},\dot{Y},\dot{Z})$ the corresponding parameters of the same electron at a plane $z=0$. We assume the beam is a solid, paraxial, axially symmetric beam. This means:

(1) The transverse position and transverse velocity components of that part of the beam which has significant charge density is small.

(2) The spread of longitudinal velocity, at a plane z, of particles in that part of the beam which has significant charge density is small.

(3) Over the transverse dimensions in which there is a significant charge density, the electric and magnetic fields vary linearly with distance from the axis and are axially symmetric.

Under the above assumptions, it is shown in Appendix A, Eqs. (A13), (A16), (A17), and (A22) that one may define complex variables w, u, W, U related to the position and velocity components of a particle by

$$w=(x+iy)e^{-i\chi}, \quad W=X+iY, \tag{2.1}$$

$$u=(\dot{x}+i\dot{y})e^{-i\chi}-i\omega_L w, \quad U=(\dot{X}+i\dot{Y})-i\Omega_L W, \tag{2.2}$$

$$\left.\begin{aligned}&\Phi_0(z)=2\eta\phi_0(z), \quad \omega_L=\tfrac{1}{2}\eta B_0(z),\\ &\Omega_L=\tfrac{1}{2}\eta B_0(0), \quad \chi=\int_0^z \omega_L(z)\Phi_0(z)^{-\frac{1}{2}}dz\end{aligned}\right\}, \tag{2.3}$$

where η is the ratio of charge to mass, $B_0(z)$ is the axial magnetic field on the axis at the plane z, and $\phi_0(z)$ is the electrostatic potential on the axis relative to the cathode. In Eqs. (2.1)–(2.3), the lower case letters refer to the values at the initial plane $z=0$.

From Eqs. (A24) and (A28), the parameters w, u, $\dot{z}$ of an electron at the plane z are related to their value W, U, $\dot{Z}$ at the initial plane by the laws

$$\begin{pmatrix} w \\ u \end{pmatrix}=\begin{pmatrix} M & S \\ \dot{M} & \dot{S} \end{pmatrix}\begin{pmatrix} W \\ U \end{pmatrix}, \tag{2.4}$$

$$|u+i\omega_L(z)w|^2+\dot{z}^2-\Phi_0(z)+\Phi_2(z)|w|^2 =\dot{Z}^2+|U+i\Omega_L W|^2. \tag{2.5}$$

Here M, S, $\dot{M}$, $\dot{S}$ are real functions of z satisfying Eqs. (A20) and (A27), so that, from Eq. (A30),

$$M\dot{S}-\dot{M}S=1, \tag{2.6}$$

while Φ_2 is given, from Eq. (A25), by

$$\Phi_2(z)=\tfrac{1}{2}\eta[\rho_0(z)+\phi''(z)]. \tag{2.7}$$

At a cathode ρ_0 and $\phi''(z)$ both become infinite. However, the right-hand side of Eq. (2.7) is proportional to

[9] P. A. Sturrock, *Static and Dynamic Electron Optics* (Cambridge University Press, New York, 1955).

the derivative with respect to r of the radial electric field. Now if the cathode is chosen as an equipotential at $z=0$, this term is zero, so that $\Phi_0(0)$ and $\Phi_2(0)$ are both zero. From now on we only consider the transformation laws of Eqs. (2.4)–(2.6). The relation between the functions M, $\dot{M}$, S, $\dot{S}$, ω_L, Ω_L, Φ_2, Φ_0 and the electromagnetic fields are not considered. However, we might note that if no magnetic field threads the cathode, $\Omega_L=0$, while if there is no magnetic field at the plane z, $\omega_L=0$.

The Phase-Space Density Distribution at the Cathode

The probability that a particle at a particular point (X,Y) of the cathode $z=0$, leaves the cathode with velocity components in the ranges $(\dot{X},\dot{X}+d\dot{X})$, $(\dot{Y},\dot{Y}+d\dot{Y})$, $(\dot{Z},\dot{Z}+d\dot{Z})$ is $P(\dot{X},\dot{Y},\dot{Z})d\dot{X}d\dot{Y}d\dot{Z}$, where P has the form

$$P(\dot{X},\dot{Y},\dot{Z})=2\lambda^3\pi^{-\frac{1}{2}}\exp[-\lambda^2(\dot{Y}^2+\dot{Y}^2+\dot{Z}^2)], \quad (2.8)$$

and λ is given by

$$\lambda=[m/(2kT)]^{\frac{1}{2}}, \quad (2.9)$$

where m is the mass of the electron, T the cathode temperature, and k is Boltzman's constant.

If the beam is axially symmetric, the total number of particles emitted from an area $dXdY$ of cathode about the point X, Y may be expressed in the form $N_T(R)dXdY$ where R is the distance of (X,Y) from the axis. The function N_T is related to the axial current density at the cathode. In terms of N_T, the number density distribution of particles in $(X,Y,\dot{X},\dot{Y},\dot{Z})$ phase space is N, where

$$N(X,Y,\dot{X},\dot{Y},\dot{Z})=2\lambda^3\pi^{-\frac{3}{2}}N_T(R)\times\exp[-\lambda^2(\dot{X}^2+\dot{Y}^2+\dot{Z}^2)]. \quad (2.10)$$

If the cathode current density is $i_c(R)$, then N_t is related to i_c by the expression

$$i_c(R)=e\int_0^\infty \dot{Z}d\dot{Z}\iint_{X,Y=-\infty}^{\infty} N(X,Y,\dot{X},\dot{Y},\dot{Z})d\dot{X}d\dot{Y}. \quad (2.11)$$

Substituting Eq. (2.11) into Eq. (2.10) we see that

$$i_c(R)=eN_t(R)\pi^{-\frac{1}{2}}/\lambda. \quad (2.12)$$

Hence N becomes

$$N(X,Y,\dot{X},\dot{Y},\dot{Z})=2\lambda^4/(\pi e)i_c(R)\times\exp[-\lambda^2(\dot{X}^2+\dot{Y}^2+\dot{Z}^2)]. \quad (2.13)$$

Now our transformation laws are written not in $(x,y,z,\dot{x},\dot{y},\dot{z})$ space but in $(w,z,u,\dot{z})$ space. In this space the density is NJ, where J is the Jacobian of the transformation from the $(x,y,\dot{x},\dot{y})$ system to the (w,u) system. This Jacobian is unity, because the transformation of $x+iy$ to w is a simple rotation, while that from $(\dot{x}+i\dot{y})$ to u is a simple rotation plus an added term independent of u. In order to determine the number density distribution of electrons omitted from the cathode $|U|^2$ must be related to $\dot{X}^2$ and $\dot{Y}^2$. From Eq. (2.2) we see that

$$\dot{X}^2+\dot{Y}^2=|U+i\Omega W|^2. \quad (2.14)$$

Using Eq. (2.13) we now see that the number density of particles omitted from W with transverse velocity U and normal velocity $\dot{Z}$ is $N(W,U,\dot{Z})$ where

$$N(W,U,\dot{Z})=2\lambda^4(\pi e)i_c(R)\times\exp-\lambda^2(|U+i\Omega_L W|^2+\dot{Z}^2). \quad (2.15)$$

Equation (2.15) is required in later sections to derive expressions for the current density at planes other than the cathode.

The Axial Current-Density Distribution

We are now in a position to find the axial current density distribution. From Liouville's theorem, the phase space density n is preserved for an ensemble of particles in an electromagnetic field. Hence, if a particle at (x,y,z), with velocity components $(\dot{x},\dot{y},\dot{z})$, comes from the point (X,Y) at the cathode with initial velocity components $(\dot{X},\dot{Y},\dot{Z})$, the number density in phase space $n(x,y,z,\dot{x},\dot{y},\dot{z})$ is equal to $n(X,Y,0,\dot{X},\dot{Y},\dot{Z})$. In accordance with our notation, the phase-space density at the cathode is denoted by $N(X,Y,\dot{X},\dot{Y},\dot{Z})$. Using the conservation of phase-space density and Eq. (2.13), the axial current density is $i(x,y,z)$, where i is given by

$$i(x,y,z)=\iiint_{\dot{x}\,\dot{y}\,\dot{z}} en(x,y,z,\dot{x},\dot{y},\dot{z})\dot{z}d\dot{x}d\dot{y}d\dot{z}$$

$$=\iiint_{\dot{x}\,\dot{y}\,\dot{z}} eN(X,Y,\dot{X},\dot{Y},\dot{Z})\dot{z}d\dot{x}d\dot{y}d\dot{z}$$

$$=\frac{2\lambda^4}{\pi}\iiint_{\dot{x}\,\dot{y}\,\dot{z}} \dot{z}i_c(R)\times\exp[-\lambda^2(\dot{X}^2+\dot{Y}^2+\dot{Z}^2)]d\dot{x}d\dot{y}d\dot{z}. \quad (2.16)$$

The regions of $(\dot{x},\dot{y},\dot{z})$ space over which Eq. (2.16) must be integrated is defined by several boundaries. If the cathode is finite, or if there are apertures in the systems, then this imposes bounds on (x,y) at some plane z_0, possibly the cathode. From Eqs. (2.1)–(2.4), these bounds impose conditions on u, v and hence $\dot{x}$, $\dot{y}$; these conditions do not, however, effect the permissible $\dot{z}$. For a particle to arrive at (x,y,z), it must have, at all z between the cathode and the plane of interest, positive axial velocity $\dot{z}$. From the energy balance equation, Eq. (2.5), this implies a condition on $\dot{z}$ of the form

$$\dot{z}\geq\dot{z}_{\min}(x,y,\dot{x},\dot{y}). \quad (2.17)$$

The exact expressions for $z_{\min}$ is discussed later. If a constraint of the type of Eq. (2.17) is applied, Eq. (2.16) may be integrated, using the energy balance equation Eq. (2.5)

$$i(x,y,z)=\frac{2\lambda^4}{\pi}\iint_{\dot{x},\dot{y}} i_c(R)\int_{\dot{z}=\dot{z}_{\min}}^{\infty} \dot{z}\exp[-\lambda^2(\dot{x}^2+\dot{y}^2+\dot{z}^2-\Phi_0+\Phi_2r^2)]d\dot{x}d\dot{y}d\dot{z}$$
$$=\frac{\lambda^2}{\pi}\iint_{\dot{x},\dot{y}} i_c(R)\exp[-\lambda^2(\dot{x}^2+\dot{y}^2+\dot{z}_{\min}^2-\Phi_0+\Phi_2r^2)]d\dot{x}d\dot{y}. \tag{2.18}$$

In substituting for $\dot{z}$ in Eq. (2.16) we have used the connection between $\dot{x}$, $\dot{y}$ and u, w, and replaced $|w|^2$ by r^2 the distance from the axis. The evaluation of Eq. (2.18), the choice of $\dot{z}_{\min}$ and the bounds of integration form the subject of this paper.

In general it is best to put everything in terms of the independent variables (x,y) of the point of interest in the z plane, (X,Y) the point on the cathode from which the particles came, and $\dot{Z}$ the initial axial velocity. The integration over $\dot{x}$, $\dot{y}$, $\dot{z}$ must then be replaced by one over X, Y, $\dot{Z}$ using the Jacobian of the transformation developed in Appendix B. From Eqs. (2.4) and (2.6) we see that

$$u=(\dot{S}w-W)/S,\quad U=(w-MW)/S. \tag{2.19}$$

Using Eqs. (2.1), (2.2), and (2.19), the relation between $\dot{x}$, $\dot{y}$, x, y, X, Y can be written

$$\begin{aligned}\dot{x}+i\dot{y}&=e^{i\chi}(u+i\omega_Lw)\\&=e^{i\chi}[(\dot{S}+i\omega_LS)w-W]/S\\&=e^{i\chi}[(\dot{S}+i\omega_LS)w-(X+iY)]/S.\end{aligned} \tag{2.20}$$

The transformation from $(\dot{x},\dot{y},\dot{z})$ to $(X,Y,\dot{Z})$ is seen from Eqs. (2.5) and (2.20) to have the form of Eq. (B1) with

$$q_1=\dot{x},\quad q_2=\dot{y},\quad q_3=\dot{z},\quad Q_1=X,\quad Q_2=Y,\quad Q_3=\dot{Z},\quad C=e^{i\chi}/S. \tag{2.21}$$

Hence the volume elements in the $(\dot{x},\dot{y},\dot{z})$ and $(X,Y,\dot{Z})$ systems are related, from Eq. (B6), by the expression

$$d\dot{x}d\dot{y}d\dot{z}=\dot{Z}/(S^2\dot{z})dXdYd\dot{Z}. \tag{2.22}$$

An alternative expression for $i(x,y,z)$ is, therefore, from Eq. (2.16)

$$i(x,y,z)=\frac{2\lambda^4}{\pi S^2}\iint_{X,Y} i_c(R)\exp[-\lambda^2(\dot{X}^2+\dot{Y}^2)]dXdY\int_{\dot{Z}} e^{-\lambda^2\dot{z}^2}\dot{Z}d\dot{Z}$$
$$=\frac{2\lambda^4}{\pi S^2}\iint_{X,Y} i_c(R)\exp[-(\lambda/S)^2|w-(M-i\Omega_LS)W|^2]dXdY\int_{\dot{Z}} \dot{Z}e^{-\lambda^2\dot{z}^2}d\dot{Z}. \tag{2.23}$$

In Eq. (2.23) we have used Eqs. (2.2) and (2.19); (W,w) are related to (X,Y,x,y) by Eq. (2.1).

The condition that $\dot{Z}$, $\dot{z}>0$ may be written, instead of as Eq. (2.17), in the form

$$\dot{Z}\geq\dot{Z}_{\min}(x,y,\dot{x},\dot{y}), \tag{2.24}$$

where $z_{\min}$ and $\dot{Z}_{\min}$ are related, from Eq. (2.5), by

$$\dot{Z}_{\min}^2=\max[0,\,-\dot{X}^2-\dot{Y}^2+\dot{x}^2+\dot{y}^2+\dot{z}_{\min}^2-\Phi_0(z)+\Phi_2(z)r^2]. \tag{2.25}$$

Using this bound on $\dot{Z}_{\min}$, Eq. (2.23) becomes

$$i(x,y,z)=\frac{\lambda^2}{S^2}\iint_{X,Y} i_c(R)\times\exp-\lambda^2[|w-(M-i\Omega_LS)W|^2+\dot{Z}_{\min}^2]dXdY. \tag{2.26}$$

Again the region of integration of X, Y and the form of $\dot{Z}_{\min}$ is discussed later.

The transformation leading to Eq. (2.26) is only possible as long as $S\neq0$. If $S=0$, we are at a cathode image plane, and the integral becomes degenerate. We may use the fact that one definition of the Dirac delta function $\delta(a)$ is

$$\delta(a)=\lim_{S\to0}[e^{-(a/S)^2}/(\pi S)^{\frac{1}{2}}]. \tag{2.27}$$

Hence the limiting form of Eq. (2.23) is

$$i(x,y,z)=(1/M^2)i_c(r/M)e^{-\lambda^2\dot{Z}_{\min}^2}. \tag{2.28}$$

In Eq. (2.28) we have used the fact that

$$|w-(M-i\Omega_LS)W|$$

can be zero, as $S\to0$, only if $WM=w$, so that $R=r/M$.

The Region of Integration

Physically there are several restraints on the range of the variables $(\dot{x},\dot{y},\dot{z})$ of Eq. (2.18) or $(X,Y,\dot{Z})$ of Eq. (2.23). The restraints we consider may be put in three categories: (i) due to the finite nature of the cathode; (ii) due to an axially symmetric aperture in the system; and (iii) those imposed by energy considerations.

The restraint of type (i) may be included in the choice of $i_c(R)$, the current density distribution at the cathode. However, it is often convenient to choose for $i_c(R)$ as simple a function as possible. When the cathode current density is constant, the integrals of Eqs. (2.18) and (2.26) may be evaluated explicitly in terms of known functions. For this reason it may be preferable to allow for a cathode of radius a by putting the restraint

$$R<a. \tag{2.29}$$

This restraint is not discussed further, for the moment, since it is a special case of type (ii). If x_0, y_0, r_0, $\dot{x}_0$, u_0 etc. refer to the values of x, y, r, $\dot{x}$, u etc. of a particle at z_0, and there exists a circular aperture radius a_0 at z_0, then this restraint may be written, from Eq. (2.1),

$$r_0=|w_0|\leq a_0. \tag{2.30}$$

It is clear that Eq. (2.29) is a special case of Eq. (2.30). If the transformation from the plane z_0 to z is as Eq. (2.4) namely with M_0, S_0, w_0, etc. replacing M, S, W, etc.:

$$\begin{pmatrix} w \\ u \end{pmatrix}=\begin{pmatrix} M_0 & S_0 \\ \dot{M}_0 & \dot{S}_0 \end{pmatrix}\begin{pmatrix} w_0 \\ u_0 \end{pmatrix}. \tag{2.31}$$

Equation (2.30) may be expressed in the form

$$|\dot{M}_0w-\dot{S}_0u|\leq a_0. \tag{2.32}$$

Clearly if we are using (W,U) coordinates, and Eq. (2.31) denotes the matrix of transformation from (w_0,u_0) to (W,U) then Eq. (2.32) again holds with (W,U) replacing (w,u). Depending on the variation of M, $\dot{M}$, S, $\dot{S}$ with z, there is a limitation of the form of Eq. (2.32) at each axial plan z. We usually assume, however, that the limitation at one particular plane z_0 is the most stringent.

The third type of restraint requires that

$$\dot{z}\geq 0,\quad \dot{z}_0\geq 0, \tag{2.33}$$

where $\dot{z}_0$ is the value of $\dot{z}$ at the plane z_0. Using Eq. (2.5) with ω_{L0}, Φ_{00}, Φ_{20}, etc., denoting the values of ω_L, Φ_0, Φ_2 at z_0, etc., we see that energy balance requires that at each plane z_0

$$|u+i\omega_Lw|^2+\dot{z}^2-\Phi_0+\Phi_2|w|^2 =|u_0+i\omega_{L0}w_0|^2+\dot{z}_0^2-\Phi_{00}+\Phi_{20}|w_0|^2. \tag{2.34}$$

For the particle to arrive at all at z, it is necessary that for all z_0 in the interval $0\leq z_0\leq z$, $\dot{z}\geq 0$ so that Eq. (2.34) gives

$$\dot{z}^2\geq\dot{z}_{\min,0}\equiv|u_0+i\omega_{L0}w_0|^2-|u+i\omega_Lw|^2 -\Phi_{00}+\Phi_0+\Phi_{20}|w_0|^2-\Phi_2|w|^2. \tag{2.35}$$

For each z_0 there exists a $\dot{z}_{\min,0}$ which depends on u, w as given by Eq. (2.35). For each w, u it is possible to define a $\dot{z}_{\min}$ by

$$\dot{z}_{\min}=\max(\dot{z}_{\min,0}). \tag{2.36}$$

The $\dot{z}_{\min}$ of Eq. (2.36) is the one which should be used in Eqs. (2.18) or (2.26). However, we usually *assume* that if a particle left the cathode with positive axial velocity and reaches the plane z with positive axial velocity, then it has positive $\dot{z}_0$ at all intermediate planes z_0. This condition is equivalent to putting

$$\dot{z}_{\min}^2=\max[0,\ |U+i\omega_LW|^2-|u+i\omega_Lw|^2 +\Phi_0-\Phi_2|w|^2]. \tag{2.37}$$

Both Eqs. (2.35) and (2.37) can be put in terms of only (w,u) by using the relations between (w,u) and (w_0,u_0) or (W,U) of Eqs. (2.31) or (2.4).

It may be noted that the convenient bounds of integration are not on $(\dot{x},\dot{y})$ or (X,Y) but on u or W. We may change coordinates from an integral over $(\dot{x},\dot{y})$ to one over the u plane by using Eqs. (2.2), (B1) and (B6) with $q=(\dot{x}+i\dot{y})$, $Q=u$, $C=e^{i\chi}$, $D=ie^{i\chi}\omega_Lw$, $E=0$, $q_3=Q_3$, so that

$$d\dot{x}d\dot{y}=du_xdu_y \tag{2.38}$$

and an integral over the u plane is identical to one over the $(\dot{x},\dot{y})$ plane. Also, an integral over the (X,Y) plane is identical to one over the W plane. Hence, using Eqs. (2.2) and (2.38), Eq. (2.18) may be written

$$i(x,y,z)=\frac{\lambda^2}{\pi}\iint_u i_c(R) \times\exp\{-\lambda^2[|u+i\omega_Lw|^2+z_{\min}^2-\Phi_0+\Phi_2|w|^2]\}du_xdu_y. \tag{2.39}$$

It is unnecessary to rewrite Eq. (2.26), since a restriction on W is automatically one on (X,Y).

It now remains only to integrate Eqs. (2.26) and (2.39) under the restrictions of the form of Eqs. (2.32) and (2.35) or (2.37).

3. THE CURRENT DENSITY ON THE AXIS FOR UNIFORM CATHODE CURRENT DENSITY

If the current density at the cathode is uniform, it is possible to evaluate the integrals of Sec. 2 explicitly under certain assumptions. In this section we restrict ourselves to evaluating the current density on the axis; the algebra is much simpler for this case, and yet the essential ideas are well illustrated. In this case we may put $w=0$, $i=i(0,0,z)$ in the equation of Sec. 2, and take

i_c outside the integral sign of Eq. (2.39), to obtain

$$i(0,0,z)=i_c\frac{\lambda^2}{\pi}\iint_u \exp[-\lambda^2(|u|^2+\dot{z}_{\min}^2-\Phi_0)]du_x du_y. \quad (3.1)$$

If we further assume that the field is such that if a particle leaves the cathode and reaches z with positive axial velocity, it will never have zero axial velocity between these two planes, then Eq. (2.37) is satisfied and gives

$$\dot{z}_{\min}^2=\max\{0,\ (M^2+\Omega_L^2S^2-1)|u|^2+\Phi_0\}. \quad (3.2)$$

Equation (3.2) is derived from Eq. (2.37) by using Eq. (3.1) and putting $w=0$.

If in addition there is an aperture a_0 at a plane z_0 with matrix of transformation as given by Eq. (2.31), then on the axis the possible range of u is given, from Eq. (2.32) by

$$|\dot{S}_0 u|\leq a_0. \quad (3.3)$$

Equation (3.1) must, therefore, be integrated over the u specified by Eq. (3.3) with the $\dot{z}_{\min}$ of Eq. (3.2). Writing

$$u=|u|e^{i\gamma} \quad (3.4)$$

Eq. (3.1) may be immediately integrated over γ to give

$$i(0,0,z)=i_c\lambda^2\int_{|u|}2|u|\ \exp[-\lambda^2(|u|^2+\dot{z}_{\min}^2-\Phi_0)]d|u|. \quad (3.5)$$

There are now several cases, as $(M^2+\Omega_L^2S^2-1)\lesseqgtr 0$ and $a_0/|\dot{S}_0|\lesseqgtr\Phi_0/(M^2+\Omega_L^2S^2-1)$; the first condition determines whether the beam is highly convergent for the given magnetic field, the second the relation between the aperture size, the energy and the convergence.

Clearly one important parameter is $M^2+\Omega_L^2S^2$; writing for brevity

$$\alpha^2=M^2+\Omega_L^2S^2, \quad (3.6)$$

we may deduce from Eq. (3.2) that

$$\left.\begin{array}{l}\text{if}\quad \alpha>1,\quad \text{or if}\quad \alpha<1 \text{ and } |u|^2<\Phi_0/(1-\alpha^2),\\ \qquad\qquad \dot{z}_{\min}^2=\Phi_0+(\alpha^2-1)|u|^2\\ \text{if}\quad \alpha<1 \text{ and } |u|^2>\Phi_0/(1-\alpha^2),\quad \dot{z}_{\min}=0.\end{array}\right\} \quad (3.7)$$

Combining Eqs. (3.7) and (3.3), Eq. (3.5) can be integrated to give

if $\alpha>1$, or if $\alpha<1$ and $a_0^2/\dot{S}_0^2<\Phi_0/(1-\alpha^2)$,

$$i(0,0,z)/i_c=[1-\exp(-\lambda^2a_0^2\alpha^2/\dot{S}_0^2)]/\alpha^2. \quad (3.8)$$

If $\alpha<1$ and $a_0^2/\dot{S}_0^2>\Phi_0/(1-\alpha^2)$,

$$i(0,0,z)/i_c=\{1-(1-\alpha^2)\exp[-\lambda^2\Phi_0\alpha^2/(1-\alpha^2)]\}/\alpha^2 -\exp[-\lambda^2(a_0^2/\dot{S}_0^2-\Phi_0)]. \quad (3.9)$$

In deriving Eqs. (3.8) and (3.9) we have assumed Φ_0 positive, which is the case of most practical interest.

Under the conditions of Eq. (3.8), it is seen that energy considerations do not affect the axial current. It is not possible to obtain infinite axial current density by compressing the beam—i.e., letting $\alpha\rightarrow 0$. Under such conditions Eqs. (3.8) and (3.9) give

$$\text{if}\quad a_0^2/S_0^2<\Phi_0,\quad \alpha\rightarrow 0,\quad i(0,0,z)/i_c\rightarrow\lambda^2a_0^2/\dot{S}_0^2, \quad (3.10)$$

$$\text{if}\quad a_0^2/S_0^2>\Phi_0,\quad \alpha\rightarrow 0,\quad i(0,0,z)/i_c\rightarrow 1+\lambda^2\Phi_0 -\exp[-\lambda^2(a_0^2/\dot{S}_0^2-\Phi_0)]. \quad (3.11)$$

The expression of Eq. (3.11), with $a_0\rightarrow\infty$, is that derived by Pierce[2] from more elementary considerations for the maximum current density at a point. It is easy to derive an expression for maximum beam which can arrive within a cone of half-angle β, neglecting energy restrictions. For this it is only necessary to repeat the integration for $\dot{z}_{\min}=|u|\tan\beta$. This expression has also been derived by Pierce, but is probably of less importance than the results of Eqs. (3.10) and (3.11) which give the maximum current density at an anode plane if there is an aperture a_0 placed at an arbitrary position between the cathode and anode planes.

It is possible to derive many physical properties of Eqs. (3.8)–(3.11), but we only stress that the principal parameter governing the current density is not M, the paraxial magnification of the beam, but $(M^2+\Omega_L^2S)$, i.e., α. Thus unless the beam is very stiff, i.e., unit transverse velocity produces little deviation from the axis so that S is small, and there is little magnetic field threading cathode so that Ω_L is small, it may not help much to reduce the magnification.

If there are several apertures a_i at z_i with transfer matrix element M_i, $\dot{S}_i$ etc. then $a_0/\dot{S}_0$ in Eqs. (3.8)–(3.11) must be defined by

$$a_0/\dot{S}_0=\min(a_i/\dot{S}_i). \quad (3.12)$$

Clearly a finite cathode of radius a is considered by considering an aperture of radius a at the cathode plane $z=0$.

4. CURRENT DENSITY OFF THE AXIS FOR UNIFORM CATHODE CURRENT DENSITY

The results of the previous section are extended to give the longitudinal current density off the axis in the presence of an aperture and the energy restrictions of Sec. 3. Again a uniform current density at the cathode is assumed.

The energy balance condition, Eq. (2.37) can be written, using Eq. (2.4),

$$\begin{aligned}\dot{z}_{\min}^2&=\max\{0,\ |(-\dot{M}+i\Omega_L\dot{S})w+(M-i\Omega_LS)u|^2 -|u+i\omega_Lw|^2+\Phi_0-\Phi_2|w|^2\}\\ &=\max\{0,\alpha^2|u-\beta_1w|^2-|u+i\omega_Lw|^2 +\Phi_0-\Phi_2|w|^2\},\end{aligned} \quad (4.1)$$

where α is given by Eq. (3.6) and

$$\beta_1=(\dot{M}-i\Omega_L\dot{S})/(M-i\Omega_LS). \quad (4.2)$$

If we have an aperture a_0 at a plane z_0 as in Eq. (2.30), the possible range of u is given by Eq. (2.32), namely,

$$|u-(\dot{M}_0/\dot{S}_0)w| \leq a_0/\dot{S}_0. \tag{4.3}$$

Now over one range of u Eq. (2.39) must be integrated with $\dot{z}_{\min}=0$; over another $\dot{z}_{\min}$ is given by the second term of Eq. (4.1). In order to perform the integration, it is necessary to determine the boundary curve where $\dot{z}_{\min}$ changes from one value to the other. This occurs where

$$\alpha^2|u-\beta_1 w|^2-|u+i\omega_L w|^2+\Phi_0-\Phi_2|w|^2=0,$$

which may be written

$$|u-C_1 w|=b_1, \tag{4.4}$$

where

$$\left.\begin{aligned} C_1&=(\alpha^2\beta_1+i\omega_L)/(\alpha^2-1),\\ b_1^2&=\frac{\Phi_0}{1-\alpha^2}+|w|^2\left[\frac{|\alpha^2\beta_1+i\omega_L|^2}{(1-\alpha^2)^2}+\frac{\alpha^2|\beta_1^2|-\omega_L^2-\Phi_2}{1-\alpha^2}\right]\end{aligned}\right\}. \tag{4.5}$$

It is to be noted that if $w=0$ as in Sec. 3, Eq. (4.4) gives the circle $|u|^2=\Phi_0/(1-\alpha^2)$ as the dividing curve. If b_1^2, as defined by Eq. (4.5), is negative, then $\dot{z}_{\min}=0$ for all u, if $\alpha<1$, and $\dot{z}_{\min}$ is the second term of Eq. (4.1) for all u if $\alpha>1$. The region over which Eq. (2.39) must be integrated is bounded by Eq. (4.3) which may be written

$$|u-C_2 w|\leq b_2,\quad C_2=\dot{M}_0/\dot{S}_0,\quad b_2=a_0/\dot{S}_0. \tag{4.6}$$

Substituting the appropriate $z_{\min}$ and boundaries, one may obtain different values of $i(x,y,z)$ depending on the relative values of b_1, b_2. For example, if the aperture is large and the beam sufficiently near the axis so that

$$b_1^2>0,\quad b_2>b_1+|C_1-C_2|,\quad \alpha<1, \tag{4.7}$$

then Eq. (2.39) gives

$$\frac{i(x,y,z)}{i_c}=\frac{\lambda^2}{\pi}\iint_0^{|u-C_1w|=b_1}\exp(-\lambda^2\alpha^2|u-\beta_1 w|^2)du_x du_y$$
$$+\frac{\lambda^2}{\pi}\exp[-\lambda^2(-\Phi_0+\Phi_2|w|^2)]$$
$$\times\iint_{|u-C_1w|=b_1}^{|u-C_2w|=b_2}\exp[-(\lambda^2|u+i\Omega_L w|^2)]du_x du_y. \tag{4.8}$$

Now the integrals of Eq. (4.8) are of the type of Eq. (C3) with a scale change, and are to be integrated over the region of Eq. (C10). Hence, using the results of Eqs. (C9)–(C13),

$$\begin{aligned} i(x,y,z)/i_c=&(1/\alpha^2)[1-I(\Delta_1,\lambda\alpha b_1)]\\ &+\exp[-\lambda^2(-\Phi_0+|w|^2\Phi_2)]\\ &\times[I(\Delta_2,\lambda b_1)-I(\Delta_3,\lambda b_2)], \end{aligned}\tag{4.9}$$

where

$$\Delta_1=|C_1-\beta_1|\alpha\lambda|w|,\quad \Delta_2=|C_1+i\Omega_L|\lambda|w|,\quad \Delta_3=|C_2+i\Omega_L|\lambda|w|. \tag{4.10}$$

And $I(\Delta,b)$ is the function defined in Eq. (C3) and shown in Figs. 2 and 3.

If the energy is so high that

$$b_1^2>0,\ b_1>b_2+|C_1-C_2|,\ \alpha<1 \text{ or } b_1^2<0,\ \alpha>1, \tag{4.11}$$

then it is easily seen that $\dot{z}_{\min}$ is always given by the second term of Eq. (4.1) and

$$i(x,y,z)/i_c=(1/\alpha^2)[1-I(\Delta_3\alpha,\lambda b_2\alpha)]. \tag{4.12}$$

By using Eq. (C8) it is easy to see that on the axis, where $|w|=0$ and, therefore, $\Delta_1=\Delta_2=\Delta_3=0$, the i/i_c of Eqs. (4.9) and (4.12) reduce to those of Eqs. (3.8) and (3.9). It should be noted that the current density only takes the comparatively simple forms of Eqs. (4.9) and (4.12) when Eqs. (4.7) or (4.11) are satisfied. This implies the circles $|u-C_1w|=b_1$ and $|U-C_2w|=b_2$ do not intersect.

In Refs. 3–5 expressions have been given for the current density variation neglecting the effect of longitudinal velocities. The results obtained may be shown to be the limiting case of this section when Φ_0 becomes large, so that Eq. (4.11) is satisfied. The current density is then exactly that of Eq. (4.12). In Refs. 3–5 there was no magnetic field, so that the comparison must be made with $\omega_L=\Omega_L=0$. In these references there was a finite cathode and no aperture hence, S_0, a_0, z_0 refer to the cathode plane.

If the relative magnitudes of $|w|$, Φ_0, and u_0 are such that neither Eq. (4.7) nor Eq. (4.11) is satisfied, it is still possible to evaluate the integrals concerned, but the results are rather more complicated. This situation would arise if for a particular point in the beam the curve defining the dividing line between $\dot{z}_{\min}=0$ and $\dot{Z}_{\min}=0$, i.e., Eq. (4.4), passes outside the permissible range of u, i.e., Eq. (4.6). In that case it is no longer possible to take the second integral of Eq. (C6) from Eq. (C7), hence the extra complication.

5. NONUNIFORM CATHODE CURRENT DENSITY AND OTHER EXTENSIONS OF THE METHOD

If the current density is still axially symmetric, but not uniform, it is no longer possible to take i_c outside the integral sign in Eq. (2.39)—unless we are at a cathode image plane so that $S=0$. If this is not the case, it is more convenient to use W, w as independent coordinates, and integrate Eq. (2.26). In this case Eq. (2.25) can be written

$$\dot{Z}_{\min}^2=\max[0,-|w+(i\Omega_L S-M)W|^2+|(\dot{S}+i\omega_L S)w-W|^2-\Phi_0+\Phi_2|w|^2], \tag{5.11}$$

and if M_0, S_0, etc. is the matrix from the cathode to the

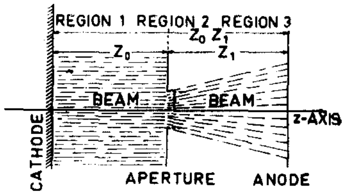

FIG. 1. Schematic of a pencil beam through a defining aperture.

aperture a_0 at z_0, the aperture can be expressed by

$$|M_0W-S_0U|<a_0,$$

which can be written

$$|(M_0S-S_0M)W-S_0w|<a_0S. \quad (5.2)$$

It is now possible to carry out exactly the same procedure as in the preceding section. Again the curve separating the two values of $\dot{Z}_{\min}$ can be written in the form

$$|W-C_3w|=b_3,$$

while the boundary over which W may range is given by Eq. (5.2) and is, therefore, of the same form. Equation (2.26) may now be evaluated in exactly the same way as Eq. (2.39) of the last section. The only difference which now occurs is that instead of the simple $I(\Delta,b)$ functions of the last section one is now required to integrate

$$\frac{1}{\pi}\int\int i_c(|W|)\exp-\alpha^2|W-Lw|^2dW_xdW_y.$$

An example of this technique when the "aperture" was the finite size of the cathode has been given in Ref. 5. In this case $M_0=1$, $S_0=1$ and Eq. (5.2) is simply the obvious $|W|<a_0$. We do not, however, go through the algebra required for nonuniform cathode emission.

It is to be noted that the methods of this paper can equally well be applied to axially symmetric beams with curvilinear ray axes. In these cases the differential equations for the matrix elements must be slightly varied from Eq. (A20), and the transformation law of Eq. (A29) must be replaced by that of Eq. (A31) where $\epsilon=\dot{Z}$. For sheet beams the $\epsilon W/R$ term need only be replaced by ϵW. These changes of transformation do somewhat complicate the algebra, and no example is given in this paper.

6. A SIMPLE EXAMPLE

As an illustration of the method, let us consider thermal effects in a pencil beam which starts as a parallel flow beam from an infinite planar cathode. After length z_0 it has reached a potential ϕ_{00}, and then passes through an aperture a. This aperture is also a magnetic shield, and immediately after the aperture the beam comes into an axial magnetic field B such that the cyclotron frequency is just equal to the plasma frequency of the beam, and in which there is no electric field. Assuming the cathode has temperature T and the beam is emitted with uniform current density at the cathode, we investigate the longitudinal current density variation across the beam at anode plane distant z_1 from the aperture. The physical situation is sketched in Fig. 1.

Clearly the first important problem is to evaluate the matrix elements M, $\dot{M}$, S, $\dot{S}$ and M_0, $\dot{M}_0$, S_0, $\dot{S}_0$ of the transformations from the cathode to the anode and from the aperture to the anode. For this purpose we divide the cathode anode region into 3 parts; the first is from the cathode to the aperture, the second passing through the aperture, the third is the drift region beyond. In all three regions w must satisfy Eq. (A20) so that

$$dw/dz=\Phi_0^{-\frac{1}{2}}u,\quad du/dz=-(\Phi_2+\omega_L^2)\Phi_0^{-\frac{1}{2}}w. \quad (6.1)$$

However, if $\mathbf{A}_i$ is the transfer matrix,

$$\mathbf{A}_i=\begin{pmatrix} M_i & S_i \\ \dot{M}_i & \dot{S}_i \end{pmatrix} \quad (6.2)$$

for the ith region, it is easily seen from the definitions of the elements that the transfer matrix from after the aperture to z, is $\mathbf{A}_3$, while that from the cathode to z, is $\mathbf{A}_3\mathbf{A}_2\mathbf{A}_1$.

In the first region, because the beam is electrostatic and obeys the Child–Langmuir equations from a space-charge-limited cathode,

$$0\leq z\leq z_0,\quad \Omega_L=\omega_L=0,\quad \phi_0=\phi_{00}(z/z_0)^{\frac{4}{3}},$$
$$\rho_0=-\epsilon_0(4/9)(\phi_{00}/z_0^2)(z/z_0)^{-\frac{2}{3}} \quad (6.3)$$

so that

$$0\leq z\leq z_0,\quad \omega_L=0,\quad \Phi_{00}=2\eta\phi_{00},$$
$$\Phi_0=\Phi_{00}(z/z_0)^{\frac{4}{3}},\quad \Phi_2=0. \quad (6.4)$$

There is a sharp discontinuity in ϕ_0' in crossing the aperture. On one side it is $\frac{4}{3}(z/z_0)^{\frac{1}{3}}(\phi_{00}/z_0)$ on the other zero. Therefore, in region 2 we may say that the change in Φ_0', $\Delta\Phi_0'$, is given by

$$\Delta\Phi_0'=-\tfrac{4}{3}(\phi_{00}/z_0). \quad (6.5)$$

In the drift region there is no electric field or change in ρ_0 so that, using Eq. (A4),

$$z_0\leq z\leq z_0+z_1,\quad \Phi_0=\Phi_{00},\quad \Phi_2=-\tfrac{1}{9}(\Phi_{00}/z_0^2). \quad (6.6)$$

Now in this region the applied magnetic field is just sufficient to counteract the space-charge spreading force. Hence, using Eq. (A18)

$$z_0\leq z\leq z_1\quad \omega_L^2=-\Phi_2=\tfrac{1}{9}(\Phi_{00}/z_0^2)\equiv\omega_{L0}^2. \quad (6.7)$$

Using the Φ_2 and ω_L of Eq. (6.4) in Eq. (6.1) it is seen that

$$0\leq z\leq z_0,\quad dw/dz=(u/z_0)(z_0/z)^{\frac{2}{3}}/(3\omega_{L0}),\quad du/dz=0, \quad (6.8)$$

which may be integrated to give

$$0\leq z\leq z_0,\quad \begin{pmatrix} w_1 \\ u_1 \end{pmatrix}=\begin{pmatrix} 1 & (z/z_0)^{\frac{1}{3}}/\omega_{L0} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} W \\ U \end{pmatrix}, \quad (6.9)$$

where W, U are the values of (w,u) at $z=0$.

As one crosses region 2, the plane of the aperture, Eq. (6.1) can be integrated, using Eqs. (6.5), (6.7), and (A4) to give a change in w, u of Δw, Δu in crossing the aperture of

$$\Delta w=0,\quad \Delta u=-\tfrac{1}{4}w\Phi_{00}^{-\frac{1}{2}}\Delta\Phi. \quad (6.10)$$

Hence, if w_1, u_1 are the values of (w,u) before the aperture, and w_2, u_2 the values after then the transformation across the aperture is given by

$$\begin{pmatrix} w_2 \\ u_2 \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ \omega_{L0} & 0 \end{pmatrix}\begin{pmatrix} w_1 \\ u_1 \end{pmatrix}. \quad (6.11)$$

Finally, in the third region Eqs. (6.6) and (6.7) are satisfied so that

$$z_0\leq z\leq z_1+z_0,\quad dw/dz=(u/z_0)/(3\omega_{L0}),\quad du/dz=0, \quad (6.12)$$

which may be integrated to give

$$z_0\leq z\leq z_0+z_1 \begin{pmatrix} w_3 \\ u_3 \end{pmatrix}=\begin{pmatrix} 1 & (z-z_0)/(3z_0\omega_{L0}) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} w_2 \\ u_2 \end{pmatrix}$$

where w_2, u_2 are the values of w, u at the plane z_0. We are now in a position to find the transfer matrices from the aperture to the plane (z_0+z_1) and from the cathode to the plane (z_0+z_1). From the definition of these matrices, Eqs. (2.31) and (A28), and from Eqs. (6.9), (6.11), and (6.13) it is seen that

$$\begin{pmatrix} M_0 & S_0 \\ \dot{M}_0 & \dot{S}_0 \end{pmatrix}=\begin{pmatrix} 1 & (z_1/z_0)/(3\omega_{L0}) \\ 0 & 1 \end{pmatrix} \quad (6.14)$$

and

$$\begin{pmatrix} M & S \\ \dot{M} & \dot{S} \end{pmatrix}=\begin{pmatrix} 1 & (z_1/z_0)/(3\omega_{L0}) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \omega_{L0} & 1 \end{pmatrix}\begin{pmatrix} 1 & 1/\omega_{L0} \\ 0 & 1 \end{pmatrix}$$
$$=\begin{pmatrix} 1+(z_1/z_0)/3 & (3+2z_1/z_0)/(3\omega_{L0}) \\ \omega_{L0} & 2 \end{pmatrix}. \quad (6.15)$$

Only because we chose such a simple example is it possible to evaluate the matrices analytically. It is to be noted that all the component matrices which make up the right hand side of Eqs. (6.14) and (6.15) have unit determinant, i.e., satisfy Eq. (A29).

Now that the matrix elements have been found, the current density distribution at the plane (z_0+z_1) may be found by directly applying the formulation of Sec. 4.

Using Eqs. (3.6) and (4.2), and remembering, from Eq. (6.3) that $\Omega_L=0$, it is seen that

$$\alpha=1+(z_1+z_0)/3,\quad \beta_1=\omega_{L0}/\alpha, \quad (6.16)$$

so that from Eqs. (4.5), (4.6), (4.10), (6.7), and (6.14)

$$C_2=0,\quad C_1=(\alpha+i)\omega_{L0}/(\alpha^2-1),\quad b_2=a_0,$$
$$b_1^2=-\frac{\Phi_{00}}{\alpha^2-1}\left[1-\frac{r^2}{9z_0^2}\left(\frac{\alpha^2+1}{\alpha^2-1}\right)\right],\quad \Delta_3=0. \quad (6.17)$$

In this case $\alpha>1$, hence if we are sufficiently near the axis for b^2 to be negative which implies, from Eqs. (6.7) and (6.17)

$$9z_0^2>r^2[\alpha^2+1/(\alpha^3-1)], \quad (6.18)$$

then from Eq. (4.10) and (4.12) the current density at radius r, $i(r,z)/i_c$ is given by

$$i(r,z)/i_c=(1/\alpha^2)[1-I(0,\lambda a_0\alpha)]$$
$$=1(/\alpha^2)[1-e^{-\lambda^2 a_0^2\alpha^2}]. \quad (6.19)$$

In Eq. (6.19) α is given by Eq. (6.16) and λ by Eq. (2.9).

This example was not supposed to be intrinsically interesting, it was merely to illustrate the method. In general the matrix elements must be found by integrating an equation such as Eq. (A20) numerically. However, once the matrix elements have been found, the application of the formulae of Sec. 4 to a simple case is not difficult.

APPENDIX A. THE MATRIX FORMULATION FOR THE TRANSFORMATION OF TRANSVERSE COORDINATES

In this appendix the equations of motion for an axially symmetric solid beam are derived to the paraxial approximation. It is shown that transverse position and velocity at one axial plane are related to those at another by a 2×2 matrix with complex coefficients. By suitable choice of variables, this matrix is shown to have unit determinant.

Inside a steady beam, Poisson's equation is

$$\mathbf{E}=-\nabla\phi,\quad \nabla^2\phi=-\rho/\epsilon_0, \quad (A1)$$

where $\mathbf{E}$ is the electric field, ϕ is the electrostatic potential, ρ the space-charge density, and ϵ_0 the dielectric constant. To the paraxial approximation, the effect of transverse variations in ρ on ϕ may be neglected. To this approximation, for axially symmetric solid beams, Eq. (A1) can be shown to give, for small r,

$$\phi\approx\phi_0(z)-\tfrac{1}{4}r^2[\phi_0''(z)+\rho_0(z)/\epsilon_0], \quad (A2)$$

where ϕ_0, ρ_0 are the potential and charge density on the axis, and prime "′" denotes d/dz.

If ϕ is given by Eq. (A2), the electric field (E_x,E_y,E_z) has the form, from Eq. (A1),

$$\mathbf{E}=(2x\phi_2,\ 2y\phi_2,\ -\phi_0'), \quad (A3)$$

where ϕ_2 denotes the expression

$$\phi_2=\tfrac{1}{4}[\phi''(z)+\rho_0(z)/\epsilon_0]. \quad (A4)$$

In the same way, if we ignore the self-magnetic field of the beam, then the magnetic equations are

$$\mathbf{B}=\nabla\psi,\quad \nabla^2\psi=0, \tag{A5}$$

where $\mathbf{B}$ is the magnetic field and ψ the magnetic scalar potential. Again for small r in an axially symmetric solid beam Eq. (A5) has the solution

$$\psi=\psi_0(z)-\tfrac{1}{4}r^2\psi_0''(z), \tag{A6}$$

so that the magnetic field (B_x,B_y,B_z) is given by

$$\mathbf{B}=[-\tfrac{1}{2}xB_0'(z),\ -\tfrac{1}{2}yB_0'(z),\ B_0(z)], \tag{A7}$$

where B_0 is the longitudinal magnetic field on the axis ψ_0'. While it is possible to add a constant azimuthal field B_θ, with extra components $(-B_\theta y/r,\ B_\theta x/r,\ 0)$, this field cannot exist without a central conductor, and is, therefore, ignored.

If $\mathbf{E}$ and $\mathbf{B}$ are given by Eqs. (A3) and (A7), we are in a position to solve Lorentz's equation

$$d\mathbf{v}/dt=-\eta[\mathbf{E}+(\mathbf{v}\times\mathbf{B})], \tag{A8}$$

where $\mathbf{v}$ is the velocity of the particle, and η is $|e|/m$. A first integral of Eq. (A8) gives the energy equation

$$v^2-2\eta\phi=\text{const.} \tag{A9}$$

Substitution of the expressions

$$\Phi_0(z)=2\eta\phi_0,\quad \Phi_2(z)=2\eta\phi_2 \tag{A10}$$

and of Eq. (A2) into Eq. (A9) yields the energy balance equation in the form

$$v^2-\Phi_0(z)+\Phi_2(z)r^2=\text{const.} \tag{A11}$$

The transverse equations of motion may be written, using the expressions for $\mathbf{E}$, $\mathbf{B}$, of Eqs. (A3) and (A7) in Eq. (A8),

$$\begin{aligned}\ddot{x}&=-\eta(2x\phi_2+\dot{y}B_0+\tfrac{1}{2}\dot{z}B_0'y),\\ \ddot{y}&=-\eta(2y\phi_2-\dot{x}B_0-\tfrac{1}{2}\dot{z}B_0'x).\end{aligned} \tag{A12}$$

We may now define the Larmor frequency ω_L by

$$\omega_L=\tfrac{1}{2}\eta B_0, \tag{A13}$$

then the time derivative of ω_L, $\dot{\omega}_L$, is given by

$$\dot{\omega}_L=\tfrac{1}{2}\eta\dot{z}B_0'. \tag{A14}$$

To this order of approximation, since $\dot{\omega}_L$ only appears in the product $(\dot{\omega}_L y)$, $\dot{\omega}_L$ may be considered as constant for all particles. By some algebra, Eq. (A12) can be written in the form:

$$(\ddot{x}+i\ddot{y})=(-\Phi_2+i\dot{\omega}_L)(x+iy)+2i\omega_L(\dot{x}+i\dot{y}). \tag{A15}$$

Equation (A15) has been derived previously by other authors, e.g., Herrman. Since the equation is linear in $(x+iy)$ and its derivatives, a linear transformation could be derived relating the $(x+iy)$, $(\dot{x}+i\dot{y})$ at one time, and, therefore, plane, with those at another. However, the solutions of Eq. (A15) are complex, and, therefore, it is first convenient to make the transformation

$$w=(x+iy)e^{-i\chi}, \tag{A16}$$

where χ is the rotation through the Larmor angle

$$\chi=\int\omega_L dt. \tag{A17}$$

In terms of these variables, Eq. (A15) becomes

$$\ddot{w}+(\Phi_2+\omega_L^2)w=0. \tag{A18}$$

If we prefer z as the independent variable, we may use the relation

$$d/dt=\dot{z}(d/dz)=\Phi_0^{\frac{1}{2}}(d/dz) \tag{A19}$$

to first order, to obtain from Eq. (A18) the relation

$$dw/dz=\Phi_0^{-\frac{1}{2}}u,\quad du/dz=-(\Phi_2+\omega_L^2)\Phi_0^{-\frac{1}{2}}w, \tag{A20}$$

where u is the transverse velocity $\dot{w}$ in the rotated coordinate system.

It is to be noted that Eq. (A18) is linear in w—whether or not there is magnetic flux threading the cathode (i.e., $\omega_L\neq 0$ at $z=0$). The paraxial equations which may be derived to give the beam boundary are nonlinear (in fact have a singularity as $w\to 0$); the equations given here are quite different, however. They are the equations for the motion of particles inside the beam. However, the equations assume we know Φ_2 which is related, via Eqs. (A4) and (A10), to the space-charge density. If the total current in the beam is I, and the beam radius $a(z)$, in the zero-temperature approximation, then the space-charge density $\rho_0(z)$ is given by

$$\rho_0=I\Phi_0^{-\frac{1}{2}}/(\pi a^2). \tag{A21}$$

Hence in order to set up Eqs. (A18) and (A20), it is first necessary to solve the paraxial equation for the beam edge (c.f. Ref. 10).

The energy balance equation, Eq. (A11) involved the square of the velocity v. It is important to relate this quantity to u. From Eqs. (A16), (A17), (A19), and (A20) we see that if the velocity is $(\dot{x},\dot{y},\dot{z})$ then

$$(\dot{x}+i\dot{y})=(d/dt)(we^{i\chi})=(u+i\omega_L w)e^{i\chi}. \tag{A22}$$

Hence we may deduce that v^2 is given by

$$v^2=\dot{x}^2+\dot{y}^2+\dot{z}^2=|u+i\omega_L w|^2+\dot{z}^2. \tag{A23}$$

Now $|w|$ is simply the distance r from the axis; hence the energy balance equation, Eq. (A11), may be written in the form

$$|u+i\omega_L w|^2+\dot{z}^2-\Phi_0(z)+\Phi_2|w|^2=\text{const.} \tag{A24}$$

The solutions of Eq. (A20) are real if the initial values of w, u are real. It is easily verified that if w_2, u_1 and w_2, u_2 are any two solutions of Eq. (A20), then

$$d(w_1u_2-w_2u_1)/dz=0. \tag{A25}$$

[10] P. T. Kirstein, J. Electron. Control 8, 207 (1960).

Let us choose $w=M(z)$, $v=\dot{M}(z)$, and $w=S(z)$, $v=\dot{S}(z)$ as the solutions of Eq. (A20) satisfying

$$M(0)=1,\quad \dot{M}(0)=0,\quad S(0)=0,\quad \dot{S}(0)=1. \tag{A26}$$

Then it is seen that $M(z)$ is the *magnification* of the beam, being the change in distance from the axis of a particle starting at unit distance from the axis with zero transverse velocity; $S(z)$ is the *stiffness* of the beam, being the distance from the axis reached by a particle starting on the axis with unit transverse velocity. It is to be noted that M, S, $\dot{M}$, $\dot{S}$ do not, to this order of approximation, depend on $\dot{Z}$, the longitudinal velocity at the cathode.

If now a particle has initial position and velocity components such that

$$w(0)=W,\quad u(0)=U, \tag{A27}$$

then these coordinates transform so that at any other plane,

$$\begin{pmatrix} w \\ u \end{pmatrix}=\begin{pmatrix} M & S \\ \dot{M} & \dot{S} \end{pmatrix}\begin{pmatrix} W \\ U \end{pmatrix}. \tag{A28}$$

From Eqs. (A25) and (A26), we see that the matrix in Eq. (A29) has unit determinant

$$M\dot{S}-\dot{M}S=1. \tag{A29}$$

Because the ray axis of the paraxial system is straight (in this case it is the z axis), the change in transverse position due to small change in initial longitudinal velocity ϵ would be of the order ϵW, ϵU, and so could be neglected to this order of approximation. By identical arguments it can be shown that, for hollow beams with a curvilinear ray axis, the transformation laws have the form

$$\begin{pmatrix} w \\ u \end{pmatrix}=\begin{pmatrix} M & S \\ \dot{M} & \dot{S} \end{pmatrix}\begin{pmatrix} W \\ U \end{pmatrix}+\frac{\epsilon W}{R}\begin{pmatrix} \Delta \\ \dot{\Delta} \end{pmatrix}, \tag{A30}$$

where w, W, u, U, ϵ are as above, R is the instantaneous radius of the ray axis, and Δ, $\dot{\Delta}$ depend only on z through the electromagnetic fields. For hollow beams with a straight ray axis, Δ, $\dot{\Delta}$ are zero, though M, $\dot{M}$, S, $\dot{S}$, are related to the fields through slightly different formulae.

For sheet beams with no variation in the y direction, the $\epsilon W/R$ term becomes ϵW, but formulae similar to Eq. (A30) arise. Again if the ray axis is straight, Δ, $\dot{\Delta}$ are zero.

APPENDIX B. CHANGE OF COORDINATES AND THE JACOBIAN

In the body of the paper we often wish to transform variables from an orthogonal coordinate system(q_1,q_2,q_3) to one (Q_1,Q_2,Q_3) where these are related by the transformations

$$q=q_1+iq_2,\quad Q=Q_1+iQ_2,$$
$$q=CQ+D,\quad q_3^2=Q_3^2+E, \tag{B1}$$

where C, D are independent of Q_1, Q_2, Q_3 and E is independent of Q_3. Now the volume element $dq_1dq_2dq_3$ in the q_i system is related to that in the Q_i system $dQ_1dQ_2dQ_3$ by the expression

$$dq_1dq_2dq_3=JdQ_1dQ_2dQ_3, \tag{B2}$$

where J is the Jacobian of the transformation, and is given by the determinant

$$J=\begin{vmatrix} \partial q_1/\partial Q_1 & \partial q_1/\partial Q_2 & \partial q_1/\partial Q_3 \\ \partial q_2/\partial Q_1 & \partial q_2/\partial Q_2 & \partial q_2/\partial Q_3 \\ \partial q_3/\partial Q_1 & \partial q_3/\partial Q_2 & \partial q_3/\partial Q_3 \end{vmatrix}. \tag{B3}$$

Now from Eq. (B1) we have the expressions

$$\partial q_1/\partial Q_3=\partial q_2/\partial Q_3=0,\quad \partial q_3/\partial Q_3=Q_3/q_3, \tag{B4}$$

and

$$(\partial q_1/\partial Q_1)\partial q_2/\partial Q_2-(\partial q_1/\partial Q_2)\partial q_2/\partial Q_1$$
$$=|\partial q/\partial Q|^2=|C|^2. \tag{B5}$$

Hence the Jacobian has the value $Q_3|C|^2/q_3$, and Eq. (B2) becomes

$$dq_1dq_2dq_3=Q_3|C|^2/q_3dQ_1dQ_2dQ_3. \tag{B6}$$

This result is required in the text. Note that the transformation is impossible if the Jacobian vanishes, i.e., $|C|$ is zero.

APPENDIX C. A SPECIAL INTEGRAL

We consider a function p which may take complex values, such that

$$|p-C|\geq b, \tag{C1}$$

where C is complex so that

$$C=\Gamma e^{i\gamma}, \tag{C2}$$

and Γ, γ, b are real. An integral which occurs frequently in this paper is

$$I(\Delta,b)=\frac{1}{\pi}\int\int \exp-|p-D|^2dp_xdp_y, \tag{C3}$$

where p_x, p_y may range over all values defined by Eq. (C1), and Δ is defined by

$$C-D=\Delta e^{i\delta}. \tag{C4}$$

Substituting

$$p=Qe^{iq}+C \tag{C5}$$

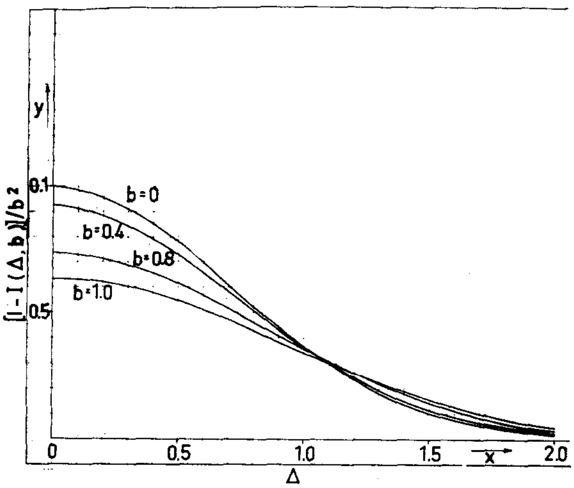

FIG. 2. Plot of $[1-I(\Delta,b)]/b^2$ vs Δ for $b\leqslant 1$.

into Eq. (C1) and (C3) it is seen that

$$
\begin{aligned}
I(\Delta,b)
&=\frac{1}{\pi}\int_b^\infty QdQ\int_0^{2\pi}\exp-|Q+\Delta e^{i(\delta-q)}|^2 dq\\
&=\frac{1}{\pi}\int_b^\infty QdQ\int_0^{2\pi}\exp-[Q^2+\Delta^2+2Q\Delta\cos(\delta-q)]dq\\
&=\int_b^\infty 2Q\exp-[Q^2+\Delta^2]I_0(2Q\Delta)dQ. \qquad \text{(C6)}
\end{aligned}
$$

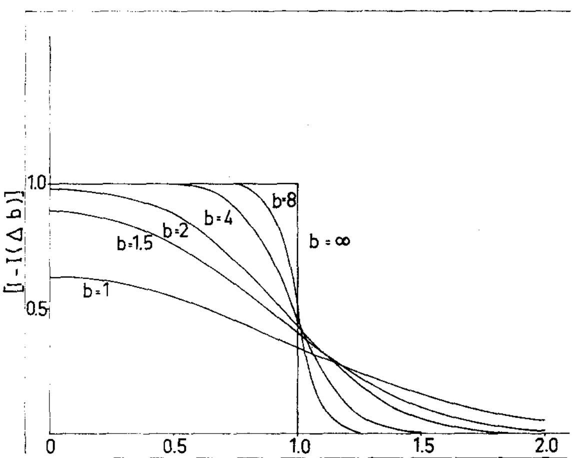

FIG. 3. Plot of $1-I(\Delta,b)$ vs Δ/b for $b\geqslant 1$.

In Eq. (C6) I_0 is the modified Bessel function of order zero, and we have used the relation

$$I_0(x)=\frac{1}{2\pi}\int_0^{2\pi}\exp(x\cos\theta)d\theta. \qquad \text{(C7)}$$

$I(\Delta,b)$ is plotted versus Δ, b in Figs. 2 and 3. When Δ is zero, i.e., $C=D$, Eq. (C6) takes the very simple form

$$I(0,b)=\int_b^\infty 2Qe^{-Q^2}dQ=e^{-b^2}, \qquad \text{(C8)}$$

while if $b\rightarrow 0$ it may be shown that

$$
\begin{aligned}
&[1-I(\Delta,b)]/b^2\\
&\quad=\int_0^b (2/b^2)Qe^{-(Q^2+\Delta^2)}I_0(2Q\Delta)dQ\rightarrow e^{-\Delta^2}. \qquad \text{(C9)}
\end{aligned}
$$

Finally if $b=\infty$,

$$I(\Delta,\infty)=0. \qquad \text{(C10)}$$

If the integral of $\exp-(p-D)^2$ is required in the region between the nonintersective circles

$$|p-C_1|=b_1,\quad |p-C_2|=b_2, \qquad \text{(C11)}$$

where the circle with suffix 2 completely surrounds that with suffix 1, then it is clear from Eq. (C6) that the integral is

$$
\begin{aligned}
I=\frac{1}{\pi}\iint_{\substack{|p-C_2|<b_2\\ |p-C_1|>b_1}}&\exp-(p-\Delta)^2dp_xdp_y\\
&=I(\Delta_1,b_1)-I(\Delta_2,b_2), \qquad \text{(C12)}
\end{aligned}
$$

where Δ_1, Δ_2 are given by

$$|C_1-D|=\Delta_1,\quad |C_2-D|=\Delta_2. \qquad \text{(C13)}$$

When the circles intersect, Eq. (C12) is more complicated. It is still possible to eliminate one of the integrals, but, for simplicity, we do not consider this case in this paper. The conditions for nonintersection are that

$$|C_2-C_1|<b_2-b_1 \qquad \text{(C14)}$$

and Eq. (C14) is only valid if this condition is satisfied.