Symplectic Topology of Projective Space: Lagrangians, Local Systems and Twistors

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I, Momchil Preslavov Konstantinov, confirm that the work presented in this thesis is my own, except for the content of section 3.1 which is in collaboration with Jack Smith. Where information has been derived from other sources, I confirm that this has been indicated in the work.

На семейството ми.

Abstract

In this thesis we study monotone Lagrangian submanifolds of $\mathbb{CP}^n$. Our results are roughly of two types: identifying restrictions on the topology of such submanifolds and proving that certain Lagrangians cannot be displaced by a Hamiltonian isotopy.

The main tool we use is Floer cohomology with high rank local systems. We describe this theory in detail, paying particular attention to how Maslov 2 discs can obstruct the differential. We also introduce some natural unobstructed subcomplexes.

We apply this theory to study the topology of Lagrangians in projective space. We prove that a monotone Lagrangian in $\mathbb{CP}^n$ with minimal Maslov number $n+1$ must be homotopy equivalent to $\mathbb{RP}^n$ (this is joint work with Jack Smith). We also show that, if a monotone Lagrangian in $\mathbb{CP}^3$ has minimal Maslov number 2, then it is diffeomorphic to a spherical space form, one of two possible Euclidean manifolds or a principal circle bundle over an orientable surface. To prove this, we use algebraic properties of lifted Floer cohomology and an observation about the degree of maps between Seifert fibred 3-manifolds which may be of independent interest.

Finally, we study Lagrangians in $\mathbb{CP}^{2n+1}$ which project to maximal totally complex submanifolds of $\mathbb{HP}^n$ under the twistor fibration. By applying the above topological restrictions to such Lagrangians, we show that the only embedded maximal Kähler submanifold of $\mathbb{HP}^n$ is the totally geodesic $\mathbb{CP}^n$ and that an embedded, non-orientable, superminimal surface in $S^4 = \mathbb{HP}^1$ is congruent to the Veronese $\mathbb{RP}^2$. Lastly, we prove some non-displaceability results for such Lagrangians. In particular, we show that, when equipped with a specific rank 2 local system, the Chiang Lagrangian $L_\Delta \subseteq \mathbb{CP}^3$ becomes wide in characteristic 2, which is known to be impossible to achieve with rank 1 local systems. We deduce that L_Δ and $\mathbb{RP}^3$ cannot be disjoined by a Hamiltonian isotopy.

Impact Statement

The research carried out in this thesis impacts several related areas of geometry.

On the one hand, it contributes to its primary domain – symplectic topology – with new results, namely: a calculation which showcases a rarely used method (high rank local systems in monotone Floer theory) and two classification results in Lagrangian topology. One of these, done in collaboration with Jack Smith, builds upon recent work of several other authors to give a satisfactory partial answer to a question asked in the field more than ten years ago. The other relies on results in low-dimensional topology and relates to the study of minimal surfaces in the four dimensional sphere.

On the other hand, we give applications of these results to examples coming from algebraic and Riemannian geometry. In particular, our symplectic methods allow us to deduce some uniqueness results about Legendrian varieties. These varieties arise naturally in the study of quaternion-Kähler manifolds and are related to a famous open problem in Riemannian geometry – the LeBrun-Salamon conjecture. The relation of symplectic geometry to this problem has not been explored in the literature and one may hope that it could lead to new insights.

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Chapter 1

Introduction

1.1 History and context

1.1.1 Origins

Symplectic geometry was born in the early 19th century through the works of Lagrange and Poisson on celestial mechanics (see [Mar09]). Since then, it has evolved into a large and deep field whose reach and importance in mathematics are well beyond the author's competence, so we leave it to one of the people to whom the field owes much of its current prominence – Vladimir Igorevich Arnold – to elucidate:

> Just as every skylark must display its crest, so every area of mathematics will ultimately become symplecticised. ([Arn92])

From a purely topological point of view, this pervasiveness of symplectic geometry can be attributed to the rather flexible nature of the symplectic structure which allows it to exist on – and be exploited for the study of – an enormous class of manifolds. Beginning with even dimensional vector spaces and tori, moving on to orientable surfaces, then arbitrary cotangent bundles, then Kähler manifolds and in particular all smooth complex projective varieties, the list grows large. Moreover, through ingenious constructions like symplectic reduction ([MW74]), blow up, fibre connected sum ([Gom95]) etc., one can quickly build new examples with more and more interesting topology. While this gives symplectic topologists an immense body of examples to study, the same flexibility leaves us not knowing the answers to basic questions about some of the simplest symplectic manifolds. This thesis focuses on one such manifold – arguably the simplest non-aspherical symplectic manifold – complex projective space.

In the study of symplectic manifolds, Lagrangian submanifolds play a role which is difficult to overstate. It was summarised by Weinstein in his symplectic creed ([Wei81]): "Everything is a Lagrangian submanifold." While this statement may appear too general, Weinstein's paper contains his concrete vision of a symplectic category, in which objects are symplectic manifolds and morphisms from (M,ω) to (M',ω') are Lagrangian submanifolds of the product $(M\times M',(-\omega)\oplus\omega')$, generalising (graphs of) symplectomorphisms.[1] In such a category, the Lagrangian submanifolds of a symplectic manifold M become its "elements" – morphisms from a point into M. Thus the following has become a central question in symplectic topology:

Question 0. Given a well-known symplectic manifold (M, ω) what can we say about its Lagrangian submanifolds?

The three main directions in which this question is explored are: finding restrictions on the topology that a Lagrangian in M can have; classifying (smooth / Lagrangian / Hamiltonian) isotopy classes of Lagrangians of a fixed topological type; and understanding the intersection patterns of the Lagrangians of M. The developments along these three lines are inextricably intertwined and a large part of the progress is due to the theory of pseudoholomorphic curves, as we explain next.

1.1.2 Pseudoholomorphic curves and Lagrangian topology

1.1.2.1 Gromov's pseudoholomorphic curves

We already alluded to the great amount of flexibility that is present in symplectic topology, but what makes the subject truly interesting is that it also displays a lot of rigidity which manifests itself in surprising ways. While some examples of rigidity were known before that (e.g. [Eli87][2]), the foundational breakthrough in this direction was made by Gromov in [Gro85] with his introduction of pseudoholomorphic curves as a way of probing the geometry of a symplectic manifold. Gromov's paper not only introduces the techniques which at present underlie the main tools in symplectic topology, but also proves several key results which gave rise to some of the field's subdomains[3]: his non-squeezing theorem and packing inequalities laid the foundations for quantitative symplectic topology (see the recent survey [Sch18] and the references therein), his results on the homotopy type of certain symplectomorphism groups opened a door to these notoriously unapproachable objects (see the survey [McD04] and the references therein for some older results, or [Sei08b],[Eva11] and [SS17] for some newer ones) and his theorem that a compact Lagrangian L in $\mathbb{C}^n$ must have $H^1(L;\mathbb{R}) \neq 0$ initiated the study of Lagrangian topology.

The last-mentioned result is particularly pertinent to this thesis. From a modern standpoint one can view this theorem as an example of the "principle of Lagrangian non-intersection", formulated by Biran in [Bir06]: the fact that a Lagrangian can be displaced from itself by a Hamiltonian isotopy puts strong restrictions on its topology. While Gromov's proof was entirely geometric, nowadays the principle of Lagrangian non-intersection usually appears as a consequence of a remarkably rich structure of algebraic invariants into which pseudoholomorphic curves are organised.

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[1] Making rigorous sense of Weinstein's category in the modern framework of symplectic topology is a major research avenue, see for example [WW10], [LL13], [Bot15],[Fuk17].

[2] The results in this paper seem to have been published much later than they were announced.

[3] None of the reference lists that we give here are anywhere close to complete.

1.1.2.2 Floer theory

The first algebraic symplectic invariant – Hamiltonian Floer cohomology – was introduced by Andreas Floer in [Flo87] as an approach to the Arnold conjecture [Arn04, Problem 1972-33] on the lower bound for the number of fixed points of a Hamiltonian symplectomorphism. In line with "the creed", this conjecture is but a special case of a statement about intersections of a pair of Lagrangian submanifolds (the special case being where one of the Lagrangians is the diagonal in $(M \times M, (-\omega) \oplus \omega)$ and the other is the graph of a Hamiltonian symplectomorphism) and so Floer generalised his methods to produce an invariant which would detect such intersections ([Flo88a]), realising another of Arnold's dreams ([Arn04, Problem 1981-27]). When the two Lagrangians L^0 and L^1 intersect transversely, their *Lagrangian intersection Floer cohomology* $HF(L^0, L^1)$ is the homology of a chain complex $CF(L^0, L^1)$, which is generated by the intersection points and whose differential counts rigid pseudoholomorphic strips with boundary on L^0 and L^1 connecting such points. It is invariant under arbitrary Hamiltonian perturbations of either Lagrangian and so a lower bound on the rank of $HF(L^0, L^1)$ gives a lower bound on the number of intersection points between the two Lagrangians up to such perturbations. Moreover, this invariance allows one to define $HF(L, L)$ – the *self-Floer cohomology* of L – by taking the homology of the complex of intersections between L and a generic Hamiltonian push-off of L.

Floer originally constructed $HF(L^0, L^1)$ only for Lagrangians which are weakly exact, that is, such that ω integrates to zero over any class in $\pi_2(M, L^i)$, and in this case he proved that $HF(L, L)$ is isomorphic to the Morse cohomology of L. Note that this already suffices to reprove Gromov's theorem on compact Lagrangians in $\mathbb{C}^n$: supposing that such a Lagrangian satisfies $H^1(L; \mathbb{R}) = 0$ implies that it is (weakly) exact and hence $HF(L, L) \cong H^*(L) \neq 0$, contradicting the fact that L can be displaced from itself by a Hamiltonian isotopy. While in principle this argument is equivalent to Gromov's original proof, it highlights nicely the way in which introducing extra algebraic structure can be used to generalise the original theorem: the topological restriction is imposed by the vanishing of $HF(L, L)$ and if we could infer this vanishing through different methods (that is, without appealing to the geometric displaceability of the Lagrangian), the same topological conclusions would follow.[4] This is illustrative of the kind of *algebraic* principle of Lagrangian non-intersection which is the main argument we use in chapter 3 to derive topological restrictions on Lagrangians.

Shortly after Floer's original construction, Oh ([Oh93]) relaxed the assumption that L should be weakly exact and extended Lagrangian Floer cohomology to *monotone* Lagrangians (roughly, Lagrangians for which the class of ω and the Maslov class are positively proportional in $H^2(M, L; \mathbb{R})$; see section 2.1.1 for a thorough discussion). The analogous generalisation on the Hamiltonian side had already been done by Floer in [Flo89]. One important feature which arises only in the La-

[4] For example, we now know that Gromov's theorem is true if we replace $\mathbb{C}^n$ by any Liouville manifold whose *symplectic cohomology* (a variant of Hamiltonian Floer cohomology for open manifolds) vanishes: symplectic cohomology is a unital ring and $HF(L, L)$ is a module over it, so it needs to vanish as well.

grangian theory is the issue of *obstruction*: the differential on the complex $CF(L^0, L^1)$ squares to zero only when the so-called *obstruction numbers* $m_0(L^0)$, $m_0(L^1)$ agree (note in particular that the self-Floer cohomology of a monotone Lagrangian is always well-defined). These numbers are an algebraic count of the pseudoholomorphic discs of Maslov index 2 with boundary on the respective Lagrangian and passing through a generic point. Thus, the obstruction here is not a technical feature of the theoretical setup, but rather carries essential geometric information about the Lagrangians.

Since these first definitions of Floer cohomology groups, a plethora of algebraic structures surrounding them has been constructed. For example, while the Hamiltonian Floer cohomology of a monotone symplectic manifold M is additively isomorphic to its singular cohomology, the former carries a "pair of pants" product ([Sch95]) which usually differs from the classical cup product. The resulting associative, unital, (graded-)commutative ring is now known as the *quantum cohomology* of M.[5] On the other hand, Donaldson observed that by counting pseudoholomorphic triangles one can equip Lagrangian Floer cohomology with a composition operation

$$HF(L^1, L^2) \otimes HF(L^0, L^1) \to HF(L^0, L^2),$$

which exhibits the set of compact (monotone) Lagrangians of M as the objects of a category where morphism spaces are Floer cohomology groups. In particular, the self-Floer cohomology $HF(L,L)$ is a unital (in general non-commutative) ring and for any other Lagrangian L', the cohomology $HF(L,L')$ is a module over this ring. Fukaya then generalised this idea, introducing composition operations on the chain level which count pseudoholomorphic polygons with boundaries on any number of Lagrangians ([Fuk93]). The resulting structure, known as the *Fukaya category* of M, is an A_∞ category whose quasi-equivalence type is an invariant of the symplectic manifold, encoding intricate information about the Lagrangians in M and the way they intersect.

Remark 1.1.1. At this point we should mention that while in the present text we work exclusively in the monotone setting, many of the structures we described above have been defined in much greater generality. For such developments on the Hamiltonian side see for example [HS95], [Ono95], [LT98], [FO99]. The definitive reference for Lagrangian Floer theory for general closed symplectic manifolds is the monumental work [FOOO09] by Fukaya-Oh-Ohta-Ono. //

Apart from the appearance of obstructions which we described above, another major difference between Floer theory for weakly exact Lagrangians and its generalisation to monotone ones is that in the latter case the self-Floer cohomology of a Lagrangian need not be isomorphic to its singular cohomology (indeed, the unit circle in $\mathbb{C}$ is a monotone Lagrangian whose Floer cohomology

[5] A deformation of the cup product on the singular cohomology of a symplectic manifold was constructed in [RT95] (see also [MS94]) using counts of certain pseudoholomorphic spheres and this deformed algebra is called quantum cohomology. The fact that it is isomorphic to Hamiltonian Floer cohomology with the pair of pants product is a celebrated result of Piunikhin, Salamon and Schwartz ([PSS96]).

vanishes). Instead, the two are related by a spectral sequence

$$H^*(L) \implies HF(L,L),$$

which was first constructed by Oh ([Oh96]) and later by Biran ([Bir06, Section 5]) using a more algebraic approach. This spectral sequence is one of the few general tools for computing Floer cohomology.

Another important development in monotone Floer theory was the construction of the *pearl complex* carried out by Biran and Cornea in [BC07b]. This machinery greatly simplifies the calculation of self-Floer cohomology because it avoids the need to perturb the Lagrangian or to introduce time-dependent almost complex structures, both of which are needed for Oh's original construction. Instead, the complex is generated by the critical points of a Morse function f on the Lagrangian and the differential counts the so-called *pearly trajectories* which are, roughly, gradient flowlines for f which may be interrupted by the boundaries of finitely many pseudoholomorphic discs. Crucially, the homology of this complex is isomorphic to the self-Floer cohomology of the Lagrangian and by decomposing the differential according to the total Maslov index of the discs involved in different pearly trajectories, one recovers the Oh-Biran spectral sequence in a natural way. Moreover, by counting more elaborate types of pearly trajectories, Biran and Cornea define various kinds of algebraic operations on the homology of their complex, including a product (which corresponds to the Floer product through the above isomorphism) and an action of the quantum cohomology of the ambient manifold. One curious feature of all known calculations of self-Floer cohomology for a monotone Lagrangian is that it either vanishes, or the Oh-Biran spectral sequence degenerates on the first page and Floer cohomology is isomorphic to singular cohomology (this is true for computations with coefficients in a field, or, more generaly in an irreducible local system of vector spaces over the Lagrangian; otherwise counterexamples exist). The Lagrangian is called *narrow* in the former case and *wide* in the latter.

We mention one more general result which has proven extremely useful in monotone Floer theory and features prominently in this thesis, namely the Auroux-Kontsevich-Seidel (AKS) criterion ([Aur07, Proposition 6.8], [She16, Corollary 2.10]). Roughly, it says that, if a monotone Lagrangian $L \subseteq M$ has non-vanishing self-Floer cohomology, then its obstruction number $m_0(L)$ is an eigenvalue of quantum multiplication by the first Chern class of M. In particular, this means that over an algebraically closed field, the Fukaya category of a monotone symplectic manifold splits into orthogonal summands, indexed by the eigenvalues of this quantum multiplication.

Remark 1.1.2. We have deliberately not discussed the coefficient rings over which the above theories are defined. As stated what we have described is approximately correct when one works over rings of characteristic two. The theory also works over different characteristic, as long as the Lagrangians satisfy certain additional hypotheses which will be discussed later on.

Another important technical detail which we haven't mentioned is that in our entire discussion of monotone Floer theory above one must assume that all Lagrangians have minimal Maslov number at least 2. Monotone Lagrangians which bound discs of Maslov index 1 are in general inaccessible to current Floer-theoretic techniques due to compactness issues for the spaces of trajectories used in the definition of the theory. //

1.1.2.3 Lagrangian topology

Given a symplectic manifold (M,ω), even a very simple one, the task of determining the topology of all of its compact Lagrangian submanifolds is extremely difficult. Note that by Darboux's theorem, every compact n-manifold which admits a Lagrangian embedding in $\mathbb{C}^n$ also admits one in every symplectic manifold of dimension $2n$, so it makes sense for one to study this local question first.

However, even for $(\mathbb{C}^n, \omega_{\text{std}})$, classifying the topological type of Lagrangians quickly becomes intractable. Already for $n=2$, where one can still rely on the complete classification of closed surfaces, the question turns out to be very hard and its answer is somewhat surprising: the only closed surfaces which admit a Lagrangian embedding in $(\mathbb{C}^2, \omega_{\text{std}})$ are T^2 and connected sums of the form $(\mathbb{RP}^2)^{\#(4k+2)}$ for $k>0$ ([Giv86], [Aud88], [She09], [Nem09]). Moving one dimension up to $\mathbb{C}^3$, one already sees why it is not reasonable to expect strong topological classification results in higher dimensions without imposing simplifying assumptions. A well-known result of Gromov and Lees ([Gro71], [Lee76]) says that any compact n-manifold L whose complexified tangent bundle is trivial admits a Lagrangian immersion in $\mathbb{C}^n$. Polterovich ([Pol91]) developed a surgery technique to eliminate the double points of such an immersion, thus showing that for some integer k the connected sum $L\#(S^1 \tilde{\times} S^{n-1})^{\#k}$ of L with k copies of the twisted S^{n-1}-bundle over S^1 admits a Lagrangian embedding in $\mathbb{C}^n$. Moreover, if n is odd, one can use the product $S^1 \times S^{n-1}$ instead of the twisted bundle. Finding lower bounds for the number k is an interesting problem which was posed by Polterovich and was addressed in recent work of Ekholm-Eliashberg-Murphy-Smith [EEMS13]. One of their results is that in dimension 3 one can always take $k=1$ and so, if L is a closed, orientable 3-manifold then $L\#(S^1\times S^2)$ admits a Lagrangian embedding in $(\mathbb{C}^3, \omega_{\text{std}})$.[6]

In light of these facts, if one wants to find meaningful restrictions on the topology of a closed Lagrangian $L \subseteq \mathbb{C}^3$, it makes sense to ask for L to be a prime 3-manifold, that is, one which cannot be expressed as a non-trivial connected sum. This restricted problem was answered completely in a landmark theorem of Fukaya ([Fuk06, Theorem 11.1]) which states that a closed, orientable, prime Lagrangian submanifold of $\mathbb{C}^3$ is diffeomorphic to $S^1 \times \Sigma$ for some orientable surface Σ (the theorem is sharp since all such products do admit Lagrangian embeddings).

In Fukaya's work this is a corollary of a more general result about Lagrangian embeddings of Eilenberg-MacLane spaces ([Fuk06, Theorem 12.1]). The theorem states, roughly, that if an orientable, spin, aspherical Lagrangian $L \subseteq (M,\omega)$ is displaceable, then $\pi_1(L)$ contains a non-trivial

[6] Recall that closed, orientable 3-manifolds are parallelisable ([Sti35]), so by the Gromov-Lees theorem they always admit Lagrangian immersions.

element with finite index centraliser. The proof of this theorem is highly technical but the general idea behind it is to use the boundaries of pseudoholomorphic discs in order to construct classes in the homology of the free loop space of L and then exploit the deep combinatorial relations that such classes must satisfy. This combination of holomorphic curve theory and string topology is a powerful idea which has found many applications in sympletic topology (see e.g. [Vit97], [BC07a], [Abo11] and the survey [LO15]). One problem with this approach and what makes Fukaya's theorem so technically difficult, however, is that it relies on high dimensional moduli spaces of pseudoholomorphic curves and transversality and good compactness properties for these are hard to achieve.

Here again the monotonicity assumption can be used to significantly simplify the situation, while still allowing one to record some homotopy data about the paths traced on a Lagrangian by the boundaries of holomorphic curves. An important development in this direction is Damian's *lifted Floer cohomology* ([Dam12]; a similar idea appears also earlier in [Sul02]). Using this theory, Damian showed (among many other things) that, if an orientable, monotone Lagrangian $L \subseteq (M, \omega)$ satisfies $HF(L,L) = 0$ and the odd homology groups of the universal cover of L vanish, then $\pi_1(L)$ contains a non-trivial element with finite index centraliser which is the boundary of a Maslov 2 disc. This allows for a simpler proof of Fukaya's theorem in the monotone case (see [Dam15], [EK14]), and in addition implies that an orientable monotone Lagrangian in $\mathbb{C}^3$ is necessarily prime (and hence, a product). Lifted Floer cohomology has found many other applications in the study of monotone Lagrangians (some of which we discuss later on) and is one of the central tools we use in this thesis.

As the above discussion indicates, studying Lagrangians in $\mathbb{C}^n$ is already a rich and difficult subject. Moving to other symplectic manifolds, one must impose some conditions which ensure that the Lagrangians one considers are, in some sense, global.

For example, if one works in an exact symplectic manifold, then it makes sense to try and classify exact Lagrangians there (this condition forces such Lagrangians to not be contained in any Darboux ball, by Gromov's theorem). The most famous problem in this area is Arnold's *nearby Lagrangian conjecture* which posits that a compact exact Lagrangian L in the cotangent bundle $(T^*Q, \omega_{\text{can}})$ of a compact manifold Q is Hamiltonian-isotopic to the base. The full statement is only known to be true for $Q = S^1, S^2, \mathbb{RP}^2$ and T^2 ([Hin12], [DRGI16]). The current state of the art in the general case asserts that the projection $\pi\colon T^*Q \to Q$ always induces a simple homotopy equivalence $\pi|_L : L \to Q$ ([AK18], building on [FSS08], [Abo12], [Kra13]) and is another testament of the successful interplay between holomorphic curves and loop space methods.

Apart from the exact case, another possibility is to study the topology of monotone Lagrangians in monotone symplectic manifolds. This is a vast topic and very rich in results but these are inevitably very specific to the particular symplectic manifold one studies. Thus, rather than trying to give a general overview, we now discuss some of what is known about the Lagrangians in arguably the simplest monotone symplectic manifold – and the focus of this thesis – complex projective space.

1.1.3 Monotone Lagrangians in $\mathbb{CP}^n$

The two most well known examples of monotone Lagrangian submanifolds in n-dimensional complex projective space are the Clifford torus T^n_{Cl} and the real projective space $\mathbb{RP}^n$. The existence of the latter already shows that, unlike linear symplectic spaces, $\mathbb{CP}^n$ can contain Lagrangians whose first homology is finite. Such Lagrangians are *global* in a very strong sense (they cannot be isotoped to lie in a Darboux ball) and so one can expect that they "see" a lot of the symplectic topology of $\mathbb{CP}^n$. This makes them also rather rare: all examples that the author is aware of are homogeneous spaces and while some of them occur in infinite families, there are many which appear only in certain dimensions. We now discuss some of the known facts about Lagrangians in $\mathbb{CP}^n$ with finite first homology. Note that these are necessarily monotone which makes them perfect ground for Floer-theoretic explorations.

Given that $H_1(\mathbb{RP}^n;\mathbb{Z}) \cong \mathbb{Z}/2$, that is, the smallest non-trivial group, it is natural to ask whether there exist Lagrangians in $\mathbb{CP}^n$ whose *integral* homology vanishes. The answer is "no" and was first proved by Seidel in [Sei00]. In fact Seidel showed that any Lagrangian L in $\mathbb{CP}^n$ must have $H^1(L;\mathbb{Z}/(2n+2)\mathbb{Z}) \neq 0$ and, if L is monotone, then its minimal Maslov number must satisfy

$$1 \leq N_L \leq n+1.$$

Note that these bounds are sharp – for each $n \geq 2$ there do exist monotone Lagrangians in $\mathbb{CP}^n$ with minimal Maslov number 1 (we'll briefly mention some of these below) and $N_{\mathbb{RP}^n} = n+1$. Seidel further showed that, if $H^1(L;\mathbb{Z}/(2n+2)\mathbb{Z})$ is 2-torsion (which implies $N_L = n+1$), then there is an isomorphism of graded $\mathbb{Z}/2$-vector spaces $H^*(L;\mathbb{Z}/2) \cong H^*(\mathbb{RP}^n;\mathbb{Z}/2)$. In particular, if $L \subset \mathbb{CP}^n$ is a Lagrangian satisfying $2H_1(L;\mathbb{Z}) = 0$ then L is additively a $\mathbb{Z}/2$-homology $\mathbb{RP}^n$.

Later, Biran–Cieliebak ([BC01, Theorem B]) reproved the first part of Seidel's result by introducing the important *Biran circle bundle* construction, which associates to a monotone Lagrangian in $\mathbb{CP}^n$ a displaceable one in $\mathbb{C}^{n+1}$ and then uses the vanishing of the Floer cohomology of the latter to constrain the topology of the former via the Gysin sequence. Combining this construction with the Oh-Biran spectral sequence, Biran ([Bir06, Theorem A]) then reproved the second part of Seidel's result – the $\mathbb{Z}/2$-homology isomorphism – but under the hypothesis that $L \subset \mathbb{CP}^n$ is monotone and of minimal Maslov number $n+1$ (he states the assumption that $H_1(L;\mathbb{Z})$ is 2-torsion but only uses the monotonicity and minimal Maslov consequences).[7] With the introduction of the pearl complex in [BC07b], Biran and Cornea gave another proof of these results which was more algebraic in flavour – rather than using the circle bundle construction, they relied on the action of the quantum cohomology $QH(\mathbb{CP}^n)$ on $HF(L,L)$ to reach the same conclusions.

All these results lead to a natural question, first asked by Biran and Cornea in [BC07b, Section 6.2.5], which is still open: for $n \geq 2$, is the standard $\mathbb{RP}^n$ the only (up to Hamiltonian isotopy or, at

[7] Note that, in conjunction with the classification of surfaces, this result already shows that for $n = 2$ the Lagrangian must be diffeomorphic to $\mathbb{RP}^2$.

least, homeomorphism) Lagrangian in $\mathbb{CP}^n$ whose first integral homology is 2-torsion?

In dimensions one and two, the answer is as strong as possible: any such Lagrangian is Hamiltonian isotopic to $\mathbb{RP}^n$. This is trivial for $n = 1$, whilst the $n = 2$ case follows from recent work of Borman–Li–Wu ([BLW14, Theorem 1.3]). In higher dimensions, Damian ([Dam12, Theorem 1.8 c)]) applied his lifted Floer theory to the circle bundle construction to show that when n is odd and $2H_1(L;\mathbb{Z}) = 0$, L must be homotopy equivalent to $\mathbb{RP}^n$. In recent work ([KS18]) Jack Smith and the author proved that the same is true also for even n and in fact one only needs to assume that L is monotone with $N_L = n+1$. This completes the classification up to homotopy of monotone Lagrangians in $\mathbb{CP}^n$ whose minimal Maslov number is as large as possible.

When one allows the minimal Maslov number to decrease, many more examples appear (still, the author only knows of homogeneous ones for $N_L > 2$). If n is odd, the next-largest value that N_L can take is $(n+1)/2$.[8] In this thesis we study one family of monotone Lagrangians which satisfy this condition – we call them the *subadjoint Lagrangians*. There is one infinite sequence of them, appearing in $\mathbb{CP}^n$ for each odd $n \geq 5$ and five exceptional examples in dimensions $n = 3, 13, 19, 31$ and 55.

The 3-dimensional example is known as the *Chiang Lagrangian*, after River Chiang who discovered it as a Lagrangian orbit of a Hamiltonian SU(2)-action on $\mathbb{CP}^3$. It is a rational homology 3-sphere with minimal Maslov number 2 and its first homology group is $\mathbb{Z}/4$. The Floer cohomology of the Chiang Lagrangian was computed by Evans and Lekili in [EL15], where they introduced several general techniques for getting control on the holomorphic discs with boundary on homogeneous Lagrangians.[9] The Chiang Lagrangian and the results of Evans-Lekili will feature prominantly in this thesis.

Remark 1.1.3. We should mention also that the Chiang Lagrangian belongs to another interesting family of four Lagrangians, the other three of which however live in different Fano 3-folds. These are called the *Platonic Lagrangians* for their connection with the Platonic solids. By generalising and extending the techiniques of Evans-Lekili, Jack Smith computed the Floer cohomomology of all Platonic Lagrangians in [Smi15]. //

Letting the minimal Maslov number decrease further still (ignoring the dimensional coincidence that the Chiang Lagrangian had $N_L = 2$), there is another well-studied family of Lagrangians in $\mathbb{CP}^n$ – the Amarzaya-Ohnita-Chiang family ([AO03], [Chi04]). These are actually several related families, who all have *intermediate* Maslov number, roughly at the order of $\sqrt{n}$. Their Floer cohomology has been investigated in [Iri17], [EL19] and [Smi17].

Remark 1.1.4. The definitive reference for examples of homogeneous Lagrangians in projective space is the paper [BG08] by Bedulli and Gori. There they classify all Lagrangians in $\mathbb{CP}^n$ which

[8] The minimal Maslov number must divide $2(n+1)$, that is, twice the minimal Chern number of $\mathbb{CP}^n$.

[9] Here, the word "homogeneous" is used to mean that the Lagrangian is an orbit of a compact group acting on a Kähler manifold by holomorphic isometries.

are orbits of Hamiltonian actions of simple Lie groups, including all the examples we mentioned above. The Floer theory of many of these Lagrangians remains unexplored. //

When the minimal Maslov number is equal to 2, the examples are too numerous to describe. Topologically, the only Lagrangian in $\mathbb{CP}^2$ with $N_L = 2$ is the 2-torus but there are infinitely many Hamiltonian non-isotopic monotone Lagrangian tori (see [Via16] and the references therein). Moreover, given any orientable, monotone Lagrangian $L \subseteq \mathbb{CP}^n$, one can obtain a monotone Lagrangian lift $\hat{L} \subseteq \mathbb{CP}^{n+1}$ with $N_{\hat{L}} = 2$ by a careful application of the Biran circle bundle construction ([BC09b, Section 6.4]). The same is also true if one starts with a Lagrangian contained in a quadric hypersurface in $\mathbb{CP}^{n+1}$ – see [OU16] for many explicit examples.

Finally, as we mentioned earlier, Lagrangians of minimal Maslov number 1 are not amenable to Floer theory and consequently very little is known about them in general. However, already in $\mathbb{CP}^2$ there is a surprising example of a monotone Lagrangian with $N_L = 1$. It is diffeomorphic to $\left(\mathbb{RP}^2\right)^{\#6}$ and was constructed by Abreu and Gadbled in [AG17]. To produce examples in higher dimension, one can again rely on the circle bundle construction. Indeed for any non-orientable, monotone Lagrangian $L \subseteq \mathbb{CP}^n$, the associated lift $\hat{L} \subseteq \mathbb{CP}^{n+1}$ is also monotone and has minimal Maslov number equal to one.

We have undoubtedly forgotten to include many examples but hopefully the above discussion illustrates the great variety which is present among monotone Lagrangians in $\mathbb{CP}^n$. We end this section by mentioning one general result on the topology of such Lagrangians. The theorem in question is due to Simon Schatz ([Sch15]) and states that if $L \subseteq \mathbb{CP}^n$ is an orientable monotone Lagrangian whose universal cover has vanishing homology in odd degrees, then $N_L = 2$ and $\pi_1(L)$ contains a non-trivial element with finite index cetraliser (the actual result in [Sch15] is much more general and applies to monotone Lagrangians in many other Kähler manifolds besides $\mathbb{CP}^n$). The proof is based on a combination of the Biran circle bundle construction with lifted Floer theory and the "neck-stretching" arguments from [BK13].

Remark 1.1.5. The above-cited theorem of Schatz implies, for example, that any monotone Lagrangian torus in $\mathbb{CP}^n$ has minimal Maslov number equal to 2. This result was already proved by Damian in [Dam12, Theorem 1.6], but in fact nowadays something much stronger is known: the statement is true for any Lagrangian torus in $\mathbb{CP}^n$ without the monotonicity assumption. Whether this is the case was a well known question by Audin, resolved in the affirmative by Cieliebak-Mohnke in [CM18] (see also [Fuk06, Theorem 11.4]). //

1.2 Overview of this thesis

1.2.1 A motivating example

Chronologically, the starting point of this thesis lies with two monotone Lagrangians in $\mathbb{CP}^3$: real projective space $\mathbb{RP}^3$ and the Chiang Lagrangian L_Δ. There is one glaring feature which they have in common, namely they are both rational homology spheres. Moreover, their first integral homologies are the smallest ones allowed – isomorphic to $\mathbb{Z}/2$ and $\mathbb{Z}/4$, respectively (recall that H_1 is not allowed to vanish and an easy Maslov class calculation shows that, if the first homology of a Lagrangian in $\mathbb{CP}^3$ is finite, then it cannot have odd cardinality). While this should already suggest that there must be some connection between these two Lagrangians, an even more compelling piece of evidence is the following equation

$$2+2=4. \tag{1.1}$$

Let us explain. In [Joy02], Joyce proposed a conjectural invariant of (almost) Calabi-Yau 3-folds which counts the number of special Lagrangian rational homology spheres, weighting each Lagrangian by the size of its first integral homology group. The weighting is needed because, while special Lagrangians deform smoothly with small variations of the Kähler metric and holomorphic volume form ([McL98], [Joy05]), large variations give rise to wall-crossing phenomena in which some special Lagrangians disappear (become singular) and others appear in their place. Joyce conjectures that whenever this happens for special Lagrangian rational homology 3-spheres, the sum of the sizes of the H_1's of the manifolds counted before and after the wall-crossing occurs should remain unchanged. Something similar happens with L_Δ and $\mathbb{RP}^3$. The Chiang Lagrangian is special Lagrangian in the complement of a discriminantal divisor (the divisor cut out by the discriminant of a cubic polynomial in one variable). One can then deform this divisor until it breaks up into the union of two quadric hypersurfaces in whose complement live two special Lagrangian real projective spaces. Therefore, during the deformation L_Δ undergoes some kind of surgery and transforms into two copies of $\mathbb{RP}^3$. And indeed, by equation (1.1), we have $|H_1(\mathbb{RP}^3;\mathbb{Z})|+|H_1(\mathbb{RP}^3;\mathbb{Z})| = |H_1(L_\Delta;\mathbb{Z})|$, as one would expect from Joyce's conjecture.

As this discussion indicates, L_Δ and $\mathbb{RP}^3$ are quite closely related. Their relationship is what ties the seemingly different parts of the thesis together. More precisely, we investigate the following three questions:

Question 1. Are L_Δ and $\mathbb{RP}^3$ the only Lagrangian rational homology spheres in $\mathbb{CP}^3$?

Question 2. What exactly is the relationship between L_Δ and $\mathbb{RP}^3$ and are there analogues in higher dimension?

Question 3. Can L_Δ and $\mathbb{RP}^3$ be disjoined by a Hamiltonian isotopy of $\mathbb{CP}^3$?

We now discuss the research that has spun out of these questions, describe the techniques that we use and state the main results.

1.2.2 Methods and results

1.2.2.1 High rank local systems

The main tool used in this thesis to infer information about Lagrangian submanifolds is monotone Floer theory with coefficients in local systems of rank higher than one. The objects we study are pairs $(L,\mathcal{E})$ where L is some closed monotone Lagrangian and $\mathcal{E}$ is a local system of R-modules on L (an "R-local system") for some commutative ring R (which has characteristic 2 unless L is also equipped with additional structure like an orientation or a (relative) (s)pin structure). The Floer complex $CF((L^0,\mathcal{E}^0),(L^1,\mathcal{E}^1))$ of two such objects is then generated by the R-linear maps between the fibres of the local systems living over points in $L^0 \cap L^1$ (assuming this intersection is tranverse) and the Floer differential counts the usual pseudoholomorphic strips but uses their boundaries on L^0 and L^1 for parallel transport. This is exactly analogous to the rank 1 case but there is one crucial difference – if a local system $\mathcal{E}$ on L has rank higher than one, then even the self-Floer complex $CF((L,\mathcal{E}),(L,\mathcal{E}))$ may be obstructed. This obstruction is governed by an endomorphism $m_0(\mathcal{E})$ of the local system $\mathcal{E}$, which we call the *obstruction section*. Equivalently, $m_0(\mathcal{E})$ can be viewed as a central element of the group algebra $R[\pi_1(L,x)/(\ker\rho_{\mathcal{E}})]$ where $\rho_{\mathcal{E}}$ is the monodromy representation of $\mathcal{E}$ at some point $x \in L$. In fact, using the local system $\mathcal{E}_{\mathrm{reg}}$ which corresponds to the regular representation of $\pi_1(L)$, one obtains a universal obstruction $m_0(L,x) \in R[\pi_1(L,x)]$ which is defined as the sum of boundaries of Maslov 2 pseudoholomorphic discs passing through x (counted with appropriate signs, if $\mathrm{char}(R) \neq 2$). The obstruction $m_0(\mathcal{E})$ is then obtained by reducing $m_0(L,x)$ modulo $\ker\rho_{\mathcal{E}}$.

This is precisely the obstruction observed by Damian in his lifted Floer theory ([Dam12]). The fact that $m_0(L,x)$ is a *central* element of the group algebra (see [Dam15, Section 1.2] or Proposition 2.2.3 below) provides one with a useful alternative – either $m_0(L,x)$ is a multiple of the identity, in which case lifted Floer cohomology is defined, or there exists an element in $\pi_1(L,x)$ whose centraliser has finite index and which is the boundary of a Maslov 2 disc. As we saw in section 1.1.3 and will further demonstrate in this thesis, this dichotomy can be used to obtain strong topological restrictions on monotone Lagrangians, especially in dimension three, where the fundamental group controls much of the topology of a manifold.

The fact that $m_0(L,x)$ is central also allows us to define a new variant of self-Floer cohomology which we call *monodromy Floer cohomology* of L and denote $HF_{\mathrm{mon}}(L;R)$. While we do not provide concrete applications of this invariant, we observe that its non-vanishing implies that L cannot be displaced from itself by a Hamiltonian isotopy, while if $HF_{\mathrm{mon}}(L;R)$ is zero, then the Floer cohomology of $(L,\mathcal{E})$ vanishes for all R-local systems $\mathcal{E}$ on L. In this sense, HF_{mon} is more refined than the other invariants considered in this text.

The usefulness of the local systems formalism is that it fits well with the established algebraic operations in Floer theory such as products or the quantum module action. For example, note that Damian's lifted Floer cohomology (when it exists) is the same as the group $HF(L,(L,\mathcal{E}_{\mathrm{reg}}))$ and so it is a (right) module over the ring $HF(L,L)$. In fact, similarly to the rank one case, one can enlarge the (monotone) Fukaya category by adding pairs $(L,\mathcal{E})$ with $\mathcal{E}$ of arbitrary rank as objects. Even when the complex $CF((L,\mathcal{E}),(L,\mathcal{E}))$ is obstructed, one can define the endomorphism space of $(L,\mathcal{E})$ in this bigger category to be the maximal unobstructed subcomplex (that is, the kernel of the square of the Floer "differential"), which we denote $\overline{CF}((L,\mathcal{E}),(L,\mathcal{E}))$. One can do the same for the hom-spaces between local systems living over different Lagrangians or different local systems over the same Lagrangian. These subcomplexes are preserved by all the A_∞ operations (again by the fact that m_0 commutes with parallel transport maps) and so one obtains a well-defined enlarged Fukaya category. While passing to the maximal unobstructed subcomplex is certainly very artificial, there are many cases in which one does not have to resort to doing so, for example, if the minimal Maslov number of L is greater than 2 or if one chooses an appropriate local system which makes the obstruction vanish.

In the case of exact manifolds (where there are no obstructions), a similar extended Fukaya category was used by Abouzaid in [Abo12] to prove that a compact exact Lagrangian with vanishing Maslov class in a cotangent bundle must be homotopy equivalent to the base. In this thesis we give some evidence that enlarging the Fukaya category by allowing high rank local systems can also be useful in the monotone case, even when a Lagrangian has minimal Maslov number 2. In particular, this technique allows us to give a negative answer to Question 3 (see Theorem E below).

Remark 1.2.1. High rank local systems have also been incorporated in a de Rham model for the Fukaya category in [Bae17]. //

Remark 1.2.2. There are many other ways in which one can "twist" the coefficients of Lagrangian Floer cohomology. For example, the pearl complex of a Lagrangian $L \subseteq M$ can be defined with coefficients in the group ring $R[H_2(M,L)]$ or the local system whose fibre over each point $x \in L$ is given by $R[\pi_1(M,L,x)]$ (see [BC07b]). In this way one can record the entire relative homology or homotopy classes of the holomorphic discs which contribute to the differential, rather than just the classes of their boundaries. Moreover, keeping track of such homotopy classes of discs is essential for Zapolsky's definition of the canonical Floer and pearl complexes, which in turn can be twisted further using an even more general notion of local coefficients (roughly, a local system on the space of paths in M with endpoints on L). For these constructions see [Zap15] and [Smi17, Appendix A]. Note however that all of these generalisations apply to the *self*-Floer complex of a single Lagrangian and it is not quite clear how to use them for pairs of Lagrangians or, more generally, how to incorporate them into the Fukaya category (although the concept of *B-fields* gives one possibility, see [Smi17, Section 4.3]). For this reason we confine ourselves to the more standard local coefficient systems described above. //

1.2.2.2 Lagrangians which look like $\mathbb{RP}^n$

An obvious subquestion of Question 1 is whether $\mathbb{RP}^3$ is the only Lagrangian rational homology sphere in $\mathbb{CP}^3$ which has minimal Maslov number equal to 4. The general problem of classifying monotone Lagrangians in $\mathbb{CP}^n$ of minimal Maslov number $n+1$ was considered by Jack Smith and the author in our joint work [KS18], where we prove the following.

Theorem A (Theorem 3.1.1)**.** *Let $L \subseteq \mathbb{CP}^n$ be a closed, connected, monotone Lagrangian submanifold with minimal Maslov number $n+1$. Then L is homotopy equivalent to $\mathbb{RP}^n$.*

Using a result of Livesay ([Liv62, Theorem 3]) about $\mathbb{Z}/2$-actions on S^3, this theorem implies that a monotone Lagrangian in $\mathbb{CP}^3$ (in particular, a rational homology sphere) of minimal Maslov number 4 is diffeomorphic to $\mathbb{RP}^3$. In higher dimensions we cannot easily upgrade homotopy equivalence to diffeomorphism since there exist smooth manifolds which are homotopy equivalent but not homeomorphic to $\mathbb{RP}^n$ (see [CS76] for the case of dimension four and [HM64] for dimension at least five). Moreover, even in dimension 3 the question whether a monotone Lagrangian of minimal Maslov number 4 must be Hamiltonian isotopic to the standard $\mathbb{RP}^3$ remains wide open.

The proof of Theorem A resembles that of [Dam12, Theorem 1.8c)] in that we also use lifted Floer theory with coefficients in $\mathbb{Z}$ to get a handle on the fundamental group of L. Unlike Damian however, we do not invoke the circle bundle construction and instead rely on the algebraic structure of Floer cohomology, in particular the action of quantum cohomology $QH(\mathbb{CP}^3)$. In this thesis we only give the proof of Theorem A in the case when n is odd. When n is even, the Lagrangian is non-orientable which makes Floer theory over $\mathbb{Z}$ difficult to define. Such a theory was developed by Zapolsky in [Zap15], where he introduced the so-called canonical Floer and pearl complexes which are well-defined over an arbitrary ground ring, provided the second Stiefel-Whitney class of L satisfies a mild vanishing property known as *Assumption (O)* (see page 93 below). Using Floer theory over $\mathbb{F}_2$, it is not difficult to show that a monotone Lagrangian in $\mathbb{CP}^{2m}$ of minimal Maslov number $2m+1$ satisfies Assumption (O) and then, using Zapolsky's theory with $\mathbb{Z}$ coefficients, the proof of Theorem A proceeds much like in the odd-dimensional case (see [KS18]). However, since the only applications of Theorem A that we need in this thesis are in odd dimensions, we do not give the full details of the even-dimensional case.

1.2.2.3 Monotone Lagrangians in $\mathbb{CP}^3$

Focusing on dimension three, Question 1 brings us to another more general problem: the topological classification of monotone Lagrangians in $\mathbb{CP}^3$. Note that the minimal Maslov number N_L of such a Lagrangian can only take the values $\{1,2,4\}$. We have already dealt with the case $N_L = 4$ and, as we explained earlier, Lagrangians of minimal Maslov number 1 are not amenable to Floer theory, so we focus on the case $N_L = 2$. We prove the following theorem which substantially narrows down the possible topology that a monotone Lagrangian in $\mathbb{CP}^3$ can have (we include the case $N_L = 4$ for a more complete statement).

Theorem B (Proposition 3.2.1, Theorem 3.2.11 and Corollary 3.2.12)**.** *Let $L \subseteq \mathbb{CP}^3$ be a closed, connected, monotone Lagrangian submanifold. Assume that L is orientable or, equivalently, that its minimal Maslov number N_L is at least 2. Then $N_L \in \{2,4\}$ and:*

a) if $N_L = 4$, then L is diffeomorphic to $\mathbb{RP}^3$;

b) if $N_L = 2$, then one has the following exclusive cases:

b1) L is diffeomorphic to a quotient of S^3 by a discrete subgroup $\Gamma \leq \mathrm{SO}(4)$, where Γ is either a cyclic group of order divisible by 4 or a product of a dihedral group of order $2^k(2n+1)$ for some $k \geq 2$, $n \geq 1$ and a cyclic group of order coprime to $2^k(2n+1)$;

b2) L is diffeomorphic to $S^1 \times S^2$;

b3) L is diffeomorphic to T^3 or the mapping torus of an order 3 diffeomorphism of T^2;

b4) L is a non-Euclidean principal circle bundle over an orientable, aspherical surface and the Euler class of this bundle is divisible by 4.

In particular, if $H_1(L;\mathbb{Q}) = 0$, then either L is diffeomorphic to $\mathbb{RP}^3$, or it is one of the spherical space forms from case b1).

Let us explain the different subcases of Theorem B b) and where our contribution lies. Our approach to dealing with this case is similar to the one of [Dam15] or [EK14], namely to use lifted Floer theory in order to deduce that the fundamental group of the Lagrangian contains a non-trivial element which has finite index centraliser and which bounds a Maslov 2 disc.

Now note that such information is redundant, if the fundamental group of L is already finite. These are precisely the manifolds considered in case b1). By the famous Elliptisation Theorem (proved by Grigori Perelman, see [MT07]) orientable 3-manifolds with finite fundamental group are necessarily quotients of S^3 by a discrete group $\Gamma \leq \mathrm{SO}(4)$. Soft observations from the properties of the Maslov class tell us that $H_1(L;\mathbb{Z})$ must contain an element of order 4 and then the restrictions on the group Γ follow from Milnor's classification of finite subgroups of $\mathrm{SO}(4)$ which act freely on the 3-sphere ([Mil57, Theorem 2]). The only known Lagrangian of $\mathbb{CP}^3$ which falls in this category is precisely the Chiang Lagrangian L_Δ whose fundamental group is the binary dihedral group of order twelve.

Now, if the fundamental group of L is not finite, then knowing that $\pi_1(L)$ contains a non-trivial element with finite index centraliser is essential for constraining the topology of L. In fact, we can already infer the existence of such an element by Schatz's result ([Sch15]) since the odd homology groups of the universal cover of L vanish. However, we choose to give a different argument (Proposition 3.2.17) which avoids the use of the circle bundle and relies instead on the algebraic structure of Floer cohomology and the AKS criterion. This approach is essential for dealing with case b4), as we explain below.

Once one has the existence of a non-trivial element $\gamma \in \pi_1(L)$ with finite index centraliser, it is not hard to show (relying again on some heavy 3-manifold theorems, in particular the Elliptisation Theorem, the Seifert fibred space theorem and Scott's rigidity theorem from [Sco83b]) that L must be a prime and Seifert fibred 3-manifold.[10] Knowing that L is prime, orientable and has infinite

[10] After the first draft of these results was written, the author learned that the same facts (that L is prime and Seifert fibred,

fundamental group implies that either it is diffeomorphic to $S^1 \times S^2$ or it is an Eilenberg-MacLane space. The first possibility is covered by case b2). Up to Hamiltonian isotopy, there is one known example of a monotone Lagrangian $S^1 \times S^2$ in $\mathbb{CP}^3$, constructed as a Biran circle bundle over a Lagrangian sphere in a quadric hypersurface (see [BC09b, Section 6.4] and also [OU16]). The case of an aspherical L is then split into two subcases as follows.

Case b3) deals with Euclidean manifolds. There are only 6 diffeomorphism types of orientable Euclidean 3-manifolds and the fact that the loop γ is the boundary of a Maslov 2 disc allows us to rule out 4 of them, leaving us with the possibility that L is either a 3-torus or the so-called *tricosm* – a T^2 bundle over S^1 with monodromy of order 3. The 3-torus, of course, does embed as a monotone Lagrangian in $\mathbb{CP}^3$, while the tricosm is not known to admit such an embedding but our methods cannot rule it out.

Finally, the most interesting case is b4). We know that L is Seifert fibred and if we assume that it is aspherical and non-Euclidean, the fact that γ bounds a Maslov 2 disc tells us that the base of the Seifert fibration must be orientable. To show that this Seifert fibration has no singular fibres (i.e. that L is a principal circle bundle), we consider the evaluation map $\mathrm{ev}\colon \mathcal{M}_{0,1}(2,L) \to L$, where $\mathcal{M}_{0,1}(2,L)$ is the moduli space of pseudoholomorphic Maslov 2 discs with boundary on L and one boundary marked point. This is a map from a principal circle bundle over a surface to an aspherical, Seifert fibred 3-manifold. We use a result of Yongwu Rong ([Ron93]) in order to prove a crucial lemma (Lemma 3.2.9) which tells us that the degree of such a map must be divisible by the multiplicities of all singular fibres of the target. A combination of this result, together with the AKS criterion and lifted Floer cohomology with coefficients in a field of *odd* characteristic are then used to finish the proof of Theorem B. There are no known examples of non-Euclidean circle bundles over aspherical surfaces which admit a monotone Lagrangian embedding in $\mathbb{CP}^3$, however, in analogy with monotone Lagrangians in $\mathbb{C}^3$, it is not unlikely that at least products $S^1 \times \Sigma$ could admit such an embedding.

As is evident, Theorem B does not give a complete answer to Question 1. However, it does rule out the possibility of embedding an aspherical rational homology 3-sphere as a monotone Lagrangian in $\mathbb{CP}^3$ and leaves only the (still infinite) list of spherical space forms to be considered.

1.2.2.4 The twistor fibration

Our next line of inquiry is motivated by Question 2 and investigates the interplay between the symplectic geometry of $\mathbb{CP}^{2n+1}$ and the natural projection $\Pi\colon \mathbb{CP}^{2n+1} \to \mathbb{HP}^n$ from complex to quaternionic projective space. The map Π is a fibration with fibre $\mathbb{CP}^1$ and exhibits $\mathbb{CP}^{2n+1}$ as the *twistor space* of $\mathbb{HP}^n$, where the latter is viewed as a quaternion-Kähler manifold. When one equips $\mathbb{CP}^{2n+1}$ and $\mathbb{HP}^n$ with their respective Fubini-Study metrics, Π becomes a Riemannian submersion with totally geodesic fibres which are called *twistor lines*. We give some more background on the general

Corollaries 3.2.18 and 3.2.19 in this thesis) are proved in Simon Schatz's PhD thesis [Sch16, Chapitre 5.5].

theory of twistor spaces for quaternion-Kähler manifolds in section 4.1.1 but for now we focus on the main question that interests us, namely:

Question 4. How does a Lagrangian $L \subseteq \mathbb{CP}^{2n+1}$ project to $\mathbb{HP}^n$?

Twistor Lagrangians

We mainly concentrate on the most degenerate situation in which the image $\Pi(L)$ is an embedded $2n$-dimensional submanifold of $\mathbb{HP}^n$ and the restricted projection map $\Pi|_L : L \to \Pi(L)$ is a circle bundle. It turns out that there is only one way that this can happen which we now explain (cf. Theorems 4.1.22 and 4.1.23).

Note that the fibres of $\Pi\colon \mathbb{CP}^{2n+1} \to \mathbb{HP}^n$ are symplectic and so the Fubini-Study form induces a splitting $T\mathbb{CP}^{2n+1} = \mathcal{V} \oplus \mathcal{H}$ into a vertical and a horizontal bundle. It is known that the horizontal bundle defines a holomorphic contact structure on $\mathbb{CP}^{2n+1}$, that is, $\mathcal{H}$ is locally given as the kernel of a holomorphic 1-form α such that $\alpha \wedge (d\alpha)^n$ is nowhere vanishing. This implies that a maximal integral manifold of $\mathcal{H}$ is a complex manifold of complex dimension n. The projective varieties of complex dimension n which are everywhere tangent to $\mathcal{H}$ are called *Legendrian subvarieties* of $\mathbb{CP}^{2n+1}$ and have been extensively studied (see e.g. [Buc09] and the many references therein). Their projections to $\mathbb{HP}^n$ are known as (immersed) *superminimal surfaces*, if $n = 1$, and (immersed) *maximal totally complex* (MTC) submanifolds, if $n \geq 2$. These objects have also been the subject of a lot of research, starting with the celebrated paper [Bry82] where Bryant showed that every compact Riemann surface Σ admits a conformal and (super)minimal immersion in $\mathbb{HP}^1 = S^4$ by exhibiting an embedding of Σ into $\mathbb{CP}^3$ as a Legendrian curve. For results on MTC submanifolds in $\mathbb{HP}^n$ for $n \geq 2$ (and more general quaternion-Kähler manifolds) see for example [Tsu85], [Tak86], [AM05] and the references therein.

Here is how this story relates to symplectic geometry. Since the fibres of Π are isometric to round spheres, one can associate to each point $x \in \mathbb{CP}^{2n+1}$ its *opposite equator*, defined as the geodesic circle in the twistor line through x which is at maximal distance from x. It turns out that, if one applies this procedure to each point on a smooth Legendrian subvariety, one obtains an immersed, minimal Lagrangian submanifold of $\mathbb{CP}^{2n+1}$. We call this phenomenon the *Legendrian – Lagrangian correspondence*. It has been observed under different guises by many authors: for example, in dimension $n = 1$ it appears in [BDVV96] but also implicitly in [Eji86, Section 15]; for $n \geq 2$ it is proved in [ET05] and used in [BGP09]. The present author also discovered it independently.

Now, since there are many Legendrian varieties, one can use the Legendrian – Lagrangian correspondence to obtain a plethora of immersed minimal Lagrangians in $\mathbb{CP}^{2n+1}$. However, if one wants to construct an embedded Lagrangian this way, the Legendrian variety $X \subseteq \mathbb{CP}^{2n+1}$ that one starts with must satisfy exactly one of the following conditions (cf. [AM05, Definition 5.3]):

1) The restricted projection $\Pi|_X : X \to \mathbb{HP}^n$ is an embedding. If X satisfies this property, we call it a Type 1 Legendrian variety.

2) For each twistor line ℓ, the intersection $X \cap \ell$ is either empty or consists of two points which are antipodal on ℓ. In this case we call X a Type 2 Legendrian variety.

Note that these are also the only cases in which the image $\Pi(X)$ is an embedded submanifold of $\mathbb{HP}^n$. If X satisfies one of these properties, we obtain a corresponding embedded, minimal Lagrangian submanifold $Z_X \subseteq \mathbb{CP}^{2n+1}$ which we call a *Type 1* or *Type 2 twistor Lagrangian*, respectively. Note that a minimal Lagrangian in $\mathbb{CP}^m$ is necessarily monotone by a result of Cieliebak-Goldstein [CG04], so we can use monotone Floer theory to study twistor Lagrangians and, consequently, Legendrian subvarieties of $\mathbb{CP}^{2n+1}$.

The easiest example of a twistor Lagrangian is the standard $\mathbb{RP}^{2n+1} \subseteq \mathbb{CP}^{2n+1}$. It is of Type 1 and its corresponding Legendrian variety is a linear $\mathbb{CP}^n$. With this in mind, we can finally state our first result on this topic.

Theorem C (Theorem 4.1.29). *If $X \subseteq \mathbb{CP}^{2n+1}$ is a smooth Type* 1 *Legendrian subvariety, then X is a linear $\mathbb{CP}^n$.*

This theorem is a rather straightforward consequence of Theorem A, after noticing that a Type 1 twistor Lagrangian in $\mathbb{CP}^{2n+1}$ must have minimal Maslov number $2n+2$. In the case $n = 1$, the result is well-known (for example from the main formula in [Fri84]) and follows from an easy Chern class computation (see the proof of Theorem 4.1.36). Note that, if one further assumes that X is a rational curve, one obtains Ernst Ruh's ([Ruh71]) classical theorem that the only embedded minimal 2-sphere in S^4 is the equator.[11]

Type 2 Legendrian varieties are much more interesting although only a handful of smooth examples are known and they are all homogeneous (see page 140 or [Tsu85]). There is one infinite family $X_{(1,m)} \cong \mathbb{CP}^1 \times \mathbf{Q}_m \subseteq \mathbb{CP}^{2m+3}$ for $m \geq 1$, where $\mathbf{Q}_m$ is the (complex) m-dimensional quadric, and 5 exceptional examples which appear in the projective spaces $\mathbb{CP}^{2n+1}$ for $n = 1, 6, 9, 13$ and 27. These Legendrian varieties are well-known from representation theory and are called *subadjoint varieties* (see e.g. [Muk98], [LM02], [Buc08b]). We denote them by X_1, X_6, X_9, X_{13} and X_{27}, respectively. We denote the corresponding twistor Lagrangians by $Z_{(1,m)}$ and $Z_1, Z_6, Z_9, Z_{13}, Z_{27}$ and call them the *subadjoint Lagrangians*. Note that $X_{(1,m)}$, X_1, X_6, X_9, X_{13} and X_{27} are homogeneous for the groups $\mathrm{SU}(2) \times \mathrm{SO}(m+2)$, $\mathrm{SU}(2)$, $\mathrm{Sp}(3)$, $\mathrm{SU}(6)$, $\mathrm{SO}(12)$ and E_7, respectively.

The variety $X_1 \subseteq \mathbb{CP}^3$ is a twisted cubic and Z_1 is precisely the Chiang Lagrangian (from this point of view, this space was observed already in [CDVV96] but it was only viewed as a totally real immersion of S^3 into $\mathbb{CP}^3$ without mention that the image of this immersion is actually an embedded Lagrangian). In fact all subadjoint Lagrangians are themselves homogeneous (for the same groups as the corresponding Legendrian varieties, see [BGP09]) and $Z_1, Z_6, Z_9, Z_{13}, Z_{27}$ appear in [BG08, Table 1] on rows 6, 11, 7, 16 and 20, respectively.

[11] It is known that a minimal 2-sphere in S^4 is necessarily superminimal. See [Bry82, Theorem C].

We conjecture that the subadjoint varieties are the only smooth Legendrian varieties of Type 2 in complex projective space. Our next result proves this conjecture in dimension one.

Theorem D (Theorem 4.1.36)**.** *If $X \subseteq \mathbb{CP}^3$ is a smooth Type* 2 *Legendrian curve, then X is a twisted cubic and there exists a linear transformation $A \in \mathrm{Sp}(2)$ whose associated projective transformation $F_A \colon \mathbb{CP}^3 \to \mathbb{CP}^3$ satisfies $F_A(X_1) = X$. Equivalently, if $\Sigma \subseteq S^4(1/2)$ is a smooth, embedded, non-orientable, superminimal surface, then Σ is congruent to the Veronese surface.*

To prove this result we use Theorem B to show that a Type 2 Legendrian curve must be rational and then we argue that such a curve must have degree 3 by appealing to a result of Massey about normal bundles of embedded, non-orientable surfaces in S^4 ([Mas69]).

From the above discussion, we see that both L_Δ and $\mathbb{RP}^3$ belong to the family of twistor Lagrangians. In fact, by exhibiting a Legendrian degeneration of the twisted cubic X_1 to the union of two Legendrian lines (one of which is double covered in the limit), we give an explicit 1-parameter family of immersed twistor Lagrangians which interpolates between L_Δ and two copies of $\mathbb{RP}^3$ (see section 4.1.6). Note that this is *not* the wall-crossing phenomenon which we explained in section 1.2.1 because none of the intermediate Lagrangians in our interpolating family are embedded. It remains an interesting open problem to understand the surgery that occurs when one deforms the discriminantal divisor in whose complement L_Δ is a special Lagrangian.

General Lagrangians

One can also consider the generic setting for Question 4, that is when the restricted projection $\Pi|_L \colon L \to \mathbb{HP}^n$ is an immersion on some non-empty open set of L. Something that one might want to know, for example, is whether it is possible for $\Pi|_L$ to be an immersion at all points of a compact Lagrangian L. At the time of writing the author has no idea. Relatedly, since the symplectic geometry of $\mathbb{CP}^{2n+1}$ is completely determined by the quaternion-Kähler structure of $\mathbb{HP}^n$, in principle one should be able to reconstruct the Lagrangian L from just local (tangential and normal) information on $\Pi(L) \subseteq \mathbb{HP}^n$, at least over the images of points where $\Pi|_L$ is an immersion. Is there some natural geometric interpretation of this local information?

We briefly explore this last question in dimension $n = 1$, in which case we show that a Lagrangian lift of a 3-ball $B^3 \subseteq \mathbb{HP}^1$ to $\mathbb{CP}^3$ corresponds to a unit vector field on B^3 which satisfies a particular differential equation involving the second fundamental form (Proposition 4.2.3). As an example, we observe that the Clifford torus $T^3_{Cl} \subseteq \mathbb{CP}^3$ projects onto an equatorial $S^3 \subseteq S^4 = \mathbb{HP}^1$ and is encoded by a 1-dimensional geodesic foliation of $S^3 \setminus \{\text{Hopf link}\}$. While the theoretical value of this observation is probably questionable, it allows us to truly "see" the standard Lagrangian embedding of the Clifford 3-torus in $\mathbb{CP}^3$ – see figure 4.1.

Given a Lagrangian $L \subseteq \mathbb{CP}^{2n+1}$ one may also want to know what other Lagrangians L' there are which satisfy $\Pi(L') = \Pi(L)$. Note that there is an anti-symplectic involution $\mathcal{X} \colon \mathbb{CP}^{2n+1} \to \mathbb{CP}^{2n+1}$ given by the antipodal map on each twistor line, so we can always choose $L' = \mathcal{X}(L)$ (note that in the case of Type 2 twistor Lagrangians or the standard Clifford torus T_{Cl}^{2n+1} one has $\mathcal{X}(L) = L$), but are there any others? We can rule out one obvious potential source of non-uniqueness, namely Hamiltonian flows which preserve the fibres: we show that any function $f\colon \mathbb{CP}^{2n+1} \to \mathbb{R}$ whose Hamiltonian vector field is tangent to the twistor lines must be constant (Proposition 4.3.1). On the other hand, uniqueness certainly fails if the image $\Pi(L)$ is invariant under the action of some positive-dimensional subgroup of $\mathrm{Sp}(n+1)$. For example, by translating T_{Cl}^3 using the natural lift of the rotation action of $\mathrm{SO}(5)$ on S^4 which preserves $\Pi(T_{Cl}^3) = S^3$, we can find a copy of T_{Cl}^3 which is contained in $\Pi^{-1}(S^3)$ and passes through any given point there.

1.2.2.5 Non-displaceability

Finally, we address some non-diplaceability problems for the known twistor Lagrangians. We start with Question 3 – whether L_Δ and $\mathbb{RP}^3$ can be Hamiltonianly displaced from eachother – which was first asked by Evans and Lekili ([EL15, Remark 1.6]). In *op. cit.* the authors computed the Floer cohomology of L_Δ and observed a strange phenomenon: L_Δ is wide in characteristic 5 but narrow over fields of any other characteristic. In fact, they show something much stronger: by equipping L_Δ with each of the four possible rank one $\mathbb{F}_5$-local systems $\{\beta_\zeta\colon \zeta \in \{1,2,3,4\}\}$, one obtains an object (L_Δ, β_ζ) of each of the four summands of the Fukaya category of $\mathbb{CP}^3$ over $\mathbb{F}_5$ and this object generates the summand (see [EL15, Section 8]). In particular, L_Δ cannot be displaced from itself or from the Clifford torus by a Hamiltonian isotopy. However, as Evans and Lekili observed, standard Floer cohomology (even with rank 1 local systems) cannot be used to address Question 3: it is well-known that $\mathbb{RP}^3$ has non-vanishing self-Floer cohomology only in characteristic 2, while the calculation in [EL15] shows that the obstruction number of L_Δ is non-zero in this characteristic (even if L_Δ carries a rank 1 local system, see Remark 5.1.7) and so the Floer complex of $\mathbb{RP}^3$ and L_Δ is obstructed (recall that $N_{\mathbb{RP}^3} = 4$, so $m_0(\mathbb{RP}^3) = 0$).

As it turns out, high rank local systems provide a solution to this problem. More precisely, following a suggestion of Evans, we show:

Theorem E (Proposition 5.1.9, Corollary 5.1.11)**.** *There exists an $\mathbb{F}_2$-local system $\mathcal{W}^D$ on L_Δ of rank 2 such that $m_0(\mathcal{W}^D) = 0$ and $HF^*((L_\Delta, \mathcal{W}^D), (L_\Delta, \mathcal{W}^D)) \cong (\mathbb{F}_2)^4$. In particular $(L_\Delta, \mathcal{W}^D)$ is a non-zero object in the enlarged monotone Fukaya category of $\mathbb{CP}^3$ over $\mathbb{F}_2$. Since this category is split-generated by the standard $\mathbb{RP}^3$, we have $HF^*(\mathbb{RP}^3, (L_\Delta, \mathcal{W}^D)) \neq 0$ and so $\mathbb{RP}^3$ and L_Δ cannot be disjoined by a Hamiltonian diffeomorphism of $\mathbb{CP}^3$.*

The first part of this theorem is proved by an explicit calculation using the Biran-Cornea pearl complex and the enumeration of holomorphic discs with boundary on L_Δ from [EL15]. It is a theorem of Tonkonog [Ton18, Proposition 1.1] that for every positive integer m the Fukaya category of $\mathbb{CP}^m$ over $\mathbb{F}_2$ is split-generated by $\mathbb{RP}^m$ and his proof still applies when one allows Lagrangians with high rank local systems as objects. Applying this to dimension $m = 3$, we obtain the desired non-displaceability.

As far as the author knows, Theorem E is the only result to date, where high rank local systems are used in order to turn a Lagrangian with vanishing Floer cohomology over a field of given characteristic into a non-zero object of the respective Fukaya category. However, Jack Smith has constructed some examples of monotone Lagrangians in products of projective spaces which are narrow over any field and with any local system of any rank, yet their Floer cohomology becomes non-zero when one deforms the differential by a so-called *B-field* – see [Smi17, Theorems 2 and 3].

Our last result concerns the other subadjoint Lagrangians. One can show (Lemma 4.1.30) that a Type 2 twistor Lagrangian Z in $\mathbb{CP}^{2n+1}$ has minimal Maslov number $n+1$. In particular, if $n \geq 2$, there are no obstructions for Floer cohomology with high rank local systems. Moreover, if $X \subseteq \mathbb{CP}^{2n+1}$ is the Type 2 Legendrian variety associated to Z, then Z is double-covered by the circle bundle $S(\mathcal{O}_X(2))$. Applying lifted Floer theory for this cover to each of the orientable subadjoint Lagrangians, we show:

Theorem F (Proposition 5.2.2, Corollary 5.2.3)**.** *Let Z denote any of the subadjoint Lagrangians $Z_{(1,2k)}$, Z_9, Z_{15} or Z_{27} and let d_Z denote the dimension of Z. Then $HF^*(Z,Z;\mathbb{F}_2) \neq 0$ and so Z cannot be displaced from $\mathbb{RP}^{d_Z}$ or $T_{Cl}^{d_Z}$ by a Hamiltonian diffeomorphism of $\mathbb{CP}^{d_Z}$. Moreover, Z_{15} split-generates the Fukaya category $\mathcal{F}(\mathbb{CP}^{31};\overline{\mathbb{F}}_2)$, where $\overline{\mathbb{F}}_2$ denotes the algebraic closure of $\mathbb{F}_2$.*

This theorem is proved by considering the Oh-Biran spectral sequence which converges to the lifted Floer cohomology of Z corresponding to the double cover $S(\mathcal{O}_X(2))$. The $\mathbb{F}_2$-cohomology of each such cover can be computed easily from that of the subadjoint variety X, which in turn is known ([MT91]). A dimension count (which does not work for the non-orientable subadjoint Lagrangians $Z_{(1,2k+1)}$ and Z_6) shows that the spectral sequence cannot converge to zero. The non-displaceability claims then follow from Tonkonog's theorem [Ton18, Proposition 1.1] and the fact that the Fukaya category of projective space (over any characteristic) is split-generated by a full subcategory whose objects are different rank one local systems on the Clifford torus (see e.g. [EL19, Corollary 1.3.1]). Finally, the fact that Z_{15} split-generates the Fukaya category of $\mathbb{CP}^{31}$ over $\overline{\mathbb{F}}_2$ follows from [EL19, Corollary 7.2.1] and is related to the fact that the minimal Chern number of $\mathbb{CP}^{31}$ is 32 which is a power of $2 = \mathrm{char}(\overline{\mathbb{F}}_2)$.

1.2.3 Structure of the thesis

Chapter 2 is devoted to establishing the machinery that we use throughout the thesis, namely monotone Floer theory with high rank local systems. Virtually all concepts and results there (apart maybe from the monodromy Floer complex and its properties) are well-known to experts but we present them in some detail since they haven't appeared in the literature quite in the form that we need. Section 2.1 recalls the basic definitions of monotonicity and local systems and establishes notation. In section 2.2 we spell out the definition of Lagrangian Floer cohomology with local systems and the properties of the obstruction, while section 2.3 is devoted to some of the algebraic properties of the theory and, in particular, explains how to add Lagrangians with high rank local systems to the monotone Fukaya category, mimicking [Abo12]. Section 2.4 discusses some the same concepts from the point of view of Biran and Cornea's pearl complex. The reader familiar with monotone Floer theory is invited to skip chapter 2 altogether and refer to it only for some of the notation which is used throughout the thesis. The monodromy Floer complex and some of its properties are discussed in sections 2.2.3 and 2.4.3 in the Hamiltonian and pearly models, respectively.

In chapter 3 we prove our results on the topology of monotone Lagrangians in $\mathbb{CP}^n$. The chapter begins with a short discussion of Floer theory in characteristic other than two, followed by section 3.1 which is based on joint work with Jack Smith and contains the proof of Theorem A in the orientable case. Section 3.2 is devoted to Lagrangians in $\mathbb{CP}^3$. The necessary background on 3-manifolds is discussed in section 3.2.1, while the proof of Theorem B is confined to section 3.2.2.

In chapter 4 we study the fibration $\mathbb{CP}^{2n+1} \to \mathbb{HP}^n$ with the main results on twistor Lagrangians contained in section 4.1. After a short discussion of the general theory of quaternion-Kähler manifolds and their twistor spaces, we prove the Legendrian–Lagrangian correspondence for $\mathbb{CP}^{2n+1}$ in section 4.1.2. This section is written mostly for the author's benefit, since all results there are either explicit checks of well-known facts, or are contained in the paper [ET05]. It is followed by section 4.1.4 which contains the proof of Theorem C and section 4.1.5 in which we describe some topological properties of general Type 2 twistor Lagrangians. In section 4.1.6 we focus on dimension 3, describe the splitting of L_Δ into two $\mathbb{RP}^3$'s and give the proof of Theorem D.

The last two sections of chapter 4 are completely independent from the rest of the thesis. Section 4.2 has a somewhat exploratory nature and describes the local correspondence between Lagrangians in $\mathbb{CP}^3$ and vector fields on their projections to $S^4 = \mathbb{HP}^1$, giving the Clifford torus as an example. In section 4.3 we prove the non-existence of non-trivial vertical Hamiltonian flows on $\mathbb{CP}^{2n+1}$.

Chapter 5 contains our Floer cohomology calculations for the orientable subadjoint Lagrangians. In section 5.1 we prove Theorem E with the help of many pictures. The final section 5.2 contains the proof of Theorem F.

The thesis ends with two appendices. Appendix A contains calculations in stereographic coordinates for S^4, needed in section 4.3. In appendix B we give a classification of the representations of $\pi_1(L_\Delta)$ over $\mathbb{F}_2$ which are used for calculations with local systems in section 5.1.5.

Chapter 2

High rank local systems in monotone Floer theory

2.1 Preliminaries

2.1.1 The Maslov class and monotonicity

The main subject of this thesis are monotone Lagrangian submanifolds of monotone symplectic manifolds. Thus we begin with a quick overview of the Maslov class and the monotonicity condition and make some general topological observations. In this section all homology and cohomology groups are considered with $\mathbb{Z}$ coefficients, unless explicitly specified otherwise.

Let (M,J) be an almost complex manifold of real dimension $2n$ and let $L \subset M$ be a properly embedded totally real submanifold of dimension n. Then we have an isomorphism $TL \otimes \mathbb{C} \cong TM|_L$ and so the bundle $\Lambda^n_{\mathbb{R}} TL$ is naturally a rank 1 real subbundle of $\Lambda^n_{\mathbb{C}} TM\big|_L$. The bundle pair $(\Lambda^n_{\mathbb{C}} TM, \Lambda^n_{\mathbb{R}} TL)$ over (M,L) is then classified by a map

$$\phi : (M,L) \to (B\mathrm{U}(1), B(\mathbb{Z}/2)),$$

where we view the pair $(B\mathrm{U}(1), B(\mathbb{Z}/2))$ as

$$B(\mathbb{Z}/2) \cong \mathbb{RP}^\infty = \mathrm{Gr}_{\mathbb{R}}(1,\mathbb{R}^\infty) \xhookrightarrow{\otimes\mathbb{C}} \mathrm{Gr}_{\mathbb{C}}(1,\mathbb{C}^\infty) = \mathbb{CP}^\infty \cong B\mathrm{U}(1).$$

Recall that the cohomology $H^*(\mathbb{CP}^\infty)$ is a polynomial ring, generated by the unique element $c_1 \in H^2(\mathbb{CP}^\infty)$ which pairs to 1 with the image of the fundamental class of $\mathbb{CP}^1$ under the inclusion $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^\infty$. Its pull-back under ϕ is the first Chern class $c_1(TM)$. We are now interested in a related relative class in $H^2(\mathbb{CP}^\infty, \mathbb{RP}^\infty)$ which will be characteristic for the bundle pair $(\Lambda^n_{\mathbb{C}} TM, \Lambda^n_{\mathbb{R}} TL)$.

Note that the inclusion map $(\mathbb{CP}^2, \mathbb{RP}^2) \hookrightarrow (\mathbb{CP}^\infty, \mathbb{RP}^\infty)$ induces an isomorphism on second relative homology and it is not hard to check that $H_2(\mathbb{CP}^2, \mathbb{RP}^2) \cong \mathbb{Z}$, for example by observing that the disc $\{z \in \mathbb{C} : \ \mathrm{Im}(z) \geq 0\} \cup \{\infty\} \to \mathbb{CP}^2$, $z \mapsto [1:0:z]$ defines a non-torsion (positive symplectic area) class in $H_2(\mathbb{CP}^2, \mathbb{RP}^2)$ whose boundary generates $H_1(\mathbb{RP}^2)$. Thus $H_2(\mathbb{CP}^\infty, \mathbb{RP}^\infty) \cong \mathbb{Z}$ and so $H^3(\mathbb{CP}^\infty, \mathbb{RP}^\infty)$ is torsion-free. Using this and the long exact sequence in cohomology for the pair

$(\mathbb{CP}^\infty, \mathbb{RP}^\infty)$ we see that $H^2(\mathbb{CP}^\infty, \mathbb{RP}^\infty)$ is isomorphic to $\mathbb{Z}$ and that its generator maps to $2c_1$ in $H^2(\mathbb{CP}^\infty)$. This generator is called the *Maslov class*, and we denote it by μ. Its pullback $\mu_L := \phi^*\mu \in H^2(M,L)$ via the classifying map is the Maslov class of L. It is clear from this description that, if $j^*\colon H^2(M,L) \to H^2(M)$ is the natural restriction map, then

$$j^*(\mu_L) = 2c_1(TM). \tag{2.1}$$

We will write $I_{\mu_L}\colon H_2(M,L) \to \mathbb{Z}$ and $I_{c_1}\colon H_2(M) \to \mathbb{Z}$ for the group homomorphisms given by pairing with μ_L and $c_1(TM)$ respectively. We call I_{μ_L} the Maslov homomorphism and I_{c_1} the Chern homomorphism.

Recall also that the cohomology $H^*(\mathbb{RP}^\infty;\mathbb{Z}/2)$ is isomorphic to $\mathbb{Z}/2[w_1]$, where $w_1 \in H^1(\mathbb{RP}^\infty;\mathbb{Z}/2)$ is the unique non-trivial element. By definition $w_1(TL) := \phi^*w_1$ is the first Stiefel-Whitney class of L. Now observe that since μ restricts to $2c_1$ in $H^2(\mathbb{CP}^\infty;\mathbb{Z})$, its mod 2 reduction $\bar{\mu}$ restricts to zero in $H^2(\mathbb{CP}^\infty;\mathbb{Z}/2)$. Hence the coboundary map induces an isomorphism $\partial^*\colon H^1(\mathbb{RP}^\infty;\mathbb{Z}/2) \to H^2(\mathbb{CP}^\infty, \mathbb{RP}^\infty;\mathbb{Z}/2)$, i.e. $\partial^* w_1 = \bar{\mu}$. Pulling back by ϕ, we see that the mod 2 reduction of μ_L equals $\partial^* w_1(TL)$. Hence, for any class $A \in H_2(M,L)$, we have the congruence

$$\langle w_1(TL), \partial A\rangle = I_{\mu_L}(A) \mod 2, \tag{2.2}$$

which tells us that if L is orientable then I_{μ_L} has image in $2\mathbb{Z}$. Conversely, if $I_{\mu_L}(H_2(M,L)) \leq 2\mathbb{Z}$ and the boundary map $\partial\colon H_2(M,L) \to H_1(L)$ is surjective (e.g. if $H_1(M) = 0$), then L is orientable.

Now let $H_2^D(M,L)$ and $H_2^S(M)$ denote the images of the Hurewicz homomorphisms

$$\pi_2(M,L) \to H_2(M,L) \quad \text{and} \quad \pi_2(M) \to H_2(M)$$

and let $j_*\colon H_2(M) \to H_2(M,L)$ be the natural map. Define the integers N_L^π, N_L^H, N_M^π and N_M^H to be the non-negative generators of the $\mathbb{Z}$-subgroups $I_{\mu_L}(H_2^D(M,L))$, $I_{\mu_L}(H_2(M,L))$, $I_{c_1}(H_2^S(M))$, $I_{c_1}(H_2(M))$, respectively. Using (2.1) and the fact that $j_*(H_2^S(M)) \leq H_2^D(M,L)$, it is easy to see that there exist non-negative integers k_L, k_M, m_π, m_H such that:

$$N_L^\pi = k_L N_L^H, \quad N_M^\pi = k_M N_M^H, \quad 2N_M^\pi = m_\pi N_L^\pi, \quad 2N_M^H = m_H N_L^H.$$

In the literature on holomorphic curves, the numbers N_M^π and N_L^π are usually the ones referred to as the *minimal Chern number* of M and the *minimal Maslov number* of L, respectively. This can potentially cause confusion since these numbers are not the same as N_M^H and N_L^H in general. However, if M is simply-connected (for example, if it is a projective Fano variety—see [Bes08, Theorem 11.26]), then these numbers coincide. Indeed, we have the commutative diagram

$$\begin{tikzcd}
\pi_2(M) \arrow[r] \arrow[d] & \pi_2(M,L) \arrow[r] \arrow[d] & \pi_1(L) \arrow[r] \arrow[d] & 1 \arrow[d] \\
H_2(M) \arrow[r] & H_2(M,L) \arrow[r] & H_1(L) \arrow[r] & 0
\end{tikzcd}$$

in which the third vertical arrow is a surjection by Hurewicz. If M is simply connected, then the first vertical arrow is also a surjection, again by Hurewicz, so $H_2^S(M) = H_2(M)$. A diagram chase in the spirit of the 5-lemma (or alternatively, noticing that $\pi_1(M,L) = 0$ and applying the relative Hurewicz theorem) then shows that the second vertical arrow must also be a surjection, i.e. $H_2^D(M,L) = H^2(M,L)$. Thus $N_M^\pi = N_M^H$ and $N_L^\pi = N_L^H$. In this case there is therefore no ambiguity, and we denote the common values simply by N_M and N_L respectively.

Consider now the case when (M,ω) is symplectic and L is a Lagrangian submanifold. Then L is totally real with respect to any almost complex structure compatible with the symplectic form and we denote by $\mu_L \in H^2(M,L)$ the corresponding Maslov class. In this setting we also have the *area homomorphisms* $I_\omega \colon H_2(M) \to \mathbb{R}$, $I_{\omega,L} \colon H_2(M,L) \to \mathbb{R}$ given by integration of the symplectic form. The manifold (M,ω) is called *monotone* if there exists a positive constant λ such that

$$I_\omega|_{H_2^S(M)} = 2\lambda\, I_{c_1}|_{H_2^S(M)}\,.$$

For example, $(\mathbb{CP}^n, \omega_{FS})$ is monotone with $\lambda = \pi/2(n+1)$ when the Fubini-Study form is normalised so that a line has area π. In turn, the Lagrangian submanifold L is called *monotone* if

$$I_{\omega,L}|_{H_2^D(M,L)} = \lambda'\, I_{\mu_L}|_{H_2^D(M,L)}$$

for some positive constant λ'. Note that if $I_{c_1}|_{H_2^S(M)} \neq 0$ then (2.1) implies that a monotone Lagrangian can only exist if M itself is monotone and λ' coincides with λ.

Suppose now that I_ω and $2\lambda I_{c_1}$ agree on the whole of $H_2(M)$ (e.g. if M is monotone and simply-connected) and that the image $\partial H_2^D(M,L) \leq H_1(L)$ is torsion (e.g. if $H^1(L) = 0$). Then L is automatically monotone. Indeed, in that case for any element $A \in H_2^D(M,L)$, there exists a positive integer k such that $\partial(kA) = 0$ and so $kA = j_* v$ for some $v \in H_2(M)$. Then from (2.1) we have

$$kI_{\mu_L}(A) = 2I_{c_1}(j_*A) = \frac{2}{2\lambda} I_\omega(A) = \frac{k}{\lambda} I_{\omega,L}(A). \tag{2.3}$$

The concept of monotonicity extends to pairs of Lagrangian submanifolds (see [Poz99, Section 3.3.2]). Given two Lagrangians L^0, L^1 in M, the area and Maslov homomorphisms can be evaluated on (homotopy classes of) continuous maps $u\colon S^1 \times [0,1] \to M$ with $u(S^1 \times \{0\}) \subseteq L^0$ and $u(S^1 \times \{1\}) \subseteq L^1$. For the area homomorphism this evaluation is just integration of ω, while for the Maslov homomorphism it corresponds to pairing $\phi_u^* \mu$ with the relative fundamental class $[S^1 \times [0,1]] \in H^2(S^1 \times [0,1], S^1 \times \partial[0,1])$, where $\phi_u \colon (S^1 \times [0,1], S^1 \times \partial[0,1]) \to (\mathbb{CP}^\infty, \mathbb{RP}^\infty)$ is the classifying map for the bundle pair $(u^* \Lambda_{\mathbb{C}}^n TM,\, u|_{S^1 \times \{0\}}^* \Lambda_{\mathbb{R}}^n TL^0 \sqcup u|_{S^1 \times \{1\}}^* \Lambda_{\mathbb{R}}^n TL^1)$. We denote these extensions by I_{ω,L^0,L^1} and I_{μ,L^0,L^1} respectively. Then we call (L^0,L^1) a *monotone pair* of Lagrangians if $I_{\omega,L^0,L^1} = \lambda I_{\mu,L^0,L^1}$ for some positive constant λ. It is not hard to see that if (L^0,L^1) is a monotone pair, then each of the two Lagrangians is monotone (with constant λ) and the pair $(\psi(L^0),L^1)$ is monotone for any Hamiltonian diffeomorphism $\psi\colon M \to M$. Another useful fact is that if L^0 and L^1 are monotone Lagrangians and for at least one $j \in \{0,1\}$ the image of $\pi_1(L^j)$ in $\pi_1(M)$ under the map induced by inclusion is trivial, then (L^0, L^1) is a monotone pair (see [Poz99, Remark 3.3.2], [Oh93, Proposition 2.7]).

2.1.2 Local systems

We now set up some notation and recall the basics of local systems. Let R be a commutative ring (in this thesis R will be either $\mathbb{Z}$ or a field) and L be a smooth manifold. A local system of R-modules, or an *R-local system* on L is a functor $\mathcal{E}\colon \Pi_1 L \to R\text{-}\mathbf{mod}$, where $\Pi_1 L$ is the fundamental groupoid of the manifold L and $R\text{-}\mathbf{mod}$ is the category of (left) R-modules. If we want to emphasize which ground ring we are working on, we will write $\mathcal{E}^R$.

More concretely, an R-local system on L is an assignment of an R-module $\mathcal{E}_x$ for each point $x \in L$ and an isomorphism $P_\gamma\colon \mathcal{E}_{s(\gamma)} \to \mathcal{E}_{t(\gamma)}$ for each homotopy class γ of paths in L with source $s(\gamma)$ and target $t(\gamma)$, in a manner which is compatible with concatenation of paths. As is customary, we call these isomorphisms *parallel transport maps*. In case the R-module $\mathcal{E}_x$ is free for some (hence every) $x \in L$, its rank is called the rank of the local system $\mathcal{E}$.

In analogy with vector bundles, we will sometimes write $\mathcal{E} \to L$ to denote such a local system, the notation being a shorthand for the map

$$\begin{array}{rcl} \bigsqcup_x \mathcal{E}_x & \to & L \\ v \in \mathcal{E}_x & \mapsto & x. \end{array}$$

Similarly, by a section $\sigma\colon L \to \mathcal{E}$ we mean a section of this map. We will call such a section *parallel* if for every path γ on L one has $P_\gamma(\sigma(s(\gamma))) = \sigma(t(\gamma))$.

As with vector bundles, one can add, dualise and take tensor products of local systems on the same space in the obvious way. One notational point we want to make is that given two local systems $\mathcal{E}^0$, $\mathcal{E}^1$ on L, we will write $\mathscr{H}\!om(\mathcal{E}^0, \mathcal{E}^1)$ for the local system given by $\mathscr{H}\!om(\mathcal{E}^0, \mathcal{E}^1)_x := \mathrm{Hom}_R(\mathcal{E}^0_x, \mathcal{E}^1_x)$ and $\mathrm{Hom}(\mathcal{E}^0, \mathcal{E}^1)$ for the space of *morphisms of local systems* between $\mathcal{E}^0$ and $\mathcal{E}^1$, that is, the space of natural transformations between the two functors. Similarly for $\mathscr{E}\!nd(\mathcal{E})$ and $\mathrm{End}(\mathcal{E})$. Observe that an element of $\mathrm{Hom}(\mathcal{E}^0, \mathcal{E}^1)$ is the same thing as a parallel section of $\mathscr{H}\!om(\mathcal{E}^0, \mathcal{E}^1)$, so we will use these notions interchangeably.

A local system of R-modules on L is essentially the same data as a representation of the fundamental group of L. More precisely, let $x \in L$ be a point and write $\Pi_1(L, x)$ for the full subcategory of $\Pi_1 L$ with x as its only object. Then, since L is path-connected, the inclusion $\Pi_1(L, x) \hookrightarrow \Pi_1 L$ induces an equivalence of categories and so we get an equivalence

$$\mathrm{Fun}(\Pi_1 L, R\text{-}\mathbf{mod}) \simeq \mathrm{Fun}(\Pi_1(L,x), R\text{-}\mathbf{mod}) \cong R[\pi_1(L,x)^{\mathrm{Opp}}]\text{-}\mathbf{mod}. \tag{2.4}$$

Note that our conventions are such that concatenation of paths will be written from left to right, while compositions of maps, as usual, from right to left. Since this can cause headaches in explicit computations, let us spell-out concretely how the above equivalence plays out in practice. Given two points $x, y \in L$ we write $\Pi_1 L(x,y) := \mathrm{Hom}_{\Pi_1 L}(x,y)$ for the set of homotopy classes of paths connecting x to y. To go from left to right in (2.4), one can associate to each local system $\mathcal{E} \to L$, a right representation of the fundamental group $\pi_1(L,x)$, by considering the action of $\Pi_1 L(x,x) \cong \pi_1(L,x)^{\mathrm{Opp}}$ on the fibre $\mathcal{E}_x$.

To go the other way, suppose we are given a representation $\rho\colon \pi_1(L,x)^{\mathrm{Opp}} \to \mathrm{Aut}_R(V)$ for some R-module V. For each point $y \in L$ choose an element $\varepsilon_{xy} \in \Pi_1 L(x,y)$ with ε_{xx} equal to the constant path. We will call these *identification paths*. Now define a functor $\mathcal{E}\colon \Pi_1 L \to R\text{-}\mathbf{mod}$ by putting

$$\begin{aligned} \mathcal{E}(y) &= V \quad \forall y \in L \\ \mathcal{E}(\gamma) &= \rho(\varepsilon_{xy} \cdot \gamma \cdot \varepsilon_{xz}^{-1}) \quad \forall \gamma \in \Pi_1 L(y,z) \quad \forall y,z \in L. \end{aligned} \tag{2.5}$$

It is easy to check that this is indeed a functor and that any similar functor defined by a different choice of identification paths is canonically isomorphic to the above.

Local systems were introduced as coefficients for (cellular) (co)homology by Steenrod in [Ste43] and then Eilenberg extended the definition to singular (co)homology (see [Eil47, Chapter 5]). Given a local system $\mathcal{E} \to L$ we will write $H^*(L;\mathcal{E})$ to denote the singular cohomology of L with coefficients in $\mathcal{E}$. We will not give the general definitions here, since we don't actually need any of the details. However, in the cases we consider L will be a smooth manifold and we will often use a Morse model for computing $H^*(L;\mathcal{E})$, so let us now briefly sketch that construction.

Let $\mathscr{D} = (f,g)$ be a Morse-Smale pair of a smooth function and a Riemannian metric on L. We denote by $\mathrm{Crit}(f)$ the set of critical points of f and for each $x \in \mathrm{Crit}(f)$ we write $\mathrm{ind}(x)$ for the index and $W^a(x)$, $W^d(x)$ for the ascending and descending manifolds of x, respectively. If the ground ring R does not have characteristic 2, we also choose an orientation for $W^d(x)$ for each $x \in \mathrm{Crit}(f)$. The Morse cochain complex with coefficients in $\mathcal{E}$ is then defined to be

$$C^k_f(L;\mathcal{E}) := \bigoplus_{\substack{x \in \mathrm{Crit}(f) \\ \mathrm{ind}(x)=k}} \mathcal{E}_x. \tag{2.6}$$

Given $x, y \in \mathrm{Crit}(f)$ with $\mathrm{ind}(x) = \mathrm{ind}(y) + 1$, we write $\widetilde{\mathcal{L}}(x,y) := W^d(x) \cap W^a(y)$ for the set of downward gradient flowlines of f, connecting x to y and $\mathcal{L}(x,y)$ for the quotient of $\widetilde{\mathcal{L}}(x,y)$ by the natural $\mathbb{R}$-action. To every element $\gamma \in \mathcal{L}(x,y)$ we can associate two pieces of data:

1. an element in $\Pi_1 L(x,y)$, which we also denote by γ,

2. a sign $\varepsilon_\gamma \in \{-1,1\}$, which is irrelevant if $\mathrm{char}(R) = 2$.

The sign ε_γ is determined as follows. Let $\tilde{\gamma} \in \widetilde{\mathcal{L}}(x,y)$ be a representative of γ. Then the Morse-Smale condition gives the following exact sequence of vector spaces

$$0 \longrightarrow \mathrm{Span}_{\mathbb{R}}(\dot{\tilde{\gamma}}(0)) \longrightarrow T_{\tilde{\gamma}(0)} W^d(x) \longrightarrow T_{\tilde{\gamma}(0)} L / T_{\tilde{\gamma}(0)} W^a(y) \longrightarrow 0 \tag{2.7}$$

Using the differential of the downward gradient flow of f (and taking limits) we can identify $T_{\tilde{\gamma}(0)}L$ with T_yL and $T_{\tilde{\gamma}(0)}W^a(y)$ with $T_yW^a(y)$. Thus the third term of (2.7) is identified with $T_yL/T_yW^a(y) \cong T_yW^d(y)$ and taking top exterior powers, we see that we have an isomorphism

$$\Lambda^{\mathrm{ind}(x)}T_{\tilde{\gamma}(0)}W^d(x) \cong \mathrm{Span}_{\mathbb{R}}(\dot{\tilde{\gamma}}(0)) \otimes \Lambda^{\mathrm{ind}(y)}T_yW^d(y),$$

Then the sign ε_γ is $+1$ if this isomorphism preserves orientations and -1 if it does not (here $\mathrm{Span}_{\mathbb{R}}(\dot{\tilde{\gamma}}(0))$ is naturally oriented by $\dot{\tilde{\gamma}}(0)$).

Once we have this information, the differential $\partial^{\mathscr{D}}\colon C_f^*(L;\mathcal{E}) \to C_f^{*+1}(L;\mathcal{E})$ is defined as follows: for all $y \in \mathrm{Crit}(f)$ and all $v \in \mathcal{E}_y$

$$\partial^{\mathscr{D}} v := \sum_{\substack{x\in\mathrm{Crit}(f)\\ \mathrm{ind}(x)=\mathrm{ind}(y)+1}} \varepsilon_\gamma P_\gamma^{-1}(v) \in \mathcal{E}_x. \tag{2.8}$$

Standard results in Morse theory imply that $(\partial^{\mathscr{D}})^2 = 0$ and that the resulting cohomology $HM^*(L;\mathcal{E})$ is independent of the choice of Morse-Smale pair $\mathscr{D}$. It is shown in [Abo12, Appendix B] that $HM^*(L;\mathcal{E})$ is isomorphic to $H^*(L;\mathcal{E})$, the singular cohomology of L with local coefficients in $\mathcal{E}$.

While cohomology with local coefficients is in general hard to compute, there are some general results in cases when the local system arises in some natural geometric way. One such source of local systems on a space L comes from covers of L. If $p\colon L' \to L$ is a covering space, then to every point $x \in L$ one associates a free R-module $\mathcal{E}^R_{L',x}$ with basis labelled by the elements of $p^{-1}(x)$. Given a path $\gamma \in \Pi_1L(x,y)$, the parallel transport map P_γ sends a basis element corresponding to a lift $\tilde{x} \in p^{-1}(x)$ to the basis element corresponding to $t(\tilde{\gamma}_{\tilde{x}}) \in p^{-1}(y)$, where $\tilde{\gamma}_{\tilde{x}}$ is any lift of γ with $s(\tilde{\gamma}_{\tilde{x}}) = \tilde{x}$. We denote the resulting local system by $\mathcal{E}^R_{L'}$. In case L' is the universal cover of L we denote the corresponding local system by $\mathcal{E}^R_{\mathrm{reg}}$, since it corresponds to the regular representation of $\pi_1(L)$ on $R[\pi_1(L)]$. We will make frequent use of the following fact:

Proposition 2.1.1. *([Hat02, Proposition 3H.5]) Suppose L is a finite CW-complex. Then for all integers k, $H^k(L;\mathcal{E}^R_{L'})$ is isomorphic to $H^k_c(L';R)$, the singular cohomology of L' with compact support.* □

Notation 2.1.2. Sometimes we will use two different local systems $\mathcal{E}^j \to L$, $j \in \{0,1\}$ on the same space. We shall write $P_{j,\gamma}\colon \mathcal{E}^j_{s(\gamma)} \to \mathcal{E}^j_{t(\gamma)}$ to distinguish between the parallel transport maps. //

2.1.3 Pre-complexes

Throughout this chapter we will often encounter obstructed candidate chain complexes. We call these *pre-complexes*. That is, for us a pre-complex is just an R-module V together with a linear endomorphism $d\colon V \to V$ (we will ignore any notion of grading for the better part of this chapter). Given a pre-complex, one automatically has the maximal unobstructed subcomplex $\overline{V} := \ker d^2 \leq V$ which is now an honest complex. Similarly, by a chain map between pre-complexes (V,d^V), (W,d^W) we mean an R-linear map $F\colon V \to W$ such that $F \circ d^V = d^W \circ F$; such a map induces an honest chain map $\overline{F}\colon \overline{V} \to \overline{W}$. Finally, a homotopy between two chain maps $F,G\colon V \to W$ is a linear map $H\colon V \to W$ such that $H \circ d^V + d^W \circ H = F - G$. Observe that if $v \in \overline{V}$ then

$$\begin{aligned}(d^W)^2(H(v)) &= d^W(F(v) - G(v) - H(d^V(v))) \\ &= F(d^V(v)) - G(d^V(v)) - [F(d^V(v)) - G(d^V(v)) - H(d^V(d^V(v)))] = 0.\end{aligned}$$

Thus H induces a map $\overline{H}\colon \overline{V} \to \overline{W}$ which is a chain homotopy between $\overline{F}$ and $\overline{G}$.

2.2 Floer cohomology and local systems

From now on, we let (M,ω) be a symplectic manifold which is closed or convex at infinity. All Lagrangian submanifolds will be assumed compact, connected and without boundary. In this chapter we set the ground ring to be $R = \mathbb{F}$, where $\mathbb{F}$ is a field of characteristic 2. In particular, we will not deal with any issues involving orientations (or grading for that matter) for now.

In this section we discuss the construction of a Floer-theoretic invariant $\overline{HF}^*((L^0,\mathcal{E}^0),(L^1,\mathcal{E}^1))$, associated to a monotone pair of Lagrangians (L^0,L^1) in M, equipped with local systems $\mathcal{E}^0 \to L^0$ and $\mathcal{E}^1 \to L^1$ (of $\mathbb{F}$-vector spaces, according to our standing convention) of arbitrary rank. This follows the well-known construction of Floer cohomology with coefficients in a rank 1 local system, but for higher rank ones, we need to bypass some obstructions caused by Maslov 2 disc bubbles. That is, we construct a pre-complex $CF^*((L^0,\mathcal{E}^0),(L^1,\mathcal{E}^1))$ for which $\overline{HF}^*((L^0,\mathcal{E}^0),(L^1,\mathcal{E}^1))$ would be the homology of the maximal unobstructed subcomplex. The failure of the differential on $CF^*((L^0,\mathcal{E}^0),(L^1,\mathcal{E}^1))$ to square to zero is captured by the so-called *obstruction sections* $m_0(\mathcal{E}^j)\colon L^j \to \mathscr{End}(\mathcal{E}^j)$ for $j \in \{0,1\}$. In order to describe the obstruction section in detail, we concentrate on a single monotone Lagrangian $L \subseteq M$ of minimal Maslov number $N_L^\pi \geq 2$, equipped with a local system $\mathcal{E} \to L$.

2.2.1 The obstruction section

We begin by making our setup precise and establishing some notation. Let $\mathcal{J}(M,\omega)$ denote the space of ω-compatible almost complex structures on M, that is, the space of sections J of $\mathrm{End}(TM)$ which satisfy $J^2 = -\mathrm{Id}$ and such that $g_J(\cdot,\cdot) := \omega(\cdot, J\cdot)$ is a Riemannian metric on M. Let D^2 denote the standard closed unit disc in $\mathbb{C}$. Given $J \in \mathcal{J}(M,\omega)$, we will be concerned with J-holomorphic discs with boundary on L, i.e. smooth maps $u\colon (D^2,\partial D^2) \to (M,L)$, which satisfy the Cauchy-Riemann equation

$$du + J(u) \circ du \circ \mathbf{i} = 0. \tag{2.9}$$

Such a disc is called *simple* if there exists an open and dense subset $S \subseteq D^2$ such that for all $z \in S$ one has $u^{-1}(u(z)) = \{z\}$ and $d_z u \neq 0$. Now let us introduce the following pieces of notation.

- Let $\pi_2^{\mathrm{f}}(M,L)$ denote the set of free homotopy classes of discs with boundary on L. For any class $C \in \pi_2^{\mathrm{f}}(M,L)$ and any $k \in \mathbb{Z}$ we set

$$\begin{aligned}
\widetilde{\mathcal{M}}^C(L;J) &:= \left\{u \in C^\infty((D^2,\partial D^2),(M,L)) : \ du + J(u)\circ du \circ \mathbf{i} = 0, \ [u] = C\right\}, \\
\widetilde{\mathcal{M}}(k,L;J) &:= \bigcup_{\substack{C\in\pi_2^{\mathrm{f}}(M,L)\\ I_{\mu_L}(C)=k}} \widetilde{\mathcal{M}}^C(L;J), \\
\mathcal{M}^C(L;J) &:= \widetilde{\mathcal{M}}^C(L;J)/G, \\
\mathcal{M}(k,L;J) &:= \widetilde{\mathcal{M}}(k,L;J)/G,
\end{aligned}$$

where $G \cong \mathrm{PSL}(2,\mathbb{R})$ is the reparametrisation group of the disc acting by precomposition. We will write $q_G\colon \widetilde{\mathcal{M}}^C(L;J) \to \mathcal{M}^C(L;J)$ for the quotient map.

- We further set

$$\begin{aligned}
\mathcal{M}_{0,1}^C(L;J) &:= \widetilde{\mathcal{M}}^C(L;J) \times_G \partial D^2, \\
\mathcal{M}_{0,1}(k,L;J) &:= \widetilde{\mathcal{M}}(k,L;J) \times_G \partial D^2,
\end{aligned}$$

where an element $\phi \in G$ acts by $\phi\cdot(u,z) = (u\circ\phi^{-1},\phi(z))$. We shall denote the corresponding quotient map again by q_G.

- The above moduli spaces come with natural evaluation maps,

$$\widetilde{\mathrm{ev}}\colon \widetilde{\mathcal{M}}^C(L;J)\times\partial D^2 \to L, \quad \widetilde{\mathrm{ev}}(u,z) := u(z),$$

which clearly descend to maps $\mathrm{ev}\colon \mathcal{M}_{0,1}^C(L;J) \to L$.

- For any point $p \in L$ we then write $\mathcal{M}_{0,1}^C(p,L;J)$ and $\mathcal{M}_{0,1}(p,k,L;J)$ for the set $\mathrm{ev}^{-1}(\{p\})$, where the evaluation map is restricted to $\mathcal{M}_{0,1}^C(L;J)$ and $\mathcal{M}_{0,1}(k,L;J)$, respectively. We also set

$$\begin{aligned}
\widetilde{\mathcal{M}}_{0,1}^C(p,L;J) &:= q_G^{-1}(\mathcal{M}_{0,1}^C(p,L;J)) \subseteq \widetilde{\mathcal{M}}^C(L;J)\times\partial D^2 \\
\widetilde{\mathcal{M}}_{0,1}(p,k,L;J) &:= q_G^{-1}(\mathcal{M}_{0,1}(p,k,L;J)) \subseteq \widetilde{\mathcal{M}}(k,L;J)\times\partial D^2.
\end{aligned}$$

- We shall decorate any of the above sets with a superscript $*$ to denote the subset, consisting of simple discs. For example $\widetilde{\mathcal{M}}^{C,*}(L;J) := \left\{u \in \widetilde{\mathcal{M}}^C(L;J) \ : \ u \text{ is simple}\right\}$ and $\mathcal{M}_{0,1}^{C,*}(L;J) := \widetilde{\mathcal{M}}^{C,*}(L;J)\times_G \partial D^2$.

All these spaces are equipped with the C^∞–topology which they inherit from $C^\infty((D^2,\partial D^2),(M,L))$. When there is no danger of confusion we shall sometimes simply write $u \in \mathcal{M}_{0,1}(p,k,L;J)$ for the equivalence class $[u,z] = q_G(u,z)$ and $\partial u \in \pi_1(L,p)$ for the based homotopy class of the loop $\mathbb{R}/\mathbb{Z} \to L$, $s \mapsto u\left(ze^{2\pi is}\right)$.

By standard transversality arguments (see [MS12, Chapter 3]) it follows that there exists a Baire subset $\mathcal{J}_{\mathrm{reg}}(L) \subseteq \mathcal{J}(M,\omega)$ such that for all $J \in \mathcal{J}_{\mathrm{reg}}(L)$ and any class $C \in \pi_2^{\mathrm{f}}(M,L)$, the space

$\widetilde{\mathcal{M}}^{C,*}(L;J)$ has the structure of a smooth manifold of dimension $n+I_{\mu_L}(C)$ and the evaluation map $\widetilde{\mathrm{ev}}\colon \widetilde{\mathcal{M}}^{C,*}(L;J)\times\partial D^2\to L$ is smooth. Since the reparametrisation action on $\widetilde{\mathcal{M}}^{C,*}(L;J)$ is free and proper, one has that $\mathcal{M}^C(L;J)$ is a smooth manifold of dimension $n+I_{\mu_L}(C)-3$ and the quotient map q_G is everywhere a submersion (in particular the map $\mathrm{ev}\colon \mathcal{M}^{C,*}_{0,1}(L;J)\to L$ is also smooth). Further transversality arguments (i.e. the Lagrangian boundary analogue of [MS12, Proposition 3.4.2]) show that for any smooth map of manifolds $F\colon X\to L$, there exists a Baire subset $\mathcal{J}_{\mathrm{reg}}(L|F)\subseteq\mathcal{J}_{\mathrm{reg}}(L)$ such that for every $J\in\mathcal{J}_{\mathrm{reg}}(L|F)$ the maps $F\colon X\to L$ and $\mathrm{ev}\colon \mathcal{M}^*_{0,1}(k,L;J)\to L$ are everywhere transverse. When X is a submanifold of L and F is the inclusion map we shall write simply $\mathcal{J}_{\mathrm{reg}}(L|X)$. Results by Kwon-Oh and Lazzarini ([KO00, Laz00]) yield that, when L is monotone, one has $\mathcal{M}(N_L^\pi,L;J)=\mathcal{M}^*(N_L^\pi,L;J)$ and so $\mathcal{M}(N_L^\pi,L;J)$ is a smooth manifold. An application of Gromov compactness for holomorphic discs ([Fra08]) then ensures that the manifold $\mathcal{M}(N_L^\pi,L;J)$ is actually compact. In particular if $N_L^\pi\geq 2$ then $\mathcal{M}_{0,1}(2,L;J)$ is a compact manifold (possibly empty) of dimension $\dim(\widetilde{\mathcal{M}}(2,L;J)\times\partial D^2)-\dim(G)=n+2+1-3=n$. Therefore for any $p\in L$ and $J^p\in\mathcal{J}_{\mathrm{reg}}(L|p)$ the manifold $\mathcal{M}_{0,1}(p,2,L;J^p)$ consists of a finite number of points. We are now ready to define the obstruction section.

Definition 2.2.1. Let $\mathcal{E}$ be an $\mathbb{F}$-local system on a monotone Lagrangian submanifold $L\subseteq(M,\omega)$ with $N_L^\pi\geq 2$. The *obstruction section* for $\mathcal{E}$ is a section of the local system $\mathscr{E}\!nd(\mathcal{E})$, defined as follows. For every point $p\in L$ we choose an almost complex structure $J^p\in\mathcal{J}_{\mathrm{reg}}(L|p)$ and set

$$m_0(p,\mathcal{E};J^p):=\sum_{u\in\mathcal{M}_{0,1}(p,2,L;J^p)}P_{\partial u}\quad\in\quad\mathrm{End}(\mathcal{E}_p).\tag{2.10}$$

The obstruction section is then

$$\begin{array}{rcl} m_0(\mathcal{E})\colon L & \to & \mathscr{E}\!nd(\mathcal{E})\\ p & \mapsto & m_0(p,\mathcal{E};J^p).\end{array}\qquad\diamondsuit$$

Remark 2.2.2. Note that when $\mathcal{E}$ is trivial and of rank one, $m_0(p,\mathcal{E};J^p)$ is just the $\mathbb{F}$–degree of the map $\mathrm{ev}\colon\mathcal{M}_{0,1}(2,L;J^p)\to L$. //

As stated, the obstruction section appears to depend on the choices of almost complex structures J^p. This is not the case, as the following proposition shows.

Proposition 2.2.3. *The following invariance properties hold:*

i) For any $p\in L$ and $J,J'\in\mathcal{J}_{\mathrm{reg}}(L|p)$ one has $m_0(p,\mathcal{E};J)=m_0(p,\mathcal{E};J')$;

ii) $m_0(\mathcal{E})$ is a parallel section of $\mathscr{E}\!nd(\mathcal{E})$, that is, an element of $\mathrm{End}(\mathcal{E})$*;*

iii) if $\psi\colon M\to M$ is any symplectomorphism, then for every point $p\in L$, one has

$$m_0(\mathcal{E})(p)=m_0(\psi_*\mathcal{E})(\psi(p)).$$

These invariance properties are well-known to experts and an explanation which does not even mention local systems can be found for example in [Dam15]. However, since the obstruction section is one of the main ingredients to all results in this thesis, we choose to give a more detailed proof here.

Proof. In the remaining part of this section we prove Proposition 2.2.3. We will make repeated use of the following lemma.

Lemma 2.2.4. *Let $\mathcal{C} = C^0((D^2,\partial D^2),(M,L))$ equipped with the compact-open topology. Let $\gamma\colon [0,1] \to L$ be a continuous path and define $\mathcal{C}_\gamma := \{(t,u,z) \in [0,1]\times\mathcal{C}\times\partial D^2 \,:\, \gamma(t) = u(z)\}$. Further let $\nu\colon [0,1] \to \mathcal{C}_\gamma$ be a continuous path and write $\nu(s) = (t(s),u_s,z_s)$. Then the loops $\delta_{\nu(0)}\colon [0,1] \to L$, $\delta_{\nu(0)}(s) := u_0(z_0e^{2\pi is})$ and $\delta_{\nu(1)}\colon [0,1]\to L$,*

$$\delta_{\nu(1)}(s) := \begin{cases} \gamma(t(3s)), & s\in[0,1/3] \\ u_1\left(z_1e^{2\pi i(3s-1)}\right), & s\in[1/3,2/3] \\ \gamma(t(3-3s)), & s\in[2/3,1] \end{cases}$$

are homotopic based at $u_0(z_0)$.

Proof. An explicit homotopy is given by $H\colon [0,1]\times[0,1]\to L$,

$$H(s,r) = \begin{cases} \gamma(t(3s)), & s\in[0,r/3],\, r\in[0,1] \\ u_r\left(z_re^{2\pi i\frac{3s-r}{3-2r}}\right), & s\in[r/3,1-r/3],\, r\in[0,1] \\ \gamma(t(3-3s)), & s\in[1-r/3,1],\, r\in[0,1] \end{cases}$$

Continuity of H follows from that of ν and of the evaluation map $\mathcal{C}\times\partial D^2 \to L$. □

To establish part i) of Proposition 2.2.3, we need to consider a homotopy of almost complex structures, interpolating between J and J'. Let us write $C^\infty([0,1],\mathcal{J}(M,\omega))$ for the space of smooth sections $J\in C^\infty(M\times[0,1],\mathrm{End}(pr_M^*TM))$ such that for each $t\in[0,1]$ one has $J_t := J(\,\cdot\,,t\,)\in\mathcal{J}(M,\omega)$. Then standard transversality and compactness arguments imply the following.

Theorem 2.2.5. *Suppose L is monotone with $N_L^\pi \geq 2$ and let $p\in L$ and J, $J'\in\mathcal{J}_{\mathrm{reg}}(L|p)$. Then there exists a Baire subset $\mathcal{J}_{\mathrm{reg}}(J,J')\subseteq C^\infty([0,1],\mathcal{J}(M,\omega))$ such that for every $\hat{J}\in\mathcal{J}_{\mathrm{reg}}(J,J')$ one has $\hat{J}(0) = J$, $\hat{J}(1) = J'$ and if we set*

$$\begin{aligned} \widetilde{\mathcal{M}}_{0,1}(p,2,L;\hat{J}) &:= \{(\lambda,u,z)\in[0,1]\times C^\infty((D^2,\partial D^2),(M,L))\times\partial D^2 \,:\, du+\hat{J}(\lambda)\circ du\circ\mathbf{i} = 0, \\ &\qquad I_{\mu_L}([u]) = 2, u(z) = p\} \quad \text{and} \\ \mathcal{M}_{0,1}(p,2,L;\hat{J}) &:= \widetilde{\mathcal{M}}_{0,1}(p,2,L;\hat{J})/G, \end{aligned}$$

*then $\widetilde{\mathcal{M}}_{0,1}(p,2,L;\hat{J})$ is a smooth 4-dimensional manifold with boundary. Further, $\mathcal{M}_{0,1}(p,2,L;\hat{J})$ is a **compact** 1-dimensional manifold with boundary*

$$\partial\mathcal{M}_{0,1}(p,2,L;\hat{J}) = (\{0\}\times\mathcal{M}_{0,1}(p,2,L;J)) \sqcup \left(\{1\}\times\mathcal{M}_{0,1}(p,2,L;J')\right)$$

and the quotient map $q_G \colon \widetilde{\mathcal{M}}_{0,1}(p,2,L;\hat{J}) \to \mathcal{M}_{0,1}(p,2,L;\hat{J})$ *is everywhere a submersion.* □

From this theorem it follows that the elements of $\partial\mathcal{M}_{0,1}(p,2,L;\hat{J})$ are naturally paired up as opposite endpoints of closed intervals. Let $(\lambda,[u,z])$ and $(\lambda',[u',z'])$ be such a pair with $\lambda \le \lambda'$ (note that $\lambda,\ \lambda' \in \{0,1\}$) and let $\bar{v}\colon [0,1] \to \mathcal{M}_{0,1}(p,2,L;\hat{J})$ be any parametrisation of the interval which connects them. Choose a lift $v\colon [0,1] \to \widetilde{\mathcal{M}}_{0,1}(p,2,L;\hat{J})$ of $\bar{v}$. Since $\widetilde{\mathcal{M}}_{0,1}(p,2,L;\hat{J})$ embeds continuously into $\mathcal{C}_p$ (this is notation from Lemma 2.2.4, where we let γ be the constant path at p), we can apply Lemma 2.2.4 to obtain $P_{\partial u} = P_{\partial u'}$. We then have

$$m_0(p,\mathcal{E};J) - m_0(p,\mathcal{E};J') = \sum_{(\lambda,[u,z])\in\partial\mathcal{M}_{0,1}(p,2,L;\hat{J})} P_{\partial u} = 0,$$

because every term in the sum appears an even number of times. This proves part i) of Proposition 2.2.3 and so we are justified to use the notation $m_0(\mathcal{E})(p)$ without reference to a specific almost-complex structure. We adopt this notation and move on to proving part ii).

Let $p,q \in L$ and let $\gamma\colon [0,1] \to L$ be any smooth path with $\gamma(0) = p$, $\gamma(1) = q$. Then, by what we explained above about achieving transversality of the evaluation map with any other map, there exists a Baire subset $\mathcal{J}_{\mathrm{reg}}(L|\gamma) \subseteq \mathcal{J}_{\mathrm{reg}}(L|p) \cap \mathcal{J}_{\mathrm{reg}}(L|q)$ such that for every $J \in \mathcal{J}_{\mathrm{reg}}(L|\gamma)$, the space

$$\widetilde{\mathcal{M}}_{0,1}(\gamma,2,L;J) := \left\{(s,u,z) \in [0,1] \times \widetilde{\mathcal{M}}(2,L;J) \times \partial D^2 \ :\ u(z) = \gamma(s)\right\}$$

is a smooth 4-dimensional manifold with boundary. Further, the manifold $\mathcal{M}_{0,1}(\gamma,2,L;J) := \widetilde{\mathcal{M}}_{0,1}(\gamma,2,L;J)/G$ is a 1-dimensional compact manifold with boundary

$$\partial\mathcal{M}_{0,1}(\gamma,2,L;J) = (\{0\} \times \mathcal{M}_{0,1}(p,2,L;J)) \sqcup (\{1\} \times \mathcal{M}_{0,1}(q,2,L;J)).$$

Thus again the elements of $\mathcal{M}_{0,1}(p,2,L;J) \sqcup \mathcal{M}_{0,1}(q,2,L;J)$ are naturally paired up as endpoints of intervals. Let N be the number of such intervals and choose parametrisations $\bar{v}_1,\dots,\bar{v}_N\colon [0,1] \to \mathcal{M}_{0,1}(\gamma,2,L;J)$ and corresponding lifts $v_1,\dots,v_N\colon [0,1] \to \widetilde{\mathcal{M}}_{0,1}(\gamma,2,L;J)$ with $v_i(s) = (t^i(s),u^i_s,z^i_s)$ such that for all $1 \le i \le N$ one has $t^i(0) \le t^i(1)$ (recall $t^i(0),t^i(1) \in \{0,1\}$). Since $\widetilde{\mathcal{M}}_{0,1}(\gamma,2,L;J)$ embeds continuously in $\mathcal{C}_\gamma$ then by applying Lemma 2.2.4 to v_i, we obtain that

$$P_{\partial u^i_0} = P_{\delta_{v_i(1)}} \quad \text{for all } 1 \le i \le N. \tag{2.11}$$

Let $N_1,N_2 \in \{1,\dots,N+1\}$ be such that:

1. for all $1 \le i \le N_1 - 1$ we have $t^i(0) = 0$, $t^i(1) = 0$; in this case the loop $\delta_{v_i(1)}$ is based at p and lies in the homotopy class $\partial u^i_1 \in \pi_1(L,p)$; applying (2.11) we have $P_{\partial u^i_0} = P_{\partial u^i_1} \in \mathrm{End}(\mathcal{E}_p)$ for all $1 \le i \le N_1 - 1$;

2. for all $N_1 \le i \le N_2 - 1$ we have $t^i(0) = 0$, $t^i(1) = 1$; in this case $\delta_{v_i(1)}$ is again based at p but now lies in the class $\gamma \cdot \partial u^i_1 \cdot \gamma^{-1} \in \pi_1(L,p)$; applying (2.11) we have $P_{\partial u^i_0} = P_\gamma^{-1} \circ P_{\partial u^i_1} \circ P_\gamma \in \mathrm{End}(\mathcal{E}_p)$ for all $N_1 \le i \le N_2 - 1$;

3. for all $N_2 \leq i \leq N$ we have $t^i(0) = 1$, $t^i(1) = 1$; then $\delta_{v_i(1)}$ is based at q and lies in the class $\partial u_1^i \in \pi_1(L,q)$; by (2.11) this gives $P_{\partial u_0^i} = P_{\partial u_1^i} \in \mathrm{End}(\mathcal{E}_q)$ for all $N_2 \leq i \leq N$.

We thus have:

$$\begin{aligned}
m_0(\mathcal{E})(p) - P_\gamma^{-1} \circ m_0(\mathcal{E})(q) \circ P_\gamma &= \sum_{i=1}^{N_1-1} \left(P_{\partial u_0^i} + P_{\partial u_1^i}\right) + \sum_{i=N_1}^{N_2-1} P_{\partial u_0^i} \\
&\quad + P_\gamma^{-1} \circ \left(\sum_{i=N_1}^{N_2-1} P_{\partial u_1^i} + \sum_{i=N_2}^{N} \left(P_{\partial u_0^i} + P_{\partial u_1^i}\right) \right) \circ P_\gamma \\
&= \sum_{i=1}^{N_1-1} \left(P_{\partial u_0^i} + P_{\partial u_1^i}\right) + \sum_{i=N_1}^{N_2-1} \left(P_{\partial u_0^i} + P_\gamma^{-1} \circ P_{\partial u_1^i} \circ P_\gamma\right) \\
&\quad + P_\gamma^{-1} \circ \left(\sum_{i=N_2}^{N} \left(P_{\partial u_0^i} + P_{\partial u_1^i}\right) \right) \circ P_\gamma \\
&= 0.
\end{aligned}$$

This concludes the proof of part ii) of Proposition 2.2.3.

Finally, part iii) is an easy consequence of part i). Indeed, we know that we are free to choose $J \in \mathcal{J}_{\mathrm{reg}}(L|p)$ to compute $m_0(\mathcal{E})(p)$ and $J' \in \mathcal{J}_{\mathrm{reg}}(\psi(L)|\psi(p))$ to compute $m_0(\psi_*\mathcal{E})(\psi(p))$. So let J be any element of $\mathcal{J}_{\mathrm{reg}}(L|p)$ and set $J' = \psi_* J$. Then, almost tautologically, we have that $J' \in \mathcal{J}_{\mathrm{reg}}(\psi(L)|\psi(p))$ (compatibility with ω is ensured by the fact that ψ is a symplectomorphism). It is then clear that $m_0(\psi(p), \psi_*\mathcal{E}; \psi_* J) = m_0(p, \mathcal{E}; J)$ and this completes the proof of Proposition 2.2.3. $\square$

2.2.2 Definition, obstruction and invariance

In this section we define the pre-complex $CF^*((L^0,\mathcal{E}^0),(L^1,\mathcal{E}^1))$ and we see how the obstruction sections control the failure of the differential to square to zero. To make the exposition more accessible, we first recall without proof some basics of Floer theory.

Let L^0, L^1 be two compact Lagrangian submanifolds of (M,ω). To keep the explicit connection to some of the older literature that we rely on, we assume for now that L^0 and L^1 intersect transversely (we will later drop this assumption in favour of the more modern approach using "Floer data"). Letting $\mathcal{E}^0 \to L^0$ and $\mathcal{E}^1 \to L^1$ be $\mathbb{F}$-local systems, we then make the following definition.

Definition 2.2.6. The Floer cochain groups of L^0 and L^1 with coefficients in the local systems $\mathcal{E}^0$ and $\mathcal{E}^1$ are defined to be

$$CF^*((L^0,\mathcal{E}^0),(L^1,\mathcal{E}^1)) := \bigoplus_{p \in L^0 \cap L^1} \mathrm{Hom}_{\mathbb{F}}(\mathcal{E}^0_p, \mathcal{E}^1_p).$$

Where no confusion can arise we will drop L^0 and L^1 from the notation and just write $CF^*(\mathcal{E}^0,\mathcal{E}^1)$. Given an element $a \in CF^*(\mathcal{E}^0,\mathcal{E}^1)$, we write $a = (\langle a,p \rangle)_{p \in L^0 \cap L^1}$ where $\langle a,p \rangle \in \mathrm{Hom}_{\mathbb{F}}(\mathcal{E}^0_p,\mathcal{E}^1_p)$ is the corresponding component of a. $\diamondsuit$

To define the Floer differential on these groups, we need an additional piece of data, namely a family of almost-complex structures. Given $J \in C^\infty([0,1], \mathcal{J}(M,\omega))$, one defines a *J-holomorphic strip* with boundary on L^0 and L^1 to be a smooth map $u\colon \mathbb{R}\times[0,1]\to M$ which satisfies the Cauchy-Riemann equation (rewritten here with respect to the global conformal coordinates (s,t) on $\mathbb{R}\times[0,1]$):

$$\bar{\partial}_J(u) := \partial_s u + J_t(u)\partial_t u = 0 \tag{2.12}$$

and is subject to the boundary constraints $u(s,j)\in L^j$ for $j\in\{0,1\}$ for all $s\in\mathbb{R}$. The *energy* of such a map is defined to be

$$E(u) := \int_0^1\int_{\mathbb{R}} ||\partial_s u||^2_{g_J}\, ds\, dt.,$$

where $g_J(\cdot,\cdot) = \omega(J\cdot,\cdot)$. Note in particular that $E(u)=0$ if and only if u is a constant map. Floer showed in [Flo88c] that the condition $E(u)<\infty$ is equivalent to the existence of intersection points $p,q\in L^0\cap L^1$ such that $\lim_{s\to-\infty}u(s,t)=p$ and $\lim_{s\to+\infty}u(s,t)=q$ for all $t\in[0,1]$. Thus we have a partition of the set

$$\widetilde{\mathcal{M}}(L_0,L_1;J) := \{u\in C^\infty(\mathbb{R}\times[0,1],M) :\ \bar{\partial}_J(u)=0, u(s,j)\in L^j\ \forall\, s\in\mathbb{R},\ j\in\{0,1\},\ E(u)<\infty\}$$

into the sets

$$\widetilde{\mathcal{M}}(p,q;J) := \{u\in C^\infty(\mathbb{R}\times[0,1],M) :\ \bar{\partial}_J(u)=0,\ u(s,j)\in L^j\ \forall\, s\in\mathbb{R},\ j\in\{0,1\},$$
$$\lim_{s\to-\infty}u(s,t)=p,\ \lim_{s\to+\infty}u(s,t)=q\}.$$

Let us write $\pi_2(M,L^0,L^1,p,q)$ for the set of homotopy classes of maps $\hat{u}\colon[0,1]\times[0,1]\to M$ which satisfy $\hat{u}(s,j)\in L^j$ for $j\in\{0,1\}$, $s\in[0,1]$, $\hat{u}(0,t)=p$, $\hat{u}(1,t)=q$ for all $t\in[0,1]$ and where the homotopies are required to preserve these conditions. We will write $I^{MV}_\mu : \pi_2(M,L^0,L^1,p,q)\to\mathbb{Z}$ for the so-called *Maslov-Viterbo index* (see [Vit87] or [Flo88b, equation (2.6)] for the definition). Now, any map $u\in\widetilde{\mathcal{M}}(p,q;J)$ has a unique continuous extension to the domain $[-\infty,+\infty]\times[0,1]$ which defines a class $[u]$ in $\pi_2(M,L^0,L^1,p,q)$. Thus we have a further partition of each set $\widetilde{\mathcal{M}}(p,q;J)$ into sets $\widetilde{\mathcal{M}}^A(p,q;J)=\{u\in\widetilde{\mathcal{M}}(p,q;J) :\ [u]=A\in\pi_2(M,L^0,L^1,p,q)\}$. A coarser partition is provided by the sets $\widetilde{\mathcal{M}}(p,q,k;J) := \cup_{I^{MV}_\mu(A)=k}\widetilde{\mathcal{M}}^A(p,q;J)$ as k ranges through $\mathbb{Z}$.

The Cauchy-Riemann equation (2.12) implies that for each $u\in\widetilde{\mathcal{M}}(p,q;J)$ one has $E(u)=\int u^*\omega$. It follows that energy depends only the class $[u]\in\pi_2(M,L^0,L^1,p,q)$ and is therefore constant on the sets $\widetilde{\mathcal{M}}^A(p,q;J)$ (although a priori not on $\widetilde{\mathcal{M}}(p,q,k;J)$). Note also that since (2.12) is translation invariant in the variable s, there is a natural $\mathbb{R}$-action on $\widetilde{\mathcal{M}}(L^0,L^1;J)$ preserving the sets $\widetilde{\mathcal{M}}^A(p,q;J)$. Dividing by this action, we set $\mathcal{M}(L^0,L^1;J) := \widetilde{\mathcal{M}}(L^0,L^1;J)/\mathbb{R}$, $\mathcal{M}(p,q;J) := \widetilde{\mathcal{M}}(p,q;J)/\mathbb{R}$, $\mathcal{M}^A(p,q;J) := \widetilde{\mathcal{M}}^A(p,q;J)/\mathbb{R}$ and $\mathcal{M}(p,q,k;J) := \widetilde{\mathcal{M}}(p,q,k;J)/\mathbb{R}$. One then has the following theorem of Floer:

Theorem 2.2.7. *([Flo88b] and [Oh93, Appendix],[Oh97, Theorem III])*
Let L^0, L^1 be two compact Lagrangian submanifolds, intersecting transversely at the points $p,q\in$

$L^0 \cap L^1$. Then there exists a Baire subset $\mathcal{J}^1_{\mathrm{reg}}(p,q) \subseteq C^\infty([0,1],\mathcal{J}(M,\omega))$ such that for every $J \in \mathcal{J}^1_{\mathrm{reg}}(p,q)$ the set $\widetilde{\mathcal{M}}(p,q;J)$ has locally the structure of a smooth manifold whose dimension near $u \in \widetilde{\mathcal{M}}(p,q;J)$ equals $I^{MV}_\mu(u)$. □

In particular, note that each connected component of $\mathcal{M}(p,q,1;J)$ is just a point. We would like to "count" these points and so we need to know that $\mathcal{M}(p,q,1;J)$ is a finite set or, in other words, that it is compact. For this to work, one first needs a version of Gromov compactness for J-holomorphic strips which in turn requires a priori bounds on the energy. Then one has to analyse the possible "bubbling" scenarios and rule them out, which in this case means good control on pseudoholomorphic spheres in M and pseudoholomorphic discs with boundary on L_0 or L_1. It is in these aspects that the monotonicity assumption becomes important. In particular, one has the following theorem:

Theorem 2.2.8. *([Oh93]) If L_0 and L_1 are two monotone Lagrangians intersecting transversely at the points $p,q \in L_0 \cap L_1$, then there exists a Baire subset $\mathcal{J}^2_{\mathrm{reg}}(p,q) \subseteq \mathcal{J}^1_{\mathrm{reg}}(p,q)$ such that for each $J \in \mathcal{J}^2_{\mathrm{reg}}(p,q)$ the set $\mathcal{M}^A(p,q;J)$ is a finite set for every class $A \in \pi_2(M,L_0,L_1,p,q)$ with $I^{MV}_\mu(A) = 1$. Further, if the pair (L_0,L_1) is monotone, then $\mathcal{M}(p,q,1;J)$ is also a finite union of points, i.e. there are only finitely many classes $A \in \pi_2(M,L_0,L_1,p,q)$ with $I^{MV}_\mu(A) = 1$ and $\mathcal{M}^A(p,q;J) \neq \emptyset$.* □

We are now ready to define a candidate differential on our cochain groups $CF^*((L^0,\mathcal{E}^0),(L^1,\mathcal{E}^1))$. For every $u \in \mathcal{M}(p,q;J)$ and $j \in \{0,1\}$ we write $\gamma^j_u \colon [-\infty,+\infty] \to L^j$ for the paths $\gamma^j_u(s) = u((-1)^j s, j)$ with $\gamma^0_u(-\infty) = p = \gamma^1_u(+\infty)$ and $\gamma^0_u(+\infty) = q = \gamma^1_u(-\infty)$.

Definition 2.2.9. We define a map $d^J \colon CF^*(\mathcal{E}^0,\mathcal{E}^1) \to CF^*(\mathcal{E}^0,\mathcal{E}^1)$ as follows: for all intersection points $q \in L^0 \cap L^1$ and all linear maps $\alpha \in \mathrm{Hom}_{\mathbb{F}}(\mathcal{E}^0_q,\mathcal{E}^1_q)$

$$d^J \alpha := \sum_{p \in L^0 \cap L^1} \sum_{u \in \mathcal{M}(p,q,1;J)} P_{\gamma^1_u} \circ \alpha \circ P_{\gamma^0_u}. \qquad \diamondsuit$$

Remark 2.2.10. In this definition we are assuming that the time-dependent ω-compatible almost complex structure J is chosen generically enough so that the above sum is in fact finite. In light of Theorem 2.2.8 this amounts to asking that $J \in \bigcap_{p,q \in L^0 \cap L^1} \mathcal{J}^2_{\mathrm{reg}}(p,q)$, which is again a Baire subset of $C^\infty([0,1],\mathcal{J}(M,\omega))$ since L^0 and L^1 are assumed to intersect transversely and thus in a finite number of points. //

We shall see below (equation (2.13)) that we don't necessarily have $(d^J)^2 = 0$ and that the failure of this to hold is measured by the obstruction sections $m_0(\mathcal{E}^j) \colon L^j \to \mathscr{E}nd(\mathcal{E}^j)$. Thus $(CF^*(\mathcal{E}^0,\mathcal{E}^1),d^J)$ is a priori just a pre-complex. To bypass the obstructions, we consider the maximal unobstructed subcomplex.

Definition 2.2.11. We define the *central Floer complex* of $(L^0,\mathcal{E}^0)$ and $(L^1,\mathcal{E}^1)$ to be

$$\overline{CF}^*((L^0,\mathcal{E}^0),(L^1,\mathcal{E}^1),d^J) := \{a \in CF^*(\mathcal{E}^0,\mathcal{E}^1) : (d^J)^2 a = 0\}.$$

We call its cohomology the *central Floer cohomology* of the monotone pair $((L^0,\mathcal{E}^0),(L^1,\mathcal{E}^1))$ and denote it by $\overline{HF}^*((L^0,\mathcal{E}^0),(L^1,\mathcal{E}^1))$. $\diamondsuit$

We shall write $\overline{HF}^*(\mathcal{E}^0,\mathcal{E}^1)$ as a shorthand when the Lagrangians are understood. Further, if $\overline{CF}^*(\mathcal{E}^0,\mathcal{E}^1) = CF^*(\mathcal{E}^0,\mathcal{E}^1)$, we will drop the bar from the notation and call $HF^*(\mathcal{E}^0,\mathcal{E}^1)$ the Floer cohomology of $(L^0,\mathcal{E}^0)$ and $(L^1,\mathcal{E}^1)$. This is consistent with the standard definition of Floer cohomology with trivial or rank 1 local systems. Still, this notation only makes sense as long as these cohomology groups are invariant under changes of J. When the local systems are assumed trivial or rank 1, this is a well-known consequence of Floer's continuation map argument. The same proofs apply to our case just as well. Essentially the only interesting phenomenon which enters the picture when one considers higher rank local systems is condition (2.13) for $(d^J)^2 = 0$, which involves the obstruction sections $m_0(\mathcal{E}^0)$ and $m_0(\mathcal{E}^1)$. To make these statements precise we package them in the following theorem, consisting mainly of well-known facts:

Theorem 2.2.12. *Let (M,ω) be a monotone symplectic manifold and let (L^0,L^1) be a monotone pair of closed Lagrangian submanifolds with $N^{\pi}_{L^j} \geq 2$ for $j \in \{0,1\}$, equipped with $\mathbb{F}$-local systems $\mathcal{E}^j \to L^j$. There exists a Baire subset $\mathcal{J}_{\mathrm{reg}}(L^0,L^1) \subseteq C^\infty([0,1],\mathcal{J}(M,\omega))$ of time-dependent ω-compatible almost complex structures such that:*

A) For all $J \in \mathcal{J}_{\mathrm{reg}}(L^0,L^1)$

i) (well-defined) the map d^J is well-defined;

ii) (obstruction) for every point $p \in L^0 \cap L^1$ and every linear map $\alpha \in \mathrm{Hom}_{\mathbb{F}}(\mathcal{E}^0_p,\mathcal{E}^1_p)$ one has

$$(d^J)^2\alpha = \alpha \circ m_0(\mathcal{E}^0)(p) - m_0(\mathcal{E}^1)(p) \circ \alpha \tag{2.13}$$

B) (invariance) Let $H\colon [0,1] \times M \to \mathbb{R}$ be a (time-dependent) Hamiltonian and $\psi_t\colon M \to M$ be its corresponding flow[1]*. Suppose that $\psi_1(L^0) \pitchfork L^1$ and let $J \in \mathcal{J}_{\mathrm{reg}}(L^0,L^1)$, $J' \in \mathcal{J}_{\mathrm{reg}}(\psi_1(L^0),L^1)$. Then there exists a chain map of pre-complexes*

$$\Psi\colon CF^*((\psi_1(L^0),(\psi_1)_*\mathcal{E}^0),(L^1,\mathcal{E}^1);d^{J'}) \to CF^*((L^0,\mathcal{E}^0),(L^1,\mathcal{E}^1);d^J),$$

inducing a homotopy equivalence

$$\overline{\Psi}\colon \overline{CF}^*((\psi_1(L^0),(\psi_1)_*\mathcal{E}^0),(L^1,\mathcal{E}^1);d^{J'}) \to \overline{CF}^*((L^0,\mathcal{E}^0),(L^1,\mathcal{E}^1);d^J).$$

In particular, the isomorphism type of $\overline{HF}^*((L^0,\mathcal{E}^0),(L^1,\mathcal{E}^1))$ does not depend on the choice of $J \in \mathcal{J}_{\mathrm{reg}}(L^0,L^1)$.

Remark 2.2.13. The minus sign in equation (2.13) appears for consistency with later chapters where we work in characteristic different from 2. //

[1] Defined by the ODE $\frac{d}{dt}\psi_t = X_t \circ \psi_t$, where $i_{X_t}\omega = -dH_t$ and the initial condition $\psi_0 = \mathrm{id}_M$.

Remark 2.2.14. By part B), it is clear that if $\overline{HF}^*((L^0,\mathcal{E}^0),(L^1,\mathcal{E}^1)) \neq 0$ for some local systems $\mathcal{E}^0$, $\mathcal{E}^1$ then, for every Hamiltonian diffeomorphism ψ, one has $\psi(L^0)\cap L^1 \neq \emptyset$, i.e. L^0 and L^1 cannot be displaced by a Hamiltonian isotopy. //

Before giving a sketch proof of Theorem 2.2.12, we explain a different point of view on the pre-complex $CF^*((L^0,\mathcal{E}^0),(L^1,\mathcal{E}^1);d^J)$ which is particularly useful for understanding the invariance properties of the cohomology $\overline{HF}^*$ and for the construction of the monotone Fukaya category in section 2.3 below. Let (L^0,L^1) be a monotone pair of Lagrangians, not necessarily intersecting transversely, in particular we allow $L^0 = L^1$. A *regular Floer datum* for (L^0,L^1), as defined in [Sei08a, 8)], is a pair (H,J), where

1. $H\colon [0,1]\times M \to \mathbb{R}$ is a *regular Hamiltonian* for (L^0,L^1), i.e. a smooth function whose Hamiltonian flow ψ_t satisfies $\psi_1(L^0) \pitchfork L^1$,
2. $J \in C^\infty([0,1],\mathcal{J}(M,\omega))$ is a time-dependent almost complex structure such that the push-forward $(\psi_*J)_t := (\psi_t)_*J_t$ defines an element of $\mathcal{J}_{\mathrm{reg}}(\psi_1(L^0),L^1)$.

We write

$$\mathcal{X}_H(L^0,L^1) := \{x\colon [0,1]\to M :\ x(0)\in L^0, x(1)\in L^1, x(t) = \psi_t(x(0))\}$$

for the set of time-one Hamiltonian chords connecting L^0 to L^1. Given $x,y \in \mathcal{X}_H(L^0,L^1)$, a parametrised *Floer trajectory* from x to y is a smooth map $v\colon \mathbb{R}\times[0,1]\to M$, satisfying the following conditions:

- $v(s,j)\in L^j$ for $s\in\mathbb{R}$ and $j\in\{0,1\}$,
- $\lim_{s\to-\infty} v(s,t) = x(t)$ and $\lim_{s\to+\infty} v(s,t) = y(t)$ uniformly in t,
- v is a solution of the Floer equation
$$\partial_s v + J_t(\partial_t v - X_t(v)) = 0, \tag{2.14}$$
where X_t is the Hamiltonian vector field of H.

We denote the space of such maps by $\widetilde{\mathcal{R}}_{1:1}(x:y;H,J)$. Quotienting out by $\mathbb{R}$-translations, we have the space $\mathcal{R}_{1:1}(x:y;H,J)$ of unparametrised Floer trajectories. Note that we have one-to-one correspondences

$$\begin{aligned} \psi_1(L^0)\cap L^1 &\longrightarrow \mathcal{X}_H(L^0,L^1) && (2.15)\\ q &\longmapsto x_q, \quad x_q(t) := \psi_t(\psi_1^{-1}(q)) \\ \widetilde{\mathcal{M}}(p,q;H,\psi_*J) &\longrightarrow \widetilde{\mathcal{R}}_{1:1}(x_p:x_q;H,J) && (2.16)\\ u &\longmapsto v_u, \quad v_u(s,t) := \psi_t(\psi_1^{-1}(u(s,t))). \end{aligned}$$

In particular, the connected components of $\mathcal{R}_{1:1}(x:y;H,J)$ are also naturally manifolds and we can write $\mathcal{R}^d_{1:1}(x:y;H,J)$ for the union of the d–dimensional components. Each Floer trajectory $v\in$

$\mathcal{R}_{1:1}(x:y;H,J)$ gives rise to paths $\gamma_v^0 \in \Pi_1 L^0(x(0),y(0))$, $\gamma_v^0(s) := v(s,0)$ and $\gamma_v^1 \in \Pi_1 L^1(y(1),x(1))$, $\gamma_v^1(s) := v(-s,1)$. Using these one can form a pre-complex

$$CF^*((L^0,\mathcal{E}^0),(L^1,\mathcal{E}^1);H,J) := \bigoplus_{x\in\mathcal{X}_H(L^0,L^1)} \mathrm{Hom}_{\mathbb{F}}(\mathcal{E}^0_{x(0)},\mathcal{E}^1_{x(1)}),$$

whose differential $d^{(H,J)}$ acts on an element $\alpha \in \mathrm{Hom}_{\mathbb{F}}(\mathcal{E}^0_{y(0)},\mathcal{E}^1_{y(1)})$ by

$$d^{(H,J)}\alpha := \sum_{x\in\mathcal{X}_H(L^0,L^1)} \sum_{v\in\mathcal{R}^0_{1:1}(x:y;H,J)} P_{\gamma_v^1}\circ\alpha\circ P_{\gamma_v^0}.$$

The correspondences (2.15) and (2.16) give an isomorphism of pre-complexes

$$CF^*((\psi_1(L^0),(\psi_1)_*\mathcal{E}^0),(L^1,\mathcal{E}^1);d^{\psi_*J}) \to CF^*((L^0,\mathcal{E}^0),(L^1,\mathcal{E}^1);H,J). \tag{2.17}$$

Setting $\overline{CF}^*((L^0,\mathcal{E}^0),(L^1,\mathcal{E}^1);H,J)$ to be the maximal unobstructed subcomplex, we get an isomorphism of cochain complexes

$$\overline{CF}^*((\psi_1(L^0),(\psi_1)_*\mathcal{E}^0),(L^1,\mathcal{E}^1);d^{\psi_*J}) \to \overline{CF}^*((L^0,\mathcal{E}^0),(L^1,\mathcal{E}^1);H,J).$$

From this point of view, part B) of Theorem 2.2.12 can be strengthened to say that for every pair of regular Floer data (H,J) and (H',J') there is a canonical (up to homotopy) map of pre-complexes

$$\Psi^{H,J}_{H',J'}\colon CF^*((L^0,\mathcal{E}^0),(L^1,\mathcal{E}^1);H',J') \to CF^*((L^0,\mathcal{E}^0),(L^1,\mathcal{E}^1);H,J),$$

which induces homotopy equivalence on maximal unobstructed subcomplexes. In particular, the isomorphism type of the cohomology $\overline{HF}^*((L^0,\mathcal{E}^0),(L^1,\mathcal{E}^1);H,J)$ does not depend on the choice of regular Floer data. Thus the following definition makes sense.

Definition 2.2.15. Let $L \subseteq M$ be a compact monotone Lagrangian such that (L,L) is a monotone pair. Let $\mathcal{E}^0 \to L$, $\mathcal{E}^1 \to L$ be $\mathbb{F}$-local systems on L. Then we define the central Floer cohomology of $(L,\mathcal{E}^0)$ and $(L,\mathcal{E}^1)$ to be $\overline{HF}^*((L,\mathcal{E}^0),(L,\mathcal{E}^1);H,J)$ for some choice of regular Floer datum (H,J) for (L,L). $\diamondsuit$

Remark 2.2.16. There is a small technical subtlety here. For the above definition to work as stated, we need not only for L to be monotone but also for (L,L) to be a monotone pair. This second condition can be relaxed as long as in the definition of the pre-complex $CF^*((L,\mathcal{E}^0),(L,\mathcal{E}^1);H,J)$ we restrict ourselves to only consider Hamiltonian chords $x\colon([0,1],\partial[0,1]) \to (M,L)$ which define the trivial element in $\pi_1(M,L)$ (see [Oh93, Proposition 2.10]). However, *if* the pair (L,L) is monotone, then the cohomologies of the larger and smaller complexes would agree, since we are free to choose H sufficiently C^1-small, so that all its time-one chords connecting L to itself are indeed contractible relative L. //

Notation 2.2.17. Given an element of $a \in CF^*((L^0,\mathcal{E}^0),(L^1,\mathcal{E}^1);H,J)$, we write $a = (\langle a,x\rangle)_{x\in\mathcal{X}_H(L^0,L^1)}$ with $\langle a,x\rangle \in \mathrm{Hom}_{\mathbb{F}}(\mathcal{E}^0_{x(0)},\mathcal{E}^1_{x(1)})$. If we are given local systems $\mathcal{V} \to L^0$, $\mathcal{W} \to L^1$ and morphisms of local systems $F \in \mathrm{Hom}(\mathcal{V},\mathcal{E}^0)$, $G \in \mathrm{Hom}(\mathcal{E}^1,\mathcal{W})$, we will write

$$a\circ F := (\langle a,x\rangle \circ F(x(0)))_{x\in\mathcal{X}_H(L^0,L^1)} \quad \in \quad CF^*((L^0,\mathcal{V}),(L^1,\mathcal{E}^1);H,J)$$

and similarly

$$G \circ a := (G(x(1)) \circ \langle a,x \rangle)_{x \in \mathcal{X}_H(L^0,L^1)} \quad \in \quad CF^*((L^0,\mathcal{E}^0),(L^1,\mathcal{W});H,J). \qquad /\!\!/$$

We now observe that part ii) of Theorem 2.2.12 allows us to give a more natural description of the central Floer complex which will hopefully explain our choice of name for it. Given $a \in CF^*((L^0,\mathcal{E}^0),(L^1,\mathcal{E}^1);H,J)$, we have that a lies in $\overline{CF}^*((L^0,\mathcal{E}^0),(L^1,\mathcal{E}^1);H,J)$ if and only if $\left\langle \left(d^{(H,J)}\right)^2 a,\, x \right\rangle = 0$ for all $x \in \mathcal{X}_H(L^0,L^1)$. That is, if and only if

$$\sum_{y \in \mathcal{X}_H(L^0,L^1)} \left\langle \left(d^{(H,J)}\right)^2 \langle a,y \rangle,\, x \right\rangle = 0$$

for all $x \in \mathcal{X}_H(L^0,L^1)$. On the other hand, given $x \in \mathcal{X}_H(L^0,L^1)$ and $\alpha \in \mathrm{Hom}(\mathcal{E}^0_{x(0)},\mathcal{E}^1_{x(1)})$, the isomorphism (2.17) translates equation (2.13) into

$$\left(d^{(H,J)}\right)^2 \alpha = \alpha \circ m_0(\mathcal{E}^0)(x(0)) - m_0(\mathcal{E}^1)(x(1)) \circ \alpha, \tag{2.18}$$

where we have used Proposition 2.2.3 iii). From this we have

$$\left\langle \left(d^{(H,J)}\right)^2 \langle a,y \rangle, x \right\rangle = \begin{cases} \langle a,x \rangle \circ m_0(\mathcal{E}^0)(x(0)) - m_0(\mathcal{E}^1)(x(1)) \circ \langle a,x \rangle, & x = y \\ 0, & x \neq y. \end{cases}$$

In the notation 2.2.17 this reads

$$\left(d^{(H,J)}\right)^2 a = a \circ m_0(\mathcal{E}^0) - m_0(\mathcal{E}^1) \circ a.$$

Thus $a \in \overline{CF}^*((L^0,\mathcal{E}^0),(L^1,\mathcal{E}^1);H,J)$ if and only if

$$a \circ m_0(\mathcal{E}^0) = m_0(\mathcal{E}^1) \circ a. \tag{2.19}$$

Note in particular that when $L^0 = L^1$ and $\mathcal{E}^0 = \mathcal{E}^1 = \mathcal{E}$, the pre-complex $CF^*(\mathcal{E},\mathcal{E};H,J)$ is unobstructed if and only if $m_0(\mathcal{E})$ is a scalar operator. This condition is always satisfied when $\mathcal{E}$ has rank 1 and this special case of Floer theory with local coefficients is widely used, especially in topics related to mirror symmetry. On the other hand $CF^*((L,\mathcal{E}^0),(L,\mathcal{E}^1);H,J)$ can be obstructed when the local systems have higher rank or when $\mathcal{E}^0 \neq \mathcal{E}^1$. It is precisely this point that we will exploit in chapter 3 to obtain restrictions on the topology of monotone Lagrangians.

For the remaining part of this section we will give sketch proofs of the different parts of Theorem 2.2.12. As mentioned above, for all statements apart from ii) one only needs to translate classical results to our setting with local coefficients. We shall give the needed references and indicate how to insert local coefficients in the respective arguments.

Proof of Theorem 2.2.12: We already observed in Remark 2.2.10 that, in order for Theorem 2.2.12 i) to hold, we need to choose $J \in \bigcap_{p,q \in L^0 \cap L^1} \mathcal{J}^2_{\mathrm{reg}}(p,q)$. As we shall see, part ii) of

Theorem 2.2.12 imposes stronger restrictions on J and these will determine the set $\mathcal{J}_{\mathrm{reg}}(L^0,L^1) \subseteq \bigcap_{p,q\in L^0\cap L^1} \mathcal{J}^2_{\mathrm{reg}}(p,q)$. Let us first introduce some more notation.

If L^0 and L^1 are two Lagrangians which intersect transversely, then:

- for every pair of intersection points $r,q \in L^0\cap L^1$ we set

$$B(r,q;J) := \bigcup_{p\in L^0\cap L^1} \mathcal{M}(r,p,1;J)\times\mathcal{M}(p,q,1;J);$$

- for every intersection point $q\in L^0\cap L^1$ we set

$$B(q;J) := \mathcal{M}_{0,1}(q,2,L^0;J_0)\cup\mathcal{M}_{0,1}(q,2,L^1;J_1)\cup B(q,q;J);$$

- for any pair of *distinct* intersection points $r,q\in L^0\cap L^1$ we set

$$\overline{\mathcal{M}(r,q,2;J)} := \mathcal{M}(r,q,2;J)\cup B(r,q;J)$$

- for any single intersection point $q\in L^0\cap L^1$ and we set

$$\overline{\mathcal{M}(q,q,2;J)} := \mathcal{M}(q,q,2;J)\cup B(q;J).$$

With these notions in place, Gromov compactness and gluing for moduli spaces of strips and discs yield the following:

Theorem 2.2.18. *([Oh93]) Let (L^0,L^1) be a monotone pair of Lagrangians, which intersect transversely in M and with $N^{\pi}_{Lj}\geq 2$ for $j\in\{0,1\}$. Then for every pair of intersection points $r,q\in L^0\cap L^1$ (not necessarily distinct) there exists a Baire subset $\mathcal{J}^3_{\mathrm{reg}}(r,q)\subseteq\mathcal{J}^2_{\mathrm{reg}}(r,q)$ such that for every $J\in\mathcal{J}^3_{\mathrm{reg}}(r,q)$ one has $J_0\in\mathcal{J}_{\mathrm{reg}}(L^0|\{r,q\})$, $J_1\in\mathcal{J}_{\mathrm{reg}}(L^1|\{r,q\})$ and the set $\overline{\mathcal{M}(r,q,2;J)}$ has the structure of a compact 1-dimensional manifold with boundary. Further $\partial\overline{\mathcal{M}(r,q,2;J)} = B(r,q;J)$ when $r\neq q$ and $\partial\overline{\mathcal{M}(q,q,2;J)} = B(q;J)$.* □

We now set $\mathcal{J}_{\mathrm{reg}}(L^0,L^1) := \bigcap_{r,q\in L^0\cap L^1}\mathcal{J}^3_{\mathrm{reg}}(r,q)$. The proof of part ii) is then confined to the following proposition:

Proposition 2.2.19. *Let $J\in\mathcal{J}_{\mathrm{reg}}(L^0,L^1)$. Then for all intersection points $q\in L^0\cap L^1$ and all maps $\alpha\in\mathrm{Hom}_{\mathbb{F}}(\mathcal{E}^0_q,\mathcal{E}^1_q)$ we have*

$$\left(d^J\right)^2\alpha = \alpha\circ m_0(q,\mathcal{E}^0;J_0) - m_0(q,\mathcal{E}^1;J_1)\circ\alpha. \tag{2.20}$$

Proof. We have:

$$\left(d^J\right)^2\alpha \;=\; \sum_{r\in L^0\cap L^1}\left[\sum_{p\in L^0\cap L^1}\sum_{\substack{u\in\mathcal{M}(r,p,1;J)\\ v\in\mathcal{M}(p,q,1;J)}} P_{\gamma^1_v\cdot\gamma^1_u}\circ\alpha\circ P_{\gamma^0_u\cdot\gamma^0_v}\right],$$

where the dot denotes concatenation of paths. Thus, for every intersection point r the corresponding element in $\mathrm{Hom}_{\mathbb{F}}(\mathcal{E}^0_r, \mathcal{E}^1_r)$ appearing in $\left(d^J\right)^2 \alpha$ can be rewritten as

$$\langle \left(d^J\right)^2 \alpha, r\rangle = \sum_{\bar{u}\in B(r,q;J)} P_{\gamma^1_{\bar{u}}} \circ \alpha \circ P_{\gamma^0_{\bar{u}}}, \tag{2.21}$$

where for $\bar{u} = (u,v) \in B(r,q;J)$ we define $\gamma^0_{\bar{u}} := \gamma^0_u \cdot \gamma^0_v$ and $\gamma^1_{\bar{u}} := \gamma^1_v \cdot \gamma^1_u$. One now observes that whenever $r \neq q$ we have that the elements in $B(r,q;J)$ are naturally paired-up as opposite ends of the closed intervals which are the connected components of the compactified 1-dimensional moduli space $\overline{\mathcal{M}(r,q,2;J)}$. Let $\{\bar{u},\bar{u}'\} \subseteq B(r,q;J)$ be such a pair. It follows (see e.g. [Dam09], Lemma 3.16) that $\gamma^0_{\bar{u}} = \gamma^0_{\bar{u}'} \in \Pi_1 L^0(r,q)$ and $\gamma^1_{\bar{u}} = \gamma^1_{\bar{u}'} \in \Pi_1 L^1(q,r)$. Thus we have the identity

$$P_{\gamma^1_{\bar{u}}} \circ \alpha \circ P_{\gamma^0_{\bar{u}}} = P_{\gamma^1_{\bar{u}'}} \circ \alpha \circ P_{\gamma^0_{\bar{u}'}}.$$

Since all isolated broken strips (u,v) from r to q come in such pairs, every summand in the right-hand side of (2.21) appears twice, yielding $\langle \left(d^J\right)^2 \alpha, r\rangle = 0$.

We now consider the case when $r = q$. In that case the boundary of the Gromov compactification $\overline{\mathcal{M}(q,q;J)}$ is $B(q;J)$. For elements $\bar{u} \in B(q;J) \setminus B(q,q;J)$ we set $\gamma^0_{\bar{u}} = \partial u$, $\gamma^1_{\bar{u}} \equiv q$, if $\bar{u} = u \in \mathcal{M}_{0,1}(q,2,L^0;J_0)$ and $\gamma^0_{\bar{u}} \equiv q$, $\gamma^1_{\bar{u}} = \partial u$, if $\bar{u} = u \in \mathcal{M}_{0,1}(q,2,L^1;J_1)$. Again the elements of $B(q;J)$ are paired-up as end points of closed intervals and when $\{\bar{u},\bar{u}'\}$ is such a pair, we have $\gamma^j_{\bar{u}} = \gamma^j_{\bar{u}'} \in \Pi_1 L^j(q,q)$, hence

$$P_{\gamma^1_{\bar{u}}} \circ \alpha \circ P_{\gamma^0_{\bar{u}}} = P_{\gamma^1_{\bar{u}'}} \circ \alpha \circ P_{\gamma^0_{\bar{u}'}}.$$

Thus $\sum_{\bar{u}\in B(q;J)} P_{\gamma^1_{\bar{u}}} \circ \alpha \circ P_{\gamma^0_{\bar{u}}} = 0$, again since every summand appears twice. Expanding the left-hand side yields

$$\sum_{\bar{u}\in B(q,q;J)} P_{\gamma^1_{\bar{u}}} \circ \alpha \circ P_{\gamma^0_{\bar{u}}} \quad + \sum_{u\in\mathcal{M}_{0,1}(q,2,L^0;J_0)} \alpha \circ P_{\partial u} \quad - \sum_{u\in\mathcal{M}_{0,1}(q,2,L^1;J_1)} P_{\partial u} \circ \alpha \quad = \quad 0.$$

This can be rewritten as

$$\langle \left(d^J\right)^2 \alpha, q\rangle + \alpha \circ m_0(q,\mathcal{E}^0;J_0) - m_0(q,\mathcal{E}^1;J_1) \circ \alpha = 0,$$

which proves the proposition. □

The proof of part B) is standard and is based on Floer's original idea of *continuation maps*. It is best seen from the point of view of the complex $CF^*((L^0,\mathcal{E}^0),(L^1,\mathcal{E}^1);H,J)$, generated by linear maps between fibres of the local systems over start and end points of Hamiltonian chords. Given two pieces of regular Floer data (H,J), (H',J') one considers a path of Floer data $\{(H^s,J^s)\}_{s\in\mathbb{R}}$ which agrees with (H,J) when $s << 0$ and with (H',J') when $s >> 0$. Then one studies strips which satisfy the following version of the Floer equation

$$\partial_s v + J^s_t(\partial_t v - X^s_t(v)) = 0$$

which is not translation-invariant. The condition $N^{\pi}_{L^j} \geq 2$ is used here to establish compactness for moduli spaces of such maps of index 0 and 1. The boundaries of these strips can be used for parallel transport. Using these, one constructs chain maps of pre-complexes

$$\Psi^{H,J}_{H',J'} \colon CF^*((L^0,\mathcal{E}^0),(L^1,\mathcal{E}^1);H',J') \longrightarrow CF^*((L^0,\mathcal{E}^0),(L^1,\mathcal{E}^1);H,J). \tag{2.22}$$

Then, considering homotopies of paths of Floer data, one constructs chain homotopies between the above chain maps and concludes that the map $\Psi^{H,J}_{H',J'}$ is independent (up to homotopy) of the choice of path of Floer data. Finally, one can show that given a triple of Floer data one has that the maps $\Psi^{H,J}_{H',J'} \circ \Psi^{H',J'}_{H'',J''}$ and $\Psi^{H,J}_{H'',J''}$ are also chain homotopic. It then follows that $\overline{\Psi}^{H,J}_{H',J'}$ is always a homotopy equivalence.

Since the proof does not depend in any way on the rank and/or triviality of the local systems we refer the reader to [Oh93, Theorem 5.1] (see also [AD14, Chapter 11] for a detailed description of the same argument for Hamiltonian Floer homology). □

2.2.3 The monodromy Floer complex

In this section we introduce monodromy Floer cohomology. It is an $\mathbb{F}$-vector space $HF^*_{\mathrm{mon}}(L;\mathbb{F})$, which is canonically associated to a single monotone Lagrangian L and whose non-vanishing implies that L cannot be displaced from itself by a Hamiltonian isotopy. On the other hand, we will see in section 2.3 below that if $HF^*_{\mathrm{mon}}(L;\mathbb{F}) = 0$, then $\overline{HF}^*((L,\mathcal{E}^0),(L^1,\mathcal{E}^1)) = 0$ for all Lagrangians L^1 and all local systems $\mathcal{E}^0 \to L$, $\mathcal{E}^1 \to L^1$.

In order to describe $HF^*_{\mathrm{mon}}(L;\mathbb{F})$, we begin again with an $\mathbb{F}$-local system $\mathcal{E}$ on a monotone Lagrangian L with $N^{\pi}_L \geq 2$. For each pair of points x and y on L we set

$$\mathrm{Hom}_{\mathrm{mon}}(\mathcal{E}_x,\mathcal{E}_y) := \mathrm{Span}_{\mathbb{F}}\{P_\gamma \colon \mathcal{E}_x \to \mathcal{E}_y \,:\, \gamma \in \Pi_1 L(x,y)\}.$$

and we write $\mathrm{End}_{\mathrm{mon}}(\mathcal{E}_x) := \mathrm{Hom}_{\mathrm{mon}}(\mathcal{E}_x,\mathcal{E}_x)$. Observe that the space $\mathrm{End}_{\mathrm{mon}}(\mathcal{E}_x)$ is precisely the image of $\mathbb{F}[\pi_1(L,x)^{\mathrm{Opp}}] \to \mathrm{End}(\mathcal{E}_x)$ under the monodromy representation. Now let (H,J) be a regular Floer datum for L. We make the following definition.

Definition 2.2.20. The monodromy Floer cochain complex of $\mathcal{E} \to L$ is

$$CF^*_{\mathrm{mon}}(\mathcal{E};H,J) := \bigoplus_{x \in \mathcal{X}_H(L,L)} \mathrm{Hom}_{\mathrm{mon}}(\mathcal{E}_{x(0)},\mathcal{E}_{x(1)}). \qquad \diamondsuit$$

By Proposition 2.2.3 we know that $m_0(\mathcal{E})$ is a parallel section of $\mathscr{End}(\mathcal{E})$ and so we have $P_\gamma \circ m_0(\mathcal{E})(x(0)) = m_0(\mathcal{E})(x(1)) \circ P_\gamma$ for every $\gamma \in \Pi_1 L(x(0),x(1))$. It follows that any element $a \in CF^*_{\mathrm{mon}}(\mathcal{E};H,J)$ satisfies condition (2.19) and so we have

$$CF^*_{\mathrm{mon}}(\mathcal{E};H,J) \subseteq \overline{CF}^*(\mathcal{E},\mathcal{E};H,J).$$

Since the Floer differential $d^{(H,J)}$ and continuation maps $\Psi^{H,J}_{H',J'}$ are defined using pre- and post-composition by parallel transport maps, it is clear that $CF^*_{\mathrm{mon}}(\mathcal{E};H,J)$ is in fact a *subcomplex* of

$\overline{CF}^*(\mathcal{E},\mathcal{E};H,J)$ and that the maps $\Psi^{H,J}_{H',J'}$ restrict to give chain-homotopy equivalences between monodromy cochain complexes for different Floer data. We then make the following definition.

Definition 2.2.21. The monodromy Floer cohomology of $\mathcal{E} \to L$ is defined to be

$$HF^*_{\mathrm{mon}}(\mathcal{E}) := H^*(CF^*_{\mathrm{mon}}(\mathcal{E};H,J),d^{(H,J)})$$

for some choice of regular Floer data (H,J). ♢

Remark 2.2.22. By the independence of choice of Floer data, it follows that if $HF^*_{\mathrm{mon}}(\mathcal{E}) \neq 0$ for some $\mathcal{E} \to L$, then L cannot be displaced from itself by a Hamiltonian isotopy. //

Recall that we use $\mathcal{E}^{\mathbb{F}}_{\mathrm{reg}}$ to denote the local system induced by the right regular representation of $\pi_1(L)$ on $\mathbb{F}[\pi_1(L)]$. We then make the following definition.

Definition 2.2.23. The monodromy Floer complex of L over the field $\mathbb{F}$ is

$$CF^*_{\mathrm{mon}}(L;H,J) := CF^*_{\mathrm{mon}}(\mathcal{E}^{\mathbb{F}}_{\mathrm{reg}};H,J).$$

We call its cohomology the monodromy Floer cohomology of L and denote it by $HF^*_{\mathrm{mon}}(L;\mathbb{F})$. ♢

The complex $CF^*_{\mathrm{mon}}(\mathcal{E};H,J)$ depends in a very limited way on the choice of local system $\mathcal{E} \to L$. To formulate this precisely we introduce the following notion.

Definition 2.2.24. Let $\mathcal{E}^0,\mathcal{E}^1 \to L$ be $\mathbb{F}$-local systems on L. Let $p \in L$ be a base point and for $j \in \{0,1\}$ let $\rho_j\colon \mathbb{F}[\pi_1(L,p)^{\mathrm{Opp}}] \to \mathrm{End}(\mathcal{E}^j_p)$ denote the monodromy representation associated to $\mathcal{E}^j$. We say that $\mathcal{E}^0$ *dominates* $\mathcal{E}^1$ if $\ker\rho_0 \subseteq \ker\rho_1$. ♢

We then have the following relation.

Proposition 2.2.25. *Let $\mathcal{E}^0,\mathcal{E}^1 \to L$ be local systems on L and suppose that $\mathcal{E}^0$ dominates $\mathcal{E}^1$. Then there is a surjective chain map*

$$\Phi\colon CF^*_{\mathrm{mon}}(\mathcal{E}^0;H,J) \to CF^*_{\mathrm{mon}}(\mathcal{E}^1;H,J).$$

If also $\mathcal{E}^1$ dominates $\mathcal{E}^0$, the map Φ is an isomorphism of complexes.

Terminology 2.2.26. The map Φ will be called the *domination map*. //

Remark 2.2.27. Note that Definition 2.2.24 requires a containment of kernels at the level of group ring homomorphisms rather than group homomorphisms. This is necessary for Proposition 2.2.25 to hold. See Remark 5.1.12. //

Proof. Let $p \in L$ be a base point and write $G = \pi_1(L,p)^{\mathrm{Opp}}$ and $\rho_j\colon \mathbb{F}[G] \to \mathrm{End}(\mathcal{E}^j_p)$ for the monodromy representations. The condition $\ker\rho_0 \subseteq \ker\rho_1$ allows us to define for any $x,y \in L$ a surjective linear map

$$\begin{aligned} \phi_{xy}\colon \mathrm{Hom}_{\mathrm{mon}}(\mathcal{E}^0_x,\mathcal{E}^0_y) &\longrightarrow \mathrm{Hom}_{\mathrm{mon}}(\mathcal{E}^1_x,\mathcal{E}^1_y) \\ P_{0,\gamma} &\longmapsto P_{1,\gamma} \end{aligned} \tag{2.23}$$

To see that this is indeed well-defined, choose paths $\varepsilon_{px} \in \Pi_1 L(p,x)$, $\varepsilon_{py} \in \Pi_1 L(p,y)$ and use them to identify $\mathrm{Hom}_{\mathrm{mon}}(\mathcal{E}_x^j, \mathcal{E}_y^j)$ with $\mathrm{End}_{\mathrm{mon}}(\mathcal{E}_p^j) = \rho_j(\mathbb{F}[G])$. Under this identification, the map (2.23) becomes the map $\rho_0(\mathbb{F}[G]) \to \rho_1(\mathbb{F}[G])$, $\rho_0(g) \mapsto \rho_1(g)\ \forall g \in G$. But this is just the composition

$$\rho_0(\mathbb{F}[G]) \xrightarrow{\cong} \mathbb{F}[G]/\ker\rho_0 \longrightarrow \frac{\mathbb{F}[G]/\ker\rho_0}{\ker\rho_1/\ker\rho_0} \xrightarrow{\cong} \mathbb{F}[G]/\ker\rho_1 \xrightarrow{\cong} \rho_1(\mathbb{F}[G]). \tag{2.24}$$

This also shows that the map ϕ_{xy} is surjective in general and an isomorphism when $\ker\rho_0 = \ker\rho_1$. Further, since all maps in the above composition preserve the ring structure, we also have that if $X \in \mathrm{Hom}_{\mathrm{mon}}(\mathcal{E}_x^0, \mathcal{E}_y^0)$ and $Y \in \mathrm{Hom}_{\mathrm{mon}}(\mathcal{E}_y^0, \mathcal{E}_z^0)$ then

$$\phi_{xz}(Y \circ X) = \phi_{yz}(Y) \circ \phi_{xy}(X). \tag{2.25}$$

Putting these maps together we can now define

$$\Phi := \bigoplus_{x \in \mathcal{X}_H(L,L)} \phi_{x(0)x(1)} : CF^*_{\mathrm{mon}}(\mathcal{E}^0; H, J) \longrightarrow CF^*_{\mathrm{mon}}(\mathcal{E}^1; H, J)\ . \tag{2.26}$$

Further, since $d^{(H,J)}$ involves only pre- and post-composition by parallel transport maps, we see from (2.25) that Φ commutes with the Floer differential. Explicitly, if $x \in \mathcal{X}_H(L,L)$ and $\gamma \in \Pi_1 L(x(0), x(1))$ then

$$\begin{aligned}
d^{(H,J)}(\phi_{x(0)x(1)}(P_{0,\gamma})) &= \sum_{y \in \mathcal{X}_H(L,L)} \sum_{v \in \mathcal{R}^0_{1:1}(y:x)} P_{1,\gamma_v^1} \circ \phi_{x(0)x(1)}(P_{0,\gamma}) \circ P_{1,\gamma_v^0} \\
&= \sum_{y \in \mathcal{X}_H(L,L)} \sum_{v \in \mathcal{R}^0_{1:1}(y:x)} \phi_{x(1)y(1)}(P_{0,\gamma_v^1}) \circ \phi_{x(0)x(1)}(P_{0,\gamma}) \circ \phi_{y(0)x(0)}(P_{0,\gamma_v^0}) \\
&= \phi_{y(0)y(1)} \left(\sum_{y \in \mathcal{X}_H(L,L)} \sum_{v \in \mathcal{R}^0_{1:1}(y:x)} P_{0,\gamma_v^1} \circ P_{0,\gamma} \circ P_{0,\gamma_v^0} \right) \\
&= \phi_{y(0)y(1)} \left(d^{(H,J)}(P_{0,\gamma}) \right).
\end{aligned}$$

$\square$

Now, since for the local system $\mathcal{E}^{\mathbb{F}}_{\mathrm{reg}}$ the corresponding ring map $\rho_{\mathrm{reg}} \colon \mathbb{F}[G] \to \mathrm{End}(\mathbb{F}[G])$ is injective, $\mathcal{E}^{\mathbb{F}}_{\mathrm{reg}}$ dominates every $\mathbb{F}$-local system $\mathcal{E} \to L$ and so we have the maps

$$CF^*_{\mathrm{mon}}(L; H, J) \xrightarrow{\Phi} CF^*_{\mathrm{mon}}(\mathcal{E}; H, J) \hookrightarrow \overline{CF}^*(\mathcal{E}, \mathcal{E}; H, J). \tag{2.27}$$

2.3 The monotone Fukaya category

The next standard Floer-theoretic construction to which we add local systems of higher rank is the monotone Fukaya category. A systematic treatment of high-rank local systems for the wrapped Fukaya category was developed by Abouzaid in [Abo12] where a modified version of the split-generation criterion (again due to Abouzaid [Abo10]) was used to infer information about the fundamental group of exact Lagrangians in cotangent bundles. In the following, we describe how to construct such an extended monotone Fukaya category and prove appropriate modifications of the theorems which we require for our application – the AKS criterion and Abouzaid's split-generation criterion. We do so following closely the exposition in [She16], [BC14], [Abo12] and [Sei08a], to which we refer the reader for more details.

2.3.1 Setup

Given a compact monotone symplectic manifold (M,ω) we associate to it an $\mathbb{F}$-linear A_∞ category $\mathcal{F}(M)$ whose objects are pairs $(L,\mathcal{E})$ where L is a compact monotone Lagrangian submanifold with $N_L \geq 2$ and $\mathcal{E} \to L$ is a local system of finite rank over $\mathbb{F}$. The morphism spaces between two objects are central Floer complexes and the A_∞ operations are defined using counts of punctured, (perturbed) pseudoholomorphic discs, with the operation μ^1 being the Floer differential on the central Floer complex. For simplicity (and since this is what we need for applications) let us only construct a full subcategory of $\mathcal{F}(M)$ with a finite set of Lagrangians $\mathcal{L} = \{L^i\}$. Since we will be counting pseudoholomorphic curves with many boundary components on different Lagrangians, one needs an analogue of monotonicity-for-pairs to hold for n-tuples of Lagrangians (this is needed to ensure that curves with the same Maslov index have the same energy so that one can apply Gromov compactness). Rather than try and formulate what monotonicity for n-tuples might mean, we will assume that for each $L \in \mathcal{L}$, the map $\iota_*\colon \pi_1(L) \to \pi_1(M)$ induced by inclusion has trivial image (cf. [BC14, Assumption (8)]). Under this assumption, the uniform energy bounds hold and the construction of the Fukaya category can be carried out. We will additionally require the L^i to be orientable although this condition is only needed for Theorem 2.3.8.

For every ordered pair (L^i, L^j) (i and j not necessarily distinct) of elements of $\mathcal{L}$ choose a regular Hamiltonian $H^{ij}\colon [0,1] \times M \to \mathbb{R}$ with corresponding flow ψ^{ij} (so $\psi_1^{ij}(L^i) \pitchfork L^j$) and then for every L^i choose $J_{L_i} \in \mathcal{J}_{\mathrm{reg}}(L^i | \cup_j ((L^i \cap (\psi_1^{ij})^{-1}(L^j)) \cup (\psi_1^{ji}(L^j) \cap L^i)))$ (recall that this notation means that evaluation maps from simple, J_{L^i}–holomorphic discs with one boundary marked point are transverse to L^i at all start and end points of Hamiltonian chords for the chosen H^{ij}). Complete H^{ij} to a regular Floer datum by choosing $J^{ij} \in C^\infty([0,1], \mathcal{J}(M,\omega))$ such that $J_0^{ij} = J_{L^i}$ and $J_1^{ij} = J_{L^j}$. We now define the morphism spaces in $\mathcal{F}(M)$ to be

$$hom_{\mathcal{F}(M)}((L^i,\mathcal{E}^i),(L^j,\mathcal{E}^j)) := \overline{CF}^*((L^i,\mathcal{E}^i),(L^j,\mathcal{E}^j);H^{ij},J^{ij})$$

and the first A_∞ operation μ^1 to consist of the differentials $d^{(H^{ij},J^{ij})}$ on all these complexes. Having fixed all Floer data, we now drop it from the notation. We shall write $\mathcal{X}(L^i,L^j)$ for the set of Hamiltonian chords from L^i to L^j for the fixed regular Hamiltonian H^{ij}.

The construction of the higher A_∞ operations is well-established, at least in the case of rank 1 local systems (see e.g. [She16, Section 2.3], based on the constructions for exact manifolds from [Sei08a]). In the wrapped setting, higher rank local systems have been described in detail by Abouzaid [Abo12]. Thus, throughout this discussion we omit a lot of technical details (mostly from [Sei08a, Section 9]), in particular the fact that Floer and perturbation data can be chosen in such a way that all moduli spaces which appear are smooth manifolds of the correct dimensions and admitting the correct compactifications. That this is possible (i.e. that modified proofs from [Sei08a] apply) is an artefact of monotonicity. The only new observation is that these operations preserve the central complexes, which we verify in Proposition 2.3.1 below.

Let us now give a brief description of the construction. For every $d \geq 2$ and any $(d+1)$-tuple of objects $\{(L^j, \mathcal{E}^j)\}_{0\leq j\leq d}$ there is a linear map

$$\mu^d : \overline{CF}^*(\mathcal{E}^{d-1}, \mathcal{E}^d) \otimes \cdots \otimes \overline{CF}^*(\mathcal{E}^0, \mathcal{E}^1) \longrightarrow \overline{CF}^*(\mathcal{E}^0, \mathcal{E}^d),$$

which is defined by counting isolated perturbed pseudoholomorphic polygons with boundary on the Lagrangians $L^0, L^1, \ldots, L^d$ and using their boundary components for parallel transport. More precisely, let $\{\zeta_0, \zeta_1, \ldots, \zeta_d\}$ be a counterclockwise cyclicly ordered set of points on ∂D^2 which are labelled either positive (also called incoming) or negative (outgoing). We call each ζ_j a positive, respectively negative puncture. A choice of strip-like ends for $(D^2, \zeta_0, \ldots, \zeta_d)$ is a collection of pairwise disjoint open neighbourhoods $\zeta_j \in U_j \subseteq D^2$, together with holomorphic diffeomorphisms $\varepsilon_j : \mathbb{R}^{\pm} \times [0,1] \to U_j \setminus \{\zeta_j\}$, satisfying $\varepsilon_j^{-1}(\partial D^2 \cap (U_j \setminus \{\zeta_j\})) = \mathbb{R}^{\pm} \times \{0,1\}$ and $\lim_{s\to\pm\infty} \varepsilon_j(s,t) = \zeta_j$, where $\mathbb{R}^+ = (0,+\infty)$, $\mathbb{R}^- = (-\infty, 0)$ and the choice between the two domains is determined by whether the corresponding puncture is labelled positive or negative.

Consider an ordered list of objects $\{(L^j, \mathcal{E}^j)\}_{0\leq j\leq d}$ and Hamiltonian chords $x_0 \in \mathcal{X}(L^0, L^d)$ and $\{x_j\}_{1\leq j\leq d}$, with $x_j \in \mathcal{X}(L^{j-1}, L^j)$. Let $(D^2, \zeta_0, \ldots, \zeta_d)$ be as above with ζ_0 labelled negative and all other punctures labelled positive and assume one has made a choice of strip-like ends. Then any continuous map $u : D^2 \setminus \{\zeta_0, \ldots, \zeta_d\} \to M$, mapping the boundary arc between ζ_j and ζ_{j+1} to L^j (with $\zeta_{d+1} := \zeta_0$) and satisfying $\lim_{s\to\pm\infty} u(\varepsilon_j(s,t)) = x_j(t)$ uniformly in t, gives rise to a linear map

$$\mu_u : \operatorname{Hom}_{\mathbb{F}}(\mathcal{E}^{d-1}_{x_d(0)}, \mathcal{E}^d_{x_d(1)}) \otimes \cdots \otimes \operatorname{Hom}_{\mathbb{F}}(\mathcal{E}^0_{x_1(0)}, \mathcal{E}^1_{x_1(1)}) \longrightarrow \operatorname{Hom}_{\mathbb{F}}(\mathcal{E}^0_{x_0(0)}, \mathcal{E}^d_{x_0(1)})$$

$$\mu_u(\alpha_d \otimes \alpha_{d-1} \otimes \cdots \otimes \alpha_1) = P_{\gamma_u^d} \circ \alpha_d \circ P_{\gamma_u^{d-1}} \circ \alpha_{d-1} \circ \cdots \circ P_{\gamma_u^1} \circ \alpha_1 \circ P_{\gamma_u^0}, \tag{2.28}$$

where $\gamma_u^0 \in \Pi_1 L^0(x_0(0), x_1(0))$, $\gamma_u^d \in \Pi_1 L^d(x_d(1), x_0(1))$ and $\gamma_u^j \in \Pi_1 L^j(x_j(1), x_{j+1}(0))$, $1 \leq j \leq d-1$ are the compactified images under u of the arcs between ζ_j and ζ_{j+1} (see Figure 2.1).

For $d \geq 2$, we now consider the moduli space of smooth maps $u : D^2 \setminus \{\zeta_0, \ldots, \zeta_d\} \to M$ as above which are required to satisfy a suitably perturbed Cauchy-Riemann equation and where the positions of the points $\{\zeta_0, \ldots, \zeta_d\}$ are allowed to vary up to biholomorphisms of D^2. We denote this moduli space by $\mathcal{R}_{1:d}(x_0 : x_1, \ldots, x_d)$ and its k-dimensional component by $\mathcal{R}^k_{1:d}(x_0 : x_1, \ldots, x_d)$ (for this to make sense one needs to first make consistent choices of strip-like ends for the universal families $\mathcal{R}_{1:d}$ of abstract holomorphic discs with d positive punctures and one negative and then make choices of perturbation data for these families which is consistent with gluing, ensures transversality and agrees with the chosen Floer data on the strip-like ends – see [Sei08a, (9g),(9h),(9i)]; this ensures that the connected components of $\mathcal{R}_{1:d}(x_0 : x_1, \ldots, x_d)$ are indeed manifolds and admit the desired compactifications; similar procedures need to be applied to all moduli spaces we discuss in

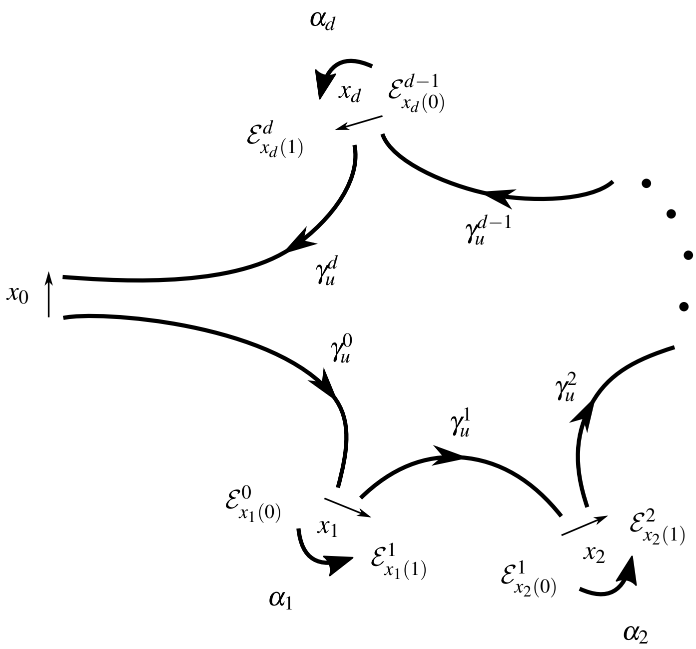


Figure 2.1: The structure maps

this section). For the case $d = 1$, the space $\mathcal{R}_{1:1}(x_0 : x_1)$ is just the space of Floer trajectories which we defined in section 2.2.2. One then defines the A_∞ operations by setting:

$$\mu^d : \overline{CF}^*(\mathcal{E}^{d-1}, \mathcal{E}^d) \otimes \cdots \otimes \overline{CF}^*(\mathcal{E}^0, \mathcal{E}^1) \longrightarrow \overline{CF}^*(\mathcal{E}^0, \mathcal{E}^d), \tag{2.29}$$

$$\mu^d := \sum_{\substack{x_0 \in \mathcal{X}(L^0, L^d) \\ (x_1, \ldots, x_d) \in \Pi_{j=1}^d \mathcal{X}(L^{j-1}, L^j)}} \sum_{u \in \mathcal{R}^0_{1:d}(x_0 : x_1, \ldots, x_d)} \mu_u.$$

Note that μ^1 is indeed built out of the differentials $d^{(H^{ij}, J^{ij})}$. We call an object $(L, \mathcal{E})$ of $\mathcal{F}(M)$ *essential* whenever the cohomology of its endomorphism space $H^*(hom_{\mathcal{F}(M)}(\mathcal{E}, \mathcal{E}), \mu^1) = \overline{HF}^*(\mathcal{E}, \mathcal{E})$ is non-zero. For (2.29) to make sense we need to check the following.

Proposition 2.3.1. *Let $a_d \otimes \cdots \otimes a_1 \in \overline{CF}^*(\mathcal{E}^{d-1}, \mathcal{E}^d) \otimes \cdots \otimes \overline{CF}^*(\mathcal{E}^0, \mathcal{E}^1)$. Then $\mu^d(a_d \otimes \cdots \otimes a_1)$ is an element of $\overline{CF}^*(\mathcal{E}^0, \mathcal{E}^d)$.*

Proof. Let $x_j \in \mathcal{X}(L^{j-1}, L^j)$, $1 \le j \le d$. Writing $\alpha_j = \langle a_j, x_j \rangle$, we know from (2.19) that

$$\alpha_j \circ m_0(\mathcal{E}^{j-1})(x_j(0)) = m_0(\mathcal{E}^j)(x_j(1)) \circ \alpha_j.$$

Further, since $m_0(\mathcal{E}^j)$ is parallel for each $0 \le j \le d$ we have that if $u \in \mathcal{R}_{1:d}(x_0 : x_1, \ldots, x_d)$, then

$$\begin{aligned}
P_{\gamma_u^0} \circ m_0(\mathcal{E}^0)(x_0(0)) &= m_0(\mathcal{E}^0)(x_1(0)) \circ P_{\gamma_u^0} \\
P_{\gamma_u^j} \circ m_0(\mathcal{E}^j)(x_j(1)) &= m_0(\mathcal{E}^j)(x_{j+1}(0)) \circ P_{\gamma_u^j}, \quad \forall 1 \le j \le d-1 \\
P_{\gamma_u^d} \circ m_0(\mathcal{E}^d)(x_d(1)) &= m_0(\mathcal{E}^d)(x_0(1)) \circ P_{\gamma_u^d}.
\end{aligned}$$

Thus, from (2.28) we have that $\mu_u(\alpha_d,\ldots,\alpha_1)\circ m_0(\mathcal{E}^0)(x_0(0)) = m_0(\mathcal{E}^d)(x_0(1))\circ\mu_u(\alpha_d,\ldots,\alpha_1)$, i.e. $\mu_u(\alpha_d,\ldots,\alpha_1)\in\overline{CF}^*(\mathcal{E}^0,\mathcal{E}^d)$. Since $\mu^d(a_d\otimes\cdots\otimes a_1)$ consists of linear combinations of such terms, we see that it also lies in $\overline{CF}^*(\mathcal{E}^0,\mathcal{E}^d)$. □

The A_∞ associativity relations

$$\sum_{j=1}^{d}\sum_{i=0}^{d-j}\mu^{d-j+1}(a_d,\ldots,a_{i+j+1},\mu^j(a_{i+j},\ldots,a_{i+1}),a_i\ldots,a_1)=0$$

are shown to hold by considering the Gromov compactification $\overline{\mathcal{R}}^1_{1:d}(x_0:x_1,\ldots,x_d)$ of the one-dimensional component of such moduli spaces (see [Sei08a, (9l)]) and using the fact that the paths used for parallel transport, which are determined by configurations of broken curves appearing at opposite ends of an interval in $\overline{\mathcal{R}}^1_{1:d}(x_0:x_1,\ldots,x_d)$ are homotopic (for an example of a similar argument see Figure 2.2 below).

Remark 2.3.2. As we remarked before, monotonicity, together with the assumption that the images $\iota^i_*(\pi_1(L^i))\subseteq\pi_1(M)$ be trivial, ensures uniform energy bounds on pseudoholomorphic maps belonging to spaces of the same expected dimension, so that Gromov compactness applies. In particular zero-dimensional moduli spaces are compact, so that all sums ranging over such spaces are finite. Disc and sphere bubbles do not appear in any of the constructions apart from μ^1 and we discussed these at length in sections 2.2.1 and 2.2.2 above. This is because all other constructions involve only zero- and one-dimensional moduli spaces of solutions to a perturbed Cauchy-Riemann equation which does not admit an $\mathbb{R}$–action and so they are governed by Fredholm problems of index 0 and 1. The conditions $N^\pi_M\geq 1$ and $N^\pi_{L_i}\geq 2$ ensure that any sphere or disc bubble would reduce the sum of the Fredholm indices governing the remaining components by at least 2, making them all negative and thus contradicting transversality. //

This finishes the setup of the extended monotone Fukaya category $\mathcal{F}(M)$, which is now allowed to contain any set of objects $\{(L^i,\mathcal{E}^i)\}$. Note that for any object $(L,\mathcal{E})$ of this category, the structure maps μ^* make $\overline{CF}^*(\mathcal{E},\mathcal{E})$ into an A_∞ algebra. In fact, since the structure maps involve only compositions with parallel transport maps, $(CF^*_{\mathrm{mon}}(\mathcal{E}),\mu^*)$ is an A_∞ subalgebra of $\overline{CF}^*(\mathcal{E},\mathcal{E})$. Further, if $\mathcal{E}^0,\mathcal{E}^1\to L$ are two local systems and $\mathcal{E}^0$ dominates $\mathcal{E}^1$ in the sense of Definition 2.2.24, then the map $\Phi\colon CF^*_{\mathrm{mon}}(\mathcal{E}^0)\to CF^*_{\mathrm{mon}}(\mathcal{E}^1)$ from Proposition 2.2.25 is an A_∞ morphism (with vanishing higher order terms). This is proved in the same manner as one shows that Φ is a chain map, namely by repeated application of (2.25), using the fact that the structure maps μ^* involve only compositions with parallel transport maps.

Consider the associated homology category $H(\mathcal{F}(M))$ which has the same objects as $\mathcal{F}(M)$ but whose morphism spaces are

$$hom_{H(\mathcal{F}(M))}((L^i,\mathcal{E}^i),(L^j,\mathcal{E}^j)) := \overline{HF}^*((L^i,\mathcal{E}^i),(L^j,\mathcal{E}^j)).$$

Composition of morphisms in $H(\mathcal{F}(M))$ is induced by the operation μ^2 and we denote it by the symbol $*$. The A_∞ relations imply that $*$ is well-defined and associative. In particular $(HF^*_{\text{mon}}(\mathcal{E}),*)$ and $(\overline{HF}^*(\mathcal{E},\mathcal{E}),*)$ are associative $\mathbb{F}$-algebras and there is an algebra homomorphism $HF^*_{\text{mon}}(\mathcal{E}) \to \overline{HF}^*(\mathcal{E},\mathcal{E})$ induced by the inclusion of A_∞ algebras at the chain level. Similarly, if $\mathcal{E}^0$ and $\mathcal{E}^1$ are two local systems on L and $\mathcal{E}^0$ dominates $\mathcal{E}^1$, we have an algebra map $H(\Phi)\colon HF^*_{\text{mon}}(\mathcal{E}^0) \to HF^*_{\text{mon}}(\mathcal{E}^1)$.

Remark 2.3.3. The above constructions depend on choices of strip-like ends and regular Floer and perturbation data. It is a fact that different choices yield quasi-equivalent categories (see [Sei08a], (10a)). We will not need this here but we will use a much weaker fact: the algebra structure on $HF^*_{\text{mon}}(\mathcal{E})$ and $\overline{HF}^*(\mathcal{E},\mathcal{E})$, induced by the μ^2 operation, are preserved by the continuation maps (2.22). A proof of this fact (without local systems, but as we have seen, adding such does not alter the arguments) can be found e.g. in [DS98]. //

2.3.2 Units and morphisms of local systems

It is a non-trivial fact that for each object $(L,\mathcal{E})$ of the category $H(\mathcal{F}(M))$ there is an identity morphism $e_{\mathcal{E}} = e_{(L,\mathcal{E})} \in hom_{H(\mathcal{F}(M))}((L,\mathcal{E}),(L,\mathcal{E})) = \overline{HF}^*(\mathcal{E},\mathcal{E})$. This makes $H^*(\mathcal{F}(M))$ into an honest $\mathbb{F}$-linear category and in particular, the algebra $(\overline{HF}^*(\mathcal{E},\mathcal{E}),*)$ is unital. We now give a brief description of the unit, point out some easy vanishing results and use the unit to convert morphisms of local systems into morphisms in $\mathcal{F}(M)$.

Given a chord $x \in \mathcal{X}(L,L)$ consider the moduli space of perturbed pseudoholomorphic discs with one outgoing puncture asymptotic to x. We denote this space by $\mathcal{R}_{1:0}(x)$ and the union of its k–dimensional components by $\mathcal{R}^k_{1:0}(x)$. Each element $u \in \mathcal{R}_{1:0}(x)$ defines a map $P_{\partial u}\colon \mathcal{E}_{x(0)} \to \mathcal{E}_{x(1)}$ by parallel transport along the boundary. We then define the element

$$\tilde{e}_{\mathcal{E}} := \sum_{x \in \mathcal{X}(L,L)} \sum_{u \in \mathcal{R}^0_{1:0}(x)} P_{\partial u} \in CF^*_{\text{mon}}(\mathcal{E}) \subseteq \overline{CF}^*(\mathcal{E},\mathcal{E}). \tag{2.30}$$

By considering the Gromov compactification of $\mathcal{R}^1_{1:0}(x)$ one shows that $\mu^1(\tilde{e}_{\mathcal{E}}) = 0$. By abuse of notation we denote by $e_{\mathcal{E}}$ the cohomology class of $\tilde{e}_{\mathcal{E}}$ in both $HF^*_{\text{mon}}(\mathcal{E})$ and $\overline{HF}^*(\mathcal{E},\mathcal{E})$ (this abuse is not entirely harmless because it can happen that $\tilde{e}_{\mathcal{E}}$ is not exact in $CF^*_{\text{mon}}(\mathcal{E})$ but is exact in $\overline{CF}^*(\mathcal{E},\mathcal{E})$ – see chapter 5, section 5.1.5; for our current discussion however, this is irrelevant).

Showing that $e_{\mathcal{E}}$ is indeed a unit involves introducing a specific 1-parameter family of perturbations to the Floer equation for strips in order to construct a map $h\colon \overline{CF}^*(\mathcal{E},\mathcal{E}') \to \overline{CF}^*(\mathcal{E},\mathcal{E}')$ for every other object $(L',\mathcal{E}')$, such that one has the identity $\mu^2(a,\tilde{e}_{\mathcal{E}}) = a + \mu^1(h(a)) + h(\mu^1(a))$ for every $a \in \overline{CF}^*(\mathcal{E},\mathcal{E}')$. This shows that right multiplication by $\tilde{e}_{\mathcal{E}}$ is homotopic to the identity. A similar argument proves the same for left multiplication. For more details, see [She16, Section 2.4]. Note that, in the case when $\mathcal{E}' = \mathcal{E}$ the map H preserves $CF^*_{\text{mon}}(\mathcal{E})$ since it is defined by using parallel transport along the boundaries of perturbed Floer trajectories. In particular, this discussion implies the following:

Lemma 2.3.4.

a) The element $e_{\mathcal{E}} \in HF^*_{\mathrm{mon}}(\mathcal{E})$ is a unit for the algebra $(HF^*_{\mathrm{mon}}(\mathcal{E}), *)$ and the ring map $HF^*_{\mathrm{mon}}(\mathcal{E}) \to \overline{HF}^*(\mathcal{E},\mathcal{E})$ is unital.

b) If $\mathcal{E}^0$ and $\mathcal{E}^1$ are local systems on L and $\mathcal{E}^0$ dominates $\mathcal{E}^1$, then one has $\Phi(\tilde{e}_{\mathcal{E}^0}) = \tilde{e}_{\mathcal{E}^1}$. In particular $H(\Phi)\colon HF^*_{\mathrm{mon}}(\mathcal{E}^0) \to HF^*_{\mathrm{mon}}(\mathcal{E}^1)$ is a unital algebra homomorphism. □

These properties have the following immediate consequences.

Proposition 2.3.5.

1) If $\overline{HF}^*(\mathcal{E},\mathcal{E}) = 0$, then $\overline{HF}^*((L,\mathcal{E}),(L',\mathcal{E}')) = 0$ for any other object $(L',\mathcal{E}')$ in $\mathcal{F}(M)$.

2) If $HF^*_{\mathrm{mon}}(\mathcal{E}) = 0$ then $\overline{HF}^*(\mathcal{E},\mathcal{E}) = 0$.

3) If $\mathcal{E}^0, \mathcal{E}^1 \to L$ are local systems, $\mathcal{E}^0$ dominates $\mathcal{E}^1$ and $HF^*_{\mathrm{mon}}(\mathcal{E}^0) = 0$ then $HF^*_{\mathrm{mon}}(\mathcal{E}^1) = 0$.

4) If $HF^*_{\mathrm{mon}}(L) = 0$ then $\overline{HF}^*((L,\mathcal{E}),(L',\mathcal{E}')) = 0$ for any Lagrangian L' such that (L,L') is a monotone pair and for any local systems $\mathcal{E} \to L$, $\mathcal{E}' \to L'$.

Proof. Part 1) holds because $\overline{HF}^*((L,\mathcal{E}),(L',\mathcal{E}')) = hom_{H(\mathcal{F}(M))}(\mathcal{E},\mathcal{E}')$ is a unital right module over the ring $hom_{H(\mathcal{F}(M))}(\mathcal{E},\mathcal{E}) = \overline{HF}^*(\mathcal{E},\mathcal{E}) = 0$. Parts 2) and 3) hold because of the unital algebra maps $HF^*_{\mathrm{mon}}(\mathcal{E}) \to \overline{HF}^*(\mathcal{E},\mathcal{E})$ and $H(\Phi)\colon HF^*_{\mathrm{mon}}(\mathcal{E}^0) \to HF^*_{\mathrm{mon}}(\mathcal{E}^1)$, respectively. Part 4) follows from 1), 2), 3) and the fact that $\mathcal{E}_{\mathrm{reg}}$ dominates every other local system $\mathcal{E} \to L$. □

We can use the unit to turn morphisms of local systems on the same Lagrangian into morphisms in the extended Fukaya category $\mathcal{F}(M)$. More precisely, we have the following lemma (for completeness we state it in a rather general form but only the identity (2.31) will be used in the sequel).

Lemma 2.3.6. *Let $\{(L,\mathcal{E}^0), (L,\mathcal{E}^1)\}, \{(K^j,\mathcal{W}^j)\}_{1\le j\le r}, \{(N^k,\mathcal{V}^k)\}_{1\le k\le s}$ be sets of objects in $\mathcal{F}(M)$ and let $F\colon \mathcal{E}^0 \to \mathcal{E}^1$ be a morphism of local systems. Suppose we are given elements $a_1 \in \overline{CF}^*(\mathcal{E}^1,\mathcal{W}^1)$, $a_j \in \overline{CF}^*(\mathcal{W}^{j-1},\mathcal{W}^j)$ for $2 \le j \le r$, $b_{|1} \in \overline{CF}^*(\mathcal{V}^1,\mathcal{E}^0)$, $b_{|k} \in \overline{CF}^*(\mathcal{V}^k,\mathcal{V}^{k-1})$ for $2 \le k \le s$ and $c_i \in \overline{CF}^*(\mathcal{E}^i,\mathcal{E}^i)$ for $i \in \{0,1\}$. Then we have*

a) The elements $c_1 \circ F \in CF^*(\mathcal{E}^0,\mathcal{E}^1)$ and $F \circ c_0 \in CF^*(\mathcal{E}^0,\mathcal{E}^1)$ satisfy

$$\begin{aligned}
\mu^{r+1+s}(a_r,\dots,a_1,(c_1\circ F),b_{|1},\dots,b_{|s}) &= \mu^{r+1+s}(a_r,\dots,a_1,c_1,(F\circ b_{|1}),\dots,b_{|s}) \\
\mu^{r+1+s}(a_r,\dots,a_1,(F\circ c_0),b_{|1},\dots,b_{|s}) &= \mu^{r+m+s}(a_r,\dots,(a_1\circ F),c_0,b_{|1},\dots,b_{|s})
\end{aligned}$$

b) in the setting above if $r = 0$, then

$$\begin{aligned}
\mu^{1+s}(F\circ c_0,b_{|1},\dots,b_{|s}) &= F\circ\mu^{1+s}(c_0,b_{|1},\dots,b_{|s}) \\
\mu^{s}(F\circ b_{|1},\dots,b_{|s}) &= F\circ\mu^{s}(b_{|1},\dots,b_{|s})
\end{aligned}$$

and if $s = 0$, then

$$\begin{aligned}\mu^{r+1}(a_r,\dots,a_1,c_1\circ F) &= \mu^{r+1}(a_r,\dots,a_1,c_1)\circ F\\ \mu^{r}(a_r,\dots,a_1\circ F) &= \mu^{r}(a_r,\dots,a_1)\circ F.\end{aligned}$$

Consequently $c_1\circ F\in\overline{CF}^*(\mathcal{E}^0,\mathcal{E}^1)$ and $F\circ c_0\in\overline{CF}^*(\mathcal{E}^0,\mathcal{E}^1)$.

c) *Writing $\tilde e_i = \tilde e_{\mathcal{E}^i}$ for $i\in\{0,1\}$, we have that $\tilde e_1\circ F$ equals $F\circ\tilde e_0$ and is a closed element of $\overline{CF}^*(\mathcal{E}^0,\mathcal{E}^1)$.*

d) *The map $F\circ - \colon \overline{CF}^*(\mathcal{V}^1,\mathcal{E}^0)\to\overline{CF}^*(\mathcal{V}^1,\mathcal{E}^1)$ is a chain map, homotopic to the map*

$$\mu^2(\tilde e_1\circ F,-)\colon \overline{CF}^*(\mathcal{V}^1,\mathcal{E}^0)\to\overline{CF}^*(\mathcal{V}^1,\mathcal{E}^1).$$

Similarly, the map $-\circ F : \overline{CF}^*(\mathcal{E}^1,\mathcal{W}^1)\to\overline{CF}^*(\mathcal{E}^0,\mathcal{W}^1)$ is a chain map, homotopic to

$$\mu^2(-,\tilde e_1\circ F) : \overline{CF}^*(\mathcal{E}^1,\mathcal{W}^1)\to\overline{CF}^*(\mathcal{E}^0,\mathcal{W}^1).$$

e) *If $\mathcal{E}^2\to L$ is another local system with corresponding unit cochain $\tilde e_2$ and $G\colon\mathcal{E}^1\to\mathcal{E}^2$ is a morphism of local systems, then for the cohomology classes $[\tilde e_1\circ F]\in\overline{HF}^*(\mathcal{E}^0,\mathcal{E}^1)$ and $[\tilde e_2\circ G]\in\overline{HF}^*(\mathcal{E}^1,\mathcal{E}^2)$ we have*

$$[\tilde e_2\circ G]*[\tilde e_1\circ F] = [\tilde e_2\circ(G\circ F)]. \tag{2.31}$$

Proof. Let $x_{|k}\in\mathcal{X}(N^k,N^{k-1})$ for $2\le k\le s$, $x_{|1}\in\mathcal{X}(N^1,L)$, $x_0\in\mathcal{X}(L,L)$, $x_1\in\mathcal{X}(L,K^1)$, $x_j\in\mathcal{X}(K^{j-1},K^j)$ for $2\le j\le r$. We write $\alpha_j=\langle a_j,x_j\rangle$, $\beta_{|k}=\langle b_{|k},x_{|k}\rangle$ and $\varsigma_i=\langle c_i,x_0\rangle$. Finally, let $z\in\mathcal{X}(N^s,K^r)$. Then for every $u\in\mathcal{R}^0_{1:r+s+1}(z:x_{|s},\dots,x_{|1},x_0,x_1,\dots,x_r)$ we have

$$\begin{aligned}&\mu_u(\alpha_r,\dots,\alpha_1,\varsigma_1\circ F(x_0(0)),\beta_{|1},\dots,\beta_{|s}) =\\ &= P_{\gamma_u^{s+r+1}}\circ\cdots\circ P_{\gamma_u^{s+2}}\circ\alpha_1\circ P_{1,\gamma_u^{s+1}}\circ\varsigma_1\circ F(x_0(0))\circ P_{0,\gamma_u^s}\circ\beta_{|1}\circ P_{\gamma_u^{s-1}}\cdots P_{\gamma_u^1}\circ\beta_{|s}\circ P_{\gamma_u^0}\\ &= P_{\gamma_u^{s+r+1}}\circ\cdots\circ P_{\gamma_u^{s+2}}\circ\alpha_1\circ P_{1,\gamma_u^{s+1}}\circ\varsigma_1\circ P_{1,\gamma_u^s}\circ F(x_{|1}(1))\circ\beta_{|1}\circ P_{\gamma_u^{s-1}}\cdots P_{\gamma_u^1}\circ\beta_{|s}\circ P_{\gamma_u^0}\\ &= \mu_u(\alpha_r,\dots,\alpha_1,\varsigma_1,F(x_{|1}(1))\circ\beta_{|1},\dots,\beta_{|s}).\end{aligned}$$

Summing this identity over all possible relevant Hamiltonian chords and all rigid pseudoholomorphic discs yields the first claim in a). The second one is done analogously. To prove the first identity in b) we note that

$$\begin{aligned}\mu_u(F(x(1))\circ\varsigma_0,\beta_{|1},\dots,\beta_{|s}) &= P_{1,\gamma_u^{s+1}}\circ F(x_0(1))\circ\varsigma_0\circ P_{0,\gamma_u^s}\circ\beta_{|1}\circ P_{\gamma_u^{s-1}}\cdots P_{\gamma_u^1}\circ\beta_{|s}\circ P_{\gamma_u^0}\\ &= F(z(1))\circ P_{0,\gamma_u^{s+1}}\circ\varsigma_0\circ P_{0,\gamma_u^s}\circ\beta_{|1}\circ P_{\gamma_u^{s-1}}\cdots P_{\gamma_u^1}\circ\beta_{|s}\circ P_{\gamma_u^0}\\ &= F(z(1))\circ\mu_u(\varsigma_0,\beta_{|1},\dots,\beta_{|s}).\end{aligned}$$

Again, summing over all chords and disks yields the claim and the other identities follow similarly. Applying the claim twice with $r=s=0$ and using the fact that $c_1\in\overline{CF}^*(\mathcal{E}^1,\mathcal{E}^1)$ yields

$$\mu^1(\mu^1(c_1\circ F)) = \mu^1(\mu^1(c_1))\circ F = 0,$$

i.e. $c_1\circ F\in\overline{CF}^*(\mathcal{E}^0,\mathcal{E}^1)$. Similarly, $F\circ c_0\in\overline{CF}^*(\mathcal{E}^0,\mathcal{E}^1)$.

For part c) we are in the situation $c_0 = \tilde{e}_0$. Then from the above we have

$$\mu^1(F \circ \tilde{e}_0) = F \circ \mu^1(\tilde{e}_0) = 0.$$

Further, from the definitions of $\tilde{e}_0$ and $\tilde{e}_1$ we have

$$\begin{aligned} F \circ \tilde{e}_0 &= \textstyle\sum_{x \in \mathcal{X}(L,L)} \sum_{u \in \mathcal{R}^0_{1:0}(x)} F(x(1)) \circ P_{0,\partial u} \\ &= \textstyle\sum_{x \in \mathcal{X}(L,L)} \sum_{u \in \mathcal{R}^0_{1:0}(x)} P_{1,\partial u} \circ F(x(0)) \\ &= \tilde{e}_1 \circ F. \end{aligned}$$

For part d), note that $F \circ - \colon \overline{CF}^*(\mathcal{V}^1, \mathcal{E}^0) \to \overline{CF}^*(\mathcal{V}^1, \mathcal{E}^1)$ is indeed a chain map, because by part b) one has $\mu^1(F \circ b_{|1}) = F \circ \mu^1(b_{|1})$ for every $b_{|1} \in \overline{CF}^*(\mathcal{V}^1, \mathcal{E}^0)$. On the other hand, the map $\mu^2(\tilde{e}_1 \circ F, -) \colon \overline{CF}^*(\mathcal{V}^1, \mathcal{E}^0) \to \overline{CF}^*(\mathcal{V}^1, \mathcal{E}^1)$ is a chain map because $\mu^1(\tilde{e}_1 \circ F) = 0$. Further, by part a), these two maps fit into the commutative diagram

$$\begin{tikzcd} \overline{CF}^*(\mathcal{V}^1, \mathcal{E}^0) \arrow[r, "F\circ -"] \arrow[dr, "\mu^2(\tilde{e}_1 \circ F, -)"'] & \overline{CF}^*(\mathcal{V}^1, \mathcal{E}^1) \arrow[d, "\mu^2(\tilde{e}_1, -)"] \\ & \overline{CF}^*(\mathcal{V}^1, \mathcal{E}^1) \end{tikzcd}$$

in which the vertical arrow is homotopic to the identity. The claim follows. Similarly for the map $- \circ F \colon \overline{CF}^*(\mathcal{E}^1, \mathcal{W}^1) \to \overline{CF}^*(\mathcal{E}^0, \mathcal{W}^1)$.

Finally, we prove (2.31):

$$\begin{aligned} [\tilde{e}_2 \circ G] * [\tilde{e}_1 \circ F] &= [\mu^2(\tilde{e}_2 \circ G, \tilde{e}_1 \circ F)] \\ &= [\mu^2(\tilde{e}_2, G \circ \tilde{e}_1 \circ F)] && \text{by part a)} \\ &= [\mu^2(\tilde{e}_2, \tilde{e}_2 \circ (G \circ F))] && \text{by part c)} \\ &= [\mu^2(\tilde{e}_2, \tilde{e}_2) \circ (G \circ F)] && \text{by part b)} \\ &= [\tilde{e}_2 \circ (G \circ F) + \mu^1(c) \circ (G \circ F)] && \text{for some } c \in \overline{CF}^*(E^2, E^2) \\ &= [\tilde{e}_2 \circ (G \circ F) + \mu^1(c \circ (G \circ F))] && \text{by part b)} \\ &= [\tilde{e}_2 \circ (G \circ F)] \end{aligned}$$

□

2.3.3 Closed-open string map and the AKS theorem

Recall that our main objective in this section is to verify that a version of Abouzaid's split-generation criterion holds in the setting of the extended monotone Fukaya category. A key role in the split-generation criterion is played by the so-called closed-open string map

$$\mathcal{CO}^* \colon QH^*(M) \to HH^*(\overline{CF}^*((L, \mathcal{E}), (L, \mathcal{E})), \tag{2.32}$$

whose definition in different settings can be found in [FOOO09], [RS17], [She16]. Its domain is (in our case) the ungraded small quantum cohomology ring of M, whose underlying vector space is simply $H^*(M; \mathbb{F})$ but whose ring structure is deformed by "quantum contributions" arising from counts of pseudoholomorphic spheres (for a brief account see e.g. [She16, Section 2.2]; full details are given in [MS12, Chapter 11]). We denote this product by $\star$. It is a fact that $\star$ is associative, commutative (graded commutative when one works over characteristic different from 2 and QH^* is graded) and together with the Poincaré pairing makes $QH^*(M)$ into a Frobenius algebra, i.e.

$$\langle a \star b, c \rangle = \langle a, b \star c \rangle.$$

Further, the usual unit $1 \in H^*(M;\mathbb{F})$ is also a unit for the $\star$ product.

In general the target of $\mathcal{CO}^*$ is the Hochschild cohomology of the entire A_∞ category $\mathcal{F}(M)$. For our application however it suffices to focus only on the endomorphism A_∞ algebra of a single object $(L,\mathcal{E})$. The Hochschild cochain complex of this endomorphism algebra is defined to be $CC^*(\overline{CF}^*(\mathcal{E},\mathcal{E})) := \Pi_{d\geq 0} CC^*_c(\overline{CF}^*(\mathcal{E},\mathcal{E}))^d$, where

$$CC^*_c(\overline{CF}^*(\mathcal{E},\mathcal{E}))^d := \mathrm{Hom}_{\mathbb{F}}(\overline{CF}^*(\mathcal{E},\mathcal{E})^{\otimes d}, \overline{CF}^*(\mathcal{E},\mathcal{E})),$$

equipped with the differential

$$\begin{aligned} \delta((\phi^0,\phi^1,\dots))^d(\alpha_d,\dots,\alpha_1) &= \sum_{j=0}^{d}\sum_{i=0}^{d-j} \mu^{d-j+1}(\alpha_d,\dots,\alpha_{i+j+1},\phi^j(\alpha_{i+j},\dots,\alpha_{i+1}),\alpha_i\dots,\alpha_1) \\ &+ \sum_{j=1}^{d}\sum_{i=0}^{d-j} \phi^{d-j+1}(\alpha_d,\dots,\alpha_{i+j+1},\mu^j(\alpha_{i+j},\dots,\alpha_{i+1}),\alpha_i\dots,\alpha_1). \end{aligned}$$

Given an element $\beta \in QH^*(M)$ whose Poincaré dual is represented by a pseudocycle $f\colon B \to M$, one defines a corresponding Hochschild cochain $\mathcal{CO}^*(\beta;f) = (\mathcal{CO}^*(\beta;f)^d)_{d\geq 0}$ as follows. For every tuple of Hamiltonian chords $(x,\vec{x}) := (x,x_1,\dots,x_d)$ in $\mathcal{X}(L,L)$ one considers the moduli space $\mathcal{R}_{1:d;1}(x:\vec{x};f)$ of perturbed pseudoholomorphic maps u from a disc with d positive boundary punctures, asymptotic to $\vec{x}$, one negative boundary puncture which is asymptotic to x and an internal marked point which is mapped to $f(B)$. Every $u \in \mathcal{R}_{1:d;1}(x:\vec{x};f)$ defines a map μ_u as in equation (2.28). One then sets

$$\mathcal{CO}^*(\beta;f)^d := \sum_{(x,\vec{x})\in\mathcal{X}(L,L)^{d+1}} \sum_{u\in\mathcal{R}^0_{1:d;1}(x:\vec{x};f)} \mu_u.$$

The facts that the resulting element is δ–closed and that its cohomology class is independent of the choice of pseudocycle f are proved for rank 1 local systems in [She16, Section 2.5] and the proofs hold just as well in our case (we review a similar argument for the open-closed string map in more detail below).

By inspecting the definition of the differential δ one sees that the length-zero projection $CC^*(\overline{CF}^*(\mathcal{E},\mathcal{E})) \to \overline{CF}^*(\mathcal{E},\mathcal{E})$, $(\phi^0,\phi^1,\dots) \mapsto \phi^0$ is a chain map. Composing $\mathcal{CO}^*$ with this projection at the level of cohomology gives the map

$$\mathcal{CO}^0\colon QH^*(M) \to \overline{HF}^*(\mathcal{E},\mathcal{E}).$$

Observe that $\mathcal{CO}^0(\beta,f)$ lies in $CF^*_{\mathrm{mon}}(\mathcal{E})$ for any pseudocycle f, representing the Poincaré dual of a cohomology class β. Thus, we actually have the diagram

$$\begin{tikzcd} QH^*(M) \arrow[r, "\mathcal{CO}^0"] \arrow[rr, bend left, "\mathcal{CO}^0"] \arrow[rrr, bend left=40, "\mathcal{CO}^0"] & HF^*_{\mathrm{mon}}(L) \arrow[r, "H(\Phi)"] & HF^*_{\mathrm{mon}}(\mathcal{E}) \arrow[r] & \overline{HF}^*(\mathcal{E},\mathcal{E}) \end{tikzcd} \quad (2.33)$$

Proposition 2.3.7. *All maps in diagram* (2.33) *are unital algebra homomorphisms.*

Proof. Unitality is easy to see already at the chain level since for any local system $\mathcal{E} \to L$, the element

$$\mathcal{CO}^0(1;M) = \sum_{x\in\mathcal{X}(L,L)} \sum_{u\in\mathcal{R}^0_{1:0;1}(x;M)} \mu_u,$$

is precisely the unit cochain $\tilde{e}_{\mathcal{E}} \in CF^*_{\mathrm{mon}}(\mathcal{E})$ (when one chooses the same perturbation data to define the moduli spaces). Thus the content of this proposition is that $\mathcal{CO}^0\colon QH^*(M) \to HF^*_{\mathrm{mon}}(L)$ intertwines the products $\star$ and $*$. More generally, the Hochschild cohomology $HH^*(CF^*_{\mathrm{mon}}(L))$ itself is an algebra when equipped with the so-called Yoneda product (see e.g. [She16, equation (A.4.1)]) and the length-zero projection to $HF^*_{\mathrm{mon}}(L)$ is an algebra homomorphism. The proposition then follows from the fact that the full map $\mathcal{CO}^*\colon QH^*(M) \to HH^*(CF^*_{\mathrm{mon}}(L))$ is an algebra homomorphism. The proof of this fact is a straightforward adaptation of [She16, Proposition 2.1]. □

We end this subsection by recalling an appropriate version of the Auroux–Kontsevich–Seidel theorem (cf. [Aur07, Proposition 6.8], [She16, Lemma 2.7]).

Theorem 2.3.8. *Let L be an orientable, monotone Lagrangian in a closed, monotone symplectic manifold (M,ω). Then the map $\mathcal{CO}^0\colon QH^*(M) \to HF^*_{\mathrm{mon}}(L)$ satisfies*

$$\mathcal{CO}^0(c_1(TM)) = [m_0(\mathcal{E}_{\mathrm{reg}}) \circ \tilde{e}_L].$$

A proof for rank 1 local systems is given in [She16, Lemma 2.7]. Strictly speaking, this proof applies only when one works over characteristic different from 2 but the assumption that L is orientable can be used to remove this restriction (the idea is that if L is orientable one can choose a pseudocycle Poincarè dual to $c_1(TM)$ and disjoint from L; see [Ton18, Theorem 1.10] and the discussion immediately after). Further, checking that the proof applies when L is equipped with the local system $\mathcal{E}_{\mathrm{reg}}$ amounts once again to using the fact that $m_0(\mathcal{E}_{\mathrm{reg}})$ is a morphism of local systems. Note that together with diagram (2.33), Theorem 2.3.8 implies that if $\mathcal{E} \to L$ is any local system, then $\mathcal{CO}^0(c_1(TM)) = [m_0(\mathcal{E}) \circ \tilde{e}_{\mathcal{E}}]$.

Consider now the endomorphism of quantum cohomology given by multiplication by the first Chern class $c_1(TM)\star\colon QH^*(M) \to QH^*(M)$. For $\lambda \in \mathbb{F}$ denote by $QH^*(M)_\lambda$ the generalised λ–eigenspace of this map (which is trivial if λ is not an eigenvalue). Then one has the following

Corollary 2.3.9. *Let $\mathcal{E} \to L$ be a local system and let $\mu \in \mathbb{F}$. If $m_0(\mathcal{E}) - \mu\mathrm{Id}_{\mathcal{E}}$ is an invertible endomorphism of $\mathcal{E}$, then $\mathcal{CO}^0\colon QH^*(M) \to \overline{HF}^*(\mathcal{E},\mathcal{E})$ vanishes on $QH^*(M)_\mu$.*

Proof. Put $\sigma = m_0(\mathcal{E}) - \mu\mathrm{Id}_{\mathcal{E}}$. By Theorem 2.3.8 we have $\mathcal{CO}^0(c_1(TM) - \mu) = [\sigma \circ \tilde{e}_{\mathcal{E}}]$. Let $a \in QH^*(M)_\mu$, i.e. there exists $k \geq 0$ such that $(c_1(TM) - \mu)^{\star k} \star a = 0$. Then we have

$$\begin{aligned} \mathcal{CO}^0(a) = [\tilde{e}_{\mathcal{E}}] * \mathcal{CO}^0(a) &= [\sigma^{-k} \circ \tilde{e}_{\mathcal{E}}] * [\sigma^k \circ \tilde{e}_{\mathcal{E}}] * \mathcal{CO}^0(a) \qquad \text{[by (2.31)]} \\ &= [\sigma^{-k} \circ \tilde{e}_{\mathcal{E}}] * \mathcal{CO}^0((c_1(TM) - \mu)^{\star k} \star a) \;=\; 0, \end{aligned}$$

as we wanted. □

A well-known consequence of Corollary 2.3.9 is that if one allows only rank 1 local systems and $\mathbb{F}$ is algebraically closed, then the Fukaya category splits into summands, indexed by the eigenvalues of $c_1(TM)\star$. We now discuss this splitting in the setting of the extended Fukaya category.

2.3.4 Decomposing $\mathcal{F}(M)$

For each $\lambda \in \mathbb{F}$, let us denote by $\mathcal{F}(M)_\lambda$ the full subcategory of $\mathcal{F}(M)$ whose objects are pairs $(L,\mathcal{E})$ with $m_0(\mathcal{E}) = \lambda \mathrm{Id}_{\mathcal{E}}$. Further, we denote by $\mathcal{F}(M)_\lambda^{\mathrm{nil}}$ the larger subcategory where we require that objects $(L,\mathcal{E})$ satisfy $(m_0(\mathcal{E}) - \lambda \mathrm{Id}_{\mathcal{E}})^{k_{\mathcal{E}}} = 0$ for some integer $k_{\mathcal{E}}$. Then the following easy lemma shows that there are no non-zero morphisms between objects belonging to $\mathcal{F}(M)_{\lambda_0}^{\mathrm{nil}}$ and $\mathcal{F}(M)_{\lambda_1}^{\mathrm{nil}}$ for $\lambda_0 \neq \lambda_1$.

Lemma 2.3.10. *Let $\mathcal{E}^0 \to L^0$ and $\mathcal{E}^1 \to L^1$ be local systems such that there exist $\lambda_0, \lambda_1 \in \mathbb{F}$ and $k_0, k_1 \in \mathbb{N}$ such that $(m_0(\mathcal{E}^j) - \lambda_j \mathrm{Id}_{\mathcal{E}^j})^{k_j} = 0$ for $j \in \{0,1\}$. Then $\overline{CF}^*(\mathcal{E}^0,\mathcal{E}^1) = 0$ unless $\lambda_0 = \lambda_1$.*

Proof. Let $a \in \overline{CF}^*(\mathcal{E}^0,\mathcal{E}^1;H,J)$, $x \in \mathcal{X}_H(L^0,L^1)$ and write $\alpha := \langle a,x\rangle$. Further, if we write $T_0 := m_0(\mathcal{E}^0)(x(0))$, $T_1 := m_0(\mathcal{E}^1)(x(1))$, condition (2.19) says that $\alpha \circ T_0 = T_1 \circ \alpha$. It then follows that

$$(T_1 - \lambda_0)^{k_0} \circ \alpha = \alpha \circ (T_0 - \lambda_0)^{k_0} = 0.$$

Assume for a contradiction that $\alpha \neq 0$, i.e. there exists $v \in \mathcal{E}^0_{x(0)}$ such that $\alpha(v) \neq 0$. Substituting into the above yields $(T_1 - \lambda_0)^{k_0}(\alpha(v)) = 0$. Then there exists a unique non-negative integer $k < k^0$ such that $w := (T_1 - \lambda_0)^k(\alpha(v)) \neq 0$ but $(T_1 - \lambda_0)^{k+1}(\alpha(v)) = 0$. Then we must have $T_1 w = \lambda_0 w$ and thus $(\lambda_0 - \lambda_1)^{k_1} w = (T_1 - \lambda_1)^{k_1} w = 0$ which forces $\lambda_0 = \lambda_1$. □

Observe also that direct sum decompositions of local systems induce such decompositions for the central Floer complexes. Indeed, if (L^0,L^1) is a monotone pair, then for local systems $\mathcal{E}^{01}$, $\mathcal{E}^{02}$ on L^0 and $\mathcal{E}^{11}$, $\mathcal{E}^{12}$ on L^1 and a chord $x \in \mathcal{X}_H(L^0,L^1)$, one has the splitting

$$\mathrm{Hom}((\mathcal{E}^{01} \oplus \mathcal{E}^{02})_{x(0)}, (\mathcal{E}^{11} \oplus \mathcal{E}^{12})_{x(1)}) = \bigoplus_{\substack{i\in\{1,2\}\\ j\in\{1,2\}}} \mathrm{Hom}(\mathcal{E}^{0j}_{x(0)}, \mathcal{E}^{1i}_{x(1)}). \tag{2.34}$$

It is then convenient to represent an element $\alpha \in \mathrm{Hom}((\mathcal{E}^{01} \oplus \mathcal{E}^{02})_{x(0)}, (\mathcal{E}^{11} \oplus \mathcal{E}^{12})_{x(1)})$ as a matrix $\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$ with $\alpha_{ij} \in \mathrm{Hom}(\mathcal{E}^{0j}_{x(0)}, \mathcal{E}^{1i}_{x(1)})$. When similarly represented as matrices, the parallel transport maps for $\mathcal{E}^{01} \oplus \mathcal{E}^{02}$ and $\mathcal{E}^{11} \oplus \mathcal{E}^{12}$ have block-diagonal from. Since the Floer differential involves only pre- and post-composing elements α by such block-diagonal matrices, it follows that $d^{(H,J)}$ preserves the decomposition

$$CF^*((\mathcal{E}^{01} \oplus \mathcal{E}^{02}),(\mathcal{E}^{11} \oplus \mathcal{E}^{12});H,J) = \bigoplus_{\substack{i\in\{1,2\}\\ j\in\{1,2\}}} CF^*(\mathcal{E}^{0j},\mathcal{E}^{1i};H,J),$$

induced from (2.34). Taking maximal unobstructed subcomplexes, we then have

$$\overline{CF}^*((\mathcal{E}^{01}\oplus\mathcal{E}^{02}),(\mathcal{E}^{11}\oplus\mathcal{E}^{12});H,J) = \bigoplus_{\substack{i\in\{1,2\}\\ j\in\{1,2\}}} \overline{CF}^*(\mathcal{E}^{0j},\mathcal{E}^{1i};H,J).$$

Suppose now that $\mathbb{F}$ is algebraically closed. Then we can decompose each finite rank local system $\mathcal{E}^j \to L^j$ into generalised eigen-subsystems for $m_0(\mathcal{E}^j)$. That is, there exist finite collections of scalars $\mathrm{Spec}\,(m_0(\mathcal{E}^j)) \subseteq \mathbb{F}$ and for each $\lambda \in \mathrm{Spec}\,(m_0(\mathcal{E}^j))$ there is a positive integer $k_{j,\lambda}$ and a non-zero local subsystem $\mathcal{E}^{j,\lambda} \leq \mathcal{E}^j$ such that

$$\begin{aligned}\mathcal{E}^j &= \bigoplus_{\lambda\in\mathrm{Spec}\,(m_0(\mathcal{E}^j))} \mathcal{E}^{j,\lambda}\\ \left(m_0(\mathcal{E}^{j,\lambda}) - \lambda \mathrm{Id}_{\mathcal{E}^{j,\lambda}}\right)^{k_{j,\lambda}} &= 0.\end{aligned}$$

It follows from our observations above that we then have:

$$\overline{CF}^*(\mathcal{E}^0,\mathcal{E}^1) = \bigoplus_{\lambda\in\mathrm{Spec}\,(m_0(\mathcal{E}^0))\cap\mathrm{Spec}\,(m_0(\mathcal{E}^1))} \overline{CF}^*(\mathcal{E}^{0,\lambda},\mathcal{E}^{1,\lambda}). \tag{2.35}$$

Thus, if $\mathbb{F}$ is algebraically closed, we lose no information by restricting ourselves to work only in a particular summand $\mathcal{F}(M)^{\mathrm{nil}}_{\lambda}$ for some fixed $\lambda \in \mathbb{F}$.

Consider now the decomposition of $QH^*(M)$ into generalised eigenspaces for quantum multiplication by the first Chern class:

$$QH^*(M) = \oplus_{\lambda\in\mathrm{Spec}(c_1(TM)\star)} QH^*(M)_\lambda. \tag{2.36}$$

We write 1_λ for the component of $1 \in QH^*(M)$ in $QH^*(M)_\lambda$. The fact that $(QH^*(M),\star)$ is a commutative Frobenius algebra implies that (2.36) is in fact a decomposition of algebras and 1_λ is a unit for $(QH^*(M)_\lambda,\star)$. From Corollary 2.3.9, we now have the following version of the AKS criterion for higher rank local systems.

Proposition 2.3.11. *Let $(L,\mathcal{E})$ be an object of $\mathcal{F}(M)^{\mathrm{nil}}_{\lambda}$. Then the map $\mathcal{CO}^0\colon QH^*(M)\to HF^*_{\mathrm{mon}}(\mathcal{E})$ vanishes on $QH^*(M)_\mu$ for all $\mu \neq \lambda$. In particular, if $HF^*_{\mathrm{mon}}(\mathcal{E}) \neq 0$, then $\lambda \in \mathrm{Spec}(c_1(TM)\star)$ and*

$$\mathcal{CO}^0_\lambda := \mathcal{CO}^0\big|_{QH^*(M)_\lambda} : (QH^*(M)_\lambda, 1_\lambda) \to (HF^*_{\mathrm{mon}}(\mathcal{E}), e_{\mathcal{E}})$$

is unital.

Proof. Suppose $\mu \neq \lambda$. Then $\sigma := m_0(\mathcal{E}) - \mu\mathrm{Id}_{\mathcal{E}} = (\lambda-\mu)\mathrm{Id}_{\mathcal{E}} + (m_0(\mathcal{E}) - \lambda\mathrm{Id}_{\mathcal{E}})$ is an invertible endomorphism of $\mathcal{E}$, since $m_0(\mathcal{E}) - \lambda\mathrm{Id}_{\mathcal{E}}$ is nilpotent. Moreover, $\sigma^{-1}\circ\tilde{e}_{\mathcal{E}}$ defines an element of $CF^*_{\mathrm{mon}}(\mathcal{E})$. So the argument from Corollary 2.3.9 tells us that $\mathcal{CO}^0$ vanishes on $QH^*(M)_\mu$. On the other hand, if $HF^*_{\mathrm{mon}}(\mathcal{E}) \neq 0$ then, since $\mathcal{CO}^0$ is unital, it cannot vanish identically. Thus we must have that $\lambda \in \mathrm{Spec}(c_1(TM))$. Unitality of $\mathcal{CO}^0_\lambda$ is clear since $\mathcal{CO}^0$ is unital and it vanishes on the other eigensummands. $\square$

We thus have that the only potentially non-trivial summands of $\mathcal{F}(M)$ are those $\mathcal{F}(M)_\lambda^{\mathrm{nil}}$ for which λ is an eigenvalue of $c_1(TM)\star$. This is in parallel with the well-known situation for rank 1 local systems.

2.3.5 Split-generation

Finally, we discuss a generalisation of Abouzaid's split-generation criterion [Abo10] to our setting involving higher rank local systems. Such an extension has already been proved in [Abo12] for the wrapped Fukaya category and our situation is in fact a lot simpler since we won't have to deal with infinite-dimensional Hom–spaces. On the other hand, the restriction to only finite-rank local systems gives us the freedom to allow for the possibility that both the generating and the generated objects of $\mathcal{F}(M)$ are equipped with higher rank local systems.

Recall first that if $\mathcal{A}$ is any cohomologically unital A_∞ category then an object E is said to split-generate an object W if W is quasi-isomorphic to an object in the smallest triangulated (in the A_∞ sense) and idempotent-closed subcategory of $\Pi(Tw\mathcal{A})$ containing E, where $\Pi(Tw\mathcal{A})$ denotes the split-closure of the category $Tw(\mathcal{A})$ of twisted complexes over $\mathcal{A}$ (see [Sei08a, (3l), (4c)]). Split-generation is important for computations in Fukaya categories but in the present work we are interested only in the following well-known consequence.

Fact 2.3.12. *Suppose that W is split-generated by E and $H^*(hom_{\mathcal{A}}(W,W),\mu^1) \neq 0$. Then $H^*(hom_{\mathcal{A}}(E,W),\mu^1) \neq 0$. In particular, if $(L,\mathcal{E})$ and $(K,\mathcal{W})$ are objects of $\mathcal{F}(M)$ and $(K,\mathcal{W})$ is split-generated by $(L,\mathcal{E})$, then $\overline{HF}^*(\mathcal{E},\mathcal{W}) \neq 0$ and hence the Lagrangians K and L cannot be displaced by a Hamiltonian isotopy.* $\square$

Abouzaid's criterion gives a sufficient condition for $(K,\mathcal{W})$ to be split-generated by $(L,\mathcal{E})$. As we saw in section 2.3.4, a necessary condition would be that both objects lie in the same summand $\mathcal{F}(M)_\lambda^{\mathrm{nil}}$. For technical reasons (in particular, the proof of Lemma 2.3.15 below) we require the stronger condition that both objects $(L,\mathcal{E})$ and $(K,\mathcal{W})$ are contained in the smaller subcategory $\mathcal{F}(M)_\lambda$. From now on, we impose this as a standing assumption. Note that in this case we have $\overline{CF}^*(\mathcal{E},\mathcal{E}) = CF^*(\mathcal{E},\mathcal{E})$, $\overline{CF}^*(\mathcal{W},\mathcal{W}) = CF^*(\mathcal{W},\mathcal{W})$ and $\overline{CF}^*(\mathcal{E},\mathcal{W}) = CF^*(\mathcal{E},\mathcal{W})$. Thus we drop the bars from the notation.

The version of the split-generation criterion we need is the following:

Theorem 2.3.13. *Let $(L,\mathcal{E})$ be an object of $\mathcal{F}(M)_\lambda$. If the map*

$$\mathcal{CO}_\lambda^* : QH^*(M)_\lambda \to HH^*(CF^*(\mathcal{E},\mathcal{E}))$$

is injective, then any other object $(K,\mathcal{W}) \in \mathcal{F}(M)_\lambda$ is split-generated by $(L,\mathcal{E})$.

This theorem is due to Abouzaid ([Abo10]) in the case of exact Lagrangians in an exact symplectic manifold and when $\mathcal{E}$ and $\mathcal{W}$ are trivial of rank 1. The case of a general symplectic manifold is work in progress by Abouzaid-Fukaya-Oh-Ohta-Ono [AFO$^+$]. Still in the exact case, the paper

[Abo12] proves a version in which $\mathcal{W}$ is allowed to be non-trivial and possibly of infinite rank. This last requirement is the cause of several algebraic complications which we avoid here. The proof for the monotone setting and with $\mathcal{E}$ and $\mathcal{W}$ of rank 1 (though possibly non-trivial) is treated in [She16, Section 2.11]. We include a sketch of that proof, modified to incorporate local systems of any finite rank. In our application to the Chiang Lagrangian we shall only use the split-generation criterion in the case when $\mathcal{E}$ is trivial of rank 1 (although $\mathcal{W}$ isn't) but for completeness we treat the slightly more general case here.

While the statement of Theorem 2.3.13 concerns only the closed-open string map, its proof relies on several other maps relating quantum cohomology of M with Hochshild invariants of the objects $(L,\mathcal{E})$ and $(K,\mathcal{W})$. More precisely, these are:

- the open-closed string map

$$\mathcal{OC}_*\colon HH_*(CF^*(\mathcal{E},\mathcal{E}))\to QH^*(M)$$

  from the Hochschild homology of the A_∞ algebra $CF^*(\mathcal{E},\mathcal{E})$ to quantum cohomology of the ambient manifold,

- the evaluation map

$$H(\mu)\colon HH_*(CF^*(\mathcal{E},\mathcal{E}),P_{\mathcal{W}}(\mathcal{E}))\to HF^*(\mathcal{W},\mathcal{W}),$$

  from Hochschild homology of $CF^*(\mathcal{E},\mathcal{E})$ with coefficients in the A_∞ bimodule $P_{\mathcal{W}}(\mathcal{E}) := CF^*(\mathcal{E},\mathcal{W})\otimes CF^*(\mathcal{W},\mathcal{E})$ to the Floer cohomology of $(K,\mathcal{W})$,

- the coproduct map

$$HH_*(\Delta)\colon HH_*(CF^*(\mathcal{E},\mathcal{E}))\to HH_*(CF^*(\mathcal{E},\mathcal{E}),P_{\mathcal{W}}(\mathcal{E})).$$

In the following three sections we describe these maps and the objects they relate.

2.3.5.1 Hochschild homology and the open-closed string map

For any A_∞ bimodule $\mathcal{N}$ over the A_∞ algebra $(CF^*(\mathcal{E},\mathcal{E}),\mu^*)$ there is a Hochschild homology group $HH_*(CF^*(\mathcal{E},\mathcal{E}),\mathcal{N})$. It is the homology of the complex

$$CC_*(CF^*(\mathcal{E},\mathcal{E}),\mathcal{N}) := \bigoplus_{d\geq 0}\mathcal{N}\otimes CF^*(\mathcal{E},\mathcal{E})^{\otimes d}$$

with respect to the A_∞ cyclic bar differential

$$\begin{aligned} b(\underline{n},\alpha_d,\dots,\alpha_1) &= \sum_{\substack{r\geq 0,s\geq 0\\ r+s\leq d}} \mu_{\mathcal{N}}^{r|1|s}(\alpha_r,\dots,\alpha_1,\underline{n},\alpha_d,\dots,\alpha_{d-s+1})\otimes\alpha_{d-s}\otimes\cdots\otimes\alpha_{r+1}\\ &+ \sum_{\substack{i\geq 0,j\geq 1\\ i+j\leq d}} \underline{n}\otimes\alpha_d\otimes\cdots\otimes\alpha_{i+j+1}\otimes\mu^j(\alpha_{i+j},\dots,\alpha_{i+1})\otimes\alpha_i\otimes\cdots\otimes\alpha_1, \end{aligned}$$

where $\mu_{\mathcal{N}}^{\cdot|1|\cdot}$ denote the bimodule structure maps for $\mathcal{N}$. Substituting $\mathcal{N} = CF^*(\mathcal{E},\mathcal{E})$ one obtains the group $HH_*(CF^*(\mathcal{E},\mathcal{E}))$, which is the source of the open-closed string map $\mathcal{OC}_*$.

In the case of rank 1 local systems, this map has been heavily studied by many authors ([FOOO09], [Abo10], [Gan12], [She16], [RS17] etc.). To incorporate local systems of higher finite rank one needs to algebraically modify the construction using a trace map. We now give a brief description of how the construction works.

Following [She16, Section 2.6], we define the open-closed string map in terms of a pairing

$$(\mathcal{OC}_*(-),-)\colon HH_*(CF^*(\mathcal{E},\mathcal{E}))\otimes H_*(M;\mathbb{F}) \to \mathbb{F}. \tag{2.37}$$

Given a generator

$$\begin{aligned}\underline{\alpha}\otimes\alpha_d\otimes\cdots\otimes\alpha_1 \quad &\in \quad \mathrm{Hom}_{\mathbb{F}}(\mathcal{E}_{\underline{x}(0)},\mathcal{E}_{\underline{x}(1)})\otimes\mathrm{Hom}_{\mathbb{F}}(\mathcal{E}_{x_d(0)},\mathcal{E}_{x_d(1)})\otimes\cdots\otimes\mathrm{Hom}_{\mathbb{F}}(\mathcal{E}_{x_1(0)},\mathcal{E}_{x_1(1)})\\ &\leq \quad CC_*(CF^*(\mathcal{E},\mathcal{E}))\end{aligned}$$

and a pseudocycle f, representing a homology class a, we consider the moduli space $\mathcal{R}_{0:d+1;1}(\underline{x},\vec{x};f)$, consisting of perturbed pseudoholomorphic discs asymptotic to $\underline{x}$ and $\vec{x} := (x_1,\ldots,x_d)$ at the boundary punctures and mapping the boundary to L and the internal marked point to $im(f)$. We define

$$(\mathcal{OC}_*(\underline{\alpha}\otimes\alpha_d\otimes\cdots\otimes\alpha_1),a;f) := \sum_{u\in\mathcal{R}^0_{0,d+1;1}(\underline{x},\vec{x};f)} \mathrm{tr}(P_{\gamma_u^d}\circ\alpha_d\circ P_{\gamma_u^{d-1}}\circ\alpha_{d-1}\circ\cdots\circ P_{\gamma_u^1}\circ\alpha_1\circ P_{\gamma_u^0}\circ\underline{\alpha}),$$

where on the right hand side one takes the trace of the element in brackets which is an endomorphism of $\mathcal{E}_{\underline{x}(0)}$. For index reasons, the boundary of the Gromov compactification of the 1-dimensional component $\mathcal{R}^1_{0,d+1;1}(\underline{x},\vec{x};f)$ consists only of strip breakings at the incoming punctures and configurations of pairs of discs, one of which carries the internal marked point and the other carries at least two punctures. With the correct choice of perturbation data, these are precisely the moduli spaces contributing to the composition

$$CC_*(CF^*(\mathcal{E},\mathcal{E})) \xrightarrow{\quad b \quad} CC_*(CF^*(\mathcal{E},\mathcal{E})) \xrightarrow{(\mathcal{OC}_*(-),a;f)} \mathbb{F}\,. \tag{2.38}$$

We claim that this implies $(\mathcal{OC}_*(b(\underline{\alpha}\otimes\alpha_d\otimes\cdots\otimes\alpha_1)),a;f)=0$. Let us illustrate this by an example. Suppose that $d=4$ and the two broken configurations in Figure 2.2 appear as opposite boundary points of a connected component of $\overline{\mathcal{R}}^1_{0:5;1}(\underline{x},x_1,x_2,x_3,x_4;f)$.

Their contributions to the composition (2.38) are given by:

$$\mathrm{tr}\left(P_{\gamma_u^2}\circ\alpha_3\circ P_{\gamma_u^1}\circ\alpha_2\circ P_{\gamma_u^0}\circ\left(P_{\gamma_v^3}\circ\alpha_1\circ P_{\gamma_v^2}\circ\underline{\alpha}\circ P_{\gamma_v^1}\circ\alpha_4\circ P_{\gamma_v^0}\right)\right)$$

and

$$\mathrm{tr}\left(P_{\gamma_{u'}^3}\circ\left(P_{\gamma_{v'}^2}\circ\alpha_4\circ P_{\gamma_{v'}^1}\circ\alpha_3\circ P_{\gamma_{v'}^0}\right)\circ P_{\gamma_{u'}^2}\circ\alpha_2\circ P_{\gamma_{u'}^1}\circ\alpha_1\circ P_{\gamma_{u'}^0}\circ\underline{\alpha}\right).$$

Since there is a 1-parameter family of glued curves interpolating between the two broken configurations, we have that for every $0\leq j\leq 4$ the two paths connecting $x_j(1)$ to $x_{j+1}(0)$ (where $x_0 = x_5 = \underline{x}$)

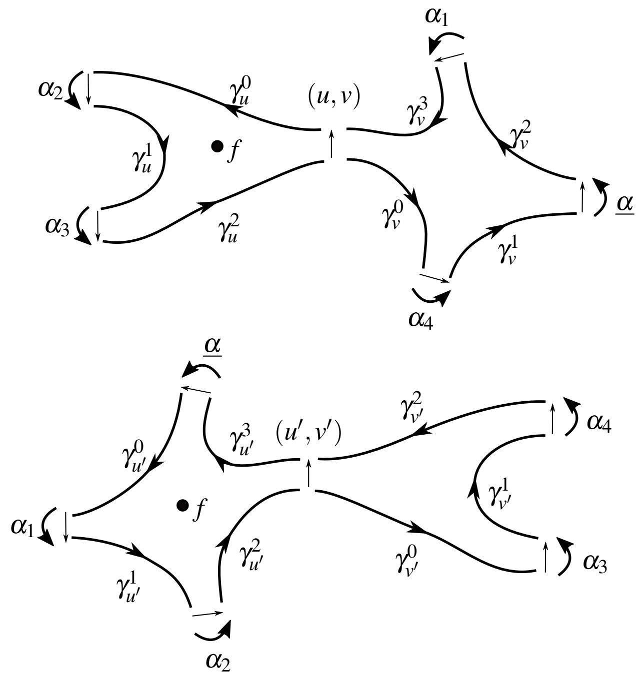


Figure 2.2: Evaluating $\mathcal{OC}_*$ on a Hochschild boundary

arising from (u,v) and (u',v') are homotopic. In particular $\gamma_v^3 \cdot \gamma_u^0 = \gamma_{u'}^1 \in \Pi_1 L(x_1(1), x_2(0))$, $\gamma_u^2 \cdot \gamma_v^0 = \gamma_{v'}^1 \in \Pi_1 L(x_3(1), x_4(0))$, $\gamma_v^1 = \gamma_{v'}^2 \cdot \gamma_{u'}^3 \in \Pi_1 L(x_4(1), \underline{x}(0))$ and $\gamma_u^1 = \gamma_{u'}^2 \cdot \gamma_{v'}^0 \in \Pi_1 L(x_2(1), x_3(0))$. Using this we see that the two expressions of which we are taking the trace are cyclic permutations of compositions of the same maps and hence the traces agree. Since all broken configurations contributing to (2.38) come in such pairs, we conclude that the composition vanishes altogether.

On the other hand, given a Hochschild chain φ and two pseudocycles f, g representing a, then by considering moduli spaces of discs with asymptotics determined by φ and which map the internal marked point to a homology between f and g one can show (see [She16, Section 2.6]) that $(\mathcal{OC}_*(\varphi), a; f) + (\mathcal{OC}_*(\varphi), a; g)$ depends only on $b(\varphi)$ and so vanishes when φ is a Hochschild cycle. One thus obtains a well defined pairing (2.37) which defines the map $\mathcal{OC}_*$.

2.3.5.2 The bimodule $P_{\mathcal{W}}(\mathcal{E})$ and the evaluation map $H(\mu)$

Let us consider for a moment a purely algebraic setup. Let $\mathcal{A}$ be an A_∞ category and let E be an object of $\mathcal{A}$. Then for every object W one can consider the space $P_W(E) := hom_{\mathcal{A}}(E,W) \otimes hom_{\mathcal{A}}(W,E)$ which is an A_∞ bimodule over $hom_A(E,E)$ with structure maps

$$\begin{aligned} \mu^{r|1|0} \colon hom_{\mathcal{A}}(E,E)^{\otimes r} \otimes P_W(E) &\to P_W(E) \\ \mu^{r|1|0}(\alpha_r, \dots, \alpha_1, f \otimes g) &= f \otimes \mu^{r+1}(\alpha_r, \dots, \alpha_1, g), \end{aligned}$$

$$\begin{aligned} \mu^{0|1|s} \colon P_W(E) \otimes hom_{\mathcal{A}}(E,E)^{\otimes s} &\to P_W(E) \\ \mu^{0|1|s}(f \otimes g, \alpha_{|1}, \dots, \alpha_{|s}) &= \mu^{s+1}(f, \alpha_{|1}, \dots, \alpha_{|s}) \otimes g \end{aligned}$$

and $\mu^{r|1|s} = 0$ for $r \neq 0 \neq s$. Thus one has a Hochschild homology group $HH_*(hom_{\mathcal{A}}(E,E), P_W(E))$. There is a natural evaluation map:

$$H(\mu)\colon HH_*(hom_{\mathcal{A}}(E,E), P_W(E)) \to H^*(hom_{\mathcal{A}}(W,W), \mu^1),$$

induced on the chain level by the map:

$$\begin{aligned} C(\mu)\colon CC_*(hom_{\mathcal{A}}(E,E), P_W(E)) &\to hom_{\mathcal{A}}(W,W) \\ C(\mu)\colon (f \otimes g) \otimes \alpha_d \otimes \cdots \otimes \alpha_1 &\mapsto \mu^{d+2}(f, \alpha_d, \ldots, \alpha_1, g). \end{aligned}$$

In this setting one has the following lemma of Abouzaid:

Lemma 2.3.14. *([Abo10, Lemma 1.4]) Let $\mathcal{A}$ be a cohomologically unital A_∞ category and E, W be objects in $\mathcal{A}$. If the unit $e_W \in H^*(hom_{\mathcal{A}}(W,W), \mu^1)$ lies in the image of the evaluation map $H(\mu)$, then W is split-generated by E.* □

Let us now specialise to the case where $\mathcal{A}$ is the category $\mathcal{F}(M)_\lambda$ from section 2.3.1 above. The bimodule is then $P_{\mathcal{W}}(\mathcal{E}) = P_{(K,\mathcal{W})}(L,\mathcal{E}) = CF^*(\mathcal{E},\mathcal{W}) \otimes CF^*(\mathcal{W},\mathcal{E})$. Note that this can be rewritten as

$$\begin{aligned} CF^*(\mathcal{E},\mathcal{W}) \otimes CF^*(\mathcal{W},\mathcal{E}) &= \left(\bigoplus_{y \in \mathcal{X}(L,K)} \mathrm{Hom}(\mathcal{E}_{y(0)}, \mathcal{W}_{y(1)}) \right) \otimes \left(\bigoplus_{z \in \mathcal{X}(K,L)} \mathrm{Hom}(\mathcal{W}_{z(0)}, \mathcal{E}_{z(1)}) \right) \\ &= \bigoplus_{\substack{y \in \mathcal{X}(L,K) \\ z \in \mathcal{X}(K,L)}} \mathrm{Hom}(\mathcal{E}_{y(0)}, \mathcal{W}_{y(1)}) \otimes \mathrm{Hom}(\mathcal{W}_{z(0)}, \mathcal{E}_{z(1)}) \\ &= \bigoplus_{\substack{y \in \mathcal{X}(L,K) \\ z \in \mathcal{X}(K,L)}} \mathcal{E}^\vee_{y(0)} \otimes \mathcal{W}_{y(1)} \otimes \mathcal{W}^\vee_{z(0)} \otimes \mathcal{E}_{z(1)} \\ &= \bigoplus_{\substack{y \in \mathcal{X}(L,K) \\ z \in \mathcal{X}(K,L)}} \mathrm{Hom}(\mathcal{W}_{z(0)}, \mathcal{W}_{y(1)}) \otimes \mathrm{Hom}(\mathcal{E}_{y(0)}, \mathcal{E}_{z(1)}), \end{aligned}$$

where we have crucially used the fact that $\mathcal{E}$ and $\mathcal{W}$ have finite rank. So one can write the elements of the components of $P_{\mathcal{W}}(\mathcal{E})$ as linear combinations of terms of one of the following two kinds:

- $\hat{f}_y \otimes \hat{g}_z \in \mathrm{Hom}(\mathcal{E}_{y(0)}, \mathcal{W}_{y(1)}) \otimes \mathrm{Hom}(\mathcal{W}_{z(0)}, \mathcal{E}_{z(1)})$
- $f_{zy} \otimes g_{yz} \in \mathrm{Hom}(\mathcal{W}_{z(0)}, \mathcal{W}_{y(1)}) \otimes \mathrm{Hom}(\mathcal{E}_{y(0)}, \mathcal{E}_{z(1)})$.

We will find the second description more useful. We then need an expression for the output of the evaluation map $C(\mu)$, when it is applied to elements of the form $f_{zy} \otimes g_{yz}$.

Lemma 2.3.15. *For elements $f_{zy} \otimes g_{yz} \in \mathrm{Hom}(\mathcal{W}_{z(0)}, \mathcal{W}_{y(1)}) \otimes \mathrm{Hom}(\mathcal{E}_{y(0)}, \mathcal{E}_{z(1)}) \leq P_{\mathcal{W}}(\mathcal{E})$ and $\alpha_d \otimes \cdots \otimes \alpha_1 \in \mathrm{Hom}(\mathcal{E}_{x_d(0)}, \mathcal{E}_{x_d(1)}) \otimes \cdots \otimes \mathrm{Hom}(\mathcal{E}_{x_1(0)}, \mathcal{E}_{x_1(1)}) \leq CF^*(\mathcal{E},\mathcal{E})^{\otimes d}$, the evaluation map $C(\mu)$ is given by*

$$C(\mu)((f_{zy} \otimes g_{yz}) \otimes \alpha_d \otimes \cdots \otimes \alpha_1) = \tag{2.39}$$

$$\sum_{w\in\mathcal{X}(K,K)}\sum_{u\in\mathcal{R}^0_{1:d+2}(w:z,x_1,\dots,x_d,y)} \mathrm{tr}(P_{\gamma_u^{d+1}}\circ\alpha_d\cdots\circ P_{\gamma_u^2}\circ\alpha_1\circ P_{\gamma_u^1}\circ g_{yz})P_{\gamma_u^{d+2}}\circ f_{zy}\circ P_{\gamma_u^0},$$

where one takes the trace of the element in brackets which is an endomorphism of $\mathcal{E}_{y(0)}$.

Proof. Note that the contribution of every disc $u\in\mathcal{R}^0_{1:d+2}(w:z,x_1,\dots,x_d,y)$ to

$$C(\mu)((\hat{f}_y\otimes\hat{g}_z)\otimes\alpha_d\otimes\cdots\otimes\alpha_1)=\mu^{d+2}(\hat{f}_y,\alpha_d,\dots,\alpha_1,\hat{g}_z)$$

is obtained by applying the composition map:

$$\mathrm{Hom}(\mathcal{E}_{y(0)},\mathcal{W}_{y(1)})\otimes\mathrm{Hom}(\mathcal{E}_{z(1)},\mathcal{E}_{y(0)})\otimes\mathrm{Hom}(\mathcal{W}_{z(0)},\mathcal{E}_{z(1)})\xrightarrow{-\circ-\circ-}\mathrm{Hom}(\mathcal{W}_{z(0)},\mathcal{W}_{y(1)}) \tag{2.40}$$

to the element $\hat{f}_y\otimes T\otimes\hat{g}_z$, where $T=P_{\gamma_u^{d+1}}\circ\alpha_d\circ\cdots\circ\alpha_1\circ P_{\gamma_u^1}$. Using again that our local systems have finite ranks, we have

$$\begin{aligned}&\mathrm{Hom}(\mathcal{E}_{y(0)},\mathcal{W}_{y(1)})\otimes\mathrm{Hom}(\mathcal{E}_{z(1)},\mathcal{E}_{y(0)})\otimes\mathrm{Hom}(\mathcal{W}_{z(0)},\mathcal{E}_{z(1)})\\ =\;&\mathcal{E}^\vee_{y(0)}\otimes\mathcal{W}_{y(1)}\otimes\mathcal{E}^\vee_{z(1)}\otimes\mathcal{E}_{y(0)}\otimes\mathcal{W}^\vee_{z(0)}\otimes\mathcal{E}_{z(1)}\\ =\;&\mathrm{Hom}(\mathcal{E}_{z(1)},\mathcal{E}_{y(0)})\otimes\mathrm{Hom}(\mathcal{E}_{y(0)},\mathcal{E}_{z(1)})\otimes\mathrm{Hom}(\mathcal{W}_{z(0)},\mathcal{W}_{y(1)}).\end{aligned}$$

We then see that the composition map (2.40) coincides with the map

$$\begin{aligned}\mathrm{Hom}(\mathcal{E}_{z(1)},\mathcal{E}_{y(0)})\otimes\mathrm{Hom}(\mathcal{E}_{y(0)},\mathcal{E}_{z(1)})\otimes\mathrm{Hom}(\mathcal{W}_{z(0)},\mathcal{W}_{y(1)})&\to\mathrm{Hom}(\mathcal{W}_{z(0)},\mathcal{W}_{y(1)})\\ T\otimes g_{yz}\otimes f_{zy}&\mapsto\mathrm{tr}(T\circ g_{yz})f_{zy},\end{aligned} \tag{2.41}$$

as both are given by performing all possible contractions of dual tensor factors in the product

$$\mathcal{E}^\vee_{y(0)}\otimes\mathcal{W}_{y(1)}\otimes\mathcal{E}^\vee_{z(1)}\otimes\mathcal{E}_{y(0)}\otimes\mathcal{W}^\vee_{z(0)}\otimes\mathcal{E}_{z(1)}.$$ □

2.3.5.3 The coproduct map Δ

Following [Abo10, Section 3.3 and 4.2], [Abo12, Section 5.1], [She16, Section 2.11], we relate $CF^*(\mathcal{E},\mathcal{E})$ to the bimodule $P_{\mathcal{W}}(\mathcal{E})$ via an A_∞ bimodule homomorphism obtained from counts of pseudoholomorphic discs with two outgoing boundary punctures. More precisely, one defines a coproduct map

$$\Delta\colon CF^*(\mathcal{E},\mathcal{E})\to P_{\mathcal{W}}(\mathcal{E})$$

as follows. Consider holomorphic discs with two negative boundary punctures ζ_{01}, ζ_{02} and one positive ζ_1, appearing in this cyclic order counterclockwise around the boundary of the disc. For every choice of Hamiltonian chords $x\in\mathcal{X}(L,L)$, $y\in\mathcal{X}(L,K)$ and $z\in\mathcal{X}(K,L)$, one has the moduli space $\mathcal{R}_{2:1}(z,y:x)$ of perturbed pseudoholomorphic discs u which are asymptotic at ζ_{01}, ζ_{02} and ζ_1 to z, y and x, respectively, and which map the boundary arc between ζ_{01} and ζ_{02} to K and the remaining two arcs to L. Every map $u\in\mathcal{R}_{2:1}(z,y:x)$ defines paths $\gamma_u^0\in\Pi_1K(z(0),y(1))$, $\gamma_u^1\in\Pi_1L(y(0),x(0))$, $\gamma_u^1\in\Pi_1L(x(1),z(1))$ which are the images of the boundary arcs connecting ζ_{01} to ζ_{02}, ζ_{02} to ζ_1 and ζ_1 to ζ_{01}, respectively. The map Δ is then defined by setting for every $\alpha\in\mathrm{Hom}(\mathcal{E}_{x(0)},\mathcal{E}_{x(1)})$

$$\Delta(\alpha)=\sum_{\substack{y\in\mathcal{X}(L,K)\\ z\in\mathcal{X}(K,L)}}\sum_{u\in\mathcal{R}^0_{2:1}(z,y:x)}P_{\gamma_u^0}\otimes(P_{\gamma_u^2}\circ\alpha\circ P_{\gamma_u^1})$$

with $P_{\gamma_u^0} \otimes (P_{\gamma_u^2} \circ \alpha \circ P_{\gamma_u^1}) \in \mathrm{Hom}(\mathcal{W}_{z(0)}, \mathcal{W}_{y(1)}) \otimes \mathrm{Hom}(\mathcal{E}_{y(0)}, \mathcal{E}_{z(1)}) \leq P_{\mathcal{W}}(\mathcal{E})$.

One can now extend the map Δ to a homomorphism of A_∞ bimodules. That is, for every $r \geq 0$, $s \geq 0$ one defines an operation

$$\Delta^{r|1|s} \colon CF^*(\mathcal{E},\mathcal{E})^{\otimes r} \otimes CF^*(\mathcal{E},\mathcal{E}) \otimes CF^*(\mathcal{E},\mathcal{E})^{\otimes s} \to P_{\mathcal{W}}(\mathcal{E})$$

by considering discs with two negative punctures and $r+1+s$ positive ones. Given chords $\vec{x} = (x_1, \ldots, x_r)$, $\underline{x}$, $\vec{x}_| = (x_{|s}, \ldots, x_{|1})$, all connecting L to L, and elements $\alpha_i \in \mathrm{Hom}(\mathcal{E}_{x_i(0)}, \mathcal{E}_{x_i(1)})$, $\underline{\alpha} \in \mathrm{Hom}(\mathcal{E}_{\underline{x}(0)}, \mathcal{E}_{\underline{x}(1)})$, $\alpha_{|i} \in \mathrm{Hom}(\mathcal{E}_{x_{|i}(0)}, \mathcal{E}_{x_{|i}(1)})$ one sets

$$\Delta^{r|1|s}(\alpha_r, \ldots, \alpha_1, \underline{\alpha}, \alpha_{|1}, \ldots, \alpha_{|s}) =$$

$$\sum_{\substack{y \in \mathcal{X}(L,K) \\ z \in \mathcal{X}(K,L)}} \sum_{u \in \mathcal{R}^0_{2:r+1+s}(z,y:\vec{x}_|,\underline{x},\vec{x})} P_{\gamma_u^0} \otimes (P_{\gamma_u^{r+s+2}} \circ \alpha_r \circ P_{\gamma_u^{r+s+1}} \circ \cdots \circ P_{\gamma_u^{s+2}} \circ \underline{\alpha} \circ P_{\gamma_u^{s+1}} \circ \cdots \circ P_{\gamma_u^2} \circ \alpha_{|s} \circ P_{\gamma_u^1}),$$

where γ_u^0 is again the image of the arc between the two negative punctures, which is mapped to K and the other arcs are ordered counterclockwise around the boundary of the disc. Note that $\Delta^{0|1|0}$ is the initially defined coproduct map. The fact that Δ is indeed an A_∞ bimodule homomorphism (i.e. satisfies [Abo10, Equation (4.13)]) is verified again by considering the Gromov compactification of the one-dimensional component $\mathcal{R}^1_{2:r+1+s}(z,y:\vec{x}_|,\underline{x},\vec{x})$. It follows that Δ induces a map $HH_*(\Delta)$ in Hochschild homology. It is defined on the chain level by using all cyclic shifts of arguments of $\Delta^{\cdot|1|\cdot}$. That is, given an element $\underline{\alpha} \otimes \alpha_d \otimes \cdots \otimes \alpha_1 \in CC_*(CF^*(\mathcal{E},\mathcal{E}))$, one has

$$CC_*(\Delta)(\underline{\alpha} \otimes \alpha_d \otimes \cdots \otimes \alpha_1) = \sum_{r+s \leq d} \Delta^{r|1|s}(\alpha_r \otimes \cdots \otimes \alpha_1 \otimes \underline{\alpha} \otimes \alpha_d \otimes \cdots \otimes \alpha_{d-s+1}) \otimes \alpha_{d-s} \otimes \cdots \otimes \alpha_{r+1}.$$

2.3.5.4 Proof of the split-generation criterion

We are now in a position to give a sketch proof of Theorem 2.3.13. It follows from the following two facts:

Proposition 2.3.16. *(cf. [She16, Corollary 2.5, Proposition 2.6]) There exists a perfect pairing*

$$HH^*(CF^*(\mathcal{E},\mathcal{E})) \otimes HH_*(CF^*(\mathcal{E},\mathcal{E})) \to \mathbb{F}. \tag{2.42}$$

Further, the diagram

$$\begin{tikzcd} QH^*(M) \arrow[r, "\cong"] \arrow[d, "\mathcal{CO}^*"] & QH^*(M)^\vee \arrow[d, "\mathcal{OC}_*^\vee"] \\ HH^*(CF^*(\mathcal{E},\mathcal{E})) \arrow[r, "\cong"] & HH_*(CF^*(\mathcal{E},\mathcal{E}))^\vee \end{tikzcd} \tag{2.43}$$

commutes, where the top isomorphism is given by the Poincaré pairing and the bottom one comes from (2.42).

Proposition 2.3.17. *(cf. [Abo12, Proposition 4.1], [She16, Lemma 2.15]) The following diagram commutes:*

$$\begin{tikzcd} HH_*(CF^*(\mathcal{E},\mathcal{E})) \arrow[r, "HH_*(\Delta)"] \arrow[d, "\mathcal{OC}_*"] & HH_*(CF^*(\mathcal{E},\mathcal{E}),P_{\mathcal{W}}(\mathcal{E})) \arrow[d, "H(\mu)"] \\ QH^*(M) \arrow[r, "\mathcal{CO}^0"] & HF^*(\mathcal{W},\mathcal{W}). \end{tikzcd} \tag{2.44}$$

Assuming these facts we have:

Proof of Theorem 2.3.13. Since $QH^*(M)$ is a Frobenius algebra we have that $c_1(TM)\star$ is symmetric with respect to the Poincaré pairing $<,>$ and so its generalised eigenspaces are orthogonal. From Proposition 2.3.16 we thus have the commutative diagram

$$\begin{tikzcd} QH^*(M)_\lambda \arrow[r, "\cong"] \arrow[d, "\mathcal{CO}^*_\lambda"] & (QH^*(M)_\lambda)^\vee \arrow[d, "\mathcal{OC}^\vee_*"] \\ HH^*(CF^*(\mathcal{E},\mathcal{E})) \arrow[r, "\cong"] & HH_*(CF^*(\mathcal{E},\mathcal{E}))^\vee \end{tikzcd}.$$

Hence, if $\mathcal{CO}^*_\lambda$ is injective, then $\mathcal{OC}_*$ surjects onto $QH^*(M)_\lambda$ and in particular $1_\lambda \in \mathcal{OC}_*(HH_*(CF^*(\mathcal{E},\mathcal{E}))$. By Proposition 2.3.11 we know that $\mathcal{CO}^0(1_\lambda) = e_{\mathcal{W}}$ and so $e_{\mathcal{W}}$ lies in the image of $\mathcal{CO}^0 \circ \mathcal{OC}_*$. By Proposition 2.3.17 we then have that $e_{\mathcal{W}}$ lies in the image of $H(\mu)$ and applying Lemma 2.3.14 yields that $(K,\mathcal{W})$ is split-generated by $(L,\mathcal{E})$. $\square$

Proof of Proposition 2.3.16. The construction of the pairing (2.42) and the proof that it is perfect can be taken directly from [She16, Lemma 2.4 & Corollary 2.5]. The only extra input needed to deal with local systems of higher finite rank is a linear algebra argument, analogous to Lemma 2.3.15 above (the proof of [She16, Lemma 2.4] uses the coproduct map Δ; as seen above, the output of Δ lies in a slightly awkward tensor product of spaces of linear maps; one needs to rearrange the tensor factors to make this output more manageable). We omit the details of this proof here.

The fact that diagram (2.43) commutes is proved in [She16, Proposition 2.6]. $\square$

We now give a sketch proof of Proposition 2.3.17, following [She16, Section 2.11].

Proof of Proposition 2.3.17. Given Hamiltonian chords $\{\underline{x}, x_1, \ldots, x_d\} \in \mathcal{X}(L,L)$ and $w \in \mathcal{X}(K,K)$, consider the moduli space

$$\mathcal{D}(w : \underline{x}, x_1, \ldots, x_d) := \{(u,v) \in \mathcal{R}_{1:0;1}(w;M) \times \mathcal{R}_{0:d+1;1}(\underline{x}, x_1, \ldots, x_d; M) : \ \mathrm{ev}(u) = \mathrm{ev}(v)\},$$

which consists of pairs of discs, connected at an internal node and asymptotic to the prescribed chords at their boundary punctures. One can use the component $\mathcal{D}^0(w : \underline{x}, x_1, \ldots, x_d)$ of rigid such configurations to define a map:

$$\chi : CC_*(CF^*(\mathcal{E},\mathcal{E})) \to CF^*(\mathcal{W},\mathcal{W})$$

$$\chi(\underline{\alpha} \otimes \alpha_d \otimes \ldots \otimes \alpha_1) = \sum_{w \in \mathcal{X}(K,K)} \sum_{\substack{(u',v') \in \\ \mathcal{D}^0(w:\underline{x},x_1,\ldots,x_d)}} \mathrm{tr}\left(P_{\gamma^d_{v'}} \circ \alpha_d \circ \cdots \circ P_{\gamma^0_{v'}} \circ \underline{\alpha}\right) P_{\partial u'}. \tag{2.45}$$

By considering the boundary of the Gromov compactification of the one-dimensional component $\mathcal{D}^1(w:\underline{x},x_1,\dots,x_d)$, one shows that χ is a chain map. As a preparatory step for proving Proposition 2.3.17 one needs the following lemma:

Lemma 2.3.18. *Let $H(\chi)\colon HH_*(CF^*(\mathcal{E},\mathcal{E}))\to HF^*(\mathcal{W},\mathcal{W})$ denote the induced map on homology. Then $H(\chi)=\mathcal{CO}^0\circ\mathcal{OC}_*$.*

Proof. Let $\{e_1,\dots,e_m\}$ be a basis for $H_*(M;\mathbb{F})$ elements of pure degree and let $\{e^1,\dots,e^m\}\subseteq H^*(M;\mathbb{F})$ denote its dual basis. Further set $\varepsilon_i=\mathrm{PD}(e^i)$ and $\varepsilon^i=\mathrm{PD}(e_i)$. Choose pseudocycles f_i, g_i representing e_i and ε_i respectively. Then, given a Hochschild cycle $\varphi=\sum_j\lambda_j\,\underline{\alpha}_j\otimes\alpha_{jd}\otimes\dots\otimes\alpha_{j1}$, one has

$$\mathcal{CO}^0(\mathcal{OC}_*(\varphi))=\left[\sum_j\lambda_j\sigma\left(\underline{\alpha}_j\otimes\alpha_{jd}\otimes\dots\otimes\alpha_{j1};\{f_i\},\{g_i\}\right)\right],$$

where the square brackets denote the cohomology class in $HF^*(\mathcal{W},\mathcal{W})$ and

$$\sigma\left(\underline{\alpha}\otimes\alpha_d\otimes\dots\otimes\alpha_1;\{f_i\},\{g_i\}\right)\ :=\ \sum_{i=1}^m\langle\mathcal{OC}_*(\underline{\alpha}\otimes\alpha_d\otimes\dots\otimes\alpha_1),e_i;f_i\rangle\,\mathcal{CO}^0(e^i;g_i)$$

$$=\sum_{w\in\mathcal{X}(K,K)}\left(\sum_{\substack{(u,v)\in\\ \bigsqcup_{i=1}^m\mathcal{R}^0_{1:0;1}(w;g_i)\times\mathcal{R}^0_{0:d+1;1}(\underline{x},x_1,\dots,x_d;f_i)}}\mathrm{tr}(P_{\gamma_v^d}\circ\alpha_d\circ\dots\circ P_{\gamma_v^0}\circ\underline{\alpha})P_{\partial_u}\right).$$

Now, given Hamiltonian chords $\{\underline{x},x_1,\dots,x_d\}\in\mathcal{X}(L,L)$, $w\in\mathcal{X}(K,K)$ and a bordism $h\colon B\to M\times M$, realising a homology between $\sum_{i=1}^l e_i\times\varepsilon_i$ and the diagonal, consider the moduli space

$$\mathcal{H}(w:\underline{x},x_1,\dots,x_d;h):=\{(u,v)\in\mathcal{R}_{1:0;1}(w;M)\times\mathcal{R}_{0:d+1;1}(\underline{x},x_1,\dots,x_d;M)\ :\ (\mathrm{ev}(u),\mathrm{ev}(v))\in im(h)\}.$$

Then $\bigsqcup_{i=1}^m\mathcal{R}^0_{1:0;1}(w;g_i)\times\mathcal{R}^0_{0:d+1;1}(\underline{x},x_1,\dots,x_d;f_i)$ and the zero-dimensional component of discs connected at a node $\mathcal{D}^0(w:\underline{x},x_1,\dots,x_d)$ form part of the boundary of the Gromov compactification of the 1-dimensional component $\mathcal{H}^1(w:\underline{x},x_1,\dots,x_d;h)$. By analysing the remaining boundary components of this compactification and using again that the homotopy classes of the paths involved in parallel transport remain invariant in 1-parameter families, one finds that the sum

$$\left(\sum_{\substack{(u,v)\in\\ \bigsqcup_{i=1}^m\mathcal{R}^0_{1:0;1}(w;g_i)\times\mathcal{R}^0_{0:d+1;1}(\underline{x},x_1,\dots,x_d;f_i)}}\mathrm{tr}(P_{\gamma_v^d}\circ\alpha_d\circ\dots\circ P_{\gamma_v^0}\circ\underline{\alpha})P_{\partial_u}\right)$$

$$+\left(\sum_{\substack{(u',v')\in\\ \mathcal{D}^0(w:\underline{x},x_1,\dots,x_d)}}\mathrm{tr}(P_{\gamma_{v'}^d}\circ\alpha_d\circ\dots\circ P_{\gamma_{v'}^0}\circ\underline{\alpha})P_{\partial u'}\right)\quad\in\quad\mathrm{Hom}_{\mathbb{F}}(\mathcal{W}_{w(0)},\mathcal{W}_{w(1)})$$

depends linearly on $b(\underline{\alpha},\alpha_d,\dots,\alpha_1)$ up to a term which is the $\mathrm{Hom}_{\mathbb{F}}(\mathcal{W}_{w(0)},\mathcal{W}_{w(1)})$-component of a μ^1-exact element. $\square$

To prove Proposition 2.3.17, it remains to be shown that $H(\chi) = H(\mu) \circ HH_*(\Delta)$. This is implied by the following lemma.

Lemma 2.3.19. *The maps χ and $C(\mu) \circ CC_*(\Delta)$ are chain-homotopic.*

Proof. Following [Abo12, Section 5.3], we construct such a homotopy by considering a moduli space of perturbed pseudoholomorphic maps, whose domain is an annulus $A_r = \{z \in \mathbb{C} :\ 1 \leq |z| \leq r\}$ (for some r) with $d+1$ positive punctures $\{\underline{\zeta} = r, \zeta_1, \ldots, \zeta_d\}$ on the outer circle and one negative puncture on the inner circle, constrained to lie at -1. Given chords $\{\underline{x}, x_1, \ldots, x_d\} \in \mathcal{X}(L,L)$ and $w \in \mathcal{X}(K,K)$, we denote by $\mathcal{C}^{-}_{1:d+1}(w : \underline{x}, x_1, \ldots, x_d)$ the moduli space of maps as above, which are furthermore required to map the boundary component $\{z \in A_r :\ |z| = 1\}$ to K, the remaining boundary components $\{z \in A_r :\ |z| = r\}$ to L and which are asymptotic to w at -1 and to $\{\underline{x}, x_1, \ldots, x_d\}$ at $\{\underline{\zeta} = r, \zeta_1, \ldots, \zeta_d\}$. The boundary of the Gromov compactification of the one-dimensional component $\mathcal{C}^{-,1}_{1:d+1}(w : \underline{x}, x_1, \ldots, x_d)$ consist of the following four types of configurations (see [Abo12, Equations (5.18), (5.19), (5.20)]):

1. a strip breaking at the outgoing puncture; connected components of this stratum are given by products
$$\mathcal{R}^0_{1:1}(w : w') \times \mathcal{C}^{-,0}_{1:d+1}(w' : \underline{x}, x_1, \ldots, x_d)$$
for some $w' \in \mathcal{X}(K,K)$.

2. a strip or a stable disc component (i.e. a disc carrying at least two punctures) breaking off at a positive puncture; connected components of this stratum are given by products
$$\mathcal{C}^{-,0}_{1:d-s-r+1}(w : \underline{x}', x_{r+1}, \ldots, x_{d-s}) \times \mathcal{R}^0_{1:r+s+1}(\underline{x}' : x_{d-s+1}, \ldots, x_d, \underline{x}, x_1, \ldots, x_r)$$
for some $\underline{x}' \in \mathcal{X}(L,L)$ and
$$\mathcal{C}^{-,0}_{1:d-j+2}(w : \underline{x}, x_1, \ldots, x_i, x', x_{i+j+1}, \ldots, x_d) \times \mathcal{R}^0_{1:j}(x' : x_{i+1}, \ldots, x_{i+j})$$
for some $x' \in \mathcal{X}(L,L)$.

3. a degeneration of the conformal modulus of the annulus as $r \to 1$; components of the boundary at $r = 1$ are given by products
$$\mathcal{R}^0_{1:d-r-s+2}(w : z, x_{r+1}, \ldots, x_{d-s}, y) \times \mathcal{R}^0_{2:r+s+1}(z, y : x_{d-s+1}, \ldots, x_d, \underline{x}, x_1, \ldots, x_r)$$
for some $y \in \mathcal{X}(L,K)$, $z \in \mathcal{X}(K,L)$.

4. a degeneration of the conformal modulus of the annulus as $r \to +\infty$; the boundary at $r = +\infty$ is the moduli space $\mathcal{D}^0(w : \underline{x}, x_1, \ldots, x_d)$ of pairs of discs, connected at a node.

Observe that the degenerations of types 3 and 4 are precisely the ones which account for the $\mathrm{Hom}_{\mathbb{F}}(\mathcal{W}_{w(0)},\mathcal{W}_{w(1)})$-component of $C(\mu)\circ CC_*(\Delta)(\underline{\alpha}\otimes\alpha_d\otimes\cdots\otimes\alpha_1)$ and $\chi(\underline{\alpha}\otimes\alpha_d\otimes\cdots\otimes\alpha_1)$, respectively. Further, from the description of $C(\mu)$ in Lemma 2.3.15 one can see that both $\chi(\underline{\alpha}\otimes\alpha_k\otimes\ldots\otimes\alpha_1)$ and $C(\mu)\circ CC_*(\Delta)(\underline{\alpha}\otimes\alpha_k\otimes\ldots\otimes\alpha_1)$ weight the parallel transport map along the boundary component mapping to K by the trace of the the loop of linear maps, obtained by composing the elements α_i with the parallel transport along the boundary components mapped to L. On the other hand, each $a\in\mathcal{C}^{-}_{d+1}(w:\underline{x},x_1,\ldots,x_d)$ defines paths $\gamma^j_a\in\Pi_1L(x_j(1),x_{j+1}(0))$, $0\leq j\leq d$, which are the images of the boundary arcs connecting ζ_j to ζ_{j+1} (again the notation means $\zeta_0=\zeta_{d+1}=\underline{\zeta}$ and $x_0=x_{d+1}=\underline{x}$) and $\gamma_a\in\Pi_1K(w(0),w(1))$, which is the image of the inner boundary circle, oriented clockwise. We then define a map

$$h\colon CC_*(CF^*(\mathcal{E},\mathcal{E}))\to CF^*(\mathcal{W},\mathcal{W})$$

$$h(\underline{\alpha}\otimes\alpha_d\otimes\cdots\otimes\alpha_1)=\sum_{w\in\mathcal{X}(K,K)}\sum_{a\in\mathcal{C}^{-,0}_{d+1}(w:\underline{x},x_1,\ldots,x_d)}\mathrm{tr}(P_{\gamma^d_a}\circ\alpha_d\circ\cdots\circ P_{\gamma^0_a}\circ\underline{\alpha})P_{\gamma_a}.$$

This is analogous to [Abo12, Equation (5.22)], except that we weight the parallel transport on K by the trace of the loop on L. Looking at the remaining types of boundary components of the compactification of $\mathcal{C}^{-,1}_{1:d+1}(w:\underline{x},x_1,\ldots,x_d)$, we see that the degenerations of types 1 and 2 account for the $\mathrm{Hom}(\mathcal{W}_{w(0)},\mathcal{W}_{w(1)})$-component of $\mu^1(h(\underline{\alpha}\otimes\alpha_d\otimes\cdots\otimes\alpha_1))$ and $h(b(\underline{\alpha}\otimes\alpha_d\otimes\cdots\otimes\alpha_1))$, respectively. Using again that all these terms are paired-off as boundary points of closed intervals we conclude that $C(\mu)\circ CC_*(\Delta)+\chi+\mu^1\circ h+h\circ b=0$, i.e. h is a chain-homotopy between χ and $C(\mu)\circ CC_*(\Delta)$. □

2.4 The pearl complex

We now recall an alternative approach to calculating self-Floer cohomology of a single monotone Lagrangian, namely Biran and Cornea's pearl complex (see [BC09a] for an extensive account of this theory or [BC07b] for the full details). This is precisely the machinery which we use in all subsequent chapters in order to compute Floer cohomology. It provides a much nicer setting for doing so, because it does not require the introduction of Hamiltonian perturbations or time-dependent complex structures. In this section we explain how to adapt Biran and Cornea's theory in order to incorporate local coefficients.

2.4.1 Definition and obstruction

Let $L\subseteq(M,\omega)$ be a closed monotone Lagrangian submanifold whose minimal Maslov number satisfies $N^{\pi}_L\geq 2$. Further, let L be equipped with a pair of $\mathbb{F}$-local systems $\mathcal{E}^0$, $\mathcal{E}^1$. Choose a Morse function $f\colon L\to\mathbb{R}$ and a Riemannian metric g, such that $\mathscr{D}:=(f,g)$ is a Morse-Smale pair. The underlying vector space of the pearl complex is basically just the Morse complex $C^*_f(L;\mathscr{H}om(\mathcal{E}^0,\mathcal{E}^1))$

(recall (2.6)) but the differential is deformed by contributions from pseudoholomorphic discs with boundary on L. We will see that this new differential meets the same obstruction to squaring to zero as the one on $CF^*(\mathcal{E}^0,\mathcal{E}^1)$ and that the two maximal unobstructed subcomplexes are homotopy equivalent. One difference between $C_f^*(L;\mathscr{H}om(\mathcal{E}^0,\mathcal{E}^1))$ and $CF^*(\mathcal{E}^0,\mathcal{E}^1)$ is that the former comes with a $\mathbb{Z}$-grading by Morse index and it will be convenient to keep track of it.[2] As we shall see, the deformed differential respects this grading (i.e. has degree 1) only modulo N_L^π, so to keep the grading absolute we follow Biran-Cornea and define:

Definition 2.4.1. The pearl pre-complex of the pair $(L,\mathcal{E}^0)$, $(L,\mathcal{E}^1)$ with respect to the Morse function f is

$$C_f^*(\mathcal{E}^0,\mathcal{E}^1) := C_f^*(L;\mathscr{H}om(\mathcal{E}^0,\mathcal{E}^1)) \otimes_{\mathbb{F}} \mathbb{F}[T^{\pm 1}],$$

where $\mathbb{F}[T^{\pm 1}]$ is the ring of Laurent polynomials in a formal variable T of degree N_L^π. ♢

Notation 2.4.2. We write $C_f^*(L,(L,\mathcal{E}^1))$ and $C_f^*((L,\mathcal{E}^0),L)$ in case $\mathcal{E}^0$ or $\mathcal{E}^1$ is trivial of rank one and $C_f^*(L,L;\mathbb{F})$ when both are. We also set $C_f^{r,s}(\mathcal{E}^0,\mathcal{E}^1) := C_f^s(L;\mathscr{H}om(\mathcal{E}^0,\mathcal{E}^1)) \cdot T^r$. Given an element $a = \sum_{y\in\mathrm{Crit}(f)} \sum_{r\in\mathbb{Z}} \alpha_{y,r} \otimes T^r \in C_f^*(\mathcal{E}^0,\mathcal{E}^1)$ and a critical point $x \in \mathrm{Crit}(f)$, we write

$$\langle a,x\rangle = \sum_{r\in\mathbb{Z}} \alpha_{x,r} \otimes T^r \in \mathrm{Hom}_{\mathbb{F}}(\mathcal{E}_x^0,\mathcal{E}_x^1) \otimes \mathbb{F}[T^{\pm 1}]$$

to denote the respective component. If we have local systems $\mathcal{V},\mathcal{W} \to L$ and morphisms of local systems $F \in \mathrm{Hom}(\mathcal{V},\mathcal{E}^0)$, $G \in \mathrm{Hom}(\mathcal{E}^1,\mathcal{W})$, we will write

$$a \circ F := \sum_{y\in\mathrm{Crit}(f)} \sum_{r\in\mathbb{Z}} (\alpha_{y,r} \circ F) \otimes T^r \in C_f^*(\mathcal{V},\mathcal{E}^1),$$

$$G \circ a := \sum_{y\in\mathrm{Crit}(f)} \sum_{r\in\mathbb{Z}} (G \circ \alpha_{y,r}) \otimes T^r \in C_f^*(\mathcal{E}^0,\mathcal{W}). \qquad /\!/$$

Remark 2.4.3. It is standard in the construction of algebraic invariants from holomorphic curves to work over a *Novikov ring* whose purpose is to record the areas of the curves that are being counted. In more general situations this is a necessity as otherwise the counts are not finite, but in the monotone case the use of $\mathbb{F}[T^{\pm 1}]$ is more of a convenience which allows us to keep track of gradings. In particular, we are free to set the variable T equal to 1 and thus obtain a complex which is only $\mathbb{Z}/N_L^\pi$-graded or indeed to forget the grading altogether. //

In order to define the appropriate candidate differential, one chooses a *time-independent*, ω-compatible almost complex structure $J \in \mathcal{J}(M,\omega)$. Then one considers the following moduli spaces of *pearly trajectories*.

Definition 2.4.4. For any pair of critical points $x,y \in \mathrm{Crit}(f)$ a parametrised pearly trajectory from y to x is defined to be a configuration $\mathbf{u} = (u_1,\dots,u_r)$ of J-holomorphic discs

$$u_\ell \colon (D^2,\partial D^2) \to (M,L), \quad du_\ell + J \circ du_\ell \circ \mathbf{i} = 0,$$

[2] There is also a natural way to put a corresponding grading on $CF^*(\mathcal{E}^0,\mathcal{E}^1)$ (see e.g. [BC07b, Section 5.6]) but it is less straightforward and we will not discuss it here.

such that if $\phi \colon \mathbb{R} \times L \to L$ denotes the negative gradient flow of f with respect to the metric g, then there exist elements $\{t_1, \ldots, t_{r-1}\} \subseteq (0,\infty)$ such that

1. $\lim_{t \to -\infty} \phi_t(u_1(-1)) = y$;
2. for all $1 \leq \ell \leq r-1$, $\phi_{t_\ell}(u_\ell(1)) = u_{\ell+1}(-1)$;
3. $\lim_{t \to +\infty} \phi_t(u_r(1)) = x$. $\diamondsuit$

The relevant moduli spaces now are:

- For any vector $\mathbf{A} = (A_1, \ldots, A_r) \in \left(H_2^D(M,L) \setminus \{0\}\right)^r$ we denote by $\widetilde{\mathcal{P}}(y,x,\mathbf{A};\mathscr{D},J)$ the set of all parametrised pearly trajectories $\mathbf{u} = (u_1, \ldots, u_r)$ such that $[u_i] = A_i$ for all $1 \leq i \leq r$;

- For any positive integer k we define

$$\widetilde{\mathcal{P}}(y,x,kN_L^\pi;\mathscr{D},J) := \bigcup_{\substack{\mathbf{A} \\ \mu(\mathbf{A}) = kN_L^\pi}} \widetilde{\mathcal{P}}(y,x,\mathbf{A};\mathscr{D},J),$$

  where the length r of the vector $\mathbf{A}$ is allowed to vary and $\mu(\mathbf{A}) := \sum_{i=1}^r I_{\mu_L}(A_i)$.

- We impose the following equivalence relation between such tuples of J–holomorphic discs: $\mathbf{u} = (u_1, u_2, \ldots, u_r) \sim \mathbf{u}' = (u_1', u_2', \ldots, u_{r'}')$ if and only if $r = r'$ and there exist elements $\varphi_\ell \in G_{-1,1} := \{g \in \mathrm{PSL}(2,\mathbb{R}) : \ g(-1) = -1, \ g(1) = 1\}$ such that $u_\ell \circ \varphi_\ell = u_\ell'$. We now set

$$\begin{aligned} \mathcal{P}(y,x,\mathbf{A};\mathscr{D},J) &:= \widetilde{\mathcal{P}}(y,x,\mathbf{A};\mathscr{D},J)/\sim \\ \mathcal{P}(y,x,kN_L;\mathscr{D},J) &:= \widetilde{\mathcal{P}}(y,x,kN_L^\pi;\mathscr{D},J)/\sim \end{aligned}$$

  These definitions extend naturally to the case when $\mathbf{A}$ is the empty vector, in which case one defines $\mathcal{P}(y,x,\emptyset;\mathscr{D},J) = \mathcal{P}(y,x,0;\mathscr{D},J)$ to be the space of unparametrised negative gradient trajectories of f connecting y to x.

- We also declare the following to be standing notation:

$$\delta(y,x,\mathbf{A}) := \operatorname{ind} y - \operatorname{ind} x + \mu(\mathbf{A}) - 1,$$

$$\delta(y,x,kN_L^\pi) := \operatorname{ind} y - \operatorname{ind} x + kN_L^\pi - 1.$$

These moduli spaces of pearly trajectories have natural descriptions as pre-images of certain submanifolds of products of L under suitable evaluation maps and are thus endowed with a topology. That is, given a vector $\mathbf{A} \neq \emptyset$ as above, one considers the map

$$\mathrm{ev}_{\mathbf{A}} \colon \mathcal{M}^{A_1}(L;J) \times \cdots \times \mathcal{M}^{A_r}(L;J) \to L^{2r},$$

$$\mathrm{ev}_{\mathbf{A}}(u_1, \ldots, u_r) := (u_1(-1), u_1(1), u_2(-1), u_2(1), \ldots, u_r(-1), u_r(1)).$$

Then, putting $Q := \{(x, \phi_t(x)) \in L \times L : \ t > 0, \ x \in L \setminus \mathrm{Crit}(f)\}$, we have that

$$\widetilde{\mathcal{P}}(y,x,\mathbf{A};\mathscr{D},J) = \mathrm{ev}_{\mathbf{A}}^{-1}(W^d(y) \times Q^{r-1} \times W^a(x)).$$

Note that from this description and our discussion about dimensions of moduli spaces of discs in section 2.2.1, it follows that the expected dimension of the space $\mathcal{P}(y,x,\mathbf{A};\mathscr{D},J)$ is exactly $\delta(y,x,\mathbf{A})$.

Following [BC07b], one can also use these descriptions to exhibit $\mathcal{P}(y,x,\mathbf{A};\mathscr{D},J)$ as a topological subspace of the much larger space $\mathcal{L}$, defined as follows. Let $\mathcal{P}_L$ denote the space of continuous paths $\{\gamma\colon [0,b] \to L : \ b \geq 0\}$ (with the compact-open topology) and let $\mathcal{P}_{\mathscr{D}} \subseteq \mathcal{P}_L$ denote the subspace consisting of paths γ which parametrise negative gradient flowlines of f in the unique way such that $f(\gamma(t)) = f(\gamma(0)) - t$. Then $\mathcal{P}(y,x,\mathbf{A};\mathscr{D},J)$ embeds continuously into the space

$$\mathcal{L} := \mathcal{P}_{\mathscr{D}} \times (\widetilde{\mathcal{M}}^{A_1}(L;J)/G_{-1,1}) \times \mathcal{P}_{\mathscr{D}} \times (\widetilde{\mathcal{M}}^{A_2}(L;J)/G_{-1,1}) \times \cdots \times (\widetilde{\mathcal{M}}^{A_r}(L;J)/G_{-1,1}) \times \mathcal{P}_{\mathscr{D}}.$$

Now let $\mathbf{u} = (u_1, u_2, \ldots, u_r) \in \mathcal{P}(y,x,\mathbf{A};\mathscr{D},J)$ be a pearly trajectory connecting y to x and let $(\tau_0, u_1, \tau_1, u_2, \ldots, \tau_{r-1}, u_r, \tau_r)$ be the corresponding element of $\mathcal{L}$. For any $1 \leq \ell \leq r$ and $j \in \{0,1\}$ define $\gamma^j_{u_\ell}\colon [0,1] \to L$, $\gamma^j_{u_\ell}(t) = \tilde{u}_\ell(e^{i\pi(j+t+1)})$ for some parametrisation $\tilde{u}_\ell \in \widetilde{\mathcal{M}}^{A_\ell}(L;J)$ of the disc u_ℓ. That is, $\gamma^0_{u_\ell}$ parametrises the image of the "bottom" half-circle, traversed counter clockwise, while $\gamma^1_{u_\ell}$ parametrises the "top" half-circle. Since we will only be interested in the homotopy classes of these paths, the particular choice of parametrisations of the discs are irrelevant. We now define the following two paths, which are the "bottom side" and "top side" of the pearly trajectory, respectively:

$$\begin{aligned} \gamma^0_{\mathbf{u}} &:= \tau_0 \cdot \gamma^0_{u_1} \cdot \tau_1 \cdots \gamma^0_{u_r} \cdot \tau_r \in \Pi_1 L(y,x) \\ \gamma^1_{\mathbf{u}} &:= \tau_r^{-1} \cdot \gamma^1_{u_r} \cdot \tau_{r-1}^{-1} \cdots \gamma^1_{u_1} \cdot \tau_1^{-1} \in \Pi_1 L(x,y). \end{aligned} \tag{2.46}$$

We then get corresponding parallel transport maps $P_{0,\gamma^0_{\mathbf{u}}}\colon \mathcal{E}^0_y \to \mathcal{E}^0_x$ and $P_{1,\gamma^1_{\mathbf{u}}}\colon \mathcal{E}^1_x \to \mathcal{E}^1_y$. Whenever we have $\mathcal{E}^0 = \mathcal{E}^1 = \mathcal{E}$ we will just write $P_{\gamma^j_{\mathbf{u}}} = P_{j,\gamma^j_{\mathbf{u}}}$ as before.

Remark 2.4.5. If one considers a function f which is small-enough in the C^1-norm, then the graph L_f of df can be assumed to lie in a Weinstein neighbourhood of L and so is a Hamiltonian deformation of L in M. Transverse intersection points of L with L_f correspond precisely to non-degenerate critical points of f. This point of view has been explored already by Floer who showed that, if f is in fact sufficiently small in the C^2-norm, then for a specific almost-complex structure J, there is a one-to-one correspondence between finite-energy J-holomorphic strips between L and L_f which do not leave the prescribed Weinstein neighbourhood and gradient flowlines of f with respect to the metric $g_J = \omega(\cdot, J\cdot)$. Moreover, this correspondence is given simply by $u \mapsto u(\cdot,0)$. That is, intuitively, low-energy strips can be "collapsed" to gradient flowlines on L. Extending this analogy, one can think of the pearly trajectories defined above as "collapsed" strips, where the "excess energy" which allows some strips to leave the Weinstein neighbourhood has concentrated in the J-holomorphic discs. From this intuitive point of view the paths defined in (2.46) are analogous to the ones we used in Definition 2.2.9. //

We wish to define a candidate differential on $C_f^*(\mathcal{E}^0,\mathcal{E}^1)$ by using parallel transport maps along the paths (2.46) corresponding to isolated pearly trajectories. The relevant theorem, guaranteeing that this is possible is the following.

Theorem 2.4.6. *([BC07b, Proposition 3.1.3]) For any Morse-Smale pair $\mathscr{D}$, there exists a Baire subset $\mathcal{J}_{\mathrm{reg}}(\mathscr{D}) \subseteq \mathcal{J}(M,\omega)$ such that for every $J \in \mathcal{J}_{\mathrm{reg}}(\mathscr{D})$ and every pair of points $x,y \in \mathrm{Crit}(f)$ the set $\mathcal{P}(y,x,kN_L^\pi;\mathscr{D},J)$ has naturally the structure of a smooth manifold of dimension $\delta(y,x,kN_L^\pi)$, whenever $\delta(y,x,kN_L^\pi) \leq 1$. Furthermore, when $\delta(y,x,kN_L^\pi) = 0$ the space $\mathcal{P}(y,x,kN_L^\pi;\mathscr{D},J)$ is compact and hence consists of a finite number of points.* $\square$

We can now define the candidate differential:

Definition 2.4.7. Fix a Morse-Smale pair $\mathscr{D}$ on L and an almost complex structure $J \in \mathcal{J}_{\mathrm{reg}}(\mathscr{D})$. We define a degree 1 map $d^{(\mathscr{D},J)}\colon C_f^*(\mathcal{E}^0,\mathcal{E}^1) \to C_f^*(\mathcal{E}^0,\mathcal{E}^1)$ to be the unique $\mathbb{F}[T^{\pm1}]$-linear map which satisfies the following: for every $x \in \mathrm{Crit}(f)$ and every $\alpha \in \mathrm{Hom}_{\mathbb{F}}(\mathcal{E}_x^0,\mathcal{E}_x^1)$,

$$d^{(\mathscr{D},J)}(\alpha) = \sum_{k\in\mathbb{N}} \sum_{\substack{y\in\mathrm{Crit}(f)\\ \delta(y,x,kN_L^\pi)=0}} \sum_{\mathbf{u}\in\mathcal{P}(y,x,kN_L^\pi;\mathscr{D},J)} (P_{1,\gamma_{\mathbf{u}}^1} \circ \alpha \circ P_{0,\gamma_{\mathbf{u}}^0}) \cdot T^k.$$

We write $\bar{d}^{(\mathscr{D},J)}\colon C_f^*(L;\mathscr{H}\!om(\mathcal{E}^0,\mathcal{E}^1)) \to C_f^*(L;\mathscr{H}\!om(\mathcal{E}^0,\mathcal{E}^1))$ for the map induced by $d^{(\mathscr{D},J)}$ after setting $T=1$. $\diamondsuit$

Propositions 5.1.2 and 5.6.2 in [BC07b] assert that (for a possibly smaller Baire subset of almost complex structures, still denoted $\mathcal{J}_{\mathrm{reg}}(\mathscr{D})$) the above map is a differential whenever the local systems $\mathcal{E}^0$ and $\mathcal{E}^1$ are assumed trivial of rank 1, and the resulting cohomology is canonically isomorphic to the Floer cohomology $HF^*(L,L)$ (after setting $T=1$ or equipping $HF^*(L,L)$ with a grading). With higher rank local systems, we don't necessarily have $\left(d^{(\mathscr{D},J)}\right)^2 = 0$ and we are thus forced to pass to the maximal unobstructed subcomplex $\overline{C}_f^*(\mathcal{E}^0,\mathcal{E}^1)$. Still, most of the content of [BC07b, Propositions 5.1.2 and 5.6.2] applies to our setting just as well and we summarise these results in the following theorem.

Theorem 2.4.8. *Let (M,ω) be a closed monotone symplectic manifold and let $L \subseteq M$ be a closed monotone Lagrangian submanifold with $N_L^\pi \geq 2$, equipped with a pair of $\mathbb{F}$-local systems $\mathcal{E}^0,\mathcal{E}^1$ and a Morse-Smale pair $\mathscr{D} = (f,g)$. Then there exists a Baire subset $\mathcal{J}_{\mathrm{reg}}(\mathscr{D}) \subseteq \mathcal{J}(M,\omega)$ such that for every $J \in \mathcal{J}_{\mathrm{reg}}(\mathscr{D})$:*

A) i) the map $d^{(\mathscr{D},J)}$ is well-defined;

ii) for every $a \in C_f^*(\mathcal{E}^0,\mathcal{E}^1)$ one has

$$\left(d^{(\mathscr{D},J)}\right)^2 a = (a \circ m_0(\mathcal{E}^0) - m_0(\mathcal{E}^1) \circ a) \cdot T. \tag{2.47}$$

In particular, if $N_L^\pi > 2$, then $\left(d^{(\mathscr{D},J)}\right)^2 = 0$.

B) Let $\mathscr{D} = (f,g)$ and $\mathscr{D}' = (f',g')$ be two Morse-Smale pairs for L and $J \in \mathcal{J}_{\text{reg}}(\mathscr{D})$, $J' \in \mathcal{J}_{\text{reg}}(\mathscr{D}')$ be regular almost complex structures. Then there exists a canonical up to homotopy map of pre-complexes

$$\Psi_{\mathscr{D},J}^{\mathscr{D}',J'} : \left(C_f^*(\mathcal{E}^0,\mathcal{E}^1), d^{(\mathscr{D},J)}\right) \to \left(C_{f'}^*(\mathcal{E}^0,\mathcal{E}^1), d^{(\mathscr{D}',J')}\right) \tag{2.48}$$

which induces a homotopy equivalence

$$\overline{\Psi}_{\mathscr{D},J}^{\mathscr{D}',J'} : \left(\overline{C}_f^*(\mathcal{E}^0,\mathcal{E}^1), d^{(\mathscr{D},J)}\right) \to \left(\overline{C}_{f'}^*(\mathcal{E}^0,\mathcal{E}^1), d^{(\mathscr{D}',J')}\right).$$

C) Let $\left(H,\hat{J}\right)$ be a regular Floer datum for L. Then there is a canonical up to homotopy map of pre-complexes

$$(\Psi_{\text{PSS}})_{\mathscr{D},J}^{H,\hat{J}} : \left(C_f^*(L;\mathscr{H}om(\mathcal{E}^0,\mathcal{E}^1)), \bar{d}^{(\mathscr{D},J)}\right) \longrightarrow CF^*\left((L,\mathcal{E}^0),(L,\mathcal{E}^1);H,\hat{J}\right), \tag{2.49}$$

inducing a homotopy equivalence of maximal unobstructed subcomplexes.

Notation 2.4.9. We will write $\overline{HF}{}_{\text{BC}}^*(\mathcal{E}^0,\mathcal{E}^1)$ for the cohomology $H^*(\overline{C}_f^*(\mathcal{E}^0,\mathcal{E}^1), d^{(\mathscr{D},J)})$, which is a $\mathbb{Z}$-graded $\mathbb{F}[T^{\pm 1}]$-module. If the obstruction (2.47) vanishes, we will drop the bar from the notation. In light of part C), we have an isomorphism

$$\overline{HF}{}_{\text{BC}}^*(\mathcal{E}^0,\mathcal{E}^1) \otimes_{\mathbb{F}[T^{\pm 1}]} \mathbb{F} \xrightarrow{\cong} \overline{HF}{}^*(\mathcal{E}^0,\mathcal{E}^1)$$

so will will not distinguish these notationally. //

Let us now give a sketch proof of Theorem 2.4.8, emphasising part ii) which is the only place where higher rank local systems make a difference.

Proof of part A) i): This is an immediate consequence of Theorem 2.4.6 above.

Proof of part A)ii): We need to show that for every $x \in \text{Crit}(f)$ and $\alpha \in \text{Hom}_{\mathbb{F}}(\mathcal{E}_x^0,\mathcal{E}_x^1)$ one has

$$\left(d^{(\mathscr{D},J)}\right)^2 \alpha = (\alpha \circ m_0(\mathcal{E}^0)(x) - m_0(\mathcal{E}^1)(x) \circ \alpha) \cdot T.$$

The proof of this fact relies on analysing the natural Gromov compactifications of the spaces $\mathcal{P}(y,x,\mathbf{A};\mathscr{D},J)$ when $\delta(y,x,\mathbf{A}) = 1$.

These compactifications are described in detail by Biran and Cornea in [BC07b, Lemma 5.1.3], where they also prove that $d^2 = 0$ in the case of trivial rank 1 local systems (we have dropped the decoration $(\mathscr{D},J)$ from the differential to alleviate notation). Generalising the same arguments to the case of arbitrary local systems yields that for any *distinct* $x,y \in \text{Crit}(f)$ and each element $\alpha \in \text{Hom}_{\mathbb{F}}(\mathcal{E}_x^0,\mathcal{E}_x^1) \otimes \mathbb{F}[T^{\pm 1}]$ one has $\langle d^2(\alpha), y\rangle = 0$.

However, some care needs to be taken when evaluating $\langle d^2(\alpha), x\rangle$. To that end we consider the space of twice marked discs $\widetilde{\mathcal{M}}^A(L;J)/G_{-1,1}$ for some $A \in \pi_2^{\text{f}}(M,L)$ with $I_{\mu_L}(A) = 2$. Since $N_L^\pi \geq 2$, the Gromov compactification $\overline{\widetilde{\mathcal{M}}^A(L;J)/G_{-1,1}}$ is obtained by adding equivalence classes of *stable maps* (see [Fra08, Definition 2.3]) with two components: one is a disc in the class A, while the other is a constant disc component and contains the two marked points. We distinguish these configurations into two types, depending on the cyclic order of the special points on the constant component. That is, with the marked points fixed at -1 and 1, we have (up to equivalence of stable maps) two possibilities for the nodal point: we write

$$\partial\left(\overline{\widetilde{\mathcal{M}}^A(L;J)/G_{-1,1}}\right) = \mathcal{D}^-(A) \cup \mathcal{D}^+(A),$$

where $\mathcal{D}^-(A)$ consists of equivalence classes with the nodal point of the constant component at $-\mathbf{i}$, while $\mathcal{D}^+(A)$ consists of the ones with the nodal point at $\mathbf{i}$. Since both marked points lie on the constant component, the extended evaluation map $\overline{\mathrm{ev}}_{(A)} : \overline{\widetilde{\mathcal{M}}^A(L;J)/G_{-1,1}} \to L^2$ maps $\mathcal{D}^-(A) \cup \mathcal{D}^+(A)$ to the diagonal. We shall write $\mathcal{D}^-(A,x) := \mathcal{D}^-(A) \cap \overline{\mathrm{ev}}_{(A)}^{-1}(x,x)$, $\mathcal{D}^+(A,x) := \mathcal{D}^+(A) \cap \overline{\mathrm{ev}}_{(A)}^{-1}(x,x)$ and $\mathcal{D}^{\mp}(A,x) := \mathcal{D}^-(A,x) \cup \mathcal{D}^+(A,x)$ for any point $x \in L$. Then, one has the following addendum to [BC07b, Lemma 5.1.3]:

Theorem 2.4.10. *There exists a Baire subset $\mathcal{J}_{\mathrm{reg}}(\mathscr{D}) \subseteq \mathcal{J}(M,\omega)$ such that for every $J \in \mathcal{J}_{\mathrm{reg}}(\mathscr{D})$ one has that for each $x \in \mathrm{Crit}(f)$ and $A \in \pi_2^{\mathrm{f}}(M,L)$ with $I_{\mu_L}(A) = 2$, the Gromov compactification $\overline{\mathcal{P}(x,x,(A);\mathscr{D},J)}$ has naturally the structure of a compact 1-dimensional manifold with boundary. Furthermore, the boundary is given by*

$$\begin{aligned}\partial\overline{\mathcal{P}(x,x,(A);\mathscr{D},J)} \;=\; & \left[\bigcup_{\substack{z\in\mathrm{Crit}(f)\\ \mathrm{ind}(z)=\mathrm{ind}(x)-1}} \mathcal{P}(x,z,\emptyset;\mathscr{D},J)\times\mathcal{P}(z,x,(A);\mathscr{D},J)\right] \;\cup \\ & \left[\bigcup_{\substack{z\in\mathrm{Crit}(f)\\ \mathrm{ind}(z)=\mathrm{ind}(x)+1}} \mathcal{P}(x,z,(A);\mathscr{D},J)\times\mathcal{P}(z,x,\emptyset;\mathscr{D},J)\right] \;\cup\; \mathcal{D}^{\mp}(A,x). \qquad \square\end{aligned}$$

The above description of the boundary $\partial\overline{\mathcal{P}(x,x,(A);\mathscr{D},J)}$ is not explicitly mentioned in [BC07b] since a natural bijection between $\mathcal{D}^-(A,x)$ and $\mathcal{D}^+(A,x)$ is implicitly used there to glue the two spaces together and thus treat them as points in the interior of $\overline{\mathcal{P}(x,x,(A);\mathscr{D},J)}$. An explicit description of this idea can be found in [Zap15, Section 6.2].

We claim that the above theorem suffices to prove part ii) of Theorem 2.4.8. Indeed, consider

$$w = [\{\tilde{u}_\alpha,\tilde{u}_\beta\},\, \{z_{\alpha\beta} = \mp\mathbf{i},\, z_{\beta\alpha}\}, \{(\alpha,-1),(\alpha,1)\}] \in \mathcal{D}^{\mp}(A,x),$$

the notation being $[\{\text{maps}\},\{\text{nodal points}\},\{\text{marked points}\}]$; note in particular that $\tilde{u}_\alpha$ is constant. Write $\partial\tilde{u}_\beta \in \Pi_1 L(x,x)$ for the boundary of $\tilde{u}_\beta$, viewed as a loop based at $x = \tilde{u}_\beta(z_{\beta\alpha})$. If $w \in \mathcal{D}^-(A,x)$, we define $\gamma_w^0 := \partial\tilde{u}_\beta$ and γ_w^1 to be the constant path at x, while, if $w \in \mathcal{D}^+(A,x)$ we define $\gamma_w^1 := \partial\tilde{u}_\beta$ and γ_w^0 to be the constant path at x. Note that there are obvious bijections $\mathcal{D}^-(A,x) \cong \mathcal{M}_{0,1}^A(x,L;J)$ and $\mathcal{D}^+(A,x) \cong \mathcal{M}_{0,1}^A(x,L;J)$ given by

$$w = [\{\tilde{u}_\alpha,\tilde{u}_\beta\},\{z_{\alpha\beta} = \mp\mathbf{i},\, z_{\beta\alpha}\},\{(\alpha,-1),(\alpha,1)\}] \quad\longmapsto\quad u_w := [\tilde{u}_\beta, z_{\beta\alpha}].$$

Observe that if $w \in \mathcal{D}^-(A,x)$, then $\gamma_w^0 = \partial u_w$ and if $w \in \mathcal{D}^+(A,x)$, then $\gamma_w^1 = \partial u_w$. From this it is immediate (at least when $J \in \mathcal{J}_{\mathrm{reg}}(L|x)$) that for every $\alpha \in \mathrm{Hom}_{\mathbb{F}}(\mathcal{E}_x^0, \mathcal{E}_x^1)$ we have

$$m(\mathcal{E}^1)(x) \circ \alpha - \alpha \circ m(\mathcal{E}^0)(x) = \sum_{\substack{A \in \pi_2^{\mathrm{f}}(M,L),\\ I_{\mu_L}(A)=2}} \sum_{w \in \mathcal{D}^{\mp}(A,x)} P_{\gamma_w^1} \circ \alpha \circ P_{\gamma_w^0}. \tag{2.50}$$

Note now that we also have

$$\langle d^2\alpha, x\rangle = \left(\sum_{\substack{z \in \mathrm{Crit}(f) \\ \mathrm{ind}(z) = \mathrm{ind}(x)-1}} \sum_{\substack{(\mathbf{u},\mathbf{v}) \in \\ \mathcal{P}(x,z,0;\mathscr{D},J) \times \mathcal{P}(z,x,2;\mathscr{D},J)}} P_{1,\gamma_{\mathbf{v}}^1 \cdot \gamma_{\mathbf{u}}^1} \circ \alpha \circ P_{0,\gamma_{\mathbf{u}}^0 \cdot \gamma_{\mathbf{v}}^0} + \sum_{\substack{z \in \mathrm{Crit}(f) \\ \mathrm{ind}(z) = \mathrm{ind}(x)+1}} \sum_{\substack{(\mathbf{u},\mathbf{v}) \in \\ \mathcal{P}(x,z,2;\mathscr{D},J) \times \mathcal{P}(z,x,0;\mathscr{D},J)}} P_{1,\gamma_{\mathbf{v}}^1 \cdot \gamma_{\mathbf{u}}^1} \circ \alpha \circ P_{0,\gamma_{\mathbf{u}}^0 \cdot \gamma_{\mathbf{v}}^0} \right) \cdot T.$$

Multiplying equation (2.50) by T and adding it to the above, we obtain what we were after:

$$\langle d^2\alpha, x\rangle + (m_0(\mathcal{E}^1)(x) \circ \alpha - \alpha \circ m_0(\mathcal{E}^0)(x)) \cdot T = 0,$$

where the right-hand side vanishes since, by Theorem 2.4.10, the sum runs over all boundary points of the compact 1-dimensional manifold $\overline{\mathcal{P}(x,x,2;\mathscr{D},J)}$. This completes the proof of part ii) of Theorem 2.4.8.

Proofs of part B) and part C): These are proved for trivial rank 1 local systems in [BC07b, Section 5.1.2] and [BC07b, Proposition 5.6.2], respectively. Straightforward generalisations of these arguments to the case of higher rank local systems yield the results. □

2.4.2 The spectral sequence and comparison with Morse cohomology

In this section we compare the cohomology of the pearl complex $C_f^*(\mathcal{E}^0, \mathcal{E}^1)$ with the singular (Morse) cohomology of L with coefficients in an approriate local system. The key tool here is the Oh-Biran spectral sequence (constructed in [Oh96], [Bir06, Section 5.2] for different models for Floer cohomology and in the present context in [BC09b]). This spectral sequence will be used extensively for computations in subsequent chapters so we now briefly describe it.

Observe that the complex $\overline{C}_f^*(\mathcal{E}^0, \mathcal{E}^1)$ comes equipped with a decreasing filtration by increasing powers of T:

$$\cdots \supseteq F^{-1}\overline{C}_f^* \supseteq F^0\overline{C}_f^* \supseteq F^1\overline{C}_f^* \supseteq \cdots \tag{2.51}$$

$$F^p\,\overline{C}_f^*(\mathcal{E}^0, \mathcal{E}^1) := \overline{C}_f^*(\mathcal{E}^0, \mathcal{E}^1) \cap \bigoplus_{\substack{r,s \in \mathbb{Z} \\ r \geq p}} C_f^{r,s}(\mathcal{E}^0, \mathcal{E}^1).$$

This filtration is preserved by the differential and in fact the map $d^{(\mathscr{D},J)}$ decomposes as

$$d^{(\mathscr{D},J)} = \partial_0 \otimes 1 + \partial_1 \otimes T + \partial_2 \otimes T^2 + \cdots, \tag{2.52}$$

where $\partial_k \colon C_f^*(L;\mathcal{Z}_{m_0}(\mathcal{E}^0,\mathcal{E}^1)) \to C_f^{*+1-kN_L^\pi}(L;\mathcal{Z}_{m_0}(\mathcal{E}^0,\mathcal{E}^1))$ sends $\alpha \in \mathcal{Z}_{m_0}(\mathcal{E}^0,\mathcal{E}^1)_x$ for some $x \in \mathrm{Crit}(f)$ to

$$\partial_k \alpha = \sum_{\substack{y\in \mathrm{Crit}(f)\\ \mathrm{ind}(y)=\mathrm{ind}(x)+1-kN_L}} \sum_{\mathbf{u}\in \mathcal{P}(y,x,kN_L^\pi;\mathscr{D},J)} P_{1,\gamma_{\mathbf{u}}^1} \circ \alpha \circ P_{0,\gamma_{\mathbf{u}}^0}.$$

Therefore the standard machinery of filtered complexes (see e.g. [McC01]) gives rise to a spectral sequence whose first page is the homology of the complex $\left(\overline{C}_f^*(\mathcal{E}^0,\mathcal{E}^1),\partial_0\right)$ and which converges to $\overline{HF}_{\mathrm{BC}}^*(\mathcal{E}^0,\mathcal{E}^1)$. To unravel this, note that since $m_0(\mathcal{E}^0)$ and $m_0(\mathcal{E}^1)$ are morphisms of local systems, they give rise to a morphism:

$$\begin{aligned} \mathscr{H}om(\mathcal{E}^0,\mathcal{E}^1) &\longrightarrow \mathscr{H}om(\mathcal{E}^0,\mathcal{E}^1) \\ \alpha &\longmapsto \alpha\circ m_0(\mathcal{E}^0) - m_0(\mathcal{E}^1)\circ\alpha \end{aligned} \tag{2.53}$$

We denote the local system which is the kernel of (2.53) by $\mathcal{Z}_{m_0}(\mathcal{E}^0,\mathcal{E}^1)$. Since ∂_0 is just the Morse differential $\partial^{\mathscr{D}}$ as described in (2.8), we conclude that the first page of the Oh-Biran spectral sequence is given by

$$E_1^{p,q} \cong H^{p+q-pN_L^\pi}(L;\mathcal{Z}_{m_0}(\mathcal{E}^0,\mathcal{E}^1))\cdot T^p. \tag{2.54}$$

Even without the full machinery of the spectral sequence, it is useful to compare the Morse and Floer cohomology of L. In particular, observe that there is an obvious inclusion of graded vector spaces

$$C_f^*(L;\mathcal{Z}_{m_0}(\mathcal{E}^0,\mathcal{E}^1)) \xrightarrow{-\otimes 1} C_f^*(\mathcal{E}^0,\mathcal{E}^1).$$

This is not a chain map between the Morse and pearl complexes in general but it becomes one when restricted to the subcomplex $(C_{f,\mathrm{cls}}^*(L;\mathcal{Z}_{m_0}(\mathcal{E}^0,\mathcal{E}^1)),\partial^{\mathscr{D}})$, defined by

$$\begin{aligned} C_{f,\mathrm{cls}}^k(L;\mathcal{Z}_{m_0}(\mathcal{E}^0,\mathcal{E}^1)) &:= C_f^k(L;\mathcal{Z}_{m_0}(\mathcal{E}^0,\mathcal{E}^1)), \quad k\le N_L^\pi-2 \\ C_{f,\mathrm{cls}}^{N_L^\pi-1}(L;\mathcal{Z}_{m_0}(\mathcal{E}^0,\mathcal{E}^1)) &:= \ker\left(\partial_0|_{C_f^{N_L^\pi-1}(L;\mathcal{Z}_{m_0}(\mathcal{E}^0,\mathcal{E}^1))}\right)\cap\ker\left(\partial_1|_{C_f^{N_L^\pi-1}(L;\mathcal{Z}_{m_0}(\mathcal{E}^0,\mathcal{E}^1))}\right) \\ C_{f,\mathrm{cls}}^k(L;\mathcal{Z}_{m_0}(\mathcal{E}^0,\mathcal{E}^1)) &:= 0, \quad k\ge N_L^\pi. \end{aligned}$$

It is easy to see that the inclusion of $(C_{f,\mathrm{cls}}^*(L;\mathcal{Z}_{m_0}(\mathcal{E}^0,\mathcal{E}^1)),\partial^{\mathscr{D}})$ into the full Morse complex with coefficients in $\mathcal{Z}_{m_0}(\mathcal{E}^0,\mathcal{E}^1)$ induces an inclusion on homology and we denote its image by $H_{\mathrm{cls}}^*(L;\mathcal{Z}_{m_0}(\mathcal{E}^0,\mathcal{E}^1))$. By the preceding discussion, we have a map

$$\mathrm{q}_L\colon H_{\mathrm{cls}}^*(L;\mathcal{Z}_{m_0}(\mathcal{E}^0,\mathcal{E}^1)) \to \overline{HF}_{\mathrm{BC}}^*(\mathcal{E}^0,\mathcal{E}^1), \tag{2.55}$$

induced by $-\otimes 1$.

Remark 2.4.11. The notation we have used here is not standard. The abbreviation “cls” stands for “classical” and the map q_L is meant to be seen as “quantising” classical cohomology. If we were working with the Hamiltonian, rather than the pearly model for $\overline{HF}^*(\mathcal{E}^0,\mathcal{E}^1)$, the existence of the map q_L is a non-trivial fact and the map itself is known as the “Lagrangian PSS morphism” [Alb08]. However, as we saw, once the pearl complex machinery is set up and the full power of the Lagrangian

PSS map Ψ_{PSS} is used to identify the homology of the pearl complex with Floer cohomology, the definition of q_L is basically trivial. //

2.4.3 The monodromy pearl complex

We now consider the pearl model for the monodromy complex of section 2.2.3. Note that the spaces $\text{End}_{\text{mon}}(\mathcal{E}_x) := \text{Hom}_{\text{mon}}(\mathcal{E}_x, \mathcal{E}_x)$ for varying $x \in L$ fit together to form a local system of rings on L which we denote by $\mathscr{E}nd_{mon}(\mathcal{E})$ (in the terminology of [Ste43], it is called a *system of operator rings*). We can then define

Definition 2.4.12. The *monodromy pearl complex* of E with respect to the Morse function f is

$$C^*_{f,\text{mon}}(\mathcal{E}) := C^*_f(L; \mathscr{E}nd_{mon}(\mathcal{E})) \otimes \mathbb{F}[T^{\pm 1}]$$

The monodromy pearl complex of L over $\mathbb{F}$ is defined by setting $\mathcal{E} = \mathcal{E}^{\mathbb{F}}_{\text{reg}}$ in the above and denoted $C^*_{f,\text{mon}}(L;\mathbb{F})$. ♢

The same kind of arguments as in section 2.2.3 show that $(C^*_{f,\text{mon}}(\mathcal{E}), d^{(\mathscr{D},J)})$ is a subcomplex of the unobstructed complex $(\overline{C}^*_f(\mathcal{E},\mathcal{E}), d^{(\mathscr{D},J)})$, that its homology is invariant under changes of $\mathscr{D}$ or J and that the PSS morphism induces an isomorphism

$$H^*\left(C^*_f(L; \mathscr{E}nd_{mon}(\mathcal{E})), \bar{d}^{(\mathcal{F},J)}\right) \cong HF^*_{\text{mon}}(\mathcal{E}).$$

We will write $HF^*_{\text{BC,mon}}(\mathcal{E})$ for the cohomology of $(C^*_{f,\text{mon}}(\mathcal{E}), d^{(\mathscr{D},J)})$ and $HF^*_{\text{BC,mon}}(L;\mathbb{F})$ instead of $HF^*_{\text{BC,mon}}(\mathcal{E}^{\mathbb{F}}_{\text{reg}})$. Note that if $\mathcal{E}^0$ dominates $\mathcal{E}^1$, then we still get domination maps

$$\Phi\colon C^*_f(L; \mathscr{E}nd_{mon}(\mathcal{E}^0)) \longrightarrow C^*_f(L; \mathscr{E}nd_{mon}(\mathcal{E}^1)) \tag{2.56}$$

$$H(\Phi)\colon HF^*_{\text{BC,mon}}(\mathcal{E}^0) \longrightarrow HF^*_{\text{BC,mon}}(\mathcal{E}^1) \tag{2.57}$$

simply by relabelling parallel transport maps as in section 2.2.3.

The filtration (2.51) restricts to a filtration of the monodromy pearl complex and we again get a Oh-Biran spectral sequence converging to $HF^*_{\text{BC,mon}}(\mathcal{E})$. Its first page is built out of the cohomology $H^*(L; \mathscr{E}nd_{mon}(\mathcal{E})) \otimes \mathbb{F}[T^{\pm 1}]$. For a general local system $\mathcal{E}$, this is hard to interpret geometrically but when $\mathcal{E} = \mathcal{E}^{\mathbb{F}}_{\text{reg}}$, the cohomology $H^*(L; \mathscr{E}nd_{mon}(\mathcal{E}^{\mathbb{F}}_{\text{reg}}))$ can be identified with the singular cohomology with compact support of a particular (generally non-regular) cover of L.

Define $\widehat{L} := \widetilde{L} \times_{\pi_1(L)} \pi_1(L)$, where $\widetilde{L}$ denotes the universal cover of L and the twisted product is formed using the deck transformation action of $\pi_1(L)$ on $\widetilde{L}$ and the *conjugation* action of $\pi_1(L)$ on itself (here $\pi_1(L)$ is given the discrete topology). Noting that the local system $\mathscr{E}nd_{mon}(\mathcal{E}^{\mathbb{F}}_{\text{reg}})$ is precisely the one associated to the (right) conjugation action of $\pi_1(L)$ on $\mathbb{F}[\pi_1(L)]$, we see that we have an isomorphism of local systems $\mathscr{E}nd_{mon}(\mathcal{E}^{\mathbb{F}}_{\text{reg}}) \cong \mathcal{E}_{\widehat{L}}$. Thus we conclude that the first page of the Oh-Biran spectral sequence computing $HF^*_{\text{BC,mon}}(L;\mathbb{F})$ is given by

$$E_1^{p,q} \quad \cong \quad H_c^{p+q-pN^{\pi}_L}(\,\widehat{L}\,;\mathbb{F}) \cdot T^p. \tag{2.58}$$

The cover $\widehat{L}$ has previously appeared in the work of Fukaya [Fuk06]. It has the following properties which are easy to derive from the definition. First, its connected components are in one to one correspondence with free homotopy classes of loops on L or, equivalently, conjugacy classes in $\pi_1(L)$. In particular, it is disconnected unless $\pi_1(L)$ is trivial. The connected component corresponding to the conjugacy class of an element $\gamma \in \pi_1(L)$ is homeomorphic to the (left) quotient $C(\gamma)\backslash\widetilde{L}$, where $C(\gamma)$ is the centraliser of γ in $\pi_1(L)$. Further, if $\mathcal{L}_L$ denotes the free loop space of L, then the evaluation map $\mathcal{L}_L \to L$ factors through $\widehat{L}$ and the fibres of the map $\mathcal{L}_L \to \widehat{L}$ are connected. In fact, the fibres of this map can easily be identified with the connected components of the based loop space of L. In particular, if L is aspherical, then the map $\mathcal{L}_L \to \widehat{L}$ has contractible fibres and therefore is a homotopy equivalence (cf. [Fuk06, Lemma 12.5]). Combining this with Poincaré duality one obtains that for an aspherical monotone Lagrangian L of dimension n, there is a spectral sequence with first page

$$E_1^{p,q} \quad \cong \quad H_{n+pN_L^\pi-(p+q)}(\,\mathcal{L}_L\,;\mathbb{F}) \,\cdot\, T^p \tag{2.59}$$

converging to $HF^*_{\mathrm{BC,mon}}(L;\mathbb{F})$.

2.4.4 Algebraic structures

In [BC07b, Sections 5.2, 5.3] Biran and Cornea also define a product and an action of quantum cohomology on HF^*_{BC}, when L carries the trivial rank 1 local system. Moreover, with these structures in place, the map Ψ_{PSS} becomes a unital homomorphism of $QH^*(M)$-algebras, as shown by Zapolsky in [Zap15, Section 5.2.4]. These constructions generalise in a straightforward way to incorporate high rank local systems. We now give a very brief account, citing the literature for proofs of the statements without local systems and leaving the generalisations to the reader.

Given a triple of local systems $\mathcal{E}^0$, $\mathcal{E}^1$, $\mathcal{E}^2$ on L, the product is a degree zero, $\mathbb{F}[T^{\pm 1}]$-linear map

$$\overline{HF}^*_{\mathrm{BC}}(\mathcal{E}^1,\mathcal{E}^2)\otimes\overline{HF}^*_{\mathrm{BC}}(\mathcal{E}^0,\mathcal{E}^1) \longrightarrow \overline{HF}^*_{\mathrm{BC}}(\mathcal{E}^0,\mathcal{E}^2). \tag{2.60}$$

It comes from a map of pre-complexes

$$\mathcal{C}^*_{f'}(\mathcal{E}^1,\mathcal{E}^2)\otimes\mathcal{C}^*_{f}(\mathcal{E}^0,\mathcal{E}^1) \longrightarrow \mathcal{C}^*_{f''}(\mathcal{E}^0,\mathcal{E}^2),$$

defined by parallel transport along the "sides" (paths defined analogously to (2.46)) of Y-shaped pearly configurations. The product (2.60) gives rise to a graded, $\mathbb{F}[T^{\pm 1}]$-linear category $\langle L\rangle_{\mathbb{F}}$, whose objects are $\mathbb{F}$-local systems on L and the morphism space between $\mathcal{E}^0$ and $\mathcal{E}^1$ is $\overline{HF}^*_{\mathrm{BC}}(\mathcal{E}^0,\mathcal{E}^1)$. That is, the product satisfies the appropriate associativity property ([BC07b, Lemma 5.2.6]) and has a unit ([BC07b, Lemma 5.2.4]). The unit is most easily defined by picking a Morse function f with a unique minimum $x_{\min} \in \mathrm{Crit}(f)$, in which case the element $\tilde{e}_{\mathcal{E}} := \mathrm{Id}\otimes 1 \in \mathrm{End}(\mathcal{E}_{x_{\min}})\otimes\mathbb{F}[T^{\pm 1}]$ is a cochain representative of the unit $e_{\mathcal{E}} \in \overline{HF}^0_{\mathrm{BC}}(\mathcal{E},\mathcal{E})$. Note also, that since the product is defined by parallel transport, it makes $HF^*_{\mathrm{BC,mon}}(\mathcal{E})$ into a unital algebra and the domination map (2.57) is a unital algebra homomorphism. It was verified by Zapolsky in [Zap15, Section 5.2.4] that the map

Ψ_{PSS} induces algebra isomorphisms

$$\begin{aligned} HF^*_{\mathrm{BC,mon}}(\mathcal{E}) \otimes_{\mathbb{F}[T^{\pm 1}]} \mathbb{F} &\xrightarrow{\cong} HF^*_{\mathrm{mon}}(\mathcal{E}) \\ \overline{HF}^*_{\mathrm{BC}}(\mathcal{E},\mathcal{E}) \otimes_{\mathbb{F}[T^{\pm 1}]} \mathbb{F} &\xrightarrow{\cong} \overline{HF}^*(\mathcal{E},\mathcal{E}). \end{aligned} \tag{2.61}$$

Recall from section 2.3.3 that the length zero closed-open string map $\mathcal{CO}^0$ makes the rings on the right-hand side of (2.61) into $QH^*(M)$-algebras. To define a similar structure on $HF^*_{\mathrm{BC,mon}}(\mathcal{E})$ and $\overline{HF}^*_{\mathrm{BC}}(\mathcal{E},\mathcal{E})$ without collapsing the grading, one also needs to keep track of grading on $QH^*(M)$. Since the quantum product $\star$ preserves the grading on $H^*(M;\mathbb{F})$ only modulo $2N^\pi_M$, in order to keep the grading absolute one considers the vector space $H^*(M,\mathbb{F}) \otimes \mathbb{F}[q^{\pm 1}]$, where q is a formal variable of degree $2N^\pi_M$. The quantum product naturally defines a product (still denoted by $\star$) on this space, making it into a graded, associative, (graded-)commutative algebra. Using the identification $q = T^{(2N^\pi_M/N^\pi_L)}$ and extending $\star$ to be $\mathbb{F}[T^{\pm 1}]$-linear, we have a graded algebra $QH^*_{\mathrm{BC}}(M;\mathbb{F}) := (H^*(M;\mathbb{F}) \otimes \mathbb{F}[T^{\pm 1}], \star)$. Then, the construction from [BC07b, Section 5.3] gives rise to the quantum module action

$$\begin{aligned} QH^*_{\mathrm{BC}}(M;\mathbb{F}) \otimes HF^*_{\mathrm{BC,mon}}(\mathcal{E}) &\longrightarrow HF^*_{\mathrm{BC,mon}}(\mathcal{E}) \\ QH^*_{\mathrm{BC}}(M;\mathbb{F}) \otimes \overline{HF}^*_{\mathrm{BC}}(\mathcal{E},\mathcal{E}) &\longrightarrow \overline{HF}^*_{\mathrm{BC}}(\mathcal{E},\mathcal{E}), \end{aligned} \tag{2.62}$$

which makes the rings on the right-hand side into graded, unital $QH^*_{\mathrm{BC}}(M;\mathbb{F})$-algebras. We will write $\mathcal{CO}^0_{\mathrm{BC}} \colon QH^*_{\mathrm{BC}}(M;\mathbb{F}) \to HF^*_{\mathrm{BC,mon}}(\mathcal{E})$ for the algebra homomorphism, obtained by acting on the unit $e_{\mathcal{E}} \in HF^0_{\mathrm{BC,mon}}(\mathcal{E})$. Again, after setting $T = 1$ and collapsing the grading, the action (2.62) is identified via the PSS map with the action we considered in section 2.3.3 (this was also checked in [Zap15, Section 5.2.4]).

Now note that there is an obvious inclusion of graded vector spaces $q_M \colon H^*(M;\mathbb{F}) \to QH^*_{\mathrm{BC}}(M;\mathbb{F})$, given by just tensoring with $1 \in \mathbb{F}[T^{\pm 1}]$. It will be important to us that when one takes trivial rank 1 local systems on L, the closed-open string map $\mathcal{CO}^0_{\mathrm{BC}} \colon QH^*(M;\mathbb{F}) \to HF^*_{\mathrm{BC}}(L,L;\mathbb{F})$ is related to the classical restriction $i^* \colon H^*(M;\mathbb{F}) \to H^*(L;\mathbb{F})$ via the diagram

$$\begin{tikzcd} H^k(M;\mathbb{F}) \arrow[r, "i^*"] \arrow[d, "q_M"'] & H^k_{\mathrm{cls}}(L;\mathbb{F}) \arrow[d, "q_L"] \\ QH^k_{\mathrm{BC}}(M;\mathbb{F}) \arrow[r, "\mathcal{CO}^0_{\mathrm{BC}}"] & HF^k_{\mathrm{BC}}(L,L;\mathbb{F}) \end{tikzcd} \qquad \forall\, 0 \le k \le N^\pi_L - 1. \tag{2.63}$$

Note that the image of $H^{N^\pi_L - 1}(M;\mathbb{F})$ under i^* is contained in $H^{N^\pi_L - 1}_{\mathrm{cls}}(L;\mathbb{F})$ since at chain level i^* and $\mathcal{CO}^0_{\mathrm{BC}}$ coincide on chains of degree strictly less than N^π_L and $\mathcal{CO}^0$ is a chain map with respect to the Morse differential on M and the pearl differential on L. Observe also that since $N^\pi_L \ge 2$, there is always a map $q_L \colon H^0(L;\mathbb{F}) \to HF^0_{\mathrm{BC}}(L,L;\mathbb{F})$ which sends the unit for classical cohomology to the unit $e_L \in HF^0_{\mathrm{BC}}(L,L;\mathbb{F})$.

Finally, one more crucial fact which we will need is that the Auroux-Kontsevich-Seidel theorem

2.3.8 still holds in the pearly setting. That is, if L is orientable, then

$$\mathcal{CO}^0_{\mathrm{BC}}(c_1(TM)) = [(m_0(\mathcal{E}) \circ \tilde{e}_{\mathcal{E}}) \cdot T] \quad \in \quad HF^2_{\mathrm{BC,mon}}(\mathcal{E}). \tag{2.64}$$

This can be proved directly using the pearl models for $\mathcal{CO}^0_{\mathrm{BC}}$ and $HF^*_{\mathrm{BC,mon}}$ or by relying on the fact that $\Psi_{\mathrm{PSS}} \colon HF^*_{\mathrm{BC,mon}}(\mathcal{E}) \to HF^*_{\mathrm{mon}}(\mathcal{E})$ is a unital homomorphism, which intertwines $\mathcal{CO}^0_{\mathrm{BC}}$ and $\mathcal{CO}^0$.

Chapter 3

Topological restrictions on monotone Lagrangians in $\mathbb{CP}^n$

In this chapter we use Floer theory with local coefficients in order to derive some restrictions on the topology of closed manifolds which admit a monotone Lagrangian embedding in $\mathbb{CP}^n$. The local systems which we will use are very limited – we will only be interested in the cohomology of the complex $C^*_f(L,(L,\mathcal{E}_{L'}))$ for different covers L' of L for which the complex is unobstructed. This is exactly the "lifted" Floer cohomology, introduced by Damian in [Dam12]. However, we will heavily use the algebraic structures discussed in section 2.4.4, in particular, the quantum module action and the AKS criterion. Moreover, we will need to work with coefficients in rings of characteristic different from 2 so we now make a small interlude to discuss this issue.

Signs

The Biran-Cornea machinery with coefficients in $\mathbb{Z}$ (when it is well-defined) contains much more information than the theory over characteristic 2 but requires more work to set up. It also requires some additional assumptions on L for the constructions to be possible. The reason is that one needs to be able to orient the various moduli spaces and do so coherently, so that configurations corresponding to opposite ends of a connected component in a 1-dimensional moduli space come with opposite signs. Note that in all our constructions with local systems in chapter 2, we only used the fact that contributions from such configurations cancel out. Thus, if there exists an orientation scheme which makes the theory with trivial local systems well-defined over characteristic other than 2, then the same scheme can be used to make the high rank theory work, as our proofs above will carry over, after being modified to incorporate the appropriate signs.

There are several orientation schemes in the literature and it is not entirely clear to what extent they agree with each other. However, we will not need the actual specifics of such a scheme, only the fact that it exists. To be more concrete, consider first the following definition:

Definition 3.0.13. Let L be an oriented, monotone Lagrangian and let $\mathcal{E} \to L$ be an R-local system for some commutative ring R. Let J be a generic almost complex structure and assume that $\mathcal{M}_{0,1}(2,L;J)$

is an **oriented** manifold. Then, if $p \in L$ is a regular value for $\mathrm{ev}\colon \mathcal{M}_{0,1}(2,L;J) \to L$, we define

$$m_0(\mathcal{E})(p) := \sum_{u \in \mathrm{ev}^{-1}(p)} \deg_u(\mathrm{ev}) P_{\partial u} \quad \in \quad \mathrm{End}(\mathcal{E}_p),$$

where $\deg_u(\mathrm{ev})$ denotes the local degree of ev at u. ♢

In this chapter we will rely on the following assumption with respect to the above definition:

Assumption 3.0.14. If $L \subseteq M$ is a monotone Lagrangian with $N_L^\pi \geq 2$, equipped with an orientation and a relative spin structure, then:

a) The orientation and relative spin structure on L induce orientations on the moduli spaces $\mathcal{M}_{0,0}(2,L;J)$ and $\mathcal{M}_{0,1}(2,L;J)$ for any generic J. With these orientations and $m_0(\mathcal{E})$ defined as in Definition 3.0.13, the invariance properties of Proposition 2.2.3 hold. That is, $m_0(\mathcal{E})$ is an element of $\mathrm{End}(\mathcal{E})$ which is independent of the choice of generic J.

b) The moduli spaces of pearly trajectories can be oriented and their isolated points counted with signs in such a way that the following hold:

   i) for any pair of R-local systems $\mathcal{E}^0, \mathcal{E}^1 \to L$ the resulting pre-complex $(C_f^*(\mathcal{E}^0,\mathcal{E}^1), d^{(\mathscr{D},J)})$ satisfies

   $$\forall a \in C_f^*(\mathcal{E}^0,\mathcal{E}^1) \qquad \left(d^{(\mathscr{D},J)}\right)^2 a = \pm(a \circ m_0(\mathcal{E}^0) - m_0(\mathcal{E}^1) \circ a) \cdot T\,;$$

   ii) the product (2.60) gives rise to a unital, graded, $R[T^{\pm 1}]$-linear category $\langle L \rangle_R$ of R-local systems on L, where the space of morphisms from $\mathcal{E}^0$ to $\mathcal{E}^1$ is $\overline{HF}{}^*_{\mathrm{BC}}(\mathcal{E}^0,\mathcal{E}^1)$;

   iii) for any R-local system $\mathcal{E} \to L$ the map $\mathcal{CO}^0_{\mathrm{BC}}\colon QH^*_{\mathrm{BC}}(M;R) \to HF^*_{\mathrm{BC,mon}}(\mathcal{E})$ is a unital $R[T^{\pm 1}]$-algebra homomorphism and it satisfies

   $$\mathcal{CO}^0_{\mathrm{BC}}(c_1(TM)) = [(m_0(\mathcal{E}) \circ \tilde{e}_{\mathcal{E}}) \cdot T],$$

   where $\tilde{e}_{\mathcal{E}} \in HF^0_{\mathrm{BC,mon}}(\mathcal{E})$ is any cocycle representing the unit;

   iv) in the special case when $\mathcal{E}$ is the trivial, rank one R-local system, there is a commutative diagram

   $$\begin{array}{ccc} H^k(M;R) & \xrightarrow{i^*} & H^k_{\mathrm{cls}}(L;R) \\ {\scriptstyle q_M}\big\downarrow & & \big\downarrow{\scriptstyle q_L} \\ QH^k_{\mathrm{BC}}(M;R) & \xrightarrow{\mathcal{CO}^0_{\mathrm{BC}}} & HF^k_{\mathrm{BC}}(L,L;R) \end{array} \qquad \forall\, 0 \leq k \leq N_L^\pi - 1; \qquad (3.1)$$

   v) the ∂_0-part of the differential $d^{(\mathscr{D},J)}$ can be identified with the Morse differential $\partial^{\mathscr{D}}$ from (2.8); in particular, for any pair of R-local systems $\mathcal{E}^0,\mathcal{E}^1 \to L$, the first page of the Oh-Biran spectral sequence which computes $\overline{HF}{}^*_{\mathrm{BC}}(\mathcal{E}^0,\mathcal{E}^1)$ is identified with the singular cohomology of L with coefficients in the R-local system $\mathcal{Z}_{m_0}(\mathcal{E}^0,\mathcal{E}^1)$. //

Let us reiterate that as long as there exists an orientation scheme for which Assumption 3.0.14 is satisfied with all local systems trivial and of rank one, then it is automatically satisfied with arbitrary local systems. It is known that a choice of orientation and relative spin structure on a Lagrangian L determines an orientation on all moduli spaces of pseudoholomorphic discs with boundary on L ([FOOO09, Chapter 8], [Sei08a, Lemmas 11.7, 11.17], [Zap15, Section 7.1]). Using this fact, Biran and Cornea give an orientation procedure for pearly moduli spaces in [BC12, Appendix A] and sketch the verification of b) ii), iv) and v).

A different approach to orientations for the pearl complex was introduced by Zapolsky in [Zap15]. Given a monotone Lagrangian L with $N_L^\pi \geq 2$, he imposes the following restriction:

Assumption(O): For some (equivalently, any) point $q \in L$, the second Stiefel-Whitney class $w_2(TL)$ vanishes on the image of $\pi_3(M,L,q)$ in $\pi_2(L,q)$ under the boundary map.

When this assumption is satisfied, Zapolsky constructs a complex, called *the canonical pearl complex* of L with coefficients in an arbitrary commutative ring R. We will denote this complex by $C^*_{f,\mathrm{Zap}}(L,L;R)$. If $R = \mathbb{F}_2$ (or any other ring of characteristic 2), then Assumption (O) can be dropped and in fact the Biran-Cornea complex $C^*_f(L,L;\mathbb{F}_2)$ is naturally a quotient of $C^*_{f,\mathrm{Zap}}(L,L;\mathbb{F}_2)$. If R does not have characteristic 2 however, Assumption (O) is necessary in order to define the chain groups $C^*_{f,\mathrm{Zap}}(L,L;R)$.

These groups are obtained by attaching a particular free R-module with basis indexed by $\pi_2(M,L,x)$ to each critical point $x \in \mathrm{Crit}(f)$. For each rigid pearly trajectory, Zapolsky defines an isomorphism between the modules attached to its endpoints and the sum of these isomorphisms gives an endomorphism $d^{(\mathscr{D},J)}_{\mathrm{Zap}}$ of $C^*_{f,\mathrm{Zap}}(L,L;R)$. He then carefully checks that this map squares to zero and in particular in [Zap15, Section 6.2] he verifies a generalised version of our Assumption 3.0.14 b) i). Zapolsky also defines a product on the cohomology $HF^*_{\mathrm{Zap}}(L,L;R)$, a graded-commutative algebra $QH^*_{\mathrm{Zap}}(M;R)$ (which we may call the canonical quantum cohomology of M) and the corresponding map $\mathcal{CO}^0_{\mathrm{Zap}} \colon QH^*_{\mathrm{Zap}}(M;R) \to HF^*_{\mathrm{Zap}}(L,L;R)$ which he shows to be a unital algebra homomorphism (in Zapolsky's paper this is expressed in terms of quantum module action of QH^*_{Zap} on HF^*_{Zap} without explicit mention of a closed-open map). One can verify directly from the definitions that the analogue of Assumption 3.0.14 b) iv) holds for the canonical complexes (that is, after replacing the abbreviation "BC" by "Zap" in diagram (3.1)). There is again a spectral sequence whose first page contains only topological information (it is the cohomology of L with coefficients in a particular graded local system of free R-modules, see [KS18, Section 3.1]) and which converges to $HF^*_{\mathrm{Zap}}(L,L;R)$. Zapolsky's complex can also be twisted by a rather general version of local system (see [Smi17, Appendix A]) of which the local systems we consider in this thesis are a special case. The obstruction issues for the differential to square to zero are more delicate but are still controlled Maslov 2 discs. In particular, if $N_L^\pi \geq 3$, then there are no obstructions.

Assumption (O) is the weakest condition on a monotone Lagrangian $L \subseteq M$ under which a version of Floer theory has been defined over an arbitrary ground ring. In particular, if L is relatively pin (recall that this means that either $w_2(TL)$ or $w_2(TL)+w_1(TL)^2$ lies in the image of the restriction map $H^2(M;\mathbb{F}_2) \to H^2(L;\mathbb{F}_2)$), then L satisfies Assumption (O) (see [Zap15, Section 7], [KS18, Section 3.4]). Note that this does not require L to be orientable but, if L is orientable, then being relatively pin is the same as being relatively spin which puts us in the setting of Assumption 3.0.14. As far as the author knows, there are no known examples of monotone Lagrangians which satisfy Assumption (O) but are not relatively pin (Zapolsky erroneously gives $\mathbb{RP}^5 \subseteq \mathbb{CP}^5$ as an example but, as we shall see below, $\mathbb{RP}^5$ is relatively spin). On the other hand, in chapter 4, section 4.1.5 we give for each $k \geq 1$ an example of a non-orientable Lagrangian in $\mathbb{CP}^{4k+1}$ whose minimal Maslov number is $2k+1$ and which does not satisfy Assumption (O) (see Lemma 4.1.34).

Here is how this story relates to the construction we need. As explained by Zapolsky in [Zap15, Section 7], if one fixes a relative pin structure on L, one can take natural quotients of $C^*_{f,\mathrm{Zap}}(L,L;R)$ in order to obtain simpler versions of Floer cohomology which still satisfy all the desired algebraic properties. In particular, if L is orientable and carries a relative spin structure, then there exists a quotient complex of $C^*_{f,\mathrm{Zap}}(L,L;R)$ which can be given a natural structure of a module over $R[T^{\pm 1}]$ (with T of degree N_L^{π}) and which satisfies all the properties in Assumption 3.0.14b). Thus, from now on, we declare that whenever we refer to the Biran-Cornea pearl complex with coefficients in a ground ring of characteristic different from 2, we mean exactly this quotient of $C^*_{f,\mathrm{Zap}}(L,L;R)$.

Remark 3.0.15. Strictly speaking, the quotient complex may depend on the choice of relative spin structure on L and so can the structure of its cohomology $HF^*_{\mathrm{BC}}(L,L;R)$ as a module over $QH^*_{\mathrm{BC}}(M;R)$. This dependence will not enter into our discussion, since we will only be using the properties listed in Assumption 3.0.14 and these hold for all choices of relative spin structure. //

Some facts about $\mathbb{CP}^n$

For the remaining part of this thesis we will be concerned almost exclusively with the symplectic manifold $(\mathbb{CP}^n;\omega_{\mathrm{FS}})$ for $n \geq 1$. Here ω_{FS} denotes the Fubini-Study symplectic form, normalised so that a line $\mathbb{CP}^1 \subseteq \mathbb{CP}^n$ has area π. We now collect some facts about this manifold, most of which are well-known and which we will use repeatedly in many of the arguments that follow. The notation established here will also be used in the rest of this work.

Let R denote any commutative ring. Recall first that the cohomology ring of $\mathbb{CP}^n$ is

$$H^*(\mathbb{CP}^n;R) = \frac{R[H]}{(H^{n+1})},$$

where the generator is $H := \mathrm{PD}_{\mathbb{CP}^n}(\mathbb{CP}^{n-1})$ is called the *hyperplane class*. With our chosen normalisation of the symplectic form, one has the equality $[\omega_{\mathrm{FS}}] = \pi H$ in $H^2(\mathbb{CP}^n;\mathbb{R})$. The first Chern class of $\mathbb{CP}^n$ is $c_1(T\mathbb{CP}^n) = (n+1)H$ and so we have

$$[\omega_{\mathrm{FS}}] = 2\frac{\pi}{2(n+1)}c_1(T\mathbb{CP}^n).$$

In particular, $\mathbb{CP}^n$ is a monotone symplectic manifold and any monotone Lagrangian $L \subseteq \mathbb{CP}^n$ has monotonicity constant $\lambda = \frac{\pi}{2(n+1)}$. Also, if $L \subseteq \mathbb{CP}^n$ is a Lagrangian with $H^1(L;\mathbb{R}) = 0$, then L is automatically monotone (recall (2.3)).

We denote the standard integrable complex structure on $\mathbb{CP}^n$ by J_0. It is compatible with ω_{FS} and so defines a Kähler metric g_{FS} on $\mathbb{CP}^n$. This metric is also Einstein, making $(\mathbb{CP}^n, g_{\mathrm{FS}}, J_0)$ into a Kähler-Einstein manifold. In particular, if $L \subseteq \mathbb{CP}^n$ is a Lagrangian submanifold, which is minimal with respect to g_{FS}, then L is monotone by a result of Cieliebak and Goldstein [CG04].

The manifold $\mathbb{CP}^n$ is simply connected and has $\pi_2(\mathbb{CP}^n) \cong \mathbb{Z}$. In particular, there is no ambiguity about the definition of minimal Chern number and one has $N_{\mathbb{CP}^n} = n+1$. If $L \subseteq (\mathbb{CP}^n, J_0)$ is an n-dimensional totally real submanifold, there is also no ambiguity about its minimal Maslov number N_L and one has that N_L divides $2(n+1)$. Moreover, L is orientable if and only if N_L is even. One can also make the following topological observation.

Lemma 3.0.16. *Let $L \subset \mathbb{CP}^n$ be a totally real submanifold with minimal Maslov number N_L and let $i\colon L \to \mathbb{CP}^n$ be the inclusion. Then one has $2(n+1)i^*H = 0 \in H^2(L;\mathbb{Z})$. Moreover, if $H^1(L;\mathbb{Z}) = 0$, then i^*H has order exactly $\frac{2(n+1)}{N_L}$ in $H^2(L;\mathbb{Z})$.*

Proof. The long exact sequence in cohomology for the pair $(\mathbb{CP}^n, L)$ yields the exact sequence

$$H^1(L;\mathbb{Z}) \longrightarrow H^2(\mathbb{CP}^n, L;\mathbb{Z}) \xrightarrow{j^*} \mathbb{Z}\langle H\rangle \xrightarrow{i^*} H^2(L;\mathbb{Z}).$$

Since the Maslov class $\mu_L \in H^2(\mathbb{CP}^n, L;\mathbb{Z})$ satisfies $j^*(\mu_L) = 2c_1(T\mathbb{CP}^n) = 2(n+1)H$, we immediately get $i^*(2(n+1)H) = 0$. Now, if $H^1(L;\mathbb{Z}) = 0$, we have that $H^2(\mathbb{CP}^n, L;\mathbb{Z})$ injects into $\mathbb{Z}\langle H\rangle$ and so it must be freely generated by some class $g \in H^2(\mathbb{CP}^n, L;\mathbb{Z})$, which is non-zero since μ_L is non-zero. By the universal coefficients theorem there exists a class $u \in H_2(\mathbb{CP}^n, L;\mathbb{Z})$ with which g pairs to 1, and so we must have $\mu_L = N_L g$. Applying j^* to both sides, we get $2(n+1)H = N_L j^*(g)$ and hence $j^*(g) = \frac{2(n+1)}{N_L} H$. By exactness of the above sequence, it follows that i^*H has order $\frac{2(n+1)}{N_L}$ in $H^2(L;\mathbb{Z})$. ☐

After these topological preliminaries, let us note some of the crucial properties of $\mathbb{CP}^n$ related to holomorphic curve theory. The quantum cohomology of $\mathbb{CP}^n$ over an arbitrary commutative ring R is

$$QH^*(\mathbb{CP}^n;R) = \frac{R\left[H, q^{\pm 1}\right]}{(H^{n+1} - q)},$$

where q is a formal variable of degree $2(n+1)$ which we may sometimes implicitly set to 1, if we are not interested in the grading. Note the crucial fact that $H \in QH^2(\mathbb{CP}^n;R)$ is an invertible element. Using this, together with the action of quantum cohomology on Lagrangian Floer cohomology, one can show that, if $L \subseteq \mathbb{CP}^n$ is a monotone Lagrangian, then

$$1 \leq N_L \leq n+1. \tag{3.2}$$

This bound was first established by Seidel in [Sei00, Theorem 3.1] but a proof along the lines of our above discussion can be found in [BC09b, Lemma 6.1.1].

Finally, we observe that combining Lemma 3.0.16 with the algebraic structures from section 2.4.4, has some immediate consequences for monotone Lagrangians in $\mathbb{CP}^n$ with high minimal Maslov number.

Lemma 3.0.17. *Let L be a monotone Lagrangian in $\mathbb{CP}^n$ with minimal Maslov number $N_L \geq 3$. Let R be any commutative ring and if R has characteristic different from 2, assume that L is orientable and relatively spin. Then $HF^*_{\mathrm{BC}}(L,L;R)$ is $2(n+1)$-torsion. Further, if $H^1(L;\mathbb{Z}) = 0$, then $HF^*_{\mathrm{BC}}(L,L;R)$ is $(2(n+1)/N_L)$-torsion.*

Proof. Since $N_L \geq 3$, we can specialise diagram (3.1) to obtain

$$\begin{tikzcd} H^2(\mathbb{CP}^n;R) \arrow[r, "i^*"] \arrow[d, "q_{\mathbb{CP}^n}"'] & H^2_{\mathrm{cls}}(L;R) \arrow[d, "q_L"] \\ QH^2_{\mathrm{BC}}(\mathbb{CP}^n;R) \arrow[r, "\mathcal{CO}^0_{\mathrm{BC}}"] & HF^2_{\mathrm{BC}}(L,L;R), \end{tikzcd}$$

from which we see that $\mathcal{CO}^0_{\mathrm{BC}}(H) = q_L(i^*(H))$. By Lemma 3.0.16, this implies that we have $\mathcal{CO}^0_{\mathrm{BC}}(2(n+1)H) = 0$ and, if $H^1(L;\mathbb{Z}) = 0$, then $\mathcal{CO}^0_{\mathrm{BC}}((2(n+1)/N_L)H) = 0$. Since H is invertible in $QH^*_{\mathrm{BC}}(\mathbb{CP}^n;R)$ and $\mathcal{CO}^0_{\mathrm{BC}}$ is a unital algebra map, we obtain the result we wanted. $\square$

Remark 3.0.18. Note that if L is not orientable and relatively spin, Lemma 3.0.17 can still be useful if $H^1(L;\mathbb{Z}) = 0$ and $2(n+1)/N_L$ is odd, because in this case it tells us that $HF^*_{\mathrm{BC}}(L,L;\mathbb{F}_2)$ vanishes. //

3.1 Lagrangians which look like $\mathbb{RP}^n$

This section is based on the paper [KS18] which is joint work with Jack Smith and is devoted to proving Theorem A. What we prove in fact is the following equivalent statement:

Theorem 3.1.1. *Let $L \subseteq \mathbb{CP}^n$ be a closed, connected monotone Lagrangian submanifold with minimal Maslov number $N_L = n+1$. Then the fundamental group of L is isomorphic to $\mathbb{Z}/2$ and the universal cover of L is homeomorphic to S^n.*

To see that this is equivalent to Theorem A, note that if L is homeomorphic to the quotient of S^n by a free $\mathbb{Z}/2$-action, then [HM64, Lemma 3] tells us that L is homotopy equivalent to $\mathbb{RP}^n$. Conversely, if L is homotopy equivalent to $\mathbb{RP}^n$, then its fundamental group is $\mathbb{Z}/2$ and its universal cover $\widetilde{L}$ is a homotopy sphere. But then the Poincaré conjecture (which is used in the proof of Theorem 3.1.1) implies that $\widetilde{L}$ is homeomorphic to S^n.

Remark 3.1.2. We will only give the full details of the proof in the case when n is odd. The case of even n is completely analogous but requires the use of Floer theory over $\mathbb{Z}$ for non-orientable Lagrangians and thus relies on Zapolsky's machinery. It is carried out in [KS18]. //

From now until the end of this section we let L denote a monotone Lagrangian in $\mathbb{CP}^n$ of minimal Maslov number $N_L = n+1$. We also assume that $n \geq 2$, since the case $n = 1$ of Theorem 3.1.1 is trivial. Lemma 3.0.17 is the crucial observation which allows us to prove the desired result but to be able to apply it we need to know that L is relatively pin. To this end we use some known results of Biran-Cornea about the Floer cohomology of L over $\mathbb{F}_2$ ([BC09b, Lemmas 6.1.3, 6.1.4]). Since Biran-Cornea state their results under an a priori stronger assumption on L than the one we are imposing here, we give the full proofs. We begin with a simple topological observation.

Lemma 3.1.3. *We have $H^1(L;\mathbb{F}_2) \neq 0$.*

Proof. The long exact sequence in homology for the pair $(\mathbb{CP}^n, L)$ gives the exact sequence

$$H_2(\mathbb{CP}^n;\mathbb{Z}) \longrightarrow H_2(\mathbb{CP}^n, L;\mathbb{Z}) \longrightarrow H_1(L;\mathbb{Z}) \longrightarrow 0.$$

Applying the left-exact functor $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{F}_2)$ we obtain the exact sequence

$$0 \longrightarrow H^1(L;\mathbb{Z}/2) \xrightarrow{\alpha} \mathrm{Hom}_{\mathbb{Z}}(H_2(\mathbb{CP}^n, L;\mathbb{Z}),\mathbb{F}_2) \xrightarrow{\beta} \mathrm{Hom}_{\mathbb{Z}}(H_2(\mathbb{CP}^n;\mathbb{Z}),\mathbb{F}_2),$$

and the penultimate term contains the mod 2 reduction I'_{μ_L} of $I_{\mu_L}/(n+1)$. Since $I_{\mu_L}/(n+1)$ restricts to $2I_{c_1}/(n+1)$ on $H_2(\mathbb{CP}^n;\mathbb{Z})$, and this is always even, we deduce that $\beta(I'_{\mu_L})$ is zero. This means that I'_{μ_L} is in the image of α, and as I'_{μ_L} itself is non-zero, we must have $H^1(L;\mathbb{F}_2) \neq 0$. $\square$

Now consider the Floer cohomology of L over $\mathbb{F}_2$. The p^{th} column of the first page of the associated spectral sequence is then $H^*(L;\mathbb{F}_2)[-pn]$, and for degree reasons the only potentially non-zero differentials in the whole spectral sequence map from $H^n(L;\mathbb{F}_2)[-(p-1)n] \cong \mathbb{F}_2$ to $H^0(L;\mathbb{F}_2)[-pn] \cong \mathbb{F}_2$ on this page. By construction these maps are independent of p, so it suffices to understand the case $p = 0$.

From this spectral sequence and the action of $\mathcal{CO}^0_{\mathrm{BC}}(H)$ we obtain the following two lemmas, which are basically [BC09b, Lemmas 6.1.3, 6.1.4].

Lemma 3.1.4. *There is an isomorphism of graded $\mathbb{F}_2$-vector spaces*

$$HF^*_{\mathrm{BC}}(L,L;\mathbb{F}_2) \cong \bigoplus_{p=-\infty}^{\infty} H^*(L;\mathbb{F}_2)[-p(n+1)].$$

That is $HF^k(L,L;\mathbb{F}_2) \cong \oplus_{p=-\infty}^{\infty} H^{k+(n+1)p}(L;\mathbb{F}_2) \cong H^{\ell_k}(L;\mathbb{F}_2)$, where $0 \leq \ell_k \leq n$ and $k \equiv \ell_k$ mod $(n+1)$. *Further, $H^k(L;\mathbb{F}_2) \cong \mathbb{F}_2$ for all $0 \leq k \leq n$.*

Proof. By the preceding discussion we see that to prove the first part, it is enough to show that the differential

$$d_1 : H^n(L;\mathbb{F}_2)[n] \to H^0(L;\mathbb{F}_2)$$

on the first page of the spectral sequence vanishes. Since the codomain comprises just 0 and the classical unit, we are done if the latter is not in the image of d_1. But the classical unit is also the unit e_L for the Floer product and so we simply need to check that $HF^*_{\mathrm{BC}}(L,L;\mathbb{F}_2)$ is non-zero. To see that this is indeed the case, observe that $H^1(L;\mathbb{F}_2)$ survives the spectral sequence and is non-zero by Lemma 3.1.3.

We thus have that $HF^*_{\mathrm{BC}}(L,L;\mathbb{F}_2) \cong \bigoplus_{p=-\infty}^{\infty} H^*(L;\mathbb{F}_2)[-p(n+1)]$. In particular, we see that $HF^0_{\mathrm{BC}}(L,L;\mathbb{F}_2) \cong H^0(L;\mathbb{F}_2) \cong \mathbb{F}_2$ and $HF^{-1}_{\mathrm{BC}}(L,L;\mathbb{F}_2) \cong H^n(L;\mathbb{F}_2) \cong \mathbb{F}_2$. But by invertibility of the hyperplane class H in quantum cohomology, Floer multiplication by $\mathcal{CO}^0_{\mathrm{BC}}(H)$ gives an isomorphism $HF^k_{\mathrm{BC}}(L,L;\mathbb{F}_2) \cong HF^{k+2}_{\mathrm{BC}}(L,L;\mathbb{F}_2)$ for every $k \in \mathbb{Z}$ and so we must have $HF^k_{\mathrm{BC}}(L,L;\mathbb{F}_2) \cong \mathbb{F}_2$ for all $k \in \mathbb{Z}$. This finishes the proof. □

Observe that Lemma 3.1.4 allows us to immediately complete the $n = 2$ case of Theorem 3.1.1 since, by the classification of surfaces, $\mathbb{RP}^2$ is the only closed surface whose first cohomology group over $\mathbb{F}_2$ is isomorphic to $\mathbb{F}_2$. It also allows us to deduce the following.

Lemma 3.1.5. *The group $H^2(L;\mathbb{F}_2)$ is isomorphic to $\mathbb{F}_2$ and is generated by i^*H, where $i : L \to \mathbb{CP}^n$ is the inclusion. In particular, L is relatively pin.*

Proof. By the above lemma we already know that $H^2(L;\mathbb{F}_2)$ is isomorphic to $\mathbb{F}_2$ and in fact, during the proof we saw that the following must hold:

1. Floer multiplication by $\mathcal{CO}^0_{\mathrm{BC}}(H)$ gives an isomorphism $HF^0_{\mathrm{BC}}(L,L;\mathbb{F}_2) \xrightarrow{\sim} HF^2_{\mathrm{BC}}(L,L;\mathbb{F}_2)$;

2. $HF^0_{\mathrm{BC}}(L,L;\mathbb{F}_2)$ is a copy of $\mathbb{F}_2$, generated by the unit e_L;

3. the map $q_L \colon H^2(L;\mathbb{F}_2) \to HF^2(L,L;\mathbb{F}_2)$ is an isomorphism (this map is well-defined when $n = 2$, since even then $d_1 : H^2(L;\mathbb{F}_2) \to H^0(L;\mathbb{F}_2)$ vanishes).

By the first two items, we conclude that $HF^2_{\mathrm{BC}}(L,L;\mathbb{F}_2)$ is a copy of $\mathbb{F}_2$, generated by $\mathcal{CO}^0_{\mathrm{BC}}(H) * e_L = \mathcal{CO}^0_{\mathrm{BC}}(H)$. Now, specialising diagram (3.1) gives the commutative diagram

$$\begin{tikzcd} H^2(\mathbb{CP}^n;\mathbb{F}_2) \arrow[r, "i^*"] \arrow[d, "q_{\mathbb{CP}^n}"'] & H^2(L;\mathbb{F}_2) \arrow[d, "q_L"] \\ QH^2_{\mathrm{BC}}(\mathbb{CP}^n;\mathbb{F}_2) \arrow[r, "\mathcal{CO}^0_{\mathrm{BC}}"] & HF^2_{\mathrm{BC}}(L,L;\mathbb{F}_2). \end{tikzcd}$$

The left-hand vertical map is an isomorphism between copies of $\mathbb{F}_2$, and the preceding discussion shows that the same is true for the right-hand vertical map and the bottom horizontal map. Hence the top horizontal map is also an isomorphism, which is what we wanted. □

At this point we make the assumption that L is orientable, which is equivalent to assuming that n is odd, since $\mathbb{CP}^n$ is simply-connected. By Lemma 3.1.5, we know that in this case L is relatively spin, so we can unleash the full power of Floer theory over $\mathbb{Z}$. Further, since $N_L = n+1 \geq 3$, there is no obstruction to using local systems of arbitrary rank for coefficients. For any cover L' of L, let $\mathcal{E}_{L'}$ denote the corresponding $\mathbb{Z}$-local system. Then each column on the first page of the Oh-Biran spectral sequence which computes $HF^*_{\mathrm{BC}}(L,(L,\mathcal{E}_{L'}))$ is isomorphic to a shifted copy of $H^*(L;\mathcal{E}_{L'})$

which by Proposition 2.1.1 is simply $H^*_c(L';\mathbb{Z})$. Further, all of the intermediate cohomology groups (meaning $0 < * < n$) survive the spectral sequence for degree reasons. The key result is the following:

Proposition 3.1.6. *For any cover L' of L the compactly-supported cohomology groups $H^k_c(L';\mathbb{Z})$ for $0 < k < n$ are 2-torsion and 2-periodic.*

Proof. Since these intermediate cohomology groups survive to $HF^*_{\mathrm{BC}}(L,(L,\mathcal{E}_{L'});\mathbb{Z})$, they are acted upon by the invertible element $\mathcal{CO}^0_{\mathrm{BC}}(H)$ of degree 2. This gives us 2-periodicity.

To prove 2-torsion, first let $L' = L$ and note that since $N_L \geq 3$, Lemma 3.0.17 tells us that $HF^*_{\mathrm{BC}}(L,L;\mathbb{Z})$ is $2(n+1)$-torsion. Since $HF^*_{\mathrm{BC}}(L,L;\mathbb{Z})$ contains the intermediate cohomology of L, then the latter is also torsion which, in particular, tells us that $H^1(L;\mathbb{Z}) = 0$ since the first cohomology must always be torsion-free. Now, by the second part of Lemma 3.0.17, we see that $HF^*_{\mathrm{BC}}(L,L;\mathbb{Z})$ is 2-torsion. Since for each cover L' the cohomology $HF^*_{\mathrm{BC}}(L,(L,\mathcal{E}_{L'});\mathbb{Z})$ is a (right) unital module over the ring $HF^*_{\mathrm{BC}}(L,L;\mathbb{Z})$, the former must also be 2-torsion. This in turn means that the intermediate compactly-supported cohomology groups of each L' are 2-torsion. □

We can now complete the proof of Theorem 3.1.1, by showing that L has fundamental group $\mathbb{Z}/2$ and universal cover homeomorphic to S^n.

Proof of Theorem 3.1.1 for n odd. Apply Proposition 3.1.6 to every connected cover L' of L to see that for every such cover the group $H^{n-1}_c(L';\mathbb{Z})$ is 2-torsion. Since L is orientable, Poincaré duality tells us that $H^{n-1}_c(L;\mathbb{Z})$ is isomorphic to $H_1(L;\mathbb{Z})$ and so the latter is 2-torsion.

By the Hurewicz theorem, this means that every subgroup of $\pi_1(L)$ has 2-torsion abelianisation. In particular, by considering the cyclic subgroups, we see that every element of $\pi_1(L)$ has order 2, so the group is abelian (every commutator $aba^{-1}b^{-1}$ is square $(ab)^2$ and hence equal to the identity). We deduce that $\pi_1(L)$ is isomorphic to $H_1(L;\mathbb{Z})$ and is 2-torsion. It is also finitely-generated (since L is compact) and so $\pi_1(L) \cong (\mathbb{Z}/2)^k$ for some $k \in \mathbb{N}$. On the other hand, by Lemma 3.1.4, we know that $H^1(L;\mathbb{F}_2) \cong \mathbb{F}_2$ and so $k = 1$.

Consider now the universal cover $\widetilde{L}$ of L, which is compact by the above discussion. By the Hurewicz and universal coefficients theorems $H^1(\widetilde{L};\mathbb{Z})$ vanishes and $H^2(\widetilde{L};\mathbb{Z})$ is torsion-free. Then, by Proposition 3.1.6 we see that $\widetilde{L}$ is an integral homology sphere. Since $\widetilde{L}$ is also simply-connected, the homology Whitehead theorem (see e.g. [May83]) and the Poincaré conjecture imply that $\widetilde{L}$ is homeomorphic to S^n. □

Remark 3.1.7. The proof when n is even proceeds along the same lines but one needs to replace "BC" by "Zap" everywhere. We are allowed to do this since by Lemma 3.1.5 L is relatively pin. There is also a small difference in the last part of the proof in order to apply Poincaré duality correctly. //

3.2 Monotone Lagrangians in $\mathbb{CP}^3$

In this section we focus on monotone Lagrangians in $\mathbb{CP}^3$ and prove Theorem B. Observe that we have $N_{\mathbb{CP}^3} = 4$ and so the possible values for the minimal Maslov number of a monotone Lagrangian are 1, 2 and 4. Since we cannot apply monotone Floer theory to Lagrangians with $N_L = 1$, we restrict our attention to the cases $N_L = 2$ and $N_L = 4$ in which we would like to classify Lagrangians up to diffeomorphism (recall that in dimension 3 the smooth and topological categories are equivalent). While in the case $N_L = 2$ we remain far from this goal, Theorem 3.1.1 allows us to deal with the case $N_L = 4$ straight away:

Proposition 3.2.1. *([KS18, Corollary 3]) If $L \subseteq \mathbb{CP}^3$ is a monotone Lagrangian with minimal Maslov number 4, then L is diffeomorphic to $\mathbb{RP}^3$.*

Proof. By Theorem 3.1.1, we know that L is homeomorphic to a quotient of S^3 by a free involution. By a result of Livesay ([Liv62, Theorem 3]) any such involution is conjugate to the antipodal map by a homeomorphism. Thus L is homeomorphic (equivalently, diffeomorphic) to $\mathbb{RP}^3$. □

From now on we focus on the case $N_L = 2$. Note that this implies that L is orientable and by a well-known theorem of Stiefel [Sti35], all closed, orientable 3-manifolds are parallelisable. In particular, they are spin, so we are free to use Floer theory with coefficients in any commutative ring. The main Floer-theoretic results we rely on in this section are the properties of the obstruction section from Assumption 3.0.14 a) and the following easy corollary of the AKS criterion:

Lemma 3.2.2. *Let $L \subseteq \mathbb{CP}^3$ be a monotone Lagrangian with $N_L = 2$ and let $\mathbb{K}$ be any field of characteristic different from 2. If $m_0(L;\mathbb{K}) = 0$, then $HF^*_{\mathrm{BC}}(L,L;\mathbb{K}) = 0$.*

Proof. Suppose $m_0(L;\mathbb{K}) = 0$. By the AKS theorem we have

$$\mathcal{CO}^0_{\mathrm{BC}}(4H) = \mathcal{CO}^0_{\mathrm{BC}}(c_1(T\mathbb{CP}^3)) = m_0(L;\mathbb{K})e_L = 0 \in HF^*_{\mathrm{BC}}(L,L;\mathbb{K}).$$

Since H is invertible in $QH^*_{\mathrm{BC}}(\mathbb{CP}^3;\mathbb{K})$ and $\mathbb{K}$ is a field of charateristic different from 2, then $4H$ is also invertible. But then, since $\mathcal{CO}^0_{\mathrm{BC}}$ is a unital algebra map, we see that 0 must be invertible in $HF^*_{\mathrm{BC}}(L,L;\mathbb{K})$ and thus $HF^*_{\mathrm{BC}}(L,L;\mathbb{K}) = 0$. □

In what follows we use the above result and the geometry of the moduli spaces of holomorphic discs in order to restrict the topology that a monotone Lagrangian in $\mathbb{CP}^3$ of minimal Maslov number 2 can have. Before we can start applying any Floer theory however, we need some preliminary observations about the topology of certain 3-manifolds. Note that throughout this section a 3-manifold will always be assumed compact and without boundary.

Notation 3.2.3. For a group G and an element $g \in G$ we write $C(g)$ for the centraliser of g, $\langle g \rangle$ for the cyclic subgroup generated by g and $[g]$ for the image of g in the abelianisation of G. If R is a commutative ring and $R[G]$ is the group algebra of G over R, we write $\varepsilon\colon R[G] \to R$ for the augmentation sending all elements of G to $1 \in R$. For an element $a \in R[G]$ we define the *support* of a to be the set $\operatorname{supp}(a) \subseteq G$ of elements which appear with non-zero coefficients in a. //

3.2.1 Preliminaries on 3-manifolds

Recall first that a closed 3-manifold N is called *prime* if the only way to write it as a connected sum of two manifolds is if one of them is the 3-sphere. The Prime Decomposition Theorem of Kneser and Milnor states that any closed orientable 3-manifold M can be decomposed as $M = N_1 \# N_2 \# \cdots \# N_r$, where each N_i is a prime 3-manifold and this decomposition is unique, up to rearranging the factors.

Observe that if a 3-manifold is not prime, then the belt spheres of the connecting tubes in the prime decomposition are embedded and homotopically non-trivial 2-spheres. A 3-manifold is called *irreducible* if every embedded 2-sphere bounds a 3-ball. Thankfully, the distinction between prime and irreducible 3-manifolds can be settled very easily: the only orientable 3-manifold which is prime but not irreducible is $S^1 \times S^2$ ([Hem76, Lemma 3.13]).

Papakyriakopoulos's Sphere Theorem ([Hem76, Theorem 4.3]) tells us that if N is an orientable, irreducible 3-manifold, then $\pi_2(N) = 0$. Thus, if N is orientable and irreducible and $\pi_1(N)$ is infinite, it follows by an application of the Hurewicz theorem to the universal cover of N that N is aspherical. Therefore, orientable, prime 3-manifolds come in three groups: the ones with finite fundamental group, $S^1 \times S^2$ and the aspherical ones.

3.2.1.1 Spherical manifolds

The 3-manifolds with finite fundamental group are the subject of the famous Elliptisation Theorem. It is a generalisation of the Poincaré conjecture and states that every closed 3-manifold with finite fundamental group is a quotient of the round S^3 by a free action of a finite group of isometries. The Elliptisation Theorem is part of the Geometrisation Theorem, proved by Grigori Perelman and we assume it in this work. The finite subgroups Γ of $\mathrm{SO}(4)$ which can act freely on S^3 have been listed by Milnor [Mil57] and fall into the following classes:

1. the trivial group,
2. $Q_{8n} = \langle x, y : x^2 = (xy)^2 = y^{2n} \rangle$ with abelianisation $\mathbb{Z}/2 \oplus \mathbb{Z}/2$,
3. $P_{48} = \langle x, y : x^2 = (xy)^3 = y^4, x^4 = 1 \rangle$ with abelianisation $\mathbb{Z}/2$,
4. $P_{120} = \langle x, y : x^2 = (xy)^3 = y^5, x^4 = 1 \rangle$ with trivial abelianisation,
5. $D_{2^k(2n+1)} = \langle x, y : x^{2^k} = 1, y^{2n+1} = 1, xyx^{-1} = y^{-1} \rangle$, $k \geq 2$, $n \geq 1$ with abelianisation $\mathbb{Z}/2^k$,
6. $P'_{8\cdot 3^k} = \langle x, y, z, : x^2 = (xy)^2 = y^2, zxz^{-1} = y, zyz^{-1} = xy, z^{3k} = 1 \rangle$ with abelianisation $\mathbb{Z}/3^k$,
7. the product of any of the above with a cyclic group C_m with m co-prime to the order.

We note for later that the only cases in which the abelianisation (i.e. the first homology group of the corresponding 3-manifold) contains an element of order 4 are:

$$\Gamma \cong C_{4n} \text{ or } \Gamma \cong D_{2^k(2n+1)} \times C_m \text{ for } n \geq 1,\ k \geq 2,\ \gcd(2^k(2n+1), m) = 1. \tag{3.3}$$

3.2.1.2 Seifert fibred manifolds

A larger class of 3-manifolds, with which we will mostly be concerned, are the so-called Seifert fibrable manifolds. We now define them and briefly describe their basic properties. For more details we refer the reader to [JN83].

Definition 3.2.4. A *Seifert fibration* of a closed, oriented[1] 3-manifold M is a smooth map $q\colon M \to \Sigma$ to a closed (possibly non-orientable) surface Σ such that for each point $x \in \Sigma$ there exists a neighbourhood $x \in U_x \subseteq \Sigma$, a pair of coprime integers α, α' with $\alpha \neq 0$ and a commuting diagram

$$\begin{tikzcd} q^{-1}(U_x) \arrow[r] \arrow[d, "q"] & S^1 \times D^2 \arrow[d] & (e^{i\theta}, re^{i\varphi}) \arrow[d, mapsto] \\ U_x \arrow[r, "x \mapsto 0"] & D^2 & r^{\alpha} e^{i(\alpha\varphi + \alpha'\theta)} \end{tikzcd}$$

where the horizontal arrows are diffeomorphisms. The set $q^{-1}(U_x)$ is called a *model neighbourhood* of the fibre $q^{-1}(x)$ and $|\alpha|$ is called the *multiplicity* of that fibre. If $|\alpha| > 1$, we call $q^{-1}(x)$ a *singular* fibre, otherwise we call it a *regular* fibre. Note that the only fibre in $q^{-1}(U_x)$ which is potentially singular is $q^{-1}(x)$.

Two Seifert fibrations $q_1\colon M \to \Sigma_1$, $q_2\colon M \to \Sigma_2$ are called isomorphic if there exists a commuting diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d, "q_1"] & M \arrow[d, "q_2"] \\ \Sigma_1 \arrow[r] & \Sigma_2 \end{tikzcd}$$

where the horizontal arrows are diffeomorphisms and the top arrow is orientation preserving.

We call an oriented 3-manifold M *Seifert fibrable*, if it admits a Seifert fibration. If an isomorphism class of such fibrations is understood, we call M a *Seifert fibred* manifold. $\diamondsuit$

Note that since only the central fibre in any model neighbourhood can be singular and the base Σ is compact, there can only be a finite number of singular fibres. Seifert fibrable manifolds are completely classified in the following sense: there is a complete set of invariants of a Seifert fibration (called the normalised Seifert invariants) which determine it up to isomorphism and, for manifolds admitting more than one isomorphism class of Seifert fibrations, the invariants of all such classes are known ([JN83, Theorem 5.1], [GL18]).

The (non-normalised) Seifert invariants of a given Seifert fibration are tuples of the form $(g; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$, where g is an integer and (α_i, β_i) is a pair of coprime integers for each

[1] There is a definition of a Seifert fibration for non-orientable manifolds as well but we won't need it in this work.

$1 \le i \le n$. From this data one can construct a Seifert fibration whose base is an oriented genus g surface, if $g \ge 0$, or a connected sum of $|g|$ copies or $\mathbb{RP}^2$, if g is negative. The resulting Seifert fibred 3-manifold will be denoted by $M(g;(\alpha_1,\beta_1),\dots,(\alpha_n,\beta_n))$. The details of how the construction goes and how to normalise the invariants can be found in [JN83]. For our purposes all we need are the following facts:

1. The Seifert fibred manifold $M(g;(\alpha_1,\beta_1),\dots,(\alpha_n,\beta_n))$ has at most n singular fibres and their multiplicities are among the numbers $\{|\alpha_1|,|\alpha_2|,\dots,|\alpha_n|\}$. The remaining α_i are all equal to ± 1.

2. Let $M = M(g;(\alpha_1,\beta_1),\dots,(\alpha_n,\beta_n))$ and let $x \in M$ be a point which lies on a regular fibre. If $g \ge 0$, then $\pi_1(M,x)$ has the presentation

$$\Big\langle a_1,b_1,a_2,b_2,\dots,a_g,b_g,q_1,q_2,\dots,q_n,h \mid \\ h \text{ central},\ q_i^{\alpha_i}h^{\beta_i},\ q_1q_2\dots q_n[a_1,b_1]\dots[a_n,b_n]\Big\rangle,$$

while, if $g < 0$, then $\pi_1(M,x)$ has the presentation

$$\Big\langle a_1,a_2,\dots,a_{|g|},q_1,q_2,\dots,q_n,h \mid \\ a_jha_j^{-1} = h^{-1},\ [h,q_i],\ q_i^{\alpha_i}h^{\beta_i},\ q_1q_2\dots q_na_1^2a_2^2\dots a_{|g|}^2\Big\rangle. \quad (3.4)$$

In these presentations h denotes the class of the regular fibre through x.

A Seifert fibration $q\colon M \to \Sigma$ gives rise to two important subgroups of $\pi_1(M)$ which we now define.

Definition 3.2.5. Let Σ^+ denote the minimal orientable cover of Σ (i.e. Σ itself, if it is orientable, or its orientable double cover, otherwise) and let $\pi^+\colon \Sigma^+ \to \Sigma$ be the covering map. The subgroup $C := q_*^{-1}(\pi_*^+(\pi_1(\Sigma^+)) \le \pi_1(M)$ is called the *canonical subgroup* of $\pi_1(M)$. Let $N := \langle h\rangle \le \pi_1(M)$ denote the subgroup generated by the class of a regular fibre. The group N is called the *Seifert fibre subgroup* of $\pi_1(M)$. $\diamondsuit$

Note that C is the whole of $\pi_1(M)$ when Σ is orientable and has index 2, otherwise. In particular, it is always a normal subgroup. Also, it can be seen from the presentations that N is also normal. This justifies the omission of base points in the above definition.

Let us now briefly discuss Seifert fibrable manifolds from the point of view of the 3-dimensional geometries. Recall that a 3-dimensional model geometry is a pair (X,G), where X is a simply connected (not necessarily closed) 3-manifold and G is a Lie group of diffeomorphisms of X, acting transitively on X with compact stabilisers and maximal amongst such groups of diffeomorphisms. One says that a closed 3-manifold M is modelled on (X,G) if M is diffeomorphic to X/Γ for some discrete subgroup $\Gamma \le G$ acting freely on X. Thurston (see e.g. [Thu97, Theorem 3.8.4]) has classified all 3-dimensional model geometries for which there exists at least one closed 3-manifold modelled on (X,G) and they are: $\mathbb{E}^3$ (Euclidean), S^3 (spherical), $\mathbb{H}^3$ (hyperbolic), $S^2\times\mathbb{E}^1$, $\mathbb{H}^2\times\mathbb{E}^1$, $\widetilde{\mathrm{SL}(2,\mathbb{R})}$, Nil and Sol (in each case the group G is the group of isometries of the given space, when the space is equipped with the obvious Riemannian metric, in the case of the first five, or any left-invariant metric, in the case of the last three). A 3-manifold is called *geometric* if it is modelled on one of these 8 geometries. It is a fact (see [Sco83a, Theorem 5.2]) that if a 3-manifold is geometric, then it can be modelled on exactly one of these geometries.

All Seifert fibrable 3-manifolds are geometric ([Sco83a, Theorem 5.3]) and the model geometry is determined by two invariants of the Seifert fibration – its Euler number and the orbifold Euler characteristic of the base. Moreover, 6 of the 8 geometries – all apart from $\mathbb{H}^3$ and Sol – are populated only by Seifert fibrable manifolds. It is also known that Seifert fibrable spaces which admit more than one isomorphism class of Seifert fibrations necessarily possess one of the geometries S^3, $S^2\times\mathbb{E}^1$ or $\mathbb{E}^3$ ([Sco83a, Theorem 3.8]). In particular, a non-Euclidean, aspherical Seifert fibrable manifold admits a unique Seifert fibration up to isomorphism.

We now discuss the Euclidean 3-manifolds in a little more detail, because they play an important role in our partial classification of monotone Lagrangians in $\mathbb{CP}^3$.

3.2.1.3 The chiral platycosms

It is well known that there are only 10 diffeomorphism classes of closed 3-manifolds which admit a Euclidean geometry (see [Sco83a, Table 4.4] and the discussion thereafter). We will adopt Conway's terminology from the paper [CR03] which refers to these as *platycosms* and gives names to all of them. Of the 10 platycosms, exactly 6 are orientable – the *chiral* platycosms – and they are:

1. the *torocosm* $T^3 := S^1\times S^1\times S^1$, a.k.a. the 3-torus,
2. the *dicosm* $L_2 = M(-2;(1,0))$; it is the only Euclidean circle bundle over the Klein bottle with orientable total space and it also admits a Seifert fibration over the sphere with invariants $M(0;(2,1),(2,1),(2,-1),(2,-1))$ (see [JN83, Theorem 5.1])
3. the *tricosm* $L_3 = M(0;(1,-1),(3,1),(3,1),(3,1))$ which is the mapping torus of an order 3 diffeomorphism of T^2,
4. the *tetracosm* $L_4 = M(0;(1,-1),(2,1),(4,1),(4,1))$ which is the mapping torus of an order 4 diffeomorphism of T^2,
5. the *hexacosm* $L_6 = M(0;(1,-1),(2,1),(3,1),(6,1))$ which is the mapping torus of an order 6 diffeomorphism of T^2,
6. the *didicosm* $L_{22} = M(-1;(1,-1),(2,1),(2,1))$, a.k.a. the Hantzsche -Wendt manifold, which is the only 3-dimensional Euclidean rational homology sphere.

All platycosms are quotients of $\mathbb{E}^3$ by a crystallographic group Γ, i.e. a discrete and co-compact subgroup of the group of isometries of 3-dimensional Euclidean space. By a well-known theorem of Bieberbach, the subgroup $\Lambda \leq \Gamma$, consisting of all translations in Γ, is isomorphic to $\mathbb{Z}^3$ and has finite index. Moreover, any other abelian subgroup of Γ with finite index is contained in Λ (see e.g. [Szc12, Lemma 2.6]). This means that for each platycosm L, there exists a minimal torus cover $p_{\min}\colon T^3 \to L$ such that any other cover $p\colon T^3 \to L$ factors through $p_{\min}$.

What will be important to us is the map that $p_{\min}$ induces on first homology groups. Using [CR03, Table 6], we now describe this map for all chiral platycosms:

1. For the 3-torus, the map $p_{\min}$ is just the identity.

2. For the dicosm L_2, one has $H_1(L_2;\mathbb{Z}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}$ and the map $(p_{\min})_*\colon H_1(T^3;\mathbb{Z}) \to H_1(L_2;\mathbb{Z})$ can be represented by the matrix

$$\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\begin{pmatrix}1&0&0\\0&1&0\\0&0&2\end{pmatrix}} \mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}\ . \tag{3.5}$$

3. For the tricosm L_3, one has $H_1(L_3;\mathbb{Z}) \cong \mathbb{Z}/3 \oplus \mathbb{Z}$ and the map $(p_{\min})_*\colon H_1(T^3;\mathbb{Z}) \to H_1(L_3;\mathbb{Z})$ can be represented by the matrix

$$\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\begin{pmatrix}1&1&0\\0&0&3\end{pmatrix}} \mathbb{Z}/3\oplus\mathbb{Z}\ . \tag{3.6}$$

4. For the tetracosm L_4, one has $H_1(L_4;\mathbb{Z}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}$ and the map $(p_{\min})_*\colon H_1(T^3;\mathbb{Z}) \to H_1(L_4;\mathbb{Z})$ can be represented by the matrix

$$\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\begin{pmatrix}1&1&0\\0&0&4\end{pmatrix}} \mathbb{Z}/2\oplus\mathbb{Z}\ . \tag{3.7}$$

5. For the hexacosm L_6, one has $H_1(L_6;\mathbb{Z}) \cong \mathbb{Z}$ and the map $(p_{\min})_*\colon H_1(T^3;\mathbb{Z}) \to H_1(L_6;\mathbb{Z})$ can be represented by the matrix

$$\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\begin{pmatrix}0&0&6\end{pmatrix}} \mathbb{Z}\ . \tag{3.8}$$

6. For the didicosm L_{22}, one has $H_1(L_{22};\mathbb{Z}) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/4$ and the map $(p_{\min})_*\colon H_1(T^3;\mathbb{Z}) \to H_1(L_{22};\mathbb{Z})$ can be represented by the matrix

$$\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\begin{pmatrix}2&0&2\\0&2&2\end{pmatrix}} \mathbb{Z}/4\oplus\mathbb{Z}/4\ . \tag{3.9}$$

3.2.1.4 Dominating maps from circle bundles to Seifert fibred manifolds

Finally in this preliminary section we prove a useful property of maps from a principal circle bundle to an aspherical Seifert fibred manifold – the degree of such a map must be divisible by the multiplicities of all singular fibres of the target. This will be proved in Lemma 3.2.9. The main tool we rely on is the classification up to homotopy of maps between aspherical Seifert fibred manifolds, proved by Yongwu Rong ([Ron93, Theorem 3.2]). To be able to use his results however, we first need a couple of definitions.

Definition 3.2.6. Let $f\colon M \to N$ be a map between 3-manifolds and let $K_1, K_2, \dots, K_s$ be a collection of knots in N. We say that f is a *branched covering*, branched over $K_1, K_2, \dots, K_s$, if the following two conditions hold:

1. $f|_{M\setminus f^{-1}(\cup_{i=1}^s K_i)} : M \setminus f^{-1}(\cup_{i=1}^s K_i) \to N \setminus (\cup_{i=1}^s K_i)$ is a covering map,
2. for each $1 \le i \le s$ the preimage $f^{-1}(K_i)$ is a collection of disjoint closed curves $C_{i1}, C_{i2}, \dots, C_{ik_i}$ in M, such that for each $1 \le j \le k_i$ there exist tubular neighbourhoods

$$\begin{aligned} \varphi_{ij}\colon S^1 \times D^2 &\to U_{ij} \subseteq M \\ S^1 \times \{0\} &\to C_{ij} \\ \psi_{ij}\colon S^1 \times D^2 &\to V_{ij} \subseteq N \\ S^1 \times \{0\} &\to K_i \end{aligned}$$

of C_{ij} and K_i, a pair of positive integers m_{ij}, n_{ij} and an integer q_{ij} such that $f(U_{ij}) = V_{ij}$ and the diagram

$$\begin{tikzcd} S^1 \times D^2 \arrow[r, "(u,z)\mapsto(u^{n_{ij}},\, u^{q_{ij}} z^{m_{ij}})"] \arrow[d, "\varphi_{ij}"] & S^1 \times D^2 \arrow[d, "\psi_{ij}"] \\ U_{ij} \arrow[r, "f"] & V_{ij} \end{tikzcd}$$

commutes. The integer m_{ij} is called the *branch multiplicity* of f at C_{ij}. $\diamondsuit$

In [Ron93] Rong defines an operation called a *vertical pinch* which transforms one Seifert fibred manifold into another, where the base of the second one has potentially smaller genus. We will only need to apply vertical pinches to circle bundles, so we only give a definition in this simplified case.

Definition 3.2.7. Let $q\colon M \to \Sigma$ be an oriented S^1 bundle over an oriented closed surface Σ. Let $C \subseteq \Sigma$ be a simple closed curve which separates Σ into Σ_1 and Σ_2. Let $M_1 := q^{-1}(\Sigma_1)$, $M_2 := q^{-1}(\Sigma_2)$ and suppose that there exists a map $f\colon M_2 \to S^1 \times D^2$ such that

$$f|_{\partial M_2} : \partial M_2 \to S^1 \times \partial D^2$$

is a homeomorphism. Then there exists a degree 1 map

$$\pi = \mathrm{id} \cup f\colon M = M_1 \cup_{q^{-1}(C)} M_2 \longrightarrow \overline{M} := M_1 \cup_{f|_{\partial M_2}} (S^1 \times D^2).$$

We call π a *vertical pinch* of M along C. $\diamondsuit$

We now observe that applying a vertical pinch to a circle bundle produces another circle bundle.

Lemma 3.2.8. *In the setting of Definition 3.2.7 and with $\overline{\Sigma} := \Sigma_1 \cup_\partial D^2$ one has that $\overline{M}$ naturally admits a circle bundle structure $\bar{q}\colon \overline{M} \to \overline{\Sigma}$ which extends $q|_{M_1} : M_1 \to \Sigma_1$.*

Proof. We have $\overline{M} = M_1 \cup_{f|_{\partial M_2}} (S^1 \times D^2)$ and on M_1 we still have the circle bundle structure $q|_{M_1} : M_1 \to \Sigma_1$. We need to extend this map to $S^1 \times D^2$. Since $q|_{M_2} : M_2 \to \Sigma_2$ is an oriented circle bundle over an oriented surface with boundary, it is trivial and so we can choose a section $\Sigma_2 \hookrightarrow M_2$, identifying M_2 with $S^1 \times \Sigma_2$. Now let the action of $f|_{\partial M_2} : S^1 \times \partial\Sigma_2 \to S^1 \times \partial D^2$ on first homology be given by a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$, where we have chosen the pairs $([S^1 \times \mathrm{pt}], [\mathrm{pt} \times \partial\Sigma_2])$ and $([S^1 \times \mathrm{pt}], [\mathrm{pt} \times \partial D^2])$ as bases for $H_1(S^1 \times \partial\Sigma_2;\mathbb{Z})$ and $H_1(S^1 \times \partial D^2;\mathbb{Z})$, respectively. We have a commuting diagram

$$\begin{tikzcd}
\mathbb{Z}\oplus\mathbb{Z} \arrow[r, equal] \arrow[d, "\begin{pmatrix} a & b \\ c & d \end{pmatrix}"'] & H_1(S^1 \times \partial\Sigma_2;\mathbb{Z}) \arrow[r] \arrow[d, "f|_{\partial M_2}"] & H_1(S^1 \times \Sigma_2;\mathbb{Z}) \arrow[d, "f"] \\
\mathbb{Z}\oplus\mathbb{Z} \arrow[r, equal] & H_1(S^1 \times \partial D^2;\mathbb{Z}) \arrow[r] & H_1(S^1 \times D^2;\mathbb{Z})
\end{tikzcd}$$

Since $[\mathrm{pt} \times \partial\Sigma_2]$ vanishes in $H_1(S^1 \times \Sigma_2;\mathbb{Z})$, we see from the diagram that we must have $b = 0$ and so $a = d = \pm 1$. In particular, $[S^1 \times \mathrm{pt}]$ is mapped by f to a generator of $H_1(S^1 \times D^2;\mathbb{Z})$. Hence the fibration $q \circ \left(f|_{S^1 \times \partial\Sigma_2}\right)^{-1} : S^1 \times \partial D^2 \to C$ extends to a circle fibration $\bar{q}|_{S^1 \times D^2} : S^1 \times D^2 \to D^2$ which agrees with $q|_{M_1} : M_1 \to \Sigma_1$ on the boundary. So these two maps glue to define a circle fibration $\bar{q}\colon \overline{M} \to \Sigma_1 \cup_\partial D^2$. ☐

With these notions at hand, we can now use the main result of [Ron93] and prove the lemma that we need.

Lemma 3.2.9. *Let M be an oriented circle bundle over an oriented, closed surface of positive genus and let L be an oriented, aspherical Seifert fibred manifold. Let $f\colon M \to L$ be a map of degree d. Then d is divisible by the multiplicity of each singular fibre of the Seifert fibration of L.*

Remark 3.2.10. By taking double covers of the domain and target if necessary, one can relax the requirement that M be an *oriented bundle*. The conclusion remains the same, except that if L has singular fibres of order 2, this does not force the degree of f to be divisible by 2: consider for example the identity map on the dicosm, when the target is given the Seifert fibration over S^2 with four singular fibres of multiplicity 2. //

Proof. If $d = 0$, there is nothing to prove, so assume $d \neq 0$. Then by [Ron93, Theorem 3.2] the map f is homotopic to a composition $p \circ \bar{f} \circ \pi$, where $\pi\colon M \to \overline{M}$ is a composition of finitely many vertical pinches, $\bar{f}\colon \overline{M} \to \overline{L}$ is a fibre-preserving branched covering, branched over fibres and $p\colon \overline{L} \to L$ is a covering. Here $\overline{M}$ is given the circle bundle structure provided by iterated application of Lemma 3.2.8.

In fact, as pointed out in [Ron93, Remark 2], the covering p can be taken to be the identity, if L is not a Euclidean manifold. It is not hard to see (by an argument analogous to the proof of

Lemma 3.2.20 below) that, if p cannot be taken to be the identity, then $\overline{L}$ is a torus and so p factors through the minimal torus covering $p_{\min}$. It can be checked directly that the degree of $p_{\min}$ is always divisible by all multiplicities of singular fibres of a Seifert fibration of a chiral platycosm. So from now on, we assume that L is not Euclidean and that p is the identity. Then, since $\deg(\pi) = 1$, we have $\deg(\bar{f}) = \deg(f) = d$. We now compute the degree of $\bar{f}$.

Let $K \subseteq L$ denote a fibre of multiplicity α of the Seifert fibration of L. Choose an orientation of K, let $x \in K$ be a point and let $r \in \pi_1(L,x)$ denote the corresponding class of K. From the description of the local neighbourhood of a Seifert fibre, we see that the element $h := r^\alpha$ generates the Seifert fibre subgroup of $\pi_1(L,x)$. Now fix $y \in \overline{M}$ such that $\bar{f}(y) = x$ and let $t \in \pi_1(\overline{M}, y)$ denote the class of the circle fibre passing through y. Since L is non-Euclidean, [Ron93, Lemma 2.1] (or see the proof of Lemma 3.2.20 below) implies that $\bar{f}_*(t) = h^n$ for some $n \in \mathbb{Z}$.

Since $\bar{f}$ is a fibre-preserving branched covering, branched over fibres, we have that $\bar{f}^{-1}(K) = C_1 \sqcup C_2 \sqcup \ldots \sqcup C_\ell$, where each C_i is a fibre of $\overline{M}$ at which $\bar{f}$ has some branch multiplicity $m_i \in \mathbb{N}_{>0}$. We claim that each restriction $\bar{f}\big|_{C_i} : C_i \to K$ is a finite covering of degree $|n\alpha|$.

To see this, assume without loss of generality that $y \in C_1$ and set $k_1 = \deg(\bar{f}\big|_{C_1})$. Then we have the equation $\bar{f}_*(t) = r^{k_1}$ at the level of fundamental groups. However, we know that $\bar{f}_*(t) = h^n$ and $h = r^\alpha$. So the above equation becomes $r^{k_1} = r^{n\alpha}$. Since L is aspherical, $\pi_1(L,x)$ is torsion-free, so we must have $k_1 = n\alpha$. Now for $i \in \{2, \ldots, \ell\}$, let $y_i \in C_i$ be a pre-image of the base point $x \in L$ and let $t_i \in \pi_1(\overline{M}, y_i)$ denote the class of C_i. Writing $k_i = \deg(\bar{f}\big|_{C_i})$, we have $\bar{f}_*(t_i) = r^{k_i}$. But we know that $\bar{f}_*(t_i)$ is a conjugate of $\bar{f}_*(t) = h^n$ (by an element of $\pi_1(L,x)$ which is the homotopy class of the image under $\bar{f}$ of some path in $\overline{M}$ which connects y_i and y). From the presentations (3.4), we see that the conjugacy class of h^n is either $\{h^n\}$ or $\{h^n, h^{-n}\}$. Thus we have $\bar{f}_*(t_i) = h^{\pm n}$ and the same argument as above shows that $k_i = \pm n\alpha$. We have shown that the restriction $\bar{f}\big|_{C_i} : C_i \to K$ is indeed a finite covering of degree $|n\alpha|$ for every $i \in \{1, \ldots, \ell\}$.

Now let $x' \in L$ be a regular value of $\bar{f}$ which is close to x. Then, from the definition of branched covering, we see that $\bar{f}^{-1}(\{x'\})$ has cardinality $\sum_{i=1}^{\ell} |n||\alpha| m_i$. Since the local degree of a branched covering does not change sign, we must have $|d| = |\deg(\bar{f})| = |n\alpha(\sum_{i=1}^{\ell} m_i)|$ which is divisible by $|\alpha|$, as we wanted to show. □

3.2.2 Proof of the main result

We are now ready to prove the following result, which together with Proposition 3.2.1 yields Theorem B.

Theorem 3.2.11. *Let $L \subseteq \mathbb{CP}^3$ be a closed, monotone Lagrangian with $N_L = 2$. Then L satisfies exactly one of the following:*

a) L is diffeomorphic to a quotient of S^3 by a discrete subgroup $\Gamma \leq \mathrm{SO}(4)$ with $\Gamma \cong C_{4k}$ for $k \geq 1$ or $\Gamma \cong D_{2^k(2n+1)} \times C_m$ for $k \geq 2$, $n \geq 1$, $gcd(2^k(2n+1), m) = 1$;

b) L is diffeomorphic to $S^1 \times S^2$;

c) L is diffeomorphic to T^3 or the tricosm L_3;

d) L is a non-Euclidean principal circle bundle over an orientable, aspherical surface and the Euler class of this bundle is divisible by 4.

Before we go into the proof, let us extract an immediate corollary regarding rational homology spheres:

Corollary 3.2.12. *Let $L \subseteq \mathbb{CP}^3$ be a closed Lagrangian submanifold. If $H_1(L;\mathbb{Q}) = 0$, then either L is diffeomorphic to $\mathbb{RP}^3$ or to one of the manifolds in case a) of Theorem 3.2.11.*

Proof. The condition $H_1(L;\mathbb{Q}) = 0$ implies that L is monotone. Further, since L is a closed 3-manifold, it has vanishing Euler characteristic and so we must have $H_2(L;\mathbb{Q}) = 0$, $H_3(L;\mathbb{Q}) \cong \mathbb{Q}$. Hence L is a rational homology sphere, in particular it is orientable. So either L has minimal Maslov number 4, in which case it is diffeomorphic to $\mathbb{RP}^3$ by Proposition 3.2.1, or it has minimal Maslov number 2 and we see that the only rational homology spheres which are allowed by Theorem 3.2.11 are the spherical space forms in case a). $\square$

We now begin the proof of Theorem 3.2.11, starting with some simple topological observations which already allow us to deal with case a).

3.2.2.1 Soft observations

In this short section we make some observations about the topology of L which follow simply from the existence of a Maslov 2 class in $H_2(\mathbb{CP}^3, L;\mathbb{Z})$, satisfying certain conditions. In what follows, L always denotes a closed Lagrangian (in fact, it suffices for L to be totally real) submanifold of $\mathbb{CP}^3$ with $N_L = 2$. Note that we have a well-defined surjective homomorphism $\frac{1}{2}I_{\mu_L}\colon H_2(\mathbb{CP}^3, L;\mathbb{Z}) \to \mathbb{Z}$ and, if $j_*\colon H_2(\mathbb{CP}^3;\mathbb{Z}) \to H_2(\mathbb{CP}^3, L;\mathbb{Z})$ is the natural map, then $\frac{1}{2}I_{\mu_L}(j_*(H_2(\mathbb{CP}^3;\mathbb{Z}))) = 4\mathbb{Z}$. Hence, by the long exact sequence in homology for the pair $(\mathbb{CP}^3, L)$, we get a well-defined homomorphism $I^{\dagger}_{\mu_L}\colon H_1(L;\mathbb{Z}) \to \mathbb{Z}/4$ which fits into the commutative diagram

$$\begin{tikzcd}
H_2(L;\mathbb{Z}) \arrow[r, "0"] & H_2(\mathbb{CP}^3;\mathbb{Z}) \arrow[r, "j_*"] \arrow[d, equal] & H_2(\mathbb{CP}^3, L;\mathbb{Z}) \arrow[r, "\partial"] \arrow[d, "\frac{1}{2}I_{\mu_L}"] & H_1(L;\mathbb{Z}) \arrow[r] \arrow[d, "I^{\dagger}_{\mu_L}"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "4"] & \mathbb{Z} \arrow[r] & \mathbb{Z}/4 \arrow[r] & 0
\end{tikzcd} \tag{3.10}$$

We use the homomorphism $I^{\dagger}_{\mu_L}$ to prove the following three easy lemmas.

Lemma 3.2.13. *Let $[u] \in H_2(\mathbb{CP}^3, L;\mathbb{Z})$ be a Maslov* 2 *class with boundary $\partial[u] \in H_1(L;\mathbb{Z})$. If there is a class $[r] \in H_1(L;\mathbb{Z})$ and an integer m such that $\partial[u] = m[r]$, then m is odd (in particular, $m \neq 0$). Further, if $\partial[u]$ has finite order k in $H_1(L;\mathbb{Z})$, then* 4 *divides k. In particular, if $H_1(L;\mathbb{Z})$ is finite, then it contains an element of order* 4.

Proof. If $[u] \in H_2(\mathbb{CP}^3, L; \mathbb{Z})$ is a Maslov 2 class, then $I^{\dagger}_{\mu_L}(\partial[u]) = 1$. Thus, if $\partial[u] = m[r]$, we have $mI^{\dagger}_{\mu_L}([r]) = 1 \in \mathbb{Z}/4$, so m is odd. On the other hand, if $\partial[u]$ has finite order k in $H_1(L;\mathbb{Z})$, then we have $0 = kI^{\dagger}_{\mu_L}(\partial[u]) = k$ in $\mathbb{Z}/4$, so 4 divides k. $\square$

This already suffices to see where case a) of Theorem 3.2.11 comes from. Indeed, if $\pi_1(L)$ is finite, then by the Elliptisation theorem L must be diffeomorphic to the quotient of S^3 by one of the groups listed in Section 3.2.1.1. From Lemma 3.2.13, we see that the only possibilities are the ones given in (3.3).

The next lemma will be used in combination with Lemma 3.2.9 in order to prove Proposition 3.2.22 below.

Lemma 3.2.14. *Suppose that L is Seifert fibred and let $h \in \pi_1(L)$ denote a generator of the Seifert fibre subgroup. Suppose further that there exist a Maslov 2 class $[u] \in H_2(\mathbb{CP}^3, L; \mathbb{Z})$ and a (necessarily odd) integer n such that $\partial[u] = n[h]$, where $[h] \in H_1(L;\mathbb{Z})$ denotes the homology class of h. Then the base of the Seifert fibration is orientable and the multiplicities of all singular fibres are odd.*

Proof. It can be seen from the presentation (3.4) of the fundamental group of L, that if the base of the Seifert fibration were non-orientable, then $2[h] = 0$ and so $2\partial[u] = 0$, which contradicts Lemma 3.2.13. Hence the base is orientable. Now let $[r] \in H_1(L;\mathbb{Z})$ denote the homology class of a fibre of multiplicity α. It follows by the description of the model neighbourhood, that $[h] = \pm\alpha[r]$ and so $\partial[u] = \pm n\alpha[r]$. By Lemma 3.2.13 α must be odd. $\square$

The next lemma will be used to show that $\mathbb{CP}^3$ does not contain monotone Lagrangian chiral platycosms, other than T^3 and, potentially, the tricosm.

Lemma 3.2.15. *Suppose L is a chiral platycosm, other than T^3 or the tricosm. Fix a Seifert fibration on L, where if L is the dicosm, we are free to choose either of the two isomorphism classes of Seifert fibrations. Let $[u] \in H_2(\mathbb{CP}^3, L; \mathbb{Z})$ be a Maslov 2 class. Then $\partial[u]$ does not lie in the image of the Seifert fibre subgroup under the Hurewicz homomorphism $\pi_1(L) \to H_1(L;\mathbb{Z})$. Further, if $p\colon T^3 \to L$ is a covering, then $\partial[u]$ is not contained in $p_*H_1(T^3;\mathbb{Z})$.*

Proof. The possible Seifert fibrations of chiral platycosms other than T^3 were given in Section 3.2.1.3. Except in the case of the tricosm, each of these Seifert fibrations either has fibres of even multiplicity or is over a non-orientable base (or both, in the case of the didicosm). It follows by Lemma 3.2.14 that $\partial[u]$ cannot be contained in the Hurewicz image of the Seifert fibre subgroup.

Suppose now that $p\colon T^3 \to L$ is a covering and $\partial[u] \in p_*H_1(T^3;\mathbb{Z})$. Since p must factor through the minimal torus covering $p_{\min}\colon T^3 \to L$, we have in particular that $\partial[u] \in (p_{\min})_*H_1(T^3;\mathbb{Z})$. However, by inspecting (3.5), (3.7), (3.8) and (3.9), we see that

$$I^{\dagger}_{\mu_L}\big((p_{\min})_*H_1(T^3;\mathbb{Z})\big) \subseteq 2(\mathbb{Z}/4).$$

This contradicts the fact that $I^{\dagger}_{\mu_L}(\partial[u]) = 1$. □

3.2.2.2 The main argument

We are finally ready to dive into Floer theory and prove Theorem 3.2.11. From now on we let L denote a closed, connected, monotone Lagrangian in $\mathbb{CP}^3$ with $N_L = 2$. As we already explained in the previous section, if L has finite fundamental group, then it must be one of the manifolds in case a) of Theorem 3.2.11. Therefore, we also assume that L has infinite fundamental group.

In order to obtain information about the fundamental group of L we will use Floer theory with local coefficients. In fact, the only local system we will need is $\mathcal{E}^R_{\mathrm{reg}} \to L$ but it will be important to work over different ground rings R. We make the following observation:

Lemma 3.2.16. *Let $\mathbb{K}$ be a field and suppose that the pearl complex for the pair $(L,(L,\mathcal{E}^{\mathbb{K}}_{\mathrm{reg}}))$ is unobstructed. Then either $\mathbb{K}$ has characteristic 2 or $m_0(L;\mathbb{K}) \neq 0$.*

Proof. Since we are assuming that L has infinite fundamental group, its universal cover is non-compact and so, by Proposition 2.1.1 and Poincaré duality, we have

$$H^0(L;\mathcal{E}^{\mathbb{K}}_{\mathrm{reg}}) = H^0_c(\widetilde{L};\mathbb{K}) \cong H_3(\widetilde{L};\mathbb{K}) = 0,$$

$$H^2(L;\mathcal{E}^{\mathbb{K}}_{\mathrm{reg}}) = H^2_c(\widetilde{L};\mathbb{K}) \cong H_1(\widetilde{L};\mathbb{K}) = 0,$$

$$H^3(L;\mathcal{E}^{\mathbb{K}}_{\mathrm{reg}}) = H^3_c(\widetilde{L};\mathbb{K}) \cong H_0(\widetilde{L};\mathbb{K}) \cong \mathbb{K}$$

for any field $\mathbb{K}$. Hence, if the pearl complex for the pair $(L,(L,\mathcal{E}^{\mathbb{K}}_{\mathrm{reg}}))$ is unobstructed, the Oh-Biran spectral sequence which computes its cohomology degenerates on the first page and we obtain $HF^*_{\mathrm{BC}}(L,(L,\mathcal{E}^{\mathbb{K}}_{\mathrm{reg}})) \neq 0$. Since this group is a unital module over $HF^*_{\mathrm{BC}}(L,L;\mathbb{K})$, the latter must also be non-zero. It follows from Lemma 3.2.2 that either $\mathbb{K}$ has characteristic 2 or $m_0(L;\mathbb{K})$ is non-zero. □

Now let us fix a point $x \in L$. Then we can identify the fibre of $\mathcal{E}^R_{\mathrm{reg}}$ over x with the group ring $R[\pi_1(L,x)]$ and the monodromy representation of $\pi_1(L,x)$ is given by right multiplication. In particular, we can identify the endomorphism $m_0(\mathcal{E}^R_{\mathrm{reg}})(x) \in \mathrm{End}(\mathcal{E}^R_{\mathrm{reg},x})$ with an element $m_0(L,x;R) \in R[\pi_1(L,x)]$ by evaluating it on the unit $1 \in R[\pi_1(L,x)]$. The fact that $m_0(\mathcal{E}^R_{\mathrm{reg}})$ commutes with parallel transport translates to the fact that $m_0(L,x;R)$ lies in the centre of the group ring $R[\pi_1(L,x)]$.

The first step to constraining the topology of L is to find an element in $\pi_1(L)$ with finite index centraliser. Following Damian [Dam15], we do this by showing that $m_0(L,x;\mathbb{C})$ is non-zero.

Proposition 3.2.17. *If $x \in L$ is any point, then $\mathrm{supp}(m_0(L,x;\mathbb{Z}))$ is non-empty and all its elements are non-trivial in $\pi_1(L,x)$. In particular $\pi_1(L,x)$ contains a non-trivial element whose centraliser has finite index.*

Proof. If $\mathrm{supp}(m_0(L,x;\mathbb{Z})) = \emptyset$, then $m_0(L,x;\mathbb{C}) = 0$ and hence $m_0(\mathcal{E}^{\mathbb{C}}_{\mathrm{reg}}) = 0$ and $m_0(L;\mathbb{C}) = \varepsilon(m_0(L,x;\mathbb{C})) = 0$. So the pearl complex for $(L,(L,\mathcal{E}^{\mathbb{C}}_{\mathrm{reg}}))$ is unobstructed, but this immediately contradicts Lemma 3.2.16. The fact that $\mathrm{supp}(m_0(L,x;\mathbb{C}))$ does not contain the unit $1 \in \pi_1(L,x)$ follows from Lemma 3.2.13. □

The existence of the element with finite index centraliser allows us to conclude the following.

Corollary 3.2.18. *L is a prime 3-manifold.*

Proof. Suppose for a contradiction that L is not prime, i.e. $L = N_1 \# N_2$ for some other manifolds N_1, N_2, neither of which is homeomorphic to S^3. Then, by the Poincaré-Perelman theorem, we know that N_1 and N_2 cannot be simply connected and hence $\pi_1(L) \cong G * H$ for some non-trivial groups G and H.

On the other hand, Proposition 3.2.17 shows that there exists a non-trivial element $a \in \pi_1(L)$ whose centraliser $C(a)$ has finite index in $\pi_1(L)$. It is shown in [MKS66, Corollaries 4.1.4, 4.1.5, 4.1.6] that the centraliser of any non-trivial element in $G * H$ is either infinite cyclic or contained in some conjugate of G or H. Since a conjugate of a free factor can never have finite index in a free product of non-trivial groups, we must have that $C(a) \cong \mathbb{Z}$.

We claim that this implies $G \cong H \cong \mathbb{Z}/2$. Suppose that this is not the case and without loss of generality let x and y be distinct non-trivial elements of G and z be a non-trivial element of H. Consider the elements $\hat{x} = xz$ and $\hat{y} = yz$. Since $[\pi_1(L) \colon C(a)] < \infty$, the pigeonhole principle implies that there exist $k, l \in \mathbb{N}_{>0}$ such that $\hat{x}^k \in C(a)$, $\hat{y}^l \in C(a)$. Since $C(a)$ is abelian, $\hat{x}^k$ and $\hat{y}^l$ must then commute. However, substituting $\hat{x} = xz$ and $\hat{y} = yz$ into the equality $\hat{x}^k \hat{y}^l = \hat{y}^l \hat{x}^k$, we obtain an equality of elements of $G * H$, which are expressed as different reduced sequences in the sense of [MKS66, Chapter 4]. This contradicts [MKS66, Theorem 4.1].

Thus we must have $\pi_1(L) \cong \mathbb{Z}/2 * \mathbb{Z}/2$. But then $H_1(L;\mathbb{Z}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ is finite but does not contain an element of order 4. This contradicts Lemma 3.2.13. □

We now know that L is prime. If L is not irreducible, then it is diffeomorphic to $S^1 \times S^2$ which is case b) of Theorem 3.2.11. The remaining possibility is that L is irreducible and since its fundamental group is infinite we have that L is aspherical. We then have the following corollary of Proposition 3.2.17.

Corollary 3.2.19. *Let $L \subseteq \mathbb{CP}^3$ be an orientable, aspherical, monotone Lagrangian with $N_L = 2$. Then L is Seifert fibrable.*

Proof. This follows from some heavy theorems about the topology of 3-manifolds. Again, let $a \in \pi_1(L,x)$ denote a non-trivial element with finite index centraliser. Let $\overline{L}$ denote the finite cover of L with $\pi_1\left(\overline{L}\right) \cong C(a)$. Then $\overline{L}$ is a compact, aspherical 3-manifold whose fundamental group has non-trivial centre and so by the Seifert Fibre Space Conjecture (now a theorem, see e.g. the survey [Prfrm[o]–4] and the references therein) $\overline{L}$ must be Seifert fibrable. Since a finite cover of L is Seifert fibrable and $\pi_1(L)$ is infinite, it follows by Scott's rigidity theorem (see the last paragraph on p.35 of [Sco83b]) that L itself is Seifert fibrable. $\square$

Knowing that L is Seifert fibrable gives us very good control over its fundamental group. The next lemma uses this to restrict the form that $\mathrm{supp}(m_0(L,x;\mathbb{Z}))$ can take, depending on whether L is Euclidean or not.

Lemma 3.2.20. *Let $L \subseteq \mathbb{CP}^3$ be an orientable, aspherical, monotone Lagrangian with $N_L = 2$ and let $q\colon L \to \Sigma$ be a Seifert fibration of L. Let $x \in L$ be a point and let $h \in \pi_1(L,x)$ denote a generator for the Seifert fibre subgroup. Then one of the following holds:*

a) Σ is orientable and there exists a positive integer k, non-zero integers $c_1, c_2, \dots, c_k$ and distinct odd integers $n_1 < n_2 < \cdots < n_k$, such that

$$m_0(L,x;\mathbb{Z}) = c_1 h^{n_1} + c_2 h^{n_2} + \cdots + c_k h^{n_k} \in \mathbb{Z}[\pi_1(L,x)].$$

Moreover, the multiplicities of all singular fibres of q are odd.

b) There exists a finite covering $p\colon T^3 \to L$ such that

$$\mathrm{supp}(m_0(L,x;\mathbb{Z})) \cap p_*(\pi_1(T^3,y)) \neq \emptyset,$$

where $y \in \pi^{-1}(x)$. In particular, L admits a Euclidean geometry.

Proof. Let $C, N \leq \pi_1(L,x)$ denote respectively the canonical and Seifert fibre subgroups of the fixed Seifert fibration $q\colon L \to \Sigma$. If $\mathrm{supp}(m_0(L,x;\mathbb{Z}))$ is contained in N, then we are in case a). Indeed, since $N = \langle h \rangle$ and $\mathrm{supp}(m_0(L,x;\mathbb{Z}))$ is non-empty, there exists a positive integer k and non-zero integers $c_1, c_2, \dots, c_k$ such that

$$m_0(L,x;\mathbb{Z}) = c_1 h^{n_1} + c_2 h^{n_2} + \cdots + c_k h^{n_k} \in \mathbb{Z}[\pi_1(L,x)]$$

for some distinct integers $n_1 < n_2 < \cdots < n_k$. By Lemma 3.2.13 we have that Σ is orientable and n_i is odd for all $1 \leq i \leq k$. By Lemma 3.2.14, we have that the multiplicities of the singular fibres of q are odd.

Suppose now that $\mathrm{supp}(m_0(L,x;\mathbb{Z}))$ is not contained in N and let $a \in \mathrm{supp}(m_0(L,x;\mathbb{Z}))$ be an element which lies outside the Seifert fibre subgroup. Then by [JS79, Proposition II.4.7] we know that the subgroup $H := C \cap C(a)$ is abelian and of index at most 2 in $C(a)$. Since by Proposition 3.2.17 we have that $[\pi_1(L,x)\colon C(a)] < \infty$, it follows that $[\pi_1(L,x)\colon H] < \infty$ and so the cover $p\colon \overline{L} \to L$ with fundamental group $\pi_1(\overline{L},y) \cong H$ is a finite cover (here y is any lift of x). Then $\overline{L}$ is a compact, aspherical 3-manifold with an abelian fundamental group. For cohomological reasons we must have $H \cong \mathbb{Z}^3$ and since Eilenberg–MacLane spaces are determined up to homotopy by their fundamental group, it follows that $\overline{L}$ is homotopy equivalent to T^3. By a famous theorem of Waldhausen [Wal68, Corollary 6.5], $\overline{L}$ is then diffeomorphic to T^3.

Finally, note that a must lie in C, since otherwise [JS79, Proposition II.4.7] tells us that $C(a)$ would need to be cyclic but this contradicts the fact that $C(a)$ contains a copy of $\mathbb{Z}^3$. Thus $a \in C \cap C(a) = H = p_*(\pi_1(T^3,y))$ and we are in case b). □

From this, we immediately have:

Corollary 3.2.21. *Let L be a chiral platycosm, other than T^3 or the tricosm. Then L does not admit a monotone Lagrangian embedding in $\mathbb{CP}^3$.*

Proof. Suppose that there exists such an embedding. Then Lemma 3.2.20 tells us that there must exist a Maslov 2 disc such that either ∂u lies in the Seifert fibre subgroup of $\pi_1(L)$ (for some Seifert fibration) or it lies in $p_*(\pi_1(T^3))$ for some torus cover $p\colon T^3 \to L$. Passing to homology, we obtain a contradiction with Lemma 3.2.15. □

Lemma 3.2.20 and Corollary 3.2.21 show that if we are not in cases a), b) or c) of Theorem 3.2.11, then L is non-Euclidean and we understand $m_0(L,x;\mathbb{Z})$ explicitly. Using this, we finish the proof of Theorem 3.2.11, by showing that the only remaining possibility is case d).

Proposition 3.2.22. *Let $L \subseteq \mathbb{CP}^3$ be a monotone, orientable, aspherical Lagrangian with $N_L = 2$ and suppose that L does not admit a Euclidean geometry. Then L is diffeomorphic to a principal circle bundle over an orientable surface of genus $g \geq 1$. The Euler class of this bundle is divisible by* 4.

Proof. By Corollary 3.2.19, we know that L admits a Seifert fibration $q\colon L \to \Sigma$. Further, since L is aspherical and does not admit a Euclidean geometry, we know that this Seifert fibration is unique up to isomorphism. We now use the geometry of the moduli space of Maslov 2 discs to show that q has no singular fibres.

Suppose J is a generic almost complex structure such that the point x is a regular value for $\mathrm{ev}\colon \mathcal{M}_{0,1}(2,L;J) \to L$. Let $S_1, S_2, \ldots, S_m$ denote the conjugacy classes in $\pi_1(L,x)$ corresponding to the free homotopy classes $\{\partial^{\mathrm{f}} u \;:\; u \in \mathcal{M}_{0,1}(2,L;J)\}$. Note that if u_1 and u_2 lie in the same connected component of $\mathcal{M}_{0,1}(2,L;J)$, then they can be joined by a continuous path of discs and so $\partial^{\mathrm{f}} u_1 = \partial^{\mathrm{f}} u_2$. So the decomposition of $\mathcal{M}_{0,1}(2,L;J)$ into connected components can be written as $\mathcal{M}_{0,1}(2,L;J) = \sqcup_{i=1}^{m} \sqcup_{j=1}^{\ell_i} M_{ij}$, where for all $1 \leq i \leq m$, if $u \in M_{ij} \cap \mathrm{ev}^{-1}(x)$, then $\partial u \in S_i$. We write $\mathrm{ev}_{ij}\colon M_{ij} \to L$ for the restriction of ev to the ij-th component and for each $1 \leq i \leq m$ we define

$$m_0^i := \sum_{j=1}^{\ell_i} \sum_{u \in \mathrm{ev}_{ij}^{-1}(x)} \deg_u(\mathrm{ev}_{ij}) \partial u \in \mathbb{Z}[\pi_1(L,x)]. \tag{3.11}$$

Then for each $1 \leq i \leq m$, the support of m_0^i satisfies $\mathrm{supp}(m_0^i) \subseteq S_i$ and we have

$$m_0(L,x;\mathbb{Z}) = \sum_{i=1}^{m} \sum_{j=1}^{\ell_i} \sum_{u \in \mathrm{ev}_{ij}^{-1}(x)} \deg_u(\mathrm{ev}_{ij})\, \partial u = \sum_{i=1}^{m} m_0^i. \tag{3.12}$$

On the other hand, by Lemma 3.2.20 we know that Σ is orientable and $m_0(L,x;\mathbb{Z})$ takes the form

$$m_0(L,x;\mathbb{Z}) = c_1 h^{n_1} + c_2 h^{n_2} + \cdots + c_k h^{n_k}. \tag{3.13}$$

Since Σ is orientable, we have from (3.4) that for each $1 \leq i \leq k$, the element h^{n_i} is central in $\pi_1(L,x)$ and so its conjugacy class is a singleton. Comparing the expressions (3.12) and (3.13), we see that we must have $k \leq m$ and without loss of generality we may assume $S_i = \{h^{n_i}\}$, $m_0^i = c_i h^{n_i}$ for $1 \leq i \leq k$ and $m_0^i = 0$ for $k+1 \leq i \leq m$. Finally, again since S_i is a singleton, we get that for each $1 \leq i \leq k$ and $1 \leq j \leq \ell_i$ the boundaries of any two discs $u_1, u_2 \in \mathrm{ev}_{ij}^{-1}(x)$ define the same *based* homotopy class $\partial u_1 = \partial u_2 = h^{n_i}$ in $\pi_1(L,x)$. Hence definition (3.11) simplifies and we obtain

$$c_i = \sum_{j=1}^{\ell_i} \deg(\mathrm{ev}_{ij}) \quad \forall\, 1 \leq i \leq k. \tag{3.14}$$

Observe now that the moduli space $\mathcal{M}_{0,1}(2,L;J)$ of discs with one boundary marked point is naturally a principal circle bundle over the moduli space $\mathcal{M}_{0,0}(2,L;J)$ of unmarked discs with the projection given by forgetting the marked point. That is, each connected component M_{ij} is a principal bundle over some orientable surface Σ_{ij}. Then each $\mathrm{ev}_{ij} \colon M_{ij} \to L$ is a map from a principal circle bundle over an orientable surface to an aspherical, orientable, Seifert fibred manifold. Hence, by Lemma 3.2.9, it follows that for each $1 \leq i \leq m$, $1 \leq j \leq \ell_i$ the degree $\deg(\mathrm{ev}_{ij})$ is divisible by the multiplicities of all singular fibres of the Seifert fibration of L.

Suppose for a contradiction that the Seifert fibration of L indeed has a singular fibre of multiplicity $|\alpha| > 1$. Then, by Lemma 3.2.20, α is odd and in particular, there exists an odd prime p which divides α. From equation (3.14) it follows that α divides c_i for each $1 \leq i \leq k$ and so by (3.13) we have

$$m_0(L,x;\mathbb{F}_p) = 0 \quad \in \; \mathbb{F}_p[\pi_1(L,x)].$$

Hence the pearl complex of the pair $(L,(L,\mathcal{E}_{\mathrm{reg}}^{\mathbb{F}_p}))$ is unobstructed and $m_0(L;\mathbb{F}_p) = 0$. This contradicts Lemma 3.2.16 because p is odd.

We have shown that the Seifert fibration of L over Σ has no singular fibres and since both Σ and L are orientable, it follows that L is a principal circle bundle over Σ. Since L is aspherical, we must have that the genus of Σ is at least 1. Now let $e \in H^2(\Sigma;\mathbb{Z})$ denote the Euler class of this bundle. By (3.13), we know that there exists $u \in M_{1j}$ for some $1 \leq j \leq \ell_1$ such that $[\partial u] = n_1[h] \in H_1(L;\mathbb{Z})$. Now, if $e = 0$, we clearly have that 4 divides $e([\Sigma])$ (note that in this case we also need to have $\mathrm{genus}(\Sigma) > 1$, since we are assuming that L does not admit a Euclidean geometry). On the other hand, since $[h] \in H_1(L;\mathbb{Z})$ is precisely the class of a circle fibre, we have that if $e \neq 0$, then $[h]$ has order $|\langle e,[\Sigma]\rangle|$ in $H_1(L;\mathbb{Z})$. Hence $\langle e,[\Sigma]\rangle[\partial u] = 0$ and so $[\partial u]$ has finite order in $H_1(L;\mathbb{Z})$ and that order divides $\langle e,[\Sigma]\rangle$. It follows by Lemma 3.2.13 that 4 divides $\langle e,[\Sigma]\rangle$. □

This finishes the proof of Theorem 3.2.11.

Chapter 4

Symplectic geometry of the twistor fibration $\mathbb{CP}^{2n+1} \to \mathbb{HP}^n$

In this chapter we investigate the fibration

$$\mathbb{CP}^1 \longrightarrow \mathbb{CP}^{2n+1} \xrightarrow{\Pi} \mathbb{HP}^n \tag{4.1}$$

from the point of view of symplectic geometry. Heuristically, this fibration is just

$$\begin{array}{ccccc} \text{complex lines in a} & & & & \\ \text{quaternionic line} & \longrightarrow & \text{complex lines in} & \longrightarrow & \text{quaternionic lines} \\ & & \mathbb{C}^{2n+2} \cong \mathbb{H}^{n+1} & & \text{in } \mathbb{H}^{n+1} \\ \mathbb{C}^2 \cong \mathbb{H} \leq \mathbb{H}^{n+1} & & & & \end{array}$$

and this is the perspective we take throughout most of this chapter. However, (4.1) fits into the more general picture of twistor fibrations for quaternion-Kähler manifolds (introduced by Salamon in [Sal82]) and, when $n = 1$, general twistor spaces of oriented Riemannian 4-manifolds (as defined in [AHS78], following pioneering ideas of Penrose). We focus very narrowly on the question:

Question 4. How does a Lagrangian $L \subseteq \mathbb{CP}^{2n+1}$ project to $\mathbb{HP}^n$?

In section 4.1 we explain the correspondence between smooth Legendrian subvarieties and Lagrangian submanifolds of $\mathbb{CP}^{2n+1}$. We use our results from chapter 3 in order to prove Theorems C and D from the introduction. In section 4.2 we study locally the projections to $\mathbb{HP}^1 = S^4$ of general Lagrangians in $\mathbb{CP}^3$. In section 4.3 we show that any function $f\colon \mathbb{CP}^{2n+1} \to \mathbb{HP}^n$ whose Hamiltonian vector field is vertical with respect to Π is constant.

Notation 4.0.1. We denote the round n-sphere of radius r by $S^n(r)$. If E is a vector bundle over a manifold X, we write $S(E)$ to denote the sphere bundle inside E as a topological space. If E carries a bundle metric, we denote the sphere bundle of radius r by $S_r(E)$. //

4.1 The Legendrian–Lagrangian correspondence

4.1.1 Background

We begin with some general background on the theory of quaternion-Kähler manifolds and their twistor spaces. For more details the reader is referred to [Bes08, Chapter 14] and [Sal82].

4.1.1.1 Quaternion-Kähler manifolds

An almost-quaternionic structure on a manifold M is a rank 3 subbundle $Q \leq \mathrm{End}(TM)$ which is *locally* spanned by sections I, J, K, satisfying $I^2 = J^2 = K^2 = IJK = -\mathrm{Id}$. Note that if M admits an almost quaternionic structure Q, then for every $x \in M$, the tangent space T_xM can be given the structure of a (necessarily free) left $\mathbb{H}$-module and so $\dim M = 4n$. Further, Q equips M with a preferred orientation, namely the one induced by any local almost complex structure in Q. On each tangent space T_xM there is an S^2-worth of complex structures contained in Q, parametrised by $\{aI_x + bJ_x + cK_x\colon a^2+b^2+c^2 = 1\}$. Thus the space $\mathcal{Z}(M,Q) := \{A \in Q\colon A^2 = -\mathrm{Id}\}$ is an S^2-bundle over M. It is called the *twistor space* of the almost-quaternionic manifold (M,Q). We will write $\tau\colon \mathcal{Z}(M,Q) \to M$ for the natural projection.

Remark 4.1.1. In general, to every Riemannian manifold (M,g) of even dimension, one can associate its full twistor space $\mathcal{Z}(M,g)$ which is the fibre-subbundle of $\mathrm{End}(TM)$ consisting of g-orthogonal pointwise complex structures. If M is oriented, $\mathcal{Z}(M,g)$ has two diffeomorphic connected components $\mathcal{Z}_+(M,g)$ and $\mathcal{Z}_-(M,g)$, consisting of complex structures which induce the prescribed–respectively opposite–orientation on M. Note that if Q is an almost-quaternionic structure on M, then one can always define a Riemannian metric g which makes all complex structures in Q orthogonal[1], i.e. such that $\mathcal{Z}(M,Q) \subseteq \mathcal{Z}_+(M,g)$. //

We now have the following definitions.

Definition 4.1.2. A *quaternion-Kähler structure* on a manifold M is a pair (g,Q), where g is a Riemannian metric on M and Q is an almost-quaternionic structure such that the following two conditions hold:

a) g is compatible with Q, i.e. for every $x \in M$ every complex structure $I_x \in \mathcal{Z}(M,Q)_x$ is orthogonal with respect to g_x,

b) the bundle Q is parallel with respect to the Levi-Civita connection of g. ♢

Definition 4.1.3. If M is a smooth manifold of dimension $4n$ with $n > 1$ and M is equipped with a quaternion-Kähler structure (g,Q), then the triple (M,g,Q) is called a *quaternion-Kähler manifold* (or qK-manifold for short). ♢

Quaternion-Kähler manifolds and their twistor spaces enjoy many rigidity properties. First of all, qK-manifolds are known to be automatically Einstein ([Ber55], [Ale67], [Ish74]) and in particular they have constant scalar curvature. Further, note that the twistor space $\mathcal{Z}(M,Q)$, being a submanifold of $\mathrm{End}(TM)$, inherits a Riemannian metric $g'_{\mathcal{Z}}$ from the Sasaki metric on $\mathrm{End}(TM)$ and the tangent bundle $T\mathcal{Z}(M,Q)$ splits into a horizontal and vertical component. The vertical component

[1]Let g' be any Riemannian metric and define $g(X,Y) = \frac{1}{4}(g'(X,Y) + g'(IX,IY) + g'(JX,JY) + g'(KX,KY))$ for some local basis I,J,K of Q.

at an element $I_x \in \mathcal{Z}(M,Q)_x$ is naturally identified with the vector space $\{B \in Q_x : I_xB + BI_x = 0\}$ and so it inherits a complex structure $B \mapsto I_xB$. On the other hand, the horizontal component is isomorphic to T_xM via the linearisation of the projection $\mathcal{Z}(M,Q) \to M$ and so has a tautological complex structure given by I_x. Taking the direct sum of these complex structures defines a natural almost complex structure $J_{\mathcal{Z}}$ on $\mathcal{Z}(M,Q)$. Salamon [Sal82, Theorem 4.1] and independently Bérard-Bergery have shown that $J_{\mathcal{Z}}$ is in fact integrable. Moreover, the vertical tangent bundle is a holomorphic line bundle over $\mathcal{Z}(M,Q)$ and, whenever the scalar curvature of M is non-zero, the horizontal distribution defines a holomorphic contact structure on $\mathcal{Z}(M,Q)$ (that is, it is locally the kernel of a holomorphic 1-form α such that $\alpha \wedge (d\alpha)^n$ is nowhere vanishing; see [Sal82, Theorem 4.3]). The fibres of the projection $\mathcal{Z}(M,Q) \to M$ are then all biholomorphic to $\mathbb{CP}^1$ and are known as *twistor lines*.

Remark 4.1.4. The reason that $n = 1$ is excluded from Definition 4.1.3 is that every oriented Riemannian 4-manifold M automatically admits a quaternion-Kähler structure with twistor space $\mathcal{Z}_+(M,g)$, which needn't be a complex manifold in general. However, as was shown in the seminal paper [AHS78], the twistor space of a self-dual Einstein 4-manifold *is* a complex manifold, which is why some authors choose to extend the definition of a qK-manifold to include self-dual Einstein 4-manifolds. We also adopt this convention in this work. //

In the theory of qK-manifolds, special attention is given to *positive* ones – that is, qK-manifolds of positive scalar curvature. If M is a positive qK-manifold, then [Sal82, Theorem 6.1] tells us that $(\mathcal{Z}(M,Q), g_{\mathcal{Z}}, J_{\mathcal{Z}})$ is a Kähler-Einstein manifold of positive scalar curvature, where $g_{\mathcal{Z}}$ is an appropriate rescaling of $g'_{\mathcal{Z}}$ in the vertical directions. In particular, the twistor space is a contact Fano variety. Using this fact, LeBrun and Salamon ([LS94]) have shown that for any n, there are only finitely many positive qK-manifolds of dimension $4n$, up to homothety. In fact the only known examples are certain symmetric spaces called Wolf spaces ([Wol65]). It is a long-standing conjecture of LeBrun and Salamon that these are the only positive qK-manifolds.

Note that from a symplectic point of view, positive qK-manifolds are interesting because their twistor spaces are monotone symplectic manifolds. We now briefly discuss some submanifolds of a qK-manifold M which, whenever M has positive scalar curvature, give rise to monotone Lagrangians in the twistor space.

4.1.1.2 Totally complex and Kähler submanifolds

There is extensive literature on interesting classes of submanifolds of qK-manifolds (see for example the survey [Mar06] and the references therein). The ones which are of particular interest to us are the so-called *maximal Kähler submanifolds* and *maximal totally complex* (MTC, for short) submanifolds, which we now define.[2]

[2] Some of the definitions we give may differ slightly from elsewhere in the literature. This is because we want to make a clear distinctions between embedded and immersed submanifolds.

Let X^{2d} be a smooth manifold of dimension $2d$, (M^{4n}, g, Q) be a qK-manifold and let $f\colon X \to M$ be an immersion. We say that f is a *locally almost complex* immersion if for each $x \in X$ there exists an open neighbourhood $U \subseteq X$ and a section I^U of $f^*\mathcal{Z}(M,Q)$ such that $I^U f_*TU = f_*TU$. We will drop the word "locally" from this definition if one can choose $U = X$. Note that if $n = 1$ and $d = 1$ then any immersion is locally almost complex. Given a locally almost complex immersion $f\colon X \to M$, we introduce the following terminology:

- We say that f is a *totally complex* immersion if $J_x(f_*T_xX) \perp f_*T_xX$ for each $x \in X$ and each $J_x \in \mathcal{Z}(M,Q)_x$ such that $J_x I^U_x = -I^U_x J_x$. Note that such an immersion can only exist if $d \leq n$.
- We say that f is a *locally Kähler* immersion, if for each neighbourhood U, the manifold (U, I^U, f^*g) is Kähler.
- We say that f is a *locally totally Kähler* immersion if $\widetilde{\nabla}_v I^U = 0$ for each $x \in X$ and $v \in T_xX$, where $\widetilde{\nabla}$ denotes the Levi-Civita connection on (M,g).

For the latter two items, we will again drop the word "locally" if one can choose $U = X$. The terminology "totally Kähler immersion" in this case is non-standard but we can quickly dispense of it, due to the following proposition:

Proposition 4.1.5. *Let $f\colon X^{2d} \to M^{4n}$ be a locally almost complex immersion into a qK-manifold (M^{4n}, g, Q). Then:*

a) if f is totally complex, then it is locally Kähler;

b) if $d = 1$, then f is totally complex; it is locally totally Kähler if and only if its local lifts to $\mathcal{Z}(M,Q)$, determined by the sections I^U, are horizontal. That is, a locally totally Kähler immersion of a surface in a qK-manifold is a superminimal immersion in the sense of Bryant ([Bry82]).

c) if $n > 1$, $d > 1$ and the scalar curvature of M is non-zero, then the following are equivalent:

i) f is totally complex;

ii) f is locally Kähler;

iii) f is locally totally Kähler;

iv) the local lifts of f to $\mathcal{Z}(M,Q)$, determined by the sections I^U, are horizontal.

Proof. Some parts of this proposition are simple rephrasings. For the non-trivial implications, see [AM01, Theorem 1.8]. □

In particular, observe that whenever $n \geq 2$, if $f\colon X \to M^{4n}$ is a locally Kähler immersion then $\dim_{\mathbb{R}} X \leq 2n$. Whenever we have equality, we say that f is a *maximal locally Kähler* immersion or, equivalently, a *maximal totally complex* (or MTC, for short) immersion. If $X \subseteq M$ is an embedded submanifold, we say that it is an *MTC submanifold* if the inclusion is an MTC immersion. Further, we say that X is a *maximal Kähler submanifold* if the inclusion is a Kähler immersion.

It is important to note that every locally Kähler immersion can be seen as a globally Kähler immersion, but one may need to replace the domain by a double cover. More precisely:

Proposition 4.1.6. *[Tak86, Theorem 4.1] Let $f\colon X^{2d} \to (M^{4n}, g, Q)$ be a locally totally Kähler immersion into a qK-manifold and suppose that either $d = 1$ or the scalar curvature of M is non-zero. Then there exists a Kähler manifold $(\hat{X}, \hat{g}, \hat{I})$, a Riemannian covering $\pi\colon (\hat{X}, \hat{g}) \to (X, f^*g)$ and a holomorphic, horizontal immersion $\hat{f}\colon \hat{X} \to \mathcal{Z}(M, Q)$ such that $f \circ \pi = \tau \circ \hat{f}$ and $f \circ \pi$ is a Kähler immersion.*

Sketch proof. Define the set $I_f := \{(x, I) \in f^*\mathcal{Z}(M, Q) \,:\, I(f_*T_xX) = f_*T_xX\}$. By Proposition 4.1.5, we know that f is a totally complex immersion and so we see that the natural projection $I_f \to X$ is a two-to-one covering, because the fibre above each point $x \in X$ consists of exactly two complex structures I_x and $-I_x$. Now the space I_f can have one or two connected components. One obtains the result by taking $\hat{X}$ to be a connected component of I_f and $\pi\colon \hat{X} \to X$ and $\hat{f}\colon \hat{X} \to \mathcal{Z}(M, Q)$ to be the restrictions of the corresponding natural maps $f^*\mathcal{Z}(M, Q) \to X$, $f^*\mathcal{Z}(M, Q) \to \mathcal{Z}(M, Q)$. $\square$

The immersion $\hat{f}\colon \hat{X} \to \mathcal{Z}(M, Q)$, constructed in the above proof is called the *twistor lift* of f (in case I_f is disconnected, there are two equally good choices of twistor lifts). Observe now that if $X \subseteq M^{4n}$ is an embedded MTC submanifold (or a superminimal surface, if $n = 1$), then its twistor lift is an embedded complex Legendrian submanifold of $\mathcal{Z}(M, Q)$, i.e. it is everywhere tangent to the holomorphic contact structure and has maximal possible dimension. Conversely, if $X \subseteq \mathcal{Z}(M, Q)$ is a Legendrian submanifold such that the projection $\tau|_X : X \to M$ has embedded image, then $\tau(X)$ is an MTC submanifold of M (or an embedded superminimal surface, if $n = 1$). Following Alekseevsky and Marchiafava [AM05], we say that a (connected) Legendrian submanifold $X \subseteq \mathcal{Z}(M, Q)$ is of *Type 1* if $\tau|_X : X \to M$ is an embedding and we say that it is of *Type 2* if $\tau(X)$ is embedded but $\tau|_X : X \to \tau(X)$ is a double cover.

Remark 4.1.7. Note that most Legendrian submanifolds are of neither type (contrary to what [AM05, Proposition 5.4] might lead one to believe) because, while $\tau|_X : X \to M$ is certainly an immersion, its image need not be embedded in general. //

The reason that we are interested in MTC submanifolds is because whenever the ambient qK-manifold is positive, they give rise to monotone Lagrangians in the twistor space. More generally, if M^{4n} is any qK-manifold with non-vanishing scalar curvature, and $X \subseteq M$ is a totally complex submanifold (or superminimal surface, if $n = 1$), then one can consider the set

$$\mathcal{L}(X) := \{(x, J_x) \in \mathcal{Z}(M, Q)|_X \,:\, J_x(T_xX) \perp T_xX\}.$$

Since X is totally complex, the fibre of $\mathcal{L}(X)$ above a point $x \in X$ is precisely the geodesic circle $\{J_x \in \mathcal{Z}(M, Q)_x \,:\, J_xI_x = -I_xJ_x\}$, where I_x is one of the two complex structures in Q preserving T_xX.

Thus $\mathcal{L}(X)$ is a circle bundle over X. Moreover, using the fact that the twistor lift of X is horizontal, one can show that $J^{\mathcal{Z}}(T\mathcal{L}(X)) \perp T\mathcal{L}(X)$ or, in other words, $\mathcal{L}(X)$ is isotropic with respect to the non-degenerate 2-form $\omega^{\mathcal{Z}} := g^{\mathcal{Z}}(J^{\mathcal{Z}}\cdot, \cdot)$. Recall now that when M is a positive qK-manifold, the form $\omega^{\mathcal{Z}}$ is closed and hence for any MTC submanifold $X \subseteq M$, the manifold $\mathcal{L}(X)$ is Lagrangian. The same is true whenever $n = 1$ (note that the only compact, Einstein, self-dual 4-manifolds with positive scalar curvature are S^4 and $\mathbb{CP}^2$ as shown in [FK82]), and X is a superminimal surface. Moreover, one can show that the Lagrangians constructed in this way are minimal with respect to the Riemannian metric $g^{\mathcal{Z}}$. Since $\mathcal{Z}(M,Q)$ is Kähler-Einstein, the main result of [CG04] shows that $\mathcal{L}(X)$ is monotone.

Remark 4.1.8. The existence of this minimal Lagrangian lift has been observed most recently by Ejiri and Tsukada in [ET05] but similar constructions are much older. In particular, for the case of superminimal surfaces in S^4, the construction is already present in Ejiri's paper [Eji86, Section 15] (see also [CDVV96], [BDVV96], [BSV02, Section 2]). There is an analogous idea, due to Reznikov ([Rez93]), who constructs Lagrangians in symplectic twistor spaces $\mathcal{Z}_+(M,g)$ from half-dimensional totally geodesic submanifolds $N \subseteq M$ by considering all complex structures along N which send TN to its orthogonal complement. The Floer theory of such Lagrangians in the twistor spaces of certain hyperbolic 6-manifolds has been investigated by Evans in [Eva14]. //

In order to avoid having to constantly distinguish between the cases $n = 1$ and $n \geq 2$, it is more convenient to speak of Lagrangian submanifolds of $\mathcal{Z}(M,Q)$, corresponding directly to Legendrian subvarieties of $\mathcal{Z}(M,Q)$. We refer to this as the *Legendrian–Lagrangian correspondence* and we call Lagrangians which arise this way *twistor Lagrangians*. Note that a twistor Lagrangian is embedded if and only if the corresponding Legendrian subvariety is of Type 1 or Type 2 and so we distinguish embedded twistor Lagrangians into Type 1 and Type 2 accordingly.

4.1.1.3 $\mathbb{CP}^{2n+1}$ and $\mathbb{HP}^n$

The easiest example of a positive qK-manifold is quaternionic projective space $\mathbb{HP}^n$, equipped with its standard Fubini-Study metric. The bundle Q consists of those endomorphisms of $T\mathbb{HP}^n$ which in the standard charts can be expressed by (right) multiplication by purely imaginary quaternions. Note that $\mathbb{HP}^1$ is isometric to a standard S^4 (of radius $1/2$ with our current conventions) which is Einstein and self-dual, so fits with the extended definition of a qK-manifold. The twistor space $(\mathcal{Z}(\mathbb{HP}^n,Q), g_{\mathcal{Z}}, J_{\mathcal{Z}})$ turns out to be Kähler-isometric to $(\mathbb{CP}^{2n+1}, g_{\mathrm{FS}}, J_0)$ and the fibration (4.1) is nothing but the standard projection $\mathcal{Z}(\mathbb{HP}^n,Q) \to \mathbb{HP}^n$. These facts are well-known but for our own peace of mind and in order to have a convenient setup for calculations, we will verify them explicitly in the next section.

4.1.2 Proof of the correspondence

In this section we prove the Legendrian–Lagrangian correspondence for $\mathbb{CP}^{2n+1}$. As we said, this is proved for twistor spaces of general qK-manifolds with non-vanishing scalar curvature in [ET05] but without the uniqueness statement which is Theorem 4.1.23 below. We only prove this theorem for $\mathbb{CP}^{2n+1}$ but the same argument is applicable in the general situation too.

Before we give the statements and proofs, we will make our setup precise, establish some notation and verify explicitly several well-known facts.

4.1.2.1 Setup

Let $\mathbb{H} = \mathrm{Span}_{\mathbb{R}}\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ denote the quaternion algebra and put $\mathbb{H}^\times = \mathbb{H} \setminus \{0\}$. Given a quaternion $q = a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d$ we define $\mathbb{R}\mathrm{e}(q) = a$, $\mathbb{C}\mathrm{o}(q) = a + \mathbf{i}b$, $\bar{q} = a - \mathbf{i}b - \mathbf{j}c - \mathbf{k}d$. We view $\mathbb{H}^{n+1}$ as a module over $\mathbb{H}$, where $\mathbb{H}$ acts by *right* multiplication. We equip $\mathbb{H}^{n+1}$ with the complex structure which is right multiplication by $\mathbf{i}$. Thus we get the identifications

$$\begin{array}{ccc} \mathbb{R}^{4n+4} & \longrightarrow & \mathbb{C}^{2n+2} \\ (x_0, y_0, \ldots, x_{2n+1}, y_{2n+1}) & \longmapsto & (x_0 + \mathbf{i}y_0, \ldots, x_{2n+1} + \mathbf{i}y_{2n+1}) \\ \mathbb{C}^{2n+2} & \longrightarrow & \mathbb{H}^{n+1} \\ (z_0, z_1, \ldots, z_{2n+1}) & \longmapsto & (z_0 + \mathbf{j}z_1, \ldots, z_{2n} + \mathbf{j}z_{2n+1}). \end{array} \tag{4.2}$$

Remark 4.1.9. Note that this gives the identification $\mathbb{R}^4 \to \mathbb{H}$, $(a,b,c,d) \mapsto a + \mathbf{i}b + \mathbf{j}c - \mathbf{k}d$. In particular, if we orient $\mathbb{H}$ by the complex structure, a positive basis is $\{1, \mathbf{i}, \mathbf{j}, -\mathbf{k}\}$. //

We now let $\mathbb{HP}^n := (\mathbb{H}^{n+1} \setminus \{0\}) / \mathbb{H}^\times$ denote quaternionic projective space and we write $\Pi_{\mathbb{H}} \colon \mathbb{H}^{n+1} \setminus \{0\} \to \mathbb{HP}^n$ for the quotient map. Similarly, if we just quotient by the action of $\mathbb{C}^\times$ we get a quotient map $\Pi_{\mathbb{C}} \colon \mathbb{H}^{n+1} \setminus \{0\} \to \mathbb{CP}^{2n+1}$. For all $v \in \mathbb{H}^{n+1} \setminus \{0\}$, $w \in \mathbb{H}^{n+1}$ these maps satisfy the following identities which we will use repeatedly in calculations:

$$\begin{aligned} d_v\Pi_{\mathbb{C}}(w) &= d_{v\lambda}\Pi_{\mathbb{C}}(w\lambda + v\mu) \quad \forall\, \lambda \in \mathbb{C}^\times, \mu \in \mathbb{C} \\ d_v\Pi_{\mathbb{H}}(w) &= d_{vp}\Pi_{\mathbb{H}}(wp + vq) \quad \forall\, p \in \mathbb{H}^\times,\ q \in \mathbb{H}. \end{aligned} \tag{4.3}$$

Given $v \in \mathbb{H}^{n+1} \setminus \{0\}$ we will write $v\mathbb{C} = \Pi_{\mathbb{C}}(v)$ and $v\mathbb{H} = \Pi_{\mathbb{H}}(v)$. By slight abuse of notation we will use the same expressions to denote the complex and quaternionic lines spanned by v in $\mathbb{H}^{n+1}$, i.e $v\mathbb{C} = \{vz \in \mathbb{H}^{n+1} \,:\, z \in \mathbb{C}\}$ and $v\mathbb{H} = \{vp \in \mathbb{H}^{n+1} \,:\, p \in \mathbb{H}\}$.

Our main object of study in this chapter is the map $\Pi \colon \mathbb{CP}^{2n+1} \to \mathbb{HP}^n$ which fits into the diagram

$$\begin{tikzcd} & & \mathbb{H}^{n+1} \setminus \{0\} \arrow[dl, "\Pi_{\mathbb{C}}"'] \arrow[dd, "\Pi_{\mathbb{H}}"] \\ \mathbb{CP}^1 \arrow[r] & \mathbb{CP}^{2n+1} \arrow[dr, "\Pi"] & \\ & & \mathbb{HP}^n \end{tikzcd}$$

In the homogeneous coordinates on $\mathbb{CP}^{2n+1}$ and $\mathbb{HP}^n$ which we have from (4.2), this map is given simply by

$$\Pi([z_0 : z_1 : \ldots : z_{2n} : z_{2n+1}]_{\mathbb{C}}) = [z_0 + \mathbf{j}z_1 : \ldots : z_{2n} + \mathbf{j}z_{2n+1}]_{\mathbb{H}}.$$

Throughout this chapter we will use this subscript notation $[-]_{\mathbb{C}}$, $[-]_{\mathbb{H}}$ to indicate which projective space we are working on.

In order to equip $\mathbb{CP}^{2n+1}$ and $\mathbb{HP}^n$ with their familiar geometric structures, we consider the following $\mathbb{H}$–valued sesquilinear form on $\mathbb{H}^{n+1}$:

$$\langle (p_0,\ldots,p_n),(q_0,\ldots,q_n)\rangle := \bar{p}_0 q_0 + \bar{p}_1 q_1 + \cdots + \bar{p}_n q_n.$$

It is immediate to verify that it satisfies the properties

$$\langle v, w\,p\rangle = \langle v,w\rangle\, p, \quad \langle v\,p, w\rangle = \bar{p}\,\langle v,w\rangle \quad \forall\, v,w \in \mathbb{H}^{n+1},\ p \in \mathbb{H}.$$

This pairing naturally equips $\mathbb{H}^{n+1}$ with

1. the Euclidean inner product $\mathbb{R}\mathrm{e}\langle\,\cdot\,,\,\cdot\,\rangle = dx_0^2 + dy_0^2 + \cdots + dx_{2n+1}^2 + dy_{2n+1}^2$,
2. the (real) symplectic form $\omega_{\mathrm{std}} := \mathbb{R}\mathrm{e}\langle\,\cdot\,\mathbf{i}\,,\,\cdot\,\rangle = dx_0 \wedge dy_0 + \cdots + dx_{2n+1} \wedge dy_{2n+1}$,
3. the complex symplectic form $\omega_{\mathbb{C}} := \mathbb{C}\mathrm{o}\langle\,\cdot\,\mathbf{j}\,,\,\cdot\,\rangle = dz_0 \wedge dz_1 + \cdots + dz_{2n} \wedge dz_{2n+1}$
4. the hermitian pairing $\mathbb{C}\mathrm{o}\langle\,\cdot\,,\,\cdot\,\rangle = \mathbb{R}\mathrm{e}\langle\,\cdot\,,\,\cdot\,\rangle + \mathbf{i}\,\omega_{std}$.

We endow $\mathbb{CP}^{2n+1}$ with the Fubini-Study metric g_{FS} and the corresponding symplectic form ω_{FS} in the standard way: given $v \in S^{4n+3}(1)$, we have an isomorphism $d_v\Pi_{\mathbb{C}} \colon v\mathbb{C}^{\perp} \to T_{v\mathbb{C}}\mathbb{CP}^{2n+1}$ and we define g_{FS}, ω_{FS} by the formulae

$$\begin{aligned} g_{\mathrm{FS}}(d_v\Pi_{\mathbb{C}}(w_1), d_v\Pi_{\mathbb{C}}(w_2)) &:= \mathbb{R}\mathrm{e}\langle w_1, w_2\rangle \\ \omega_{\mathrm{FS}}(d_v\Pi_{\mathbb{C}}(w_1), d_v\Pi_{\mathbb{C}}(w_2)) &:= \omega_{std}(w_1,w_2) \quad \forall v \in S^{4n+3}(1),\, w_1,w_2 \in v\mathbb{C}^{\perp}. \end{aligned} \tag{4.4}$$

The standard integrable almost complex structure J_0 on $\mathbb{CP}^{2n+1}$ is the unique one making the projection $\Pi_{\mathbb{C}} \colon \mathbb{H}^{n+1}\setminus\{0\} \to \mathbb{CP}^{2n+1}$ holomorphic, i.e.:

$$J_0(d_v\Pi_{\mathbb{C}}(w)) = d_v\Pi_{\mathbb{C}}(w\mathbf{i}) \quad \forall v \in \mathbb{H}^{n+1}\setminus\{0\},\ w \in \mathbb{H}^{n+1}.$$

We also have the isomorphism $d_v\Pi_{\mathbb{H}} \colon v\mathbb{H}^{\perp} \to T_{v\mathbb{H}}\mathbb{HP}^n$ and we equip $\mathbb{HP}^n$ with the Riemannian metric g given by the formula

$$g(d_v\Pi_{\mathbb{H}}(w_1), d_v\Pi_{\mathbb{H}}(w_2)) := \mathbb{R}\mathrm{e}\langle w_1,w_2\rangle \quad \forall v \in S^{4n+3}(1),\, w_1,w_2 \in v\mathbb{H}^{\perp}. \tag{4.5}$$

The identities (4.3), show that all these structures are well-defined. Note that g_{FS} induces a splitting $T\mathbb{CP}^{2n+1} = \mathcal{V} \oplus \mathcal{H}$, where $\mathcal{V} := \ker d\Pi$ and $\mathcal{H} := \mathcal{V}^{\perp}$ and the metric g is the unique one making Π a Riemannian submersion. This splitting is in fact symplectic, since for all $v \in \mathbb{H}^{n+1}\setminus\{0\}$ the spaces $v\mathbb{C}^{\perp} \cap v\mathbb{H}$ and $v\mathbb{H}^{\perp}$ are $\omega_{std}-$symplectic subspaces of $\mathbb{H}^{n+1}$. That is, we have a splitting

$$\omega_{\mathrm{FS}} = \omega^{\mathcal{V}} \oplus \omega^{\mathcal{H}} \tag{4.6}$$

into a vertical and a horizontal component.

Finally, we will identify $\mathbb{HP}^1$ with a round sphere of radius $1/2$ via the isometry

$$\begin{array}{rcl}\Phi\colon (\mathbb{HP}^1, g) & \longrightarrow & S^4(1/2) \subseteq \mathbb{R}^5 = \mathbb{H}\oplus\mathbb{R} \\ [p:q]_{\mathbb{H}} & \longmapsto & \dfrac{1}{2\left(|p|^2+|q|^2\right)}\left(2p\bar{q}, |p|^2-|q|^2\right).\end{array} \tag{4.7}$$

The map Φ is nothing but the composition

$$\begin{array}{ccccc}\mathbb{HP}^1 & \longrightarrow & \mathbb{H}\cup\{\infty\} & \longrightarrow & S^4(1/2) \\ [p:q]_{\mathbb{H}} & \longmapsto & pq^{-1} & & \\ & & a & \longmapsto & \frac{1}{2\left(1+|a|^2\right)}\left(2a, |a|^2-1\right),\end{array}$$

where the second map is the usual inverse stereographic projection. One can easily check that Φ is an isometry by noting that the differential of the map $\Phi\circ\Pi_{\mathbb{H}}\colon S^7(1)\to S^4(1/2)$ at a point (p,q) is an isometry between the horizontal space $(p,q)\mathbb{H}^\perp$ and $T_{\Phi([p:q]_{\mathbb{H}})}S^4(1/2) = \Phi([p:q]_{\mathbb{H}})^\perp$.

Identifying $\mathbb{CP}^{2n+1}$ and the twistor space of $\mathbb{HP}^n$

Now let us exhibit the link with twistor geometry. The almost-quaternionic structure Q on $\mathbb{HP}^n$ is defined as follows: for all $v\in\mathbb{H}^{n+1}\setminus\{0\}$

$$Q_{v\mathbb{H}} = \{A\in \mathrm{End}(T_{v\mathbb{H}}\mathbb{HP}^n) : \exists p\in \mathrm{Span}_{\mathbb{R}}\{\mathbf{i},\mathbf{j},\mathbf{k}\} \text{ such that } A(d_v\Pi_{\mathbb{H}}(w)) = d_v\Pi_{\mathbb{H}}(wp)\ \forall w\in\mathbb{H}^{n+1}\}.$$

Again using the identities (4.3) it is easy to see that this is well-defined. To see that (g,Q) defines a quaternion-Kähler structure, consider a path $\gamma\colon(-\varepsilon,\varepsilon)\to\mathbb{HP}^n$ and a section $A\colon(-\varepsilon,\varepsilon)\to Q$ of Q along γ. We need to show that $\nabla_t A(t)\in Q_{\gamma(t)}$ for all $t\in(-\varepsilon,\varepsilon)$, where ∇ is the Levi-Civita connection of $(\mathbb{HP}^n,g)$. To do this, we choose a vector field $Y\colon(-\varepsilon,\varepsilon)\to T\mathbb{HP}^n$ along γ and we pick horizontal lifts $v\colon(-\varepsilon,\varepsilon)\to S^{4n+3}(1)$, $w\colon(-\varepsilon,\varepsilon)\to\mathbb{H}^{n+1}$ of γ and Y, respectively. That is, for all $t\in(-\varepsilon,\varepsilon)$ we have $\Pi_{\mathbb{H}}(v(t))=\gamma(t)$, $\dot v(t)\in v(t)\mathbb{H}^\perp$ and $d_{v(t)}\Pi_{\mathbb{H}}(w(t)) = Y(t)$, $w(t)\in v(t)\mathbb{H}^\perp$. Now note that for each $t\in(-\varepsilon,\varepsilon)$, there exists a unique $p(t)\in\mathrm{Span}_{\mathbb{R}}\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ such that $d_{v(t)}\Pi_{\mathbb{H}}\colon v(t)\mathbb{H}^\perp\to T_{v(t)\mathbb{H}}\mathbb{HP}^n$ intertwines right multiplication by $p(t)$ and the endomorphism $A(t)$. Now, letting $\overline{\nabla}$ denote the Levi-Civita connection on $S^{4n+3}(1)$ and using the general fact that one can compute covariant derivatives on the base of a Riemannian submersion by taking horizontal lifts, differentiating and pushing back down ([Pet06, Proposition 13]), we calculate:

$$\begin{aligned}(\nabla_t A(t))Y(t) &= \nabla_t\left(A(t)Y(t)\right) - A(t)\nabla_t Y(t) \\ &= d_{v(t)}\Pi_{\mathbb{H}}(\overline{\nabla}_t\left(w(t)p(t)\right)) - A(t)d_{v(t)}\Pi_{\mathbb{H}}(\overline{\nabla}_t w(t)) \\ &= d_{v(t)}\Pi_{\mathbb{H}}(\dot w(t)p(t) + w(t)\dot p(t)) - d_{v(t)}\Pi_{\mathbb{H}}(\dot w(t)p(t)) \\ &= d_{v(t)}\Pi_{\mathbb{H}}(w(t)\dot p(t)).\end{aligned}$$

So for each $t\in(-\varepsilon,\varepsilon)$ the endomorphism $\nabla_t A(t)$ satisfies $\nabla_t A(t)(d_{v(t)}\Pi_{\mathbb{H}}(w')) = d_{v(t)}\Pi_{\mathbb{H}}(w'\dot p(t))$ for each $w'\in v(t)\mathbb{H}^\perp$. Thus $\nabla_t A(t)\in Q_{\gamma(t)}$ by the definition of Q.

The metric g induces a bundle metric on $\mathrm{End}(T\mathbb{HP}^n)$ which is given by

$$\forall A,B \in \mathrm{End}(T_{v\mathbb{H}}\mathbb{HP}^n) \qquad \{A,B\} := \sum_{s=1}^{4n} g_{v\mathbb{H}}(A(e_s),B(e_s)),$$

where $\{e_1,e_2,\ldots,e_{4n}\}$ is any orthonormal basis for $T_{v\mathbb{H}}\mathbb{HP}^n$. Let $\tau\colon \mathrm{End}(T\mathbb{HP}^n) \to \mathbb{HP}^n$ denote the projection and let $\theta \in C^\infty(\mathrm{End}(T\mathbb{HP}^n), T^*\mathrm{End}(T\mathbb{HP}^n) \otimes \tau^*\mathrm{End}(T\mathbb{HP}^n))$ denote the connection 1-form for the Levi-Civita connection of g. Define a Riemannian metric g_S on $\mathrm{End}(T\mathbb{HP}^n)$ by

$$\forall X,Y \in T\,\mathrm{End}(T\mathbb{HP}^n) \quad g_S(X,Y) = g(\tau_*X,\tau_*Y) + \frac{1}{16n}\{\theta(X),\theta(Y)\}.$$

That is, g_S is a vertical rescaling of the standard Sasaki metric on $\mathrm{End}(T\mathbb{HP}^n)$ induced by g.

Now consider the twistor space $\mathcal{Z}(\mathbb{HP}^n,Q) = \{A \in Q :\ A^2 = -\mathrm{Id}\}$. Let $g_{\mathcal{Z}}$ denote the restriction of g_S to $\mathcal{Z}(\mathbb{HP}^n,Q)$, write $T\mathcal{Z}(\mathbb{HP}^n,Q) = \mathcal{V}^{\mathcal{Z}} \oplus \mathcal{H}^{\mathcal{Z}}$ for the splitting of the tangent bundle to the twistor space into a vertical and a horizontal component.

We now define a fibre-preserving embedding

$$\begin{tikzcd} \mathbf{I}\colon \mathbb{CP}^{2n+1} \arrow[r] \arrow[dr, "\Pi"] & Q \subseteq \mathrm{End}(T\mathbb{HP}^n) \arrow[d, "\tau"] \\ & \mathbb{HP}^n \end{tikzcd}$$

by associating to each point $v\mathbb{C} \in \mathbb{CP}^{2n+1}$ a complex structure $\mathbf{I}(v\mathbb{C})\colon T_{v\mathbb{H}}\mathbb{HP}^n \to T_{v\mathbb{H}}\mathbb{HP}^n$ via the equation

$$\forall w \in v\mathbb{H}^\perp \quad \mathbf{I}(v\mathbb{C})d_v\Pi_{\mathbb{H}}(w) := d_v\Pi_{\mathbb{H}}(w\mathbf{i}). \tag{4.8}$$

Using the first identity from (4.3), it is easy to verify that this map is well-defined. It is also clear that the image of $\mathbf{I}$ is precisely $\mathcal{Z}(\mathbb{HP}^n,Q)$. We now show that $\mathbf{I}$ identifies the spaces $\mathbb{CP}^{2n+1}$ and $\mathcal{Z}(\mathbb{HP}^n,Q)$ with all their relevant structures.

Lemma 4.1.10. *The map $\mathbf{I}\colon \mathbb{CP}^{2n+1} \to \mathcal{Z}(\mathbb{HP}^n,Q)$ satisfies $\mathbf{I}^*g_{\mathcal{Z}} = g_{\mathrm{FS}}$ and it is $(J_0,J_{\mathcal{Z}})$-holomorphic.*

Proof. Since $\mathbf{I}$ sends fibres of Π to fibres of τ, it clearly satisfies $\mathbf{I}_*\mathcal{V} = \mathcal{V}^{\mathcal{Z}}$. We now show that it also satisfies $\mathbf{I}_*\mathcal{H} = \mathcal{H}^{\mathcal{Z}}$. To see this, let $\gamma\colon (-\varepsilon,\varepsilon) \to \mathbb{CP}^{2n+1}$ be a horizontal path. We want to show that $\mathbf{I}(\gamma(t))$ is horizontal, i.e. that $\nabla_t(\mathbf{I}(\gamma(t))) = 0$ for all $t \in (-\varepsilon,\varepsilon)$. In other words, for any vector field $Y\colon (-\varepsilon,\varepsilon) \to T\mathbb{HP}^n$ along $\Pi\circ\gamma$ we must show that $\nabla_t(\mathbf{I}(\gamma(t))Y(t)) = \mathbf{I}(\gamma(t))\nabla_t Y(t)$. To do this, we again pick horizontal lifts $v\colon (-\varepsilon,\varepsilon) \to S^{4n+3}(1)$, $w\colon (-\varepsilon,\varepsilon) \to \mathbb{H}^{n+1}$ of γ and Y, respectively: for all $t \in (-\varepsilon,\varepsilon)$ we have $\Pi_{\mathbb{C}}(v(t)) = \gamma(t)$, $\dot{v}(t) \in v(t)\mathbb{C}^\perp$ and $d_{v(t)}\Pi_{\mathbb{H}}(w(t)) = Y(t)$, $w(t) \in v(t)\mathbb{H}^\perp$. Crucially, since γ is horizontal with respect to Π, we can choose the lift $v(t)$ so that it satisfies $\dot{v}(t) \in v(t)\mathbb{H}^\perp$ for all $t \in (-\varepsilon,\varepsilon)$, that is, $v(t)$ is now also a *horizontal* lift of $\Pi\circ\gamma$. We can now compute:

$$\begin{aligned}
\nabla_t\big(\mathbf{I}(\gamma(t))Y(t)\big)-\mathbf{I}(\gamma(t))\nabla_t Y(t) &= \nabla_t\big(\mathbf{I}(v(t)\mathbb{C})d_{v(t)}\Pi_{\mathbb{H}}(w(t))\big)-\mathbf{I}(v(t)\mathbb{C})\nabla_t\big(d_{v(t)}\Pi_{\mathbb{H}}(w(t))\big)\\
&= \nabla_t\big(d_{v(t)}\Pi_{\mathbb{H}}(w(t)\mathbf{i})\big)-\mathbf{I}(v(t)\mathbb{C})d_{v(t)}\Pi_{\mathbb{H}}(\overline{\nabla}_t w(t))\\
&= d_{v(t)}\Pi_{\mathbb{H}}(\overline{\nabla}_t(w(t)\mathbf{i}))-\mathbf{I}(v(t)\mathbb{C})d_{v(t)}\Pi_{\mathbb{H}}(\dot w(t))\\
&= d_{v(t)}\Pi_{\mathbb{H}}(\dot w(t)\mathbf{i})-d_{v(t)}\Pi_{\mathbb{H}}(\dot w(t)\mathbf{i}) \;=\; 0,
\end{aligned}$$

which is what we wanted to show.

So the differential of $\mathbf{I}$ splits as $\mathbf{I}_*=\mathbf{I}_*^{\mathcal{V}}\oplus\mathbf{I}_*^{\mathcal{H}}\colon \mathcal{V}\oplus\mathcal{H}\to\mathcal{V}^{\mathcal{Z}}\oplus\mathcal{H}^{\mathcal{Z}}$. Since both $\Pi\colon(\mathbb{CP}^{2n+1},g_{\mathrm{FS}})\to(\mathbb{HP}^n,g)$ and $\tau\colon(\mathcal{Z}(\mathbb{HP}^n,Q),g_{\mathcal{Z}})\to(\mathbb{HP}^n,g)$ are Riemannian submersions and $\tau\circ\mathbf{I}=\Pi$, it is clear that $\mathbf{I}_*^{\mathcal{H}}$ is a linear isometry. Therefore, in order to prove $\mathbf{I}^*g_{\mathcal{Z}}=g_{\mathrm{FS}}$, it suffices to show that $\mathbf{I}_*^{\mathcal{V}}$ is a linear isometry.

One way to see this is to recall that each fibre $\Pi_{\mathbb{C}}(v\mathbb{H})$ of Π is a complex line in $(\mathbb{CP}^{2n+1},g_{\mathrm{FS}})$ and hence is isometric to a round 2-sphere of radius $1/2$. On the other hand, the corresponding fibre $\mathcal{Z}(\mathbb{HP}^n,Q)_{v\mathbb{H}}$ of the twistor space of $\mathbb{HP}^n$ is the 2-sphere in the Euclidean space $(Q_{v\mathbb{H}},\frac{1}{16n}\{\,\cdot\,,\cdot\,\})$, consisting of those elements of Q which square to $-\mathrm{Id}$. So it suffices to show that the radius of that sphere is also $1/2$, for example, by computing the length of the element $\mathbf{I}(v\mathbb{C})\in\mathcal{Z}(\mathbb{HP}^n,Q)_{v\mathbb{H}}$ with respect to the metric $\frac{1}{16n}\{\,\cdot\,,\cdot\,\}$. For future use, let us directly compute the inner product of two elements, say $\mathbf{I}(v\mathbb{C})$ and $\mathbf{I}(vq\mathbb{C})$ for some $q\in\mathbb{H}^\times$. For that purpose, we assume that $\|v\|=1$, we let $\{e_1,e_2,\dots,e_{4n}\}$ be an orthonormal basis for $v\mathbb{H}^\perp$ and then we compute, using (4.3), (4.5) and (4.8):

$$\begin{aligned}
\frac{1}{16n}\{\mathbf{I}(v\mathbb{C}),\mathbf{I}(vq\mathbb{C})\} &= \frac{1}{16n}\sum_{s=1}^{4n} g_{v\mathbb{H}}(\mathbf{I}(v\mathbb{C})d_v\Pi_{\mathbb{H}}(e_s),\mathbf{I}(vq\mathbb{C})d_v\Pi_{\mathbb{H}}(e_s))\\
&= \frac{1}{16n}\sum_{s=1}^{4n} g_{v\mathbb{H}}(d_v\Pi_{\mathbb{H}}(e_s\mathbf{i}),d_{vq}\Pi_{\mathbb{H}}(e_sq\mathbf{i}))\\
&= \frac{1}{16n}\sum_{s=1}^{4n}\mathbb{R}\mathrm{e}\langle e_s\mathbf{i},\,e_sq\mathbf{i}q^{-1}\rangle\\
&= \frac{1}{4}\,\mathbb{R}\mathrm{e}\left(-\mathbf{i}q\mathbf{i}q^{-1}\right).
\end{aligned}\tag{4.9}$$

In particular, putting $q=1$ we see that $\frac{1}{16n}\{\mathbf{I}(v\mathbb{C}),\mathbf{I}(v\mathbb{C})\}=\frac{1}{4}$, which is what we wanted.

An even more explicit approach is to directly compute the map $\mathbf{I}_*^{\mathcal{V}}$. To that end, pick any vector $w\in\mathcal{V}_{v\mathbb{C}}=d_v\Pi_{\mathbb{C}}(v\mathbb{C}^\perp\cap v\mathbb{H})$ and let $p\in\mathbb{H}$ be such that $vp\in v\mathbb{C}^\perp\cap v\mathbb{H}$ and $d_v\Pi_{\mathbb{C}}(vp)=w$. Note that the first condition implies $\mathbb{C}\mathrm{o}(p)=0$. To compute the image vector $d_{v\mathbb{C}}\mathbf{I}(w)\in Q_{v\mathbb{H}}\subseteq\mathrm{End}(T_{v\mathbb{H}}\mathbb{HP}^n)$ we consider its action on a basis element $d_v\Pi_{\mathbb{H}}(e_s)\in T_{v\mathbb{H}}\mathbb{HP}^n$ (here we have identified the vertical tangent bundle to Q with Q itself). For the vertical path $t\mapsto\Pi_{\mathbb{C}}(v+tvp)\in\Pi_{\mathbb{C}}(v\mathbb{H})$ we have the identity

$$\begin{aligned}
&\mathbf{I}((v+tvp)\mathbb{C})d_{v+tvp}\Pi_{\mathbb{H}}(e_s) &&= d_{v+tvp}\Pi_{\mathbb{H}}(e_s\mathbf{i})\\
\Leftrightarrow\quad &\mathbf{I}((v+tvp)\mathbb{C})d_v\Pi_{\mathbb{H}}(e_s(1+tp)^{-1}) &&= d_v\Pi_{\mathbb{H}}(e_s\mathbf{i}(1+tp)^{-1}).
\end{aligned}\tag{4.10}$$

Differentiating the identity $(1+tp)(1+tp)^{-1}=1$ at $t=0$ yields $\frac{d}{dt}\big|_{t=0}(1+tp)^{-1}=-p$. Then differentiating (4.10) at $t=0$ gives the identity

$$\begin{aligned} d_{v\mathbb{C}}\mathbf{I}(d_v\Pi_{\mathbb{C}}(vp))\,(d_v\Pi_{\mathbb{H}}(e_s))+\mathbf{I}(v\mathbb{C})d_v\Pi_{\mathbb{H}}(-e_sp) &= d_v\Pi_{\mathbb{H}}(-e_s\mathbf{i}p) \\ \Leftrightarrow \quad d_{v\mathbb{C}}\mathbf{I}(w)\,(d_v\Pi_{\mathbb{H}}(e_s)) &= d_v\Pi_{\mathbb{H}}(e_s(p\mathbf{i}-\mathbf{i}p)) \\ &= d_v\Pi_{\mathbb{H}}(2e_s p\mathbf{i}), \end{aligned}$$

where in the last line we have used that $\mathbb{C}\mathrm{o}(p)=0$ and so $p\mathbf{i}=-\mathbf{i}p$. From here we compute

$$\begin{aligned} \frac{1}{16n}\{d_{v\mathbb{C}}\mathbf{I}(w),d_{v\mathbb{C}}\mathbf{I}(w)\} &= \frac{1}{16n}\{d_{v\mathbb{C}}\mathbf{I}(d_v\Pi_{\mathbb{C}}(vp)),d_{v\mathbb{C}}\mathbf{I}(d_v\Pi_{\mathbb{C}}(vp))\} \\ &= \frac{1}{16n}\sum_{s=1}^{4n} g_{v\mathbb{H}}(d_v\Pi_{\mathbb{H}}(2e_s p\mathbf{i}),d_v\Pi_{\mathbb{H}}(2e_s p\mathbf{i}) \\ &= \frac{1}{16n}4n\|2p\mathbf{i}\|^2=\|p\|^2, \end{aligned}$$

On the other hand

$$\|w\|^2=g_{\mathrm{FS}}(d_v\Pi_{\mathbb{C}}(vp),d_v\Pi_{\mathbb{C}}(vp))=\|vp\|^2=\|p\|^2.$$

Thus $\mathbf{I}_*^{\mathcal{V}}\colon\mathcal{V}\to\mathcal{V}^{\mathcal{Z}}$ is indeed an isometry.

Finally, let us show that $\mathbf{I}$ intertwines the almost complex structures. Again, we check this separately for $\mathbf{I}_*^{\mathcal{V}}$ and $\mathbf{I}_*^{\mathcal{H}}$. For $\mathbf{I}_*^{\mathcal{V}}$, let v,w,p and e_s be as above and note that

$$d_{v\mathbb{C}}\mathbf{I}(J_0(w))\,(d_v\Pi_{\mathbb{H}}(e_s))=d_v\Pi_{\mathbb{H}}(2e_s p\mathbf{i}\mathbf{i})=-d_v\Pi_{\mathbb{H}}(2e_s p).$$

On the other hand, since $d_{v\mathbb{C}}\mathbf{I}(w)\in\mathcal{V}^{\mathcal{Z}}$, we have that $J_{\mathcal{Z}}(d_{v\mathbb{C}}\mathbf{I}(w))=\mathbf{I}(v\mathbb{C})\circ d_{v\mathbb{C}}\mathbf{I}(w)\in Q_{v\mathbb{H}}$. But

$$(\mathbf{I}(v\mathbb{C})\circ d_{v\mathbb{C}}\mathbf{I}(w))d_v\Pi_{\mathbb{H}}(e_s)=\mathbf{I}(v\mathbb{C})d_v\Pi_{\mathbb{H}}(2e_s p\mathbf{i})=d_v\Pi_{\mathbb{H}}(2e_s p\mathbf{i}\mathbf{i})=-d_v\Pi_{\mathbb{H}}(2e_s p),$$

as we wanted. To check that $\mathbf{I}_*^{\mathcal{H}}$ intertwines J_0 and $J_{\mathcal{Z}}$, it suffices to check that for every $w'\in v\mathbb{H}^{\perp}$ one has $(d_{\mathbf{I}(v\mathbb{C})}\tau\circ d_{v\mathbb{C}}\mathbf{I}\circ J_0)d_v\Pi_{\mathbb{C}}(w')=(\mathbf{I}(v\mathbb{C})\circ d_{\mathbf{I}(v\mathbb{C})}\tau\circ d_{v\mathbb{C}}\mathbf{I})d_v\Pi_{\mathbb{C}}(w')$. But this is immediate from the fact that $\tau\circ\mathbf{I}=\Pi$ and the definitions of J_0 and $\mathbf{I}(v\mathbb{C})$. $\square$

We have thus verified that $(\mathcal{Z}(\mathbb{HP}^n,Q),g_{\mathcal{Z}},J_{\mathcal{Z}})$ is a Kähler manifold which is Kähler-isometric to $(\mathbb{CP}^{2n+1},g_{\mathrm{FS}},J_0)$. From now on we will not distinguish the two spaces and we will refer to the fibres of Π as twistor lines. One important observation which follows from this identification is that the horizontal part $\omega^{\mathcal{H}}$ of the Fubini-Study form ω_{FS} is "tautological", i.e. we have

$$\omega^{\mathcal{H}}_{v\mathbb{C}}(w_1,w_2)=g_{v\mathbb{H}}(\mathbf{I}(v\mathbb{C})d_{v\mathbb{C}}\Pi(w_1),d_{v\mathbb{C}}\Pi(w_2))\quad\forall w_1,w_2\in T_{v\mathbb{C}}\mathbb{CP}^{2n+1}. \tag{4.11}$$

This can also be seen directly from formulae (4.4) and (4.5). The vertical part $\omega^{\mathcal{V}}$ on the other hand is simply the area form on each fibre, giving the twistor lines area π.

Antipodal points and opposite equators

Since each twistor line is isometric to a round sphere, there is a well-defined notion of antipodal point and equator.

Definition 4.1.11. For each point $x \in \mathbb{CP}^{2n+1}$ we define its *antipodal point* to be the unique point $\mathcal{X}(x)$ which lies on the twistor line through x and is at maximal distance from x. We call the map $\mathcal{X}\colon \mathbb{CP}^{2n+1} \to \mathbb{CP}^{2n+1}$ the *fibrewise antipodal map*.

We define the *equator opposite* x to be the set $S(x)$ of points which lie on the twistor line through x and are equidistant from x and $\mathcal{X}(x)$. ♢

Note that since the twistor lines are totally geodesic, we can use the exponential map of the Fubini-Study metric to give a formula for $\mathcal{X}$ and to parametrise equators. More precisely, we see that $\mathcal{X}$ is given by $\mathcal{X}(x) = \exp_{g_{\mathrm{FS}}}(x, \frac{\pi}{2}v)$ where $v \in \mathcal{V}_x$ is any vector with $\|v\| = 1$. As for the equators, we make the following definition:

Definition 4.1.12. Let X be a smooth manifold and let $\varphi\colon X \to \mathbb{CP}^{2n+1}$ be a smooth map. Let Y_φ denote the circle bundle over X defined by $Y_\varphi := \{(x,v) \in \varphi^*\mathcal{V}\colon \|v\| = \pi/4\}$. We define the *opposite equator map* corresponding to φ to be

$$\widehat{\varphi}\colon Y_\varphi \to \mathbb{CP}^{2n+1}, \quad \widehat{\varphi}(x,v) = \exp_{g_{FS}}(\varphi(x), v).$$

Whenever $X \subseteq \mathbb{CP}^{2n+1}$ is embedded and φ is the inclusion map, we will write Y_X instead of Y_φ. Further, if the image $\widehat{\varphi}(Y_X) = \bigcup_{x\in X} S(x)$ is embedded, we'll denote it by Z_X and call it the *opposite equator manifold* of X. ♢

While this definition gives a nice global parametrisation of the opposite equator manifold, it is extremely inconvenient for direct calculations. To remedy this problem, we make the following observation:

Lemma 4.1.13. *For each $v \in \mathbb{H}^{n+1} \setminus \{0\}$, the antipodal point to $v\mathbb{C}$ is $(v\mathbf{j})\mathbb{C}$ and the equator opposite $v\mathbb{C}$ can be parametrised by*

$$S(v\mathbb{C}) = \{v(1+\mathbf{j}e^{\mathbf{i}\theta})\mathbb{C} \in \mathbb{CP}^{2n+1} \;:\; \theta \in \mathbb{R}/2\pi\mathbb{Z}\}.$$

In coordinates: the fibrewise antipodal map is given by the formula

$$\begin{aligned} \mathcal{X}\colon \mathbb{CP}^{2n+1} &\longrightarrow \mathbb{CP}^{2n+1} \\ [z_0 : z_1 : \ldots : z_{2n} : z_{2n+1}]_{\mathbb{C}} &\longmapsto [-\bar{z}_1 : \bar{z}_0 : -\bar{z}_3 : \bar{z}_2 : \ldots : -\bar{z}_{2n+1} : \bar{z}_{2n}]_{\mathbb{C}} \end{aligned} \tag{4.12}$$

and the equator opposite $[z_0 : z_1 : \ldots : z_{2n} : z_{2n+1}]_{\mathbb{C}}$ is

$$\{[z_0 - e^{\mathbf{i}\theta}\bar{z}_1 : z_1 + e^{\mathbf{i}\theta}\bar{z}_0 : \ldots : z_{2n} - e^{\mathbf{i}\theta}\bar{z}_{2n+1} : z_{2n+1} + e^{\mathbf{i}\theta}\bar{z}_{2n}]_{\mathbb{C}} \;:\; \theta \in \mathbb{R}/2\pi\mathbb{Z}\}. \tag{4.13}$$

Proof. Recall that the map $\mathbf{I}$ identifies the twistor line $\Pi_{\mathbb{C}}(v\mathbb{H})$ with a sphere in the Euclidean space $(Q_{v\mathbb{H}}, \frac{1}{16n}\{\cdot,\cdot\})$. So to see that the antipodal point to $v\mathbb{C}$ is $v\mathbf{j}\mathbb{C}$, it suffices to show that $\mathbf{I}(v\mathbf{j}\mathbb{C}) = -\mathbf{I}(v\mathbb{C})$. Indeed $\forall w \in \mathbb{H}^{n+1}$ we have

$$\mathbf{I}(v\mathbf{j}\mathbb{C})(d_v\Pi_{\mathbb{H}}(w)) = \mathbf{I}(v\mathbf{j}\mathbb{C})(d_{v\mathbf{j}}\Pi_{\mathbb{H}}(w\mathbf{j})) = d_{v\mathbf{j}}\Pi_{\mathbb{H}}(w\mathbf{ji}) = d_v\Pi_{\mathbb{H}}(-w\mathbf{i}) = -\mathbf{I}(v\mathbb{C})(d_v\Pi_{\mathbb{H}}(w)).$$

Suppose now that $q \in \mathbb{H}^\times$ is such that $vq\mathbb{C}$ lies in $S(v\mathbb{C})$. This is equivalent to the equation $\{\mathbf{I}(v\mathbb{C}), \mathbf{I}(vq\mathbb{C})\} = 0$, which from (4.9) becomes $\mathbb{R}\mathrm{e}(\mathbf{i}q\mathbf{i}q^{-1}) = 0$. Putting $q = x + \mathbf{j}y$ we get the equation $|y|^2 - |x|^2 = 0$. As we are only interested in q up to right multiplication by a complex number, we may assume that $x = 1$, $y = e^{\mathbf{i}\theta}$ for some $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. So the equator opposite $v\mathbb{C}$ is given by

$$S(v\mathbb{C}) = \{v(1+\mathbf{j}e^{\mathbf{i}\theta})\mathbb{C} \in \mathbb{CP}^{2n+1} \,:\, \theta \in \mathbb{R}/2\pi\mathbb{Z}\}.$$

The formulae in homogeneous coordinates are then immediate from the identifications (4.2). □

Remark 4.1.14. Note that formula (4.13) determines a well-defined subset $S^1 \subseteq \mathbb{CP}^{2n+1}$ corresponding to the point $x = [z_0 : z_1 : \ldots : z_{2n} : z_{2n+1}]_{\mathbb{C}}$ but it does not determine a parametrisation of this circle. If one chooses a lift $\tilde{x} \in \mathbb{C}^{2n+2}$ of x however, then the formula can be used as a parametrisation. //

Holomorphic contact structure

Recall that for any qK-manifold with non-vanishing scalar curvature, the vertical and horizontal distribution on the twistor space are respectively a holomorphic line bundle and a holomorphic contact structure ([Sal82, Theorem 4.3]). In the case of $\mathbb{CP}^{2n+1}$, it follows that $\mathcal{V}$ is a line bundle which restricts to the tangent bundle on each twistor line and hence $\mathcal{V}$ is isomorphic to $\mathcal{O}_{\mathbb{CP}^{2n+1}}(2)$. Thus the projection $\mathrm{pr}_{\mathcal{V}}\colon T\mathbb{CP}^{2n+1} \to \mathcal{V}$ must be given by an $\mathcal{O}_{\mathbb{CP}^{2n+1}}(2)$-valued holomorphic 1-form $\hat{\alpha}$, whose $\mathbb{C}$-valued expression $\hat{\alpha}^U$ in each trivialising chart U for $\mathcal{O}_{\mathbb{CP}^{2n+1}}(2)$ is such that $\hat{\alpha}^U \wedge (d\hat{\alpha}^U)^n$ is nowhere vanishing. This form can be succinctly written in homogeneous coordinates as

$$\hat{\alpha}_{[z_0:z_1:\ldots:z_{2n}:z_{2n+1}]_{\mathbb{C}}} \;=\; z_0dz_1 - z_1dz_0 + z_2dz_3 - z_3dz_2 + \cdots + z_{2n}dz_{2n+1} - z_{2n+1}dz_{2n}. \tag{4.14}$$

By this we mean that for each $0 \le i \le 2n+1$ there is a trivialisation ψ_i of $\mathcal{V}$ over the standard chart $U_i = \{[z_0 : z_1 : \ldots : z_{i-1} : 1 : z_{i+1} : \ldots : z_{2n} : z_{2n+1}]_{\mathbb{C}} \,:\, z_j \in \mathbb{C}\ \forall j \neq i\}$ in which $\mathrm{pr}_{\mathcal{V}}$ is given by the $\mathbb{C}$-valued form α^i, obtained from (4.14) by formally substituting $z_i = 1$, $dz_i = 0$.

To explain why this is the case, consider the 1-form α on $\mathbb{H}^{n+1} = \mathbb{C}^{2n+2}$ given again by the expression (4.14). Then it is immediate to check that for each $v \in \mathbb{H}^{n+1} \setminus \{0\}$ and $w \in \mathbb{H}^{n+1}$ one has

$$\alpha_v(w) = \omega_{\mathbb{C}}(v,w) = \mathbb{C}\mathrm{o}\langle v\mathbf{j}, w\rangle. \tag{4.15}$$

In other words, $\alpha_v(w)$ is the complex coefficient of $v\mathbf{j}$ in the orthogonal projection of w onto $v\mathbf{j}\mathbb{C} = v\mathbb{C}^\perp \cap v\mathbb{H}$. Since $d_{\frac{v}{\|v\|}}\Pi_{\mathbb{C}}\colon v\mathbb{C}^\perp \to T_{v\mathbb{C}}\mathbb{CP}^{2n+1}$ is an isometry which maps $v\mathbb{C}^\perp \cap v\mathbb{H}$ to $\mathcal{V}_{v\mathbb{C}}$, we have the identity

$$\mathrm{pr}_{\mathcal{V}}(d_v\Pi_{\mathbb{C}}(w)) = \alpha_v(w)d_v\Pi_{\mathbb{C}}\left(\frac{1}{\|v\|^2}v\mathbf{j}\right) \quad \forall v \in \mathbb{H}^{n+1} \setminus \{0\},\ w \in \mathbb{H}^{n+1}. \tag{4.16}$$

In particular, a (real) subspace $V \leq T_{v\mathbb{C}}\mathbb{CP}^{2n+1}$ is horizontal if and only if $\alpha|_{(d_v\Pi_{\mathbb{C}})^{-1}(V)} = 0$ and a submanifold $X \subseteq \mathbb{CP}^{2n+1}$ is horizontal if and only if $\alpha|_{(\Pi_{\mathbb{C}})^{-1}(X)} = 0$. Further, if we write $\tilde{\varphi}_i\colon \mathbb{C}^{2n+1} \to \mathbb{H}^{n+1} = \mathbb{C}^{2n+2}$ for the map $\tilde{\varphi}_i(z_0, z_1, \ldots, z_{i-1}, \hat{z}_i, z_{i+1}, \ldots, z_{2n+1}) = (z_0, z_1, \ldots, z_{i-1}, 1, z_{i+1}, \ldots, z_{2n+1})$, so that $\varphi_i := \Pi_{\mathbb{C}} \circ \tilde{\varphi}_i\colon \mathbb{C}^{2n+1} \to U_i$ give the standard charts on $\mathbb{CP}^{2n+1}$, we can trivialise $\mathcal{V}|_{U_i}$ via the map

$$\psi_i\colon \mathbb{C}^{2n+1} \times \mathbb{C} \longrightarrow \mathcal{V}|_{U_i}, \quad \psi_i(z,\lambda) = \lambda\, d_{\tilde{\varphi}_i(z)}\Pi_{\mathbb{C}}\left(\frac{1}{\|\tilde{\varphi}_i(z)\|^2}\tilde{\varphi}_i(z)\mathbf{j}\right). \tag{4.17}$$

It is not hard to check that the transition maps are given by $\psi_j^{-1} \circ \psi_i(z,\lambda) = \left(\varphi_j^{-1} \circ \varphi_i(z), \lambda/z_j^2\right)$, which are exactly the transition maps for $\mathcal{O}_{\mathbb{CP}^{2n+1}}(2)$. Moreover, the identities (4.16) and (4.17) show that

$$\begin{aligned} \psi_i^{-1}(\mathrm{pr}_{\mathcal{V}}(d_z\varphi_i(w))) &= \psi_i^{-1}\left(\alpha_{\tilde{\varphi}_i(z)}(d_z\tilde{\varphi}_i(w))\, d_{\tilde{\varphi}_i(z)}\Pi_{\mathbb{C}}\left(\frac{1}{\|\tilde{\varphi}_i(z)\|^2}\tilde{\varphi}_i(z)\mathbf{j}\right)\right) \\ &= (z, (\tilde{\varphi}_i^*\alpha)_z(w)) \;=\; (z, \alpha_z^i(w)). \end{aligned}$$

$\mathbb{C}$-isotropic and Legendrian submanifolds

We now briefly describe the properties of submanifolds of $\mathbb{CP}^{2n+1}$ which are horizontal with respect to Π, that is, submanifolds which are everywhere tangent to the holomorphic contact structure. To that end, we first introduce some notation and terminology.

Notation 4.1.15. We will use the following notation: given a complex vector space W, we will write $V \leq_{\mathbb{R}} W$, $V \leq_{\mathbb{C}} W$ to denote that V is a real or complex subspace of W, respectively. If $V \leq_{\mathbb{R}} W$ is a real subspace, we write $V_{\mathbb{C}} := \mathrm{Span}_{\mathbb{C}}(V)$. //

Definition 4.1.16. A subspace $V \leq_{\mathbb{R}} \mathbb{H}^{n+1}$ is called $\omega_{\mathbb{C}}$*−isotropic* if $\omega_{\mathbb{C}}|_V = 0$. It is called $\omega_{\mathbb{C}}$*−Lagrangian* if it is $\omega_{\mathbb{C}}$−isotropic and $\dim_{\mathbb{R}} V = 2(n+1)$. ♢

We immediately note the following:

Lemma 4.1.17. *Let $V \leq_{\mathbb{R}} \mathbb{H}^{n+1}$ be a subspace. The following are equivalent*

1. *V is $\omega_{\mathbb{C}}$−isotropic;*
2. *$V_{\mathbb{C}} = V + V\mathbf{i}$ is $\omega_{\mathbb{C}}$-isotropic;*
3. *$V\mathbf{j} \perp V$ and $V\mathbf{k} \perp V$.*

In particular, if V is $\omega_{\mathbb{C}}$-isotropic, then $\dim_{\mathbb{R}}(V) \leq 2\dim_{\mathbb{C}}(V_{\mathbb{C}}) \leq 2(n+1)$ and so, if V is $\omega_{\mathbb{C}}$−Lagrangian, then $V = V_{\mathbb{C}}$ is a complex subspace of $\mathbb{H}^{n+1}$.

Proof. The equivalence of the three assertions follows from the fact that for every $v_1, v_2 \in V$, we have

$$\omega_{\mathbb{C}}(v_1,v_2) = 0 \Leftrightarrow \mathbb{C}\mathrm{o}(\langle v_1\mathbf{j}, v_2\rangle) = 0 \Leftrightarrow \mathbb{R}\mathrm{e}(\langle v_1\mathbf{j}, v_2\rangle) - \mathbf{i}\,\mathbb{R}\mathrm{e}(\langle v_1\mathbf{k}, v_2\rangle) = 0 \Leftrightarrow \{v_2, v_2\mathbf{i}\} \perp \{v_1\mathbf{j}, v_1\mathbf{k}\}.$$

The remaining conclusions follow by comparing dimensions. □

Lemma 4.1.18. *Let $V \leq_{\mathbb{R}} \mathbb{H}^{n+1}$ be an $\omega_{\mathbb{C}}-$isotropic subspace and let $v \in V$ be a non-zero vector. Then there exists a complex subspace $V' \leq_{\mathbb{C}} v\mathbb{H}^{\perp}$ such that $V_{\mathbb{C}} = v\mathbb{C} \oplus V'$.*

Proof. Let $\mathrm{pr}_{v\mathbb{H}} \colon \mathbb{H}^{n+1} \to v\mathbb{H}$ denote the projection along $v\mathbb{H}^{\perp}$. We clearly have $v\mathbb{C} \leq v\mathbb{H} \cap V_{\mathbb{C}} \leq \mathrm{pr}_{v\mathbb{H}}(V_{\mathbb{C}})$. The claim will then follow if we can show that $\dim_{\mathbb{C}}(\mathrm{pr}_{v\mathbb{H}}(V_{\mathbb{C}})) = 1$. But if that is not the case, then since $\dim_{\mathbb{C}} v\mathbb{H} = 2$ we must have $\mathrm{pr}_{v\mathbb{H}}(V_{\mathbb{C}}) = v\mathbb{H}$, so in particular there exists $v' \in V_{\mathbb{C}}$ such that $\mathrm{pr}_{v\mathbb{H}}(v') = v\mathbf{j}$. But then we have $\omega_{\mathbb{C}}(v, v') = \mathbb{C}\mathrm{o}(\langle v\mathbf{j}, v' \rangle) = \mathbb{C}\mathrm{o}(\langle v\mathbf{j}, v\mathbf{j} \rangle) = \|v\|^2 \neq 0$ which contradicts the fact that $V_{\mathbb{C}}$ is $\omega_{\mathbb{C}}-$isotropic. □

We now turn to $\mathbb{CP}^{2n+1}$. We make the following definition.

Definition 4.1.19. A real subspace $V \leq_{\mathbb{R}} T_{v\mathbb{C}}\mathbb{CP}^{2n+1}$ is called $\mathbb{C}-$*isotropic* if $d_v\Pi_{\mathbb{C}}^{-1}(V)$ is $\omega_{\mathbb{C}}-$isotropic. It is called *Legendrian* if $d_v\Pi_{\mathbb{C}}^{-1}(V)$ is $\omega_{\mathbb{C}}-$Lagrangian. A map $\varphi \colon X \to \mathbb{CP}^{2n+1}$ of smooth manifolds is called $\mathbb{C}$-isotropic if $d_x\varphi(T_xX)$ is a $\mathbb{C}$-isotropic subspace of $T_x\mathbb{CP}^{2n+1}$ for all $x \in X$. If $\dim_{\mathbb{R}} X = 2n$ and φ is an immersion, we call it a *Legendrian immersion*. If $X \subseteq \mathbb{CP}^{2n+1}$ is a submanifold and φ is the inclusion, we will call X a $\mathbb{C}$-isotropic (resp. Legendrian) submanifold. ♢

While these definitions seem to differ from the analogous situation in real contact geometry, where a submanifold is called isotropic whenever it is tangent to the contact distribution, we now show that this is actually not the case: a submanifold of $\mathbb{CP}^{2n+1}$ is $\mathbb{C}$-isotropic if and only if it is horizontal. More precisely, we have the following lemma.

Lemma 4.1.20.

a) A subspace $V \leq_{\mathbb{R}} T_{v\mathbb{C}}\mathbb{CP}^{2n+1}$ is $\mathbb{C}$-isotropic if and only if $V \leq \mathcal{H}_{v\mathbb{C}}$ and

$$\mathbf{I}(v(1+\mathbf{j}e^{\mathbf{i}\theta})\mathbb{C})(\Pi_* V) \perp \Pi_* V \quad \forall\, \theta \in \mathbb{R}/2\pi\mathbb{Z}.$$

b) For a map of smooth manifolds $\varphi \colon X \to \mathbb{CP}^{2n+1}$ the following are equivalent:

b1) φ is $\mathbb{C}$-isotropic;

b2) φ is horizontal, i.e. $d_x\varphi(T_xX) \leq \mathcal{H}_{\varphi(x)}$ for all $x \in X$;

b3) each local lift $v \colon U \to \mathbb{H}^{n+1}$, where $U \subseteq X$ is an open set and $\Pi_{\mathbb{C}} \circ v = \varphi|_U$, satisfies $v^*\alpha = 0$.

In particular, if $\varphi \colon X \to \mathbb{CP}^{2n+1}$ is a horizontal immersion, then $\dim_{\mathbb{R}} X \leq 2n$ and φ is a Legendrian immersion if and only if it is horizontal and $\dim_{\mathbb{R}} X = 2n$. Moreover, in this case X can be equipped with an almost complex structure, so that X is a complex n-manifold and φ is holomorphic.

Proof. First we prove part a). Suppose $V \leq_{\mathbb{R}} T_{v\mathbb{C}}\mathbb{CP}^{2n+1}$ is $\mathbb{C}-$isotropic and write $\widetilde{V} = d_v\Pi_{\mathbb{C}}^{-1}(V) \leq \mathbb{H}^{n+1}$. By Lemma 4.1.18 we can write $\widetilde{V}_{\mathbb{C}} = v\mathbb{C} \oplus V'$ where $V' \leq v\mathbb{H}^{\perp}$. Then $V \leq V_{\mathbb{C}} = d_v\Pi_{\mathbb{C}}(V') \leq d_v\Pi_{\mathbb{C}}(v\mathbb{H}^{\perp}) = \mathcal{H}_{v\mathbb{C}}$. Now let $w_1, w_2 \in \Pi_* V$ and let $\tilde{w}_1, \tilde{w}_2 \in v\mathbb{H}^{\perp}$ be their lifts under $d_v\Pi_{\mathbb{H}}$. Then

$\tilde{w}_1, \tilde{w}_2 \in V'$ and since V' is $\omega_{\mathbb{C}}-$isotropic, Lemma 4.1.18 again tells us that we can write $\tilde{w}_2 = \tilde{w}_1 z + \tilde{w}_2'$ for some $z \in \mathbb{C}$ and $\tilde{w}_2' \in \tilde{w}_1 \mathbb{H}^{\perp}$. Then, assuming without loss of generality that $v \in S^{4n+3}(1)$, we have:

$$\begin{aligned} g_{v\mathbb{H}}(w_1, \mathbf{I}(v(1+\mathbf{j}e^{\mathbf{i}\theta})\mathbb{C})w_2) &= g_{v\mathbb{H}}(d_v\Pi_{\mathbb{H}}(\tilde{w}_1), \mathbf{I}(v(1+\mathbf{j}e^{\mathbf{i}\theta})\mathbb{C})d_v\Pi_{\mathbb{H}}(\tilde{w}_2)) \\ &= \mathbb{R}\mathrm{e}(\langle \tilde{w}_1, \tilde{w}_2(1+\mathbf{j}e^{\mathbf{i}\theta})\mathbf{i}(1+\mathbf{j}e^{\mathbf{i}\theta})^{-1}\rangle) \\ &= \mathbb{R}\mathrm{e}(-\langle \tilde{w}_1, \tilde{w}_2\mathbf{k}e^{\mathbf{i}\theta}\rangle) \\ &= -\mathbb{R}\mathrm{e}(\|\tilde{w}_1\|^2\mathbf{k}\bar{z}e^{\mathbf{i}\theta}) - \mathbb{R}\mathrm{e}(\langle \tilde{w}_1, \tilde{w}_2'\rangle\mathbf{k}e^{\mathbf{i}\theta}) = 0. \end{aligned}$$

Conversely, suppose that $V \leq_{\mathbb{R}} \mathcal{H}_{v\mathbb{C}}$ and $\mathbf{I}(v(1+\mathbf{j}e^{\mathbf{i}\theta}))\Pi_*V \perp \Pi_*V$ for all $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. From the first assumption we have $d_v\Pi_{\mathbb{C}}^{-1}(V_{\mathbb{C}}) = v\mathbb{C} \oplus V'$ for $V' \leq_{\mathbb{C}} v\mathbb{H}^{\perp}$ and it suffices to show that V' is $\omega_{\mathbb{C}}-$isotropic. If $\tilde{w}_1, \tilde{w}_2 \in V'$, then the second assumption and the same calculation as above show that for all $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ we have

$$0 = -\mathbb{R}\mathrm{e}(\langle \tilde{w}_1, \tilde{w}_2\rangle\mathbf{k}e^{\mathbf{i}\theta}) = -\mathbb{R}\mathrm{e}(\mathbf{k}e^{\mathbf{i}\theta}\langle \tilde{w}_1, \tilde{w}_2\rangle) = \mathbb{R}\mathrm{e}(\langle \tilde{w}_1 e^{-\mathbf{i}\theta}\mathbf{k}, \tilde{w}_2\rangle).$$

In particular, putting $\theta = 0, \pi/2$ yields

$$\mathbb{R}\mathrm{e}(\langle \tilde{w}_1\mathbf{k}, \tilde{w}_2\rangle) = \mathbb{R}\mathrm{e}(\langle \tilde{w}_1\mathbf{j}, \tilde{w}_2\rangle) = 0.$$

By Lemma 4.1.17 it follows that V' is $\omega_{\mathbb{C}}-$isotropic. This concludes the proof of part a).

Let us now prove part b). The fact that b1) implies b2) follows immediately from part a). On the other hand b2) is equivalent to b3) by identity (4.16). It remains to be shown that b3) implies b1). Fix $x \in X$ and let $U \subseteq X$ be an open neighbourhood of x such that $\varphi|_U$ admits a local lift $v\colon U \to \mathbb{H}^{n+1}$. We want to show that $d_x\varphi(T_xX)$ is $\mathbb{C}$-isotropic, i.e. that $d_{v(x)}\Pi_{\mathbb{C}}^{-1}(d_x\varphi(T_xX))$ is $\omega_{\mathbb{C}}$-isotropic. Note that the equation $\Pi_{\mathbb{C}} \circ v = \varphi|_U$ implies that

$$d_{v(x)}\Pi_{\mathbb{C}}^{-1}(d_x\varphi(T_xX)) = d_xv(T_xX) + \ker(d_{v(x)}\Pi_{\mathbb{C}}) = d_xv(T_xX) + v(x)\mathbb{C}.$$

Now define the map $\hat{v}\colon U \times \mathbb{C} \to \mathbb{H}^{n+1}$, $\hat{v}(x,\lambda) = v(x)\lambda$. Since $\hat{v}_*(T_xX \oplus \mathbb{C}) = v_*(T_xX) + v(x)\mathbb{C}$, we need to show that $\hat{v}^*\omega_{\mathbb{C}} = 0$. Note that for each $\lambda, \mu \in \mathbb{C}$ and $w \in T_xX$, we have the formula $d_{(x,\lambda)}\hat{v}(w,\mu) = d_xv(w)\lambda + v(x)\mu$ and so

$$(\hat{v}^*\alpha)_{(x,\lambda)}(w,\mu) = \lambda(v^*\alpha)_x(w) + \mu\,\alpha_{v(x)\lambda}(v(x)) = 0 \,+\, \mathbb{C}\mathrm{o}\langle v(x)\lambda\mathbf{j}, v(x)\rangle = 0.$$

Thus $\hat{v}^*\alpha = 0$ and hence $\hat{v}^*\omega_{\mathbb{C}} = \frac{1}{2}\hat{v}^*d\alpha = 0$, which is what we wanted.

The dimensional restrictions on manifolds admitting a horizontal immersion into $\mathbb{CP}^{2n+1}$ follow immediately from Lemma 4.1.17, which also tells us that if φ is a Legendrian immersion, then $\varphi_*(T_xX)$ is a complex subspace of $T_{\varphi(x)}\mathbb{CP}^{2n+1}$. Hence φ^*J_0 is a well-defined almost complex structure on X which is also integrable since J_0 is integrable. This makes X into a complex n-manifold and φ becomes a holomorphic immersion. $\square$

4.1.2.2 Statement and proof

We can now give the precise statement of the Legendrian–Lagrangian correspondence. First, we define the type of Lagrangians $L \subseteq \mathbb{CP}^{2n+1}$ which we will consider, namely the ones for which the restricted projection $\Pi|_L : L \to \mathbb{HP}^n$ is locally an S^1-bundle.

Definition 4.1.21. (cf. [DRGI16]) Let $\phi: L \to \mathbb{CP}^{2n+1}$ be a Lagrangian immersion. We say that ϕ is *compatible* with the twistor fibration Π, if there exists a smooth manifold X of dimension $2n$, a submersion $\pi: L \to X$ and an immersion $\bar{\varphi}: X \to \mathbb{HP}^n$ such that

i) the map $\pi: L \to X$ gives L the structure of a smooth locally trivial circle bundle over X;

ii) $\Pi \circ \phi = \bar{\varphi} \circ \pi$.

We call $\bar{\varphi}: X \to \mathbb{HP}^n$ the *base immersion* corresponding to ϕ. $\diamondsuit$

With this definition in place, the Legendrian–Lagrangian correspondence is summarised in the following two theorems.

Theorem 4.1.22. *Let X be a complex manifold and let $\varphi: X \to \mathbb{CP}^{2n+1}$ be a Legendrian immersion. As usual, set $Y_\varphi := \{(x,v) \in \varphi^*\mathcal{V} : \|v\| = \pi/4\}$ and consider the opposite equator map*

$$\widehat{\varphi}: Y_\varphi \to \mathbb{CP}^{2n+1}, \quad \widehat{\varphi}(x,v) = \exp_{g_{FS}}(\varphi(x), v).$$

Then $\widehat{\varphi}$ is a minimal Lagrangian immersion.

Clearly $\widehat{\varphi}$ is compatible with the twistor fibration and its corresponding base immersion is $\bar{\varphi} = \Pi \circ \varphi$. The next theorem shows that this is essentially the only way that compatible Lagrangian immersions arise.

Theorem 4.1.23. *Let $\phi: L \to \mathbb{CP}^{2n+1}$ be a Lagrangian immersion which is compatible with Π, has circle bundle structure $\pi: L \to X$ and corresponding base immersion $\bar{\varphi}: X \to \mathbb{HP}^n$. Then for every sufficiently small open set $U \subseteq X$ there exists a Legendrian embedding $\varphi^U: U \to \mathbb{CP}^{2n+1}$ which lifts $\bar{\varphi}|_U$ and satisfies $\widehat{\varphi}^U(Y_{\varphi^U}) = \phi(\pi^{-1}(U))$.*

Isotropic opposite equator manifolds

We now prove Theorem 4.1.22 and Theorem 4.1.23. They will follow from two local results about $\mathbb{C}$-isotropic maps into $\mathbb{CP}^{2n+1}$ which we formulate and prove in the next two propositions.

Proposition 4.1.24. *Let X be a smooth manifold and let $\varphi: X \to \mathbb{CP}^{2n+1}$ be a smooth map. Let $\widehat{\varphi}: Y_\varphi \to \mathbb{CP}^{2n+1}$ denote its corresponding opposite equator map. Then $\widehat{\varphi}^*\omega_{\mathrm{FS}} = 0$ if and only if φ is $\mathbb{C}$-isotropic.*

Proof. Since the statement is entirely local, we may assume that X is a small ball and that we have a lift $v: X \to S^{4n+3}(1)$ such that $\varphi = \Pi_{\mathbb{C}} \circ v$. Let $Y = (\mathbb{R}/2\pi\mathbb{Z}) \times X$ and define

$$\tilde{\phi}: Y \to \mathbb{H}^{n+1}, \quad \tilde{\phi}(\theta, x) = v(x)(1 + \mathbf{j}e^{\mathbf{i}\theta}).$$

Then by Lemma 4.1.13, the map $\phi := \Pi_{\mathbb{C}} \circ \tilde{\phi}$ parametrises $\widehat{\varphi}(Y_\varphi)$ and so $\widehat{\varphi}^*\omega_{\mathrm{FS}} = 0$ if and only if $\phi^*\omega_{\mathrm{FS}} = 0$.

Suppose that φ is a $\mathbb{C}-$isotropic. We first show that the horizontal form $\omega^{\mathcal{H}}$ vanishes on $\phi_* TY$. We have

$$\begin{aligned} \omega^{\mathcal{H}}(d_{(\theta,x)}\phi \cdot T_{(\theta,x)}Y,\, d_{(\theta,x)}\phi \cdot T_{(\theta,x)}Y) &= g_{\Pi(\varphi(x))}(\mathbf{I}(\phi(\theta,x))\Pi_*\varphi_* T_x X, \Pi_*\varphi_* T_x X) \\ &= g_{v\mathbb{H}}(\mathbf{I}(v(1+\mathbf{j}e^{\mathbf{i}\theta})\mathbb{C})\Pi_{\mathbb{H}*}v_* T_x X, \Pi_{\mathbb{H}*}v_* T_x X) \\ &= 0, \end{aligned} \tag{4.18}$$

where the first line comes from (4.11) and the last line follows from Lemma 4.1.20 a) and the assumption that φ is $\mathbb{C}-$isotropic.

We now need to show that $\omega^{\mathcal{V}}$ also vanishes on $\phi_* TY$. Since $(\mathcal{V}, \omega^{\mathcal{V}})$ is a (real) rank 2 symplectic vector bundle and by construction $\phi_* T(\mathbb{R}/2\pi\mathbb{Z}) \leq V$, then $\omega^{\mathcal{V}}$ vanishes on $\phi_* TY$ if and only if $\mathrm{pr}_{\mathcal{V}}(\phi_* TY) = \phi_* T(\mathbb{R}/2\pi\mathbb{Z})$, i.e. if and only if $\mathrm{pr}_{\mathcal{V}}(\phi_* TX) \leq \phi_* T(\mathbb{R}/2\pi\mathbb{Z})$. To show this, it is enough to prove that if $x \in X$ and $w \in T_x X$, then

$$d_{(\theta,x)}\tilde{\phi}(w) \leq \left(\tilde{\phi}(\theta,x)\mathbb{H}\right)^{\perp} \oplus \tilde{\phi}(\theta,x)\mathbb{C} \oplus \mathrm{Span}_{\mathbb{R}}\left(d_{(\theta,x)}\tilde{\phi}\left(\frac{\partial}{\partial\theta}\right)\right). \tag{4.19}$$

Observe first that

$$d_{(\theta,x)}\tilde{\phi}\left(\frac{\partial}{\partial\theta}\right) = \frac{\partial}{\partial\theta}(v(x)(1+\mathbf{j}e^{\mathbf{i}\theta})) = -v(x)\mathbf{k}e^{\mathbf{i}\theta}.$$

Now since φ is $\mathbb{C}-$isotropic, we have that $\mathrm{Span}_{\mathbb{C}}(d_x v(T_x X))$ is an $\omega_{\mathbb{C}}-$isotropic subspace of $\mathbb{H}^{n+1}$. So by Lemma 4.1.18, we can write $d_x v(w) = v(x)z + w'$ for some $z \in \mathbb{C}$ and $w' \in (v(x)\mathbb{H})^{\perp}$. Then we have

$$\begin{aligned} d_{(\theta,x)}\tilde{\phi}(w) &= (d_x v(w))(1+\mathbf{j}e^{\mathbf{i}\theta}) \\ &= w'(1+\mathbf{j}e^{\mathbf{i}\theta}) + v(x)z(1+\mathbf{j}e^{\mathbf{i}\theta}) \\ &= w'(1+\mathbf{j}e^{\mathbf{i}\theta}) + v(x)(1+\mathbf{j}e^{\mathbf{i}\theta})z - v(x)\mathbf{j}e^{\mathbf{i}\theta}z + v(x)\mathbf{j}e^{\mathbf{i}\theta}\bar{z} \\ &= w'(1+\mathbf{j}e^{\mathbf{i}\theta}) + v(x)(1+\mathbf{j}e^{\mathbf{i}\theta})z + 2\,\mathrm{Im}(z)v(x)\mathbf{k}e^{\mathbf{i}\theta} \\ &\in \left(\tilde{\phi}(\theta,x)\mathbb{H}\right)^{\perp} \oplus \tilde{\phi}(\theta,x)\mathbb{C} \oplus \mathrm{Span}_{\mathbb{R}}\left(d_{(\theta,x)}\tilde{\phi}\left(\frac{\partial}{\partial\theta}\right)\right). \end{aligned}$$

Now for the converse: suppose ω_{FS} vanishes on $\phi_* TY$. We will show that $v^*\alpha = 0$. First we claim that for every $(\theta,x) \in Y$ we have

$$\mathrm{pr}_{\mathcal{V}}(d_{(\theta,x)}\phi(T_{(\theta,x)}Y)) = \mathrm{Span}_{\mathbb{R}}\left(d_{(\theta,x)}\phi\left(\frac{\partial}{\partial\theta}\right)\right).$$

Indeed, if this is not the case, then since $\dim_{\mathbb{R}} \mathcal{V}_{\phi(\theta,x)} = 2$ and $\phi_*(\partial/\partial\theta) \neq 0$, there exists a vector $w \in T_{(\theta,x)}Y$ such that $\omega^{\mathcal{V}}(\phi_*(w), \phi_*(\partial/\partial\theta)) = 1$. But since $\phi_*(\partial/\partial\theta) \in \mathcal{V}_{\phi(\theta,x)}$ we have

$$\omega_{\mathrm{FS}}\left(\phi_*(w), \phi_*\left(\frac{\partial}{\partial\theta}\right)\right) = \omega^{\mathcal{V}}\left(\phi_*(w), \phi_*\left(\frac{\partial}{\partial\theta}\right)\right) = 1$$

which contradicts the assumption that $\phi^*\omega_{\mathrm{FS}} = 0$.

Then, we must have $\mathrm{pr}_{\mathcal{V}}(\phi_* TY) = \phi_* T(\mathbb{R}/2\pi\mathbb{Z})$ and so for every $w \in T_xX$, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, we have $d_{(\theta,x)}\tilde{\phi}(w) \in \left(\tilde{\phi}(\theta,x)\mathbb{H}\right)^{\perp} \oplus \tilde{\phi}(\theta,x)\mathbb{C} \oplus \mathrm{Span}_{\mathbb{R}}\left(d_{(\theta,x)}\tilde{\phi}\left(\frac{\partial}{\partial\theta}\right)\right)$ as above. Thus we can write

$$d_x v(w)(1+\mathbf{j}e^{\mathbf{i}\theta}) = w' + v(x)(1+\mathbf{j}e^{\mathbf{i}\theta})z + \lambda v(x)\mathbf{k}e^{\mathbf{i}\theta}$$

for some $w' \in v\mathbb{H}^{\perp}$, $z \in \mathbb{C}$ and $\lambda \in \mathbb{R}$. Then we have

$$\begin{aligned} d_x v(w) &= w'\frac{(1-\mathbf{j}e^{\mathbf{i}\theta})}{2} + v(x)z + v(x)\left(\frac{\lambda}{2} - \mathrm{Im}(z)\right)\mathbf{k}e^{\mathbf{i}\theta}(1-\mathbf{j}e^{\mathbf{i}\theta}) \\ &= w'\frac{(1-\mathbf{j}e^{\mathbf{i}\theta})}{2} + v(x)\left(\mathrm{Re}(z) + \frac{\lambda}{2}\mathbf{i}\right) + v(x)\left(\frac{\lambda}{2} - \mathrm{Im}(z)\right)\mathbf{k}e^{\mathbf{i}\theta} \end{aligned}$$

Applying the 1-form α to both sides we obtain

$$\alpha_v(v_*w) = \mathbb{C}\mathrm{o}\left\langle v\mathbf{j}, \left(\frac{\lambda}{2} - \mathrm{Im}(z)\right)\mathbf{k}e^{\mathbf{i}\theta}\right\rangle = \|v\|^2\left(\mathrm{Im}(z) - \frac{\lambda}{2}\right)\mathbf{i}e^{\mathbf{i}\theta}$$

The left-hand side is independent of θ, while the right-hand side is purely imaginary for $\theta = 0$ and real for $\theta = \pi/2$. We conclude that $\alpha_v(v_*w) = 0$ which is what we wanted to show. □

Next we show that if an isotropic submanifold of $\mathbb{CP}^{2n+1}$ of dimension at least $n+1$ intersects the twistor lines in circles, then it is in fact the opposite equator manifold of a $\mathbb{C}$-isotropic submanifold. Again, since the result is local, we assume that we can lift all maps to $\mathbb{H}^{n+1}$.

Proposition 4.1.25. *Let X be a smooth manifold with $\dim X \geq n+1$ and put $Y = (\mathbb{R}/2\pi\mathbb{Z}) \times X$. Suppose $\tilde{\phi}\colon Y \to \mathbb{H}^{n+1}$ is a smooth map such that $\phi := \Pi_{\mathbb{C}} \circ \tilde{\phi}$ is an immersion, satisfying $\phi^*\omega_{\mathrm{FS}} = 0$ and $\Pi_*\phi_*(\partial/\partial\theta) = 0$. Then there exists a $\mathbb{C}$-isotropic immersion $\varphi\colon X \to \mathbb{CP}^{2n+1}$ such that $\phi(Y) = \widehat{\varphi}(Y_\varphi)$, where $\widehat{\varphi}$ is the opposite equator map of φ.*

Proof. Let us write $\bar{\phi} := \Pi \circ \phi$. Since by assumption $\bar{\phi}$ is independent of θ, it factors through a map $\bar{\varphi}\colon X \to \mathbb{HP}^n$. We first observe that we must have $\phi^*\omega^{\mathcal{V}} = 0$. Indeed, for each $w \in T_{(\theta,x)}Y$ the following holds:

$$\begin{aligned} 0 &= \phi^*\omega_{\mathrm{FS}}\left(\frac{\partial}{\partial\theta}, w\right) \\ &= \omega^{\mathcal{V}}\left(\phi_*\left(\frac{\partial}{\partial\theta}\right), \mathrm{pr}_{\mathcal{V}}(\phi_* w)\right) + g_{\bar{\varphi}(x)}\left(\mathbf{I}(\phi(\theta,x))\bar{\phi}_*\left(\frac{\partial}{\partial\theta}\right), \bar{\phi}_* w\right) \\ &= \omega^{\mathcal{V}}\left(\phi_*\left(\frac{\partial}{\partial\theta}\right), \mathrm{pr}_{\mathcal{V}}(\phi_* w)\right) \end{aligned}$$

Since ϕ is an immersion, we have that $\phi_*(\partial/\partial\theta) \neq 0$ and so we must have $\mathrm{pr}_{\mathcal{V}}(\phi_* TY) \leq \mathrm{Span}_{\mathbb{R}}(\phi_*(\partial/\partial\theta))$. Hence $\phi^*\omega^{\mathcal{V}} = 0$ and moreover $\bar{\varphi}$ is an immersion.

We now know that $\phi^*\omega^{\mathcal{H}} = \phi^*\omega_{\mathrm{FS}} = 0$ and so for any point $x \in X$ we have

$$g_{\bar{\varphi}(x)}\left(\mathbf{I}(\phi(\theta,x))\bar{\varphi}_* T_xX,\ \bar{\varphi}_* T_xX\right) = 0 \quad \forall\theta \in \mathbb{R}/2\pi\mathbb{Z}. \tag{4.20}$$

Consider the subspace $P_x := \mathrm{Span}_{\mathbb{R}}\{\mathbf{I}(\phi(\theta,x)) : \theta \in \mathbb{R}/2\pi\mathbb{Z}\} \leq Q_{\bar{\varphi}(x)}$ and let I, J, K be a basis for $Q_{\bar{\varphi}(x)}$, satisfying $I^2 = J^2 = K^2 = IJK = -\mathrm{Id}$. By (4.20), we have that $A(\bar{\varphi}_* T_xX) \perp \bar{\varphi}_* T_xX$ for each

$A \in P_x$. Since ϕ is an immersion and $\mathbf{I}$ is an embedding, we have that $\mathbf{I}_*\phi_*(\partial/\partial\theta) \neq 0$ and so P_x is at least 2-dimensional. The condition $\dim X \geq n+1$ then forces P_x to be exactly 2-dimensional: otherwise we must have $P_x = Q_{\bar{\varphi}(x)}$ and so $I(\bar{\varphi}_* T_x X) \perp \bar{\varphi}_* T_x X$, $J(\bar{\varphi}_* T_x X) \perp \bar{\varphi}_* T_x X$, $K(\bar{\varphi}_* T_x X) \perp \bar{\varphi}_* T_x X$ which implies $4\dim(\bar{\varphi}_* T_x X) \leq \dim \mathbb{HP}^n = 4n$, contradicting the fact that $\bar{\varphi}$ is an immersion.

Hence $P_x \cap \mathcal{Z}(\mathbb{HP}^n, Q)_{\bar{\varphi}(x)}$ is an equator of the twistor line and the map $\mathbf{I}(\phi(\,\cdot\,,x)) \colon \mathbb{R}/2\pi\mathbb{Z} \to P_x \cap \mathcal{Z}(\mathbb{HP}^n, Q)_{\bar{\varphi}(x)}$ is a covering. In order to pick out one of the poles opposite to this equator, we use the fixed lift $\tilde{\phi}$ of ϕ. We define a map $\varphi' \colon Y \to \mathbb{CP}^{2n+1}$ via the equation

$$\mathbf{I}(\varphi'(\theta,x))d_{\tilde{\phi}(\theta,x)}\Pi_{\mathbb{H}}(w) = d_{\tilde{\phi}(\theta,x)}\Pi_{\mathbb{H}}(w\mathbf{j}) \quad \forall(\theta,x) \in Y,\ w \in T_{(\theta,x)}Y. \tag{4.21}$$

One can then easily check that $\{\mathbf{I}(\varphi'(\theta,x)), \mathbf{I}(\phi(\theta,x))\} = 0$ for all $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. Thus φ' is independent of θ with $\mathbf{I}(\varphi'(\,\cdot\,,x))$ constant at one of the poles opposite the equator $\mathbf{I}(\phi(\mathbb{R}/2\pi\mathbb{Z},x))$. Letting $\varphi \colon X \to \mathbb{CP}^{2n+1}$ be the map through which φ' factors, it follows by construction that $\phi(Y) = \widehat{\varphi}(Y_\varphi)$. In particular $\widehat{\varphi}^*\omega_{\mathrm{FS}} = 0$ and so by Proposition 4.1.24 that map φ is $\mathbb{C}$-isotropic. It is also an immersion since $\bar{\varphi} = \Pi \circ \varphi$ is an immersion. □

Theorem 4.1.23 now follows easily.

Proof of Theorem 4.1.23. Recall that we have a Lagrangian immersion $\phi \colon L \to \mathbb{CP}^{2n+1}$ which is compatible with Π with base immersion $\bar{\varphi} \colon X \to \mathbb{HP}^n$. We want to show that $\bar{\varphi}$ admits local Legendrian lifts. So let $U \subseteq X$ be a small open set such that $\bar{\varphi}|_U$ is an embedding. Then $\phi|_{\pi^{-1}(U)}$ is also an embedding, where $\pi \colon L \to X$ is the circle fibration. After possibly shrinking U, we may trivialise the circle bundle $\pi^{-1}(U) \to U$ and choose a lift $\tilde{\phi}_U \colon \pi^{-1}(U) \to \mathbb{H}^{n+1}$ of $\phi|_{\pi^{-1}(U)}$. These maps then satisfy all hypotheses of Proposition 4.1.25 and so there exists a Legendrian lift $\varphi^U \colon U \to \mathbb{CP}^{2n+1}$ of $\bar{\varphi}|_U$ and $\phi(\pi^{-1}(U)) = \widehat{\varphi}^U(Y_{\varphi^U})$. □

Observe also that Proposition 4.1.24 establishes most of Theorem 4.1.22. That is, it tells us that, if $\varphi \colon X \to \mathbb{CP}^{2n+1}$ is a Legendrian immersion, then $\widehat{\varphi}$ is a Lagrangian immersion. It remains to be shown that $\widehat{\varphi}$ is minimal. We do this in the next section.

Minimality

Recall that the sphere $S^{4n+3}(1)$ carries a standard *real* contact structure $\xi = TS^{4n+3} \cap (TS^{4n+3})\mathbf{i}$. We call a $(2n+1)$-dimensional submanifold of the sphere $\mathbb{R}$-Legendrian if it is everywhere tangent to ξ.

Remark 4.1.26. It is not without cringing that we impose this terminology but it is necessary to avoid the clash with Legendrian subvarieties of $\mathbb{CP}^{2n+1}$ which dominate a large portion of this thesis. The reader may find consolation in the fact that $\mathbb{R}$-Legendrian (ouch) submanifolds will be mentioned only very briefly in this section. //

It follows immediately from the definition of the Fubini-Study form that every $\mathbb{R}$-Legendrian submanifold of S^{2m+1} projects to an immersed Lagrangian in $\mathbb{CP}^m$. Conversely, every Lagrangian submanifold of $\mathbb{CP}^m$ *locally* admits an $\mathbb{R}$-Legendrian lift. To prove that the opposite equator map of a Legendrian immersion in $\mathbb{CP}^{2n+1}$ is minimal, we rely on a well-known result (see e.g. [CLU06, Proposition 2.2]), which states that a Lagrangian immersion $\phi\colon M^m \to \mathbb{CP}^m$ is minimal if and only if for every $y_0 \in M$, there exists an open neighbourhood $U \subseteq M$ of y_0 and a local $\mathbb{R}$-Legendrian lift $\tilde{\phi}\colon U \to S^{2m+1}$, such that the real cone $\mathrm{Cone}_{\mathbb{R}}\left(\tilde{\phi}(U)\right) := \left\{\lambda\tilde{\phi}(y) \in \mathbb{C}^{m+1} \;:\; y \in U,\, \lambda \in \mathbb{R}^\times\right\}$ is special Lagrangian in $\mathbb{C}^{m+1}$. That is, if and only if for any frame $\{e_1,\dots,e_m\}$ of U one has that the phase map

$$U \to S^1, \quad y \mapsto \frac{\det_{\mathbb{C}}\left(\tilde{\phi}(y), \tilde{\phi}_* e_1(y), \dots, \tilde{\phi}_* e_m(y)\right)}{\left|\det_{\mathbb{C}}\left(\tilde{\phi}(y), \tilde{\phi}_* e_1(y), \dots, \tilde{\phi}_* e_m(y)\right)\right|}$$

is constant.

The following lemma provides the desired local $\mathbb{R}$-Legendrian lifts.

Lemma 4.1.27. *Let X be a complex ball with $\dim_{\mathbb{C}} X = n$ and let $v\colon X \to S^{4n+3}(1)$ be such that $\varphi := \Pi_{\mathbb{C}} \circ v\colon X \longrightarrow \mathbb{CP}^{2n+1}$ is a holomorphic Legendrian embedding. Let $Y = \mathbb{R}/2\pi\mathbb{Z} \times X$. Then the map*

$$\tilde{\phi}\colon Y \to S^{4n+3}(1), \quad \tilde{\phi}(\theta, x) = v(x)\left(\frac{e^{-\mathbf{i}\frac{\theta}{2}} + \mathbf{j}e^{\mathbf{i}\frac{\theta}{2}}}{\sqrt{2}}\right)$$

is $\mathbb{R}$-Legendrian and $\phi := \Pi_{\mathbb{C}} \circ \tilde{\phi}$ parametrises $\widehat{\varphi}(Y_\varphi)$.

Proof. The fact that $\phi = \Pi_{\mathbb{C}} \circ \tilde{\phi}$ parametrises $\widehat{\varphi}(Y_\varphi)$ is just the observation that for all $x \in X$, the equator opposite $\varphi(x)$ is

$$\begin{aligned} S(v(x)\mathbb{C}) &= \{v(x)(1 + \mathbf{j}e^{\mathbf{i}\theta})\mathbb{C} \in \mathbb{CP}^{2n+1} \;:\; \theta \in \mathbb{R}/2\pi\mathbb{Z}\} \\ &= \left\{ v(x)\left(\frac{e^{-\mathbf{i}\frac{\theta}{2}} + \mathbf{j}e^{\mathbf{i}\frac{\theta}{2}}}{\sqrt{2}}\right)\mathbb{C} \in \mathbb{CP}^{2n+1} \;:\; \theta \in \mathbb{R}/2\pi\mathbb{Z} \right\}. \end{aligned}$$

We now need to show that $\tilde{\phi}_* TY \leq TS^{4n+3} \cap (TS^{4n+3}\mathbf{i})$, or, equivalently, that for every $(\theta, x) \in Y$ we have $d_{(\theta,x)}\tilde{\phi}\,(T_{(\theta,x)}Y) \perp \mathrm{Span}_{\mathbb{C}}\left(\tilde{\phi}(\theta,x)\right)$. Since $\|\tilde{\phi}\| \equiv 1$ we only need to show that

$$\tilde{\phi}(\theta,x)\,\mathbf{i} \perp d_{(\theta,x)}\tilde{\phi}\,(T_{(\theta,x)}Y).$$

First we compute

$$d_{(\theta,x)}\tilde{\phi}\left(\frac{\partial}{\partial\theta}\right) = \frac{1}{\sqrt{2}}v(x)\left(-\frac{\mathbf{i}}{2}e^{-\mathbf{i}\frac{\theta}{2}} + \mathbf{j}\frac{\mathbf{i}}{2}e^{\mathbf{i}\frac{\theta}{2}}\right) = \frac{1}{2\sqrt{2}}v(x)\left(-e^{-\mathbf{i}\frac{\theta}{2}} + \mathbf{j}e^{\mathbf{i}\frac{\theta}{2}}\right)\mathbf{i}. \tag{4.22}$$

Hence:

$$\begin{aligned} \mathbb{R}\mathrm{e}\left(\left\langle \tilde{\phi}(\theta,x)\,\mathbf{i}, d_{(\theta,x)}\tilde{\phi}\left(\frac{\partial}{\partial\theta}\right)\right\rangle\right) &= \frac{1}{4}\,\mathbb{R}\mathrm{e}\left(\left\langle v(x)\left(e^{-\mathbf{i}\frac{\theta}{2}} + \mathbf{j}e^{\mathbf{i}\frac{\theta}{2}}\right)\mathbf{i}, v(x)\left(-e^{-\mathbf{i}\frac{\theta}{2}} + \mathbf{j}e^{\mathbf{i}\frac{\theta}{2}}\right)\mathbf{i}\right\rangle\right) \\ &= \frac{1}{4}\,\mathbb{R}\mathrm{e}\left(-\mathbf{i}\left(e^{\mathbf{i}\frac{\theta}{2}} - \mathbf{j}e^{\mathbf{i}\frac{\theta}{2}}\right)\left(-e^{-\mathbf{i}\frac{\theta}{2}} + \mathbf{j}e^{\mathbf{i}\frac{\theta}{2}}\right)\mathbf{i}\right) \\ &= 0. \end{aligned}$$

Now let $w \in T_xX$. Since φ is Legendrian, Lemma 4.1.18 tells us that there exist $w' \in v(x)\mathbb{H}^\perp$ and $z \in \mathbb{C}$ such that $d_xv(w) = v(x)z + w'$. Note further that since $\|v\| \equiv 1$, we must also have that $d_xv(w) \in$

$v(x)\mathbb{R}^{\perp}$ and so $z = \lambda \mathbf{i}$ for some $\lambda \in \mathbb{R}$. We then have:

$$\begin{aligned} \mathbb{R}\mathrm{e}\left(\left\langle \tilde{\phi}(\theta,x)\,\mathbf{i}\,,\,d_{(\theta,x)}\tilde{\phi}(w)\right\rangle\right) &= \frac{1}{2}\mathbb{R}\mathrm{e}\left(\left\langle v(x)\left(e^{-\mathbf{i}\frac{\theta}{2}}+\mathbf{j}e^{\mathbf{i}\frac{\theta}{2}}\right)\mathbf{i}\,,\,(v(x)\lambda\mathbf{i}+w')\left(e^{-\mathbf{i}\frac{\theta}{2}}+\mathbf{j}e^{\mathbf{i}\frac{\theta}{2}}\right)\right\rangle\right) \\ &= \frac{\lambda}{2}\mathbb{R}\mathrm{e}\left(-\mathbf{i}\left(e^{\mathbf{i}\frac{\theta}{2}}-\mathbf{j}e^{\mathbf{i}\frac{\theta}{2}}\right)\mathbf{i}\left(e^{-\mathbf{i}\frac{\theta}{2}}+\mathbf{j}e^{\mathbf{i}\frac{\theta}{2}}\right)\right) \\ &= 0, \end{aligned}$$

where in the second line we used that $w' \in v(x)\mathbb{H}^{\perp}$. This concludes the proof. □

We now show that the real cone over the above $\mathbb{R}$-Legendrian lift is special Lagrangian. By [CLU06, Proposition 2.2], it follows that ϕ is a minimal Lagrangian immersion, and this completes the proof of Theorem 4.1.22.

Lemma 4.1.28. *Let X, v, φ, Y, $\tilde{\phi}$ be as in Lemma 4.1.27. Then* $\mathrm{Cone}_{\mathbb{R}}\left(\tilde{\phi}(Y)\right)$ *is special Lagrangian in $\mathbb{C}^{2n+2}$.*

Proof. Fix $x \in X$ and let $\{e_1, e_2, \ldots, e_n\}$ be a unitary $\mathbb{C}$-basis for T_xX with respect to the Kähler metric $\varphi^* g_{FS}$. For each $1 \le s \le n$ we can uniquely write

$$d_x v(e_s) = \lambda_s v(x)\mathbf{i} + v_s \in \mathbb{H}^{n+1} \text{ for } \lambda_s \in \mathbb{R} \text{ and } v_s \in v(x)\mathbb{C}^{\perp}.$$

Observe that since $d_{v(x)}\Pi_{\mathbb{C}}\big|_{v(x)\mathbb{C}^{\perp}} : v(x)\mathbb{C}^{\perp} \to T_{v(x)\mathbb{C}}\mathbb{CP}^{2n+1}$ is a $\mathbb{C}-$linear isometry sending v_s to $d_x\varphi(e_s)$, we have that $\{v_1, \ldots, v_n\}$ are unitary in $\mathbb{H}^{n+1}$, i.e. $\mathbb{C}\mathrm{o}(\langle v_s, v_t\rangle) = \delta_{st}$ for all $1 \le s,t \le n$. Setting $v_0 := v(x)$, we have

$$\mathbb{C}\mathrm{o}(\langle v_i, v_j\rangle) = \delta_{ij} \quad \forall 0 \le i,j \le n. \tag{4.23}$$

Now note that $\{v_0, v_1, \ldots, v_n\}$ is a $\mathbb{C}$-basis for $T_{v(x)}\Pi_{\mathbb{C}}^{-1}(\varphi(X))$. Since φ is Legendrian, we know that $T_{v(x)}\Pi_{\mathbb{C}}^{-1}(\varphi(X))$ is $\omega_{\mathbb{C}}$-Lagrangian and so we have

$$\mathbb{C}\mathrm{o}(\langle v_i\mathbf{j}, v_j\rangle) = \omega_{\mathbb{C}}(v_i, v_j) = 0 \quad \forall 0 \le i,j \le n. \tag{4.24}$$

From (4.23) and (4.24) we obtain that $\langle v_i, v_j\rangle = 0$ for all $0 \le i,j \le n$. This is the crucial ingredient we need, since now we know that

$$\det_{\mathbb{C}}\left(v_0, v_0\mathbf{j}, v_1, v_1\mathbf{j}, \ldots, v_n, v_n\mathbf{j}\right) \;=\; \frac{1}{(n+1)!}\omega_{\mathbb{C}}^{n+1}\left(v_0, v_0\mathbf{j}, v_1, v_1\mathbf{j}, \ldots, v_n, v_n\mathbf{j}\right) \;=\; 1. \tag{4.25}$$

By slight abuse of notation, let us use right multiplication by $\mathbf{i}$ to also denote the complex structure on TX. This is justified because φ is a holomorphic map and so we have the identity

$$d_x\varphi(e_s\mathbf{i}) = d_{v(x)}\Pi_{\mathbb{C}}(v_s\mathbf{i}) \quad \forall\, 1 \le s \le n.$$

So $\{e_1, e_1\mathbf{i}, e_2, e_2\mathbf{i}, \ldots, e_n, e_n\mathbf{i}\}$ is an $\mathbb{R}-$basis of T_xX and for all $1 \le s \le n$ there exists $\lambda_s' \in \mathbb{R}$ such that

$$d_x v(e_s\mathbf{i}) = \lambda_s' v(x)\mathbf{i} + v_s\mathbf{i}.$$

Therefore, writing $u(\theta) := \left(\frac{e^{-\mathbf{i}\frac{\theta}{2}}+\mathbf{j}e^{\mathbf{i}\frac{\theta}{2}}}{\sqrt{2}}\right)$, we have the following $\mathbb{R}$-basis for $T_{\tilde{\phi}(\theta,x)}\mathrm{Cone}_{\mathbb{R}}\left(\tilde{\phi}\left(Y\right)\right)$:

$$\left\{\tilde{\phi}(\theta,x),\, d_{(\theta,x)}\tilde{\phi}\left(\frac{\partial}{\partial\theta}\right),\, d_{(\theta,x)}\tilde{\phi}(e_1),\, d_{(\theta,x)}\tilde{\phi}(e_1\,\mathbf{i}),\, \ldots\, d_{(\theta,x)}\tilde{\phi}(e_n),\, d_{(\theta,x)}\tilde{\phi}(e_n\,\mathbf{i})\right\} =$$
$$= \left\{v_0\,u(\theta); v_0\,\dot{u}(\theta); \lambda_1\,v_0\,\mathbf{i}\,u(\theta)+v_1\,u(\theta),\ \lambda_1'\,v_0\,\mathbf{i}\,u(\theta)+v_1\,\mathbf{i}\,u(\theta),\ \ldots\right.$$
$$\left.\ldots,\ \lambda_n\,v_0\,\mathbf{i}\,u(\theta)+v_n\,u(\theta),\ \lambda_n'\,v_0\,\mathbf{i}\,u(\theta)+v_n\,\mathbf{i}\,u(\theta)\right\}.$$

Observe that $\dot{u}(\theta) = -\frac{1}{2}\,\mathbf{i}\,u(\theta)$ so the second term becomes $-\frac{1}{2}v_0\,\mathbf{i}\,u(\theta)$ and using Gaussian elimination we can transform the above basis into

$$\left\{v_0\,u(\theta),\ v_0\,\mathbf{i}\,u(\theta),\ v_1\,u(\theta),\ v_1\,\mathbf{i}\,u(\theta),\ \ldots,\ v_n\,u(\theta),\ v_n\,\mathbf{i}\,u(\theta)\right\}.$$

Writing $u = u(\theta)$ and using the identity $\mathbf{i}\,u = u\,\mathbf{i} + \frac{2}{\sqrt{2}}\mathbf{k}\,e^{\mathbf{i}\frac{\theta}{2}}$, we can now just compute the desired complex determinant:

$$\begin{aligned}
&\det_{\mathbb{C}}(v_0\,u,\ v_0\,\mathbf{i}\,u,\ v_1\,u,\ v_1\,\mathbf{i}\,u,\ \ldots,\ v_n\,u,\ v_n\,\mathbf{i}\,u) = \\
= \ &\det_{\mathbb{C}}\left(v_0\,u,\ v_0\,u\,\mathbf{i}+\frac{2}{\sqrt{2}}v_0\,\mathbf{k}e^{\mathbf{i}\frac{\theta}{2}},\ v_1\,u,\ v_1\,u\,\mathbf{i}+\frac{2}{\sqrt{2}}v_1\,\mathbf{k}e^{\mathbf{i}\frac{\theta}{2}},\ \ldots,\ v_n\,u,\ v_n\,u\,\mathbf{i}+\frac{2}{\sqrt{2}}v_n\,\mathbf{k}e^{\mathbf{i}\frac{\theta}{2}}\right) \\
= \ &\det_{\mathbb{C}}\left(v_0\,u,\ \frac{2}{\sqrt{2}}v_0\,\mathbf{k}e^{\mathbf{i}\frac{\theta}{2}},\ v_1\,u,\ \frac{2}{\sqrt{2}}v_1\,\mathbf{k}e^{\mathbf{i}\frac{\theta}{2}},\ \ldots, v_n\,u,\ \frac{2}{\sqrt{2}}v_n\,\mathbf{k}e^{\mathbf{i}\frac{\theta}{2}}\right) \\
= \ &\det_{\mathbb{C}}\left(\frac{1}{\sqrt{2}}v_0\,e^{-\mathbf{i}\frac{\theta}{2}}+\frac{1}{\sqrt{2}}v_0\mathbf{j}e^{\mathbf{i}\frac{\theta}{2}},\ \frac{2}{\sqrt{2}}v_0\mathbf{j}(-\mathbf{i}e^{\mathbf{i}\frac{\theta}{2}}),\ \frac{1}{\sqrt{2}}v_1\,e^{-\mathbf{i}\frac{\theta}{2}}+\frac{1}{\sqrt{2}}v_1\,\mathbf{j}e^{\mathbf{i}\frac{\theta}{2}},\ \frac{2}{\sqrt{2}}v_1\,\mathbf{j}(-\mathbf{i}e^{\mathbf{i}\frac{\theta}{2}}),\right. \\
&\left.\ldots,\ \frac{1}{\sqrt{2}}v_n\,e^{-\mathbf{i}\frac{\theta}{2}}+\frac{1}{\sqrt{2}}v_n\mathbf{j}e^{\mathbf{i}\frac{\theta}{2}},\ \frac{2}{\sqrt{2}}v_n\,\mathbf{j}(-\mathbf{i}e^{\mathbf{i}\frac{\theta}{2}})\right) \\
= \ &\det_{\mathbb{C}}\left(v_0\,e^{-\mathbf{i}\frac{\theta}{2}},\ v_0\,\mathbf{j}(-\mathbf{i}e^{\mathbf{i}\frac{\theta}{2}}),\ v_1\,e^{-\mathbf{i}\frac{\theta}{2}},\ v_1\mathbf{j}(-\mathbf{i}e^{\mathbf{i}\frac{\theta}{2}}),\ \ldots,\ v_n\,e^{-\mathbf{i}\frac{\theta}{2}},\ v_n\mathbf{j}(-\mathbf{i}e^{\mathbf{i}\frac{\theta}{2}})\right) \\
= \ &(-\mathbf{i})^{n+1}\det_{\mathbb{C}}(v_0,\ v_0\,\mathbf{j},\ v_1,\ v_1\,\mathbf{j},\ \ldots,\ v_n,\ v_n\,\mathbf{j}) \\
= \ &(-\mathbf{i})^{n+1} \quad \text{by (4.25).}
\end{aligned}$$

This is independent of θ and x, which is what we wanted to show. □

4.1.3 Known examples

The Legendrian–Lagrangian correspondence allows one to look at submanifolds of $\mathbb{CP}^{2n+1}$ and $\mathbb{HP}^n$ from different points of view. These are summarised in Table 4.1 (note that in Table 4.1 and in the discussion that follows, we only consider *smooth* Legendrian varieties).

If one is only interested in immersed twistor Lagrangians/superminimal sufaces/MTC submanifolds, then there are plenty of examples because Legendrian subvarieties of $\mathbb{CP}^{2n+1}$ have been extensively studied. This was initiated in the seminal paper [Bry82] by Bryant, where he showed that every closed Riemann surface admits a Legendrian embedding in $\mathbb{CP}^3$. In higher dimensions it was believed that smooth Legendrian subvarieties are quite rare and for a long time the only known examples were certain homogeneous varieties known as “subadjoint varieties”. The first non-homogeneous example of a smooth Legendrian surface in $\mathbb{CP}^5$ was constructed by Landsberg and

	Holomorphic Contact Geometry of $\mathbb{CP}^{2n+1}$	**Geometry of Submanifolds of $\mathbb{HP}^n$**	**Symplectic Geometry of $\mathbb{CP}^{2n+1}$**
$\mathbf{n=1}$	Legendrian curves in $\mathbb{CP}^3$	immersed superminimal surfaces in $\mathbb{HP}^1 = S^4$	immersed twistor Lagrangians in $\mathbb{CP}^3$
	Legendrian curves in $\mathbb{CP}^3$ of Type 1 or Type 2	embedded superminimal surfaces in $\mathbb{HP}^1 = S^4$	embedded twistor Lagrangians in $\mathbb{CP}^3$
$\mathbf{n \geq 2}$	Legendrian subvarieties of $\mathbb{CP}^{2n+1}$	MTC immersions in $\mathbb{HP}^n$	immersed twistor Lagrangians in $\mathbb{CP}^{2n+1}$
	Legendrian subvarieties of $\mathbb{CP}^{2n+1}$ of Type 1 or Type 2	embedded MTC submanifolds of $\mathbb{HP}^n$	embedded twistor Lagrangians in $\mathbb{CP}^{2n+1}$

Table 4.1: Twistor correspondences for submanifolds of $\mathbb{CP}^{2n+1}$ and $\mathbb{HP}^n$

Manivel in [LM07]. This was quickly followed by a few more examples by Buczynski. Finally, in [Buc08a] Buczynski used a symplectic reduction argument to show that a generic hyperplane section of a smooth Legendrian subvariety of $\mathbb{CP}^{2n+1}$ always admits a Legendrian embedding in $\mathbb{CP}^{2n-1}$. Applying this to the subadjoint varieties leads to large families of examples in every dimension (see [Buc09, Example A.15, Theorem A.16]).

On the other hand, if one is interested only in Legendrian subvarieties of Type 1 or Type 2, that is, the ones which give rise to embedded twistor Lagrangians and MTC submanifolds, then the situation is quite different. In section 4.1.4 below we use the results from section 3.1 to show that the only Type 1 Legendrian subvarieties are horizontal linear subspaces $\mathbb{CP}^n \subseteq \mathbb{CP}^{2n+1}$. In other words – the only maximal Kähler submanifold of $\mathbb{HP}^n$ is the totally geodesic $\mathbb{CP}^n$. The corresponding twistor Lagrangian is the standard $\mathbb{RP}^{2n+1}$.

As for Type 2 Legendrians, the only known examples so far are the subadjoint varieties. In fact, all subadjoint varieties of (complex) dimension more than 1 were listed by Tsukada in [Tsu85, Corollary 6.11] who showed that their projections to $\mathbb{HP}^n$ are the only *parallel* MTC submanifolds of $\mathbb{HP}^n$, that is the only ones which have parallel second fundamental form. Later the condition of being parallel has been shown to be equivalent to the requirement that the MTC submanifold is locally Kähler-Einstein or locally reducible ([Tsu04]) and is also equivalent to being homogeneous ([BGP09]). Here is Tsukada's list:

1. One infinite family of Type 2 Legendrian varieties obtained as follows. For each $m \geq 1$ let $\mathbf{Q}_m \subseteq \mathbb{CP}^{m+1}$ denote the quadric hypersurface. Then the Segre embedding $\mathbb{CP}^1 \times \mathbb{CP}^{m+1} \to \mathbb{CP}^{2m+3}$ restricts to a Legendrian embedding $\mathbb{CP}^1 \times \mathbf{Q}_m \to \mathbb{CP}^{2m+3}$. Explicitly, putting $\mathbf{Q}_m = \{[u_0 : u_1 : \ldots : u_{m+1}]_{\mathbb{C}} \in \mathbb{CP}^{m+1} : \sum_{i=0}^{m+1} u_i^2 = 0\}$, the embedding is

$$\begin{aligned} \sigma \colon \mathbb{CP}^1 \times \mathbf{Q}_m &\longrightarrow \mathbb{CP}^{2m+3} \\ ([x:y]_{\mathbb{C}}, [u_0 : u_1 : \ldots : u_{m+1}]_{\mathbb{C}}) &\longmapsto [xu_0 : yu_0 : xu_1 : yu_1 : \ldots : xu_{m+1} : yu_{m+1}]_{\mathbb{C}}. \end{aligned}$$

It is immediate to check that $\sigma^*(\hat{\alpha}) = 0$ and

$$\mathcal{X}\left(\sigma([x:y]_{\mathbb{C}}, [u_0 : u_1 : \ldots : u_{m+1}]_{\mathbb{C}})\right) = \sigma([-\bar{y} : \bar{x}]_{\mathbb{C}}, [\bar{u}_0 : \bar{u}_1 : \ldots : \bar{u}_{m+1}]_{\mathbb{C}}).$$

We denote this Type 2 Legendrian variety by $X_{(1,m)}$ and the corresponding twistor Lagrangian by $Z_{(1,m)}$.

2. The $\omega_{\mathbb{C}}$-Lagrangian Grassmanian $\mathrm{Gr}_{\mathrm{Lag}}(\mathbb{C}^6, \omega_{\mathbb{C}}) = \frac{\mathrm{Sp}(3)}{U(3)} \subseteq \mathbb{CP}^{13}$ giving an MTC submanifold in $\mathbb{HP}^6$. We denote this variety by X_6 and the corresponding twistor Lagrangian by Z_6.

3. The complex Grassmannian $\mathrm{Gr}_{\mathbb{C}}(3,6) = \frac{\mathrm{U}(6)}{\mathrm{U}(3)\times\mathrm{U}(3)} \subseteq \mathbb{CP}^{19}$ giving an MTC submanifold of $\mathbb{HP}^9$. We denote this variety by X_9 and the corresponding twistor Lagrangian by Z_9.

4. The homogeneous space $\frac{\mathrm{SO}(12)}{\mathrm{U}(6)} \subseteq \mathbb{CP}^{31}$ giving an MTC submanifold of $\mathbb{HP}^{15}$. We denote this variety by X_{15} and the corresponding twistor Lagrangian by Z_{15}.

5. The homogeneous space $\frac{\mathrm{E}_7}{\mathrm{E}_6 \cdot T^1} \subseteq \mathbb{CP}^{55}$ giving an MTC submanifold of $\mathbb{HP}^{27}$. We denote this variety by X_{27} and the corresponding twistor Lagrangian by Z_{27}.

For the representation theory of subadjoint varieties see e.g. [Muk98] and [LM02]. See also [Buc08b] which gives the explicit equations defining the Legendrian embeddings of the subadjoint varieties. As one can see from these references, there is one other subadjoint variety apart from the ones in the above list, namely the twisted cubic $X_1 := v_3(\mathbb{CP}^1) \subseteq \mathbb{CP}^3$ (v_3 denotes the degree 3 Veronese embedding). This is again of Type 2, the superminimal surface to which it projects is the well-known Veronese surface $\mathbb{RP}^2 \subseteq S^4$ and its corresponding twistor Lagrangian Z_1 is precisely the Chiang Lagrangian $L_\Delta \subseteq \mathbb{CP}^3$ (see section 5.1.1 below). We will refer to Z_1, Z_6, Z_9, Z_{15}, Z_{27} and $Z_{(1,m)}$ for $m \geq 1$ as the *subadjoint Lagrangians*.

We now begin our systematic study of Type 1 and Type 2 twistor Lagrangians in $\mathbb{CP}^{2n+1}$.

4.1.4 Type 1 twistor Lagrangians

By projectivising an $\omega_{\mathbb{C}}$-Lagrangian subspace of $\mathbb{C}^{2n+2}$ one obtains a Type 1 Legendrian embedding $\mathbb{CP}^n \subseteq \mathbb{CP}^{2n+1}$, whose image under Π is a totally geodesic Kähler submanifold of $\mathbb{HP}^n$. Letting $X = \mathbb{CP}^n \subseteq \mathbb{CP}^{2n+1}$ denote this Type 1 Legendrian and φ denote the inclusion, we have that the opposite equator map

$$\widehat{\varphi}\colon Y_X = \{(x,v) \in \mathcal{V}|_X \ : \ \|v\| = \pi/4\} \longrightarrow \mathbb{CP}^{2n+1}$$

is an embedding. Recalling that $\mathcal{V} = \mathcal{O}_{\mathbb{CP}^{2n+1}}(2)$, we see that the twistor Lagrangian corresponding to X is diffeomorphic to the circle bundle inside $\mathcal{O}_{\mathbb{CP}^n}(2)$, that is $Z_X = \mathbb{RP}^{2n+1}$. This is the standard Lagrangian embedding of $\mathbb{RP}^{2n+1}$: to obtain exactly the copy of $\mathbb{RP}^{2n+1}$ parametrised by real homogeneous coordinates, one can take the horizontal $\mathbb{CP}^n$ cut out by the equations $z_{2k+1} = \mathbf{i}z_{2k}$,

$0 \le k \le n$, whose opposite equator manifold is the set

$$\Pi_{\mathbb{C}}(\ \{(1-\mathbf{i})e^{-\mathbf{i}\frac{\theta}{2}}(z_0+\mathbf{i}\bar{z}_0e^{\mathbf{i}\theta}, \mathbf{i}z_0+\bar{z}_0e^{\mathbf{i}\theta}, \dots, z_{2n}+\mathbf{i}\bar{z}_{2n}e^{\mathbf{i}\theta}, \mathbf{i}z_{2n}+\bar{z}_{2n}e^{\mathbf{i}\theta}) :$$
$$(z_0, z_2, \dots, z_{2n}) \in \mathbb{C}^{n+1} \setminus \{0\}, \theta \in \mathbb{R}/2\pi\mathbb{Z}\}\ ).$$

Note that the group $\mathrm{Sp}(n+1)$ acts transitively on $\omega_{\mathbb{C}}$-Lagrangian subspaces of $\mathbb{C}^{2n+2}$ so there is only one linear Legendrian $\mathbb{CP}^n \subseteq \mathbb{CP}^{2n+1}$ up to this action. We now prove that this is also the only Type 1 Legendrian variety.

Theorem 4.1.29. *If $X \subseteq \mathbb{CP}^{2n+1}$ is a Type 1 Legendrian subvariety, then X is a linear $\mathbb{CP}^n$.*

Proof. Since X is Type 1, the corresponding twistor Lagrangian Z_X is diffeomorphic to the principal circle bundle $Y_X = S(\mathcal{O}_X(2))$. Our goal is to show that $N_{Z_X} = 2n+2$ so that we can apply Theorem 3.1.1 (recall that twistor Lagrangians are automatically monotone by [CG04], because they are minimal Lagrangians in a Kähler-Einstein manifold).

Since the Hurewicz homomorphism $\pi_2(\mathbb{CP}^{2n+1}, Z_X) \to H_2(\mathbb{CP}^{2n+1}, Z_X; \mathbb{Z})$ is surjective, we can find a continuous map $u\colon (D^2, \partial D^2) \to (\mathbb{CP}^{2n+1}, Z_X)$ whose class $[u] \in H_2(\mathbb{CP}^{2n+1}, Z_X; \mathbb{Z})$ realises the minimal Maslov number, i.e. $I_{\mu_{Z_X}}(u) = N_{Z_X}$. Now consider the reflected disc $\check{u} := \mathcal{X} \circ u \circ c$, where $c\colon D^2 \to D^2$ denotes complex conjugation. Since Z_X is setwise fixed by $\mathcal{X}$, $\check{u}$ is also a disc with boundary on Z_X and since $\mathcal{X}$ is antihomolomorphic, we have that $\mathcal{X}^*\mu_{Z_X} = -\mu_{Z_X}$ and so $I_{\mu_{Z_X}}(\check{u}) = I_{\mu_{Z_X}}(u) = N_{Z_X}$. Now, since $Z_X \to X$ is a principal S^1-bundle and $\mathcal{X}$ acts on Z_X as the antipodal map on the fibres, we see that $\mathcal{X}|_{Z_X} : Z_X \to Z_X$ is homotopic to the identity map because we can use the S^1-action to rotate the fibres 180 degrees.[3] In particular then $\mathcal{X}$ acts trivially on $H_1(Z_X; \mathbb{Z})$ and so we have $\partial[u] = -\partial[\check{u}] \in H_1(Z_X; \mathbb{Z})$, where the minus sign comes from complex conjugation. Then $[u] + [\check{u}]$ must lie the image of the natural map $j_*\colon H_2(\mathbb{CP}^{2n+1}; \mathbb{Z}) \to H_2(\mathbb{CP}^{2n+1}, Z_X; \mathbb{Z})$ and hence

$$2I_{\mu_{Z_X}}([u]) = I_{\mu_{Z_X}}([u] + [\check{u}]) \ \in \ I_{\mu_{Z_X}}(j_*(H_2(\mathbb{CP}^{2n+1}; \mathbb{Z})) = 2(2n+2)\mathbb{Z}.$$

So $2n+2$ divides N_{Z_X}. By the bound (3.2), we must then have $N_{Z_X} = 2n+2$, as we wanted.

Theorem 3.1.1 now tells us that $\pi_1(Z_X) \cong \mathbb{Z}/2$ and the universal cover of Z_X is homeomorphic to S^{2n+1}. Recall that Z_X is diffeomorphic to $S(\mathcal{O}_X(2))$. So $\widetilde{Z}_X := S(\mathcal{O}_X(1))$ is a connected double cover of Z_X and hence it must be homeomorphic to S^{2n+1}. Now let $i\colon X \to \mathbb{CP}^{2n+1}$ denote the inclusion and note that the restriction of the hyperplane class $i^*H \in H^2(X; \mathbb{Z})$ is the Euler class of the circle bundle $S^1 \to \widetilde{Z}_X \to X$. Using the Gysin long exact sequence for this circle bundle together with the fact that $H^i(\widetilde{Z}_X; \mathbb{Z}) = 0$ for all $1 \le i \le 2n$, we get that the map

$$H^i(X; \mathbb{Z}) \xrightarrow{\smile (i^*H)} H^{i+2}(X; \mathbb{Z})$$

is an isomorphism for all $0 \le i \le 2n-2$. Hence the degree of X as a subvariety of $\mathbb{CP}^{2n+1}$ is

[3] Thanks to Jack Smith for pointing this out.

$\deg(X) = \int_X (i^*H)^n = 1$. It follows (see [GH94, page 173]) that X is a linear subvariety of $\mathbb{CP}^{2n+1}$ and so it is a linear horizontal $\mathbb{CP}^n$. $\square$

4.1.5 Type 2 twistor Lagrangians

Having seen that there are no interesting Type 1 twistor Lagrangians, we now move on to the ones of Type 2. In this section we derive some topological properties that any Type 2 twistor Lagrangian must have.

Let $X \subseteq \mathbb{CP}^{2n+1}$ be a Type 2 Legendrian subvariety. We will write $\overline{X} := \Pi(X) \subseteq \mathbb{HP}^n$ for the corresponding MTC submanifold (or superminimal surface, if $n = 1$). Topologically $\overline{X}$ is the quotient of X by the $\mathbb{Z}/2$-action of the fibrewise antipodal map $\mathcal{X}|_X$. As before, we write $Y_X := \{(x, v) \in \mathcal{V}|_X \ : \ \|v\| = \pi/4\}$ for the circle bundle $S(\mathcal{O}_X(2))$, and Z_X for the actual twistor Lagrangian. Note that the opposite equator map $\widehat{\varphi}\colon Y_X \to Z_X$ is a double cover.

Lemma 4.1.30. *The following hold:*

1) *$\overline{X}$ is orientable if and only if n is even.*

2) *The bundle $S^1 \to Z_X \xrightarrow{\Pi} \overline{X}$ is non-orientable. In particular the homology class of a circle fibre has order 2 in $H_1(Z_X;\mathbb{Z})$.*

3) *The manifold Z_X is orientable if and only if n is odd.*

4) *The minimal Maslov number of Z_X is $N_{Z_X} = n+1$. Moreover, if $[u_1], [u] \in H_2(\mathbb{CP}^{2n+1};\mathbb{Z})$ denote respectively the class of a hemisphere of a twistor fibre passing through Z_X and any class with Maslov index $n+1$, then $[u] - \mathcal{X}_*[u] = [u_1]$.*

Proof. The map $\mathcal{X}|_X : X \to X$ is an antiholomorphic involution and so it is orientation-preserving exactly when $n = \dim_{\mathbb{C}} X$ is even. This implies 1). To prove 2), we lift the action of $\mathcal{X}$ to $\mathcal{V}$ by setting:

$$\widetilde{\mathcal{X}}\colon \mathcal{V} \to \mathcal{V}, \quad \widetilde{\mathcal{X}}(x, v) = (\mathcal{X}(x), -d_x\mathcal{X}(v)).$$

Thinking about how the antipodal map interacts with the exponential map on a round 2-sphere of radius $1/2$, we see that we have the relation

$$\exp_{g_{\mathrm{FS}}}(\widetilde{\mathcal{X}}(x, v)) = \exp_{g_{\mathrm{FS}}}(x, v) \quad \forall x \in \mathbb{CP}^{2n+1}, v \in \mathcal{V}_x, \|v\| = \pi/4. \tag{4.26}$$

So the restriction of $\widetilde{\mathcal{X}}$ acts on Y_X, the quotient is Z_X and $\widehat{\varphi}$ is the quotient map. Now let V be a nowhere vanishing vector field on Y_X which is everywhere positively tangent to the circle fibres of the bundle $Y_X \xrightarrow{\pi} X$ (recall that this bundle is canonically oriented by the S^1-action). Then we have $\widetilde{\mathcal{X}}_* V = -V$ and so $\widetilde{\mathcal{X}}$ reverses the orientation on fibres. It follows from this and the diagram

$$\begin{tikzcd} Y_X \arrow[r, "\pi"] \arrow[d, "\widehat{\varphi}"'] & X \arrow[d, "\Pi|_X"] \\ Z_X \arrow[r, "\Pi|_{Z_X}"] & \overline{X} \end{tikzcd}$$

that the bundle $S^1 \to Z_X \to \overline{X}$ is a non-orientable fibre bundle. Now let $\delta\colon \mathbb{R}/\mathbb{Z} \to \overline{X}$ be a loop such that following the fibre of Z_X around δ reverses its orientation. Then the circle bundle $K := \delta^* Z_X \to \mathbb{R}/\mathbb{Z}$ is a Klein bottle. Let $f\colon K \to Z_X$ denote the natural map, choose a point $p \in f(K)$ and let $\gamma\colon \mathbb{R}/\mathbb{Z} \to Z_X$ be a parametrisation of the circle fibre through p. Then there exist maps $\tilde{\gamma}\colon \mathbb{R}/\mathbb{Z} \to K$, $\tilde{\delta}\colon \mathbb{R}/\mathbb{Z} \to K$ such that $f \circ \tilde{\gamma} = \gamma$, $\Pi \circ f \circ \tilde{\delta} = \delta$ and $\tilde{\gamma}(0) = \tilde{\delta}(0) \in f^{-1}(p)$. It follows that in $\pi_1(K, \tilde{\gamma}(0))$ we have the relation $\tilde{\delta} \cdot \tilde{\gamma} \cdot \tilde{\delta}^{-1} = \tilde{\gamma}^{-1}$. Applying f_* and passing to homology yields $2[\gamma] = 0 \in H_1(Z_X;\mathbb{Z})$. Now let $u_1\colon (D^2, \partial D^2) \to (\mathbb{CP}^{2n+1}, Z_X)$ be a holomorphic parametrisation of one of the hemispheres of the twistor line ℓ whose equator is parametrised by γ. Then $\check{u}_1 := \mathcal{X} \circ u_1 \circ c$ is a holomorphic parametrisation of the other hemisphere and so

$$[u_1] + [\check{u}_1] = j_*[\ell] \in H_2(\mathbb{CP}^{2n+1}, Z_X;\mathbb{Z}),$$

where $j_*\colon H_2(\mathbb{CP}^{2n+1};\mathbb{Z}) \to H_2(\mathbb{CP}^{2n+1}, Z_X;\mathbb{Z})$ as usual denotes the natural map. Applying $I_{\mu_{Z_X}}$ to the above equation and using the fact that $\mathcal{X}^*\mu_{Z_X} = -\mu_{Z_X}$, we see that $I_{\mu_{Z_X}}([u_1]) = 2n+2$. In particular, $[\gamma] = \partial[u_1]$ is non-zero and hence it has order exactly 2 in $H_1(Z_X;\mathbb{Z})$. This completes the proof of 2).

To show 3), observe that we have the exact sequence $0 \to \mathrm{Span}_{\mathbb{R}}(V) \to TY_X \to \pi^*TX \to 0$. From this we see that $\widetilde{\mathcal{X}}\Big|_{Y_X}$ is orientation preserving if and only if $\mathcal{X}|_X$ is orientation reversing which, by 1), happens exactly when n is odd. In other words, Z_X is orientable if and only if n is odd.

From the long exact sequence in homotopy for the fibre bundle $S^1 \to Z_X \xrightarrow{\Pi} \overline{X}$ we have the exact sequence of groups $\pi_1(S^1) \to \pi_1(Z_X) \xrightarrow{\Pi_*} \pi_1(\overline{X}) \to 1$. Applying the Hurewicz homomorphism and using the fact that abelianisation is right-exact, we have the short exact sequence

$$0 \to \langle [\gamma] = \partial[u_1] \rangle \to H_1(Z_X;\mathbb{Z}) \xrightarrow{\Pi_*} H_1(\overline{X};\mathbb{Z}) \to 0.$$

Now let $u\colon (D^2, \partial D^2) \to (\mathbb{CP}^{2n+1}, Z_X)$ be a disc which realises the minimal Maslov number, i.e. $I_{\mu_{Z_X}}([u]) = N_{Z_X}$. Put $\check{u} := \mathcal{X} \circ u \circ c$ and note that since $\Pi \circ \mathcal{X} = \Pi$ we have that

$$\Pi_*(\partial[u] + \partial[\check{u}]) = 0 \in H_1(\overline{X};\mathbb{Z}).$$

It follows from the above exact sequence that $\partial([u] + [\check{u}]) \in \langle \partial[u_1] \rangle$ and so $\partial(2[u] + 2[\check{u}]) = 0$. Hence there exists $[v] \in H_2(\mathbb{CP}^{2n+1};\mathbb{Z})$ such that $2[u] + 2[\check{u}] = j_*([v])$ and applying $I_{\mu_{Z_X}}$ to this gives $4N_{Z_X} \in 4(n+1)\mathbb{Z}$. It follows from the bound (3.2) that $N_{Z_X} \in \{n+1, 2n+2\}$.

Suppose for contradiction that $N_{Z_X} = 2n+2$. Then by Theorem 3.1.1 we have $\pi_1(Z_X) \cong \mathbb{Z}/2$ and since $Y_X \to Z_X$ is a connected double cover, Y_X must be simply connected. In particular, the circle fibre of the bundle $S^1 \to Y_X \to X$ is contractible in Y_X. But this contradicts the fact that there is a fibre-preserving inclusion $Y_X = S(\mathcal{O}_X(2)) \hookrightarrow S(\mathcal{O}_{\mathbb{CP}^{2n+1}}(2)) \cong \mathbb{RP}^{4n+3}$ and the circle fibre defines a non-trivial class in the fundamental group of the latter space. This is the desired contradiction and so we must have $N_{Z_X} = n+1$.

Note also that since $\partial([u] + [\check{u}]) \in \langle \partial[u_1] \rangle \cong \mathbb{Z}/2$, there exists an integer $k \in \{0,1\}$ such that $\partial([u] + [\check{u}] - k[u_1]) = 0$ and hence there exists a homology class $[w] \in H_2(\mathbb{CP}^{2n+1};\mathbb{Z})$ such that

$[u]+[\check{u}]-k[u_1]=j_*([w])$. Applying $I_{\mu_{Z_X}}$ to both sides yields $2(n+1)(1-k)=2I_{c_1}([w])\in 4(n+1)\mathbb{Z}$ which implies that $k=1$ and $[w]=0$. Therefore $[u]-\mathcal{X}_*[u]=[u]+[\check{u}]=[u_1]$. □

Applying part 3) of Lemma 4.1.30 to the twistor Lagrangians corresponding to subadjoint varieties, we see that $Z_{(1,m)}$ is orientable precisely when m is even, Z_6 is non-orientable and Z_9, Z_{15} and Z_{27} are orientable.

Next we turn our attention to the mod 2 algebraic topology of Z_X. In particular, we are interested under what circumstances Z_X is relatively pin or satisfies Assumption (O). Regarding the first point, we have the following result:

Lemma 4.1.31. *For a Type* 2 *twistor Lagrangian Z_X, the following hold:*

1) The first and second Stiefel-Whitney classes of Z_X satisfy

$$\begin{aligned} w_1(TZ_X)^2 &= 0 \\ w_2(TZ_X) &= \Pi|_{Z_X}^*(w_2(T\bar{X})). \end{aligned}$$

2) The Lagrangian Z_X is relatively pin if and only if $w_2(TZ_X)=0$.

3) If $w_2(TZ_X)=0$, then $w_2(TX)=0$, i.e. the Legendrian variety X is spin. The converse is also true, whenever $H^1(X;\mathbb{F}_2)=0$.

Remark 4.1.32. Note that it is an open question whether every smooth Legendrian subvariety of $\mathbb{CP}^{2n+1}$ for $n\geq 2$ is simply connected. //

Proof. The first part of 1) holds for any Lagrangian L in $\mathbb{CP}^{2n+1}$ because the complex structure J_0 identifies the tangent and normal bundles to L and so $w_1(TL)^2=w_2\left(T\mathbb{CP}^{2n+1}\big|_L\right)=0$ since the mod 2 reduction of $c_1(T\mathbb{CP}^{2n+1})$ vanishes. To prove the second part, recall that the tangent bundle to Z_X splits as $TZ_X=(\mathcal{V}\cap TZ_X)\oplus(\mathcal{H}\cap TZ_X)$, because $\mathrm{pr}_{\mathcal{V}}(TZ_X)=\mathcal{V}\cap TZ_X$ (see for example the proof of Proposition 4.1.25). Moreover, there is a bundle isomorphism $\mathcal{H}\cap TZ_X\cong\Pi|_{Z_X}^*T\bar{X}$. Hence we have:

$$w_1(TZ_X) = w_1(\mathcal{V}\cap TZ_X) + \Pi|_{Z_X}^*w_1(T\bar{X}) \tag{4.27}$$

$$w_2(TZ_X) = w_1(\mathcal{V}\cap TZ_X)\smile\Pi|_{Z_X}^*w_1(T\bar{X}) + \Pi|_{Z_X}^*w_2(T\bar{X}). \tag{4.28}$$

Now note that the complex structure J_0 on $\mathbb{CP}^{2n+1}$ defines an isomorphism $\mathcal{V}\cap TZ_X\cong\mathcal{V}/(\mathcal{V}\cap TZ_X)$ and so $w_2(\mathcal{V}|_{Z_X})=w_1(\mathcal{V}\cap TZ_X)^2$. On the other hand, $w_2(\mathcal{V})$ is the mod 2 reduction of the first Chern class of $\mathcal{V}\cong\mathcal{O}_{\mathbb{CP}^{2n+1}}(2)$ and hence vanishes. Thus $w_1(\mathcal{V}\cap TZ_X)^2=0$. Now, if n is odd, then Z_X is orientable and so (4.27) tells us that $w_1(\mathcal{V}\cap TZ_X)=\Pi|_{Z_X}^*w_1(T\bar{X})$. Therefore we have $w_1(\mathcal{V}\cap TZ_X)\smile\Pi|_{Z_X}^*w_1(T\bar{X})=w_1(\mathcal{V}\cap TZ_X)^2=0$ and so $w_2(TZ_X)=\Pi|_{Z_X}^*w_2(T\bar{X})$ by (4.28). On the other hand, if n is even, then $\bar{X}$ is orientable and so (4.28) tells us that $w_2(TZ_X)=\Pi|_{Z_X}^*w_2(T\bar{X})$. This finishes the proof of 1).

To prove 2), we write $M_X = \Pi^{-1}(\bar{X})$ and let $i_{M_X} : M_X \to \mathbb{CP}^{2n+1}$ and $\mathrm{inc}_{Z_X} : Z_X \to M_X$ denote the respective inclusions. Suppose that Z_X is relatively pin. Since $w_1(TZ_X)^2 = 0$ and $w_2(TZ_X) = \Pi|_{Z_X}^* w_2(T\bar{X})$, this is equivalent to the existence of $k \in \{0,1\}$ such that

$$\Pi|_{Z_X}^* w_2(T\bar{X}) = \mathrm{inc}_{Z_X}^*(i_{M_X}^*(kH)), \tag{4.29}$$

where H denotes the (mod 2 reduction of) the hyperplane class. We want to show that $k = 0$.

Using $\Pi|_{Z_X} = \Pi|_{M_X} \circ \mathrm{inc}_{Z_X}$, we see that (4.29) is equivalent to

$$\mathrm{inc}_{Z_X}^*(\Pi|_{M_X}^* w_2(T\bar{X}) + k\, i_{M_X}^* H) = 0. \tag{4.30}$$

By the long exact sequence in cohomology for the pair (M_X, Z_X) we have the exact sequence

$$H^2(M_X, Z_X; \mathbb{F}_2) \xrightarrow{\alpha} H^2(M_X; \mathbb{F}_2) \xrightarrow{\mathrm{inc}_{Z_X}^*} H^2(Z_X; \mathbb{F}_2)$$

and so (4.30) is equivalent to

$$\Pi|_{M_X}^* w_2(T\bar{X}) + k i_{M_X}^* H \;\in\; \alpha(H^2(M_X, Z_X; \mathbb{F}_2)). \tag{4.31}$$

Observe now that the space M_X/Z_X is homeomorphic to the Thom space of the bundle $\mathcal{O}_X(2)$. Explicitly, if we model the Thom space as D_X/Y_X, where $D_X := \{(x,v) \in \mathcal{V}|_X \;:\; \|v\| \leq \pi/4\}$, then the exponential map provides the desired homeomorphism $D_X/Y_X \to M_X/Z_X$. It follows that $H^2(M_X, Z_X; \mathbb{F}_2)$ is 1-dimensional and generated by the Thom class $[\phi]$ which pairs to 1 with any hemisphere of a twistor line $\ell \subseteq M_X$. Since the whole twistor line is the sum of two hemispheres, we have that $\langle \alpha([\phi]), [\ell] \rangle_{M_X;\mathbb{F}_2} = 0$. It then follows from (4.31) that:

$$\begin{aligned}
\Big\langle \Pi|_{M_X}^* w_2(T\bar{X}) \,+\, k\, i_{M_X}^* H \,,\, [\ell] \Big\rangle_{M_X;\mathbb{F}_2} &= 0 \quad \Leftrightarrow \\
\big\langle w_2(T\bar{X}) \,,\, \Pi_*[\ell] \big\rangle_{\bar{X};\mathbb{F}_2} \quad + \quad k\,\langle H \,,\, [\ell] \rangle_{\mathbb{CP}^{2n+1};\mathbb{F}_2} &= 0 \quad \Leftrightarrow \\
k &= 0,
\end{aligned}$$

which is what we wanted to show.

To prove 3), let λ denote the rank 1 subbundle of $Q|_{\bar{X}}$, consisting of endomorphisms of $T\mathbb{HP}^n$ which preserve $T\bar{X}$ and let $\lambda^\perp$ denote its orthogonal complement in $Q|_{\bar{X}}$. Then X is naturally identified with the S^0-bundle $S(\lambda) \subseteq \lambda$, while Z_X is naturally identified with the S^1-bundle $S(\lambda^\perp) \subseteq \lambda^\perp$. Since $\lambda \oplus \lambda^\perp = Q|_{\bar{X}}$ and all Stiefel-Whitney classes of Q vanish (the corresponding cohomology groups of $\mathbb{HP}^n$ are zero), we get the identities $w_1(\lambda) = w_1(\lambda^\perp)$, $w_2(\lambda^\perp) = w_1(\lambda)^2$ and $w_1(\lambda)^3 = 0$. Substituting the second identity into the mod 2 Gysin sequence for the bundle $S^1 \to Z_X \to \bar{X}$, we obtain the exact sequence

$$H^0(\bar{X}; \mathbb{F}_2) \xrightarrow{\smile w_1(\lambda)^2} H^2(\bar{X}; \mathbb{F}_2) \xrightarrow{\Pi|_{Z_X}^*} H^2(Z_X; \mathbb{F}_2). \tag{4.32}$$

On the other hand, by the mod 2 Gysin sequence of the bundle $S^0 \to X \to \bar{X}$ (a.k.a. the transfer sequence of the double cover), we have the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H^0(\bar{X};\mathbb{F}_2) \arrow[r, "\smile w_1(\lambda)"] & H^1(\bar{X};\mathbb{F}_2) \arrow[r, "\Pi|_X^*"] & H^1(X;\mathbb{F}_2) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\ & H^1(\bar{X};\mathbb{F}_2) \arrow[r, "\smile w_1(\lambda)"] & H^2(\bar{X};\mathbb{F}_2) \arrow[r, "\Pi|_X^*"] & H^2(X;\mathbb{F}_2) \end{tikzcd} \tag{4.33}$$

Now suppose that $\Pi|_{Z_X}^* w_2(T\bar{X}) = 0$. Then (4.32) tells us that $w_2(T\bar{X}) \in \mathrm{Span}_{\mathbb{F}_2}(w_1(\lambda)^2)$ and so $\Pi|_X^* w_2(T\bar{X}) = 0$ by the second line of (4.33). But $\Pi|_X^* T\bar{X} = TX$ and so X is spin.

On the other hand, suppose $H^1(X;\mathbb{F}_2) = 0$. Then by the first line of (4.33) we get that $H^1(\bar{X};\mathbb{F}_2) = \mathrm{Span}_{\mathbb{F}_2}(w_1(\lambda))$ and so $\ker\big(\Pi|_X^* : H^2(\bar{X};\mathbb{F}_2) \to H^2(X;\mathbb{F}_2)\big) = \mathrm{Span}_{\mathbb{F}_2}(w_1(\lambda)^2)$ by the second line of (4.33). Now, if X is spin, then $\Pi|_X^* w_2(T\bar{X}) = w_2(TX) = 0$ and hence $w_2(T\bar{X}) \in \mathrm{Span}_{\mathbb{F}_2}(w_1(\lambda)^2)$. From (4.32), it follows that $\Pi|_{Z_X}^* w_2(T\bar{X}) = 0$. $\square$

Let X be one of the subadjoint varieties. Since all such varieties are simply connected, part 3) of Lemma 4.1.31 tells us that Z_X is (relatively) pin if and only if X is spin. The varieties X_6, X_9, X_{15} and X_{27} are known to be spin, see [AC16, Theorems 3.25 and 3.27]. Hence Z_6 is pin and Z_9, Z_{15}, Z_{27} are spin.

On the other hand, $X_{(1,m)}$ is spin if and only if $\mathbf{Q}_m$ is spin and it is well-known that this happens precisely when m is even or equal to 1 (this follows from the adjunction formula, see equation (4.35) below). We conclude that when m is odd and strictly bigger than 1, the Lagrangian $Z_{(1,m)}$ is neither orientable, nor (relatively) pin. In fact, in this case $Z_{(1,m)}$ does not satisfy Assumption (O). To prove this, we first make the following observation:

Lemma 4.1.33. *A Type 2 twistor Lagrangian Z_X satisfies Assumption (O) if and only if $w_2(TX)$ vanishes on the image of the map $\pi_*\colon \pi_2(Y_X) \to \pi_2(X)$.*

Proof. Note first that if L is a Lagrangian in $\mathbb{CP}^m$, then the boundary map $\partial\colon \pi_3(\mathbb{CP}^m, L) \to \pi_2(L)$ is an isomorphism. This can be seen easily from the long exact sequence in homotopy for the pair $(\mathbb{CP}^m, L)$, using the fact that the Lagrangian condition forces the map $\pi_2(L) \to \pi_2(\mathbb{CP}^m)$ to vanish. In particular, L satisfies Assumption (O) exactly when $w_2(TL)$ vanishes on $\pi_2(L)$.

Now consider the double cover $\widehat{\varphi}\colon Y_X \to Z_X$. It induces an isomorphism on π_2 and also $\widehat{\varphi}^* TZ_X = TY_X$, so we get

$$\langle w_2(TZ_X), \pi_2(Z_X)\rangle = \langle w_2(TY_X), \pi_2(Y_X)\rangle. \tag{4.34}$$

On the other hand, since the projection $\pi\colon Y_X \to X$ is a principal circle bundle, the vertical bundle to Y_X is trivial and so we have $TY_X \cong \mathbb{R} \oplus \pi^* TX$. Thus $w_2(TY_X) = \pi^* w_2(TX)$ which combined with (4.34) gives the desired result. $\square$

Lemma 4.1.34. *Let $k \geq 1$. Then $Z_{(1,2k+1)}$ does not satisfy Assumption (O).*

Proof. By Lemma 4.1.33, it suffices to show that $w_2(T(\mathbb{CP}^1 \times \mathbf{Q}_{2k+1}))$ does not vanish on the image of $\pi_2(Y_{\mathbb{CP}^1 \times \mathbf{Q}_{2k+1}})$ in $\pi_2(\mathbb{CP}^1 \times \mathbf{Q}_{2k+1})$. We now compute this image.

Note that for all $m \geq 2$, $\mathbf{Q}_m$ contains a line $\mathbb{CP}^1 \cong \ell \subseteq \mathbf{Q}_m$ (it is well known that $\mathbf{Q}_2 \cong \mathbb{CP}^1 \times \mathbb{CP}^1$ and any higher dimensional quadric contains $\mathbf{Q}_2$) and when $m \geq 3$, the class of this line generates $\pi_2(\mathbf{Q}_m)$ which is a copy of $\mathbb{Z}$ (for $m > 3$ this is immediate from the Lefschetz hyperplane theorem, while the case $m = 3$ can be seen for example from the long exact sequence in homotopy groups of the fibration $\mathrm{SO}(2) \times \mathrm{SO}(3) \to \mathrm{SO}(5) \to \mathbf{Q}_3$ coming from the well-known identification $\mathbf{Q}_m = \mathrm{SO}(m+2)/(\mathrm{SO}(2) \times \mathrm{SO}(m))$). Hence, using $[\mathbb{CP}^1 \times \mathrm{pt}]$ and $[\mathrm{pt} \times \ell]$ as basis, we have an isomorphism $\pi_2(\mathbb{CP}^1 \times \mathbf{Q}_m) \cong \mathbb{Z} \times \mathbb{Z}$ and the long exact sequence of homotopy groups for the fibration $S^1 \to Y_{\mathbb{CP}^1 \times \mathbf{Q}_m} \to \mathbb{CP}^1 \times \mathbf{Q}_m$ takes the form

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & \pi_2(Y_{\mathbb{CP}^1 \times \mathbf{Q}_m}) \arrow[r] \arrow[d, equal] & \pi_2(\mathbb{CP}^1 \times \mathbf{Q}_m) \arrow[r] \arrow[d, "\cong"] & \pi_1(S^1) \arrow[r] \arrow[d, "\cong"] & \pi_1(Y_{\mathbb{CP}^1 \times \mathbf{Q}_m}) \arrow[r] \arrow[d, equal] & 0 \arrow[d, equal] \\
0 \arrow[r] & \pi_2(S(\mathcal{O}_{\mathbb{CP}^1 \times \mathbf{Q}_m}(2,2))) \arrow[r, "\alpha"] & \mathbb{Z} \times \mathbb{Z} \arrow[r, "\beta"] & \mathbb{Z} \arrow[r] & \pi_1(S(\mathcal{O}_{\mathbb{CP}^1 \times \mathbf{Q}_m}(2,2))) \arrow[r] & 0
\end{tikzcd}$$

Our goal is to compute the image of α or, equivalently, the kernel of β. To that end, consider the inclusions $i_1 \colon \mathbb{CP}^1 \cong \mathbb{CP}^1 \times \mathrm{pt} \hookrightarrow \mathbb{CP}^1 \times \mathbf{Q}_m$ and $i_2 \colon \mathbb{CP}^1 \cong \mathrm{pt} \times \ell \hookrightarrow \mathbb{CP}^1 \times \mathbf{Q}_m$. In each case, the restriction of $Y_{\mathbb{CP}^1 \times \mathbf{Q}_m}$ is a copy of $\mathbb{RP}^3$:

$$i_j^* Y_{\mathbb{CP}^1 \times \mathbf{Q}_m} = i_j^* S(\mathcal{O}_{\mathbb{CP}^1 \times \mathbf{Q}_m}(2,2)) = S(\mathcal{O}_{\mathbb{CP}^1}(2)) = \mathbb{RP}^3 \quad \text{for} \quad j \in \{1,2\}.$$

So we obtain the following diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d, equal] & \mathbb{Z} \arrow[r, "2"] \arrow[d, "\cong"] & \mathbb{Z} \arrow[r] \arrow[d, "\cong"] & \mathbb{Z}/2 \arrow[r] \arrow[d, "\cong"] & 0 \arrow[d, equal] \\
0 \arrow[r] \arrow[d, equal] & \pi_2(\mathbb{RP}^3) \arrow[r] \arrow[d] & \pi_2(\mathbb{CP}^1) \arrow[r] \arrow[d, "(i_1)_*"] & \pi_1(S^1) \arrow[r] \arrow[d, "\cong"] & \pi_1(\mathbb{RP}^3) \arrow[r] \arrow[d] & 0 \arrow[d, equal] \\
0 \arrow[r] & \pi_2(S(\mathcal{O}_{\mathbb{CP}^1 \times \mathbf{Q}_m}(2,2))) \arrow[r, "\alpha"] & \mathbb{Z} \times \mathbb{Z} \arrow[r, "\beta"] & \mathbb{Z} \arrow[r] & \pi_1(S(\mathcal{O}_{\mathbb{CP}^1 \times \mathbf{Q}_m}(2,2))) \arrow[r] & 0 \\
0 \arrow[r] \arrow[u, equal] & \pi_2(\mathbb{RP}^3) \arrow[r] \arrow[u] & \pi_2(\mathbb{CP}^1) \arrow[r] \arrow[u, "(i_2)_*"] & \pi_1(S^1) \arrow[r] \arrow[u, "\cong"] & \pi_1(\mathbb{RP}^3) \arrow[r] \arrow[u] & 0 \arrow[u, equal] \\
0 \arrow[r] \arrow[u, equal] & 0 \arrow[r] \arrow[u, equal] & \mathbb{Z} \arrow[r, "2"] \arrow[u, "\cong"] & \mathbb{Z} \arrow[r] \arrow[u, "\cong"] & \mathbb{Z}/2 \arrow[r] \arrow[u, "\cong"] & 0 \arrow[u, equal]
\end{tikzcd}$$

From this we read off that $\pi_2(S(\mathcal{O}_{\mathbb{CP}^1 \times \mathbf{Q}_m}(2,2))) \cong \mathbb{Z}$, $\pi_1(S(\mathcal{O}_{\mathbb{CP}^1 \times \mathbf{Q}_m}(2,2))) \cong \mathbb{Z}/2$ and the central row of the diagram is nothing but the sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\begin{pmatrix} -1 \\ 1 \end{pmatrix}} \mathbb{Z} \times \mathbb{Z} \xrightarrow{\begin{pmatrix} 2 & 2 \end{pmatrix}} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

That is, the image of the map $\pi_2(Y_{\mathbb{CP}^1 \times \mathbf{Q}_m}) \to \pi_2(\mathbb{CP}^1 \times \mathbf{Q}_m)$ is a copy of $\mathbb{Z}$, generated by the "antidiagonal" class $\overline{\Delta} := -[\mathbb{CP}^1 \times \mathrm{pt}] + [\mathrm{pt} \times \ell]$.

Now put $m = 2k+1$ and note that

$$
\begin{aligned}
w_2(T(\mathbb{CP}^1 \times \mathbf{Q}_{2k+1})) &= \mathrm{pr}^*_{\mathbf{Q}_{2k+1}} w_2(T\mathbf{Q}_{2k+1}) \\
&= \left[\mathrm{pr}^*_{\mathbf{Q}_{2k+1}} c_1(T\mathbf{Q}_{2k+1})\right]_2 \\
&= \left[\mathrm{pr}^*_{\mathbf{Q}_{2k+1}} (2k+1)\, H|_{\mathbf{Q}_{2k+1}}\right]_2 \\
&= \left[\mathrm{pr}^*_{\mathbf{Q}_{2k+1}}\, H|_{\mathbf{Q}_{2k+1}}\right]_2, \qquad (4.35)
\end{aligned}
$$

where $[\quad]_2$ denotes reduction mod 2, H denotes the hyperplane class in $\mathbb{CP}^{2k+2}$ and the second to last line follows from the adjunction formula for the quadric $\mathbf{Q}_{2k+1} \subseteq \mathbb{CP}^{2k+2}$. Therefore

$$\left\langle w_2(TX_{(1,2k+1)})\,,\, \overline{\Delta}\right\rangle_{X_{(1,2k+1)};\mathbb{F}_2} = \langle H\,,\, [\ell]\rangle_{\mathbb{CP}^{2k+2};\mathbb{F}_2} = 1 \neq 0.$$

This finishes the proof. □

4.1.6 Legendrian curves in $\mathbb{CP}^3$ and the Chiang Lagrangian

Finally, we turn our attention to the case of smallest dimension. That is, we consider the original Penrose twistor fibration $\mathbb{CP}^1 \to \mathbb{CP}^3 \to \mathbb{HP}^1 \cong S^4(1/2)$ and the problem of finding embedded twistor Lagrangians in $\mathbb{CP}^3$. First, let us give some examples (we will see later that these are *the only* examples).

Of course, we have the standard $\mathbb{RP}^3$ which is the twistor Lagrangian associated to a horizontal Legendrian line $\mathbb{CP}^1 \subseteq \mathbb{CP}^3$. For more interesting examples, we consider a family of twisted cubics $\{X_\lambda \;:\; \lambda \in \mathbb{C}^\times\}$, where X_λ is parametrised by:

$$
\begin{aligned}
&\varphi_\lambda : \mathbb{C} \cup \{\infty\} \to \mathbb{CP}^3 \\
&\varphi_\lambda(t) = \left[t^3 : \lambda^2 : \sqrt{3}\lambda\, t : \sqrt{3}\lambda\, t^2\right]_{\mathbb{C}}. \qquad (4.36)
\end{aligned}
$$

It is immediate to check that $\varphi_\lambda^* \hat{\alpha} = 0$ and so this defines a family $\{X_\lambda\}_{\lambda \in \mathbb{C}^\times}$ of Legendrian rational curves of degree 3. The associated circle bundle $Y_{\varphi_\lambda} \to \mathbb{CP}^1$ is easily seen to be diffeomorphic to the lens space $L(6,1)$:

$$Y_{\varphi_\lambda} \cong \varphi_\lambda^* S(\mathcal{O}_{\mathbb{CP}^3}(2)) = S(\mathcal{O}_{\mathbb{CP}^1}(6)) \cong L(6,1).$$

An easy calculation shows that

$$\mathcal{X}(\varphi_\lambda(t)) = \varphi_{1/\bar{\lambda}}(-1/\bar{t})$$

and so X_λ is $\mathcal{X}$-invariant whenever $|\lambda| = 1$. It is not hard to check that in this case, X_λ is a Legendrian of Type 2. For the rest of this thesis, X_1 will denote the Type 2 Legendrian twisted cubic

$$X_1 = \varphi_1(\mathbb{C} \cup \{\infty\}) = \left\{\left[t^3 : 1 : \sqrt{3}t : \sqrt{3}t^2\right]_{\mathbb{C}} \in \mathbb{CP}^3 \;:\; t \in \mathbb{C} \cup \{\infty\}\right\}$$

and Z_1 will denote its corresponding twistor Lagrangian. The corresponding embedded superminimal surface $\overline{X}_1 \cong \mathbb{RP}^2$ in $\mathbb{HP}^1 \cong S^4(1/2)$ is known as the *Veronese surface*. As we will explicitly verify in section 5.1.1, the Lagrangian Z_1 is precisely the Chiang Lagrangian.

The homogeneous structure of the Chiang Lagrangian has proven very valuable in the study of its Floer theory by Evans and Lekili in [EL15] and Smith in [Smi15] and is also the perspective we adopt in chapter 5. Our current perspective however is useful because it exhibits Z_1 and $\mathbb{RP}^3$ as members of the same family – they are both twistor Lagrangians (as we shall see, they are in a sense the only embedded twistor Lagrangians in $\mathbb{CP}^3$).

From this point of view, it is also easy to manipulate the Legendrian curves and exhibit an interesting transformation between these Lagrangians. To that end, we investigate the behaviour of the family $\{X_\lambda\}$ as λ tends to zero and to infinity. Let us define the following Legendrian lines:

$$\ell_1 := \{[0 : z_1 : z_2 : 0]_{\mathbb{C}} :\ (z_1,z_2) \in \mathbb{C}^2 \setminus (0,0)\}, \quad \ell_2 := \{[z_0 : 0 : z_2 : 0]_{\mathbb{C}} :\ (z_0,z_2) \in \mathbb{C}^2 \setminus (0,0)\}$$
$$\ell_3 := \{[z_0 : 0 : 0 : z_3]_{\mathbb{C}} :\ (z_0,z_3) \in \mathbb{C}^2 \setminus (0,0)\}, \quad \ell_4 := \{[0 : z_1 : 0 : z_3]_{\mathbb{C}} :\ (z_1,z_3) \in \mathbb{C}^2 \setminus (0,0)\}.$$

Note that $\ell_3 = \mathcal{X}(\ell_1)$ and $\ell_4 = \mathcal{X}(\ell_2)$. We have the following lemma.

Lemma 4.1.35. *Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{C}^\times$ with $\lim_{n\to\infty} \lambda_n = 0$. The Gromov limit of the curves X_{λ_n} consists of ℓ_1 and a double cover of ℓ_2. Similarly, if $\lim_{n\to\infty} \lambda_n = \infty$, then the Gromov limit consists of ℓ_3 and a double cover of ℓ_4.*

Proof. Consider the case $\lim_{n\to\infty} \lambda_n = 0$. The parametrisation (4.36) is not good for computing the Gromov limit since it converges to the constant map at $[1 : 0 : 0 : 0]_{\mathbb{C}}$ which is not even the nodal point $\ell_1 \cap \ell_2 = [0 : 0 : 1 : 0]_{\mathbb{C}}$ of the limit. So consider the parametrisations

$$h_\lambda(t) := \varphi_\lambda(\lambda t) = \left[\lambda t^3 : 1 : \sqrt{3}\, t : \sqrt{3}\lambda\, t^2\right]_{\mathbb{C}}.$$

Letting $\lambda_n \to 0$, the sequence h_{λ_n} converges to the map $h_0 \colon \mathbb{CP}^1 \to \mathbb{CP}^3$, $h_0(t) = \left[0 : 1 : \sqrt{3}\, t : 0\right]_{\mathbb{C}}$ which is a parametrisation of ℓ_1. To see ℓ_2 in the limit, choose a branch for the holomorphic square root which is defined on $\{\lambda_n\}_{n\in\mathbb{N}}$ and consider the parametrisations

$$k_{\lambda_n}(t) := \varphi_{\lambda_n}\left(\sqrt{\lambda_n}\, t\right) = \left[t^3 : \sqrt{\lambda_n} : \sqrt{3}\, t : \sqrt{3}\sqrt{\lambda_n}\, t^2\right]_{\mathbb{C}}.$$

Letting $\lambda_n \to 0$, the sequence k_{λ_n} converges to the map $k_0 \colon \mathbb{CP}^1 \to \mathbb{CP}^3$, $k_0(t) = \left[t^2 : 0 : \sqrt{3} : 0\right]_{\mathbb{C}}$ which is a double cover of ℓ_2. Finally, we know that this is the entire Gromov limit, because it accounts for all the energy of a twisted cubic: $E(h_0) + E(k_0) = \int_{\ell_1} \omega_{\mathrm{FS}} + 2\int_{\ell_2} \omega_{\mathrm{FS}} = 3\pi = E(\varphi_{\lambda_n})$.

Similarly, letting $\lambda_n \to \infty$ in the parametrisations h_{λ_n} and k_{λ_n}, one obtains that in this case the Gromov limit of X_{λ_n} consists of ℓ_3 and a double cover of ℓ_4. □

Now consider what happens at the level of Lagrangians as λ varies along the real axis from 0 to 1. At $\lambda = 0$, the two Legendrian lines ℓ_1 and ℓ_2 meet at the point $[0 : 0 : 1 : 0]_{\mathbb{C}}$ and intersect the twistor line $\ell := \{z_2 = z_3 = 0\}$ in the antipodal points $p_0 := [0 : 1 : 0 : 0]_{\mathbb{C}}$ and $p_\infty := [1 : 0 : 0 : 0]_{\mathbb{C}}$. Hence, the associated twistor Lagrangians are two copies of $\mathbb{RP}^3$ which intersect cleanly along two circles. Note that $p_0 = \varphi_\lambda(0)$ and $p_\infty = \varphi_\lambda(\infty)$ for all $\lambda \in \mathbb{C}^\times$, so in fact the points p_0 and p_∞ are common for all twisted cubics in the family. Thus, for $\lambda \in (0,1)$, the corresponding Lagrangian is an immersed $L(6,1)$, intersecting itself in a circle which is the equator of ℓ opposite p_0 and p_∞. Finally, when $\lambda = 1$, the immersed $L(6,1)$ collapses on itself two-to-one to give the embedded Z_1.

Let us now prove the promised uniqueness result.

Theorem 4.1.36. *Let $X \subseteq \mathbb{CP}^3$ be a Legendrian curve.*

a) If X is of Type 1, then X is a line.

b) If X is of Type 2, then X is a twisted cubic and there exists $F \in \mathrm{Sp}(2)$ such that $F(X_1) = X$.

Proof. Part a) is, of course, just a special case of Theorem 4.1.29. However, appealing to Theorem 4.1.29 in this dimension really is an overkill and in fact the result follows directly from Friedrich's formula for the Euler class of the normal bundle to immersions in S^4 with holomorphic twistor lifts ([Fri84]). Let us now give the proof, since it is short and we will need the Euler number calculation for the proof of part b) anyway.

Let $f\colon \Sigma_k \to \mathbb{CP}^3$ be a Legendrian embedding of degree d of an oriented genus k surface and let $\bar{f} := \Pi \circ f$ be the corresponding superminimal immersion in $\mathbb{HP}^1$. Let $\nu(\bar{f}) := \bar{f}^* T\mathbb{HP}^1 / T\Sigma_k$ denote the normal bundle of this immersion. Then its Euler class satisfies

$$\left\langle e(\nu(\bar{f})), [\Sigma_k] \right\rangle = 2(d+k-1). \tag{4.37}$$

To see this, note that if $\nu_{\mathcal{H}}(f) := f^*\mathcal{H}/T\Sigma_k$ denotes the horizontal normal bundle to the Legendrian curve, we have $\nu(\bar{f}) \cong \nu_{\mathcal{H}}(f)$ as oriented bundles over Σ_k. On the other hand, we have a decomposition $f^*T\mathbb{CP}^3 = f^*\mathcal{V} \oplus T\Sigma_k \oplus \nu_{\mathcal{H}}(f)$ of complex vector bundles on Σ_k. Taking the first Chern class on both sides and using the isomorphism $\mathcal{V} \cong \mathcal{O}_{\mathbb{CP}^3}(2)$ yields the equality $f^*(4H) = f^*(2H) + e(T\Sigma_k) + e(\nu_{\mathcal{H}}(f))$ in $H^2(\Sigma_k;\mathbb{Z})$. Using that f has degree d and the isomorphism $\nu(\bar{f}) \cong \nu_{\mathcal{H}}(f)$, we obtain (4.37).

Now, if f is of Type 1, then $\bar{f}$ is an embedding of an oriented surface in S^4 (equivalently, $\mathbb{R}^4$) and so $\nu(\bar{f})$ is trivial. By (4.37) this immediately gives $d = 1$, $k = 0$, which proves part a).

Suppose now that f is of Type 2 and let $X = f(\Sigma_k)$. Then $\bar{X} = \Pi(X) \subseteq S^4$ is an embedded non-orientable surface of Euler characteristic $\chi = 1-k$ and $\bar{f}\colon \Sigma_k \to \bar{X}$ is the oriented double cover. The surface $\bar{X}$ has a fundamental class $\left[\bar{X}\right] \in H_2(\bar{X};\mathcal{E}_{\mathrm{or}})$ and its normal bundle $\nu\left(\bar{X}\right)$ has an Euler class $e(\nu(\bar{X})) \in H^2(\bar{X};\mathcal{E}_{\mathrm{or}})$, where $\mathcal{E}_{\mathrm{or}} \to \bar{X}$ denotes the local system, whose fibre at each point $x \in \bar{X}$ is the free $\mathbb{Z}$-module of rank 1, generated by the two orientations of $T_x\bar{X}$, modulo the relation that they sum to zero [4]. Since $\mathcal{E}_{\mathrm{or}} \otimes \mathcal{E}_{\mathrm{or}} \cong \underline{\mathbb{Z}}$ is the trivial $\mathbb{Z}$-local system of rank 1, there is a well-defined pairing $H^2(\bar{X};\mathcal{E}_{\mathrm{or}}) \otimes H_2(\bar{X};\mathcal{E}_{\mathrm{or}}) \to \mathbb{Z}$ and so one can associate a number $\left\langle e(\nu(\bar{X})), \left[\bar{X}\right] \right\rangle \in \mathbb{Z}$ to the embedded surface $\bar{X} \subseteq S^4$. In [Mas69], Massey proved a conjecture of Whitney, which states that for any non-orientable surface of Euler characteristic χ, embedded in S^4, this number must lie in

[4] We are being a little sloppy here about the choices one needs to make for this statement to hold as given. See [Mas69, Appendix 1].

the range $\{2\chi-4, 2\chi, \ldots, 4-2\chi\}$. In our case, we have $\chi = 1-k$, so the allowed values for this number are $\{-2-2k, 2-2k, \ldots, 2+2k\}$.

Now note that since $\bar{f}\colon \Sigma_k \to \bar{X}$ is the oriented double cover and $\nu(\bar{f}) = \bar{f}^* \nu(\bar{X})$, we have the relation

$$\left\langle e(\nu(\bar{f})), [\Sigma_k] \right\rangle = 2 \left\langle e(\nu(\bar{X})), \left[\bar{X}\right] \right\rangle.$$

Using (4.37) and Massey's bounds, we see that the degree of a Type 2 Legendrian curve must satisfy

$$d \in \{-1-3k, 3-3k, 7-3k, \ldots, -1+k, 3+k\} \cap \mathbb{N}_{>0}. \tag{4.38}$$

Hence, if $k = 0$, the only possible degree is $d = 3$, that is, X is a twisted cubic. The proof that X must then be $\mathrm{Sp}(2)$-equivalent to X_1 is given in Lemma 4.1.37 below.

It remains to be shown that we can't have $k \geq 1$. Note that the bound $d \leq k+3$ is *really low* for the degree of a Legendrian curve of genus $k \geq 1$, as can be inferred for example from the results of [CM96]. In fact, [CM96, Theorem 1 and Proposition 7] imply that there are no embedded (let alone Type 2) Legendrian curves of genus 1, 2 or 3 and degree less than 7. It may well be the case that there is no embedded Legendrian curve of genus k and degree at most $k+3$ for any $k \geq 1$ but the author has not been able to find such a proof. Therefore, we focus only on Type 2 Legendrians and we finish the proof of case b) using our results on monotone Lagrangians in $\mathbb{CP}^3$.

Suppose for contradiction that $X \subseteq \mathbb{CP}^3$ is a Legendrian curve of Type 2 and genus $k \geq 1$. Then Z_X is an embedded, monotone Lagrangian. By Lemma 4.1.30, Z_X is orientable and has minimal Maslov number 2. Since $k \geq 1$, we have that $\pi_1(Z_X)$ is infinite and non-cyclic. Moreover, Z_X is not a Euclidean manifold, because its double cover $Y_X = S(\mathcal{O}_X(2))$ is a non-trivial principal circle bundle over an orientable surface and such 3-manifolds do not admit a Euclidean geometry. In particular, Z_X admits a unique up to isomorphism Seifert fibration and in fact we know exactly what it is: by construction Z_X is a non-orientable circle bundle over the non-orientable surface $\bar{X}$. Now Theorem 3.2.11 tells us that such a monotone Lagrangian in $\mathbb{CP}^3$ cannot exist and this is the contradiction we were after. □

Lemma 4.1.37. *Let $X \subseteq \mathbb{CP}^3$ be a Type 2 Legendrian twisted cubic. Then there exists a linear transformation $A \in \mathrm{Sp}(2) \subseteq \mathrm{GL}(4, \mathbb{C})$ whose associated projective transformation $F_A\colon \mathbb{CP}^3 \to \mathbb{CP}^3$ satisfies $F_A(X_1) = X$.*

Proof. Recall that $\mathrm{PGL}(4, \mathbb{C})$ acts transitively on the set of twisted cubics in $\mathbb{CP}^3$. Moreover, the stabiliser of X_1 under this action is a copy of $\mathrm{PGL}(2,\mathbb{C})$, embedded via the homomorphism[5]

$$\begin{array}{ccc} \mathrm{GL}(2,\mathbb{C}) & \longrightarrow & \mathrm{GL}(4,\mathbb{C}) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \longmapsto & \dfrac{1}{(ad-bc)^3}\begin{pmatrix} d^3 & -c^3 & \sqrt{3}c^2d & -\sqrt{3}cd^2 \\ -b^3 & a^3 & -\sqrt{3}a^2b & \sqrt{3}ab^2 \\ \sqrt{3}b^2d & -\sqrt{3}a^2c & a^2d+2abc & -b^2c-2abd \\ -\sqrt{3}bd^2 & \sqrt{3}ac^2 & -bc^2-2acd & ad^2+2bcd \end{pmatrix}. \end{array} \tag{4.39}$$

This stabilising $\mathrm{PGL}(2,\mathbb{C})$ acts triply-transitively on X_1.

Now consider again the pair of antipodal points $p_0 = [0:1:0:0]_{\mathbb{C}}$ and $p_\infty = [1:0:0:0]_{\mathbb{C}}$ on X_1. Let $\hat{p}_0$ and $\hat{p}_\infty$ be two points on the Type 2 cubic X, which satisfy $\mathcal{X}(\hat{p}_0) = \hat{p}_\infty$. From our discussion above, we know that there exists a linear map $A' \in \mathrm{GL}(4,\mathbb{C})$ whose associated projective transformation $F_{A'}$ satisfies $F_{A'}(X_1) = X$, $F_{A'}(p_0) = \hat{p}_0$ and $F_{A'}(p_\infty) = \hat{p}_\infty$. Making these choices leaves us with one degree of freedom for $F_{A'}$ and therefore two degrees of freedom for A', which we shall now fix.

Let us denote the standard complex basis of $\mathbb{C}^4$ by

$$e_0 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix},\; e_1 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix},\; e_2 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix},\; e_3 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

Under the identifications (4.2), we have $e_1 = e_0\mathbf{j}$ and $e_3 = e_2\mathbf{j}$ and so $\{e_0, e_2\}$ form a basis of $\mathbb{H}^2$ as a right $\mathbb{H}$-module. For consistency, we will express elements of $\mathbb{C}^4$ as *right* $\mathbb{C}$-linear combinations of $\{e_0, e_1, e_2, e_3\}$.

Now let us write $\hat{e}_0 := A'(e_0)$ and note that by rescaling A' if necessary, we may assume that $\|\hat{e}_0\| = 1$. By our choice of A', there exists a constant $\lambda \in \mathbb{C}^\times$ such that $A'(e_0\mathbf{j}) = \hat{e}_0\mathbf{j}\lambda$. Now observe that, if we put $b = c = 0$, $a = 1$ in (4.39) and we set d to be a third root of λ, we obtain an element $B \in \mathrm{GL}(4,\mathbb{C})$ such that $F_B(X_1) = X_1$ and $B(e_0) = e_0$, $B(e_0\mathbf{j}) = e_0\mathbf{j}\frac{1}{\lambda}$. We set $A := A' \circ B$, so that $A(e_0\mathbf{j}) = \hat{e}_0\mathbf{j}$. We aim to show that A is an element of

$$\mathrm{Sp}(2) = \mathrm{Sp}(4,\mathbb{C}) \cap \mathrm{GL}(2,\mathbb{H}) \subseteq \mathrm{GL}(4,\mathbb{C}).$$

First we show that A lies in $\mathrm{Sp}(4,\mathbb{C})$, i.e. A preserves the complex symplectic form $\omega_{\mathbb{C}} = \mathbb{C}\mathrm{o}\langle \cdot\mathbf{j}, \cdot\rangle$. To do this we use the fact that $X = F_A(X_1)$ is a Legendrian curve.

Given a curve $X_\circ \in \{X_1, X\}$ and a point $x \in X_\circ$ we denote the affine tangent space to $X_\circ$ at x by $\hat{T}_x X_\circ$. Since $X_\circ$ is a Legendrian curve in $\mathbb{CP}^3$, the affine tangent space $\hat{T}_x X_\circ$ is an $\omega_{\mathbb{C}}$-Lagrangian subspace of $\mathbb{C}^4$ for every $x \in X_\circ$. Using the parametrisations $\varphi_1(t) = [t^3 : 1 : \sqrt{3}t : \sqrt{3}t^2]_{\mathbb{C}}$ and $\check{\varphi}(t) = \varphi(1/t) = [1 : t^3 : \sqrt{3}t^2 : \sqrt{3}t]$ of the cubic X_1, we find that

$$\hat{T}_{p_0} X_1 = \mathrm{Span}_{\mathbb{C}}\{e_0\mathbf{j}, e_2\}, \qquad \hat{T}_{p_\infty} X_1 = \mathrm{Span}_{\mathbb{C}}\{e_0, e_2\mathbf{j}\},$$

[5] This comes from viewing $\mathbb{C}^4$ as the third symmetric power of the dual to the standard vector representation of $\mathrm{GL}(2,\mathbb{C})$ on $\mathbb{C}^2$. See equations (5.2) and (5.3) in chapter 5.

$$\hat{T}_{\varphi(t)}X_1 = \mathrm{Span}_{\mathbb{C}}\left\{e_0t^3 + e_0\mathbf{j} + e_2\sqrt{3}t + e_2\mathbf{j}\sqrt{3}t^2,\ e_03t^2 + e_2\sqrt{3} + e_2\mathbf{j}\,2\sqrt{3}\,t\right\}$$

and therefore

$$\hat{T}_{\hat{p}_0}X = \mathrm{Span}_{\mathbb{C}}\left\{\hat{e}_0\mathbf{j}, A(e_2)\right\}, \qquad \hat{T}_{\hat{p}_\infty}X = \mathrm{Span}_{\mathbb{C}}\left\{\hat{e}_0, A(e_2\mathbf{j})\right\},$$

$$\hat{T}_{F_A(\varphi(t))}X = \mathrm{Span}_{\mathbb{C}}\left\{\hat{e}_0t^3 + \hat{e}_0\mathbf{j} + A(e_2)\sqrt{3}t + A(e_2\mathbf{j})\sqrt{3}t^2,\ \hat{e}_03t^2 + A(e_2)\sqrt{3} + A(e_2\mathbf{j})\,2\sqrt{3}\,t\right\},$$

where we have used the fact that $A(e_0\mathbf{j}) = A(e_0)\mathbf{j} = \hat{e}_0\mathbf{j}$. Since X is Legendrian, these spaces must be $\omega_{\mathbb{C}}$-Lagrangian, which gives us the equations

$$\mathbb{C}\mathrm{o}\langle\hat{e}_0, A(e_2)\rangle = 0, \qquad \mathbb{C}\mathrm{o}\langle\hat{e}_0\mathbf{j}, A(e_2\mathbf{j})\rangle = 0 \tag{4.40}$$

$$\mathbb{C}\mathrm{o}\left\langle\left(\hat{e}_0t^3 + \hat{e}_0\mathbf{j} + A(e_2)\sqrt{3}t + A(e_2\mathbf{j})\sqrt{3}t^2\right)\mathbf{j},\ \hat{e}_03t^2 + A(e_2)\sqrt{3} + A(e_2\mathbf{j})\,2\sqrt{3}\,t\right\rangle = 0 \quad \forall t\in\mathbb{C}. \tag{4.41}$$

From (4.40) we find that there exist $a, b \in \mathbb{C}$ and $\hat{e}_2, \hat{e}_3 \in (\hat{e}_0\mathbb{H})^\perp$ such that

$$A(e_2) = \hat{e}_0\mathbf{j}a + \hat{e}_2 \quad \text{and} \quad A(e_2\mathbf{j}) = \hat{e}_0b + \hat{e}_3.$$

Substituting this in (4.41), we obtain that

$$-2a\sqrt{3}t^3 + 3(2\mathbb{C}\mathrm{o}\langle\hat{e}_2\mathbf{j}, \hat{e}_3\rangle + \mathbb{C}\mathrm{o}\langle\hat{e}_3\mathbf{j}, \hat{e}_2\rangle - ab - 1)t^2 - 2b\sqrt{3}t = 0 \quad \forall t\in\mathbb{C}$$

and so $a = b = 0$ and $\mathbb{C}\mathrm{o}\langle\hat{e}_2\mathbf{j}, \hat{e}_3\rangle = 1$.

Summarising, we now have $A(e_0) = \hat{e}_0$, $A(e_0\mathbf{j}) = \hat{e}_0\mathbf{j}$, $A(e_2) = \hat{e}_2$, $A(e_2\mathbf{j}) = \hat{e}_3$ and these satisfy $\|\hat{e}_0\| = 1$, $\{\hat{e}_2, \hat{e}_3\} \subseteq (\hat{e}_0\mathbb{H})^\perp$, $\mathbb{C}\mathrm{o}\langle\hat{e}_2\mathbf{j}, \hat{e}_3\rangle = 1$. We conclude that $A \in \mathrm{Sp}(4,\mathbb{C})$.

It remains to be shown that A lies in $\mathrm{GL}(2,\mathbb{H})$, i.e. that A commutes with right multiplication by $\mathbf{j}$. Since we already have $A(e_0\mathbf{j}) = A(e_0)\mathbf{j}$, all we need to check is that $A(e_2\mathbf{j}) = A(e_2)\mathbf{j}$, i.e. that $\hat{e}_3 = \hat{e}_2\mathbf{j}$. This will follow from the fact that X is a $\mathcal{X}$-invariant curve. Indeed, since $\mathcal{X}(X) = X$ and $F_A(X_1) = X$ we see that for each $t \in \mathbb{C}\cup\{\infty\}$, there must exist $T = T(t) \in \mathbb{C}\cup\{\infty\}$ such that $\mathcal{X}(F_A(\varphi_1(t))) = F_A(\varphi_1(T))$. In particular, for each $t \in \mathbb{C}^\times$, there exist $T, \lambda \in \mathbb{C}^\times$ such that

$$\begin{aligned} A\left(e_0t^3 + e_0\mathbf{j} + e_2\sqrt{3}t + e_2\mathbf{j}\sqrt{3}t^2\right)\mathbf{j} &= A\left(e_0T^3 + e_0\mathbf{j} + e_2\sqrt{3}T + e_2\mathbf{j}\sqrt{3}T^2\right)\lambda \\ \Leftrightarrow \qquad \hat{e}_0\mathbf{j}\bar{t}^3 - \hat{e}_0 + \hat{e}_2\mathbf{j}\sqrt{3}\bar{t} + \hat{e}_3\mathbf{j}\sqrt{3}\bar{t}^2 &= \hat{e}_0T^3\lambda + \hat{e}_0\mathbf{j}\lambda + \hat{e}_2\sqrt{3}T\lambda + \hat{e}_3\sqrt{3}T^2\lambda. \end{aligned}$$

From here we find that $\lambda = \bar{t}^3$, $T = -\mu/\bar{t}$ for some $\mu \in \left\{1, e^{2\pi\mathbf{i}/3}, e^{4\pi\mathbf{i}/3}\right\}$ and

$$\hat{e}_2\mathbf{j} + \hat{e}_3\mathbf{j}\bar{t} = \hat{e}_3\mu^2 - \hat{e}_2\mu\bar{t} \quad \forall t\in\mathbb{C}^\times.$$

Hence, we get that $\hat{e}_2\mathbf{j} = \hat{e}_3\mu^2$ and $\hat{e}_3\mathbf{j} = -\hat{e}_2\mu$ which suffices to conclude that $\mu = 1$ and thus $\hat{e}_2\mathbf{j} = \hat{e}_3$ which is what we wanted. $\square$

4.2 The Lagrangian equation for $\mathbb{CP}^3$ from a twistor perspective

In section 4.1 we studied Lagrangian immersions in $\mathbb{CP}^{2n+1}$ which were compatible with the twistor fibration. We now consider the more generic situation in which $\phi\colon L \to \mathbb{CP}^{2n+1}$ is a Lagrangian immmersion, such that $\Pi \circ \phi\colon L \to \mathbb{HP}^n$ is also an immersion. In fact, we change our perspective: we start from an immersion $\iota\colon M^{2n+1} \to \mathbb{HP}^n$ of some $(2n+1)$-dimensional manifold and we look for a Lagrangian lift $\tilde{\iota}\colon M \to \mathbb{CP}^{2n+1}$. We focus specifically on the case $n = 1$, because then M is a hypersurface in $\mathbb{HP}^1 = S^4(1/2)$ and the Lagrangian lift $\tilde{\iota}$ can be identified with a unit vector field on M, satisfying a certain linear PDE involving the second fundamental form of the immersion ι. In this way we essentially split the equation for an immersion $\tilde{\iota}$ to be Lagrangian into a coupled system of equations for an immersion ι into $\mathbb{HP}^1$ and a vector field on M. This is not surprising, given the splitting (4.6) but we still find it interesting to see exactly what equations we get.

4.2.1 The general equation

For the better part of this section, we work in a rather general setting. Namely, let (N^4, g) be an oriented Riemannian 4-manifold and let $\iota : M \to N$ be an immersion of an oriented 3-manifold. For each point $p \in M$ we have an oriented 3-dimensional subspace $\iota_* T_pM \subseteq T_{\iota(p)}N$. This determines a unit vector field $v_4 \in C^\infty(M, \iota^*TN)$ by the requirement that if $\{v_1, v_2, v_3\}$ is a positive orthonormal frame for M, then $\{v_1, v_2, v_3, v_4\}$ is a positive orthonormal frame for N along M. Before we move on, let us make a small detour through the linear algebra which underpins our subsequent discussion.

4.2.1.1 Some linear algebra

Let V be a 4-dimensional oriented vector space with fixed inner product g. Let $\{v_1, v_2, v_3, v_4\}$ be a positive orthonormal basis with dual basis $\{\phi^1, \phi^2, \phi^3, \phi^4\}$. We have the usual musical isomorphisms

$$\begin{array}{ccccc} V \otimes V & \underset{\#}{\overset{\flat}{\rightleftarrows}} & V^* \otimes V & \underset{\#}{\overset{\flat}{\rightleftarrows}} & V^* \otimes V^* \\ v_i \otimes v_j & \leftrightarrows & \phi^i \otimes v_j & \leftrightarrows & \phi^i \otimes \phi^j \end{array} \tag{4.42}$$

and the inclusions

$$\begin{array}{ccc} V \wedge V & \longrightarrow & V \otimes V \\ v_i \wedge v_j & \longmapsto & v_i \otimes v_j - v_j \otimes v_i \end{array} \qquad \begin{array}{ccc} V^* \wedge V^* & \longrightarrow & V^* \otimes V^* \\ \phi^i \wedge \phi^j & \longmapsto & \phi^i \otimes \phi^j - \phi^j \otimes \phi^i. \end{array} \tag{4.43}$$

Note that with respect to the inner products induced by g on tensor and exterior powers of V the maps (4.42) are isometries while the maps (4.43) are conformal and stretch lengths by a factor of $\sqrt{2}$. As is standard, whenever we use $\flat$ (resp. #) on 2-vectors (resp. 2-forms) we are always implicitly precomposing by the inclusions (4.43). Finally, the orientation on V determines a Hodge star operator $*$ which on 2-vectors and 2-forms squares to the identity. We denote its respective $+1$-eigenspaces by $\Lambda^2_+ V$ and $\Lambda^2_+ V^*$.

Now, any unit vector $v \in V$ determines an injective map

$$\begin{aligned} \psi_v \colon v^\perp &\to \Lambda^2_+ V^* \\ u &\mapsto u^\flat \wedge v^\flat + *(u^\flat \wedge v^\flat), \end{aligned} \tag{4.44}$$

which in this dimension is an isomorphism. Moreover:

Lemma 4.2.1. *The inverse $\psi_v^{-1} \colon \Lambda^2_+ V^* \to v^\perp$ is given by*

$$\psi_v^{-1}(\alpha) = -\alpha^\#(v). \tag{4.45}$$

Proof. Let $u \in v^\perp$. We want to show that $\psi_v(u)^\#(v) = -u$. If $u = 0$, there is nothing to prove, so we assume $u \neq 0$ and without loss of generality $v_1 = \frac{u}{\|u\|}$ and $v_4 = v$. Then

$$\begin{aligned} \psi_{v_4}(u)^\#(v) = \|u\|\psi_{v_4}(v_1)^\#(v_4) = \|u\|(\phi^1 \wedge \phi^4 + \phi^2 \wedge \phi^3)^\#(v_4) \\ = \|u\|(\phi^1 \otimes v_4 - \phi^4 \otimes v_1)(v_4) = -\|u\|v_1 = -u. \end{aligned}$$
□

Now note that the map ψ_v is conformal and multiplies lengths by $\sqrt{2}$. Therefore, any ordered pair (v,w) of unit vectors determines an orthogonal transformation

$$\begin{aligned} S_{v,w} \colon V &\to V \\ v &\mapsto w \\ u &\mapsto \psi_w^{-1}(\psi_v(u)) \quad \forall\, u \in v^\perp. \end{aligned} \tag{4.46}$$

We have the following lemma:

Lemma 4.2.2. *The map $S_{v,w}$ is given by the formula*

$$S_{v,w} = g(v,w)\mathrm{Id}_V \,+\, (v \wedge w - *(v \wedge w))^\flat. \tag{4.47}$$

Proof. Without loss of generality, we assume that $v = v_4$. Then, we have the identities $S_{v_4,w}(v_4) = w$ and, by Lemma 4.2.1, $S_{v_4,w}(u) = -\psi_{v_4}(u)^\#(w)$ for $u \in \mathrm{Span}\{v_1, v_2, v_3\}$. From these one computes that, if $w = \sum_{i=1}^4 \mu^i v_i$, then the matrix for $S_{v_4,w}$ in the frame $\{v_1, v_2, v_3, v_4\}$ is

$$\begin{pmatrix} \mu^4 & -\mu^3 & \mu^2 & \mu^1 \\ \mu^3 & \mu^4 & -\mu^1 & \mu^2 \\ -\mu^2 & \mu^1 & \mu^4 & \mu^3 \\ -\mu^1 & -\mu^2 & -\mu^3 & \mu^4 \end{pmatrix}.$$

It is easy to check that this is precisely the matrix for the operator on the right-hand side of (4.47). □

4.2.1.2 Some covariant differentiation

Now let us go back to the geometric setting. Recall that we have an immersion $\iota : M \to N$ of an oriented 3-manifold into an oriented Riemannian 4-manifold (N,g) and this determines a unit normal field $v_4 \in C^\infty(M, \iota^*TN)$ along M. Then, if $\Lambda^2_+ N := \Lambda^2_+(T^*N)$ denotes the bundle of self-dual 2-forms on N, by the previous section we get a bundle isomorphism

$$\begin{aligned} \Psi\colon TM &\to \iota^*\Lambda^2_+ N \\ v &\mapsto \psi_{v_4}(v) = v^\flat \wedge v_4^\flat + *(v^\flat \wedge v_4^\flat). \end{aligned} \tag{4.48}$$

Now consider the twistor space of N:

$$\mathcal{Z}^\flat_+(N,g) = \left\{(x,p) \in \Lambda^2_+ N : \ \|p\| = \sqrt{2}\right\},$$

that is, the $\sqrt{2}-$sphere bundle inside the bundle of self-dual 2-forms on N (in this section it is more convenient to deal with 2-forms rather than complex structures). Then the vector bundle isomorphism $\Psi\colon TM \to \iota^*\Lambda^2_+ N$ restricts to a diffeomorphism $S(TM) \to \iota^*\mathcal{Z}^\flat_+(N,g)$, where $S(TM)$ denotes the bundle of *unit* length tangent vectors to M with respect to the metric induced from N. Using this isomorphism, we identify unit vector fields on M with lifts of ι to the twistor space.

The space $\mathcal{Z}^\flat_+(N,g)$ carries a natural tautological 2-form ω^τ defined by

$$\omega^\tau_{(x,p)} = \tau^* p \qquad \forall x \in N,\ p \in \mathcal{Z}^\flat_+(N,g)_x$$

where $\tau\colon \mathcal{Z}^\flat_+(N,g) \to N$ is the projection. For each $\lambda \neq 0$, one can extend ω^τ to a non-degenerate 2-form ω^λ, given by setting $\forall v,\, w \in T_{(x,p)}\mathcal{Z}^\flat_+(N,g)$

$$\begin{aligned} \omega^\lambda_{(x,p)}(v,w) &= \omega^\tau(v,w) + \lambda\,\omega^{S^2}(\mathrm{pr}_{\mathcal{V}}(v), \mathrm{pr}_{\mathcal{V}}(w)) \\ &= p(\tau_* v, \tau_* w) + \lambda\,\omega^{S^2}(\mathrm{pr}_{\mathcal{V}}(v), \mathrm{pr}_{\mathcal{V}}(w)), \end{aligned}$$

where $\mathcal{V} = \ker \tau_*$ is the vertical tangent bundle and $\omega^{S^2} \in C^\infty(\mathcal{Z}^\flat_+(N,g), \Lambda^2\mathcal{V}^*)$ is the 2-form which restricts to the area form on each fibre of τ, giving it area 8π.

While the form ω^λ is always non-degenerate, it is very rarely closed, so it does not define a symplectic structure on the twistor space in general. Precisely this problem, for manifolds of any dimension, was addressed by Reznikov in [Rez93], where he defines a natural closed form ω_{rez} on the twistor space and studies under what conditions on (N,g) the form ω_{rez} is non-degenerate. In the case where (N,g) is hyperbolic, this was studied further by Fine-Panov in [FP09] leading to the construction of many non-Kähler compact monotone symplectic manifolds. The Floer theory of some of these manifolds and their Lagrangians was then studied by Evans in [Eva14].

However, recall that we are mainly interested in $N = \mathbb{HP}^1 \cong S^4(1/2)$ in which case we saw in section 4.1 that the form $\omega^{\frac{1}{8}}$ is closed, because $(\mathcal{Z}^\flat_+(\mathbb{HP}^1,g), \omega^{\frac{1}{8}})$ is symplectomorphic to $(\mathbb{CP}^3, \omega_{\mathrm{FS}})$ (in fact, in this case $\omega^{\frac{1}{8}}$ agrees with the Reznikov form up to an overall constant). Since we will not use the closedness of ω^λ in our arguments, we will keep the general perspective and look for the conditions that a unit vector field on M should satisfy, in order for the corresponding lift of ι to $\mathcal{Z}^\flat_+(N,g)$ to be ω^λ-Lagrangian.

To unravel these conditions, consider the splitting $T\Lambda^2_+N = \mathcal{H} \oplus \widetilde{\mathcal{V}}$ into horizontal and vertical subspaces, induced by the Levi-Civita connection $\overline{\nabla}$ on N. Using the bundle isomorphism Ψ, this translates into a splitting $TM = \widehat{\mathcal{H}} \oplus \widehat{\mathcal{V}}$, i.e. we get an affine connection on M. Let $\mathrm{pr}_{\widehat{\mathcal{V}}} \colon TM \to \widehat{\mathcal{V}}$ be the projection along $\widehat{\mathcal{H}}$ and let us denote the corresponding covariant derivative by $\widehat{\nabla}$. The following is then our main observation:

Proposition 4.2.3. *The connection $\widehat{\nabla}$ is given by the formula*

$$\widehat{\nabla}_V W = \nabla_V W + s(V) \times W \quad \forall\, V, W \in C^\infty(M, TM), \tag{4.49}$$

where

i) ∇ is the Levi-Civita connection of $(M, \iota^* g)$,

ii) $s := -\overline{\nabla} v_4 \colon TM \to TM$ is the shape operator of the immersion $\iota \colon M \to N$,

iii) the cross product of two vectors V, W in TM is defined to be $V \times W := *(V \wedge W)$, where $*$ denotes the Hodge star operator on $(M, \iota^* g)$ with respect to the fixed orientation on M.

A unit vector field Z on M defines an ω^λ-Lagrangian lift of M if and only if for all vector fields V, W on M one has

$$\mathrm{dvol}(Z, V, W) + 2\lambda \mathrm{dvol}(Z, \widehat{\nabla}_V Z, \widehat{\nabla}_W Z) = 0, \tag{4.50}$$

where dvol *denotes the volume form on $(M, \iota^* g)$. This is equivalent to the existence of a positive orthonormal frame $\{X, Y, Z\}$ on M such that*

$$\begin{aligned} \widehat{\nabla}_Z Z &= 0 \\ \widehat{\nabla}_X Z \times \widehat{\nabla}_Y Z &= -\frac{1}{2\lambda} Z. \end{aligned} \tag{4.51}$$

Proof. We break the proof into its natural three parts – first we prove formula (4.49), then we establish equation (4.50) and finally we show that it is equivalent to (4.51).

Proof of formula (4.49)*:* Let $x \in M$ be a point and $V \in T_x M$ be a tangent vector. Consider a path $\gamma\colon [0,1] \to M$ such that $\gamma(0) = x$, $\dot\gamma(0) = V$ and let W be a vector field along γ. We write $v_4(t) := v_4(\gamma(t))$. Then, by definition of covariant differentiation, we have $\widehat{\nabla}_V W = \frac{d}{dt}\big|_{t=0} \Big(\widehat{P}_\gamma(t)^{-1} W(t)\Big)$, where $\widehat{P}_\gamma$ denotes parallel transport with respect to $\widehat{\nabla}$. Using the isomorphism Ψ, this rewrites as

$$\widehat{\nabla}_V W = \frac{d}{dt}\bigg|_{t=0} \Psi^{-1}\left(\overline{P}_\gamma(t)^{-1} \Psi\left(W(t)\right)\right),$$

where $\overline{P}_\gamma$ denotes parallel transport on N with respect to $\overline{\nabla}$. Thus we have

$$\begin{aligned} \widehat{\nabla}_V W &= \frac{d}{dt}\bigg|_{t=0} \Psi^{-1}\left(\overline{P}_\gamma(t)^{-1}\left(W^\flat \wedge v_4^\flat + *(W^\flat \wedge v_4^\flat)\right)\right) \\ &= \frac{d}{dt}\bigg|_{t=0} \Psi^{-1}\left(\overline{P}_\gamma(t)^{-1}(W)^\flat \wedge \overline{P}_\gamma(t)^{-1}(v_4)^\flat + *\left(\overline{P}_\gamma(t)^{-1}(W)^\flat \wedge \overline{P}_\gamma(t)^{-1}(v_4)^\flat\right)\right) \\ &= \frac{d}{dt}\bigg|_{t=0} \left(\psi_{v_4(0)}\right)^{-1} \circ \psi_{\overline{P}_\gamma(t)^{-1}(v_4(t))}\left(\overline{P}_\gamma(t)^{-1}(W)\right) \\ &= \frac{d}{dt}\bigg|_{t=0} S_{\overline{P}_\gamma(t)^{-1}(v_4(t)), v_4(0)}\left(\overline{P}_\gamma(t)^{-1}(W)\right). \end{aligned}$$

Now, from formula (4.47) we have that the above equals the sum of the following three terms:

1. $\frac{d}{dt}\big|_{t=0} g\left(\overline{P}_\gamma(t)^{-1}(v_4(t)), v_4(0)\right) \overline{P}_\gamma(t)^{-1}(W) = g\left(\overline{\nabla}_V v_4, v_4(0)\right) W(0) + \overline{\nabla}_V W = \overline{\nabla}_V W$

2. $$\begin{aligned} \frac{d}{dt}\bigg|_{t=0} \left(\overline{P}_\gamma(t)^{-1}(v_4(t)) \wedge v_4(0)\right)^\flat \left(\overline{P}_\gamma(t)^{-1}(W)\right) &= \left(\overline{\nabla}_V v_4 \wedge v_4(0)\right)^\flat (W) \\ &= g\left(\overline{\nabla}_V v_4, W\right) v_4(0) - g\left(v_4(0), W\right) \overline{\nabla}_V v_4 \\ &= -II(V,W), \end{aligned}$$

   where $II\colon TM \otimes TM \to TM^\perp$ is the second fundamental form and the last line is the Weingarten formula;

3. $$\begin{aligned} -\frac{d}{dt}\bigg|_{t=0} \left(*\left(\overline{P}_\gamma(t)^{-1}(v_4(t)) \wedge v_4(0)\right)\right)^\flat \left(\overline{P}_\gamma(t)^{-1}(W)\right) &= -\left(*\left(\overline{\nabla}_V v_4 \wedge v_4(0)\right)\right)^\flat (W) \\ &= \left(*\left(s(V) \wedge v_4(0)\right)\right)^\flat (W). \end{aligned}$$

Summing the three terms we obtain

$$\widehat{\nabla}_V W = \nabla_V W + \left(*\left(s(V) \wedge v_4(0)\right)\right)^\flat (W) \tag{4.52}$$

Now fix a positive orthonormal frame $\{v_1, v_2, v_3\}$ for $(M, \iota^* g)$. So $\{v_1, v_2, v_3, v_4\}$ is a positive orthonormal frame for (N, g) along M and we let $\{\phi^1, \phi^2, \phi^3, \phi^4\}$ denote the dual frame. Then for any vector $Y \in T_x M$ we have

$$\begin{aligned} g\left(\left(*\left(s(V) \wedge v_4(0)\right)\right)^\flat (W), Y\right) &= *(s(V)^\flat \wedge \phi^4)(W,Y) \;=\; *\left(\sum_{i=1}^{3} \phi^i(s(V))\, \phi^i \wedge \phi^4\right)(W,Y) \\ &= \left(\phi^1(s(V))\phi^2 \wedge \phi^3 - \phi^2(s(V))\phi^1 \wedge \phi^3 + \phi^3(s(V))\phi^1 \wedge \phi^2\right)(W,Y) \\ &= (\phi^1 \wedge \phi^2 \wedge \phi^3)(s(V), W, Y) \;=\; \mathrm{dvol}(s(V), W, Y) \\ &= g\left(s(V) \times W, Y\right) \end{aligned} \tag{4.53}$$

From (4.52) and (4.53) we get

$$g\left(\widehat{\nabla}_V W, Y\right) = g\left(\nabla_V W, Y\right) + g(s(V) \times W, Y), \tag{4.54}$$

which gives formula (4.49). Note further that (4.54) shows that $\widehat{\nabla}(\iota^* g) = 0$, i.e $\widehat{\nabla}$ is a metric connection.

Establishing equation (4.50)*:* Now let Z be a unit vector field on M, such that $\Psi \circ Z\colon M \to \mathcal{Z}_+^\flat(N,g)$ is an ω^λ-Lagrangian lift of $\iota\colon M \to N$. That is, we want $Z^* \Psi^* \omega^\lambda = 0$. Unravelling this we have

$$\begin{aligned}
0 = Z^*\Psi^*\omega^\lambda(V,W) &= \omega^\lambda\Big|_{\Psi(Z)}(\Psi_* Z_* V, \Psi_* Z_* W) \\
&= \Psi(Z)(\tau_*\Psi_* Z_* V, \tau_*\Psi_* Z_* W) + \lambda\,\omega^{S^2}(\mathrm{pr}_{\mathcal{V}}\Psi_* Z_* V, \mathrm{pr}_{\mathcal{V}}\Psi_* Z_* W) \\
&= \Psi(Z)(V,W) + \lambda\Psi^*\omega^{S^2}\left(\mathrm{pr}_{\widehat{\mathcal{V}}}(Z_* V), \mathrm{pr}_{\widehat{\mathcal{V}}}(Z_* W)\right) \\
&= (Z^\flat \wedge v_4^\flat + *(Z^\flat \wedge v_4^\flat))(V,W) + 2\lambda(Z\lrcorner\,\mathrm{dvol})\left(\widehat{\nabla}_V Z, \widehat{\nabla}_W Z\right) \\
&= *(Z^\flat \wedge v_4^\flat)(V,W) + 2\lambda\mathrm{dvol}\left(Z, \widehat{\nabla}_V Z, \widehat{\nabla}_W Z\right) \\
&= \mathrm{dvol}(Z,V,W) + 2\lambda\mathrm{dvol}\left(Z, \widehat{\nabla}_V Z, \widehat{\nabla}_W Z\right).
\end{aligned}$$

Note that the factor of 2 appears because of the fibrewise scaling introduced by Ψ.

Equivalence of (4.50) *and* (4.51)*:* Let Z be a unit vector field on M. Then, since $\widehat{\nabla}$ is a metric connection, we have that $g\left(\widehat{\nabla}_V Z, Z\right) = 0$ for all $V \in TM$. That is, the image of the linear map $\widehat{\nabla}Z\colon TM \to TM$ is contained in $Z^\perp$ and, in particular, $\widehat{\nabla}Z$ has non-trivial kernel.

Suppose now that Z satisfies equation (4.50) for all $V,W \in TM$. If V is a tangent vector to M such that $\widehat{\nabla}_V Z = 0$, then (4.50) tells us that $\mathrm{dvol}(Z,V,W) = 0$ for all tangent vectors W and so $V \in \mathrm{Span}(Z)$. Thus $\ker\left(\widehat{\nabla}Z\right) \leq \mathrm{Span}(Z)$ and since we know that $\widehat{\nabla}Z$ has non-trivial kernel, we must have $\ker\left(\widehat{\nabla}Z\right) = \mathrm{Span}(Z)$. In particular $\widehat{\nabla}_Z Z = 0$.

We now complete Z to a positive orthonormal frame $\{X,Y,Z\}$. Using that $\widehat{\nabla}_Z Z = 0$, we see that equation (4.50) is satisfied for all V and W, if and only if

$$\begin{aligned}
2\lambda\mathrm{dvol}\left(Z, \widehat{\nabla}_X Z, \widehat{\nabla}_Y Z\right) &= -\mathrm{dvol}(Z,X,Y) \\
\Leftrightarrow g\left(Z, \widehat{\nabla}_X Z \times \widehat{\nabla}_Y Z\right) &= -\frac{1}{2\lambda}.
\end{aligned}$$

Since $\left\{\widehat{\nabla}_X Z, \widehat{\nabla}_Y Z\right\} \subseteq Z^\perp$ we have that $\widehat{\nabla}_X Z \times \widehat{\nabla}_Y Z \in \mathrm{Span}(Z)$ and thus

$$\widehat{\nabla}_X Z \times \widehat{\nabla}_Y Z = -\frac{1}{2\lambda}Z.$$

Hence equations (4.51) holds.

Conversely, if the system (4.51) holds for a positive frame $\{X,Y,Z\}$, then it is immediate to check that the 2-form $Z\lrcorner\,\mathrm{dvol} + 2\lambda\mathrm{dvol}\left(Z, \widehat{\nabla}Z, \widehat{\nabla}Z\right)$ is identically zero. □

4.2.1.3 Some tautologies

Suppose that Z is a unit vector field on M which defines an ω^λ-Lagrangian lift of M to $\mathcal{Z}_+^\flat(N,g)$. We now make some pointwise observations which reflect the tautological nature of such a vector field. Note that $\mathcal{Z}_+^\flat(N,g)\big|_M$ is a hypersurface in $\mathcal{Z}_+^\flat(N,g)$ and so the non-degenerate 2-form ω^λ has a 1-dimensional kernel when restricted to $\mathcal{Z}_+^\flat(N,g)\big|_M$. Similarly, for every $x \in M$ every 2-form $p \in \mathcal{Z}_+^\flat(N,g)_x$ has a 1-dimensional kernel, when restricted to $T_x M$. We then have the following fact.

Proposition 4.2.4. *For every $x \in M$, $p \in \mathcal{Z}_+^\flat(N,g)_x$, the unit vector $Z_x^p := \Psi_x^{-1}(p) \in T_x M$ generates the kernel of $p|_{T_x M}$ and its horizontal lift $\widetilde{Z}_x^p \in \mathcal{H}_{(x,p)} \leq T_{(x,p)}\mathcal{Z}_+^\flat(N,g)$ generates the kernel of*

$\omega^\lambda\big|_{\mathcal{Z}^\flat_+(N,g)|_M}$. *In particular, if* $\tilde{\imath}\colon M\to\mathcal{Z}^\flat_+(N,g)$ *is an* ω^λ*-Lagrangian lift with corresponding unit vector field* Z*, then*

$$\ker\left(\omega^\lambda\big|_{\mathcal{Z}^\flat_+(N,g)|_M}\right)\Big|_{\tilde{\imath}(M)}=\operatorname{Span}\left(\widetilde{Z}\right),\tag{4.55}$$

where $\widetilde{Z}$ *denotes the horizontal component of* $\tilde{\imath}_*Z$*.*

Proof. First we show that $p(Z^p_x,V)=0$ for every $V\in T_xM$. Let $X,Y\in T_xM$ be unit vectors such that $\{X,Y,Z^p_x\}$ is a positive orthonormal frame for T_xM. It then suffices to show that

$$p(Z^p_x,X)=p(Z^p_x,Y)=0.$$

Indeed:

$$\begin{aligned}
p(Z^p_x,X) &= \Psi_x(Z^p_x)(Z^p_x,X)\;=\;\left((Z^p_x)^\flat\wedge\nu^\flat_4+*\left((Z^p_x)^\flat\wedge\nu^\flat_4\right)\right)(Z^p_x,X)\\
&= *\left((Z^p_x)^\flat\wedge\nu^\flat_4\right)(Z^p_x,X)\;=\;(X^\flat\wedge Y^\flat)(Z^p_x,X)\\
&= g(X,Z^p_x)\,g(Y,X)-g(Y,Z^p_x)\,\|X\|^2=0
\end{aligned}$$

Similarly for $p(Z^p_x,Y)=0$. Now if $\widetilde{Z}^p_x$ denotes the horizontal lift of Z^p_x, then for any $W\in T_{(x,p)}\left(\mathcal{Z}^\flat_+(N,g)\big|_M\right)$ we have $\tau_*W\in T_xM$ and so $\omega^\lambda(\widetilde{Z}^p_x,W)=p(Z^p_x,\tau_*W)+\lambda\,\omega_{S^2}(0,\mathrm{pr}_{\mathcal{V}}W)=0$. Hence $\widetilde{Z}^p_x$ generates the kernel of $\omega^\lambda\big|_{\mathcal{Z}^\flat_+(N,g)|_M}$.

Now, if $\tilde{\imath}\colon M\to\mathcal{Z}^\flat_+(N,g)$ is an ω^λ-Lagrangian lift, then for each $x\in M$, the space $\tilde{\imath}_*T_xM$ is a ω^λ-Lagrangian subspace of $T_{\tilde{\imath}(x)}\mathcal{Z}^\flat_+(N,g)$, contained in the codimension 1 subspace $T_{\tilde{\imath}(x)}\left(\mathcal{Z}^\flat_+(N,g)\big|_M\right)$. Hence $\ker\left(\omega^\lambda\big|_{\mathcal{Z}^\flat_+(N,g)|_M}\right)\Big|_{\tilde{\imath}(M)}\le\tilde{\imath}_*TM$, that is, along the Lagrangian, the characteristic line field of the hypersurface is tangent to the Lagrangian. But now, since Z is defined by the equation $\tilde{\imath}=\Psi\circ Z$, equation (4.55) follows by our previous pointwise considerations. $\square$

4.2.2 An example: the Clifford torus

We now consider a familiar example, namely the standard Clifford torus $T^3_{Cl}\subseteq\mathbb{CP}^3$. Recall that it is the image of the product torus $(S^1)^4\subseteq\mathbb{C}^4$ under the map $\Pi_{\mathbb{C}}$, so in homogeneous coordinates we can parametrise it as $(\mathbb{R}/2\pi\mathbb{Z})^3\to\mathbb{CP}^3$, $(\theta_1,\theta_2,\theta_3)\mapsto\left[e^{\mathbf{i}\theta_1}:e^{\mathbf{i}\theta_2}:e^{\mathbf{i}\theta_3}:1\right]_{\mathbb{C}}$. From this and formula (4.12) we see that T^3_{Cl} is preserved by the fibrewise antipodal map $\mathcal{X}$. For our calculations it will be more convenient to change coordinates by setting $\theta_1=\theta,\theta_2=\varphi-\psi$, $\theta_3=\theta-\psi$. Thus, writing $T^3:=(\mathbb{R}/2\pi\mathbb{Z})^3$, our parametrisation becomes

$$\begin{aligned}
\widetilde{F}\colon T^3 &\longrightarrow \mathbb{CP}^3\\
(\theta,\varphi,\psi) &\longmapsto \left[e^{\mathbf{i}\theta}:e^{\mathbf{i}(\varphi-\psi)}:e^{\mathbf{i}(\theta-\psi)}:1\right]_{\mathbb{C}}
\end{aligned}\tag{4.56}$$

and we have $\mathcal{X}\circ\widetilde{F}(\theta,\varphi,\psi)=\widetilde{F}(\theta+\varphi+\pi,-\varphi,\psi-\varphi)$. By composing with the twistor fibration we obtain the map $F:=\Pi\circ\widetilde{F}\colon T^3\to\mathbb{HP}^1$, given by

$$F(\theta,\varphi,\psi)=\left[\frac{e^{-\mathbf{i}\varphi}+1}{2}e^{\mathbf{i}\psi}+\mathbf{j}\frac{e^{\mathbf{i}\varphi}-1}{2}e^{-\mathbf{i}\theta}:1\right]_{\mathbb{H}}.\tag{4.57}$$

From this we see that the image of the Clifford torus under the twistor fibration is the set

$$\mathbb{S} := \Pi(T^3_{Cl}) = F(T^3) = \{[p:1]_{\mathbb{H}} : \ \|p\| = 1\} \subseteq \mathbb{HP}^1$$

which is isometric to an equatorial $S^3(1/2)$ in $S^4(1/2)$ via the isometry Φ from (4.7). Moreover, observe that $F(\theta,0,\psi) = \left[e^{\mathbf{i}\psi} : 1\right]_{\mathbb{H}}$ and $F(\theta,\pi,\psi) = \left[-\mathbf{j}e^{-\mathbf{i}\theta} : 1\right]_{\mathbb{H}}$. That is, the two 2-dimensional tori[6] $\widetilde{F}(\{\varphi = 0\})$ and $\widetilde{F}(\{\varphi = \pi\})$ in T^3_{Cl} are collapsed by Π to the Hopf link $HL \subseteq \mathbb{S}$, defined by

$$\Phi(HL) = \left\{ z + \mathbf{j}w \in \mathbb{H} : \ |z|^2 + |w|^2 = \frac{1}{4},\ zw = 0 \right\} \times \{0\} \subseteq S^3(1/2) \times \{0\} \subseteq \mathbb{H} \oplus \mathbb{R}.$$

It is not hard to see that, away from these 2-tori, the map

$$\Pi|_{T^3_{Cl}} : T^3_{Cl} \setminus \widetilde{F}(\varphi \in \{0,\pi\}) \longrightarrow \mathbb{S} \setminus HL \tag{4.58}$$

is a double cover and the pre-image of every point consists of two fibrewise antipodal points on T^3_{Cl}.

Now, since the image $\Pi(T^3_{Cl}) = \mathbb{S} \subseteq \mathbb{HP}^1$ is a totally geodesic submanifold, its shape operator vanishes and so the connection $\widehat{\nabla}$ from Proposition 4.2.3 becomes just the Levi-Civita connection on $\mathbb{S}$. Hence, since the Clifford torus is Lagrangian in $\mathbb{CP}^3$, equation (4.51) tells us that T^3_{Cl} locally corresponds to a geodesic unit vector field on $\mathbb{S}$. More precisely, because of the sign ambiguity coming from the fact that (4.58) is a double cover with $\mathcal{X}$ as the deck involution, $T^3_{Cl} \setminus \widetilde{F}(\varphi \in \{0,\pi\})$ should be identified with the sphere bundle inside a rank 1 geodesic distribution on $\mathbb{S} \setminus HL$. We call this rank 1 geodesic distribution (and also its integral curves) the *Clifford foliation*. Since this foliation consists of great circles in S^3, one should be able to visualise it and thus "see" the Clifford torus embedded as a Lagrangian in $\mathbb{CP}^3$.

To determine the Clifford foliation, we consider the map

$$\Phi' \colon \mathbb{HP}^1 \setminus \{[1:0]_{\mathbb{H}}\} \longrightarrow \mathbb{H}, \qquad \Phi'([p,q]_{\mathbb{H}}) = pq^{-1}.$$

The restriction of Φ' to $\mathbb{S}$ identifies it with $\mathrm{Sp}(1) = \{p \in \mathbb{H} : \ \|p\| = 1\}$ and this identification is conformal and stretches distances by a factor of 2. We will exhibit the Clifford foliation on $\mathbb{S}$ by pushing it forward to $\mathrm{Sp}(1)$. The key is the following diagram:

$$\begin{tikzcd}
\mathbb{CP}^3 \arrow[r, "\mathbf{I}"] \arrow[drr, "\Pi"'] & \mathcal{Z}_+(\mathbb{HP}^1,g) \arrow[r, "\flat"] \arrow[dr, "\tau"] & \mathcal{Z}^\flat_+(\mathbb{HP}^1,g) \arrow[d] & \supseteq & \mathcal{Z}^\flat_+(\mathbb{HP}^1,g)\big|_{\mathbb{S}} \arrow[r, "\Psi^{-1}"] \arrow[d] & S_1(T\mathbb{S}) \arrow[r, "d\Phi'"] \arrow[dl] & S_2(T\mathrm{Sp}(1)) \arrow[d] \\
 & & \mathbb{HP}^1 & \supseteq & \mathbb{S} \arrow[rr, "\Phi'"] & & \mathrm{Sp}(1)
\end{tikzcd}$$

Recall that the definition (4.48) of the isomorphisms Ψ requires a choice of orientation on the 3-manifold, so we assume that we have chosen an orientation on $\mathbb{S}$. We denote by $N^{\mathbb{S}}$ the unit normal to $\mathbb{S}$ determined by the orientation on $\mathbb{HP}^1$ and the chosen orientation on $\mathbb{S}$. Then Lemma 4.2.1 tells us that if $x \in \mathbb{S}$ is a point and α is an element of $\mathcal{Z}^\flat_+(\mathbb{HP}^1,g)_x$, then $\Psi_x^{-1}(\alpha) = -\alpha^\#\left(N^{\mathbb{S}}_x\right)$.

[6] Getting neat equations for these tori is the reason for changing coordinates in the beginning.

Now, from the above diagram, we see that the unit tangent vector to $\mathrm{Sp}(1)$, determined by the point $\widetilde{F}(\theta,\varphi,\psi)\in T^3_{Cl}$, is given by

$$\frac{1}{2}\Phi'_*\circ\Psi^{-1}\left(\mathbf{I}\left(\widetilde{F}(\theta,\varphi,\psi)\right)^{\flat}\right)=-\frac{1}{2}\Phi'_*\left[\mathbf{I}\left(\widetilde{F}(\theta,\varphi,\psi)\right)\left(N^{\mathbb{S}}_{F(\theta,\varphi,\psi)}\right)\right]. \tag{4.59}$$

The right hand side is not difficult to compute. First, an easy calculation shows that the derivative of the map $\Pi_{\mathbb{H}}\circ\Phi'\colon\mathbb{H}^2\setminus(\mathbb{H}\times\{0\})\longrightarrow\mathbb{H}$ is given by

$$d_{(p,q)}\Pi_{\mathbb{H}}\circ\Phi'(u,v)=uq^{-1}-pq^{-1}vq^{-1}. \tag{4.60}$$

On the other hand, since $\Phi'|_{\mathbb{S}}:\mathbb{S}\longrightarrow\mathrm{Sp}(1)$ is a conformal diffeomorphism which doubles all lengths, we have that if $[p:q]_{\mathbb{H}}\in\mathbb{S}$ (i.e. $\|p\|=\|q\|$), then the unit normal to $\mathbb{S}$ at $[p:q]_{\mathbb{H}}$ satisfies $\Phi'_*\left(N^{\mathbb{S}}_{[p:q]_{\mathbb{H}}}\right)=2pq^{-1}$ (up to a sign, but this can be fixed by changing the chosen orientation on $\mathbb{S}$). Using (4.60), it is immediate to verify that $d_{[p:q]_{\mathbb{H}}}\Phi'\left(d_{(p,q)}\Pi_{\mathbb{H}}(p,-q)\right)=2pq^{-1}$ and so we conclude that

$$N^{\mathbb{S}}_{[p:q]_{\mathbb{H}}}=d_{(p,q)}\Pi_{\mathbb{H}}(p,-q). \tag{4.61}$$

Now put $p=e^{\mathbf{i}\theta}+\mathbf{j}e^{\mathbf{i}(\varphi-\psi)}$ and $q=e^{\mathbf{i}(\theta-\psi)}+\mathbf{j}$, so that $\widetilde{F}(\theta,\varphi,\psi)=\Pi_{\mathbb{C}}(p,q)$. Plugging this and (4.61) into the right-hand side of (4.59), we see that the unit vector we want to find is

$$\begin{aligned}
-\frac{1}{2}d_{[p:q]_{\mathbb{H}}}\Phi'\left(\mathbf{I}(\Pi_{\mathbb{C}}(p,q))\left(d_{(p,q)}\Pi_{\mathbb{H}}(p,-q)\right)\right) &= -\frac{1}{2}d_{[p:q]_{\mathbb{H}}}\Phi'\left(d_{(p,q)}\Pi_{\mathbb{H}}(p\mathbf{i},-q\mathbf{i})\right)\\
&= -\frac{1}{2}d_{(p,q)}\Phi'\circ\Pi_{\mathbb{H}}(p\mathbf{i},-q\mathbf{i})\\
&= -p\,\mathbf{i}\,q,
\end{aligned} \tag{4.62}$$

where we have used the definition of the map $\mathbf{I}$ and formula (4.60). Since T^3_{Cl} is $\mathcal{X}$-invariant, the sign is irrelevant. Plugging in the values of p and q into (4.62) and writing $F':=\Phi'\circ F$, we find that the Clifford foliation at $F'(\theta,\varphi,\psi)$ is spanned by the vector

$$\begin{aligned}
Z|_{F'(\theta,\varphi,\psi)} &:= \frac{e^{-\mathbf{i}(\varphi+\pi)}+1}{2}e^{\mathbf{i}\left(\psi+\frac{\pi}{2}\right)}+\mathbf{j}\frac{e^{\mathbf{i}(\varphi+\pi)}-1}{2}e^{-\mathbf{i}\left(\theta+\frac{\pi}{2}\right)}\\
&= \left(\frac{\partial}{\partial\theta}+2\frac{\partial}{\partial\varphi}+\frac{\partial}{\partial\psi}\right)F'(\theta,\varphi,\psi).
\end{aligned}$$

It is then not hard to see that

$$\begin{aligned}
\sin(\varphi/2)F'(\theta,\varphi,\psi)\,+\,\cos(\varphi/2)\,Z|_{F'(\theta,\varphi,\psi)} &\in \mathrm{Span}_{\mathbb{R}}\{\mathbf{j},\mathbf{k}\}\\
\cos(\varphi/2)F'(\theta,\varphi,\psi)\,-\,\sin(\varphi/2)\,Z|_{F'(\theta,\varphi,\psi)} &\in \mathrm{Span}_{\mathbb{R}}\{1,\mathbf{i}\}
\end{aligned}$$

and therefore for every $\varphi\notin\pi\mathbb{Z}$ one has

$$Z|_{F'(\theta,\varphi,\psi)}\in\mathrm{Span}_{\mathbb{R}}\{F'(\theta,\varphi,\psi),1,\mathbf{i}\}\cap\mathrm{Span}_{\mathbb{R}}\{F'(\theta,\varphi,\psi),\mathbf{j},\mathbf{k}\}.$$

We conclude that the Clifford foliation consists of all great circles on S^3, obtained by intersecting a great 2-sphere that contains one component of the Hopf link with one that contains the other component.

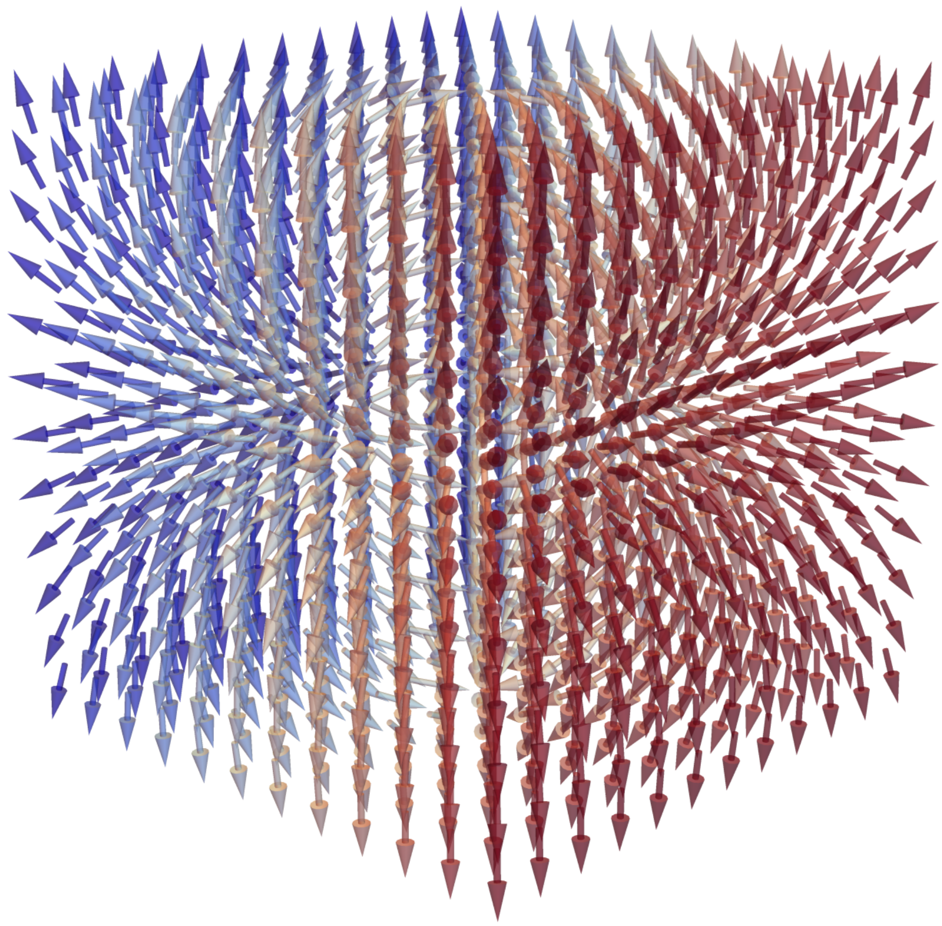


Figure 4.1: The Clifford foliation

4.3 No vertical Hamiltonians

In this section we prove the following:

Proposition 4.3.1. *Let $U \subseteq \mathbb{HP}^n$ be a connected open set. Suppose that $f : \mathbb{CP}^{2n+1} \to \mathbb{R}$ is a smooth function whose Hamiltonian vector field X^f is vertical on $\Pi^{-1}(U)$, i.e.*

$$d_x\Pi\left(X^f_x\right) = 0 \quad \forall x \in \Pi^{-1}(U).$$

Then f is constant on $\Pi^{-1}(U)$.

Proof. First we note that it suffices to prove the statement for $n = 1$. Indeed, in the setting of the proposition, suppose $\tilde{x}$ and $\tilde{y}$ are two distinct points in $\Pi^{-1}(U) \subseteq \mathbb{CP}^{2n+1}$. Suppose further that their projections $x = \Pi(\tilde{x})$ and $y = \Pi(\tilde{y})$ are distinct (the argument when $\tilde{x}$ and $\tilde{y}$ lie in the same fibre is even easier). Since U is connected, we can find a sequence of points $x = x_0, x_1, \ldots, x_N = y$ in U such that for all $i \in \{1, 2, \ldots, N\}$ the set $U_i := U \cap \mathbb{HP}^1_i$ is a connected open subset of $\mathbb{HP}^1_i$, where $\mathbb{HP}^1_i$ denotes the quaternionic line through x_{i-1} and x_i. Then X^f is vertical on $\Pi^{-1}(U_i)$ and hence, assuming the proposition holds for $n=1$, f is constant on $\Pi^{-1}(U_i)$. But then $f(\tilde{x}) = f(\tilde{x}_0) = f(\tilde{x}_1) = \cdots = f(\tilde{x}_N) = f(\tilde{y})$, where $\tilde{x}_i \in \mathbb{CP}^{2n+1}$ is any lift of x_i for $i \in \{1,2,\ldots,N\}$ and $\tilde{x}_0 = \tilde{x}$, $\tilde{x}_N = \tilde{y}$.

We now prove the case $n=1$. That is, we assume that $U \subseteq \mathbb{HP}^1$ is a connected open set and $f\colon \mathbb{CP}^3 \to \mathbb{R}$ is a function whose Hamiltonian vector field X^f is vertical on $\Pi^{-1}(U)$. Let $p \in U$ and note that it suffices to prove that f is constant $\Pi^{-1}(B)$, where $B \subseteq U$ is a small geodesic ball, centred at p.

Since the complex structure J_0 preserves the vertical bundle $\mathcal{V}$, we have that X^f is vertical at a point $x \in \mathbb{CP}^3$ if and only if $\left(\operatorname{grad}_{g_{\mathrm{FS}}} f\right)_x = -J_0(X^f_x)$ is vertical. That is, $X^f_x \in \mathcal{V}_x$ if and only if

$$d_x f(\mathcal{H}_x) = 0. \tag{4.63}$$

We now identify $\mathbb{HP}^1$ with $S^4 = S^4(1/2)$ via the isometry Φ and we identify $\mathbb{CP}^3$ with $S_{\sqrt{2}}(\Lambda^2_+ S^4)$ via $\Phi^* \circ \flat \circ \mathbf{I}$. Note that this identification preserves the splittings into horizontal and vertical bundles, so we get a function (still denoted by) $f\colon S_{\sqrt{2}}(\Lambda^2_+ S^4) \to \mathbb{R}$, whose differential annihilates the horizontal distribution at all points which project to a geodesic ball (still denoted by) $B \subseteq S^4$, centred at a point $x_0 = \Phi(p)$. We now extend f radially, setting

$$\tilde{f}\colon \Lambda^2_+(S^4) \setminus \{\text{zero section}\} \longrightarrow \mathbb{R}$$

$$\tilde{f}(\alpha) \;=\; f\left(\frac{\sqrt{2}}{\|\alpha\|}\alpha\right)$$

Note that $d\tilde{f}$ annihilates the horizontal distribution at all points which project to B.

Now choose stereographic coordinates $x = (x^1, x^2, x^3, x^4)$ on $S^4(1/2)$, centred at the point x_0, so that the geodesic ball B corresponds to a ball $B_r(0) \subseteq \mathbb{R}^4$ of some positive radius r. Let $y = (y^1, y^2, y^3)$ be fibre coordinates on $\Lambda^2_+(S^4)$ with respect to the basis

$$\{\alpha_1 := dx^1 \wedge dx^2 + dx^3 \wedge dx^4,\ \alpha_2 := dx^1 \wedge dx^3 - dx^2 \wedge dx^4,\ \alpha_3 := dx^1 \wedge dx^4 + dx^2 \wedge dx^3\}.$$

We introduce the quaternionic notation $x = x^1 + x^2\mathbf{i} + x^3\mathbf{j} + x^4\mathbf{k}$ and $y = y^1\mathbf{i} + y^2\mathbf{j} + y^3\mathbf{k}$. For the function $\tilde{f}$ we also write

$$\nabla_x \tilde{f} := \frac{\partial \tilde{f}}{\partial x^1} + \frac{\partial \tilde{f}}{\partial x^2}\mathbf{i} + \frac{\partial \tilde{f}}{\partial x^3}\mathbf{j} + \frac{\partial \tilde{f}}{\partial x^4}\mathbf{k} \quad \text{and} \quad \nabla_y \tilde{f} := \frac{\partial \tilde{f}}{\partial y^1}\mathbf{i} + \frac{\partial \tilde{f}}{\partial y^2}\mathbf{j} + \frac{\partial \tilde{f}}{\partial y^3}\mathbf{k}.$$

In appendix A we show that in these coordinates the condition that $d\tilde{f}$ annihilates the horizontal distribution above B, translates to the equation

$$\frac{1+\|x\|^2}{2}\nabla_x \tilde{f} \;=\; \left(2\nabla_y \tilde{f}\cdot y \;+\; \nabla_y \tilde{f} \times y\right)x \quad \forall x \in B_r(0),\ \forall y \in \mathbb{R}^3 \setminus \{0\}, \tag{4.64}$$

where juxtaposition of vectors denotes quaternion multiplication.

Note that the function $\tilde{f}$ was constructed so that it is scale invariant in the y-direction, i.e.

$\nabla_y \tilde{f} \cdot y = 0$. Substituting this into (4.64), we see that $\tilde{f}$ satisfies

$$\begin{aligned} \frac{1+\|x\|^2}{2} \nabla_x \tilde{f} &= \left(\nabla_y \tilde{f} \times y\right) x & (4.65) \\ \Leftrightarrow \quad \frac{1+\|x\|^2}{2} \nabla_x \tilde{f} &= \left(\nabla_y \tilde{f}\right) yx. & \\ \Rightarrow \quad \frac{1+\|x\|^2}{2} \left(\nabla_x \tilde{f}\right) x^{-1} y^{-1} &= \nabla_y \tilde{f}, \quad \forall x \neq 0. & (4.66) \end{aligned}$$

Now multiplying equation (4.65) on the right by $\bar{x}$ and rearranging we get

$$\left(\nabla_x \tilde{f}\right) \bar{x} = \frac{2\|x\|^2}{1+\|x\|^2} \nabla_y \tilde{f} \times y.$$

Taking real parts on both sides yields $\nabla_x \tilde{f} \cdot x = 0$.

Now fix a point $y \in \mathbb{R}^3 \setminus \{0\}$. Since $\nabla_x \tilde{f} \cdot x = 0$ for all $x \in B_r(0)$, we know that for all $t \in \left[0, \frac{r}{\|x\|}\right)$ we have $\tilde{f}(tx, y) = \tilde{f}(x, y)$. Setting $t = 0$ we obtain $\tilde{f}(x, y) = \tilde{f}(0, y)$ for all $x \in B_r(0)$. Thus $\tilde{f}\big|_{B_r(0) \times (\mathbb{R}^3 \setminus \{0\})}$ is actually just a function of y and so $\nabla_x \tilde{f} = 0$ for all $(x, y) \in B_r(0) \times (\mathbb{R}^3 \setminus \{0\})$. But then, by equation (4.66) and continuity, we find that $\nabla_y \tilde{f} = 0$ and hence $\tilde{f}\big|_{B_r(0) \times (\mathbb{R}^3 \setminus \{0\})}$ is constant. $\square$

Chapter 5

Non-displaceability of some twistor Lagrangians

In this section we prove that the orientable subadjoint Lagrangians are not displaceable. By work of Evans-Lekili ([EL15]), this was known for the Chiang Lagrangian $L_\Delta = Z_1 \subseteq \mathbb{CP}^3$ which displays a significantly different behaviour from the others by virtue of having minimal Maslov number equal to 2. In section 5.1, we use the results of [EL15] to compute the Floer cohomology of L_Δ with coefficients in high rank local systems and prove Theorem E.

In section 5.2, we treat the other orientable subadjoint Lagrangians and we prove Theorem F.

5.1 The Chiang Lagrangian and $\mathbb{RP}^3$

5.1.1 Identifying Z_1 and L_Δ

In section 4.1.6 we saw the twistor Lagrangian Z_1, associated to a Type 2 Legendrian twisted cubic X_1. From this description, we know that Z_1 is a monotone Lagrangian, which is orientable and spin, has minimal Maslov number 2 and admits the structure of a circle bundle over $\mathbb{RP}^2$ whose pull-back to S^2 is the lens space $L(6,1)$. We now verify that Z_1 is indeed the Chiang Lagrangian L_Δ which was described in [Chi04] as an orbit of a Hamiltonian $\mathrm{SU}(2)$ action on $\mathbb{CP}^3$. Let us first give the definition.

We view $\mathbb{CP}^3 \cong \mathrm{Sym}^3\left(\mathbb{CP}^1\right)$ as configurations of triples of points on $\mathbb{CP}^1$. The action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{CP}^1$ by Möbius transformations then defines an action on $\mathbb{CP}^3$ whose restriction to the compact form $\mathrm{SU}(2) \subseteq \mathrm{SL}(2,\mathbb{C})$ is Hamiltonian (see e.g. [Smi15, Section 3.1]). Setting

$$\Delta := \left\{[1:1]_\mathbb{C}, [\omega^2:1]_\mathbb{C}, [\omega^4:1]_\mathbb{C}\right\} \in \mathrm{Sym}^3\left(\mathbb{CP}^1\right) \cong \mathbb{CP}^3 \text{ with } \omega := e^{i\pi/3}, \tag{5.1}$$

we then have a decomposition $\mathbb{CP}^3 = W_\Delta \cup Y_\Delta$, where $W_\Delta = \mathrm{SL}(2,\mathbb{C}) \cdot \Delta$ is the orbit consisting of all triples of pairwise distinct points and Y_Δ is a compactifying divisor consisting of triples with at least two coinciding points (note then that Y_Δ is cut out by the discriminant of a cubic, which is a section of $\mathcal{O}_{\mathbb{CP}^3}(4)$; that is, Y_Δ is an anticanonical hypersurface). For a suitable identification of $\mathbb{CP}^1$ and $S^2(1)$, the set Δ corresponds to an equilateral triangle, equatorially inscribed in the sphere. From this point of view, $\mathbb{CP}^3$ is a special case of an $\mathrm{SL}(2,\mathbb{C})$-quasihomogeneous 3-fold X_C, obtained by compactifying an $\mathrm{SL}(2,\mathbb{C})$-orbit W_C of a configuration $C \in \mathrm{Sym}^n\left(\mathbb{CP}^1\right) = \mathbb{CP}^n$ of n distinct points in $\mathbb{CP}^1$. It is known by work of Aluffi and Faber ([AF93]) that $X_\Delta = \mathbb{CP}^3$ is the first of only four cases in which such a compactifiaction is smooth. The other three are the 3-folds $X_T \subseteq \mathbb{CP}^4$, $X_O \subseteq \mathbb{CP}^6$ and $X_I \subseteq \mathbb{CP}^{12}$, obtained by choosing C to consist of the vertices of a regular tetrahedron, octahedron and icosahedron, respectively. It is a fact that in all 4 cases, when C is chosen to be such a regular configuration, its orbit L_C under the action of the compact real form $\mathrm{SU}(2) \subseteq \mathrm{SL}(2,\mathbb{C})$ is a Lagrangian submanifold of X_C. The four Lagrangians L_Δ, L_T, L_O and L_I have come to be known as the *Platonic Lagrangians*. The Chiang Lagrangian is then the first (admittedly, somewhat degenerate) such Lagrangian:

Definition 5.1.1. The Chiang Lagrangian is the orbit $L_\Delta := \mathrm{SU}(2)\cdot\Delta$ in $X_\Delta = \mathbb{CP}^3$. $\diamondsuit$

To see that this is indeed the same Lagrangian Z_1 which we considered in section 4.1.6, recall that the identification

$$\mathbb{CP}^3 \cong \mathrm{Sym}^3\left(\mathbb{CP}^1\right)$$

is obtained by viewing $\mathbb{C}^4$ as the space $\mathbb{C}[x,y]_3$ of degree 3 homogeneous polynomials in 2 variables and sending each element $f \in \mathbb{C}[x,y]_3 \setminus \{0\}$ to its projectivised zero set. We make the identification of $\mathbb{C}^4$ with $\mathbb{C}[x,y]_3$ via the map

$$\begin{aligned} \mathbb{C}^4 &\longrightarrow \mathbb{C}[x,y]_3 \\ (z_0,z_1,z_2,z_3) &\longmapsto z_0 x^3 + z_1 y^3 + z_2\sqrt{3}xy^2 + z_3\sqrt{3}x^2y. \end{aligned} \tag{5.2}$$

A matrix $A \in \mathrm{SU}(2)$ acts on $f \in \mathbb{C}[x,y]_3$ by

$$(A\cdot f)(x,y) := f\left(A^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right) \tag{5.3}$$

and in this way $\mathrm{SU}(2)$ acts on the projectivised zero set of f in $\mathbb{CP}^1$ by usual projective transformations. Recall that we identify $\mathbb{C}^4$ with $\mathbb{H}^2$ via (4.2). Using (4.12) and (5.2) it is straightforward to check that the above action of $\mathrm{SU}(2)$ is right $\mathbb{H}$-linear and in fact it defines a homomorphism $\mathrm{SU}(2) \to \mathrm{U}(4)\cap\mathrm{GL}(2,\mathbb{H}) = \mathrm{Sp}(2)$. So it descends to an action of $\mathrm{SU}(2)$ on $\mathbb{HP}^1$ by isometries and the twistor fibration $\Pi\colon \mathbb{CP}^3 \to \mathbb{HP}^1$ is equivariant. For the record, if we parametrise $\mathrm{SU}(2) = \left\{\begin{pmatrix} a & -\bar{b} \\ b & \bar{a}\end{pmatrix} \;:\; a,b\in\mathbb{C},\ |a|^2+|b|^2 = 1\right\}$, then the homomorphism $\mathrm{SU}(2)\to\mathrm{Sp}(2)$ we get with our identifications is explicitly given by

$$\begin{pmatrix} a & -\bar{b} \\ b & \bar{a}\end{pmatrix} \longmapsto \begin{pmatrix} \bar{a}^3 + \mathbf{j}\bar{b}^3 & \sqrt{3}(\bar{a}b^2 + \mathbf{j}a^2\bar{b}) \\ \sqrt{3}(\bar{a}\bar{b}^2 + \mathbf{j}\bar{a}^2\bar{b}) & a(|a|^2 - 2|b|^2) - \mathbf{j}b(2|a|^2 - |b|^2)\end{pmatrix}. \tag{5.4}$$

Note that Δ is the projectivised zero set of the polynomial $f_\Delta(x,y) = x^3 - y^3$ which under the identification (5.2) corresponds to the point $(1,-1,0,0)\in\mathbb{C}^4$. Since L_Δ is the orbit of Δ under the $\mathrm{SU}(2)$ action and Π is equivariant, we have that $\Pi(L_\Delta)$ is the orbit of

$\Pi(\Delta) = [1:0]_{\mathbb{H}} \in \mathbb{HP}^1$. From (5.4) one easily finds that the stabiliser of $[1:0]_{\mathbb{H}}$ is the set $\mathrm{Stab}([1:0]_{\mathbb{H}}) = \left\{ \begin{pmatrix} e^{\mathbf{i}\theta} & 0 \\ 0 & e^{-\mathbf{i}\theta} \end{pmatrix} : \theta \in \mathbb{R}/2\pi\mathbb{Z} \right\} \cup \left\{ \begin{pmatrix} 0 & e^{\mathbf{i}\theta} \\ -e^{-\mathbf{i}\theta} & 0 \end{pmatrix} : \theta \in \mathbb{R}/2\pi\mathbb{Z} \right\}$ and so $\Pi(L_\Delta) = \mathrm{SU}(2)\cdot\Pi(\Delta) = \mathrm{SU}(2)/\mathrm{Stab}([1:0]_{\mathbb{H}}) \cong \mathbb{RP}^2$. Thus L_Δ is an embedded Lagrangian, compatible with the twistor fibration and hence by Theorem 4.1.23 we know that L_Δ is a twistor Lagrangian which projects to a (necessarily) superminimal $\mathbb{RP}^2$. The twistor lift of this $\mathbb{RP}^2$ is the unique Legendrian curve in $\mathbb{CP}^3$ which projects to it. It is not hard to find this curve: just consider the orbit $\mathrm{SU}(2)\cdot[1:0:0:0]_{\mathbb{C}}$. On the one hand, it certainly projects to the orbit $\mathbb{RP}^2 = \mathrm{SU}(2)\cdot[1:0]_{\mathbb{H}}$ because $[1:0:0:0]_{\mathbb{C}}$ is a lift of $[1:0]_{\mathbb{H}}$ and Π is equivariant. On the other hand, using the identification (5.2), $[1:0:0:0]_{\mathbb{C}}$ corresponds to the line spanned by the polynomial $f(x,y) = x^3$ and so its SU(2)-orbit is precisely the projectivisation of the set of polynomials in $\mathbb{C}[x,y]_3$ whose dehomogenisation has a triple root. This space is isomorphic to $\mathbb{CP}^1$ and can easily be parametrised by

$$\begin{aligned} \mathbb{CP}^1 = \mathbb{C}\cup\{\infty\} &\longrightarrow \mathbb{P}(\mathbb{C}[x,y]_3) = \mathbb{CP}^3 \\ t &\longmapsto \left[(tx+y)^3\right]_{\mathbb{C}} = \left[t^3 : 1 : \sqrt{3}t : \sqrt{3}t^2\right]_{\mathbb{C}}, \end{aligned}$$

where we have again used the identification (5.2). Thus, the orbit $\mathrm{SU}(2)\cdot[1:0:0:0]_{\mathbb{C}}$ is indeed a Legendrian curve which projects to the superminimal $\mathbb{RP}^2 = \mathrm{SU}(2)\cdot[1:0]_{\mathbb{H}}$ and hence it is its twistor lift. Moreover, observe that this lift is exactly the Type 2 twisted cubic which we denoted by X_1 in section 4.1.6 and so we have indeed identified L_Δ and Z_1.

Viewing Z_1 as the homogeneous space L_Δ is extremely valuable for enumerating the holomorphic discs which it bounds in $\mathbb{CP}^3$. Moreover, since L_Δ is exhibited as a finite quotient of S^3, it can be effectively visualised. Further still, identifying the SU(2)-action on $\mathbb{CP}^1$ by projective transformations with the action on $S^2(1)$ by quaternionic rotations allows one to translate problems about the geometry of L_Δ into problems about equilateral triangles inscribed in the unit sphere in Euclidean 3-space. We now combine these different points of view in order to thoroughly understand the topology of L_Δ and to compute its Floer cohomology with arbitrary local systems. This calculation is heavily based on the results of Evans and Lekili from [EL15].

5.1.2 Topology of L_Δ

The Lie algebra $\mathfrak{su}(2)$ is the real-linear span of the Pauli matrices:

$$\sigma_1 := \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix}, \ \sigma_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ \text{ and } \ \sigma_3 := \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}.$$

From now on, all occurrences of "exp" refer to the exponential map in SU(2). For a unit vector $V = (v_1, v_2, v_3) \in S^2(1)$ and $t \in \mathbb{R}$ we will write $\exp(tV)$ to mean $\exp(t(v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3))$. The action of SU(2) on $\mathbb{CP}^1$ by projective transformations can be identified with the action of SU(2) on $S^2(1)$ by quaternionic rotations, as long as we adopt the following conventions (for any other choice the two actions would, of course, be conjugate):

- for any unit vector $V \in S^2(1)$, $\exp(\theta V)$ acts on $S^2(1)$ by a right-hand rotation in the axis V by an angle of 2θ; this is the adjoint action of $\mathrm{SU}(2)$ on $S^2 \subseteq \mathfrak{su}(2)$, where we identify $\mathfrak{su}(2)$ with $\mathbb{R}^3$ via the basis $\{\sigma_1, \sigma_2, \sigma_3\}$;

- we identify $\mathbb{C} \cup \{\infty\} \cong \mathbb{CP}^1$ via $z \mapsto [z:1]_{\mathbb{C}}$, $\infty \mapsto [1:0]_{\mathbb{C}}$;

- we identify $\mathbb{C} \cup \{\infty\} \cong S^2(1)$ via $z \mapsto \left(\frac{|z|^2-1}{|z|^2+1}, \frac{2\,\mathbb{R}\mathrm{e}(\mathbf{i}z)}{|z|^2+1}, \frac{2\,\mathrm{Im}(\mathbf{i}z)}{|z|^2+1}\right)$, $\infty \mapsto (1,0,0)$, i.e. via stereographic projection from $(1,0,0)$ followed by multiplication by $-\mathbf{i}$.

The latter two identifications combine to give the diffeomorphism

$$\begin{aligned} \Phi\colon \mathbb{CP}^1 &\longrightarrow S^2(1) \\ \Phi([x:y]_{\mathbb{C}}) &= \left(\frac{|x|^2-|y|^2}{|x|^2+|y|^2}, \frac{2\,\mathbb{R}\mathrm{e}(\mathbf{i}x\bar{y})}{|x|^2+|y|^2}, \frac{2\,\mathrm{Im}(\mathbf{i}x\bar{y})}{|x|^2+|y|^2}\right). \end{aligned}$$

This way the triple $\Delta = \left\{[1:1]_{\mathbb{C}}, [\omega^2:1]_{\mathbb{C}}, [\omega^4:1]_{\mathbb{C}}\right\}$ corresponds to the equilateral triangle with vertices $V_1' := (0,0,1)$, $V_3' := (0,-\sqrt{3}/2,-1/2)$, and $V_2' := (0,\sqrt{3}/2,-1/2)$ (our choice of names for the vertices will become apparent when we discuss a particular Morse function on L_Δ below). The Chiang Lagrangian is then the manifold of all equilateral triangles, equatorially inscribed in $S^2(1)$. We call these *maximal equilateral triangles*.

Using (5.4), it is easy to see that the stabiliser of Δ under the $\mathrm{SU}(2)$ action is the binary dihedral group of order 12, given explicitly by

$$\Gamma_\Delta = \left\{\begin{pmatrix} \omega^k & 0 \\ 0 & \bar{\omega}^k \end{pmatrix} : k \in \{0,1,\dots 5\}\right\} \cup \left\{\begin{pmatrix} 0 & \mathbf{i}\omega^k \\ \mathbf{i}\bar{\omega}^k & 0 \end{pmatrix} : k \in \{0,1,\dots 5\}\right\} \subseteq \mathrm{SU}(2).$$

Abstractly, we view this group by the presentation

$$\Gamma_\Delta = \left\langle a, b \mid a^6 = 1, b^2 = a^3, ab = ba^{-1} \right\rangle,$$

the above complex representation being given by $a \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{pmatrix}$ and $b \mapsto \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}$. So we have $L_\Delta \cong \mathrm{SU}(2)/\Gamma_\Delta$ and $\mathrm{SU}(2)$ is tiled by 12 fundamental domains for the right action of Γ_Δ. Further, the quotient map $q\colon \mathrm{SU}(2) \to L_\Delta$ induces a natural isomorphism

$$\begin{aligned} \Gamma_\Delta &\to \pi_1(L_\Delta, q(\mathrm{Id}))^{\mathrm{Opp}} \\ x &\mapsto [q \circ \ell_x], \end{aligned} \tag{5.5}$$

where $\ell_x\colon [0,1] \to \mathrm{SU}(2)$ is any path with $\ell(0) = \mathrm{Id}$ and $\ell(1) = x$. In figure 5.1 below we give a schematic description of a fundamental domain for the right action of Γ_Δ on $\mathrm{SU}(2)$. The picture is essentially borrowed from [EL15] with the difference that the fundamental domain given there is (erroneously) for a left Γ_Δ-action. A detailed derivation of the domain can be found in [Smi15, Section 5.1].

Evans and Lekili also describe a Morse function on L_Δ by specifying its critical points and some of its flowlines. We shall use essentially the same Morse function (depicted in figure 5.2 below) to

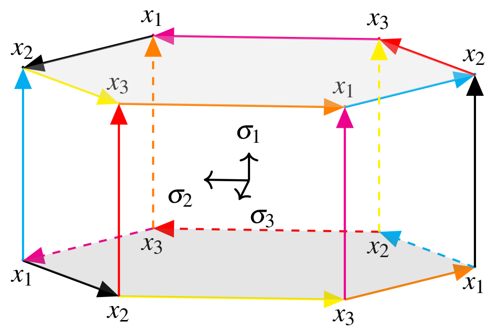


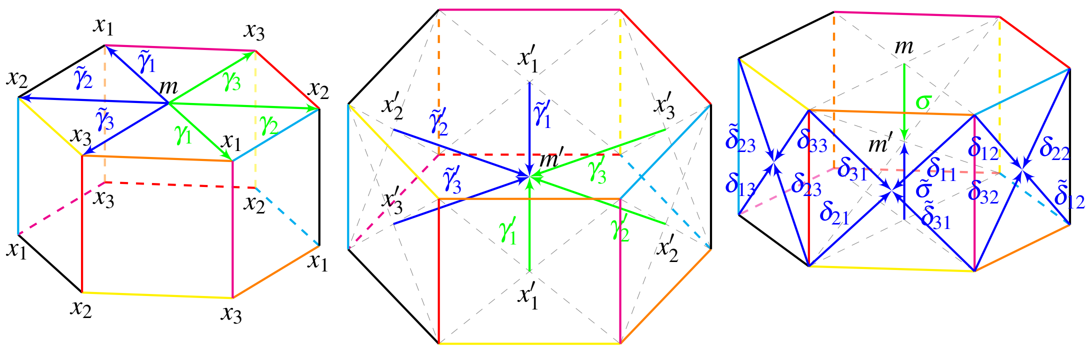
Figure 5.1: The fundamental domain for L_Δ. Opposite quadrilateral faces are identified by a 90° rotation and the two hexagonal faces are identified by a 60° rotation so that colours of edges match. The fundamental domain is viewed as sitting in $\mathrm{SU}(2)$ with Id at the center of the prism and the matrices $\{\sigma_1,\sigma_2,\sigma_3\} \subseteq T_{\mathrm{Id}}\mathrm{SU}(2)$ are given for orientation.

compute Floer cohomology but since we want to work with local coefficients, we are particularly concerned with where exactly its index 1 downward gradient flowlines (with respect to the round metric on $\mathrm{SU}(2) = S^3(1)$) pass. This is what we shall now spell out. Throughout this discussion it is useful to keep in mind the picture of rotating maximal equilateral triangles. For example, for any unit vectors $V, W \in \mathbb{R}^3$ we think of the point $q(\exp(sV)\exp(tW)) \in L_\Delta$ as the triangle, obtained from Δ by first applying a right-hand rotation by $2t$ in the axis W and then a right-hand rotation by $2s$ in the axis V.

Recall that we defined $V_1' := (0,0,1)$, $V_2' := (0,\sqrt{3}/2,-1/2)$, $V_3' := (0,-\sqrt{3}/2,-1/2)$. We now further set $h := \exp\left(\frac{\pi}{6}\sigma_1\right) \in \mathrm{SU}(2)$ and $V_1 := h \cdot V_1' = (0,-\sqrt{3}/2,1/2)$, $V_2 := h \cdot V_2' = (0,\sqrt{3}/2,1/2)$, $V_3 := h \cdot V_3' = (0,0,-1)$. We then define the Morse function $f \colon L_\Delta \to \mathbb{R}$ to have:

- one minimum at $m' := q(\mathrm{Id}) = \triangle V_1'V_2'V_3'$;
- three critical points of index 1: $x_1' := q\left(\exp\left(\frac{\pi}{4}V_1'\right)\right)$, $x_2' := q\left(\exp\left(\frac{\pi}{4}V_2'\right)\right)$ and $x_3' := q\left(\exp\left(\frac{\pi}{4}V_3'\right)\right)$. They are connected to the minimum m' via 6 flowlines whose compactified images can be parametrised for $t \in [0,1]$ by $\gamma_i'(t) := q\left(\exp\left((1-t)\frac{\pi}{4}V_i'\right)\right)$ and $\tilde{\gamma}_i'(t) := q\left(\exp\left(-(1-t)\frac{\pi}{4}V_i'\right)\right)$ for $i \in \{1,2,3\}$;
- three critical points of index 2: $x_1 := q\left(\exp\left(\frac{\pi}{4}V_1\right)h\right)$, $x_2 := q\left(\exp\left(\frac{\pi}{4}V_2\right)h\right)$ and $x_3 := q\left(\exp\left(\frac{\pi}{4}V_3\right)h\right)$;
- one maximum at $m := q(h) = \triangle V_1V_2V_3$. It connects to the index 2 critical points via 6 flowlines whose images are similarly given by $\gamma_i(t) := q\left(\exp\left(t\frac{\pi}{4}V_i\right)h\right)$ and $\tilde{\gamma}_i(t) := q\left(\exp\left(-t\frac{\pi}{4}V_i\right)h\right)$ for $t \in [0,1]$ and $i \in \{1,2,3\}$;

- there are 12 gradient flowlines connecting critical points of index 2 to critical points of index 1. For our purposes we do not need a similarly precise description of their images and the schematic description from figure 5.2 c) will do.



a) b) c)

Figure 5.2: A Morse function $f\colon L_\Delta \to \mathbb{R}$. All flowlines of index 1 are depicted with arrows pointing in the direction of downward gradient flow. Note that in diagram c) the flowlines δ_{ij} and $\tilde{\delta}_{ij}$ always go from x_i to x_j'. The index 3 flowlines σ and $\tilde{\sigma}$ and the different colouring (green and blue) of the flowlines will be used below for the calculation of parallel transport maps.

For the sake of completeness, let us now give a formula for such a function. To describe it we will use coordinates on L_Δ coming from the Hopf coordinates on S^3. Consider the following "Euler angles map":

$$G\colon (\mathbb{R}/2\pi\mathbb{Z})^3 \to \mathrm{SU}(2), \quad G(\theta,\varphi,\psi) := \exp\left(\frac{\varphi+\psi}{2}\sigma_1\right)\cdot\exp(\theta\sigma_3)\cdot\exp\left(\frac{\varphi-\psi}{2}\sigma_1\right).$$

The map G has degree 4 and its singular values (i.e. where "gimbal lock" occurs) form the standard Hopf link $\{\exp(t\sigma_1) \,:\, t\in[0,2\pi]\}\cup\{\exp(t\sigma_1)\cdot\exp((\pi/2)\sigma_3) \,:\, t\in\mathbb{R}/2\pi\mathbb{Z}\}$. Now our Morse function $f\colon L_\Delta\to\mathbb{R}$ (or rather, its pull-back under $q\circ G$) is given by

$$f(\theta,\varphi,\psi) = -\cos^4(\theta)\cos(6\varphi) - \sin^4(\theta)\cos(6\psi) \tag{5.6}$$

and one can easily check that in these coordinates its critical points are indeed $m'=(0,0,0)$, $x_1' = \left(\frac{\pi}{4},0,0\right)$, $x_2' = \left(\frac{\pi}{4},\frac{2\pi}{3},\frac{2\pi}{3}\right)$, $x_3' = \left(\frac{\pi}{4},\frac{\pi}{3},\frac{\pi}{3}\right)$, $x_1 = \left(\frac{\pi}{4},\frac{\pi}{6},\frac{\pi}{6}\right)$, $x_2 = \left(\frac{\pi}{4},\frac{5\pi}{6},\frac{5\pi}{6}\right)$, $x_3 = \left(\frac{\pi}{4},\frac{3\pi}{6},\frac{3\pi}{6}\right)$ and $m = \left(0,\frac{\pi}{6},\frac{\pi}{6}\right)$.

5.1.3 Computation of Floer cohomology with local coefficients

5.1.3.1 Morse differential

Let V be any vector space over $\mathbb{F}_2$ and let $\rho\colon \Gamma_\Delta \to \mathrm{Aut}(V)$ be a representation. Through the isomorphism $\Gamma_\Delta \cong \pi_1(L_\Delta,m')^{\mathrm{Opp}}$, ρ determines a *right* action of $\pi_1(L_\Delta,m')$ on V and so we obtain a local system $\mathcal{W}\to L_\Delta$ with fibre isomorphic to V by the recipe from (2.5). As Morse data for the pearl complex we shall use the pair $\mathscr{D}=(f,g)$, where f is the Morse function (5.6) and g is the

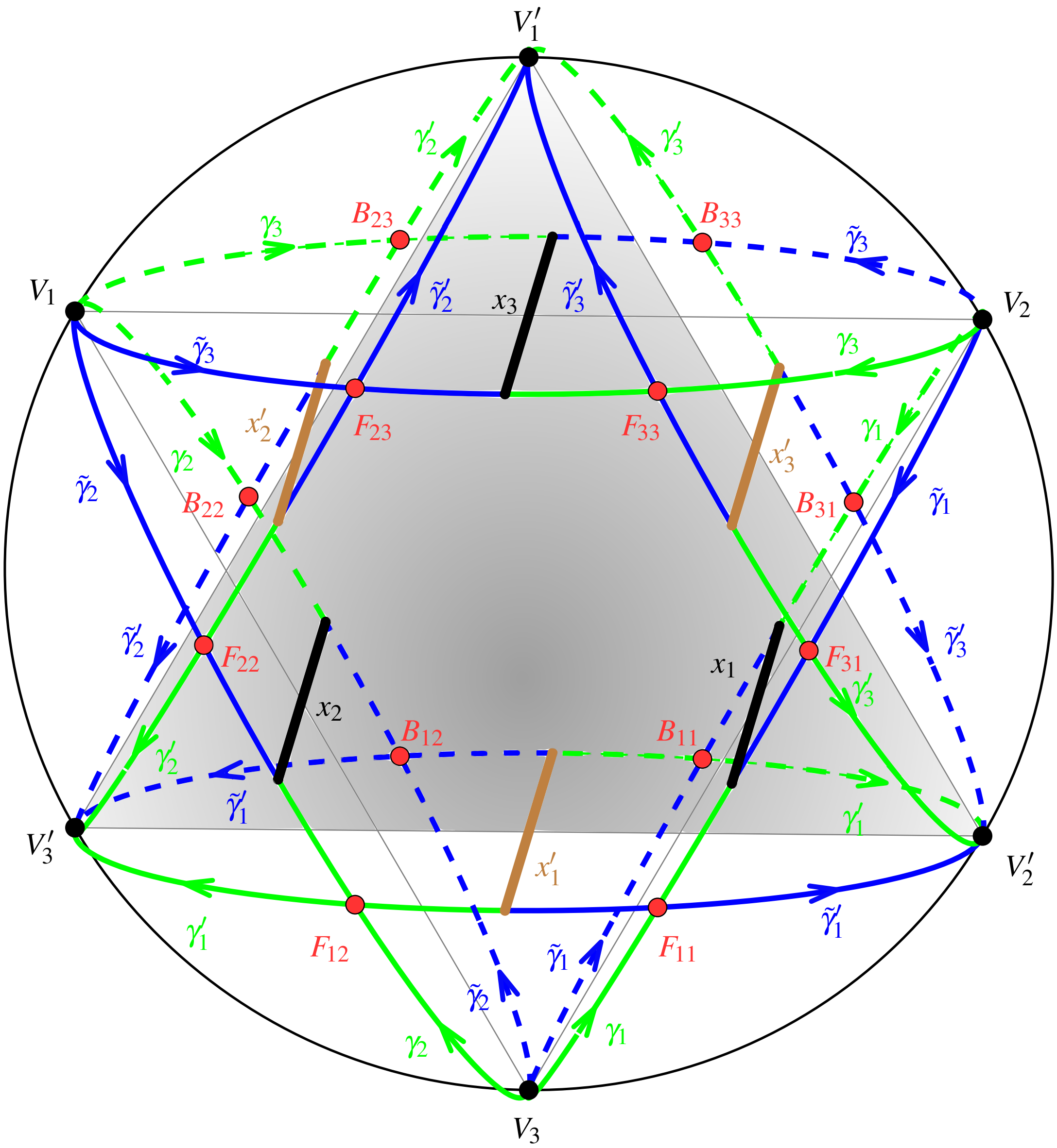


Figure 5.3: Another representation of the Morse function f. The minimum m' corresponds to the triangle $\triangle V_1'V_2'V_3'$ and the maximum m is $\triangle V_1V_2V_3$. The critical points of index one $\{x_i'\}_{1\leq i\leq 3}$ and index two $\{x_i\}_{1\leq i\leq 3}$ correspond to maximal equilateral triangles with one side along the segment with the respective label. The flowlines of index 1 through the minimum and maximum are also illustrated by the pairs of circular arcs with matching labels. Each downward flowline consists of triangles rotating around a fixed vertex, with their other two vertices tracing out the two arcs in the indicated directions. The labels of these arcs match the ones on the flowlines in figure 5.2.

round metric on SU(2). We will first explicitly compute the Morse differential $\partial^{\mathscr{D}}$ on the complex $C_f^*(L_\Delta;\mathscr{E}nd(\mathcal{W}))$.

We thus need to calculate parallel transport maps on $\mathcal{W}$ along the index 1 flow lines of f. To

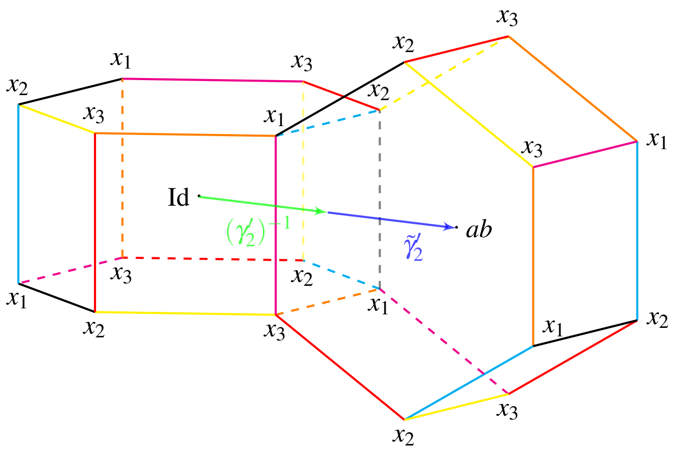


Figure 5.4: Parallel transport along $\tilde{\gamma}_2'$.

that end we first fix an identification $\mathcal{W}_{m'} \cong V$. Next, we also identify with V the fibres of $\mathcal{W}$ which lie over other critical points. We do so in the unique way so that parallel transport maps along the paths $(\gamma_1')^{-1}$, $(\gamma_2')^{-1}$, $(\gamma_3')^{-1}$, σ^{-1}, $(\sigma^{-1}\cdot\gamma_1)$, $(\sigma^{-1}\cdot\gamma_2)$ and $(\sigma^{-1}\cdot\gamma_3)$ are represented by the identity map $V \to V$. From now on we refer to the paths in this list as *identification paths* and we draw them in green on all diagrams (see also figure 5.2 above).

Suppose now that ℓ is a path from $s(\ell) \in \mathrm{Crit}(f)$ to $t(\ell) \in \mathrm{Crit}(f)$. By pre-concatenating ℓ with the identification path to $s(\ell)$ and post-concatenating it with the inverse of the identification path to $t(\ell)$ we obtain the corresponding loop $\hat{\ell}$, based at m'. We identify this loop with an element $\left[\hat{\ell}\right] \in \Gamma_\Delta$ via the isomorphism (5.5). Then, using the identifications above, the parallel transport map is given by

$$
\begin{array}{cccc}
P_\ell & : & V & \xrightarrow{\rho\left(\left[\hat{\ell}\right]\right)} & V \\
 & & \wr\Vert & & \wr\Vert \\
 & & \mathcal{W}_{s(\ell)} & & \mathcal{W}_{t(\ell)}
\end{array}
\tag{5.7}
$$

We now use this setup, together with the universal cover $\mathrm{SU}(2) \to L_\Delta$ to calculate $\partial^{\mathscr{D}}$. Note that the fundamental domain whose centre lies at $\mathrm{Id} \in \mathrm{SU}(2)$ borders 8 other fundamental domains with centres at $a = \exp(\sigma_1\pi/3)$, $a^5 = \exp(-\sigma_1\pi/3)$, $b = \exp(\sigma_3\pi/2)$, ab, a^2b, a^3b, a^4b and a^5b. The ones with centres at ab, a and a^5b are schematically depicted (after stereographic projection from $-\mathrm{Id} \in \mathrm{SU}(2)$) in figures 5.4, 5.5 and 5.6, respectively.[1]

Let us now compute $P_{\tilde{\gamma}_2'} : \mathcal{W}_{x_2'} \to \mathcal{W}_{m'}$ (the identifications with V from (5.7) will be used implicitly in this and all our subsequent calculations). The corresponding loop is $(\gamma_2')^{-1}\cdot\tilde{\gamma}_2'$. A lift of this loop at $\mathrm{Id} \in \mathrm{SU}(2)$ is shown in figure 5.4. From there we read off:

$$
P_{\tilde{\gamma}_2'} : \mathcal{W}_{x_2'} \xrightarrow{\quad AB \quad} \mathcal{W}_{m'},
$$

[1] These schematic figures (as well as figures 5.1, 5.2, 5.3, 5.8) were created using the tikz-3dplot package by Jeff Hein. See Figure 3 in [EL15] or figures 5.9, 5.10 below for accurate pictures of the stereographically projected fundamental domains.

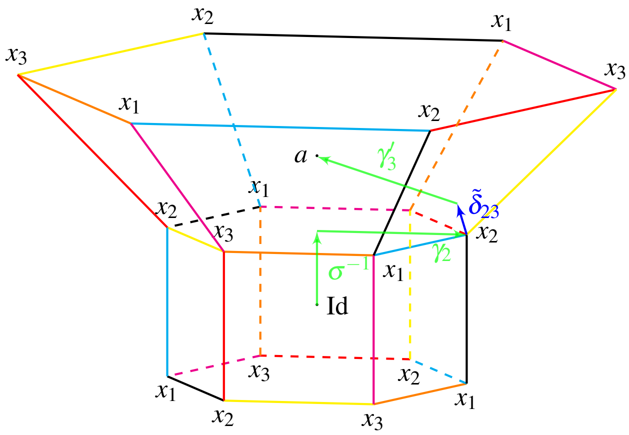


Figure 5.5: Parallel transport along $\tilde{\delta}_{23}$.

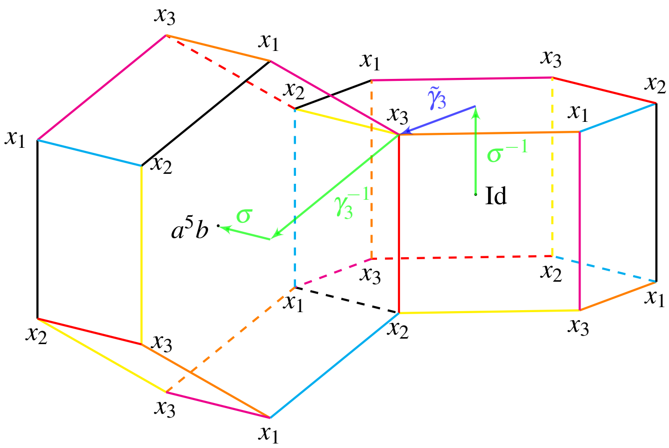


Figure 5.6: Parallel transport along $\tilde{\gamma}_3$.

where we write $A := \rho(a)$ and $B := \rho(b)$. Similarly we can compute $P_{\tilde{\delta}_{23}} : \mathcal{W}_{x_2} \to \mathcal{W}_{x_3'}$ by lifting the corresponding loop $(\sigma)^{-1} \cdot \gamma_2 \cdot \tilde{\delta}_{23} \cdot \gamma_3'$ to the universal cover. This is done in figure 5.5 and we read off:

$$P_{\tilde{\delta}_{23}} : \mathcal{W}_{x_2} \xrightarrow{\quad A \quad} \mathcal{W}_{x_3'}.$$

We compute one more example, namely $P_{\tilde{\gamma}_3} : \mathcal{W}_m \to \mathcal{W}_{x_3}$. The loop $(\sigma)^{-1} \cdot \tilde{\gamma}_3 \cdot (\gamma_3)^{-1} \cdot \sigma$ lifts as shown in figure 5.6. This yields:

$$P_{\tilde{\gamma}_3} : \mathcal{W}_m \xrightarrow{\quad A^5B \quad} \mathcal{W}_{x_3}.$$

Continuing this way we obtain all the needed maps for the calculation of $\partial^{\mathscr{D}}$, which we summarise in the following table:

$$\begin{array}{lcl|lcl|lcl|lcl|lcl} P_{\tilde{\gamma}'_1} & = & B & P_{\delta_{11}} & = & \mathrm{Id} & P_{\delta_{12}} & = & \mathrm{Id} & P_{\delta_{13}} & = & A & P_{\tilde{\gamma}_1} & = & A^3B \\ P_{\tilde{\gamma}'_2} & = & AB & P_{\delta_{21}} & = & A^2B & P_{\tilde{\delta}_{12}} & = & A & P_{\delta_{23}} & = & \mathrm{Id} & P_{\tilde{\gamma}_2} & = & A^4B \\ P_{\tilde{\gamma}'_3} & = & A^2B & P_{\delta_{31}} & = & A^2B & P_{\delta_{22}} & = & \mathrm{Id} & P_{\tilde{\delta}_{23}} & = & A & P_{\tilde{\gamma}_3} & = & A^5B \\ & & & P_{\tilde{\delta}_{31}} & = & A^3B & P_{\delta_{32}} & = & A^3B & P_{\delta_{33}} & = & \mathrm{Id} & & & \end{array} \tag{5.8}$$

Now every flowline γ connecting y to x gives us a map $\mathrm{End}(\mathcal{W}_x) \to \mathrm{End}(\mathcal{W}_y)$ by conjugation $\alpha \mapsto P_\gamma^{-1} \circ \alpha \circ P_\gamma$ and the sum of all these maps is the Morse differential $\partial^{\mathscr{D}}$ on the complex $C_f^*(L_\Delta; \mathscr{E}nd(\mathcal{W}))$. Using (5.8) we obtain:

- for every $\alpha' \in \mathrm{End}(\mathcal{W}_{m'})$:

$$\begin{aligned} \partial^{\mathscr{D}}(\alpha') &= \big(\alpha' + B^{-1}\alpha' B,\, \alpha' + (AB)^{-1}\alpha'(AB),\, \alpha' + (A^2B)^{-1}\alpha'(A^2B)\big) \\ &\in \mathrm{End}(\mathcal{W}_{x'_1}) \oplus \mathrm{End}(\mathcal{W}_{x'_2}) \oplus \mathrm{End}(\mathcal{W}_{x'_3}) \end{aligned} \tag{5.9}$$

- for every $(\alpha'_1, \alpha'_2, \alpha'_3) \in \mathrm{End}(\mathcal{W}_{x'_1}) \oplus \mathrm{End}(\mathcal{W}_{x'_2}) \oplus \mathrm{End}(\mathcal{W}_{x'_3})$:

$$\begin{aligned} \partial^{\mathscr{D}}(\alpha'_1, \alpha'_2, \alpha'_3) &= \big(\alpha'_1 + \alpha'_2 + A^{-1}\alpha'_2 A + A^{-1}\alpha'_3 A\,, \\ &\quad (A^2B)^{-1}\alpha'_1(A^2B) + \alpha'_2 + A^{-1}\alpha'_3 A + \alpha'_3\,, \\ &\quad (A^2B)^{-1}\alpha'_1(A^2B) + (A^3B)^{-1}\alpha'_1(A^3B) + (A^3B)^{-1}\alpha'_2(A^3B) + \alpha'_3\big) \\ &\in \mathrm{End}(\mathcal{W}_{x_1}) \oplus \mathrm{End}(\mathcal{W}_{x_2}) \oplus \mathrm{End}(\mathcal{W}_{x_3}) \end{aligned} \tag{5.10}$$

- and for every $(\alpha_1, \alpha_2, \alpha_3) \in \mathrm{End}(\mathcal{W}_{x_1}) \oplus \mathrm{End}(\mathcal{W}_{x_2}) \oplus \mathrm{End}(\mathcal{W}_{x_3})$:

$$\begin{aligned} \partial^{\mathscr{D}}(\alpha_1, \alpha_2, \alpha_3) &= \alpha_1 + (A^3B)^{-1}\alpha_1(A^3B) + \alpha_2 + (A^4B)^{-1}\alpha_2(A^4B) \\ &\quad + \alpha_3 + (A^5B)^{-1}\alpha_3(A^5B) \\ &\in \mathrm{End}(\mathcal{W}_m) \end{aligned} \tag{5.11}$$

5.1.3.2 Contributions from holomorphic discs

The classification of holomorphic discs of Maslov indices 2 and 4 with boundary on L_Δ (which are precisely the ones appearing in the pearly differential) has been carried out in [EL15] (see also [Smi15]). Our main goal for this section will be to trace where their boundaries pass and to determine the parallel transport maps along the paths $\gamma_{\mathbf{u}}^j$ for all relevant pearly trajectories $\mathbf{u}$.

Let us first recall the main notions and results on holomorphic discs from [EL15] which give us total control over the positions of the Maslov 2 discs.

Definition 5.1.2. Let L be a manifold and let K be a Lie group acting on L. Denote the stabiliser of a point $x \in L$ by K_x. An *x-admissible* homomorphism is defined to be any homomorphism $R\colon \mathbb{R} \to K$ such that $R(2\pi) \in K_x$. Such a homomorphism is called *primitive* if $R(\theta) \notin K_x$, $\forall \theta \in (0, 2\pi)$. $\Diamond$

Definition 5.1.3. Let X be a manifold and let K be a Lie group acting on X. Suppose further that $L \subseteq X$ is a submanifold preserved by the action. A disc $u\colon (D^2, \partial D^2) \to (X, L)$ is called *axial* if, after possibly reparametrising u, there exists a $u(1)$-admissible homomorphism R such that $u(re^{i\theta}) = R(\theta)\cdot u(r)$ for every $r \in [0,1]$, $\theta \in \mathbb{R}$. For the particular case when $K = \mathrm{SU}(2)$ we also define the *axis* of a non-constant axial disc u to be the normalised infinitesimal generator $R'(0)/\|R'(0)\| \in S^2(1)$, where again we identify $\mathfrak{su}(2)$ with $\mathbb{R}^3$ via the basis $\{\sigma_1, \sigma_2, \sigma_3\}$. $\diamondsuit$

Recall that we denote the standard (integrable) almost complex structure on $\mathbb{CP}^3$ by J_0. Using the above notions, one can summarise Evans and Lekili's classification results for J_0-holomorphic discs $u\colon (D^2, \partial D^2) \to (\mathbb{CP}^3, L_\Delta)$ in the following three theorems.

Theorem 5.1.4. *([EL15, Lemma 3.3.1]) All J_0-holomorphic discs $u\colon (D^2, \partial D^2) \to (\mathbb{CP}^3, L_\Delta)$ are* regular.[2] $\square$

Theorem 5.1.5. *([EL15, Sections 3.5, 6.1]) All J_0-holomorphic discs of Maslov index 2 with boundary on L_Δ are axial. Through every point on L_Δ there pass exactly three such discs, namely the appropriate* $\mathrm{SU}(2)$*-translates of the discs $\{u_1', u_2', u_3'\}$ with $u_i'(1) = m'$ for $i \in \{1,2,3\}$ and axes V_1', V_2' and V_3' respectively.* $\square$

Theorem 5.1.6. *(originally [EL15], but see [Smi15, Proposition 6.5] for a simpler proof) There are precisely two J_0-holomorphic discs $w_1, w_{-1}\colon (D^2, \partial D^2) \to (\mathbb{CP}^3, L_\Delta)$ of Maslov index 4 and passing through m and m'. They are both simple and axial with axes* $(1,0,0)$ *and* $(-1,0,0)$. [3] $\square$

Note first that Theorem 5.1.5 immediately allows us to compute the value of the obstruction section $m_0(\mathcal{W})$ at the point m'. Indeed, the boundaries of the three Maslov 2 discs passing through m' are given by

$$\partial u_1' = (\gamma_1')^{-1}\cdot\tilde{\gamma}_1', \quad \partial u_2' = (\tilde{\gamma}_2')^{-1}\cdot\gamma_2' \quad \text{and} \quad \partial u_3' = (\gamma_3')^{-1}\cdot\tilde{\gamma}_3'. \tag{5.12}$$

Referring to table (5.8), we have $P_{\partial u_1'} = B$, $P_{\partial u_2'} = (AB)^{-1} = A^4B$ and $P_{\partial u_3'} = A^2B$. This gives:

$$m_0(\mathcal{W})(m') = (\mathrm{Id} + A^2 + A^4)B. \tag{5.13}$$

Remark 5.1.7. Recall that, in order for the complex $CF^*((L_\Delta, \mathcal{W}), \mathbb{RP}^3)$ to be unobstructed, we need $m_0(\mathcal{W}) = 0$. Using (5.13) and the identity $(A^2 - \mathrm{Id})(A^4 + A^2 + \mathrm{Id}) = 0$, it is easy to see that $m_0(\mathcal{W}) = 0$ precisely when A^2 has no non-zero fixed vector. Now note that, due to the relation $ba^2b^{-1} = a^4$, any 1-dimensional representation of Γ_Δ must satisfy $A^2 = 1$ and so any rank 1 local system over a field $\mathbb{F}$ of characteristic 2 has non-vanishing obstruction section (the point here is that this holds not just for $\mathbb{F}_2$ but for all of its extensions as well). Since the rank is 1, this implies that the central complex is identically zero. This shows that, in order to achieve non-vanishing cohomology, it is necessary to work with higher rank local systems. //

[2] In particular $J_0 \in \mathcal{J}_{reg}(L_\Delta)$.

[3] The discs w_1 and w_{-1} are the two hemispheres of the twistor fibre through m'.

The three theorems above actually allow us to completely determine all isolated pearly trajectories which a candidate differential $d^{(\mathscr{D},J_0)}: C_f^*(\mathcal{W},\mathcal{W}) \to C_f^*(\mathcal{W},\mathcal{W})$ would count. Note that while Theorems 5.1.4, 5.1.5 and 5.1.6 give us a strong control over the moduli spaces of discs involved in $d^{(\mathscr{D},J_0)}$, we cannot a priori be sure that J_0 is an element of the set $\mathcal{J}_{\text{reg}}(\mathscr{D})$ from Theorem 2.4.8. This potential problem has been dealt with already in [EL15] and further elaborated on in [Smi15, Appendix A] and the solution is to perturb the Morse data $\mathscr{D}$ by pushing it forward through a diffeomorphism of L_Δ which can be chosen arbitrarily close to the identity. Since such a perturbation will preserve homotopy classes of paths, it will not affect any of our calculations, so we work directly with the pre-complex $C_f^*(\mathcal{W},\mathcal{W})$ and determine the candidate differential $d^{(\mathscr{D},J_0)}$. To alleviate notation we shall also temporarily drop the decorations $(\mathscr{D},J_0)$.

From equation (2.52) we know that the maps which we need to figure out are:

$$
\begin{array}{rrcl}
\partial_1: & \operatorname{End}(\mathcal{W}_{x_1'}) \oplus \operatorname{End}(\mathcal{W}_{x_2'}) \oplus \operatorname{End}(\mathcal{W}_{x_3'}) & \to & \operatorname{End}(\mathcal{W}_{m'}), \\
\partial_1: & \operatorname{End}(\mathcal{W}_{x_1}) \oplus \operatorname{End}(\mathcal{W}_{x_2}) \oplus \operatorname{End}(\mathcal{W}_{x_3}) & \to & \operatorname{End}(\mathcal{W}_{x_1'}) \oplus \operatorname{End}(\mathcal{W}_{x_2'}) \oplus \operatorname{End}(\mathcal{W}_{x_3'}), \\
\partial_1: & \operatorname{End}(\mathcal{W}_m) & \to & \operatorname{End}(\mathcal{W}_{x_1}) \oplus \operatorname{End}(\mathcal{W}_{x_2}) \oplus \operatorname{End}(\mathcal{W}_{x_3}), \\
\partial_2: & \operatorname{End}(\mathcal{W}_m) & \to & \operatorname{End}(\mathcal{W}_{m'}).
\end{array}
$$

To determine the first one of these, we are interested in pearly configurations, consisting of a single Maslov 2 disc u such that $u(-1) \in W^d(m')$ and $u(1) \in W^a(x_i')$. Since $W^d(m') = \{m'\}$ such a disc must be one of $\{u_1', u_2', u_3'\}$. From (5.12) and the table (5.8) we see that the corresponding parallel transport maps are

$$
\begin{array}{lll}
P_{\gamma^0_{u_1'}} = \mathrm{Id}, & P_{\gamma^0_{u_2'}} = (AB)^{-1} = A^4B, & P_{\gamma^0_{u_3'}} = \mathrm{Id} \\
P_{\gamma^1_{u_1'}} = B, & P_{\gamma^1_{u_2'}} = \mathrm{Id}, & P_{\gamma^1_{u_3'}} = A^2B.
\end{array}
$$

Thus for every $(\alpha_1', \alpha_2', \alpha_3') \in \operatorname{End}(\mathcal{W}_{x_1'}) \oplus \operatorname{End}(\mathcal{W}_{x_2'}) \oplus \operatorname{End}(\mathcal{W}_{x_3'})$ we have

$$
\partial_1(\alpha_1', \alpha_2', \alpha_3') = B\alpha_1' + \alpha_2'(A^4B) + (A^2B)\alpha_3' \quad \in \quad \operatorname{End}(\mathcal{W}_{m'}). \tag{5.14}
$$

Similarly, to determine $\partial_1: \operatorname{End}(\mathcal{W}_m) \to \operatorname{End}(\mathcal{W}_{x_1}) \oplus \operatorname{End}(\mathcal{W}_{x_2}) \oplus \operatorname{End}(\mathcal{W}_{x_3})$ we look for pearly configurations containing one Maslov 2 disc u such that $u(-1) \in W^d(x_i)$ and $u(1) \in W^a(m) = \{m\}$. From Theorem 5.1.5 we know that these discs must be axial. Consulting figure 5.3 we see that their axes are $\{V_1, V_2, V_3\}$ and, denoting by u_i the disc with axis V_i, we have: $\gamma^0_{u_1} = (\tilde{\gamma}_1)^{-1}$, $\gamma^1_{u_1} = \gamma_1$, $\gamma^0_{u_2} = \gamma_2^{-1}$, $\gamma^1_{u_2} = \tilde{\gamma}_2$, $\gamma^0_{u_3} = (\tilde{\gamma}_3)^{-1}$, $\gamma^1_{u_3} = \gamma_3$. Again from the table (5.8) we see

$$
\begin{array}{lll}
P_{\gamma^0_{u_1}} = (A^3B)^{-1} = B, & P_{\gamma^0_{u_2}} = \mathrm{Id}, & P_{\gamma^0_{u_3}} = (A^5B)^{-1} = A^2B \\
P_{\gamma^1_{u_1}} = \mathrm{Id}, & P_{\gamma^1_{u_2}} = A^4B, & P_{\gamma^1_{u_3}} = \mathrm{Id}.
\end{array}
$$

Thus for every $\alpha \in \operatorname{End}(\mathcal{W}_m)$ we have:

$$
\partial_1(\alpha) = \big(\alpha B, (A^4B)\alpha, \alpha(A^2B)\big) \quad \in \quad \operatorname{End}(\mathcal{W}_{x_1}) \oplus \operatorname{End}(\mathcal{W}_{x_2}) \oplus \operatorname{End}(\mathcal{W}_{x_3}). \tag{5.15}
$$

Determining $\partial_1: \operatorname{End}(\mathcal{W}_{x_1}) \oplus \operatorname{End}(\mathcal{W}_{x_2}) \oplus \operatorname{End}(\mathcal{W}_{x_3}) \to \operatorname{End}(\mathcal{W}_{x_1'}) \oplus \operatorname{End}(\mathcal{W}_{x_2'}) \oplus \operatorname{End}(\mathcal{W}_{x_3'})$ requires a bit more work. In the case of trivial local systems Evans-Lekili are able to deduce that this part of the differential must be zero by a simple algebraic argument using the fact that the whole pearly differential has to square to zero ([EL15, Corollary 7.2.4]). We cannot appeal to such an argument in our case (indeed, for specific choices of $\mathcal{W}$ this part of the differential is non-zero, see section 5.1.4 below) and so we must analyse all relevant pearly trajectories. That is, we are looking at pearly trajectories consisting of one Maslov 2 disc u, satisfying $u(-1) \in W^d(x_i')$ and $u(1) \in W^a(x_j)$, as depicted in figure 5.7. To find all such trajectories we consider again figure 5.3 and argue in terms of triangles inscribed in the unit sphere in $\mathbb{R}^3$.

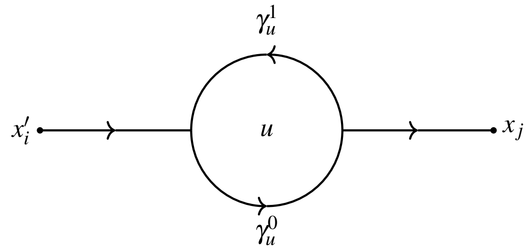


Figure 5.7: A pearly trajectory $\mathbf{u} = (u)$ connecting x_i' to x_j.

Theorem 5.1.5 and our choice of Morse function give us that each of the following three sets

- the descending manifold of x_i'
- the boundary of the Maslov 2 disc u
- the ascending manifold of x_j

consists of triangles, obtained from a single maximal equilateral triangle by applying a rotation which keeps one of its vertices fixed. In fact we know that for the descending manifold of x_i' the fixed vertex is V_i' and for the ascending manifold of x_j, it is V_j. For any unit vector $p \in S^2(1)$, let S_p^1 denote the circle, obtained by intersecting $S^2(1)$ with the plane $p^\perp - \frac{1}{2}p$. Then, since $u(-1)$ lies on the descending manifold of x_i', we have that one of the vertices of $u(-1)$ is V_i' and the other two lie on $S^1_{V_i'}$. Similarly one of the vertices of $u(1)$ is V_j and the other two lie on $S^1_{V_j}$. Let us temporarily denote the axis of u by $A \in S^2(1)$. Note then that A is a vertex which all triangles in $u(\partial D^2)$ share, in particular, it is a vertex of both $u(-1)$ and $u(1)$. So $V_i' \neq A \neq V_j$ and we see that we must have $A \in S^1_{V_i'} \cap S^1_{V_j}$. From figure 5.3, this is equivalent to $j \equiv i$ or $i+1 \pmod 3$ and $A = F_{ij}$ or $A = B_{ij}$. Let us denote by $u^{F_{ij}}$ and $u^{B_{ij}}$ the Maslov 2 axial discs with axes F_{ij} and B_{ij} respectively.

Proposition 5.1.8. *For every $i, j \in \{1,2,3\}$ with $j \equiv i$ or $i+1 \pmod 3$, there are precisely two pearly trajectories $\mathbf{u}^{F_{ij}}$ and $\mathbf{u}^{B_{ij}}$ connecting x_i' to x_j. They are given by $\mathbf{u}^{F_{ij}} = (u^{F_{ij}})$ and $\mathbf{u}^{B_{ij}} = (u^{B_{ij}})$. If $j \equiv i+2 \pmod 3$, there are no pearly trajectories connecting x_i' to x_j.*

Proof. Our discussion above already proves the uniqueness and non-existence parts of the proposition. We only need to show that $u^{F_{ij}}$ and $u^{B_{ij}}$ do indeed give rise to pearly trajectories from x_i'

to x_j when $j \equiv i$ or $i+1 \pmod 3$. That is, we need to show that both $u^{F_{ij}}(\partial D^2)$ and $u^{B_{ij}}(\partial D^2)$ intersect the ascending manifold of x_j. To rephrase this once again, we need to show that whenever $j \equiv i$ or $i+1 \pmod 3$, there exists one maximal equilateral triangle with V_j and F_{ij} among its vertices and another one with V_j and B_{ij} among its vertices. From figure 5.3, we can immediately see the only two such triangles, namely $\triangle V_j F_{ij} B_{jj}$ and $\triangle V_j F_{jj} B_{ij}$, where $j \cong i+1 \pmod 3$. $\square$

It is worth looking carefully at the rotations that one needs to apply to a triangle $\triangle V_i' F_{ii} B_{ij}$ or $\triangle V_i' F_{ij} B_{ii}$ in order to land in the family of maximal equilateral triangles with V_i or V_j as a vertex. This will allow us to parametrise the boundaries $u^{F_{ij}}(\partial D^2)$ and $u^{B_{ij}}(\partial D^2)$. Let us do this only for $u^{B_{11}}$ since all other calculations are identical by the symmetries of figure 5.3.

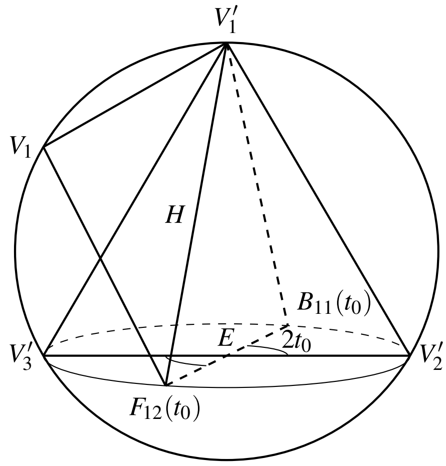


Figure 5.8: Parametrising disc boundaries.

For all $t \in (0, \pi/4]$ one of the vertices of $\exp(tV_1') \cdot \Delta$ is V_1' and the other two lie on $S^1_{V_1'}$. Let us denote these two vertices by $F_{12}(t)$ and $B_{11}(t)$, where $F_{12}(t)$ is the one with positive x-coordinate (the letters F and B are to be read as "front" and "back"; see figure 5.3). Define $c(t) := \cos(\measuredangle V_1' V_1 F_{12}(t))$. Let E denote the midpoint of the line segment $V_2'V_3'$, i.e. the centre of $S^1_{V_1'}$. Then $\measuredangle B_{11}(t) E V_2' = 2t$ and so $\measuredangle V_2' E F_{12}(t) = \pi - 2t$. By the cosine rule for $\triangle V_2' E F_{12}(t)$ we have:

$$\begin{aligned} |F_{12}(t)V_2'|^2 &= |EV_2'|^2 + |EF_{12}(t)|^2 - 2|EV_2'||EF_{12}(t)|\cos(\pi - 2t) \\ &= \frac{3}{2}(1+\cos(2t)) \;=\; 3\cos^2(t). \end{aligned}$$

Then from Pythagoras's theorem for $\triangle V_1 F_{12}(t) V_2'$ we get

$$|V_1 F_{12}(t)|^2 = 4 - 3\cos^2(t). \tag{5.16}$$

On the other hand, the cosine rule for $\triangle V_1' V_1 F_{12}(t)$ gives

$$|V_1' F_{12}(t)|^2 = |V_1' V_1|^2 + |V_1 F_{12}(t)|^2 - 2|V_1' V_1||V_1 F_{12}(t)| c(t). \tag{5.17}$$

Substituting (5.16), together with $|V_1' V_1| = 1$ and $|V_1' F_{12}(t)| = \sqrt{3}$ into (5.17), we obtain

$$c(t) = \frac{2 - 3\cos^2(t)}{2\sqrt{4 - 3\cos^2(t)}}.$$

Set $t_0 := \arccos\left(\frac{\sqrt{2}}{\sqrt{3}}\right)$ and let $S^2_{V_1' F_{12}(t_0)}$ denote the sphere in $\mathbb{R}^3$ whose diameter is the line segment $V_1' F_{12}(t_0)$. Since $c(t_0) = 0$ we have that $\measuredangle V_1' V_1 F_{12}(t_0)$ is a right angle and so

$$V_1 \in S^2_{V_1' F_{12}(t_0)} \cap S^2(1) = S^1_{B_{11}(t_0)}. \tag{5.18}$$

Put $t_1 := \frac{\pi}{2} - t_0$. Note that $\cos(\measuredangle V_1 V_1' F_{12}(t_0)) = |V_1 V_1'|/|V_1' F_{12}(t_0)| = \frac{1}{\sqrt{3}} = \cos(t_1)$. Thus we must have $\cos(\measuredangle V_1 V_1' F_{12}(t_0)) = t_1$ and so, if H is the midpoint of $V_1' F_{12}(t_0)$ (i.e. the centre of $S^1_{B_{11}(t_0)}$), then $\measuredangle F_{12}(t_0) H V_1 = 2t_1$. From this and (5.18) we deduce that a right-hand rotation by $2t_1$ in the axis $B_{11}(t_0)$ sends the point $F_{12}(t_0)$ to V_1. In other words, acting by $\exp(t_1 B_{11}(t_0))$ on the triangle $\triangle F_{12}(t_0) V_1' B_{11}(t_0) = \exp(t_0 V_1') \cdot \Delta$ gives a triangle, one of whose vertices is V_1 and hence the other two (among which is $B_{11}(t_0)$) lie on $S^1_{V_1}$. First of all, this shows that $B_{11}(t_0) \in S^1_{V_1} \cap S^1_{V_1'}$ and so $B_{11}(t_0) = B_{11}$; by symmetry $F_{12}(t_0) = F_{12}$. Second, we see that the axial Maslov 2 disc with axis B_{11} intersects the descending manifold of x_1' at the point $q(\exp(t_0 V_1'))$ (that is, $\triangle V_1' F_{12} B_{11}$) and the ascending manifold of x_1 at the point $q(\exp(t_1 B_{11}) \cdot \exp(t_0 V_1'))$ (that is, $\triangle V_1 F_{31} B_{11}$).

In this way, we can parametrise all the paths $\gamma^*_{u\square}$ for $* = 0$ or 1 and $\square = F_{ij}$ or B_{ij}. For example for $\square = F_{12}$ or B_{11} we get

$$\begin{aligned} \gamma^0_{u\square} &= q\left(\exp(t\square) \cdot \exp\left(t_0 V_1'\right)\right), \quad t \in [0, t_1] \\ \gamma^1_{u\square} &= q\left(\exp(t\square) \cdot \exp\left(t_0 V_1'\right)\right), \quad t \in [t_1, \pi/2] \end{aligned}$$

and for $\square = F_{11}$ or B_{12} we get

$$\begin{aligned} \gamma^0_{u\square} &= q\left(\exp(t\square) \cdot \exp\left(t_1 V_1'\right)\right), \quad t \in [0, t_0] \\ \gamma^1_{u\square} &= q\left(\exp(t\square) \cdot \exp\left(t_1 V_1'\right)\right), \quad t \in [t_0, \pi/2]. \end{aligned}$$

Using these and the parametrisations for the index 1 flowlines of f, described in section 5.1.2, one can plot the lifts of the paths $\gamma^*_{\mathbf{u}}$, associated with the pearly trajectories $\mathbf{u} = (u^{F_{ij}})$ or $\mathbf{u} = (u^{B_{ij}})$. Two such lifts are shown in figure 5.9 and figure 5.10 and the Mathematica code used to produce all plots can be found in the UCL repository for this thesis.

From these plots the parallel transport maps are immediate to read off. Applying this procedure to all 12 pearly trajectories $\{\mathbf{u}^{F_{ij}}, \mathbf{u}^{B_{ij}} \,:\, i \equiv j, j+1 \pmod 3\}$ we obtain the results summarised in table 5.1. We have thus computed that for every $(\alpha_1, \alpha_2, \alpha_3) \in \mathrm{End}(\mathcal{W}_{x_1}) \oplus \mathrm{End}(\mathcal{W}_{x_2}) \oplus \mathrm{End}(\mathcal{W}_{x_3})$ we have

$$\begin{aligned} \partial_1(\alpha_1, \alpha_2, \alpha_3) \quad &= \quad \Big((A^2B)\alpha_1 + \alpha_1(A^4B) + (A^2B)\alpha_2 + A^5\alpha_2(A^5B), \\ &\qquad (A^2B)\alpha_2 + \alpha_2 B + (A^3B)\alpha_3 A + \alpha_3 B, \\ &\qquad (A^4B)\alpha_3 + \alpha_3 B + A\alpha_1(A^3B) + (A^5B)\alpha_1 A^5\Big) \\ &\in \quad \mathrm{End}(\mathcal{W}_{x_1'}) \oplus \mathrm{End}(\mathcal{W}_{x_2'}) \oplus \mathrm{End}(\mathcal{W}_{x_3'}). \end{aligned} \tag{5.19}$$

Finally, let us determine $\partial_2\colon \mathrm{End}(\mathcal{W}_m) \to \mathrm{End}(\mathcal{W}_{m'})$ which counts pearly trajectories of total Maslov index 4 connecting m' to m. We claim that such a trajectory cannot consist of a pair of Maslov 2 discs (v_1, v_2). Indeed, otherwise, we would need to have $v_1(-1) = m'$, $v_2(1) = m$ and

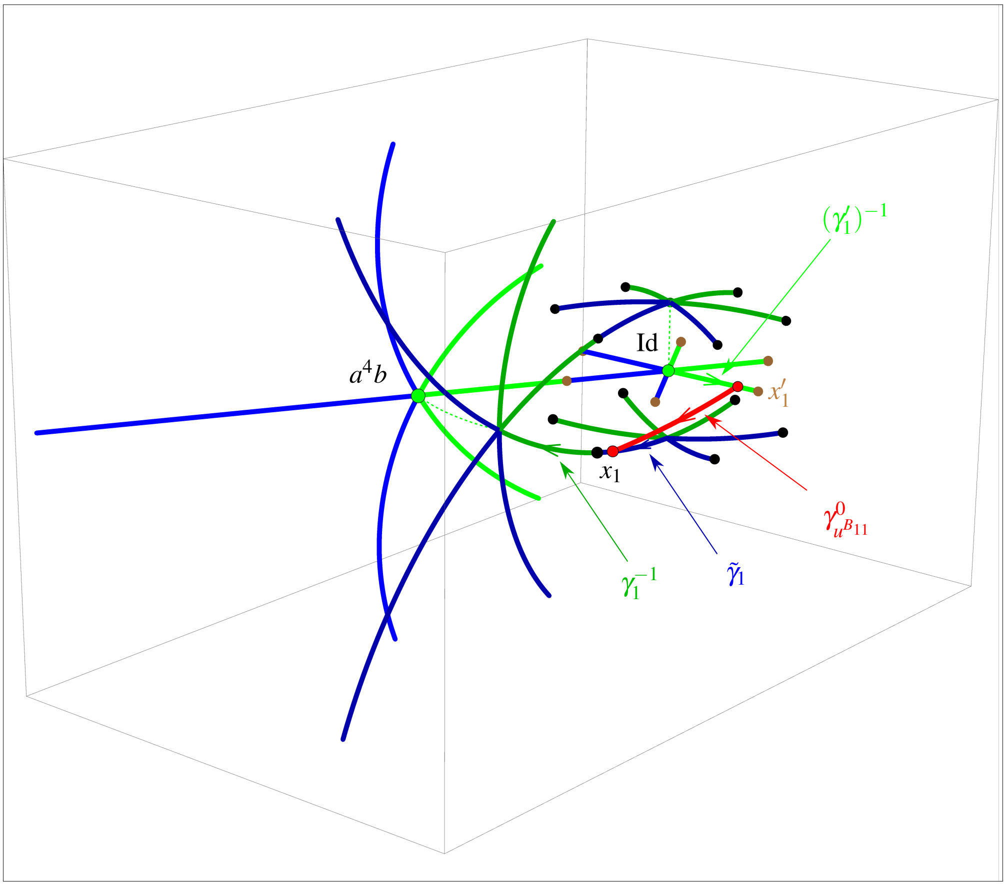


Figure 5.9: A lift of the path $\gamma^0_{u^{B_{11}}}$ together with the descending manifolds of the index 1 critical points and the ascending manifolds for index 2 critical points for the fundamental domains centred at Id and a^4b. Lifts of the index 3 flowline σ are represented by the dashed line segments. From this plot one reads off that $P_{\gamma^0_{\mathbf{u}^{B_{11}}}} = A^4B$.

$v_1(1) > v_2(-1)$. The first two conditions and Theorem 5.1.5 imply that we must have $v_1 = u_i'$, $v_2 = u_j$ for some $i, j \in \{1,2,3\}$ and so $v_1(\partial D^2) = W^d(x_i')$, $v_2(\partial D^2) = W^a(x_j)$. But this contradicts the requirement $v_1(1) > v_2(-1)$, because by formula (5.6) we have that $f(x_i') = -\frac{1}{2} < \frac{1}{2} = f(x_j)$ for all $i, j \in \{1,2,3\}$.

So a Maslov 4 pearly trajectory must consist of a single Maslov 4 disc u, satisfying $u(-1) = m'$ and $u(1) = m$. Thus u must be one of the discs $\{w_1, w_{-1}\}$ described in Theorem 5.1.6. Figure 5.3 can be used to infer that $\gamma^0_{w_1} = (\sigma)^{-1}$, $\gamma^1_{w_1} = \tilde{\sigma}$, $\gamma^0_{w_{-1}} = (\tilde{\sigma})^{-1}$ and $\gamma^1_{w_{-1}} = \sigma$. It is clear then that

$$P_{\gamma^0_{w_1}} = \mathrm{Id}, \quad P_{\gamma^1_{w_1}} = A, \quad P_{\gamma^0_{w_{-1}}} = A^5, \quad P_{\gamma^1_{w_{-1}}} = \mathrm{Id}$$

and hence for every $\alpha \in \mathrm{End}(\mathcal{W}_m)$ we have:

$$\partial_2(\alpha) = A\alpha + \alpha A^5 \quad \in \quad \mathrm{End}(\mathcal{W}_{m'}). \tag{5.20}$$

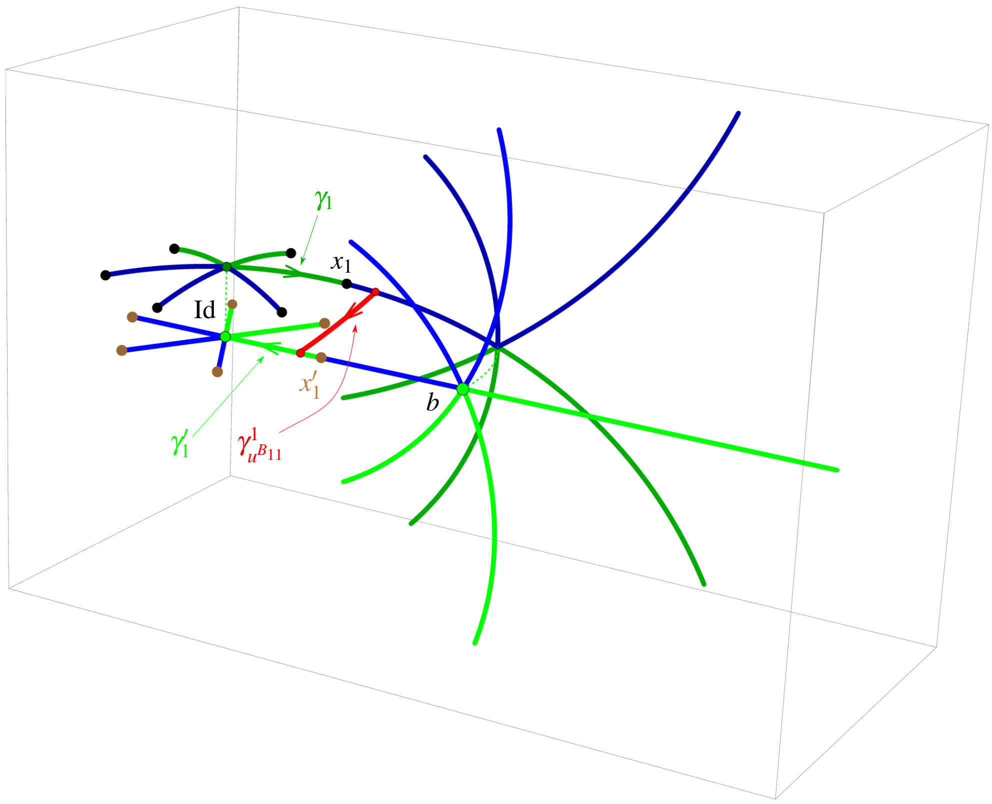


Figure 5.10: A lift of the path $\gamma^1_{u^{B_{11}}}$ with the fundamental domains centred at Id and b. From this plot one reads off that $P_{\gamma^1_{\mathbf{u}^{B_{11}}}} = \mathrm{Id}$.

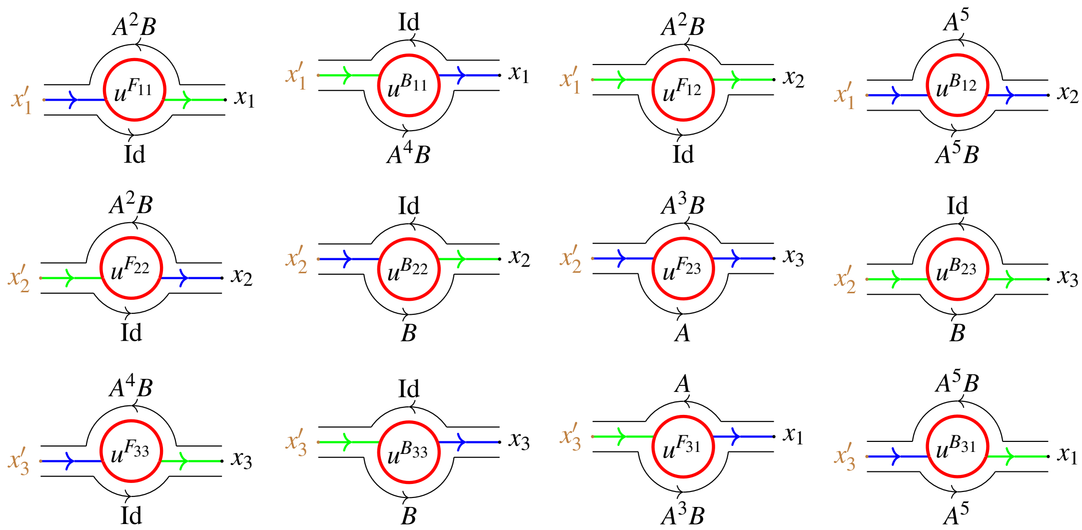


Table 5.1: Parallel transport maps for the pearly trajectories $\{\mathbf{u}^{F_{ij}}, \mathbf{u}^{B_{ij}} \ : \ i \equiv j, j+1 \pmod 3\}$.

We end this section by writing down the complete candidate Floer differential $d = d^{(\mathscr{D},J_0)}$. We set the formal variable T to 1, collapsing the $\mathbb{Z}$-grading on $C^*_f(\mathcal{W},\mathcal{W})$ to a $(\mathbb{Z}/2)$-grading and we write

$$\begin{aligned}
d^0\colon \bigoplus_{\square=m',x_1,x_2,x_3} \mathrm{End}(\mathcal{W}_\square) &\to \bigoplus_{\square=x'_1,x'_2,x'_3,m} \mathrm{End}(\mathcal{W}_\square)\\
d^1\colon \bigoplus_{\square=x'_1,x'_2,x'_3,m} \mathrm{End}(\mathcal{W}_\square) &\to \bigoplus_{\square=m',x_1,x_2,x_3} \mathrm{End}(\mathcal{W}_\square)
\end{aligned}$$

for the two components of the map d. Now from equations (5.9), (5.10), (5.11), (5.14), (5.15), (5.19) and (5.20) we have:

- for every $(\alpha',\alpha_1,\alpha_2,\alpha_3) \in \mathrm{End}(\mathcal{W}_{m'})\oplus\mathrm{End}(\mathcal{W}_{x_1})\oplus\mathrm{End}(\mathcal{W}_{x_2})\oplus\mathrm{End}(\mathcal{W}_{x_3})$

$$\begin{aligned}
d^0(\alpha',\alpha_1,\alpha_2,\alpha_3) &= \Big(\partial_0(\alpha')+\partial_1(\alpha_1,\alpha_2,\alpha_3),\, \partial_0(\alpha_1,\alpha_2,\alpha_3)\Big)\\
&= \Big(\big[\alpha'+B^{-1}\alpha' B\big]+(A^2B)\alpha_1+\alpha_1(A^4B)+(A^2B)\alpha_2+A^5\alpha_2(A^5B),\\
&\quad \big[\alpha'+(AB)^{-1}\alpha'(AB)\big]+(A^2B)\alpha_2+\alpha_2 B+(A^3B)\alpha_3 A+\alpha_3 B,\\
&\quad \big[\alpha'+(A^2B)^{-1}\alpha'(AB)\big]+(A^4B)\alpha_3+\alpha_3 B+A\alpha_1(A^3B)+(A^5B)\alpha_1 A^5,\\
&\quad \big[\alpha_1+(A^3B)^{-1}\alpha_1(A^3B)+\alpha_2+(A^4B)^{-1}\alpha_2(A^4B)+\\
&\qquad\qquad +\alpha_3+(A^5B)^{-1}\alpha_3(A^5B)\big]\Big)\\
&\in \mathrm{End}(\mathcal{W}_{x'_1})\oplus\mathrm{End}(\mathcal{W}_{x'_2})\oplus\mathrm{End}(\mathcal{W}_{x'_3})\oplus\mathrm{End}(\mathcal{W}_m).
\end{aligned} \tag{5.21}$$

- for every $(\alpha'_1,\alpha'_2,\alpha'_3,\alpha) \in \mathrm{End}(\mathcal{W}_{x'_1})\oplus\mathrm{End}(\mathcal{W}_{x'_2})\oplus\mathrm{End}(\mathcal{W}_{x'_3})\oplus\mathrm{End}(\mathcal{W}_m)$

$$\begin{aligned}
d^1(\alpha'_1,\alpha'_2,\alpha'_3,\alpha) &= \Big(\partial_1(\alpha'_1,\alpha'_2,\alpha'_3)+\partial_2(\alpha),\, \partial_0(\alpha'_1,\alpha'_2,\alpha'_3)+\partial_1(\alpha)\Big)\\
&= \Big(B\alpha'_1+\alpha'_2(A^4B)+(A^2B)\alpha'_3+\big[A\alpha+\alpha A^5\big],\\
&\quad \big[\alpha'_1+\alpha'_2+A^{-1}\alpha'_2 A+A^{-1}\alpha'_3 A\big]+\alpha B,\\
&\quad \big[(A^2B)^{-1}\alpha'_1(A^2B)+\alpha'_2+A^{-1}\alpha'_3 A+\alpha'_3\big]+(A^4B)\alpha,\\
&\quad \big[(A^2B)^{-1}\alpha'_1(A^2B)+(A^3B)^{-1}\alpha'_1(A^3B)+\\
&\qquad\qquad +(A^3B)^{-1}\alpha'_2(A^3B)+\alpha'_3\big]+\alpha(A^2B)\Big)\\
&\in \mathrm{End}(\mathcal{W}_{m'})\oplus\mathrm{End}(\mathcal{W}_{x_1})\oplus\mathrm{End}(\mathcal{W}_{x_2})\oplus\mathrm{End}(\mathcal{W}_{x_3}).
\end{aligned} \tag{5.22}$$

With these expressions at hand we are now in a position to prove Theorem E.

5.1.4 Proof of non-displaceability

In appendix B we give an explicit description of all indecomposable representations of Γ_Δ over $\mathbb{F}_2$. We find that there is a unique non-trivial irreducible representation which we denote by D. It is a two-dimensional faithful representation of the dihedral group of order 6 and its pullback to Γ_Δ is given explicitly by

$$\begin{aligned}
\rho_D\colon \Gamma_\Delta &\to \mathrm{GL}(2,\mathbb{F}_2)\\
\rho_D(a) = \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}, &\qquad \rho_D(b) = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}.
\end{aligned} \tag{5.23}$$

Let $\mathcal{W}^D$ denote the resulting local system on L_Δ. It is immediate to check from (5.13) that one has $m_0(\mathcal{W}^D) = 0$ and thus the complex $\left(C_f^*(\mathcal{W}^D, \mathcal{W}^D), d^{(\mathcal{F},J_0)}\right)$ is unobstructed. Further, since ρ_D is surjective, we have

$$C_f^*(\mathcal{W}^D, \mathcal{W}^D) = \overline{C}_f^*(\mathcal{W}^D, \mathcal{W}^D) = C_{f,\mathrm{mon}}^*(\mathcal{W}^D).$$

A direct calculation shows the following.

Proposition 5.1.9. *We have $HF_{\mathrm{BC}}^0(\mathcal{W}^D, \mathcal{W}^D) \cong HF_{\mathrm{BC}}^1(\mathcal{W}^D, \mathcal{W}^D) \cong (\mathbb{F}_2)^2$.*

Proof. We identify both $\bigoplus_{\square=m',x_1,x_2,x_3} \mathrm{End}(\mathcal{W}^D_\square)$ and $\bigoplus_{\square=x_1',x_2',x_3',m} \mathrm{End}(\mathcal{W}^D_\square)$ with $\mathrm{End}((\mathbb{F}_2)^2)^4 = (\mathrm{Mat}_{2\times 2}(\mathbb{F}_2))^4$. For $(\mathrm{Mat}_{2\times 2}(\mathbb{F}_2))^4$ we choose a basis as follows:

- set $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
- for each $l \in \mathbb{N}$ write $l = 4(l) + \langle l \rangle$ for the division with remainder of l by 4
- for each $1 \leq l \leq 16$ and $1 \leq k \leq 4$ define the matrix $\mathcal{B}_{lk} \in \mathrm{Mat}_{2\times 2}(\mathbb{F}_2)$ by

$$\mathcal{B}_{lk} := \begin{cases} e_{\langle l \rangle}, & \text{when } \langle l \rangle \neq 0 \text{ and } k = (l) + 1 \\ e_4, & \text{when } \langle l \rangle = 0 \text{ and } k = (l) \\ \text{the zero matrix}, & \text{otherwise.} \end{cases}$$

- define a basis $\mathcal{B} := \{\mathcal{B}_l \,:\, 1 \leq l \leq 16\}$ for $(\mathrm{Mat}_{2\times 2}(\mathbb{F}_2))^4$ by setting $\mathcal{B}_l := (\mathcal{B}_{l1}, \mathcal{B}_{l2}, \mathcal{B}_{l3}, \mathcal{B}_{l4})$.

For example

$$\mathcal{B}_1 = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right), \ \mathcal{B}_6 = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right).$$

Plugging (5.21) and (5.22) into a computer programme (the Mathematica code can be found in the UCL repository for this thesis), one finds that with respect to the basis $\mathcal{B}$ the operators d^0 and d^1 are given respectively by the matrices

$$D^0 = \begin{pmatrix}
0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1
\end{pmatrix}$$

$$D^1 = \begin{pmatrix}
1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0
\end{pmatrix}$$

One then computes that $\mathrm{rank}(D^0) = 6$, $\mathrm{rank}(D^1) = 8$ and thus

$$HF^0_{\mathrm{BC}}((L_\Delta, \mathcal{W}^D), (L_\Delta, \mathcal{W}^D)) \cong HF^1_{\mathrm{BC}}((L_\Delta, \mathcal{W}^D), (L_\Delta, \mathcal{W}^D)) \cong (\mathbb{F}_2)^2. \qquad \square$$

Remark 5.1.10. It is a fact (see section 5.1.5 below) that $\mathcal{W}^D$ is the minimal $\mathbb{F}_2$-local system on L_Δ for which the central Floer cohomology is non-vanishing. By this we mean that any other finite rank local system $\mathcal{W} \to L_\Delta$ with $\overline{HF}^*(\mathcal{W}, \mathcal{W}) \neq 0$ must have $\mathcal{W}^D$ as a direct summand. //

Corollary 5.1.11. *The Floer cohomology $HF^*(\mathbb{RP}^3, (L_\Delta, \mathcal{W}^D))$ is non-zero and so L_Δ and $\mathbb{RP}^3$ cannot be displaced by a Hamiltonian diffeomorphism of $\mathbb{CP}^3$.*

Proof. In the setup of the split-generation criterion (Theorem 2.3.13) set $L = \mathbb{RP}^3$ and $\mathcal{E}$ to be the trivial $\mathbb{F}_2$-local system of rank 1. By [Ton18, Proposition 1.1] the map $\mathcal{CO}^* \colon QH^*(\mathbb{CP}^3) \to HH^*(CF^*(\mathbb{RP}^3, \mathbb{RP}^3))$ is injective and so $\mathbb{RP}^3$ split-generates $\mathcal{F}(\mathbb{CP}^3)$. From Proposition 5.1.9 we have that $(L_\Delta, \mathcal{W}^D)$ is an essential object in $\mathcal{F}(\mathbb{CP}^3)$. By Fact 2.3.12 it then follows that $HF^*(\mathbb{RP}^3, (L_\Delta, \mathcal{W}^D)) \neq 0$. $\square$

5.1.5 Some additional calculations

In this section, we fully describe the behaviour of the central and monodromy Floer complexes for all indecomposable $\mathbb{F}_2$-local systems on L_Δ. In appendix B we show that there are 6 isomorphism classes of indecomposable representations of Γ_Δ over $\mathbb{F}_2$. These fall into the following 3 groups:

1. the representations V_1, V_2, V_3, V_4 (with $\dim V_j = j$) which are the indecomposable representations of the cyclic group C_4 over $\mathbb{F}_2$ with V_1 being the trivial one and V_4 being the regular representation. Since C_4 is the quotient of Γ_Δ by $C_3 = \{1, a^2, a^4\}$, these are also representations of Γ_Δ and are given by

$$\rho_1(b) = \mathrm{Id}, \quad \rho_2(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_3(b) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad \rho_4(b) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{5.24}$$

2. the irreducible representation D given by (5.23),

3. a faithful Γ_Δ-representation U_4 of dimension 4, given by

$$\rho_{U_4}(a) = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \rho_{U_4}(b) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{5.25}$$

For convenience, let us drop the notational distinction between a representation of Γ_Δ and a local system on L_Δ. Using (5.13) one computes that (L_Δ, D) and (L_Δ, U_4) are objects of $\mathcal{F}(\mathbb{CP}^3)_0$, while $\forall j \in \{1,2,3,4\}$ the pair (L_Δ, V_j) is an object of $\mathcal{F}(\mathbb{CP}^3)_1^{\mathrm{nil}}$ and $m_0(V_j)(m') = \rho_j(b)$. Since $1 \notin \mathrm{Spec}(c_1(\mathbb{CP}^3)\star) = \{0\}$, then from Proposition 2.3.10 and Proposition 2.3.11 we immediately have:

1) $\overline{CF}^*(V_j, D) \cong \overline{CF}^*(V_j, U_4) \cong 0 \quad \forall j \in \{1,2,3,4\}$

2) $HF^*_{\mathrm{mon}}(V_j) \cong \overline{HF}^*(V_j, V_j) \cong 0 \quad \forall j \in \{1,2,3,4\}$

Note further that the Floer complex is obstructed for V_j when $j \in \{2,3,4\}$ and unobstructed otherwise. Using our general expressions for the Morse and Floer differentials (5.9), (5.10), (5.11), (5.21) and (5.22), one can also compute:

3) $H^i(L_\Delta; \mathcal{Z}_{m_0}(V_j, V_j)) \cong H^i(L_\Delta; \mathscr{E}nd_{mon}(V_j)) \cong (\mathbb{F}_2)^j \quad \forall i \in \{0,1,2,3\} \quad \forall j \in \{1,2,3,4\}$

4) $H^i(L_\Delta; \mathcal{Z}_{m_0}(D, D)) \cong H^i(L_\Delta; \mathscr{E}nd(D)) \cong H^i(L_\Delta; \mathscr{E}nd_{mon}(D)) \cong \mathbb{F}_2 \quad \forall i \in \{0,1,2,3\}$

5) $H^i(L_\Delta; \mathcal{Z}_{m_0}(U_4, U_4)) \cong H^i(L_\Delta; \mathscr{E}nd(U_4)) \cong \begin{cases} (\mathbb{F}_2)^2, & i \in \{0,3\} \\ 0, & i \in \{1,3\} \end{cases}$

6) $H^i(L_\Delta; \mathscr{E}nd_{mon}(U_4)) \cong (\mathbb{F}_2)^2 \quad \forall i \in \{0,1,2,3\}$

7) $HF^i_{\mathrm{BC}}(D, D) \cong HF^i_{\mathrm{BC,mon}}(D) \cong (\mathbb{F}_2)^2 \quad \forall i \in \{0,1\}$

8) $HF^i_{\mathrm{BC}}(U_4, U_4) \cong 0 \quad \forall i \in \{0,1\}$

9) $HF^i_{\mathrm{BC,mon}}(U^4) \cong (\mathbb{F}_2)^4 \quad \forall i \in \{0,1\}$

Note first from 2), 7) and 8) that D is the only indecomposable representation of Γ_Δ whose associated local system has non-vanishing (central) Floer cohomology. This gives the minimality claim we made in Remark 5.1.10.

Second, we observe that even though U_4 dominates D (see the end of appendix B), (L_Δ, U_4) is a trivial object in $\mathcal{F}(\mathbb{CP}^3)_0$. However, $HF^*_{\mathrm{mon}}(U_4) \neq 0$, as expected from the fact that (L_Δ, D) is an essential object.

Remark 5.1.12. Another point to note here is that the discrepancy

$$H^*(L_\Delta; \mathscr{E}nd_{mon}(V_3)) \ncong H^*(L_\Delta; \mathscr{E}nd_{mon}(V_4))$$

which we get from 3) is evidence that the domination relation from Definition 2.2.24 really needs the inclusion of kernels to be at the level of group rings, rather than groups for Proposition 2.2.25 to hold (recall Remark 2.2.27). Indeed, the group homomorphisms $\Gamma_\Delta \to \mathrm{End}_{\mathbb{F}_2}(V_j)$ have the same kernels for $j \in \{3,4\}$, namely $\{1, a^2, a^4\}$. If the conclusions of Proposition 2.2.25 were true in this situation, then the map (2.56) would define an isomorphism of the Morse complexes $(C^*_f(L; \mathscr{E}nd_{mon}(V_3)), \partial^{\mathscr{D}})$ and $(C^*_f(L; \mathscr{E}nd_{mon}(V_4)), \partial^{\mathscr{D}})$ which have different homologies. //

Further, it is not hard to see that the regular representation of Γ_Δ is isomorphic to $V_4 \oplus U_4 \oplus U_4$ (see the end of appendix B) and direct calculation shows that [4]

$$\begin{aligned} H^i(L_\Delta; \mathscr{E}nd_{mon}(\mathcal{E}^{\mathbb{F}_2}_{\mathrm{reg}})) &\cong (\mathbb{F}_2)^6 \quad \forall i \in \{0,1,2,3\} \\ HF^i_{\mathrm{BC,mon}}(L_\Delta; \mathbb{F}_2) &\cong (\mathbb{F}_2)^4 \quad \forall i \in \{0,1\}. \end{aligned} \tag{5.26}$$

It is worth noting that, by 1), 2) and 8) one has $\overline{HF}^*(\mathcal{E}^{\mathbb{F}_2}_{\mathrm{reg}}, \mathcal{E}^{\mathbb{F}_2}_{\mathrm{reg}}) = 0$ but one can detect the non-displaceability of L_Δ from the non-vanishing of $HF^*_{\mathrm{mon}}(L_\Delta; \mathbb{F}_2)$.

Finally note that from 1), 2), 3) and 7) we see that we can add copies of V_j to D and this will increase the dimension of $H^*(L_\Delta; \mathcal{Z}_{m_0}(\cdot,\cdot))$, but not that of the central Floer cohomology. That is, when using such local systems, the corrections to the Morse differential on $\overline{C}^*_f(V,V)$ coming from holomorphic discs are non-trivial but they also do not kill the cohomology entirely. As (5.26) shows, the same behaviour is exhibited by the monodromy complex.

5.2 Orientable subadjoint Lagrangians

We now consider twistor Lagrangians in $\mathbb{CP}^{2n+1}$ with $n \geq 2$. By Lemma 4.1.30, their minimal Maslov number is $n+1 \geq 3$ and so there are no obstructions to Floer theory with high rank local systems. This fact, together with a simple dimension count, allows us to prove the following.

Lemma 5.2.1. *Let $n \geq 2$ and let $X \subseteq \mathbb{CP}^{2n+1}$ be a Type 2 Legendrian subvariety. Suppose that*

$$\dim(H^n(X;\mathbb{F}_2)) + \dim\left(H^{n-1}(X;\mathbb{F}_2)\right) \geq 3 + \dim\left(H^1(X;\mathbb{F}_2)\right). \tag{5.27}$$

Then $HF^*_{\mathrm{BC}}(Z_X, Z_X; \mathbb{F}_2) \neq 0$.

Proof. We consider $HF^*_{\mathrm{BC}}(Z_X, (Z_X, \mathcal{E}_{Y_X}))$, where $\mathcal{E}_{Y_X}$ is the rank two $\mathbb{F}_2$-local system, associated to the double cover $Y_X \to Z_X$. The spectral sequence which computes this Floer cohomology degenerates on the third page and its first page is built out of the cohomology of Y_X. More precisely, it is given by

$$\cdots \to H^*(Y_X;\mathbb{F}_2)[2n] \to H^*(Y_X;\mathbb{F}_2)[n] \to H^*(Y_X;\mathbb{F}_2) \to H^*(Y_X;\mathbb{F}_2)[-n] \to H^*(Y_X;\mathbb{F}_2)[-2n] \to \dots$$

(each term represents a column, the square brackets denote grading shift as usual, and the arrows represent the differential which maps horizontally from one column to the next). In particular, the

[4] Note that the cover $\widehat{L}_\Delta$ which we considered at the end of section 2.4.3 is $\widehat{L}_\Delta = L_\Delta \sqcup L_\Delta \sqcup L(6,1) \sqcup L(6,1) \sqcup \mathbb{RP}^3 \sqcup \mathbb{RP}^3$.

non-trivial piece of the zeroth row on the first page is

$$0 \longrightarrow H^{2n}(Y_X;\mathbb{F}_2) \longrightarrow H^n(Y_X;\mathbb{F}_2) \longrightarrow H^0(Y_X;\mathbb{F}_2) \longrightarrow 0.$$

Note that, if this 5-term sequence is not exact at the middle position $E_1^{-1,0} = H^n(Y_X;\mathbb{F}_2)$, then for degree reasons there is no differential on the next page of the spectral sequence which can kill the cohomology. We conclude that, if

$$\dim\left(H^n(Y_X;\mathbb{F}_2)\right) > \dim\left(H^{2n}(Y_X;\mathbb{F}_2)\right) + \dim\left(H^0(Y_X;\mathbb{F}_2)\right), \tag{5.28}$$

then $HF^*_{\mathrm{BC}}(Z_X,(Z_X,\mathcal{E}_{Y_X})) \neq 0$ and hence $HF^*_{\mathrm{BC}}(Z_X,Z_X;\mathbb{F}_2) \neq 0$. Noting that $\dim\left(H^0(Y_X;\mathbb{F}_2)\right) = 1$ and, by Poincaré duality, $\dim\left(H^{2n}(Y_X;\mathbb{F}_2)\right) = \dim\left(H^1(Y_X;\mathbb{F}_2)\right)$, we see that (5.28) is equivalent to the inequality

$$\dim\left(H^n(Y_X;\mathbb{F}_2)\right) \geq 2 + \dim\left(H^1(Y_X;\mathbb{F}_2)\right). \tag{5.29}$$

We now show that (5.29) is equivalent to (5.27).

This is done using the Gysin sequence for the circle bundle $S^1 \to Y_X = S(\mathcal{O}_X(2)) \to X$. Since the Euler class of this bundle vanishes modulo 2, the Gysin sequence gives short exact sequences

$$0 \to H^k(X;\mathbb{F}_2) \to H^k(Y_X;\mathbb{F}_2) \to H^{k-1}(X;\mathbb{F}_2) \to 0 \quad \text{for all } k \in \mathbb{Z}.$$

Putting $k = 1$, we get

$$\dim\left(H^1(Y_X;\mathbb{F}_2)\right) = \dim\left(H^1(X;\mathbb{F}_2)\right) + \dim\left(H^0(X;\mathbb{F}_2)\right) = 1 + \dim\left(H^1(X;\mathbb{F}_2)\right), \tag{5.30}$$

while, putting $k = n$, gives

$$\dim\left(H^n(Y_X;\mathbb{F}_2)\right) = \dim\left(H^n(X;\mathbb{F}_2)\right) + \dim\left(H^{n-1}(X;\mathbb{F}_2)\right). \tag{5.31}$$

Substituting (5.30) and (5.31) into (5.29), we obtain inequality (5.27). □

We can now prove non-displaceability for the orientable subadjoint Lagrangians. The reason that we restrict our attention to the orientable ones is that the argument relies on a simple dimension count which fails for $Z_{(1,2k+1)}$ and Z_6. Note also that, if Z is any of the subadjoint Lagrangians other than L_Δ and $\{Z_{(1,2k+1)} \, : \, k \geq 0\}$, and $\mathbb{K}$ is a field of characteristic different from 2, then $HF^*_{\mathrm{BC}}(Z,Z;\mathbb{K}) = 0$ (replacing "BC" by "Zap" in the case of Z_6 which is pin but non-orientable). Indeed, for any such Z one has $H^1(Z;\mathbb{Z}) = 0$ and if $\dim(Z) = 2n+1$, then $N_Z = n+1 > 3$ so Lemma 3.0.17 implies that $HF^*(Z,Z;\mathbb{K})$ must be $2(2n+2)/(n+1) = 4$–torsion.

Proposition 5.2.2. *The following hold:*

1) *For each $k \geq 1$, the Floer cohomology $HF^*_{\mathrm{BC}}\left(Z_{(1,2k)},Z_{(1,2k)};\mathbb{F}_2\right)$ is non-zero.*

2) *The Floer cohomologies $HF^*_{\mathrm{BC}}(Z_9,Z_9;\mathbb{F}_2)$, $HF^*_{\mathrm{BC}}(Z_{15},Z_{15};\mathbb{F}_2)$ and $HF^*_{\mathrm{BC}}(Z_{27},Z_{27};\mathbb{F}_2)$ are non-zero.*

Proof. The proof of this proposition amounts to a simple dimension count, using the known cohomology of the homogeneous spaces $X_{(1,2k)}$, X_9, X_{15} and X_{27}. All of these spaces are simply connected, so by Lemma 5.2.1, it suffices to show that

$$\dim\left(H^n(X_\square;\mathbb{F}_2)\right)+\dim\left(H^{n-1}(X_\square;\mathbb{F}_2)\right)\geq 3, \tag{5.32}$$

where $X_\square$ ranges through the above spaces and $n=\dim_{\mathbb{C}}(X_\square)$.

Consider first part 1), where the Legendrian variety is $X_{(1,2k)}=\mathbb{CP}^1\times\mathbf{Q}_{2k}\subseteq\mathbb{CP}^{4k+3}$. For the quadrics one has:

$$H^s(\mathbf{Q}_{2k};\mathbb{Z})=\begin{cases}\mathbb{Z} & 0\leq s\leq 4k \text{ is even and } s\neq 2k\\ \mathbb{Z}\oplus\mathbb{Z} & s=2k\\ 0 & \text{otherwise.}\end{cases} \tag{5.33}$$

From this we get $H^{2k+1}(\mathbb{CP}^1\times\mathbf{Q}_{2k};\mathbb{F}_2)\cong 0$ and

$$H^{2k}(\mathbb{CP}^1\times\mathbf{Q}_{2k};\mathbb{F}_2)\cong H^{2k-2}(\mathbf{Q}_{2k};\mathbb{F}_2)\oplus H^{2k}(\mathbf{Q}_{2k};\mathbb{F}_2)\cong(\mathbb{F}_2)^3$$

and so inequality (5.32) is satisfied.

We now move on to part 2).

First, recall that $X_9\cong\mathrm{U}(6)/(\mathrm{U}(3)\times\mathrm{U}(3))=\mathrm{Gr}_{\mathbb{C}}(3,6)$. By [MT91, Chapter III, Theorem 6.9(2)] we know that

$$H^*\left(\mathrm{Gr}_{\mathbb{C}}(3,6);\mathbb{Z}\right)\cong\frac{\mathbb{Z}[c_1,c_2,c_3]}{\left(c_1^4=3c_1^2c_2-2c_1c_3-c_2^2,\ c_1^3c_2=2c_2c_2^2-2c_2c_3+c_1^2c_3,\ c_1^3c_3=2c_1c_2c_3-c_3^2\right)},$$

where the c_i's are the Chern classes of the tautological bundle. From this, we see that the cohomology of $\mathrm{Gr}_{\mathbb{C}}(3,6)$ vanishes in odd degrees and $H^8(\mathrm{Gr}_{\mathbb{C}}(3,6);\mathbb{Z})$ is a free $\mathbb{Z}$-module, generated by $c_1^2c_2$, c_1c_3 and c_2^2. Therefore $H^9(\mathrm{Gr}_{\mathbb{C}}(3,6);\mathbb{F}_2)=0$, while $H^8(\mathrm{Gr}_{\mathbb{C}}(3,6);\mathbb{F}_2)\cong(\mathbb{F}_2)^3$ and so the dimensions of these groups satisfy inequality (5.32).

Next, consider $X_{15}\cong\mathrm{SO}(12)/\mathrm{U}(6)$. By [MT91, Chapter III, Theorem 6.11] we know that there exist elements $e_{2i}\in H^{2i}(\mathrm{SO}(12)/\mathrm{U}(6);\mathbb{Z})$ for $i\in\{1,2,3,4,5\}$ such that

$$H^*(\mathrm{SO}(12)/\mathrm{U}(6);\mathbb{Z})\cong\Delta(e_2,e_4,e_6,e_8,e_{10}).$$

Here $\Delta(e_2,e_4,e_6,e_8,e_{10})$ denotes the free $\mathbb{Z}$-module, generated by all *simple* monomials in the elements $\{e_2,e_4,e_6,e_8,e_{10}\}$, that is, the monomials without a repeated factor. The above notation encodes some of the algebra structure as well: it tells us that, if the juxtaposition (followed by rearranging the factors) of two simple monomials is again a simple monomial, then the resulting formal identity is true in the ring itself (e.g. $(e_2e_6)(e_4e_{10})=e_2e_4e_6e_{10}$; see [MT91, Volume I, p.121]). It also tells us that non-simple monomials can be expressed as a linear combination of simple ones but it does not tell us what these relations are. This is not a problem for us, since at this point we are only interested in counting dimensions. In particular, we see that the cohomology of $\mathrm{SO}(12)/\mathrm{U}(6)$ vanishes in odd degrees and $H^{14}(\mathrm{SO}(12)/\mathrm{U}(6);\mathbb{Z})$ is a free $\mathbb{Z}$-module, generated by $\{e_2e_4e_8, e_4e_{10}, e_6e_8\}$. Hence $H^{15}(\mathrm{SO}(12)/\mathrm{U}(6);\mathbb{F}_2) = 0$, $H^{14}(\mathrm{SO}(12)/\mathrm{U}(6);\mathbb{F}_2) \cong (\mathbb{F}_2)^3$ and so inequality (5.32) is satisfied.

Finally, consider $X_{27} \cong \mathrm{E}_7/(\mathrm{E}_6 \cdot T^1)$. In this case, inequality (5.32) translates into

$$\dim\left(H^{27}(X_{27};\mathbb{F}_2)\right) + \dim\left(H^{26}(X_{27};\mathbb{F}_2)\right) \geq 3.$$

We will show that $\dim\left(H^{26}(X_{27};\mathbb{F}_2)\right) = 3$. By [MT91, Chapter VII, Lemma 6.13(2)], we know that

$$H^k(\mathrm{E}_7/\mathrm{E}_6;\mathbb{F}_2) = \begin{cases} \mathbb{F}_2, & k \in \{0,10,18,27,28,37,45,55\} \\ 0, & \text{otherwise.} \end{cases}$$

Plugging this into the mod 2 Gysin sequence for the circle bundle $S^1 \to \mathrm{E}_7/\mathrm{E}_6 \to \mathrm{E}_7/(\mathrm{E}_6 \cdot T^1) \cong X_{27}$, one finds that there are isomorphisms $H^k(X_{27};\mathbb{F}_2) \xrightarrow{\cong} H^{k+2}(X_{27};\mathbb{F}_2)$ for $0 \leq k \leq 7$, $10 \leq k \leq 15$ and $18 \leq k \leq 24$. Combining these with the fact that $H^1(X_{27};\mathbb{F}_2) = 0$, one further finds short exact sequences

$$0 \longrightarrow H^8(X_{27};\mathbb{F}_2) \longrightarrow H^{10}(X_{27};\mathbb{F}_2) \longrightarrow H^{10}(\mathrm{E}_7/\mathrm{E}_6;\mathbb{F}_2) \longrightarrow 0$$

$$0 \longrightarrow H^{16}(X_{27};\mathbb{F}_2) \longrightarrow H^{18}(X_{27};\mathbb{F}_2) \longrightarrow H^{18}(\mathrm{E}_7/\mathrm{E}_6;\mathbb{F}_2) \longrightarrow 0,$$

whose penultimate terms are 1-dimensional. Starting with $H^0(X_{27};\mathbb{F}_2) \cong \mathbb{F}_2$ and chasing through these isomorphisms and exact sequences yields that indeed $H^{26}(X_{27};\mathbb{F}_2) \cong (\mathbb{F}_2)^3$. $\square$

Corollary 5.2.3. *Let Z denote any of the subadjoint Lagrangians $Z_{(1,2k)}$, Z_9, Z_{15}, Z_{27} and let d_Z denote the dimension of Z. Then Z cannot be displaced from $\mathbb{RP}^{d_Z}$ or $T^{d_Z}_{Cl}$ by a Hamiltonian diffeomorphism of $\mathbb{CP}^{d_Z}$. Moreover, Z_{15} split-generates the Fukaya category $\mathcal{F}(\mathbb{CP}^{31};\overline{\mathbb{F}}_2)$, where $\overline{\mathbb{F}}_2$ denotes the algebraic closure of $\mathbb{F}_2$.*

Proof. By Proposition 5.2.2, the Lagrangian Z defines an essential object of the monotone Fukaya category$\mathcal{F}(\mathbb{CP}^{d_Z};\mathbb{F}_2)$. By Tonkonog's theorem [Ton18, Proposition 1.1], $\mathcal{F}(\mathbb{CP}^{d_Z};\mathbb{F}_2)$ is split-generated by $\mathbb{RP}^{d_Z}$ and hence $HF^*(Z;\mathbb{RP}^{d_Z};\mathbb{F}_2) \neq 0$. As for the Clifford torus, it is a well-known result of Cho ([Cho04] but see also [Smi17, Example 3.1.4]) that T^m_{Cl} is wide over $\mathbb{F}_2$ in any dimension $m \in \mathbb{N}$ (in fact, T^m_{Cl} is wide over any field of any characteristic). It then follows from [BC09a, Corollary 8.1.2] that Z and $T^{d_Z}_{Cl}$ are not Hamiltonianly displaceable.

In fact, something more general is true. For any $m \in \mathbb{N}$, the quantum cohomology $QH^*(\mathbb{CP}^m;\overline{\mathbb{F}}_2)$ splits into a direct product of local rings (see e.g. [EL19, Example 1.3.2 and Section 4.1]) and the Fukaya category splits accordingly into orthogonal summands. The local rings in question are in one-to-one correspondence with the $(m+1)$-th roots of unity in $\overline{\mathbb{F}}_2$ and so when m is odd and $m+1$ is not a power of 2, this is a strictly finer decomposition than the one corresponding to eigenvalues of the first Chern class which we described in section 2.3.4. Now, [EL19, Corollary 1.3.1] tells us that by equipping $T^{d_Z}_{Cl}$ with different rank one $\overline{\mathbb{F}}_2$-local systems, one obtains objects of $\mathcal{F}(\mathbb{CP}^{d_Z};\overline{\mathbb{F}}_2)$ which split-generate the different summands. Hence, the direct sum $\mathcal{T}_{Cl}^{d_Z}$ of these objects split-generates $Tw(\mathcal{F}(\mathbb{CP}^{d_Z};\overline{\mathbb{F}}_2))$ and since L defines an essential object of this category, it must have non-zero Floer cohomology with at least one of the summands in $\mathcal{T}_{Cl}^{d_Z}$. This again shows that Z and $T_{Cl}^{d_Z}$ are not Hamiltonianly displaceable.

Finally, when $Z = Z_{15}$, we have that $d_Z + 1 = 32$ is a power of 2 and so $QH^*(\mathbb{CP}^{31};\overline{\mathbb{F}}_2)$ does not decompose. It then follows by [EL19, Corollary 7.2.1] that Z_{15} split-generates $\mathcal{F}(\mathbb{CP}^{31};\overline{\mathbb{F}}_2)$. $\square$

Appendix A

Vertical gradient equation on $\Lambda^2_+ S^4$

Let $S^4 = S^4(1/2)$ denote the sphere of radius $1/2$ in $\mathbb{R}^5$, equipped with the round metric and a fixed orientation and let $\Lambda^2_+ S^4$ denote the corresponding bundle of self-dual 2-forms. In this appendix we derive equation (4.64) which expresses in coordinates the condition that the differential of a function $f\colon \Lambda^2_+ S^4 \to \mathbb{R}$ annihilates the horizontal distribution.

We pick stereographic coordinates $x = (x^1, x^2, x^3, x^4)$ on S^4 such that $dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$ is a positive orientation form. In these coordinates the round metric is $g_{ij} = \frac{1}{(1+\|x\|^2)^2}\delta_{ij}$ and the Hodge star satisfies $*(dx^1 \wedge dx^2) = dx^3 \wedge dx^4$, $*(dx^1 \wedge dx^3) = -dx^2 \wedge dx^4$, $*(dx^1 \wedge dx^4) = dx^2 \wedge dx^3$. We trivialise the bundle of self-dual 2-forms over this chart using the basis

$$\{\alpha_1 := dx^1 \wedge dx^2 + dx^3 \wedge dx^4, \alpha_2 := dx^1 \wedge dx^3 - dx^2 \wedge dx^4, \alpha_3 := dx^1 \wedge dx^4 + dx^2 \wedge dx^3\}$$

and let $y := (y^1, y^2, y^3)$ be fibre coordinates on $\Lambda^2_+ S^4$ with respect to this basis.

Now let $\{\Gamma^k_{ij} \,:\, 1 \le i,j,k \le 4\}$ be the Christoffel symbols for the Levi-Civita connection on S^4, i.e. $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k}$ and let $\left\{\widetilde{\Gamma}^k_{ij} \,:\, 1 \le i \le 4,\ 1 \le j,k \le 3\right\}$ be the Christoffel symbols for the induced connection on $\Lambda^2_+ S^4$, i.e. $\nabla_{\frac{\partial}{\partial x^i}} \alpha_j = \widetilde{\Gamma}^k_{ij}\alpha_k$. Note that, since the vectors $\left\{\frac{\partial}{\partial x_i} \,:\, 1 \le 1 \le 4\right\}$ all have the same norm and are mutually orthogonal, one has $\Gamma^k_{ij} = -\Gamma^j_{ik}$ for $j \ne k$ and $\Gamma^1_{i1} = \Gamma^2_{i2} = \Gamma^3_{i3} = \Gamma^4_{i4}$. Similarly, since $\langle dx^i, dx^j\rangle = \delta^{ij}(1+\|x\|^2)^2$, one has $\langle \alpha_i, \alpha_j\rangle = 2\delta_{ij}(1+\|x\|^2)^4$ and hence $\widetilde{\Gamma}^k_{ij} = -\widetilde{\Gamma}^j_{ik}$ for $j \ne k$ and $\widetilde{\Gamma}^1_{i1} = \widetilde{\Gamma}^2_{i2} = \widetilde{\Gamma}^3_{i3}$.

Using these facts, together with the identity $\nabla_{\frac{\partial}{\partial x^i}} dx^j = -\Gamma^j_{ik} dx^k$, one computes

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial x^i}} \alpha_1 &= \nabla_{\frac{\partial}{\partial x^i}} (dx^1 \wedge dx^2 + dx^3 \wedge dx^4) \\
&= \nabla_{\frac{\partial}{\partial x^i}} dx^1 \wedge dx^2 + dx^1 \wedge \nabla_{\frac{\partial}{\partial x^i}} dx^2 + \nabla_{\frac{\partial}{\partial x^i}} dx^3 \wedge dx^4 + dx^3 \wedge \nabla_{\frac{\partial}{\partial x^i}} dx^4 \\
&= -\Gamma^1_{ik} dx^k \wedge dx^2 - dx^1 \wedge \Gamma^2_{ik} dx^k - \Gamma^3_{ik} dx^k \wedge dx^4 - dx^3 \wedge \Gamma^4_{ik} dx^k \\
&= -(\Gamma^1_{i1} + \Gamma^2_{i2}) dx^1 \wedge dx^2 + (\Gamma^4_{i1} - \Gamma^2_{i3}) dx^1 \wedge dx^3 - (\Gamma^2_{i4} + \Gamma^3_{i1}) dx^1 \wedge dx^4 + \\
&\quad (\Gamma^1_{i3} + \Gamma^4_{i2}) dx^2 \wedge dx^3 + (\Gamma^1_{i4} - \Gamma^3_{i2}) dx^2 \wedge dx^4 - (\Gamma^3_{i3} + \Gamma^4_{i4}) dx^3 \wedge dx^4 \\
&= -2\Gamma^1_{i1}\alpha_1 + (\Gamma^4_{i1} + \Gamma^3_{i2})\alpha_2 + (\Gamma^4_{i2} - \Gamma^3_{i1})\alpha_3
\end{aligned}$$

This way one obtains

$$\begin{aligned}\widetilde{\Gamma}^1_{i1}=\widetilde{\Gamma}^2_{i2}=\widetilde{\Gamma}^3_{i3} &= -2\Gamma^1_{i1}\\ \widetilde{\Gamma}^2_{i1}=-\widetilde{\Gamma}^1_{i2} &= \Gamma^4_{i1}+\Gamma^3_{i2}\\ \widetilde{\Gamma}^3_{i1}=-\widetilde{\Gamma}^1_{i3} &= \Gamma^4_{i2}-\Gamma^3_{i1}\\ \widetilde{\Gamma}^3_{i2}=-\widetilde{\Gamma}^2_{i3} &= \Gamma^2_{i1}+\Gamma^4_{i3}. \end{aligned} \tag{A.1}$$

For the Chritsoffel symbols of the round sphere we have

$$\begin{aligned}\Gamma^k_{ij} &= \frac{1}{2}g^{kl}(g_{il,j}+g_{lj,i}-g_{ij,l})\\ &= \frac{1}{2}g^{kk}(g_{ik,j}+g_{kj,i}-g_{ij,k})\\ &= \frac{1}{2}(1+\|x\|^2)^2\frac{-4}{(1+\|x\|^2)^3}(\delta_{ik}x^j+\delta_{kj}x^i-\delta_{ij}x^k)\\ &= \frac{-2}{1+\|x\|^2}(\delta_{ik}x^j+\delta_{kj}x^i-\delta_{ij}x^k).\end{aligned}$$

Thus the non-zero symbols are

$$\Gamma^j_{ij}=\frac{-2}{1+\|x\|^2}x^i \quad \forall i,j \quad \text{and} \quad \Gamma^k_{ii}=-\Gamma^i_{ik}=\frac{2}{1+\|x\|^2}x^k \quad \forall\, i\neq k. \tag{A.2}$$

Plugging (A.2) into (A.1) we obtain the formulae

$$\begin{aligned}\widetilde{\Gamma}^1_{i1}=\widetilde{\Gamma}^2_{i2}=\widetilde{\Gamma}^3_{i3} &= \frac{2}{1+\|x\|^2}2x^i\\ \widetilde{\Gamma}^2_{i1}=-\widetilde{\Gamma}^1_{i2} &= \frac{2}{1+\|x\|^2}(\delta_{i1}x^4+\delta_{i2}x^3+\delta_{i3}(-x^2)+\delta_{i4}(-x^1))\\ \widetilde{\Gamma}^3_{i1}=-\widetilde{\Gamma}^1_{i3} &= \frac{2}{1+\|x\|^2}(\delta_{i1}(-x^3)+\delta_{i2}x^4+\delta_{i3}x^1+\delta_{i4}(-x^2))\\ \widetilde{\Gamma}^3_{i2}=-\widetilde{\Gamma}^2_{i3} &= \frac{2}{1+\|x\|^2}(\delta_{i1}x^2+\delta_{i2}(-x^1)+\delta_{i3}x^4+\delta_{i4}(-x^3)).\end{aligned} \tag{A.3}$$

Now let $f\colon \Lambda^2_+S^4\to\mathbb{R}$ be a smooth function. Using the coordinates $(x^1,x^2,x^3,x^4,y^1,y^2,y^3)$, the condition that df annihilates the horizontal distribution (that is, f has vertical gradient with respect to the Sasaki metric) translates into the requirement that at each point df lies in $\mathrm{Span}\left(\langle\frac{\partial}{\partial y^1},\cdot\rangle,\langle\frac{\partial}{\partial y^2},\cdot\rangle,\langle\frac{\partial}{\partial y^3},\cdot\rangle\right)$. We know that the horizontal distribution is spanned by

$$v_1:=\frac{\partial}{\partial x^1}-\widetilde{\Gamma}^k_{1j}y^j\frac{\partial}{\partial y^k},\quad v_2:=\frac{\partial}{\partial x^2}-\widetilde{\Gamma}^k_{2j}y^j\frac{\partial}{\partial y^k},\quad v_3:=\frac{\partial}{\partial x^3}-\widetilde{\Gamma}^k_{3j}y^j\frac{\partial}{\partial y^k}\quad v_4:=\frac{\partial}{\partial x^4}-\widetilde{\Gamma}^k_{4j}y^j\frac{\partial}{\partial y^k}$$

and so

$$\begin{aligned}\left\langle\frac{\partial}{\partial y^i},\cdot\right\rangle &= \left\langle\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\right\rangle dy^j+\left\langle\frac{\partial}{\partial y^i},\frac{\partial}{\partial x^k}\right\rangle dx^k\\ &= \delta_{ij}2(1+\|x\|^2)^4dy^j+\left(\left\langle\frac{\partial}{\partial y^i},v_k\right\rangle+\left\langle\frac{\partial}{\partial y^i},\widetilde{\Gamma}^l_{kj}y^j\frac{\partial}{\partial y^l}\right\rangle\right)\\ &= 2(1+\|x\|^2)^4(dy^i+\widetilde{\Gamma}^i_{kj}y^jdx^k).\end{aligned}$$

Hence, f has vertical gradient if and only if there exist smooth functions $\lambda_1,\lambda_2,\lambda_3$ such that

$$df=\lambda_1(dy^1+\widetilde{\Gamma}^1_{kj}y^jdx^k)+\lambda_2(dy^2+\widetilde{\Gamma}^2_{kj}y^jdx^k)+\lambda_3(dy^3+\widetilde{\Gamma}^3_{kj}y^jdx^k). \tag{A.4}$$

Comparing coefficients in front of dy^i, we see that $\lambda_i = \frac{\partial f}{\partial y^i}$. Now substituting the formulae (A.3) into (A.4) and comparing coefficients on front of dx^j for each $1 \leq j \leq 4$ one obtains that the vertical gradient condition is equivalent to the following system of PDEs (we change notation from upper to lower indices for better legibility):

$$\frac{1+\|x\|^2}{2}\begin{pmatrix}\frac{\partial f}{\partial x_1}\\ \frac{\partial f}{\partial x_2}\\ \frac{\partial f}{\partial x_3}\\ \frac{\partial f}{\partial x_4}\end{pmatrix} = \begin{pmatrix}(2x_1y_1 - x_4y_2 + x_3y_3) & (x_4y_1 + 2x_1y_2 - x_2y_3) & (-x_3y_1 + x_2y_2 + 2x_1y_3)\\ (2x_2y_1 - x_3y_2 - x_4y_3) & (x_3y_1 + 2x_2y_2 + x_1y_3) & (x_4y_1 - x_1y_2 + 2x_2y_3)\\ (2x_3y_1 + x_2y_2 - x_1y_3) & (-x_2y_1 + 2x_3y_2 - x_4y_3) & (x_1y_1 + x_4y_2 + 2x_3y_3)\\ (2x_4y_1 + x_1y_2 + x_2y_3) & (-x_1y_1 + 2x_4y_2 + x_3y_3) & (-x_2y_1 - x_3y_2 + 2x_4y_3)\end{pmatrix}\begin{pmatrix}\frac{\partial f}{\partial y_1}\\ \frac{\partial f}{\partial y_2}\\ \frac{\partial f}{\partial y_3}\end{pmatrix} \tag{A.5}$$

To simplify this we introduce the quaternionic notation $x := x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}$, $y := y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k}$, $\nabla_x f := \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2}\mathbf{i} + \frac{\partial f}{\partial x_3}\mathbf{j} + \frac{\partial f}{\partial x_4}\mathbf{k}$ and $\nabla_y f := \frac{\partial f}{\partial y_1}\mathbf{i} + \frac{\partial f}{\partial y_2}\mathbf{j} + \frac{\partial f}{\partial y_3}\mathbf{k}$. One can rewrite the right-hand side of (A.5) as

$$\begin{pmatrix}
2\nabla_y f\cdot y & \det\begin{pmatrix} y_2 & y_3\\ \frac{\partial f}{\partial y_2} & \frac{\partial f}{\partial y_3}\end{pmatrix} & -\det\begin{pmatrix} y_1 & y_3\\ \frac{\partial f}{\partial y_1} & \frac{\partial f}{\partial y_3}\end{pmatrix} & \det\begin{pmatrix} y_1 & y_2\\ \frac{\partial f}{\partial y_1} & \frac{\partial f}{\partial y_2}\end{pmatrix}\\
-\det\begin{pmatrix} y_2 & y_3\\ \frac{\partial f}{\partial y_2} & \frac{\partial f}{\partial y_3}\end{pmatrix} & 2\nabla_y f\cdot y & \det\begin{pmatrix} y_1 & y_2\\ \frac{\partial f}{\partial y_1} & \frac{\partial f}{\partial y_2}\end{pmatrix} & \det\begin{pmatrix} y_1 & y_3\\ \frac{\partial f}{\partial y_1} & \frac{\partial f}{\partial y_3}\end{pmatrix}\\
\det\begin{pmatrix} y_1 & y_3\\ \frac{\partial f}{\partial y_1} & \frac{\partial f}{\partial y_3}\end{pmatrix} & -\det\begin{pmatrix} y_1 & y_2\\ \frac{\partial f}{\partial y_1} & \frac{\partial f}{\partial y_2}\end{pmatrix} & 2\nabla_y f\cdot y & \det\begin{pmatrix} y_2 & y_3\\ \frac{\partial f}{\partial y_2} & \frac{\partial f}{\partial y_3}\end{pmatrix}\\
-\det\begin{pmatrix} y_1 & y_2\\ \frac{\partial f}{\partial y_1} & \frac{\partial f}{\partial y_2}\end{pmatrix} & -\det\begin{pmatrix} y_1 & y_3\\ \frac{\partial f}{\partial y_1} & \frac{\partial f}{\partial y_3}\end{pmatrix} & -\det\begin{pmatrix} y_2 & y_3\\ \frac{\partial f}{\partial y_2} & \frac{\partial f}{\partial y_3}\end{pmatrix} & 2\nabla_y f\cdot y
\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix}$$

and it is not hard to see that this equals $(2y\cdot\nabla_y f\ -\ y\times\nabla_y f)x$, where juxtaposition of vectors denotes quaternion multiplication. Thus the final form of the vertical gradient equation is

$$\frac{1+\|x\|^2}{2}\nabla_x f = (2\nabla_y f\cdot y\ +\ \nabla_y f\times y)x.$$

Appendix B

Indecomposable representations over $\mathbb{F}_2$ of the binary dihedral group of order 12

In this appendix we describe all indecomposable $\mathbb{F}_2$-representations of the binary dihedral group of order twelve. Such a classification is, of course, not new and much more general results are proved for example in [Jan69]. Here we give a rather direct and pedestrian argument for the classification in order to make the arguments in section 5.1.5 complete and the thesis more self-contained.

We start by making the following observations. First note that by setting $c := a^2$ (giving $a = c^2b^2$) one can view the binary dihedral group

$$\Gamma_\Delta = \left\langle a, b \mid a^6 = 1, b^2 = a^3, ab = ba^5 \right\rangle$$

as the semi-direct product

$$C_3 \rtimes C_4 = \left\langle c, b \mid c^3 = 1, b^4 = 1, cb = bc^2 \right\rangle.$$

This point of view will be particularly convenient for us since we will classify representations of Γ_Δ by viewing them simultaneously as C_3-representations and C_4-representations. To that end, let us introduce some notation. We put

$$\begin{aligned} R_3 &:= \mathbb{F}_2[C_3] = \frac{\mathbb{F}_2[c]}{(c^3-1)} \\ R_4 &:= \mathbb{F}_2[C_4] = \frac{\mathbb{F}_2[b]}{(b^4-1)} = \frac{\mathbb{F}_2[b]}{(b+1)^4}. \end{aligned}$$

If V is a Γ_Δ-representation, we shall write $O_{C_3}(V)$ for the set of orbits of the C_3-action on $V \setminus \{0\}$. Note that since C_3 is a normal subgroup of Γ_Δ, we have a C_4-action on $O_{C_3}(V)$. We denote the set of orbits of this action by $O_{C_4}(O_{C_3}(V))$. For an element $\mathcal{A} \in O_{C_4}(O_{C_3}(V))$ we shall write $\operatorname{Span}\mathcal{A} := \sum_{A \in \mathcal{A}} \operatorname{Span} A \leq V$. Note that $\operatorname{Span}\mathcal{A}$ is always a Γ_Δ-subrepresentation of V. Further, given a Γ_Δ-representation V and a C_k-representation W for some $k \in \{3,4\}$, we will write $V \cong_k W$ to mean that V and W are isomorphic as representations of C_k.

Note now that the ring R_3 is semisimple with

$$R_3 \cong \mathbb{F}_2 \oplus \frac{\mathbb{F}_2[c]}{(1+c+c^2)}$$

and hence any finite-dimensional R_3-module $\widehat{V}$ can be written as

$$\widehat{V} \cong \widehat{V}_1^{\oplus k_1} \oplus \widehat{D}^{\oplus k_2},$$

where $\widehat{V}_1$ is the trivial one-dimensional R_3-module and $\widehat{D} := \mathbb{F}_2[c]/(1+c+c^2)$.

On the other hand, the ring R_4 is not semisimple but from the structure theorem for finitely-generated modules over a principal ideal domain, we know that the only indecomposable finite-dimensional R_4-modules are

$$\overline{V}_1 := \frac{\mathbb{F}_2[b]}{b+1}, \overline{V}_2 := \frac{\mathbb{F}_2[b]}{(b+1)^2}, \overline{V}_3 := \frac{\mathbb{F}_2[b]}{(b+1)^3}, \overline{V}_4 := R_4 = \frac{\mathbb{F}_2[b]}{(b+1)^4}.$$

Observe that since we have the short exact sequence $1 \to C_3 \to \Gamma_\Delta \to C_4 \to 1$, the above vector spaces are also indecomposable Γ_Δ-representations with trivial C_3-action. When we view them as such, we will lose the bar on top and denote them as V_1, V_2, V_3, V_4. In the basis $\{1, b, \dots, b^{j-1}\}$ for V_4, these are given by (5.24).

Further, since we have the short exact sequence

$$1 \longrightarrow C_2 = \{1, b^2\} \longrightarrow \Gamma_\Delta \longrightarrow D_6 = \langle c, \hat{b} \mid c^3 = 1, \hat{b}^2 = 1, c\hat{b} = \hat{b}c^2 \rangle \longrightarrow 1$$

and D_6 acts naturally on $\widehat{D} = \mathbb{F}_2[c]/(1+c+c^2)$ by $\hat{b} \cdot 1 = 1$, $\hat{b} \cdot c = c^2$, $\hat{b} \cdot c^2 = c$, we see that $\widehat{D}$ has naturally the structure of a non-faithful irreducible Γ_Δ-representation. We denote this representation by D. In the basis $\{1, c\}$ for $\mathbb{F}_2[c]/(1+c+c^2)$, it is precisely given as in (5.23).

Finally, we define the following faithful representation of Γ_Δ. Let

$$U_4 := \frac{\mathbb{F}_2[c]}{(1+c+c^2)} \oplus \frac{\mathbb{F}_2[c]}{(1+c+c^2)} x \tag{B.1}$$

and set $b \cdot 1 = 1$, $b \cdot x = 1 + cx$. Using linearity and the relation $bc = c^2 b$ this extends uniquely to an action of C_4 on U_4, thus making U_4 into a well-defined Γ_Δ-representation. In the basis $\{1, c, x, cx\}$ it is given by (5.25). It is important to note that $U_4 \cong_4 \overline{V}_4$, for example via the map

$$\begin{array}{rcl} U_4 & \longrightarrow & \dfrac{\mathbb{F}_2[b]}{(b+1)^4} \\ 1 & \mapsto & 1 + b + b^2 + b^3 \\ c & \mapsto & 1 + b^2 \\ x & \mapsto & 1 \\ cx & \mapsto & 1 + b^2 + b^3. \end{array}$$

We are now ready to state the classification.

Proposition B.0.4. *The only finite-dimensional indecomposable representations of* Γ_Δ *over* $\mathbb{F}_2$ *are* V_1, V_2, V_3, V_4, D *and* U_4.

We will prove this statement in several steps and in the course of the proof it will become apparent that all these representations are indeed indecomposable. Note that V_1 and D are the only irreducible representations since U_4 contains a copy of D (C_4 preserves the first summand in (B.1)) while $V_i \cong (b+1)^{4-i} \cdot V_j \leq V_j$ whenever $i \leq j$.

It will be useful for us to also consider the following representation. Let

$$U_8 := \frac{\mathbb{F}_2[c]}{(1+c+c^2)} \oplus \frac{\mathbb{F}_2[c]}{(1+c+c^2)}x \oplus \frac{\mathbb{F}_2[c]}{(1+c+c^2)}x^2 \oplus \frac{\mathbb{F}_2[c]}{(1+c+c^2)}x^3$$

and let $b \in C_4$ act as the cyclic permutation $(1, x, x^2, x^3)$. Again, the relation $bc = c^2 b$ allows us to extend this action making U_8 into a Γ_Δ-representation. In fact, we have $U_8 \cong U_4 \oplus U_4$ via the map

$$\begin{array}{ccc} U_8 & \longrightarrow & U_4 \oplus U_4 \\ 1 & \mapsto & (x, cx). \end{array}$$

To begin the classification, we first observe that we can restrict attention only to representations on which C_3 acts non-trivially.

Lemma B.0.5. *Let V be a Γ_Δ-representation over $\mathbb{F}_2$. Define*

$$\begin{array}{rcl} V^{C_3} & := & \{v \in V \,:\, c \cdot v = v\} \\ W & := & \{v \in V \,:\, v + c \cdot v + c^2 \cdot v = 0\}. \end{array}$$

Then V^{C_3} and W are Γ_Δ-representations and we have a decomposition $V = V^{C_3} \oplus W$.

Proof. The fact that $V \cong_3 V^{C_3} \oplus W$ is just a restatement of the fact that R_3 is semisimple. To see that V^{C_3} and W are preserved by the C_4-action note that if $v \in V^{C_3}$ then $c \cdot (b \cdot v) = b \cdot (c^2 \cdot v) = b \cdot v$, i.e. $b \cdot v \in V^{C_3}$ and if $v \in W$ then $(1+c+c^2) \cdot (b \cdot v) = b \cdot ((1+c^2+c) \cdot v) = 0$, i.e. $b \cdot v \in W$. □

We thus have that $V \cong V_1^{\oplus k_1} \oplus V_2^{\oplus k_2} \oplus V_3^{\oplus k_3} \oplus V_4^{\oplus k_4} \oplus W$, where $W^{C_3} = 0$. To prove Proposition B.0.4 it then suffices to show that the only indecomposable representations V with $V^{C_3} = 0$ are D and U_4. We do this in two steps: first, we show that these are the only indecomposable Γ_Δ-representations of dimension at most 8 and then we prove that any Γ_Δ-representation V with $V^{C_3} = 0$ and $\dim V > 8$ cannot be indecomposable.

Classifying the two-dimensional representations is easy. Indeed, if V is such, then we have $V \cong_3 \widehat{D} = \mathbb{F}_2[c]/(1+c+c^2)$ and V contains exactly 3 non-zero vectors $\{1, c, c^2\}$. Since C_4 acts on this set, we must have that either this action is trivial, or that b fixes one of the three vectors and swaps the other two. But C_4 cannot act trivially since then we would have

$$c^2 = c^2 \cdot (b \cdot 1) = b \cdot (c \cdot 1) = c,$$

a contradiction. Hence b fixes exactly one non-zero vector and without loss of generality $b \cdot 1 = 1$ and $b \cdot c = c^2$, $b \cdot c^2 = c$. Thus $V \cong D$ as Γ_Δ-representations.

In fact, the only indecomposable representations of the dihedral group D_6 over $\mathbb{F}_2$ are the trivial representation, the regular representation of C_2 and D. This is an easy special case of [Bon75, Theorem 2] and can also be proved directly by writing $D_6 = C_3 \rtimes C_2$ and using the same methods we employ here (see Remark B.0.7 below). On the other hand, it is not hard to see that the only non-trivial proper normal subgroups of Γ_Δ are $C_3 = \langle c \rangle$, $C_2 = \langle b^2 \rangle$ and $C_6 = \langle c, b^2 \rangle$ (using that, if $K \trianglelefteq C_3 \rtimes C_4$ and $c^m b^n \in K$, then $c^m = (b(c^m b^n)b^{-1})(c^m b^n)^{-1} \in K$) and we have already found all the indecomposable representations of the corresponding quotients. Thus, we can restrict ourselves to finding the faithful indecomposable Γ_Δ-representations.

So, let V be a faithful Γ_Δ-representation with $V^{C_3} = 0$ and $\dim V = 4$. Then we have

$$V \cong_3 \frac{\mathbb{F}_2[c]}{(1+c+c^2)} \oplus \frac{\mathbb{F}_2[c]}{(1+c+c^2)} x.$$

and

$$\begin{aligned} O_{C_3}(V) \quad = \quad & \left\{ \left\{1, c, c^2\right\}, \left\{x, cx, c^2x\right\}, \left\{1+x, c+cx, c^2+c^2x\right\}, \right. \\ & \left. \left\{1+cx, c+c^2x, c^2+x\right\}, \left\{1+c^2x, c+x, c^2+cx\right\} \right\}. \end{aligned}$$

Since the size of each orbit of the C_4-action on $O_{C_3}(V)$ must divide $|C_4| = 4$ and $|O_{C_3}(V)| = 5$ we see that C_4 must preserve at least one C_3-orbit. Up to a C_3-equivariant change of basis for V, we may assume that $b \cdot \{1, c, c^2\} = \{1, c, c^2\}$ and further $b \cdot 1 = 1$, $b \cdot c = c^2$, $b \cdot c^2 = c$. Since we are assuming that V is a faithful representation, the action of C_4 on $O_{C_3}(V)$ must also be faithful (otherwise b^2 must fix all elements of $O_{C_3}(V)$ and, since it commutes with c, it will then have to act trivially on V). Hence, the set

$$\mathcal{A} := \{\{x, cx, c^2x\}, \{1+x, c+cx, c^2+c^2x\}, \{1+cx, c+c^2x, c^2+x\}, \{1+c^2x, c+x, c^2+cx\}\}$$

forms a single orbit of the C_4-action on $O_{C_3}(V)$. By linearity and the relation $bc = c^2 b$, the action of b on $\mathcal{A}$ is uniquely determined by which element $b \cdot x$ is. We now have the following cases:

1. Suppose that $b \cdot x \in \{x, 1+x, 1+cx, 1+c^2x\}$. Then:
   (a) if $b \cdot x = x$ then $V = \mathrm{Span}\{1, c\} \oplus \mathrm{Span}\{x, cx\} \cong D \oplus D$ which contradicts faithfulness.
   (b) if $b \cdot x = 1+x$ then $b \cdot (c^2+x) = c+1+x = c^2+x$ and so
   $$V = \mathrm{Span}\{1, c\} \oplus \mathrm{Span}\{c^2+x, 1+cx\} \cong D \oplus D$$
   which again contradicts faithfulness.
   (c) if $b \cdot x = 1+cx$ we obtain precisely the representation U_4. It is clearly indecomposable since it is not isomorphic to $D \oplus D$.
   (d) if $b \cdot x = 1+c^2x$ then consider the C_3-equivariant change of basis for V given by the substitution $\tilde{x} = 1+c^2x$. Then we have $x = c \cdot (1+\tilde{x})$ and
   $$\begin{aligned} b \cdot \tilde{x} = b \cdot (1+c^2x) = 1 + c \cdot (b \cdot x) = 1 + c \cdot (1+c^2x) \\ = 1+c+x = 1+c+c+c\tilde{x} = 1+c\tilde{x} \end{aligned}$$
   and thus $V \cong U_4$.

2. Suppose that $b \notin \{x, 1+x, 1+cx, 1+c^2x\}$:

   (a) if $b \cdot x = c \cdot (1+\alpha x)$ for some $\alpha \in \{1, c, c^2\}$, then consider the substitution $\tilde{x} = cx$. Then $x = c^2\tilde{x}$ and $b \cdot \tilde{x} = b \cdot cx = c^2 b \cdot x = c^2 c(1+\alpha x) = 1+\alpha x = 1+\alpha c^2 \tilde{x}$ and we are back in case 1.

   (b) if $b \cdot x = c^2 \cdot (1+\alpha x)$ then, putting $\tilde{x} = c^2 x$ we obtain $b \cdot \tilde{x} = 1 + \alpha c \tilde{x}$ and again we can apply case 1.

We have thus seen that the only faithful indecomposable Γ_Δ-representation of dimension 4 is U_4.

Recall that $U_4 \cong_4 \overline{V}_4$. In order to extend the classification to higher-dimensional representations we will repeatedly use this fact, together with the following lemma.

Lemma B.0.6. *Let V be a finite-dimensional representation of Γ_Δ over $\mathbb{F}_2$ and let $U \leq V$ be a subrepresentation. Suppose that $U \cong_4 \overline{V}_4^{\oplus k}$ for some $k \geq 1$. Then there exists a subrepresentation $W \leq V$ such that $V = U \oplus W$ as representations of Γ_Δ.*

Remark B.0.7. We note here that a similar statement holds also for $\mathbb{F}_2$-representations of the dihedral group $D_6 = C_3 \rtimes C_2$. That is, if $U \leq V$ is a pair of representations of D_6 and U is isomorphic to a direct sum of copies of the regular representation of C_2, then U is actually a direct summand of V. The proof is an easier version of the proof we present below. //

The proof of Lemma B.0.6 requires a short detour. We first note the following standard fact whose proof is straightforward.

Lemma B.0.8. *Let R be a ring (not necessarily commutative) and let X be an R-module. Let $M, N \leq X$ be submodules such that $X = M \oplus N$ and let $\pi\colon X \to M$ be the projection along N. Let $M' \leq X$ be another R-submodule. Then $X = M' \oplus N$ if and only if $\pi|_{M'} : M' \to M$ is an isomorphism.* □

Using this fact, we can now make a step towards Lemma B.0.6 by first showing that copies of $\overline{V}_4$ are always direct summands of C_4-representations.

Lemma B.0.9. *Let $\overline{V}$ be an R_4-module which is finite-dimensional over $\mathbb{F}_2$. Suppose $\overline{U} \leq \overline{V}$ is a submodule with $\overline{U} \cong \overline{V}_4$. Then there exists an R_4-submodule $\overline{W} \leq \overline{V}$ such that $\overline{V} = \overline{U} \oplus \overline{W}$.*

Proof. Since $\overline{V}$ is an R_4-module, there exist non-negative integers n_1, n_2, n_3, n_4 such that

$$\overline{V} \cong \overline{V}_1^{\oplus n_1} \oplus \overline{V}_2^{\oplus n_2} \oplus \overline{V}_3^{\oplus n_3} \oplus \overline{V}_4^{\oplus n_4}. \tag{B.2}$$

Let

$$\begin{array}{rccc} \phi\colon & \dfrac{\mathbb{F}_2[b]}{(b+1)^4} & \longrightarrow & \overline{V}_1^{\oplus n_1} \oplus \overline{V}_2^{\oplus n_2} \oplus \overline{V}_3^{\oplus n_3} \oplus \overline{V}_4^{\oplus n_4} \\ & 1 & \mapsto & \vec{v}_1 + \vec{v}_2 + \vec{v}_3 + \vec{v}_4 \end{array}$$

denote the inclusion of R_4-modules obtained by restricting the isomorphism (B.2) to the submodule $\overline{U} \cong \mathbb{F}_2[b]/(b+1)^4$. Then $\phi((b+1)^3) = (b+1)^3 \cdot (\vec{v}_1 + \vec{v}_2 + \vec{v}_3 + \vec{v}_4) = (b+1)^3 \cdot \vec{v}_4$. Since this must be non-zero, there must exist an index $1 \leq j \leq n_4$ such that $(b+1)^3 \cdot v_{4j} \neq 0$, where we write $\vec{v}_4 = (v_{41}, v_{42}, \ldots, v_{4n_4}) \in \overline{V}_4^{\oplus n_4}$. Then v_{4j} generates its copy of $\overline{V}_4$ as an R_4-module. Now, let $\pi\colon \overline{V} \to \overline{V}_4$ denote the projection to the j^{th} $\overline{V}_4$-factor along the other factors in the decomposition (B.2). Then $\pi(\phi(1)) = v_{4j}$, i.e. $\pi \circ \phi$ is a map of cyclic R_4-modules sending a generator to a generator and hence, it is an isomorphism.

The existence of a complement for $\overline{U} \leq \overline{V}$ now follows from Lemma B.0.8. ☐

To finish the proof of Lemma B.0.6 we also need the following general lemma.

Lemma B.0.10. *Let G be a group with subgroups $H \trianglelefteq G$, $K \leq G$ such that $G = H \rtimes K$. Suppose that H is finite and that $\mathbb{F}$ is a field such that* $\operatorname{char}(\mathbb{F})$ *does not divide $|H|$. Let V be a representation of G over $\mathbb{F}$ and assume that there is a splitting $V = U \oplus \overline{W}$ as K-representations. Further, suppose that U is actually a G-subrepresentation of V. Then there exists a G-representation $W \leq V$ such that $V = U \oplus W$ as G-representations.*

Proof. The proof is based on the standard technique of "averaging the projection". Namely, let $\overline{\pi}\colon V \to V$ denote the projection to U along $\overline{W}$, followed by the inclusion $\iota\colon U \hookrightarrow V$. Since $\operatorname{char}(\mathbb{F})$ does not divide $|H|$ we can define

$$\begin{aligned} \pi\colon V &\longrightarrow V \\ v &\longmapsto \frac{1}{|H|}\sum_{h\in H}(h^{-1},1)\cdot\overline{\pi}((h,1)\cdot v). \end{aligned} \tag{B.3}$$

We claim that π is G-equivariant and $\pi|_U = \iota$. For the second claim note that since H preserves U and $\overline{\pi}|_U = \iota$ then for all $h \in H$ and $u \in U$ we have $\overline{\pi}((h,1)\cdot u) = (h,1)\cdot u$. Plugging this into (B.3), we see that $\pi(u) = \frac{1}{|H|}|H|u = u$ for all $u \in U$. Now, to see that that π is G-equivariant we let $v \in V$, $(h_0,k_0) \in H \rtimes K = G$ and compute

$$\begin{aligned}
\pi((h_0,k_0)\cdot v) &= \frac{1}{|H|}\sum_{h\in H}(h^{-1},1)\cdot\overline{\pi}((h,1)(h_0,k_0)\cdot v) \\
&= \frac{1}{|H|}\sum_{h\in H}(h^{-1},1)\cdot\overline{\pi}((hh_0,1)(1,k_0)\cdot v) \\
&= \frac{1}{|H|}(h_0,1)\cdot\sum_{h\in H}(hh_0,1)^{-1}\cdot\overline{\pi}((hh_0,1)(1,k_0)\cdot v) \\
&= \frac{1}{|H|}(h_0,1)\cdot\sum_{h\in H}(h^{-1},1)\cdot\overline{\pi}((h,1)(1,k_0)\cdot v) \\
&= \frac{1}{|H|}(h_0,1)\cdot\sum_{h\in H}(h^{-1},1)\cdot\overline{\pi}((1,k_0)(k_0^{-1}hk_0,1)\cdot v) \\
&= \frac{1}{|H|}(h_0,1)\cdot\sum_{h\in H}(h^{-1},1)(1,k_0)\cdot\overline{\pi}((k_0^{-1}hk_0,1)\cdot v) \quad \text{[since } \overline{\pi} \text{ is } K\text{-equivariant]} \\
&= \frac{1}{|H|}(h_0,1)(1,k_0)\cdot\sum_{h\in H}(k_0^{-1}hk_0,1)^{-1}\cdot\overline{\pi}((k_0^{-1}hk_0,1)\cdot v) \\
&= (h_0,k_0)\cdot\pi(u).
\end{aligned}$$

Putting $W := \ker\pi$ we now obtain the desired splitting $V = U \oplus W$. ☐

We are now in a position to prove Lemma B.0.6.

Proof of Lemma B.0.6. We are assuming that we have a pair of Γ_Δ-representations $U \leq V$ and that $U \cong_4 \overline{V}_4^{\oplus k}$ for some $k \geq 1$. By applying Lemma B.0.9 k times, we find a C_4-subrepresentation $\overline{W} \leq V$ such that $V \cong_4 U \oplus \overline{W}$. Now, since C_3 preserves U and $\Gamma_\Delta = C_3 \rtimes C_4$, we can apply Lemma B.0.10 to find a Γ_Δ-subrepresentation $W \leq V$ such that $V = U \oplus W$. $\square$

Armed with Lemma B.0.6, we are now ready to extend our classification of indecomposable Γ_Δ-representations to higher dimensions. So let V be a Γ_Δ-representation with $V^{C_3} = 0$ and $\dim V = 6$. We will show that V cannot be indecomposable. We observe that since $|O_{C_3}(V)| = 21$ is odd, there must exist an orbit $A \in O_{C_3}(V)$ which is fixed by the C_4-action on $O_{C_3}(V)$. Then $D_0 := \operatorname{Span} A$ is a two-dimensional Γ_Δ-subrepresentation of V and so $D_0 \cong D$ and we have a short exact sequence

$$0 \longrightarrow D_0 \longrightarrow V \xrightarrow{\pi} V/D_0 \longrightarrow 0. \tag{B.4}$$

By the semisimplicity of R_3, this is a split sequence of C_3-representations and in particular $(V/D_0)^{C_3} = 0$. Then $|O_{C_3}(V/D_0)| = 5$ and again there must be an orbit $B \in O_{C_3}(V/D_0)$ which is fixed by the C_4-action. Put $D_1 := \operatorname{Span} B \leq V/D_0$ and $U := \pi^{-1}(D_1) \leq V$. We thus obtain a composition series

$$0 \lneq D_0 \lneq U \lneq V$$

with $U/D_0 = D_1 \cong D$ and $V/U \cong D$. From our classification of the four-dimensional representations, we now have the following two possibilities.

1. Suppose that $U \cong U_4$ or $V/D_0 \cong U_4$. Then

   (a) if $U \cong U_4$ we know by Lemma B.0.6 that V is not indecomposable and in fact $V \cong U_4 \oplus D$;

   (b) if $V/D_0 \cong U_4$ then in particular $V/D_0 \cong_4 \overline{V}_4$ is a free R_4-module and hence (B.4) splits as a sequence of C_4-representations. However, Lemma B.0.10 then implies that (B.4) is also a split sequence of Γ_Δ-representations, i.e. again $V \cong D \oplus U_4$.

2. Suppose that $U \cong D \oplus D$ and $V/D_0 \cong D \oplus D$. It follows (see Remark B.0.7) that there exist Γ_Δ-equivariant sections $r\colon D_1 = U/D_0 \longrightarrow U$ and $s\colon V/U = (V/D_0)/D_1 \longrightarrow V/D_0$ of the respective quotient maps. Now let $t\colon V/D_0 \longrightarrow V$ be any $\mathbb{F}_2$-linear section of $\pi\colon V \to V/D_0$, satisfying $t|_{D_1} = r$. These maps fit into the following diagram of Γ_Δ-representations, whose

rows and columns are exact:

$$\begin{tikzcd}
& & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & D_0 \arrow[r] \arrow[d, equal] & U \arrow[r] \arrow[d] & U/D_0 = D_1 \arrow[r] \arrow[l, bend right, "r"'] \arrow[d] & 0 \\
0 \arrow[r] & D_0 \arrow[r] & V \arrow[r, "\pi"'] \arrow[d] & V/D_0 \arrow[r] \arrow[l, bend right, dotted, "t"'] \arrow[d] & 0 \\
& & V/U \arrow[r, equal] \arrow[d] & (V/D_0)/D_1 \arrow[u, bend right, "s"'] \arrow[d] & \\
& & 0 & 0 &
\end{tikzcd}$$

Observe that since π and s are C_4-equivariant, we have

$$\pi(b \cdot ts(x)) = b \cdot \pi(ts(x)) = b \cdot s(x) = s(b \cdot x). \tag{B.5}$$

Consider now the $\mathbb{F}_2$-linear splitting

$$V = D_0 \oplus t(V/D_0) = D_0 \oplus t(D_1 \oplus s(V/U)) = D_0 \oplus r(D_1) \oplus ts(V/U). \tag{B.6}$$

We claim that C_4 preserves the summand $\overline{W} := D_0 \oplus ts(V/U)$. Indeed, if $v_0 \in D_0$, $x \in V/U$, then

$$b \cdot (v_0 + ts(x)) = \Big[b \cdot v_0 + b \cdot ts(x) - ts(b \cdot x) \Big] + ts(b \cdot x)$$

and by (B.5) we see that the term in the square brackets lies in $\ker \pi = D_0$. Now, since the section r is C_4-equivariant we see that (B.6) gives rise to the splitting $V = r(D_1) \oplus \overline{W}$ of C_4-representations. On the other hand, since r is also C_3-equivariant, we have that $r(D_1)$ is a Γ_Δ-subrepresentation of V and then it follows from Lemma B.0.10 that V is not indecomposable.

We have seen that, if V is a six-dimensional Γ_Δ-representation with $V^{C_3} = 0$, then we must have $V \cong D^{\oplus 3}$ or $V \cong D \oplus U_4$. In particular, the only faithful six-dimensional Γ_Δ-representation with $V^{C_3} = 0$ is $D \oplus U_4$.

Now let V be a faithful Γ_Δ-representation with $V^{C_3} = 0$ and $\dim V = 8$. Since the representation is faithful, there exists $\mathcal{A} \in O_{C_4}(O_{C_3}(V))$ with $|\mathcal{A}| = 4$. Then $\operatorname{Span} \mathcal{A}$ is a faithful subrepresentation of V and so $\dim(\operatorname{Span} \mathcal{A}) \in \{4, 6, 8\}$. If $\dim(\operatorname{Span} \mathcal{A}) = 4$ we know that $\operatorname{Span} \mathcal{A} \cong U_4$. By Lemma B.0.6 we have that V is not indecomposable. If $\dim(\operatorname{Span} \mathcal{A}) = 6$, then must have $\operatorname{Span} \mathcal{A} \cong D \oplus U_4$; in particular $U_4 \leq V$ and again Lemma B.0.6 shows that V cannot be indecomposable. We are left with the case $\dim(\operatorname{Span} \mathcal{A}) = 8$, i.e. $\operatorname{Span} \mathcal{A} = V$. We can then write

$$\mathcal{A} = \{\{1, c, c^2\}, \{x, cx, c^2x\}, \{x^2, cx^2, c^2x^2\}, \{x^3, cx^3, c^2x^3\}\}$$

and $\{1, c, x, cx, x^2, cx^2, x^3, cx^3\}$ forms a basis for V. Hence

$$V \cong_3 \frac{\mathbb{F}_2[c]}{(1+c+c^2)} \oplus \frac{\mathbb{F}_2[c]}{(1+c+c^2)} x \oplus \frac{\mathbb{F}_2[c]}{(1+c+c^2)} x^2 \oplus \frac{\mathbb{F}_2[c]}{(1+c+c^2)} x^3$$

and $b \in C_4$ acts as the cyclic permutation $(1, x, x^2, x^3)$. That is, $V \cong U_8 \cong U_4 \oplus U_4$ is not indecomposable.

Finally, we are ready to finish the proof of Proposition B.0.4, by showing that if V is a faithful representation of Γ_Δ with $V^{C_3} = 0$ and $\dim V > 8$, then V cannot be indecomposable. Indeed, by faithfulness, there must exist $\mathcal{A} \in O_{C_4}(O_{C_3}(V))$ with $|\mathcal{A}| = 4$. Then $\operatorname{Span}\mathcal{A}$ is a faithful subrepresentation of V. In particular, we have that $(\operatorname{Span}\mathcal{A})^{C_3} = 0$ and hence $V \cong_3 \widehat{D}^{\oplus k}$. It follows that $\dim(\operatorname{Span}\mathcal{A})$ must be even and for each $A = \{v, c\cdot v, c^2 \cdot v\} \in \mathcal{A}$ we have $v + c\cdot v + c^2 \cdot v = 0$. Then $\dim(\operatorname{Span}\mathcal{A}) \leq 2|\mathcal{A}| \leq 8$. But we have seen that any faithful Γ_Δ-representation of dimension at most 8 contains a copy of U_4. Hence $U_4 \leq \operatorname{Span}\mathcal{A} \leq V$ and Lemma B.0.6 shows that V cannot be indecomposable.

Proposition B.0.4 is now proved. $\square$

We end this appendix with two quick observations. First, we note that the regular representation of Γ_Δ over $\mathbb{F}_2$ is isomorphic to $V_4 \oplus U_4^{\oplus 2}$. Indeed

$$\mathbb{F}_2[\Gamma_\Delta] = \operatorname{Span}\{1, c, c^2\} \oplus \operatorname{Span}\{b, cb, c^2 b\} \oplus \operatorname{Span}\{b^2, cb^2, c^2 b^2\} \oplus \operatorname{Span}\{b^3, cb^3, c^2 b^3\} \tag{B.7}$$

and it contains a copy of V_4, namely the ideal

$$(1 + c + c^2) = \operatorname{Span}\{1 + c + c^2, b + cb + c^2 b, b^2 + cb^2 + c^2 b^2, b^3 + cb^3 + c^2 b^3\}.$$

Now, by Lemma B.0.6, V splits off as a direct summand and the quotient $\mathbb{F}_2[\Gamma_\Delta]/V$ is manifestly isomorphic to $U_8 \cong U_4 \oplus U_4$.

Second, we note that the representation U_4 dominates D in the sense of Definition 2.2.24. Indeed, it is not hard to see, that the kernel of the ring map $\mathbb{F}_2[\Gamma_\Delta] \to \operatorname{End}\big((\mathbb{F}_2)^4\big)$ corresponding to the representation U_4 is precisely the ideal $(1 + c + c^2)$, while the kernel of the homomorphism $\mathbb{F}_2[\Gamma_\Delta] \to \operatorname{End}\big((\mathbb{F}_2)^2\big)$ corresponding to D is the ideal $(1 + c + c^2, 1 + b^2)$.

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