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Paraxial Formulation of the Equations of Electrostatic Space-Charge Flow*

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The equations of electrostatic irrotational space-charge flow are set up in a paraxial manner. Given an arbitrary trajectory, and the potential distribution along that trajectory, the variation of such quantities as potential and beam convergence in the vicinity of the specified trajectory are investigated. A detailed analysis is carried out for beams from a space-charge-limited cathode in the form of a cone. The agreement between the paraxial theory and the exact theory for particular beams is investigated. Numerical results are presented for beams from a conical cathode which predict beams with a microperveance of about 15 and a very large area convergence. It is felt that by use of this method much higher perveances and beam convergences could be obtained for hollow and sheet beams than have heretofore been possible.

1. INTRODUCTION

THE subject of curvilinear, electrostatic electron beams in which space-charge effects cannot be neglected has been treated by many authors. Some authors, e.g., Meltzer[1] and Walker,[2] have tried to find a few particular solutions to the equations for electrostatic space-charge flow. Kirstein[3,4] has found a more general, but still rather restricted, set of solutions for curvilinear flow in which the partial differential equations for the flow separated. While he was able to obtain many examples of such flow from space-charge-limited cathodes, which are, of course, of particular interest in electron gun design, the number of such solutions which were discovered was small. A recent paper by Meltzer[5] suggested forming a coordinate system in which all the electrons flow along the level lines of two of the coordinates. In practice, such a coordinate system is very difficult to construct, and this procedure has not, to the author's knowledge, been followed except in some very simple instances.

The authors so far cited have sought exact solutions to the equations of irrotational laminar flow. Sturrock,[6] on the other hand, has developed a very promising paraxial formulation to deal with the problems of beams in which the space-charge has small effect, or is constant over the beam cross section. He defines a certain trajectory, S, and specifies the potential distribution on S; he then forms a coordinate system based on S. He assumes that the electrons near this trajectory will flow more or less parallel to it, and makes paraxial predictions of such quantities as the convergence of the beam. This is rather similar to Meltzer's method of defining the coordinate system such that all electrons flow along level lines of two of the coordinates. However, it is a much simpler coordinate system to construct, and Sturrock has obtained many useful examples of electrostatic and even electromagnetic flow using this system. In this paper we combine Meltzer's suggestion with Sturrock's formalism and develop the paraxial equations of irrotational electrostatic space-charge flow, for axially symmetric hollow beams and sheet beams. We then consider in detail the problem of axially symmetric hollow beams from a conical cathode, and develop the formalism for such beams. We state the equations for one higher order approximation, but do little to solve them. We merely use them to develop an expression for the current density variation at the cathode in terms of the initial curvature of the beam. We carry out some computations on beams from a conical cathode, and show graphically how the convergence is dependent on the potential distribution of the specified trajectory. We show that the theory predicts beams of high convergence with high perveance. As an illustration, we obtain a beam of microperveance 15 with a paraxially infinite convergence. It is to be realized, of course, that the higher order terms may greatly affect this convergence; since we are dealing with hollow beams, the convergence is both in thickness and in radius; hence, a convergence of about 10 or 20 with a microperveance of about 15 seems indicated by the theory for the particular solution presented. However, it would be possible to obtain much higher perveances from this theory by particular choice of potential distribution. We feel, however, that such effort would be wasted until it has been found by experiment that it is possible to construct the more modest beams predicted in this paper.

One disadvantage of a paraxial formulation is that it is very difficult to predict how far the results are valid, since higher order terms are neglected. To obtain some idea of the limitations of the paraxial formulation, we compare the performance figures of a specific beam from a conical cathode which has been obtained elsewhere by an exact solution.[3,4] We show that for conical beams found by the exact solution the variation of current density at the cathode agrees with that obtained in this paper. We show that there is a very good agreement between such quantities as convergence and potential at

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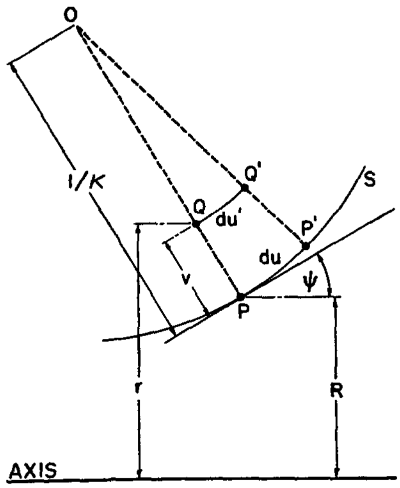

FIG. 1. Diagram of the coordinate system for the paraxial formulation.

the beam edge for results obtained by the paraxial and by the exact methods. This seems to indicate that the paraxial formulation developed in this paper should serve as a very useful basis for design of hollow electrostatic or sheet electrostatic beams with arbitrary trajectories and arbitrary potential distribution on that trajectory.

In Sec. 2 the coordinate system used is described. The expressions for important vector quantities such as ∇ are developed. In Sec. 3 the equations of irrotational electrostatic flow are described, and they are reformulated in paraxial terms in the coordinate system of Sec. 2. In Sec. 4 it is shown how this formulation is particularly convenient for the study of axially symmetric hollow beams from a conical cathode, and the relevant formulas are derived. In Sec. 5 a comparison is given between the results of an exact theory and those of the paraxial theory for conical beams. In Sec. 6 some computations of the paraxial theory are presented, showing the characteristics of beams from a conical cathode. In Sec. 7 the problem of electrode design is considered, and the conclusions which may be drawn from this paper are summarized.

2. COORDINATE SYSTEM OF THE PARAXIAL FLOW

In the paraxial method one defines a certain trajectory, and the potential along it, and considers how electrons near that specified trajectory will behave. Since the *a priori* information is given defined in terms of and along one trajectory, it seems reasonable to define our coordinate system such that one coordinate axis, which we will call the u axis, lies along this trajectory, while the orthogonal curves, $u=$const, are the normals to the trajectory. An orthogonal planar coordinate system is usually determined by the element of arc length and, in general, has the form

$$dl^2=h_1^2(du)^2+h_2^2(dv)^2. \tag{2.1}$$

The coordinate system for the paraxial method is sketched in Fig. 1. Here Q is an arbitrary point (u,v). The normal to S from Q cuts S at P. The curvature at P is κ, while the center of curvature is 0. The distance QP is v. The specified trajectory S is $v=0$. The point Q' is $(u+du, v)$, and the arc length QQ' is du'. The projection of Q' on S from 0 is P'. Then P' is the point $(u+du, 0)$ and the arc length PP' is du. From the diagram, it is seen that the arc length QQ', du', is given by

$$du'=(1/\kappa-v)\kappa du=(1-\kappa v)du. \tag{2.2}$$

Hence, the coordinate system is defined by the expression

$$dl^2=dv^2+(1-\kappa v)^2du^2. \tag{2.3}$$

Equation (2.3) defines a planar coordinate system, which has previously been given elsewhere.[6] We will, in most of this work, be interested in axially symmetric coordinate systems. Such a coordinate system is defined by

$$dl^2=(1-\kappa v)^2du^2+dv^2+r^2d\theta^2, \tag{2.4}$$

where θ is the angle between the plane through P and the axis to some fixed plane through the axis, and r is the distance of Q from that axis. In that case, if the distance of P from the axis is R, the distance of Q from the axis is given by

$$r=R+v\cos\psi, \tag{2.5}$$

where ψ is the angle between the trajectory line S and the axis. For convenience, we now define h by

$$h=1-\kappa v. \tag{2.6}$$

The expressions for the products of importance in the evaluation of the vector operators are then

$$\left.\begin{aligned} rh&=R+v(\cos\psi-\kappa R)-v^2\kappa\cos\psi \\ r/h&=R+v(\cos\psi+\kappa R)+\cdots \end{aligned}\right\}. \tag{2.7}$$

For the problem we are considering in this paper, namely, that of axially symmetric electrostatic beams with no rotation, all scalars will be independent of θ, and all vectors will have no component in the θ direction; also, each component of a vector will be independent of θ. In this case, if A_1 and A_2 are the components in the u and v directions of the vector $\mathbf{A}$, and if vectors $\mathbf{A}$ and scalars a are of the sort we require, then the expressions for the gradient, divergence, and Laplacian of scalars and vectors are given by

$$\left.\begin{aligned} \nabla\cdot\mathbf{A}&=\frac{1}{rh}\left[\frac{\partial}{\partial u}(rA_1)+\frac{\partial}{\partial v}(rhA_2)\right] \\ \nabla a&=\left(\frac{1}{h}\frac{\partial a}{\partial u},\frac{\partial a}{\partial v},0\right) \\ \nabla^2 a&=\frac{1}{rh}\left[\frac{\partial}{\partial u}\left(\frac{r}{h}\frac{\partial a}{\partial u}\right)+\frac{\partial}{\partial v}\left(rh\frac{\partial a}{\partial v}\right)\right] \end{aligned}\right\}. \tag{2.8}$$

3. EQUATIONS OF SPACE-CHARGE FLOW AND THEIR PARAXIAL FORMULATION

Many authors, e.g., Kirstein,[3,4] have stated the equations for irrotational space-charge flow. These equations are particularly appropriate for the problem we have in mind in this paper, namely, that of the formation of electrostatic beams under space-charge-limited conditions from a cathode neglecting thermal velocities. Under these conditions it is shown that the velocity may be derived from a scalar quantity W, called the action function, by

$$\mathbf{V}=\nabla W. \tag{3.1}$$

Moreover, the usual Poisson's, conservation of energy, and continuity equations yield the relations between the potential Φ, the current density $\mathbf{j}$, and the charge density ρ

$$-2\eta\phi=V^2, \tag{3.2}$$

$$\nabla^2\Phi=-\rho/\epsilon, \tag{3.3}$$

$$\nabla\cdot\mathbf{j}=0. \tag{3.4}$$

Equations (3.1), (3.2), (3.3), and (3.4) could be combined into one fourth-order nonlinear partial differential equation as has been done.[3,4] This equation is very difficult to solve. In the earlier reference this equation was solved by the method of separation of variables, and certain solutions were found. However, we will now find solutions valid near some trajectory S and will formulate these in a paraxial manner. In this way we will find an alternative method of reducing the partial differential equations of Eqs. (3.1) through (3.4) to ordinary differential equations. For this purpose, we take an action function defined by

$$W=w_0(u)+vw_1(u)+v^2w_2(u)+v^3w_3(u)\cdots, \tag{3.5}$$

where the coordinate line $v=0$ is defined as some known trajectory with known curvature κ and known slope ψ. Now, if the action function has the form of (3.5), it is seen that the velocity is given, from Eq. (2.1), by

$$\mathbf{V}=[1/h(\dot{w}_0+v\dot{w}_1+v^2\dot{w}_2+v^3\dot{w}_3+\cdots), \\ w_1+2vw_2+3v^2w_3+\cdots,0], \tag{3.6}$$

where a dot denotes differentiation with respect to the arc length u. From Eq. (3.6) we see that, if the curve $v=0$ is to be a trajectory, the velocity must lie along this curve, and, therefore, w_1 must be zero. In this case, from Eqs. (3.1) and (3.6) we obtain the formula for the potential

$$-2\eta\Phi=\phi_0+\phi_1v+\phi_2v^2+\phi_3v^3+\cdots, \tag{3.7}$$

where

$$\left.\begin{aligned}\phi_0&=\alpha^2,\\ \phi_1&=2\kappa\alpha^2,\\ \phi_2&=3\kappa^2\alpha^2+2(2w_2{}^2+\alpha\dot{w}_2)\end{aligned}\right\} \tag{3.8}$$

and

$$\dot{w}_0=\alpha. \tag{3.9}$$

In Eq. (3.8), $\dot{w}_0$ has been replaced by α to emphasize that α is a velocity previously determined on S. We have replaced h by the expressions given in the previous section. In a similar way to the derivation of Eqs. (3.7) and (3.8), we may show by some lengthy algebra that it is possible to derive from Eq. (3.3) and the equations of Sec. 2, the definition of $\mathbf{j}$, the expressions

$$2\eta\rho/\epsilon=\rho_0+v\rho_1+\cdots, \tag{3.10}$$

$$2\eta\mathbf{j}/\epsilon=(j_{10}+vj_{11}+\cdots,vj_{21}+v^2j_{22}+\cdots,0), \tag{3.11}$$

where

$$\rho_0=\frac{1}{R}\frac{d}{du}(R\phi_0)+2\phi_2+\frac{\phi_1}{R}(\cos\psi-\kappa R), \tag{3.12}$$

$$\rho_1=\frac{(\kappa R-\cos\psi)}{R}\rho_0+\frac{1}{R}\frac{d}{du}[R\phi_1+(\kappa R+\cos\psi)\phi_0] \\ +\frac{2}{R}[3\phi_3R+2(\cos\psi-\kappa R)\phi_2-\kappa\cos\psi\phi_1] \tag{3.13}$$

and

$$\left.\begin{aligned}j_{10}&=\rho_0\alpha\\ j_{11}&=\kappa\alpha\rho_0+\rho_1\alpha\\ j_{21}&=2w_2\rho_0\end{aligned}\right\}. \tag{3.14}$$

The detailed expressions of Eqs. (3.12) and (3.13) do not concern us yet; they are merely given for later reference. Finally, the equation of continuity (3.4) yields, using (3.14) and the equations of Sec. 2,

$$\left[\frac{1}{R}-\frac{v}{R^2}(\cos\psi-\kappa R)+\cdots\right] \\ \times\left\{\frac{\partial}{\partial u}[(R+v\cos\psi)(j_{10}+vj_{11}+\cdots)] \\ +\frac{\partial}{\partial v}[Rvj_{21}+\cdots]\right\}=0. \tag{3.15}$$

From considering terms of different order in v, we may now obtain differential equations for w_2, w_3, etc. These soon become very complicated, and for the purpose of this paper the terms of lowest order will suffice. These yield the equation

$$\frac{d}{du}(Rj_{10})+Rj_{21}=0. \tag{3.16}$$

Equation (3.16), with its defining equations, is the paraxial equation. The higher order equations describe the departure from paraxial conditions.

It will be shown that Eq. (3.16) is sufficient to give most of the information we require for beam design, as long as we fulfill the paraxial conditions. In particular, we shall show that it allows us to predict the convergence of the beam. It does not, however, allow us to predict the variation of current density across the cathode. This current variation requires, from Eq. (3.14), a knowledge of ρ_1. While, from Eq. (3.13), it is seen that ρ_1 depends on ϕ_3, which in turn involves the third derivative of W, w_3, we will show later that a knowledge of w_3 is not essential for a first-order prediction of ρ_1 near the cathode.

We now derive an expression for the convergence of the beam. From the definition of the coordinate system, we see that if the edge of the beam is defined by

$$v=\pm v_1(u), \tag{3.17}$$

then the ratio $v_1(0)/v_1(u)$ gives the thickness convergence of the beam, while $R(0)v_1(0)/R(u)v_1(u)$ gives the area convergence. Now from Eq. (3.6), keeping only the lowest terms in v, it is seen that

$$\frac{dv_1}{du}=\frac{V_v}{V_u}=\frac{2w_2}{\alpha}v_1, \tag{3.18}$$

using the definition of α of Eq. (3.19). From Eqs. (3.14) and (3.16), we see that

$$\frac{d}{du}(R\rho_0\alpha)=-\frac{2w_2}{\alpha}(R\rho_0\alpha). \tag{3.19}$$

Defining

$$z_2=R\rho_0\alpha, \tag{3.20}$$

we see, from Eqs. (3.18) and (3.19), that

$$v_1z_2=\text{const}$$

so that the thickness convergence is given directly by $z_2(u)/z_2(0)$. Hence we automatically obtain the convergence by solving the paraxial equation (3.16). It is to be noted that $z_2(u)$ can become infinite, leading to a prediction of infinite convergence. This prediction might not really be valid, since in this case it would be necessary to consider higher order terms in the definition of convergence. However, it would indicate infinite ρ_0, if it occurred at finite α.

It still remains to consider the boundary conditions under which the equations must be solved, and then their actual solution. The latter can only, of course, be performed numerically, however we shall obtain both qualitative and quantitative results for the performance to be expected from electrostatic beams.

One incidental but important fact should be mentioned. The formalism of this section may be applied directly to sheet beams by two simplifications of the equations. For sheet beams, all terms involving $\cos\psi$ should be omitted, and R must be independent of u.

4. ELECTROSTATIC HOLLOW BEAMS FROM A CONICAL CATHODE

The formalism of the previous section could, it seems, be applied to the formation of all electrostatic beams under the assumption that thermal velocities are negligible, and the beams come from a space-charge-limited cathode, provided the paraxial conditions apply. This last proviso inhibits the application of the formalism to solid beams. For solid beams, it would be necessary to make the axis the fundamental trajectory S, $v=0$; and, in this case, the formalism would be so much simpler that it is unnecessary to use this approach. Moreover, it is particularly simple to apply boundary conditions if the cathode is conical. In this section we shall develop the paraxial theory of space-charge-limited beams from a conical cathode for numerical choice of potential; in Sec. 6 we will give detailed results.

If the cathode is conical, it may be chosen as the curve $u=0$. Now we know that near a plane diode, under space-charge-limited conditions, the potential varies as $u^{\frac{4}{3}}$. Similarly it has been shown[3] that near a conical cathode the potential still varies as $u^{\frac{4}{3}}$, so that $\dot{w}_i$ must all vary as $u^{\frac{2}{3}}$, and w_i as $u^{5/3}$. While the w_i may not have variation as low as $u^{5/3}$, there are certainly no terms of lower degree. Moreover, it is the $u^{5/3}$ variation which will be shown to affect the current density at the cathode.

We therefore assume that the cathode is the cone of angle $\psi_0+\pi/2$, so that the initial slope of the trajectory is ψ_0. Then

$$\psi=\psi_0+\kappa u+\cdots. \tag{4.1}$$

If the axial distance is Z,

$$\left.\begin{aligned}\dot{Z}&=\cos\psi\\ \dot{R}&=\sin\psi\end{aligned}\right\}. \tag{4.2}$$

For convenience, we choose the initial value of R as unity, so that

$$\dot{R}=\sin\psi_0+\kappa u\cos\psi_0+\cdots, \tag{4.3}$$

$$R=1+u\sin\psi_0+\tfrac{1}{2}\kappa u^2\cos\psi_0+\cdots. \tag{4.4}$$

Since we wish space-charge limited emission, we shall define α as

$$\alpha=u^{\frac{2}{3}}(1+a_1u+a_2u^2). \tag{4.5}$$

To a certain extent, α may be defined arbitrarily; however, if we consider flow from a planar diode and from the inside of a cylinder, we see that a_1 is determined by the slope of the cathode. This will be demonstrated in this section. For the moment we shall consider a_1 and a_2 arbitrary.

As mentioned earlier, it is necessary for w_i to have a $u^{5/3}$ variation, since the cathode is conical; hence, we define

$$\left.\begin{aligned}w_2&=u^{5/3}(b_0+b_1u+\cdots)\\ w_3&=u^{5/3}(c_0+\cdots)\end{aligned}\right\}. \tag{4.6}$$

It would be possible, of course, to assume a u^s variation to w_i. This has been done elsewhere,[3] and it was found that s was 5/3. The constants b_i, c_i are unknown. They may be found from the equations of the previous section. In this section we shall find b_0 and show that the variation of the current density at the cathode may be determined without knowledge of c_0.

From Eqs. (3.19), (4.5), and (4.6),

$$\frac{d}{du}(R\rho_0\alpha) = -2b_0(R\rho_0\alpha)u + \cdots; \qquad (4.7)$$

hence,

$$R\rho_0\alpha = d_0(1 - b_0u^2 + \cdots). \qquad (4.8)$$

Substitution of Eq. (4.8) into (3.12) now yields

$$d_0(1 - b_0u^2 + \cdots) = \alpha\frac{d}{du}(2R\alpha\dot{\alpha}) + \alpha[2R\phi_2 + \phi_1(\cos\psi - \kappa R)]. \qquad (4.9)$$

The first term on the right of Eq. (4.9) is of order unity; the second is, from Eq. (3.8), of order u^2. Substituting Eqs. (3.8), (4.4), (4.5), and (4.6) into Eq. (4.9) and comparing different orders in u yields, from the coefficient of u^0,

$$d_0 = (4/9); \qquad (4.10)$$

from the coefficient of u,

$$\left.\begin{aligned} (4/9)a_1 + (16/9)\sin\psi_0 + (56/9)a_1 &= 0 \\ a_1 &= -(4/15)\sin\psi_0 \end{aligned}\right\}; \qquad (4.11)$$

and, from the coefficient of u^2,

$$b_0 = (67/200)\sin^2\psi_0 - (9/4)a_2 - (1/2)\kappa\cos\psi_0 - (9/16)\kappa^2. \qquad (4.12)$$

Equations (4.10) through (4.12), in combination with Eqs. (4.6) and (4.8), now allow us to solve the paraxial equation (3.16). Before we do this, however, let us consider the variation of current density across the cathode. To find this variation, we require ρ_1, which is, from Eq. (3.13), a formidable expression. However, only the first two terms contribute near the cathode, and it may be seen, almost by inspection of Eq. (3.13) and use of Eqs. (3.8), (4.6), and (4.8), that

$$\rho_1\alpha \simeq (\kappa - \cos\psi + 2\kappa + \kappa + \cos\psi)\rho_0\alpha = 4\kappa\rho_0\alpha. \qquad (4.13)$$

Using Eqs. (3.14), (4.8), and (4.10), we now see

$$j_{11} = (20/9)\kappa. \qquad (4.14)$$

We thus come to some interesting conclusions. From Eqs. (3.11), (4.10), and (4.14), we see that the current density at the cathode is given by

$$j_{\text{cathode}} = (4/9)(1 + 5\kappa v). \qquad (4.15)$$

To obtain high perveance beams, we wish to use as much cathode as possible. Hence, the initial curvature of the beam should be small.

A paraxial theory is rather unconvincing unless it is possible to give some idea of the range of its validity. This problem will be discussed in some detail in the next section.

5. COMPARISON BETWEEN THE PARAXIAL THEORY AND AN EXACT THEORY FOR BEAMS FROM A CONICAL CATHODE

In recent reports[3,4] the electrostatic equations of space-charge flow were solved under a particular set of conditions, in which the equations could be separated. Since these solutions were valid for infinite beams, they did not predict very startling characteristics from the point of view of convergence and perveance. However, since the computations on these beams have been performed in great detail, it is possible to compare the performance predicted by the paraxial method with that obtained by the exact analysis.

For this purpose, we shall first show that the relation between the variation of current density at the cathode and initial curvature is in agreement with the exact theory. In the reference quoted it was shown that the relevant variation of the physical quantities of the beam near the cathode are

$$\left.\begin{aligned} W &= (-2\eta)^{\frac{1}{2}}\tfrac{3}{5}r^m(\theta_0-\theta)^{5/3} \\ v_r &= (-2\eta)^{\frac{1}{2}}\tfrac{3}{5}mr^{m-1}(\theta_0-\theta)^{5/3} \\ v_\theta &= (-2\eta)^{\frac{1}{2}}r^{m-1}(\theta-\theta_0)^{\frac{2}{3}} \\ \Phi &= r^{2m-2}(\theta-\theta_0)^{\frac{4}{3}} \\ j_\theta &= (25/9)r^{3m-5} \end{aligned}\right\}, \qquad (5.1)$$

where r and θ are the usual variables of spherical polar coordinates, and θ_0 is the cone semiangle of the cathode. From these equations it can simply be deduced that the initial slope and curvature are given by

$$\left.\begin{aligned} \psi &= [\theta - (\pi/2)] + (v_r/v_\theta) = \theta - (\pi/2) - \tfrac{3}{5}m(\theta-\theta_0) \\ \kappa &= (1/r)(-1 + \tfrac{3}{5}m) \end{aligned}\right\}. \qquad (5.2)$$

Expanding the current density variation of Eq. (5.1) at the cathode, it is seen from the first term of the binomial expansion that this agrees with the paraxial result of Eq. (4.15), namely, that the current density variation at the cathode is proportional to the product of five times the curvature and the distance from the trajectory. Moreover, the exact expression for the current density variation gives us a good idea of how far the paraxial result of Eq. (4.15) is valid, and what modification we should make. If we take a cathode width of δ, we may compare $(1+\delta)^{5\kappa}$ with $1+5\kappa\delta$ to find out how much of the cathode we may use.

We have shown that the variation in current density of the cathode derived in the previous section agrees with that of the exact solution of beams from the conical cathode derived in the previous report. We shall now, in addition, compare the results of the paraxial and the exact methods for one particular beam. In this beam, $m=-1$, and the cone semiangle of the cathode is $\pi/2$, that is, $\psi_0 = 0°$; the curvature and potential variation could only be given numerically. The comparison between the theories is shown in Fig. 2. Here the dotted lines represent the exact theory for different beam

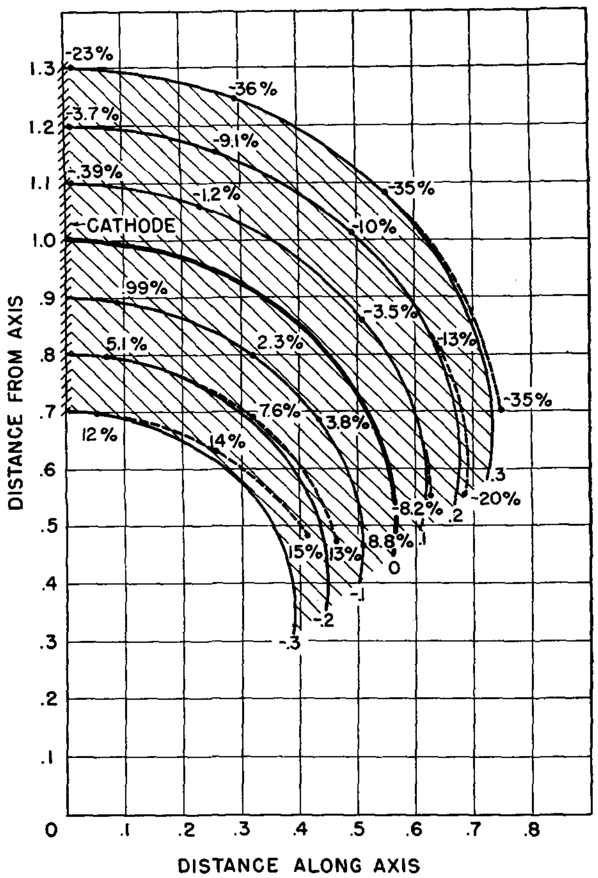

FIG. 2. Comparison of cross sections of the trajectories and potentials obtained for a beam from an annular cathode between exact and paraxial theories. Solid lines denote results of exact theory, broken lines the result of the paraxial theory for different beam thicknesses. The figures denote the percentage error in the potentials obtained by the paraxial method.

thicknesses, while the solid lines represent the results of paraxial theory. It is seen that for a thickness of about 0.1 or 0.2 of the original radius, the agreement is very good. The numbers at the edge of the beams show the percentage error of the potential as predicted by the exact and the paraxial methods. Here again the agreement is seen to be satisfactory for small thicknesses. It should be remembered that in the exact theory the potential in this beam varies as r^{-4}, the current density as r^{-8}, and the curvature is -1.6. Hence for a variation of current density at the cathode of two, one would only desire a v of ± 0.05. In this range the paraxial theory gives excellent agreement. The curves are principally shown to give an example of the agreement which can be expected by the paraxial method.

These examples illustrate several points. First, that it is not necessary for the curvature of the specified trajectory S and the potential variation on S to be given in any algebraic way, except, of course, near the cathode. It is sufficient for this variation to be given numerically. Secondly, although this is an isolated instance, the result encourages us to hope that the paraxial theory may give good agreement with an exact theory over the range of interest for hollow beams. It is unfortunately very difficult to compare the paraxial with an exact method for curvilinear beams, in general, since very few exact curvilinear solutions with convergent beams exist in the literature.

6. CONICAL CATHODE

In Sec. 4 we developed the formalism for paraxial beams, from a conical cathode, for an arbitrary trajectory and an arbitrary potential distribution with specific variation along the specified trajectory S. In the last section we compared a solution obtained by exact method with a paraxial one. In this section we shall, for the sake of simplicity, consider only a circular trajectory. It would have been equally simple to consider any other form of trajectory, but the principles of the preceding section are adequately illustrated by this example. We first give the characteristics of the different beams in which S, the specified trajectory, is circular with the potential variation on S of the previous section. We then show that, if we slightly modify the potential variation, we are able to predict beams with much more desirable characteristics. While the electrodes required to produce these beams will not be shown, some discussion on the design will be presented.

For convenience, we choose our trajectory such that S is a circle. Since we are interested in trying to converge electron beams, we shall choose a circle such that the final radius is one-fifth of its initial radius. This choice is arbitrary, and is made because with this particular radius we predict beams with favorable characteristics. Of course, it may well be possible to obtain better beams with different choices of trajectory, certainly with different choices of the radius of the circle.

If we choose the velocity variation of the previous section, namely,

$$\alpha = u^{\frac{2}{3}}(1+a_1u+a_2u^2), \tag{6.1}$$

then it was shown that a_1 was completely specified by the initial cone angle $(\pi/2)+\psi_0$ of the cathode. Since we have restricted our trajectory to be the arc of a circle, and have chosen the ratio of its final radius, when it is parallel to the axis, to its initial radius as one-fifth, we have completely specified the trajectory except for its initial slope, ψ_0. Therefore, we are left with only two variable parameters, ψ_0 and a_2. For the same a_2 we have carried out computations for different cone angles; we shall present in graphical form only those results obtained where the cathode is an annular disk, for which $\psi_0=0$, and those in which the cathode is the inside of the cylinder, $\psi_0=-\pi/2$. The results for intermediate cone angles can be interpolated between the two curves shown. Since the beams with the most desirable characteristics are those from a disk cathode, we discuss these in greater detail.

The choice of α of Eq. (6.1) yields constant current density on S at the cathode for different a_2 and ψ_0.

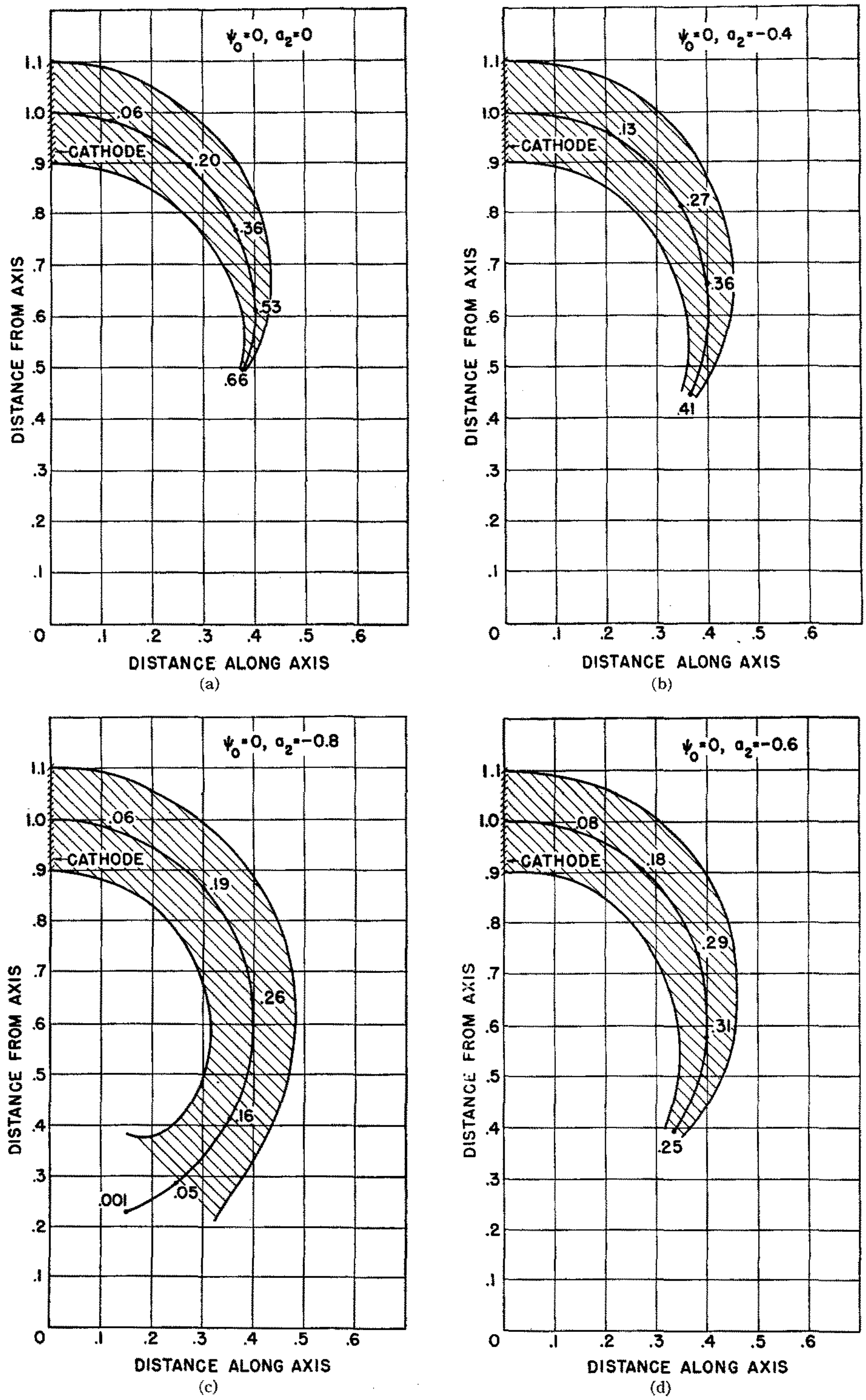

FIG. 3. The cross sections of beams from an annular cathode for different potential variations a_2. The figures denote the potentials at the corresponding points on the specified trajectory.

Hence, comparison of potentials along the same beam and along different beams serves as a comparison for the perveance of the guns predicted. Our ideal will be, of course, to obtain highly convergent beams with a minimum possible potential. The beams which result with initial trajectory angle $\psi_0=0$ are shown in Fig. 3, while those of the inside of the cylinder, $\psi_0=-\pi/2$, are shown in Fig. 4. These figures show two things. First, as the potential variation along S is reduced, that is, as a_2 is made more negative, the convergence of the beam is drastically altered. For an a_2 as large as -0.6, the beam still converges to a point. However, for a larger a_2, -0.8, the beam converges for some distance, but then as the potential starts decreasing again the beam diverges rapidly.

Second, a comparison of Figs. 3 and 4 shows that

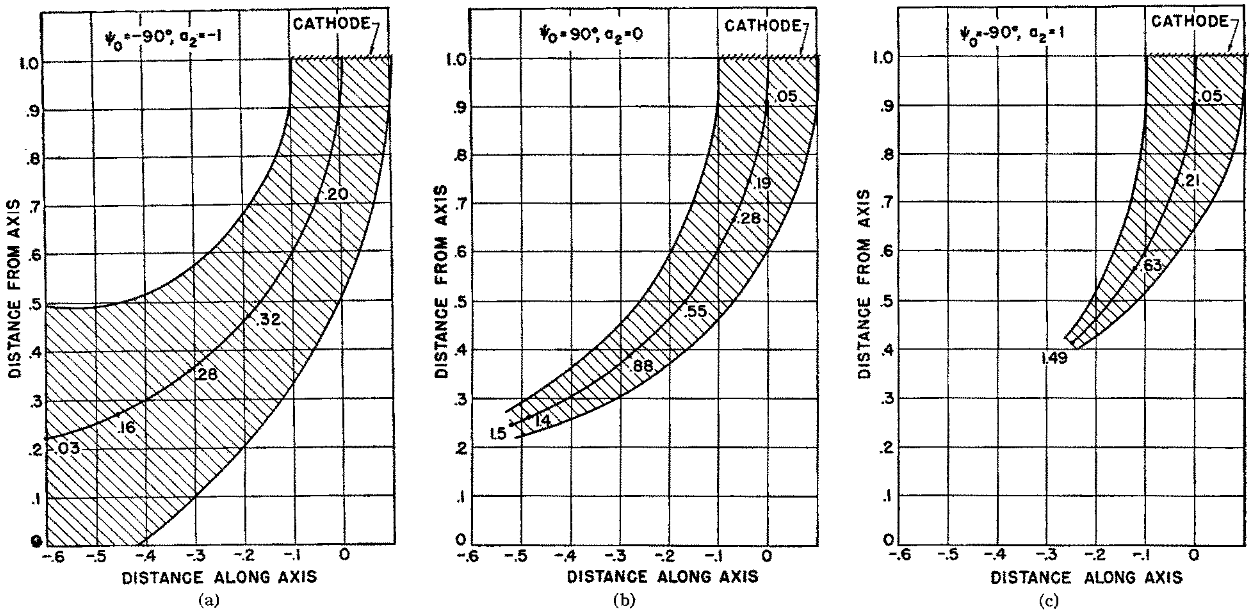

FIG. 4. Cross sections of beams from a cylindrical cathode for different potential variations a_2. The figures denote the potentials at the corresponding points on the specified trajectories.

the linear variation of potential with distance along the curve has so much effect that the potential at corresponding points of the curve of Fig. 4 is much higher than those in Fig. 3. In the previous section we showed that the current density variation across the cathode was proportional to five times the curvature of the beam at the cathode. Sections of the curves of Figs. 3 and 4 show that the curvature for the beam from the cylindrical cathode is only half that of the beam from the disk cathode, and, therefore, twice as much cathode could probably be used in the second case. However, the potential variation in Fig. 4 is even more than twice that of Fig. 3, and so disk cathodes seem more useful.

By suitable adjustment of potential variation at later points in the beam, it might be possible to obtain good characteristics even with beams from cylindrical cathodes. However, these first results indicate that beams from a disk cathode are the more promising, and therefore, the rest of the section will be concerned with such beams. It is seen in Fig. 3 that, for a beam with $a_2=-0.6$, the potential passes through a maximum. In high-power tubes, this could be an undesirable characteristic since it is usually best to have the final beam at a potential at least as high as at any intermediate points, if possible. Hence, we have performed additional computations for different potential variation; at the point at which the beam passes through a maximum, the potential variation is changed so as to be constant the rest of the way. This results in no discontinuity in the field, but would predict a discontinuity in the charge density. We feel, however, that such a discontinuity would have a very small effect on the beam. In this case, the characteristics of two beams are shown in Fig. 5. On putting in numbers, we see that the perveance of the beam of the left curve of Fig. 5, as measured by a potential on S, is, from Eqs. (3.8) and (3.11), about 15, while paraxially it converges to a point.

It seems clear that by suitable choices of potential variation and trajectory it would be possible to obtain beams with much better characteristics than these. In practice how reliable the paraxial theory may be can only be determined by experiment. The purpose of the numerical examples of this section is twofold. First, they serve as an illustration of the theory of Sec. 4. Second, they predict beams with characteristics considerably superior to those obtained in the past. However, the actual choice of trajectory and potential variation adopted in practice would depend on the application for which the gun was designed.

7. PROBLEM OF BEAM DESIGN AND CONCLUSIONS

A discussion of beams considering space-charge effects is incomplete without some consideration of the electrodes required to produce the beam. Several methods of designing electrodes, given the potential distribution at the edge of the beam, have been suggested. One method, by Picquendar,[7] involves integrating out the space charge with its images in the cathode. This method would be very laborious for the cathode geometries and potential distributions described in this section. Another, by Radley,[8] involves solving Laplace's equation under Cauchy conditions. This may lead to electrodes of very awkward shapes. Others, which employ the use of

[7] J. E. Picquendar *et al.*, *Les Effets de la charge d'espace dans les canons à électrons* (Companie Française de Thomson Houston, Paris, 1956).
[8] D. E. Radley, J. Electronics and Control 4, 125 (1958).

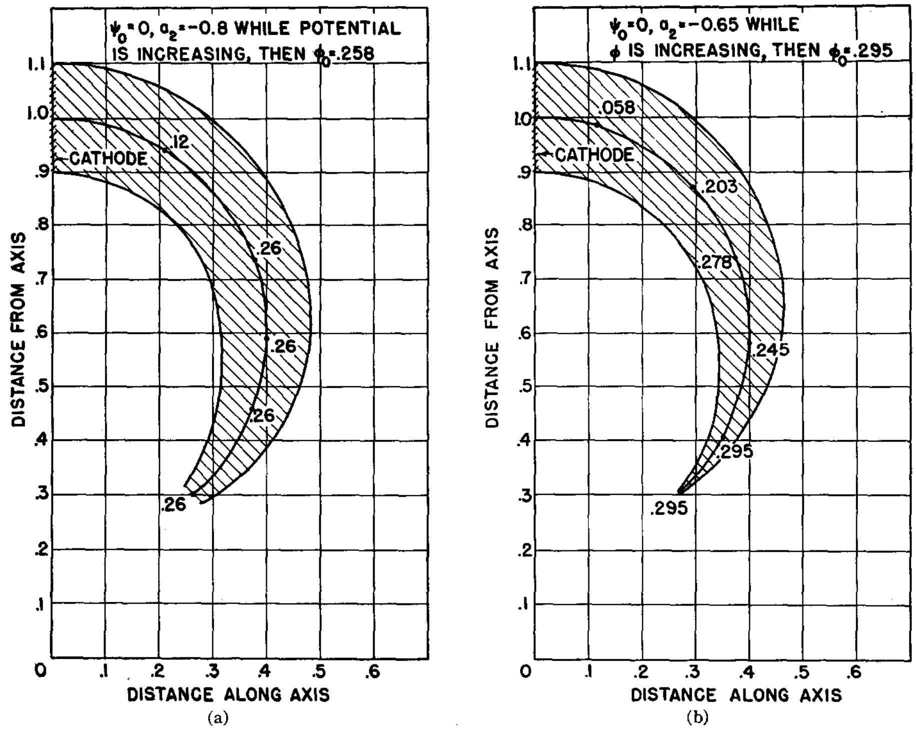

FIG. 5. Cross sections of beams from an annular cathode for different potential variations a_2 as long as the potential increases, and then constant potential. The figures denote the potentials at the corresponding points on the specified trajectories.

an electrolytic tank with current probes, would be very satisfactory if such were available. The method suggested by Cook[9] could be readily employed. In this Cook replaced the edge of the beam by a resistive sheet in an electrolytic tank. The resistivity of the sheet is so adjusted that the potential on the sheet would cause currents to flow proportional to this resistivity; in this way, the analog of the appropriate ratio of normal to transverse fields of the boundary would be obtained.

A further method which uses the paraxial nature of the flow might be possible. Knowing the potential distribution at the edge of the beam, one could continue it back to the specified trajectory, assuming a Laplacian field between the specified trajectory and the beam edge. In this way it might be possible to consider the space charge as being concentrated on the specified trajectory for the purpose of integrating out its effects. This method has not been investigated very carefully.

One of the very important conclusions which can be drawn from the work described in this paper is that it is possible to use the methods of deflection focusing to obtain guns with desirable characteristics. By forcing the beams to curve in the gun region, they can be made to converge even at low potential. However, it is better for the curvature to commence not right at the cathode but shortly beyond it. In such a way the current density at the cathode could be kept fairly constant. The choice of trajectory and potential distribution could be so tailored as to fit into slalom flow[10] or many other focusing systems.

A further factor which may affect performance of guns designed by paraxial methods is the aberrations due to imperfection in the design of the electrode systems. This problem could be well examined using the formalism developed by Sturrock.[6] The best method of verifying the results predicted by the theory presented in this paper is to build a gun. It is our intention to carry on an experimental investigation in the near future.

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[9] E. J. Cook, Proc. Inst. Radio Engrs. **46**, 497 (1958).

[10] J. S. Cook *et al.*, Proc. Inst. Radio Engrs. **45**, 1517 (1958).