

## Minimum image quality for reliable optical characterizations of soil particle shapes

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### Abstract

Remarkable advances have been seen in image-based methods for automating soil particle shape characterizations in the last decade. However, the accuracy and reliability of image-based methods has rarely been questioned. This study shows that image quality affects the computational results of particle shape descriptors, including aspect ratio, sphericity, convexity, circularity, and roundness. These descriptors display a hierarchy of resistance to the effects of low image quality. The particle length, perimeter, and area are used as controlling parameters for quantifying the influence of image quality. The minimum requirements for ensuring reliable image-based shape characterization of these parameters are established.

Keywords: optical shape characterization; granular soils; image quality

## **1. Introduction**

Particle shape is a fundamental property of granular soils. Angular and elongated particles create stronger inter-particle locking and larger coordination numbers. Therefore, they do not roll and slide as the more rounded and spherical particles. Experimental and numerical studies have shown that angular and elongated soils exhibit larger index void ratios, a larger angle of internal friction, larger dilatancy, a larger constant volume friction angle, larger compressibility, and a larger small-strain modulus than rounded and spherical soils (Alshibli and Cil 2018; Altuhafi et al. 2016; Bareither et al. 2008; Cavarretta et al. 2010; Cho et al. 2006; Jerves et al. 2016; Kandasami and Murthy 2014; Liu and Yang 2018; Nougier-Lehon et al. 2003; Shin and Santamarina 2013; Vangla et al. 2017; Zheng et al. 2017; Zheng and Hryciw 2016, 2017).

Traditionally, the particle shape is measured through either manual measurement or visual comparison to reference charts (Krumbein 1941; Krumbein and Sloss 1951; Powers 1953). The manual method is tedious and the chart method is subjective. Both methods are difficult to implement on a large number of particles (Hryciw et al. 2016).

Significant advances have been seen in image-based methods for automating particle shape measurements in the last two decades. A variety of imaging techniques have been developed and applied to particle shape analysis by geotechnical engineers (Bowman et al. 2001; Chandan et al. 2004; Gélinas and Vidal 2010; Kuo et al. 1996; Kuo and Freeman 2000; Matsushima et al. 2009; Mora and Kwan 2000; Ohm and Hryciw 2013; Sukumaran and Ashmawy 2001; Tafesse et al. 2012; Wettimuny and Penumadu 2004; Zheng et al. 2014; Zheng and Hryciw 2014, 2015, 2018; and many others), and a review of these techniques was provided by Hryciw et al. (2014). Commercial image-based particle shape analyzers, such as products from Malvern, the Sympatec QicPic, and the Camsizer, have also been used in geotechnical research (Altuhafi et al. 2013, 2016; Altuhafi and Coop 2011).

Image-based methods significantly improve accuracy and efficiency of particle shape characterizations compared to manual and visual methods. However, the reliability of image-based methods has rarely been questioned by geotechnical engineers. Whenever erroneous results are encountered by geotechnical engineers, they would rather attribute these results to the complexity and uncertainty of soil particles, not

the computer. In fact, image-based methods are not always reliable and precise as they are affected by many factors.

The image quality is the most important factor and is also the most questioned factor. For example, a particle is captured under eight different resolutions with the particle lengths ( $L$ ) in a range of 3350 pixels to 25 pixels in Figs. 1(a) to (h). The low quality image produces an aliasing effect at particle boundaries and the smaller and sharper corners are not delineated accurately. The computational geometry technique (Zheng and Hryciw 2015) has been used to determine Wadell's roundness,  $R$  (Wadell 1935). The parameter  $R$  measures the sharpness of corners on the particle as illustrated in Table 1. The lower quality images, especially when  $L$  is smaller than 130 pixels, significantly change the particle boundaries and therefore overestimate  $R$ . When soil's  $R$  changes by 0.1 (the  $R$  ranges from 0 to 1), the predicted critical state friction angle will vary by as much as 1.7 degrees (Cho et al. 2006), and the peak friction angle could vary by 2.4 degrees (Bareither et al. 2008). Such large differences in soil strength parameters would profoundly affect geotechnical design and analysis, which demonstrates the importance of reliable particle shape characterizations.

As geotechnical engineers rely increasingly on image based soil particle shape characterization, the minimum accepted image quality must be considered. However, existing studies using image based particle characterization rarely addressed this issue. Therefore, this study investigates the effects of image quality on computational results of commonly used particle shape descriptors in geotechnical engineering. Based on the results, minimum criteria for image quality that ensure accuracy and reliability of particle shape analyses are established. This research is important and original as these minimum criteria can be used as a guideline for use and development of image-based soil particle characterization techniques by geotechnical engineers. In addition, numerous correlations between particle shape descriptors and the macro-mechanical behavior of soils have been developed in the literature (Alshibli and Cil 2018; Altuhafi et al. 2016; Bareither et al. 2008; Cavarretta et al. 2010; Cho et al. 2006; Liu and Yang 2018; Shin and Santamarina 2013; Zheng et al. 2017; Zheng and Hryciw 2016, 2017). These minimum criteria will allow

geotechnical engineers to obtain reliable particle shape characterizations, which are a prerequisite to use these correlations.

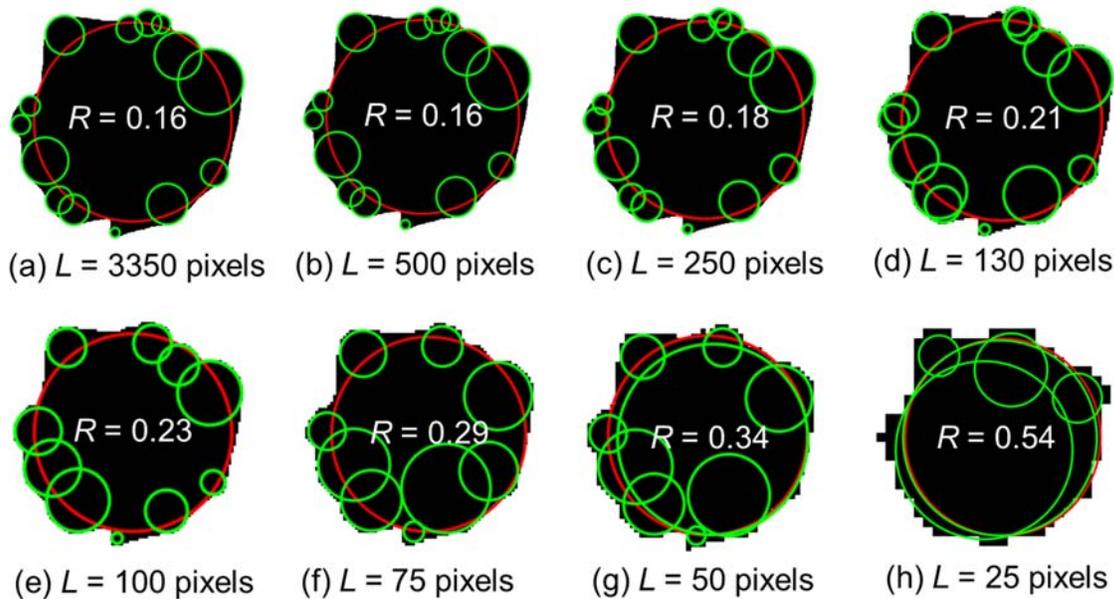


Fig. 1. Effects of image quality on particle shape analysis

## 2. Commonly used particle shape descriptors and reference particles for analysis

A variety of parameters have been proposed for quantifying particle shapes from two-dimensional particle projections. Eight commonly used shape descriptors in geotechnical engineering are summarized in Table 1. The numerical range of all these parameters is from 0 to 1. Some descriptors in Table 1 may have different terminologies, while describing the same particle property, while in other cases the same descriptor may be defined in different ways. For example, the convexity is also called solidity and sphericity has different definitions. Computations of the shape descriptors in Table 1 involve the particle's area, perimeter, convex boundary, inscribed and circumscribing circles, width, and length. These parameters can each be determined by the computational geometry technique developed by Zheng and Hryciw (2015). In particular, the computational geometry technique can automatically identify corners on the particle perimeter and fit appropriate circles to corners to determine roundness.

**Table 1. Commonly used particle shape descriptors**

Descriptors	Formula	Note	Reference
Aspect ratio (Width-to-length ratio sphericity, elongation)	$AR = \frac{W}{L}$	The ratio of the width of the particle ( $W$ ) to the length of particle ( $L$ ).	Krumbein and Sloss (1951)
Circle ratio sphericity	$S_C = \frac{D_{ins}}{D_{cir}}$	The ratio of the diameter of the largest inscribed circle of the particle ( $D_{ins}$ ) to the smallest circumscribing circle of the particle ( $D_{cir}$ ).	Santamarina and Cho, (2004)
Diameter sphericity	$S_D = \frac{D_e}{D_{cir}}$	The ratio of the diameter of a circle having the same area as the original particle ( $D_e$ ) to the diameter of the minimum circumscribing circle ( $D_{cir}$ ).	Wadell (1935)
Area sphericity	$S_A = \frac{A}{A_{cir}}$	The ratio of the area of the particle ( $A$ ) to the area of the smallest circumscribing circle ( $A_{cir}$ ).	Riley, (1941)
Perimeter sphericity	$S_P = \frac{P_e}{P}$	The ratio of the perimeter of the circle having the same area as the particle ( $P_e$ ) to the real perimeter of the particle ( $P$ ).	Kuo and Freeman (2000)
Circularity	$C = \frac{4\pi A}{P^2}$	The ratio of the area of the particle ( $A$ ) to the area of the circle having the same perimeter as the particle ( $P^2/4\pi$ ).	ISO (2008)
Convexity (solidity)	$C_x = \frac{A}{A_c}$	The ratio of the area of the particle ( $A$ ) to the area of the minimum convex boundary circumscribing the particle ( $A_c$ ).	Mora and Kwan (2000)
Roundness (angularity)	$R = \frac{\sum_{i=1}^N r_i / N}{r_{ins}}$	The ratio of the average radius of corner circles of the particles ( $r_i$ is the radius of $i$ -th corner and $N$ is the number of corners) to the radius of the maximum inscribed circle ( $r_{ins}$ ).	Wadell (1932, 1933, 1935)

It is challenging to establish a minimum criterion of image quality for particle shape characterizations due to the complexity of soil particles. The particle shape in an image may vary from very angular to well-rounded. Angular particles having complex surface structures require high resolution images to quantify, but rounded particles do not. The particle size in an image may vary over a wide range, and as a result, the

same image resolution may be sufficient to delineate larger particles' shapes, but not smaller particles. Therefore, the minimum criteria must consider both particle size and shape.

To consider effects of different particle shapes, this paper utilized twelve reference particles from the Powers' chart (Powers 1953) as shown in Fig. 2. These particles include very angular to well-rounded and elongated to spherical shapes, which represent the typically particle shapes encountered in natural, crushed, and manufactured granular soils.

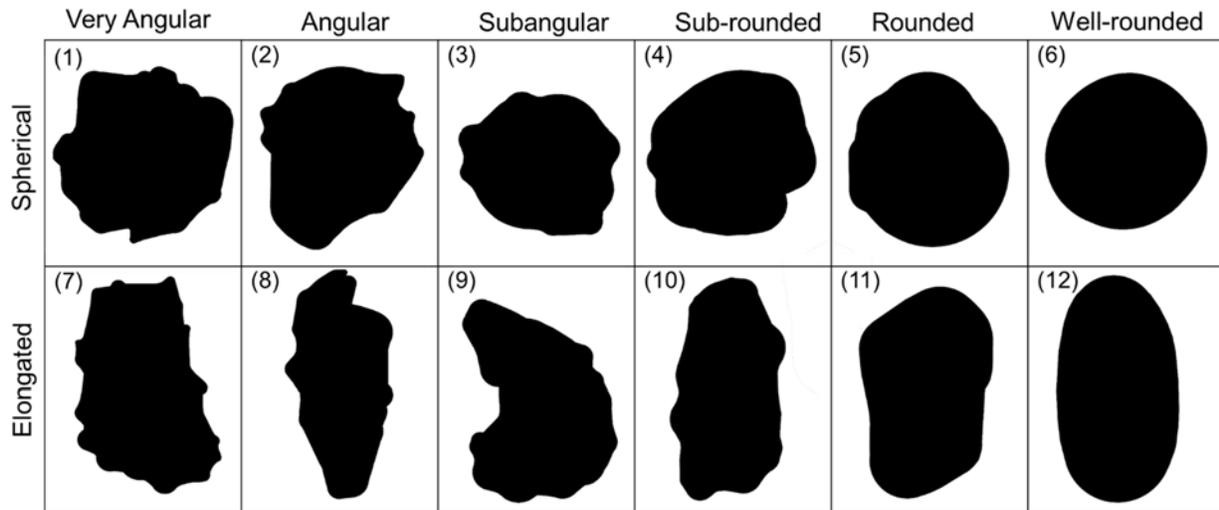


Fig. 2. The reference particles (lengths of particles are 3350 pixels)

To consider the effects of particle sizes, the image quality is quantified using particle's length ( $L$ ), perimeter ( $P$ ), and area ( $A$ ) in pixels, and not correlated to the physical dimensions of the particles. As such, the pixel-based minimum image quality criteria will be maintained regardless of the physical particle sizes. For example, if the minimum required particle length is 250 pixels for reliably characterizing roundnesses of angular particles, this criterion can be applied to all the particles with any sizes. If the actual particle size is 0.005 mm, a microscope may be required to capture this particle with a particle length of 250 pixels to meet the minimum image quality criterion. If the actual particle size is 50 mm, a cellphone camera may be adequate to capture this particle with a particle length of 250 pixels to meet the minimum image quality criterion.

### 3. Influence of image qualities on particle shape characterizations

The twelve particle projections in Fig. 2 have the same  $L$  of 3350 pixels, which represents high quality images. These particle projections are downscaled to generate new particle projections having  $L$  values of 1150, 800, 500, 250, 130, 100, 75, 50, 25, 15, and 8 pixels, which represent lower quality images. The computational geometry technique (Zheng and Hryciw 2015) was then used to compute the eight shape descriptors for each particle at different image qualities. The roundness computational results are shown in Fig. 3 (the results for  $L=1150$  pixels are not shown as they are identical to  $L=3350$  pixels). The  $R$  values are overestimated when image quality decreases because sharp and small corners of particles are missed. The minimum  $L$  ( $L_{\min}$ ) has been defined to ensure that the cumulative increases of particle shape descriptors are smaller than 2% relative to the overall measured range of that parameter for all of the particles. The typical measured range of  $R$  values of granular soils is from 0.1 to 1.0, so the  $L_{\min}$  is computed as 0.018, which is rounded to 0.20 as  $R$  values are usually reported in two decimals. The  $L_{\min}$  values are delineated as thick lines in Fig. 3. As shown, when the particles'  $L$  values are larger than  $L_{\min}$ , the image quality has a negligible impact on the  $R$  values. However, once the particles'  $L$  values are lower than  $L_{\min}$ , significant increases of the  $R$  values occur due to the changing of geometries of corners, so the image quality significantly affects the  $R$  computations. The  $L_{\min}$  establishes the minimum required particle length for particle roundness characterizations. As expected, angular particles need larger  $L_{\min}$  values than rounded particles in Fig. 3. The results of Fig. 3 are summarized in Fig. 4(h).

The same computations were repeated for the remaining seven descriptors and all the results are shown in Fig. 4. When  $L$  decreases, the shape descriptors tend to increase (the convexity slightly decreases and then shows inconsistent pattern of change below  $L_{\min}$ ). Therefore, the shape descriptors will be overestimated at lower resolution. The typical measured range of aspect ratio, sphericity descriptors and circularity of granular soils is around 0.5, and therefore the  $L_{\min}$  for these descriptors is  $2\% \times 0.5 = 0.01$ . The typical measured range of convexity of granular soils is around 0.1 and therefore the  $L_{\min}$  for convexity is  $2\% \times 0.1$

= 0.002. These  $L_{\min}$  values are also plotted in Fig. 4. The  $L_{\min}$  values for elongated and spherical particles of most of the descriptors are very close, so they are not differentiated, as they have been for roundness. Only relying on particle length may not be sufficient to define the minimum image quality because of the irregularity and complexity of soil particles. Therefore, we combine the particle's length, perimeter, and area together to establish the minimum image quality criterion. The relationships between eight shape descriptors and particle perimeter ( $P$ ) and square root of particle area ( $\sqrt{A}$ ) are plotted in Fig. 4. The  $P_{\min}$  and  $\sqrt{A}_{\min}$  are also defined similarly to the definition of  $L_{\min}$  and these values are superimposed on the figures. They again define cumulative increases of particle shape descriptors to be smaller than 2% of the overall measured range. Table 2 summarizes the minimum criteria (i.e.,  $L_{\min}$ ,  $P_{\min}$  and  $\sqrt{A}_{\min}$ ) for image quality for different shape descriptors.

$L$	3350	800	500	250	130	100	75	50
1	0.16	0.16	0.16	0.18	0.21	0.23	0.29	0.34
2	0.25	0.25	0.26	0.27	0.27	0.32	0.34	0.47
3	0.37	0.37	0.37	0.37	0.39	0.41	0.46	0.67
4	0.47	0.47	0.47	0.48	0.49	0.50	0.55	0.59
5	0.54	0.54	0.54	0.54	0.55	0.60	0.63	0.80
6	0.96	0.96	0.96	0.96	0.96	0.96	0.97	0.99
7	0.12	0.12	0.13	0.14	0.20	0.25	0.30	0.48
8	0.27	0.27	0.27	0.27	0.27	0.32	0.37	0.43
9	0.29	0.30	0.30	0.30	0.30	0.34	0.43	0.64
10	0.39	0.39	0.39	0.39	0.39	0.47	0.47	0.65
11	0.65	0.65	0.65	0.65	0.66	0.70	0.73	0.80
12	0.72	0.72	0.72	0.72	0.72	0.72	0.74	0.80

Fig. 3. Effects of image qualities on the roundness computations.

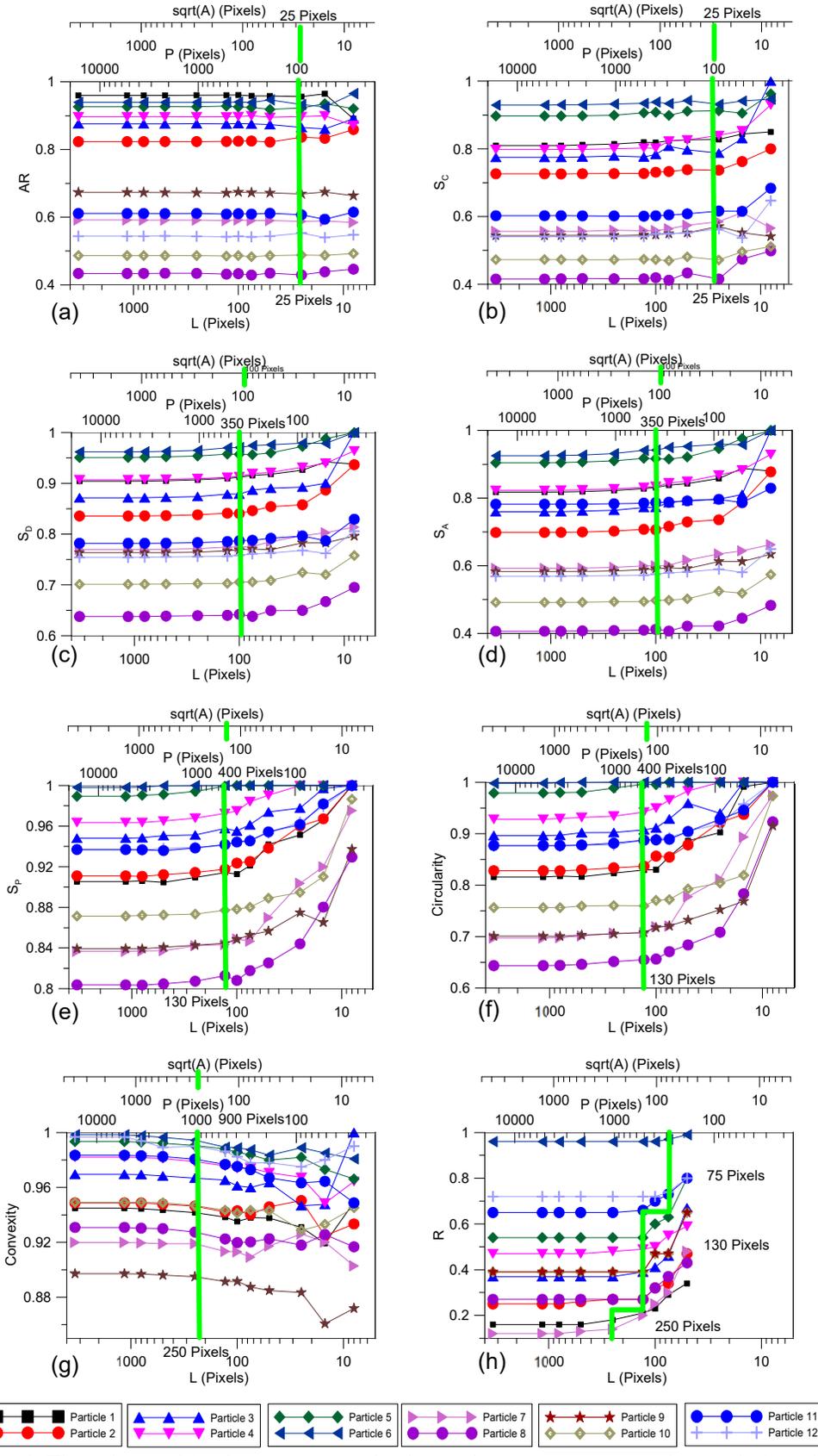


Fig. 4. The relationships between particle shape characterizations and particle lengths

**Table 2. The minimum criteria for ensuring reliable particle shape characterizations (units are pixels)**

Shape descriptors		$L_{\min}$	$P_{\min}$	$\text{Sqrt}(A)_{\min}$	Hierarchy
Aspect ratio ( $AR$ )		25	100	25	<b>Coarse descriptors</b> Evaluating principal dimensions of particles
Circle ratio sphericity ( $S_C$ )					
Diameter sphericity ( $S_D$ )		100	350	75	<b>Medium coarse descriptor</b> Related to areas of particles
Area sphericity ( $S_A$ )					
Perimeter sphericity ( $S_P$ )		130	400	100	<b>Fine descriptor</b> Related to perimeter of particles
Circularity					
Convexity		250	900	200	<b>Very fine descriptors</b> Evaluating perimeters of particles
Roundness ( $R$ )	Very angular to angular ( $0 < R < 0.17$ )				
	Angular to rounded ( $0.17 < R < 0.70$ )				
	Rounded to well-rounded ( $0.70 < R < 1.0$ )	75	350	70	

#### 4. Hierarchy of shape descriptors for resistance to the effects of low image quality

Low image qualities significantly affect the curvilinearities of perimeters (or boundaries) of particles because severe aliasing effects occur, as shown in Fig. 1. Therefore, the very fine descriptors,  $R$  and convexity, that are directly evaluating curvilinearities of particle perimeters have poor resistance to the low image quality and require the largest  $L_{\min}$ ,  $P_{\min}$  and  $\text{sqrt}(A)_{\min}$  values among the investigated descriptors. The fine descriptors,  $S_p$  and circularity, are related to perimeters of particles, but not directly analyzing them. Therefore, these two descriptors require relatively less strict criteria of image quality.

Low image qualities have limited influences on the areas of particles. Therefore, the medium coarse descriptors,  $S_D$  and  $S_A$ , which are related to areas of particles, require loose criteria. Low image qualities have minor influences on the principal dimensions (length and width) of particles. Therefore, the coarse descriptors,  $AR$  and  $S_C$ , which evaluate principal dimensions, require the loosest criteria of image quality.

## **Conclusion**

An analysis of common shape descriptors applied to standard particle images from Powers (1953) has shown that the low image resolution relative to the particle size can have a severe influence on the values of the various parameters. The particle shape descriptors will generally tend to be overestimated for low resolution images, but different shape descriptors show different resistances to low image quality, allowing a hierarchy of descriptor to be defined. The particle length, perimeter, and area are used as controlling parameters for establishing the minimum image quality for ensuring reliable optical characterizations of particle shapes, ranging from aspect ratio at the coarsest level, through various sphericities in the intermediate range to convexity and roundness at the finest. While the values of minimum required resolution do not vary much between angular and rounded particles for most descriptors, for roundness there is a clear need for better resolution as the particles become more angular.

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