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Solution to the Equations of Space-Charge Flow by the Method of the Separation of Variables*

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The equations for irrotational, electrostatic laminar space-charge flow (no thermal velocities or normal magnetic field at the cathode) are set up in terms of the action function. The resulting nonlinear partial differential equation is first reduced, by the method of the separation of variables in cylindrical polar coordinates, to the solution of a set of first-order, nonlinear, ordinary differential equations in one coordinate. It is shown that it is possible to predict axially symmetric, electrostatic, hollow beams from the inside of hollow cylindrical cathodes with space-charge-limited emission which asymptote, theoretically, to other extremely thin cylinders—though thermal velocities would, in practice, limit such a convergence. The characteristics of a particular beam, and the electrode system to produce it are shown.

The beams which result from the separation of variables in other coordinate systems are described. In spherical polar coordinates one obtains hollow axially symmetric beams from a conical cathode, which may asymptote to other cones; in a less familiar coordinate system, spiral coordinates, sheet beams from equiangular spiral cathodes result.

Finally, it is shown how the method may be extended to include magnetic fields transverse to the cathode. Some new solutions to space-charge flow in crossed electric and magnetic fields from a space-charge-limited cathode are mentioned.

I. INTRODUCTION

THERE has been much interest in recent years in the steady-state behavior of electron beams with high space-charge densities. The equations which must be satisfied for space-charge flow are complicated; no analytic solutions are known except under certain special conditions, namely, that the flow is irrotational and laminar. These conditions are satisfied in one very important physical system—that of electron guns with no magnetic field threading the cathode, in which thermal velocities are neglected. In this paper we will be concerned only with flow satisfying these conditions; we will have the boundary conditions pertaining to the problem of electron gun design continually in mind. Although it will be shown that some of the solutions to space-charge flow we obtain are valid even with magnetic fields present, we will deal mainly with the problem of electrostatic, space-charge flow.

In the past, the best known of the solutions to the equations for space-charge flow, those by Langmuir *et al.*,[1,2] have involved rectilinear trajectories. More recently, Meltzer[3] and Walker[4] have given some solutions for which the trajectories are curvilinear. However, the published solutions have not been numerous; none, other than Langmuir's, have been useful for gun design. There has been no systematic attempt to give a wide class of space-charge-flow solutions with curvilinear trajectories and variable characteristics. In this paper we will develop a method which partially remedies this situation. We will present classes of solutions which are derivable from a real cathode. By the method of the separation of variables we will reduce the partial differential equations governing the space-charge flow to ordinary differential equations.

The method of solution which will be used is applicable in a large number of coordinate systems. However, we will use as our main illustrations the solution in cylindrical polar coordinates which yield a hollow beam emitted from a cylindrical cathode. We shall also describe a gun utilizing this flow and present some of its pertinent characteristics. This gun is presently being constructed.

We will also sketch the characteristics of electrostatic beams which result from solutions in spherical polar coordinates and a less well-known coordinate system which we call equiangular spiral coordinates.

Finally, we will discuss briefly how the methods outlined in this paper may be extended to yield solutions for space-charge flow when there is magnetic field present, but with the normal component of magnetic field at the cathode zero.

II. EQUATIONS FOR SPACE-CHARGE FLOW

When thermal velocities are neglected all electrons are emitted from the cathode with zero velocity. Hence, the potential Φ is related to the velocity $\mathbf{v}$ of an electron by the equation

$$\Phi = -v^2/2\eta, \tag{1}$$

where η is the ratio of charge to mass of an electron.

Walker[4] and Gabor[5] have shown that, in the absence of magnetic field, the flow from a cathode from which all the electrons are emitted with zero velocity is

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[1] I. Langmuir, Phys. Rev. 2, 450 (1913).
[2] I. Langmuir and K. Blodgett, Phys. Rev. 22, 347 (1923); Phys. Rev. 24, 49 (1924).
[3] B. Meltzer, Proc. Phys. Soc. (London) B62, 813 (1949); Electronics 29, 118 (1956).
[4] G. B. Walker, Proc. Phys. Soc. (London) B63, 1017 (1950).
[5] D. Gabor, Proc. Inst. Radio. Engrs. 33, 792 (1945).

irrotational, i.e.,

$$\nabla \times \mathbf{v} = 0. \tag{2}$$

The velocity may, therefore, be derived from a scalar function W, called the action function, by the relation

$$\mathbf{v} = \nabla W. \tag{3}$$

Equations (1) and (3) may then be combined to yield a single equation, the Hamilton-Jacobi equation, for the action function W,

$$(\nabla W)^2 = -2\eta\Phi. \tag{4}$$

Thus, the problem of determining the flow, when the space-charge fields are negligible, becomes the problem of determining the solution of the Hamilton-Jacobi equation, under the condition that the potential satisfies Laplace's equation

$$\nabla^2\Phi = 0. \tag{5}$$

However, when the fields due to space-charge are of importance, as in space-charge-limited flow, it is necessary to solve the Hamilton-Jacobi equation with the additional conditions that Poisson's equation,

$$\nabla^2\Phi = -\rho/\epsilon, \tag{6}$$

and the equation of continuity,

$$\nabla \cdot (\rho \mathbf{v}) = 0, \tag{7}$$

be satisfied.

In the past, a number of solutions of the Hamilton-Jacobi equation for electron flow have been given. Iwata,[6] for instance, expressed W in the form $W = W_1(q_1) + W_2(q_2) + W_3(q_3)$, in the coordinate system (q_1,q_2,q_3), in order to find a number of possible space-charge-free flows. Spangenberg[7] made use of the action function in the form

$$W = W_1(x) \tag{8}$$

in Cartesian coordinates, and derived Langmuir's solution for flow between parallel plates. Meltzer[3] and Walker[4] found other space-charge solutions by postulating that either W or Φ be a function of one variable in the chosen coordinate system.

We shall demonstrate in this paper that it is possible to obtain a far more general class of solutions of the equations for space-charge flow, namely, the Hamilton-Jacobi equation, Poisson's equation, and the equation of continuity, by the consideration of the action function W in many orthogonal coordinate systems (q_1,q_2,q_3) in the form

$$W = W_1(q_1)W_2(q_2)W_3(q_3). \tag{9}$$

The solutions derived here will be such that the potential Φ, the space-charge density ρ, and each component of current j_1, j_2, and j_3 may also be written in product form. Thus, the choice of possible coordinate systems will be restricted.

III. SOLUTION OF THE SPACE-CHARGE EQUATIONS IN CYLINDRICAL POLAR COORDINATES

(A) General Formulas and the Universal Trajectories for an Infinite Beam

In this section we shall demonstrate the derivation of a circularly symmetric solution in the cylindrical polar coordinate system (r,θ,z) by writing

$$W = W_1(z)W_2(r). \tag{10}$$

It would be possible to consider W in the more general form $W = W_1(z)W_2(r)W_3(\theta)$ with little added complexity. However, the θ-independent case has more practical interest. The method of solutions for the more general case has been described.[8]

In order to carry through the separation of variables, it will be necessary to choose a comparatively simple function for one of the W_i, W_1, in such a way that not only is W in the form of Eq. (10), but also Φ has the form

$$\Phi = \Phi_1(z)\Phi_2(r). \tag{11}$$

The form of Eqs. (10) and (11) will henceforth be called "in product form."

Substitution of a W of the form of Eq. (10) into Eqs. (1) and (4) yields

$$\begin{aligned}\Phi &= -\frac{1}{2\eta}(W_1'^2 W_2^2 + W_1^2 W_2'^2)\\ &= -\frac{W_1^2 W_2^2}{2\eta}\left[\left(\frac{W_1'}{W_1}\right)^2 + \left(\frac{W_2'}{W_2}\right)^2\right],\end{aligned} \tag{12}$$

where the prime ($'$) denotes differentiation with respect to the argument.

Equation (12) has the form of Eq. (11) if, and only if, at least one of (W_1'/W_1) or (W_2'/W_2) are constant. We will assume that

$$W_1'/W_1 = n,$$

so that

$$W_1 = e^{nz}. \tag{13}$$

We could also have tried the alternative choice $W_2'/W_2 = n$. However in this coordinate system such a choice would cause the separation procedure to break down at a later stage. If W_1 has the form of Eq. (13), it can be shown that

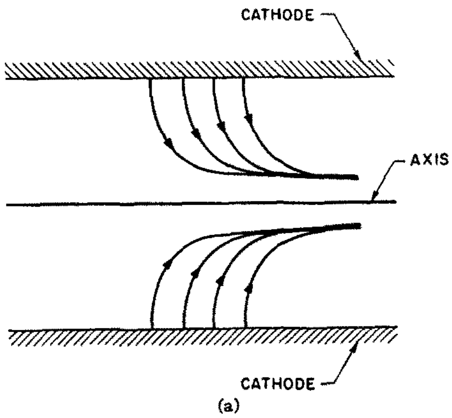
$$W = e^{nz}W_2(r), \tag{14}$$

$$\mathbf{v} = e^{nz}(nW_2, 0, W_2'), \tag{15}$$

and

$$\Phi = e^{2nz}\Phi_2(r), \tag{16}$$

[6] G. Iwata, Progr. Theoret. Phys. (Kyoto) **15**, 513 (1956).
[7] K. Spangenberg, *Vacuum Tubes* (McGraw-Hill Book Company, Inc., New York, 1948).
[8] P. T. Kirstein, Technical Report No. 440, Microwave Laboratory, Stanford University, Stanford, California (1957).



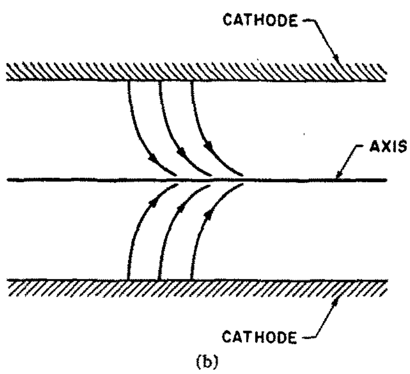


FIG. 1. Sketches of possible electron-flow patterns from a cylindrical cathode, arising from the separation of variables in cylindrical polar coordinates.

where

$$\Phi_2=-\frac{1}{2\eta}(n^2W_2^2+W_2'^2). \tag{17}$$

The possible solutions of Eq. (17) may be divided into two classes: the zero-space-charge solutions, for which $\rho=0$; and the finite-space-charge solutions, for which $\rho\neq 0$. These two cases will be considered separately.

(i) *Zero-Space-Charge Solution*, $\rho=0$

When $\rho=0$, the potential Φ must satisfy Laplace's equation. Therefore, if Φ is written in the form of Eq. (16), the term $\Phi_2(y)$ must be of the form

$$\Phi_2(y)=AJ_0(2nr)+BY_0(2nr), \tag{18}$$

where A and B are constants, and J_0 and Y_0 are the Bessel function of zero order and first and second kind, respectively. Equation (17) may therefore be written in the form

$$-\frac{1}{2\eta}(n^2W_2^2+W_2'^2)=AJ_0(2nr)+BY_0(2nr). \tag{19}$$

We may solve Eq. (19) to find W_2, and hence W, and so determine the flow in the potential field $\Phi=e^{2nz}[AJ_0(2nr)+BY_0(2nr)]$.

It is interesting to consider some of the features of this solution. We shall show, first, that all the trajectories of the solution may be derived by a displacement in the z direction of one trajectory. From Eq. (15) the slope of any trajectory is given by

$$dr/dz=W_2'/nW_2.$$

Hence, since W_2 is a function only of r, the position of any electron is given by the expression

$$z-z_0=\int\frac{nW_2}{W_2'}dr, \tag{20}$$

where z_0 is a constant. Thus, Eq. (20) shows that the z displacement of an electron from its initial value is dependent only on r, so that all trajectories may be derived from a known one by displacement in the z direction. The particular trajectory with $z_0=0$ will therefore be termed a "universal trajectory."

We may now determine the form of the trajectories. If the electrons come from a real cathode, at zero potential, the cathode must be a cylinder since the condition that $\mathbf{v}$ be zero is equivalent to

$$W_2=W_2'=0,$$

which occurs on certain cylinders arising from the solution for r of Eq. (19), with the left-hand side put equal to zero. The possible types of trajectories obtained from this solution are shown in Fig. 1.

In Fig. 1 solid lines represent electron trajectories. Two forms of trajectory result since dr/dz can be zero only if from Eq. (15), $W_2'=0$. However, if $W_2'=0$, dr/dz is zero for all z. If W_2' becomes zero, the flow pattern of Fig. 1(a) results; otherwise, that of Fig. 1(b).

In the first case, the asymptotic solution for the electron beam would be an infinitely thin cylinder coaxial with the cathode. In the second case, the trajectories would pass through the axis, yielding a solution with crossing trajectories; this feature of the solution is caused by the nature of the coordinate system, but presents no anomaly when there is no space charge present.

(ii) *The Finite-Space-Charge Solution*, $\rho\neq 0$

When $\rho\neq 0$, the potential Φ must satisfy Poisson's equation and the equation of continuity, Eqs. (6) and (7). We shall show that it is still possible, in this case, to find an ordinary differential equation for W_2 which satisfies Eq. (17), and, hence, a flow pattern in the presence of space charge.

By using Eq. (16), we may write Poisson's equation in the form

$$-\frac{\rho}{\epsilon}=e^{2nz}\left[\frac{1}{r}\left(\frac{d}{dr}r\frac{d\Phi_2}{dr}\right)+4n^2\Phi_2\right]. \tag{21}$$

We shall find it convenient to use the notation

$$\left.\begin{aligned}\sigma(r)&=\left[\frac{1}{r}\left(\frac{d}{dr}r\frac{d\Phi_2}{dr}\right)+4n^2\Phi_2\right]\frac{nW_2}{r},\\ \tau(r)&=\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\Phi_2}{dr}\right)+4n^2\Phi_2\right]\frac{W_2'}{r}.\end{aligned}\right\} \tag{22}$$

Then, Eq. (7), the equation of continuity, may now be written in the form

$$\nabla\cdot(e^{3nz}\sigma, e^{3nz}\tau)=0,$$

which becomes

$$\frac{1}{r}e^{3nz}[\tau'+3n\sigma]=0. \tag{23}$$

Since $e^{3nz}\neq 0$, the only solution to Eq. (23) is

$$\tau'+3n\sigma=0. \tag{24}$$

Equation (24), with the defining equations, (22), is an ordinary differential equation for W_2. From its solution, under appropriate boundary conditions, we may find W_2 and so, from Eqs. (15), (16), and (21), the velocity, potential, and charge density at any point.

When the flow is space-charge-limited, the boundary conditions at $r=r_0$, the cathode, for potential and potential gradient are

$$\mathbf{v}=\Phi=\mathbf{E}=0, \quad \text{at the cathode,} \tag{25}$$

which becomes

$$W_2=W_2'=\Phi_2'=0, \quad \text{at} \quad r=r_0. \tag{26}$$

The solution of Eq. (24) with these boundary conditions will then give a possible space-charge-limited flow. By varying the value of n, different flow patterns may be obtained; however in all cases the cathode is a cylinder. It should be noted that when $n=0$, a solution of this equation reduces to that of Langmuir's for flow between coaxial cylinders. The two basic types of flow that may result are of the same form as those of Figs. 1(a) and 1(b), although the patterns for any given value of n will, of course, be different from the zero-space-charge case. But it should be noted that when there is space charge present, the beam must be terminated before it reaches the axis, since the solutions given would not be valid when electrons cross, as the equation of continuity is based on the assumption of laminar single-valued flow. As before, the other trajectories may be obtained from one trajectory by a translation in the z direction.

Numerical solution of the flow for different values of n which have been obtained on an IBM 650 computer are given in Fig. 2. If $n>1.55$, it is seen that the flow pattern is substantially like that of Fig. 1(a); the beam would, theoretically, asymptote to an infinitesimally thin cylinder, though thermal velocities would, in practice, limit such a convergence. If $n<1.55$, the beam

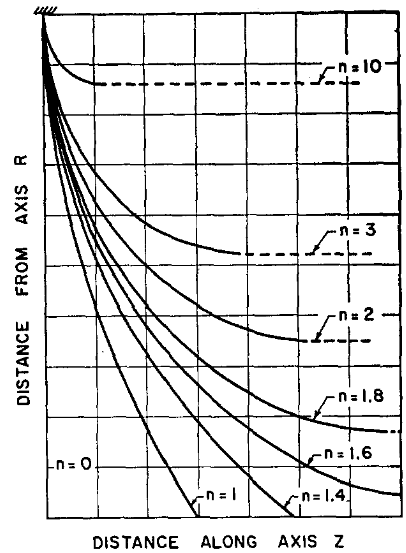


FIG. 2. Universal trajectories for beams from a cylindrical cathode with different n.

would cross the axis, and therefore would have to be terminated before it reached the axis; otherwise the implicit assumption of single stream flow would be violated. Negative values of n merely give the positive n solution reflected in the plane $z=0$.

We have stated earlier that different trajectories resulting from a solution for a particular value of n differ only by translation in the z direction. We have shown that the physical parameters of flow have the form

$$\begin{aligned}\mathbf{v}&=e^{nz}\mathbf{v}_2,\\ \Phi&=e^{2nz}\Phi_2,\\ \mathbf{E}&=e^{2nz}\mathbf{E}_2,\\ \rho&=e^{2nz}\rho_2,\end{aligned} \tag{27}$$

where $\mathbf{v}_2$, Φ_2, $\mathbf{E}_2$, and ρ_2 depend only on r and may be derived from Eqs. (15)–(17), (21), etc., for a W_2 satisfying Eq. (24). Therefore, the value of the physical parameters of flow along one trajectory differ from those along another by a simple constant factor.

(B) Finite Beam

The solutions which were derived in the previous section are those for an infinite beam. In practice we require a finite beam. For such a beam, then, we may find the potentials and fields along the two enclosing surfaces, and design an electrode system, exterior to the beam, to give the correct values of potential and field at these surfaces. The difficulties which arise when this Pierce procedure is applied to gun design with curvilinear trajectories will be discussed in Sec. III (C).

In order to design a finite beam in practice, it is necessary to know not only the way in which the different parameters vary throughout the beam, but also how the current density varies across the cathode, and what the expected value of the total current would be. We will illustrate this calculation by using the solution

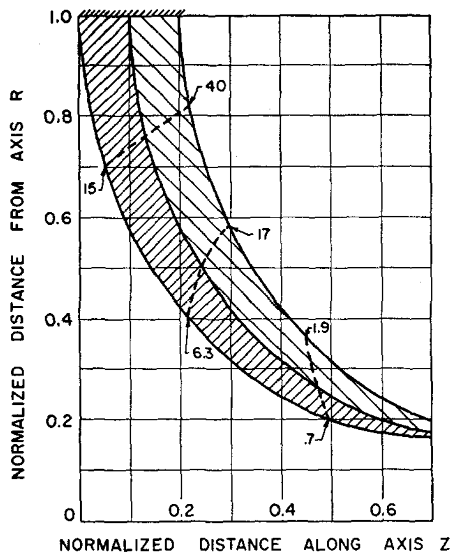


FIG. 3. Beams for two different widths of cathode from a cylindrical cathode with $n=1.8$. The numbers on the left refer to the microperveances of the heavily shaded narrow beams, terminated along equipotentials, and those on the right refer to the wide beam.

given in the first part of this paper to find the expression for the current density and total current emitted from a finite cylindrical cathode. From Eq. (27), we find that the current density at any point is of the form

$$\mathbf{j}=\rho\mathbf{v}=e^{3nz}\mathbf{j}_2, \tag{28}$$

where

$$\mathbf{j}_2=\rho_2(r)\mathbf{v}_2(r). \tag{29}$$

At the cathode, $r=\text{const}$, it can be shown that the current density $\mathbf{j}_2$ is normal to the surface. Hence, we may find the current density at the cathode, $r=r_0$, to be in the form

$$j_c=(j_2)_{r=r_0}e^{3nz}, \tag{30}$$

where j_c is defined as the cathode current density. If the cathode is the region between $z=z_1$ and $z=z_2$, then the total current in the beam is found from Eqs. (9) and (11) to be I, where

$$I=\int_{\text{cathode area}} j_c da=2\pi r_0\int_{z_1}^{z_2} e^{3nz}j_c dz$$
$$=2\pi r_0 j_c\frac{e^{3nz_2}-e^{3nz_1}}{3n}. \tag{31}$$

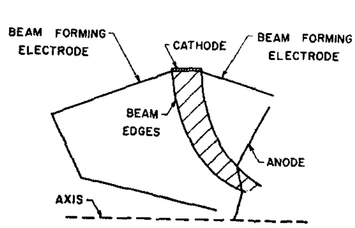


FIG. 4. An electrode system to produce one of the beams of Fig. 2, the beam from a cylindrical cathode with $n=1.8$, and cathode width 0.2 of the cathode radius.

The other characteristics are similarly derivable. The details of their derivation have been given.[8]

The finite beam which may be derived from the universal trajectories of Fig. 2 for one particular value of n, $n=1.8$, is shown in Fig. 3 for two cathode widths. Here solid lines represent trajectories and dotted lines equipotentials. From Fig. 2, it can be seen that the asymptotic form of this beam, for large z, would be an infinitely thin cylinder of radius 0.17 of the cathode radius—though thermal velocities would, in practice, limit its convergence. The curves of Fig. 3 are not extended far enough to show the asymptotic form of the trajectories. This characteristic of these beams may not be as useful as would appear at first sight; from Eqs. (27) and (28) it is seen that the perveance density K is given by

$$K=\frac{\rho_2|v_2|}{\Phi_2^{\frac{1}{2}}}\propto\frac{\rho_2}{\Phi_2}. \tag{32}$$

If the beam is traveling almost parallel to the z axis, as occurs for large z, it follows from Eq. (32) that K will

TABLE I. The variation of the physical parameters of flow along the inside of a beam from a cylindrical cathode with $m=1.8$.

$n=1.8$	Cathode width and axial distance across beam: 4 mm		Cathode radius: 2 cm	Total current: 2.18 amp	Variation of potential, fields, and charge-density across beam for same R: 2.05
Radius in cm	Axial position in cm	Potential, Φ, in kv	Radial field, E_r, in kv cm^{-1}	Longitudinal field E_z, in kv cm^{-1}	Charge density ρ, in 10^{-4} coulomb/m^3
2	0	0	0	0	∞
1.6	0.0432	0.736	2.42	− 1.32	1.91
1.2	0.179	2.19	2.84	− 3.94	1.60
0.8	0.436	4.72	1.28	− 8.50	1.94
0.6	0.641	6.83	− 2.03	−12.30	2.60
0.4	0.985	10.98	−14.04	−19.80	5.44

be almost constant. Hence, the total perveance will be inversely proportional to the area convergence.

In Table I we show how the physical parameters of flow vary along the inside of the beam of Fig. 3 for specified values of cathode radius and voltage. The characteristics of the two finite beams of Fig. 3 depend on where the beams are terminated. The figures on the left of Fig. 3 show the perveance of the heavily shaded beam, while those on the right refer to that of the whole beam. The beam is assumed terminated along an equipotential. The derivation of these figures for perveance have been given.[8] The current-density variation across the cathode for the narrow beam is 1.7, while that of the wide beam is 2.9.

(C) Electrode Design and an Electron Gun

Even though we have presented solutions for the physical parameters of the flow which satisfy all the relevant equations in the beam, it is still necessary to design electrodes to produce the required voltage and field distributions at the edge of the beam. We may assume, as in the usual Pierce[9] procedure, that if conditions are correct at the beam edges, they will also be correct throughout the beam.

In the normal Pierce procedure, the electrode may be designed by putting a dielectric into an electrolytic tank to simulate an electron beam. In the guns that have been designed in the past on the basis of an analytic solution, the trajectories are rectilinear, and, hence, the normal electric field is zero at the beam edge. The dielectric analogue is therefore satisfactory. In the curvilinear beams of this paper, the component of electric field normal to the beam edge is no longer zero, and so a dielectric slab is no longer an adequate analogue for the beam.

The problem of electrode design, in this case, has not been satisfactorily solved; however, various suggestions have been proposed. In one, according to Picquendar,[10] the field is divided up into two parts—that due to space charge and that due to applied field, the space-charge contribution being subtracted out analytically and numerically. This method has been described.[8] An electrode system to produce the beam from the cylindrical cathode of Fig. 2 has been designed by us in the electrolytic tank using Picquendar's method, and is shown in Fig. 4.

A gridded gun based on the electrode system of Fig. 4 has been built and will shortly be tested.

IV. SPACE-CHARGE FLOW SOLUTIONS IN OTHER COORDINATE SYSTEMS

(A) Spherical Polar Coordinates

The separation of variables for motion with axial symmetry is equally possible in spherical polar coordinates. In this case, the analog of Eq. (10) is

$$W = W_1(r)W_2(\theta). \tag{33}$$

With a W of this form, it can be shown, in the same way as in the previous section, that the functional form which allows the separability of variables is

$$W = r^m W_2(\theta). \tag{34}$$

This solution, when the boundary conditions of Eq. (25) are applied, leads to motion from a conical cathode, $\theta = \theta_0$, of the type shown in Fig. 5. Owing to an ambiguity of sign, flow may occur both outwards, as in Figs. 5(b) and 5(c), and inwards, as in Figs. 5(a) and 5(d).

In the motion of Fig. 5, the electrons are either asymptotic to another cone and may travel either

[9] J. R. Pierce, *Theory and Design of Electron Beams* (D. Van Nostrand Company, Inc., Princeton, New Jersey. 1954).

[10] Picquendar, Cahen, and Lapostolle, "Space-charge effects in electron guns," translated by M. W. Muller, Technical Memorandum TMO-24, Varian Associates, Palo Alto, California.

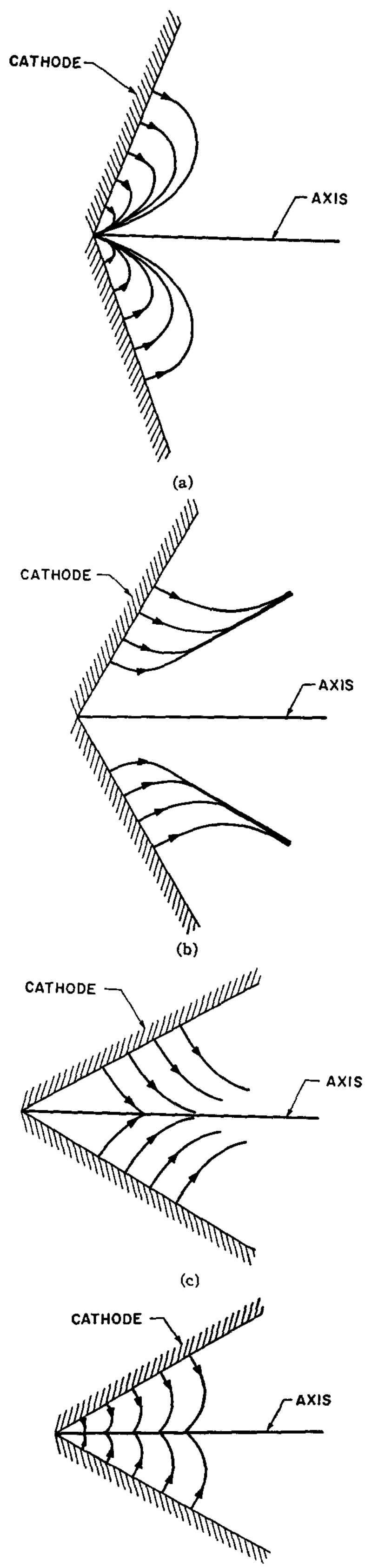


FIG. 5. Sketches of possible electron-flow patterns from a conical cathode, arising from the separation of variables in spherical polar coordinates.

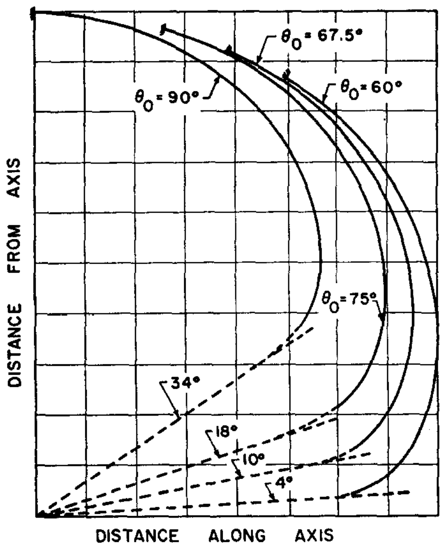


FIG. 6. Universal trajectories for beams from a conical cathode for $m=-1$ and different cone angles θ_0.

towards or away from the apex of the cone, as in Figs. 5(a) and 5(b), or else cut the axis, in which case the beam would have to be terminated before the axis, as in Figs. 5(c) and 5(d).

The motion of Figs. 5(b) and 5(c) correspond to positive m, while that of Figs. 5(a) and 5(d) to negative m. The Cone angle, θ_0 can be greater than $\pi/2$, but a solution for $\theta_1=\pi-\theta_0$ and $m_1=-m$ is merely a reflection, in the plane $\theta=\pi/2$, of the solution for θ_0 and m.

The derivation of the equations resulting from the action function of Eq. (34) has been given.[8] However, the procedure shown in Sec. III must again be followed. We now have a two-parameter family of universal trajectories—with m and the cathode cone angle θ_0 as independent parameters. It should be noted, in particular, that if the cathode cone angle θ_0 is made to tend to zero in such a way that the product $m\theta_0$ is kept constant, the solutions tend to those of Sec. III, with $m\theta_0$ proportional to the n of that section.

Some calculated universal trajectories for different values of the parameters m and θ_0 are shown in Figs. 6,

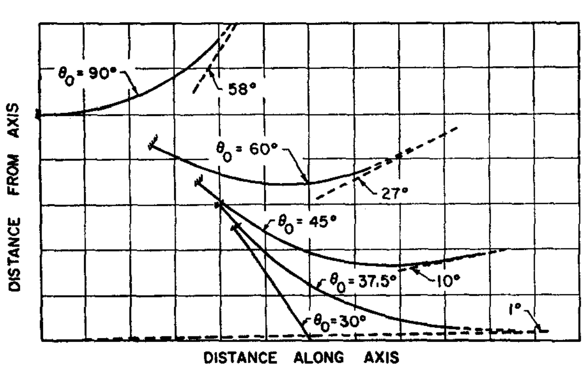


FIG. 7. Universal trajectories for beams from a conical cathode for $m=3$ and different cone angles θ_0.

7, and 8. It may be seen from these curves that hollow beams with widely varying trajectories may be designed. It may be seen that for a given value of m, the beam trajectories are substantially independent of $(\theta_0-\theta)$, by tilting the curves of Figs. 6 and 7 until the initial points are coincident. The final angle of the universal trajectory can be varied by varying θ_0 for each m, or m for each θ_0. This flexibility may allow a design to partially counterbalance the defocusing effect of an anode hole.

In Sec. III, finite beams were derived from the universal trajectories by displacement in the z direction. In the beams of Sec. IV (A), finite beams would be derived from the universal curves by magnification. Examples of two such finite beams for different cathode widths are shown in Figs. 9 and 10. These curves show how the area converges along the beam; the dotted lines in these figures denote equipotentials. The trajectories of the beam of Fig. 9 tend to follow equipotentials, since high transverse fields are needed to turn it through such large angles. The trajectories in

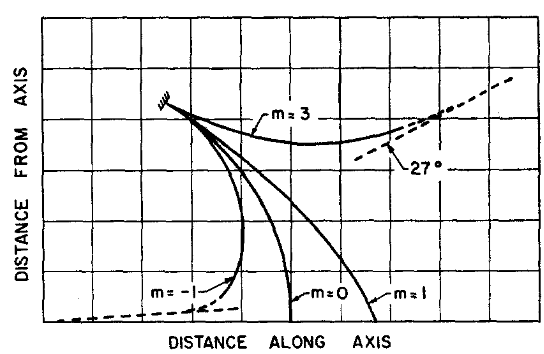


FIG. 8. Universal trajectories for beams from a conical cathode for a cone angle θ_0 of 60°, and different m.

the beam of Fig. 10, however, tend to follow the field lines more closely.

The characteristic of finite beams in this coordinate system may be derived in much the same way as those of Sec. III. Again the details of the derivation have been given.[8] As before the characteristics of the finite beams are a function of the distance along the beam at which the anode is placed. The microperveances of the heavily shaded narrow beams of Figs. 9 and 10, assumed terminated along equipotentials, are shown to the left of the beam; those of the wide beams on the right. The current-density variations across the cathode for the narrow beams are 1.8, while those for the wide beams are 3. Just as with the beams of Sec. III, it has been shown[8] that for the beams from a conical cathode the perveance density remains substantially constant along the rapidly convergent section of the beam, therefore, the value of perveance density provides a convenient figure of merit for such beams, since it is proportional to the product of area convergence and perveance. It can be seen[8] that for a current density variation of about 1.8 across the cathode the figure of merit for these beams would be approximately 14, 120, and 6, respectively. These figures are consistent, since the cylindrical beam of Fig. 3 turns through an angle intermediate between that of Figs. 9 and 10. Moreover, it is to be expected that the beam of Fig. 9 should have the best characteristics, because, as was stated earlier, it most closely follows the equipotential lines.

As yet no electrode systems have been designed for use with beams from conical cathodes.

(B) Solutions in Equiangular Spiral Coordinates

We have described, so far, the space-charge-flow solutions which may be derived for cylindrical polar and spherical polar coordinate systems. These are the only three-dimensional coordinate systems in which we have been able to derive such solutions. However, it is also possible to apply the same type of analysis to two dimensional flow to obtain solutions in Cartesian coordinates in which W is of the form

$$W=e^{mx}W_2(y) \tag{35}$$

and to obtain solutions in circular coordinates in which

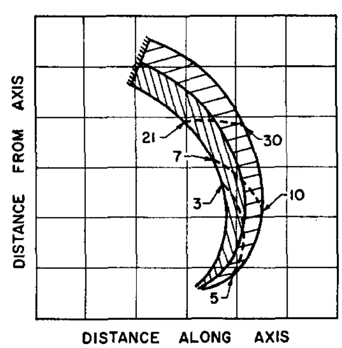


FIG. 9. Beams for two different widths of cathode from a conical cathode with cone angle θ_0 of 67.5° and $m=-1$. The numbers on the left refer to the microperveances of the heavily shaded narrow beams, terminated along equipotentials, and those on the right refer to the wide beam.

W is either of the form

$$W=r^mW_2(\theta) \tag{36}$$

or

$$W=e^{m\theta}W_2(r). \tag{37}$$

Both of these coordinate systems are, however, special cases of a more general system which we have called "equiangular spiral coordinates." In this system the coordinate lines are formed by two sets of orthogonal equiangular spirals

$$re^{(b_1/b_2)\theta}=\text{const}=\exp\{[(b_1^2+b_2^2)/b_2]q_2\}, \tag{38}$$

and

$$re^{-(b_2/b_1)\theta}=\text{const}=\exp\{[(b_1^2+b_2^2)/b_1]q_1\}, \tag{39}$$

where r and θ are the usual polar coordinates. Equations (38) and (39) may also be rewritten in the form

$$\theta=b_1q_2-b_2q_1, \tag{40}$$

and

$$r=e^{b_1q_1+b_2q_2}, \tag{41}$$

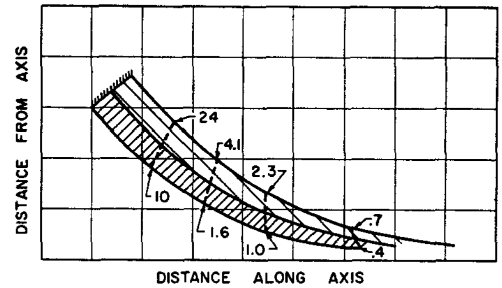


FIG. 10. Beams for two different widths of cathode from a conical cathode, with cone angle θ_0 of 37.5° and $m=3$. The numbers on the left refer to the microperveances of the heavily shaded narrow beams, terminated along equipotentials, and those on the right refer to the wide beam.

where q_1 and q_2 are the equiangular spiral coordinates. The system is illustrated in Fig. 11.

It should be noted that the equiangular coordinate system reduces to a cylindrical polar coordinate system if $b_1=0$, as may be seen from Eqs. (40) and (41). It may also be shown that the system reduces to a Cartesian coordinate system when $b_1=b_2=0$, for then the two sets of orthogonal spirals become orthogonal sets of straight lines.

The determination of the flow in equiangular spiral coordinates is carried out in a similar manner to the solution for cylindrical polar coordinates in Sec. III. As before, we write W in the form

$$W=W_1(q_1)W_2(q_2). \tag{42}$$

We may then use the expression for ∇ in equiangular spiral coordinates given elsewhere[8] to write the Hamilton-Jacobi equation, Eq. (4), in the form

$$\Phi=-\frac{1}{2\eta}W_1^2W_2^2e^{-2(b_1q_1+b_2q_2)}\left[\frac{W_1'^2}{W_1^2}+\frac{W_2'^2}{W_2^2}\right], \tag{43}$$

where the prime, as usual, denotes differentiation with respect to the argument. For Φ, in Eq. (43), to have the product form

$$\Phi=\Phi_1(q_1)\Phi_2(q_2), \tag{44}$$

it is necessary, as before, to put either $W_1=\exp(mq_1)$, or $W_2=\exp(mq_2)$.

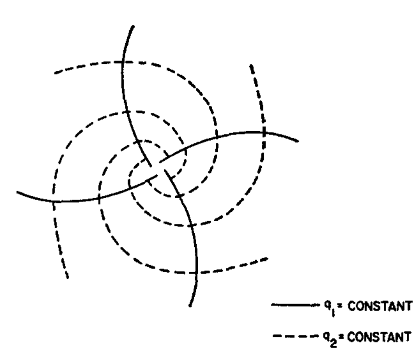


FIG. 11. Sketches of the coordinate lines in equiangular-spiral coordinates.

We may, without loss of generality, use

$$W_1 = e^{mq_1}. \tag{45}$$

If W_1 is of the form given in Eq. (45), it is then possible to derive space-charge-flow solutions in the same manner as was shown for cylindrical polar coordinates.

As before we may derive a zero space-charge solution in which the potential Φ may be shown from Laplace's equation, to be of the form

$$\Phi = A e^{2mq_1} \sin 2(m-b_1)(q_2-q_0), \tag{46}$$

where q_0 and A are arbitrary constants and the cathode is the equiangular spiral $q_2=q_0$.

Again, as before, we may derive the solutions for space-charge-limited flow in terms of a nonlinear differential equation in q_2, with the boundary conditions

$$W_2 = W_2' = \Phi_2' = \Phi_2 = 0 \tag{47}$$

at the cathode $q_2=q_0$ which is an equiangular spiral.

Thus in a spiral metric of arbitrary pitch (that is, arbitrary b_2/b_1) it is possible to find solutions for W and, hence, the flow for arbitrary m. A correct choice of the two arbitrary parameters allows the design of highly convergent sheet beams. The form of such a beam is shown in Fig. 12. As before, the trajectories from the cathode may either asymptote to a single curve, in this case another equiangular spiral, or continue out to infinity, or all pass through the origin without asymptoting to any single curve. What the behavior will be in a given case depends on the choice of m and b_2/b_1.

As in Secs. III and IV (A), for each m and b_1/b_2, it is possible to derive the flow pattern from the universal trajectory by a displacement in the q_1 direction. Some universal trajectories for one particular ratio of b_1/b_2, unity, are shown in Fig. 13. The solutions were terminated where they were for economy of machine time.

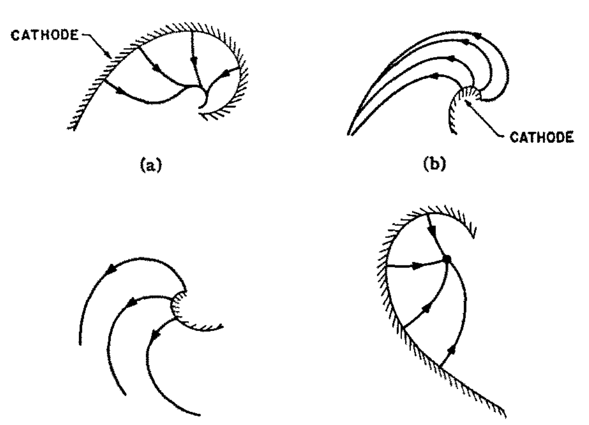


FIG. 12. Some sketches of possible electron-flow patterns from a spiral cathode, arising from the separation of variables in spiral coordinates.

A derived beam for $m=1$, $b_1/b_2=1$ is shown in Fig. 14. The choice of two parameters m, b_1, and b_2 allows a very wide range of possible beam trajectories. Some of these have very interesting properties. For instance, the trajectories with $m=0$ are the equiangular spirals $q_1=\text{const}$. For some of these solutions the potential passes through a maximum along the trajectory; hence if an electrode system is designed for use with a finite beam derived from this solution, electrodes which are at a potential greater than this maximum value will not intersect the beam, so that the problem of allowing for a hole in the anode is eliminated.

It may easily be shown that if $b_1=b_2=0$, W assumes the Cartesian form of Eq. (35). Similarly, if $b_1=0$, W assumes the cylindrical polar form of Eq. (37), and if $b_2=0$, W assumes the cylindrical polar form of Eq. (36).

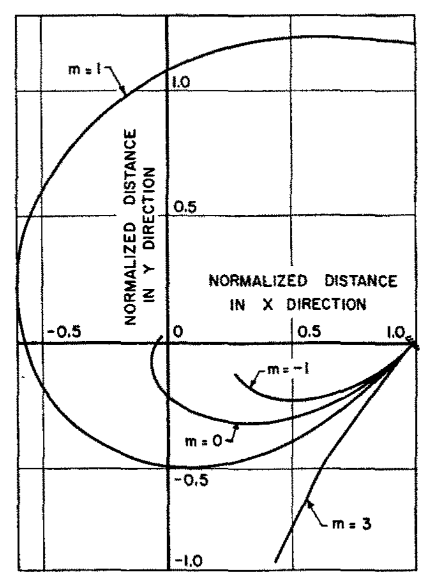


FIG. 13. Universal trajectories for beams from an equiangular spiral cathode for a spiral metric $K=1$, and different m.

Four special cases of these results have been previously reported in the literature. These are

(a) $b_1=0$, $b_2=0$, and $m=0$.

This is Langmuir's[1] solution for rectilinear motion between parallel planes.

(b) $b_1=0$, $b_2=1$, and $m=0$.

This is Langmuir's and Blodgett's solution[2] of rectilinear flow from a circular cathode.

(c) $b_1=1$, $b_2=0$, and $m=0$.

This is Meltzer's[3] solution in which the electrons move in circular trajectories.

(d) $b_1=1$, $b_2=0$, and $m=1$.

This is Walker's[4] solution for flow between inclined planes.

V. SOLUTIONS IN THE PRESENCE OF A MAGNETIC FIELD

If the magnetic field at the cathode is parallel to the cathode, it is possible to generalize the solutions which have been discussed by writing the velocity in the form

$$\mathbf{v}=\nabla W-\eta \mathbf{A}, \tag{48}$$

where $\mathbf{A}$ is the magnetic vector potential and W the scalar action function.

In order to obtain space-charge-flow solutions, however, the magnetic field must have a variation which allows the new form of the solution to be separable. It is possible to obtain the variation of magnetic field which is required in all the coordinate systems that have been described in the preceding part of this paper, so that it is possible to design guns in which there is magnetic field present. But, so far, none of these new magnetic field solutions have been evaluated numerically.

It should be noted, also, that there are special cases of importance where the magnetic field is constant. Thus, in Cartesian coordinates, the constant magnetic field solution reduces to that of Slater.[11] But in spiral coordinates there is also a constant magnetic field solution with W of the form

$$W=e^{2b_1q_1}W_2(q_2). \tag{49}$$

When $b_2=0$, this solution reduces to one for which the cathode is a plane, but which is different from that of Slater. For zero space charge the solution is the one described by Poritsky,[12] and so may be regarded as a generalization of Poritsky's solution which takes space charge into account.

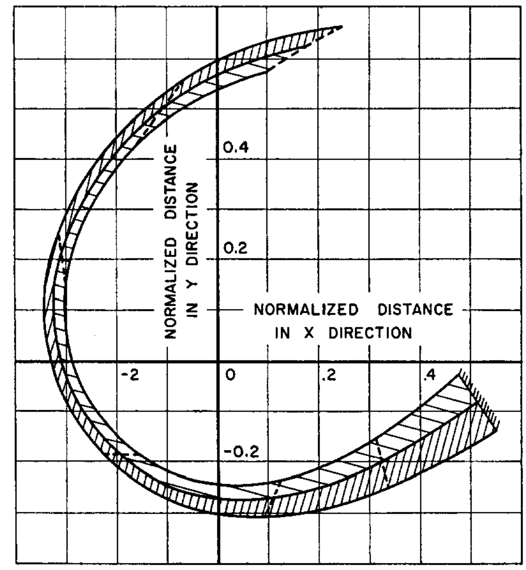


FIG. 14. Beams for two different widths of cathode, from an equiangular spiral cathode with $m=1$, and spiral metric $K=1$.

These magnetic field solutions are being investigated further, and it is hoped to publish a second paper in the near future which will discuss them in more detail.

ACKNOWLEDGMENTS

The authors would like to express their thanks to Miss J. Szego who obtained many of the numerical results of this paper, and to Mr. T. Mitford who obtained the electrode system of Fig. 4 in the electrolytic tank.

[11] J. C. Slater, *Microwave Electronics* (D. Van Nostrand Company, Inc., Princeton, New Jersey, 1954), pp. 333–336.

[12] H. Poritsky and R. P. Jerrand, J. Appl. Phys. **23**, 928 (1952).

Announcement

Effective January 1, 1959, there will be an increase in page charges from $15 to $25 per page, and an increase in subscription rates from $10 to $12 for members of the American Institute of Physics and from $12 to $14 for nonmembers. The single-copy rate will be increased from $1.50 to $1.75 and the yearly back number rate (when complete year is available) from $15 to $17. These changes have been necessitated by the continued and significant increase in expenses and material costs over the past several years. (Subscription renewals will be accepted at the old rates listed on page ii provided remittances are postmarked not later than December 31, 1958.) We hope that, in addition, it will be possible to consider some desirable changes in the Journal which, while minor, have not been financially feasible under the present circumstances.