Algebraic Aspects of Poincaré Duality

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Declaration

I, Erin Sutton, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

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Abstract

Let G be a finite group. It is an unsolved problem to classify closed connected manifolds with fundamental group G. This thesis represents a first approximation to solving this problem. We consider the universal covers of such manifolds, and require that these covers be connected up to, but not including, the middle dimension, and that they satisfy a specific formulation of Poincaré Duality originally set out by Lefschetz. Using results from homological algebra, in particular the work of Johnson and Remez in constructing diagonal resolutions for metacyclic groups, we are able to construct purely algebraic chain complexes and invariants which act as a first approximation to these universal covers for the cases G cyclic and metacyclic.

Impact Statement

The main benefits of this thesis lie in the field of academia. It uses an open question in topology as a starting point, and builds on recent results in homological algebra to explore this question, and as such might be of interest to scholars of both fields. It also touches on some aspects of number theory. As with much research in pure mathematics, it is difficult to provide immediate benefits outside of academia; external applications are usually discovered later. Existing research in the three main areas of maths considered in this thesis has found applications in fields such as physics, data science, computer science, and economics.

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Dedicated to my late brother, Stephen Sutton.
Keep on keeping on.

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Chapter 1

Introduction

1.1 Motivation

This thesis has several motivations, both algebraic and topological. The first comes from Lefschetz [15] and his original treatment of Poincaré Duality using dual cell structures. Poincaré Duality has been a fundamental part of many major results within the field of topology, and is usually stated as giving an isomorphism between the homology and cohomology groups of a space X, or a homotopy equivalence of chain groups. While this formulation has proved extremely powerful, it is also true that only defining an equivalence up to homotopy can obscure non-trivial and difficult behaviour. However by following Lefschetz's work and placing some minor restrictions upon X, one can in fact define Poincaré Duality as a strict isomorphism of chain complexes. This extremely precise formulation is beneficial when performing calculations at the chain level, and one can then ask for worthwhile applications.

The theory of manifolds is an expansive subject, and there are many ways to divide it into distinct cases. One could start by splitting into the three classes of Topological, Piecewise-Linear (PL) and Differential manifolds. One can also loosely divide the subject by dimension into manifolds M such that $\dim(M) \geq 5$, and then low-dimensional cases. From these we are interested in PL manifolds of dimension greater than 5. Finally, one can further split the subject into the study of simply connected manifolds, i.e. those manifolds M such that $\pi_1(M) = 0$, and non-simply connected manifolds, i.e. those with non-trivial fundamental group.

The development of techniques such as surgery theory allowed almost all problems in the simply connected case to be solved. However many problems in the non-simply connected case remain outstanding, such as the construction of free group actions on manifolds. Suppose you had a group G and wished to construct a free action on a specific manifold X, then this is equivalent to realising X as the universal cover of some manifold M satisfying $\pi_1(M) = G$.

The case $X = S^n$ was studied in depth by the likes of Milnor [17], Petrie [21], and Madsen, Thomas and Wall [16], and it makes sense to ask what the next stage up of complexity is from this case. We can classify the sphere as having non-zero homology only in the top and bottom dimensions and as a next step can consider the case where there exists non-zero homology in the middle dimensions too. Writing the dimension of such an X to be $2n$ or $2n + 1$, this is the same as X being $(n - 1)$ connected.

In the simply connected case and for $n \geq 3$, Wall gave a general classification of such $(n - 1)$ connected $2n$ manifolds in [25] and $(n - 1)$ connected $2n + 1$ manifolds in [26], and called them highly-connected. Barden completed the results in the odd case with a treatment of $n = 2$ in [1]. However when considering universal covers, we have the added structure of the chain groups being free $\mathbb{Z}[G]$ modules to contend with. As a first approximation to constructing M with fundamental group G and highly-connected universal cover $\tilde{M}$, we can seek to construct potential universal covers by constructing chain complexes of free $\mathbb{Z}[G]$ modules satisfying certain requirements, such as our precise formulation of Poincaré Duality.

To this end, recent developments in homological algebra have facilitated such constructions for the case $G = G(p, q)$. The metacyclic groups are defined by the presentation

$$G(p, q) = \langle x, y;\ x^p = y^q = 1,\ yx = \theta(x)y \rangle$$

where p prime, q a divisor of $p-1$ and $\theta \in \mathrm{Aut}(C_p)$ satisfies $\mathrm{ord}(\theta) = q$.

The concept of a free resolution was introduced by Hilbert (see [13]). If G is a finite group and $\Lambda = \mathbb{Z}[G]$, then every Λ-module M admits a free resolution, in other words, there exists an exact sequence of the form

$$\begin{tikzcd}[column sep=small]
 & & K_{m+1} \arrow[dr] & & & & & & K_1 \arrow[dr] & & & \\
\cdots \arrow[r] & F_{m+1} \arrow[ur] \arrow[rr] & & F_m \arrow[rr] \arrow[dr] & & F_{m-1} \arrow[r] & \cdots \arrow[r] & F_2 \arrow[ur] \arrow[rr] & & F_1 \arrow[rr] \arrow[dr] & & F_0 \arrow[r] & M \\
 & & & & K_m \arrow[ur] & & & & & & K_0 \arrow[ur] & &
\end{tikzcd}$$

where each F_i is a free module over Λ. Each K_i is a kernel and a stable invariant of the resolution, by which we mean if F, F' are two resolutions of a Λ-module M, then for each i, $K_i \oplus \Lambda^a \cong K_i' \oplus \Lambda^b$ for some $a, b \in \mathbb{N}$. The syzygy operators are then defined by $\Omega_i = [K_i]$, where $[K_i]$ denotes the stable equivalence class of K_i. There is also the notion of a projective resolution where each F_i may now be a projective module.

Johnson defines an almost free resolution of diagonal type and period $2q$ of a Λ-module M to be a resolution

$$\Delta_* = \ldots \to \Delta_{2n+1} \xrightarrow{\partial_{2n+1}} \Delta_{2n} \xrightarrow{\partial_{2n}} \Delta_{2n-1} \xrightarrow{\partial_{2n-1}} \ldots \xrightarrow{\partial_2} \Delta_1 \xrightarrow{\partial_1} \Delta_0 \to M$$

satisfying

- $\Delta_0 = \Lambda$;
- for each $k \geq 1$, $\Delta_{2k-1} = \Lambda \oplus \Lambda$ and $\Delta_{2k} = P(k) \oplus \Lambda$ where $P(k)$ is a projective module of rank 1 over Λ;
- for each $k \geq 2$ the differential ∂_k has the diagonal form $\partial_k = \begin{pmatrix} \partial_k^+ & 0 \\ 0 & \partial_k^- \end{pmatrix}$;
- $P(k+mq) = P(k)$ and $\partial_{k+mq}^{\pm} = \partial_k^{\pm}$ for all $k, m \geq 1$;
- $\bigoplus_{r=1}^{q-1} P(r) \cong \Lambda^{q-1}$ and $P(q) \cong \Lambda$.

Furthermore he defines a resolution of strongly diagonal type to be one also satisfying $\Delta_{2k} = \Lambda \oplus \Lambda$. In [7] Johnson showed that for the dihedral groups $D_{2p} = G(p, 2)$, $\mathbb{Z}$ admits a strongly diagonal resolution. Together with Remez in [9] he later showed that for general metacyclic groups $G(p, q)$, $\mathbb{Z}$ admits an almost free resolution, and in an upcoming paper they will also show that for a range of small values of p and q, $\mathbb{Z}$ admits a strongly diagonal resolution. These constructions also give a complete description of the syzygy modules.

These resolutions allow the construction of chain complexes of dimension $4k + 1$ which, in the strongly diagonal case, are possible models for our universal covers. Defining Ω^*_{2k+1} to be the dual of Ω_{2k+1} (see section 2.1), the non-trivial behaviour is encapsulated in a single boundary homomorphism

$$\partial : \Omega^*_{2k+1} \to \Omega_{2k+1}$$

where Ω^*_{2k+1} and Ω_{2k+1} are completely understood. Furthermore, the general almost free construction acts as a first approximation to such completely free constructions.

1.2 Statement of Results

In Chapter 2 we cover a range of algebraic concepts and results which are the foundation for much of the work in this paper. Particular interest is shown to the concept of a free resolution, which was introduced in the previous section. As well as the most general construction needed for its definition, we consider the case for two specific choices of $\Lambda = \mathbb{Z}[G]$. Firstly, $G = C_n$, the cyclic group of order n, where we derive the resolution from first principles, and then G metacyclic, where we give a brief overview of the results and concepts produced by Johnson and Remez in [9]. Fixing $G = G(p, q)$, we define a right Λ-module $R = \mathbb{Z}[\zeta_p]$ where ζ_p is the p^{th} root of unity with Λ-actions

$$\zeta^i \cdot x = \zeta^{i-1}$$
$$\zeta^i \cdot y = \theta^{-1}(\zeta^i)$$

and define the fixed ring of R by

$$R_0 = \{r \in R;\ \theta^{-1}(r) = r\}$$

Furthermore define $\pi \in R_0$ to be the unique (up to units) prime lying over p i.e the element satisfying

$$\pi^{\frac{p-1}{q}} = pu$$

for some $uinR_0$ a unit. Define the upper quasi-triangular matrices over A relative to I

$$\mathcal{T}_n(A, I) = \{X = (x_{ij})_{1 \leq i,j \leq n} \in M_n(A);\ x_{ij} \in I \text{ if } i > j\}$$

We can define right modules $R(k) = \{k^{th} \text{ row of } \mathcal{T}_n(A, (\pi)\}$, where

$$R(k + nq) \cong R(k)$$

for all k, n. The almost free resolution of $\mathbb{Z}$ which Johnson and Remez construct provides the odd syzygy description

$$\Omega_{2k+1}(\mathbb{Z}) = [R(k) \oplus [y - 1)]$$

where $[y - 1)$ is the right ideal generated by $y - 1$. We also note that each $R(k)$ has an alternate cyclotomic interpretation. Defining $P^k = (\zeta - 1)^k \mathbb{Z}[\zeta_p]$, it is true that

$$R(k) \cong \begin{cases} P^k & 1 \leq k \leq q - 1 \\ R & k = q \end{cases}$$

In Chapter 3 we give a proof of Poincaré Duality as originally set out by Lefschetz in [15]. We build up a number of results and constructions on ordered sets and ordered simplicial complexes from first principals, before focusing on what we term combinatorial manifolds, which are simplicial complexes which also possess the local structure of a manifold. Assuming a space X to be a combinatorial manifold we then provide a genuine geometric version of duality upon it via the use of the dual cell construction. We then formalise this result algebraically in terms of chain and cochain complexes through the use of what we call Lefschetz complexes and co-complexes. Given a combinatorial manifold X, we can associate a Lefschetz complex $\mathfrak{X}$ to it. Our dual cell construction then allows us to construct a Lefschetz co-complex $D(\mathfrak{X})$, and a chain isomorphism

$$h_* : C_*(\mathfrak{X}) \to C^{n-*}(D(\mathfrak{X}))$$

Defining $\mathcal{B}(X)$ to be the barycentric subdivision of X, and $\mathcal{B}(\mathfrak{X})$ to be the Lefschetz complex associated to $\mathcal{B}(X)$, we then improve upon this with the following strong version of Poincaré Duality:

Theorem. *There exist chain isomorphisms*

$$h_* : C_*(\mathcal{B}(\mathfrak{X})) \to C^{n-*}(\mathcal{B}(\mathfrak{X}))$$
$$\overline{h}_* : C^*(\mathcal{B}(\mathfrak{X})) \to C_{n-*}(\mathcal{B}(\mathfrak{X}))$$

We call a function f symmetric when $f^* = f$, and skew-symmetric when $f^* = -f$. There then exists a symmetry condition on each h_k, namely

$$h_k^* = (-1)^{\frac{n(n-1)}{2}} h_{n-k}$$

We then turn to studying combinatorial manifolds X with fundamental group G whose universal cover $\tilde{X}$ is a $(m-1)$ connected $2m+1$ combinatorial manifold. For these odd-dimensional manifolds, we use the above result to show their homology is essentially determined by a single boundary homomorphism. Reversing our thinking, we observe a similarity between the almost exact sequences of free Λ-modules which constitute our universal cover chain complexes, and the exact sequences of Λ-modules of free resolutions. We then use our results from Chapter 2 to construct purely algebraic chain complexes which satisfy conditions of a $(m-1)$ connected $2m+1$ combinatorial universal cover, whose homology is determined by the diagram

$$\begin{tikzcd} C^m \arrow[rrr, "\partial_{2m+1}"] \arrow[dr] & & & C_m \\ & \Omega^*_{m+1} \arrow[r, "\tilde{\partial}_{2m+1}"'] & \Omega_{m+1} \arrow[ur] & \end{tikzcd}$$

where $\tilde{\partial}^*_{m+1} = (-1)^m \tilde{\partial}_{m+1}$.

Given this diagram, we analyse $\tilde{\partial}_{2m+1} : \Omega^*_{m+1} \to \Omega_{m+1}$ at the minimal level for specific choices of Λ, starting with $\mathbb{Z}[C_n]$ in Chapter 4. For the case $m = 2k+1$, a general classification is simple. The case $m = 2k$ requires more work, but still allows a complete result. Here, the problem becomes studying $\mathbb{Z}[C_n]$-homomorphisms

$$F : \mathcal{R}^*_n \to \mathcal{R}_n$$

where

$$\mathcal{R}_n = {}^{\mathbb{Z}[C_n]}\big/_{1 + x + \ldots + x^{n-1} = 0}$$

We proceed by looking at matrix representations, and define $\xi_i \in GL_i(\mathbb{Z})$ to be the upper-triangluar square matrix consisting entirely of 1's, in other words, the matrix whose $(k,l)^{th}$ entry is defined by

$$(\xi_i)_{kl} = \begin{cases} 1 & l \geq k \\ 0 & k < l \end{cases}$$

We then define $\gamma_i \in GL_{p-1}(\mathbb{Z})$ by

$$\gamma_i = \begin{pmatrix} 0 & -\xi_i^t \\ \xi_{p-1-i} & 0 \end{pmatrix}$$

where t denotes the transose, and produce the following result:

Theorem. *Suppose $F : \mathcal{R}^*_n \to \mathcal{R}_n$ is a symmetric $\mathbb{Z}[C_n]$-homomorphism* *satisfying $F^* = F$. Then F has matrix representation*

$$F = a_{\frac{p-1}{2}}(\sum_{i=0}^{\frac{p-3}{2}} 2\gamma_i + \gamma_{\frac{p-1}{2}}) + \sum_{i=\frac{p+1}{2}}^{p-2} a_i(\gamma_i - \gamma_{p-(i+1)})$$

where $a_i \in \mathbb{Z}$.

In Chapter 5 we fix $G = G(p,q)$ and $m = 2k$, and begin by showing, at the minimal level, $\tilde{\partial}_{4k+1} : \Omega^*_{2k+1} \to \Omega_{2k+1}$ can be described in block matrix form

$$\tilde{\partial}_{2m+1} = \begin{pmatrix} F_k & G \\ G^* & H \end{pmatrix}$$

where

$$\begin{aligned} G &= (G_{k,0}, G_{k,2}, G_{k,3}, \ldots, G_{k,p-1})^t \\ G^* &= (G^*_{k,0}, G^*_{k,2}, G^*_{k,3}, \ldots, G^*_{k,p-1}) \end{aligned}$$

and, writing $\tilde{P}^k = (P^k)^*$

$$\begin{aligned} F &: \tilde{P}^k \to P^k \\ G_{k,i} &: \tilde{P}^i \to P^k \\ H &: [y-1)^* \to [y-1) \end{aligned}$$

The remainder of the chapter is then spent building a collection of results and techniques to classify F_k and $G_{k,i}$. We briefly look at the situation over $\mathbb{Q}[G(p,q)]$, before coming to the key concept that allows much of the following analysis; each P^k is completely characterised by two simple conditions. Defining the unit

$$v_X = \sum_{j=1}^{p-b} x^j$$

allows us to define two properties for a Λ-module M $(0 \le k \le q-1)$:

M(Σ): $\mathrm{rk}_{\mathbb{Z}}(M) = p-1$ and $M \cdot \Sigma_x = 0$ where $\Sigma_x = \sum_{k=0}^{p-1} x^k$

M(k): There exists $\epsilon_k \in M$ such that $\epsilon_k \cdot y = \epsilon_k \cdot (-1)^k v_X^k$ and $\mathrm{Span}_{\mathbb{Z}}\{\epsilon_k \cdot x^i\}_{0 \leq i \leq p-2} = M$

It is then true that $M \cong P^k$ if and only if M satisfies both properties. We call ϵ_k the characteristic element of P^k. Since we also have duality relations

$$\tilde{P}^k = \begin{cases} P & k = 0 \\ R & k = 1 \\ P^{q+1-k} & 2 \leq k \leq q-1 \end{cases}$$

each $\tilde{P}^k$ must possess its own characteristic element which we label $\tilde{\epsilon}_k$. It follows that any Λ-homomorphism $F_k : \tilde{P}^k \to P^k$ will be completely determined by where it sends $\tilde{\epsilon}_k$, and so can be completely described by

$$F_k(\tilde{\epsilon}_k) = \epsilon_k \cdot \alpha$$

for some $\alpha \in R$. The question then becomes how to calculate α. To do this, we produce a number of results that allow F_k to be broken down into constituent parts of duality isomorphisms, projection homomorphisms and endomorphisms, each defined on the relevant characteristic elements. Since we also require a symmetry condition on F_k, we need to understand how these parts behave under duality, and it is here we encounter our main hurdle: to calculate the dual of our duality isomorphism, we need a description of $\tilde{\epsilon}_k$ in terms of the natural dual basis.

By showing $\overline{\pi} = \pi \cdot w$ for $w \in R_0^{\times}$ a unit, it follows we can consider π as an element acting from the right. Furthermore, we show the existence of a unit $u_\pi \in R$ satisfying

$$(x^{p-1} - 1)^q u_\pi = \pi$$

Defining basis elements $p[k,i] = \epsilon_k \cdot x^i$ we then obtain:

Theorem.

$$\tilde{\epsilon}_k = \begin{cases} p[k,0]^* & k = 0 \\ p[k,0]^* \cdot x & k = 1 \\ p[k,0]^* \cdot x^k u_\pi^{-1} & 2 \leq k \leq q-1 \end{cases}$$

This result allows us to calculate how our functions behave under duality. However to make further progress, we need an explicit description for $w \in R_0^\times$, and therefore define a property of G:

G(π): $\overline{\pi} = (-1)^q \pi$

For the rest of Chapter 5, we show that $G(\pi)$ is satisfied for a large number of groups, and produce the following:

Theorem. *Suppose $G = G(p, 2j)$ for some j, or $G(p, \frac{p-1}{2})$ where $\frac{p-1}{2}$ is odd. Then Λ satisfies $G(\pi)$.*

In Chapter 6 we bring together the results of the previous chapter to give the classifications of F_k. Defining subsets of R_0

$$R_0^\pm = \{r \in R_0;\ \overline{r} = \pm r\}$$

we obtain the following:

Theorem. *Suppose $G = (p, 2r)$ and $F_k : \tilde{P}^k \to P^k$ is a symmetric Λ-isomorphism. Then*

$$F_k(\tilde{\epsilon}_k) = \begin{cases} \epsilon_k \cdot (x^{p-1} - 1)\alpha_+ & k = 0 \\ \epsilon_k \cdot (x^{p-1} - 1)^{q+1-2k} u_\pi \alpha_+ & k = 1 \\ \epsilon_k \cdot (x^{p-1} - 1)^{q+1-2k} \alpha_+ & 1 < k \leq \frac{q}{2} \\ \epsilon_k \cdot (x^{p-1} - 1)^{2(q-k)+1} u_\pi \alpha_+ & \frac{q+2}{2} \leq k < q \end{cases}$$

where $\alpha_+ \in R_0^+$, $\alpha_- \in R_0^-$.

Theorem. *Suppose $G = G(p, 2r+1)$, Λ satisfies $G(\pi)$, and $F_k : \tilde{P}^k \to P^k$ is a symmetric Λ-isomorphism. Then*

$$F_k^+(\tilde{\epsilon}_k) = \begin{cases} \epsilon_k \cdot (x^{p-1}-1)\alpha_+ & k=0 \\ \epsilon_k \cdot (x^{p-1}-1)^{q+1-2k}u_\pi\alpha_- & k=1 \\ \epsilon_k \cdot (x^{p-1}-1)^{q+1-2k}\alpha_- & 1<k<\frac{q+1}{2} \\ \epsilon_k \cdot \alpha_- & k=\frac{q+1}{2} \\ \epsilon_k \cdot (x^{p-1}-1)^{2(q-k)+1}u_\pi\alpha_+ & \frac{q+1}{2}<k<q \end{cases}$$

where $\alpha_+ \in R_0^+$, $\alpha_- \in R_0^-$.

We then provide a number of explicit matrix representations for F_k for some small values of p, q, before also using the results of Chapter 5 to classify $G_{k,i}$:

Theorem. *Suppose $G_{k,0} \in \mathrm{Hom}_\Lambda(\tilde{R}, P^k)$. Then*

$$G_{k,0}(\tilde{\epsilon}_0) = \begin{cases} \epsilon_k \cdot (x^{p-1}-1)\alpha & k=0 \\ \epsilon_k \cdot \alpha & k=1 \\ \epsilon_k \cdot (x^{p-1}-1)^{q+1-k}u_\pi\alpha & 2 \leq k \leq q-1 \end{cases}$$

for $\alpha \in R_0$

Theorem. *Suppose $2 \leq i \leq q-1$, and $G_{k,i} \in \mathrm{Hom}_\Lambda(\tilde{P}^i, P^k)$. Then*

$$G_{k,i}(\tilde{\epsilon}_i) = \begin{cases} \epsilon_k \cdot (x^{p-1}-1)^{q+1-i-k}\alpha & k<q+1-i \\ \epsilon_k \cdot (x^{p-1}-1)^{2q+1-i-k}u_\pi\alpha & k>q+1-i \\ \epsilon_k \cdot \alpha & k=q+1-i \end{cases}$$

for $\alpha \in R_0$.

Finally we consider $H : [y-1)^* \to [y-1)$. Defining

$$\alpha_n = \begin{pmatrix} 0 & I_{n-1} \\ 1 & 0 \end{pmatrix}$$

we obtain:

Theorem. *Suppose $H : [y-1)^* \to [y-1)$ is a symmetric Λ-homomorphism. Then H can be expressed as a product*

$$(h_1 \otimes h_2) : (\mathcal{R}_q^* \otimes \mathbb{Z}[C_p]^*) \to (\mathcal{R}_q \otimes \mathbb{Z}[C_p])$$

where

$$h_1 = a_{\frac{n-1}{2}}(\sum_{i=0}^{\frac{n-3}{2}} 2\gamma_i + \gamma_{\frac{n-1}{2}}) + \sum_{i=\frac{n+1}{2}}^{n-2} a_i(\gamma_i - \gamma_{n-(i+1)})$$

$$h_2 = b_0\alpha_0 + \sum_{j=1}^{\frac{p-1}{2}} b_i(\alpha_i + \alpha_{p-i})$$

$a_i, b_i \in \mathbb{Z}$.

Together, these results give a pleasingly complete algebraic description of our middle boundary homomorphisms for the case G metacyclic and satisfying $G(\pi)$

Chapter 2

Algebraic Preliminaries

2.1 Basic Concepts

Let Λ be a ring with multiplicative identity 1_Λ. A right Λ-module M consists of an abelian group $(M, +)$ and an operation $M \times \Lambda \to M$ such that

- $(x + y)\lambda_1 = x\lambda_1 + y\lambda_1$;
- $x(\lambda_1 + \lambda_2) = x\lambda_1 + x\lambda_2$;
- $x(\lambda_1\lambda_2) = (x\lambda_1)\lambda_2$;
- $x1_\lambda = x$

for all $x, y \in M$, $\lambda_1, \lambda_2 \in \Lambda$. We call a Λ-module M free when it has a basis over Λ. A Λ-lattice is a Λ-module whose underlying abelian group is finitely generated and torsion free. A module M is simple if its only submodules are $\{0\}$ and M, and is semisimple when it can be decomposed as the direct product

$$M = M_1 \oplus \ldots \oplus M_n$$

where each M_i is simple.

Suppose Λ is a commutative ring and G a finite group. The group ring $\Lambda[G]$ is defined as the formal sum

$$\Lambda[G] = \sum_{g \in G} a_g g$$

where $a_g \in \Lambda$. Defining the sum and product respectively by

$$\sum_{g\in G} a_g g + \sum_{g\in G} b_g g = \sum_{g\in G}(a_g + b_g)g$$
$$(\sum_{g\in G} a_g g) \cdot (\sum_{h\in G} b_h h) = \sum_{g\in G}(\sum_{h\in G} a_h b_{h^{-1}g})g$$

we see that $\Lambda[G]$ has the structure of a free module over Λ. If Λ is a field, say $\mathbb{F}$, then $\mathbb{F}[G]$ becomes an algebra called the group algebra. We note that the definition of semisimplicity given for modules is easily extended to algebras.

Let V be a vector field over a field $\mathbb{F}$ and G a finite group. Classically a group representation of G on $\mathbb{F}$ is defined to be the homomorphism

$$\rho : G \to GL_{\mathbb{F}}(V)$$

where $GL_{\mathbb{F}}(V)$ is the group of automorphisms of V over $\mathbb{F}$. If V has dimension n over $\mathbb{F}$, then $GL_{\mathbb{F}}(V)$ can be thought of as the group of $n \times n$ invertible matrices with coefficients in $\mathbb{F}$. We have the following famous theorems of representation theory (see [2] §3 for a complete treatment):

Theorem 2.1.1 (Wedderburn Decomposition). *Let $\mathbb{A}$ be a semisimple algebra of finite dimension over a field $\mathbb{F}$. Then there exists an isomorphism of $\mathbb{F}$-algebras*

$$\mathbb{A} \cong \prod_{i=1}^{m} M_{n_i}(\mathcal{D}_i)$$

where $n_i, m \in \mathbb{N}$, $\mathcal{D}_i$ are division algebras over $\mathbb{F}$, and $M_{n_i}(\mathcal{D}_i)$ denotes the set of $n_i \times n_i$ matrices with coefficients in $\mathcal{D}_i$. Both n_i and $\mathcal{D}_i$ are uniquely determined up to isomorphism.

Theorem 2.1.2 (Maschke). *Let G be a finite group and $\mathbb{F}$ a field with characteristic coprime to the order of G. Then $\mathbb{F}[G]$ is semisimple.*

The majority of this thesis is concerned with $\mathbb{Z}[G]$ and its modules (equivalently, its representations), where since $\mathbb{Z}$ is not a field, the above results do not hold. Representations are easily extended to free modules M over a ring

R however, and we define integral representations

$$\rho : G \to GL_{\mathbb{Z}}(M)$$

Suppose M is a free right $\mathbb{Z}[G]$-module of rank n with basis elements e_i, so that $M = \mathrm{Span}_{\mathbb{Z}}\{e_i\}_{0\leq i\leq n-1}$. We can construct explicit matrix representations via the standard relation

$$\rho(g)(e_i) = e_i \cdot g$$

for $g \in G$. Suppose

$$\rho(g)(e_i) = \sum_{j=0}^{n-1} a_{ij}e_j$$

for $a_{ij} \in \mathbb{Z}$. Then $\rho(g)$ can be expressed as a matrix $\rho(g) \in M_n(\mathbb{Z})$ via

$$\rho(x) = (a_{ji})_{ij}$$

If M is a $\mathbb{Z}[G]$ module, define the dual module

$$M^* = \mathrm{Hom}_{\Lambda}(M, \Lambda)$$

M^* is naturally a left module, which can be converted to a right module via the formal involution $\omega : \mathbb{Z}[G] \to \mathbb{Z}[G]$ defined by

$$\omega(\sum_{g\in G} a_g g) = \sum_{g\in G} a_g \omega(g) = \sum_{g\in G} a_g g^{-1}$$

Unless otherwise specified, when we write M^* we shall mean it considered as a right module. For $\lambda \in \mathbb{Z}[G]$, we introduce a shorthand for ω as

$$\omega(\lambda) = \overline{\lambda}$$

M^* has a dual representation, ρ^*, associated to it, and written as matrices we have

$$\rho^*(g) = \rho(g^{-1})^t$$

2.2 Resolutions and Syzygies

Let Λ be a ring and M a finitely generated Λ module. We can take a minimal generating set $\{\phi_i\}_{0<i\leq m_0}$ for M, i.e. every $x \in M$ can be written

$$x = \phi_1\lambda_1 + \ldots + \phi_{m_0}\lambda_{m_0}$$

for some $m_0 \in \mathbb{Z}$, $\lambda_i \in \Lambda$. Let $\{e_i\}_{0<i\leq m_0}$ be the standard basis for Λ^{m_0}, and define a surjective mapping

$$\begin{aligned} \partial_0 : \Lambda^{m_0} &\to M \\ e_i &\mapsto \phi_i \end{aligned}$$

It follows that ∂_0 has a kernel, which we denote K_1, and so generates a short exact sequence

$$0 \to K_1 \hookrightarrow \Lambda^{m_0} \twoheadrightarrow M \to 0$$

We can repeat this process for K_1 to generate a surjective mapping

$$\partial_1 : \Lambda^{m_1} \to K_1$$

and a short exact sequence

$$0 \to K_2 \hookrightarrow \Lambda^{m_1} \twoheadrightarrow K_1 \to 0$$

Continuing this process inductively, we generate a series of exact sequences

$$0 \to K_{i+1} \hookrightarrow \Lambda^{m_i} \twoheadrightarrow K_i \to 0$$

Splicing these short sequences together allows us to construct the long exact sequence

$$
\begin{tikzcd}[column sep=small]
 & & K_{m+1} \arrow[dr] & & & & & & K_1 \arrow[dr] & & & \\
\cdots \arrow[r] & F_{m+1} \arrow[rr] \arrow[ur] & & F_m \arrow[r] \arrow[dr] & F_{m-1} \arrow[r] & \cdots \arrow[r] & F_2 \arrow[r] \arrow[urr] & & F_1 \arrow[r] \arrow[dr] & F_0 \arrow[r] & M \\
 & & & & K_m \arrow[u] & & & & & K_0 \arrow[u] & &
\end{tikzcd}
$$

This construction gives us the notion of a free resolution. Formally, a free resolution of a Λ-module M is an exact sequence

$$\ldots \to F_m \xrightarrow{\partial_m} F_{m-1} \xrightarrow{\partial_{m-1}} \ldots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} M \to 0$$

where each F_i is a free Λ-module. Any such free resolution F_* is not unique; there exists another free resolution, say F'_*

$$\ldots \to F'_m \xrightarrow{\partial'_m} F'_{m-1} \xrightarrow{\partial'_{m-1}} \ldots \xrightarrow{\partial'_2} F'_1 \xrightarrow{\partial'_1} F'_0 \xrightarrow{\partial'_0} M \to 0$$

where $F_i \ncong F'_i$. However there does exist a relation between the kernels of any two resolutions. Two Λ-modules M, M' are said to be stably equivalent when there exists some $a, b \geq 0$ such that

$$M \oplus \Lambda^a \cong M' \oplus \Lambda^b$$

We denote the stable equivalence class of M by $[M]$.

Proposition 2.2.1. *Let F_*, F'_* be two free resolutions of a Λ module M. Then K_i is stably equivalent to K'_i.*

To prove this we first need a result of Schanuel (pp 165-168 in [14]):

Lemma 2.2.2 (Schanuel). *Suppose*

$$0 \to K \xrightarrow{i} F \xrightarrow{p} M \to 0$$

$$0 \to K' \xrightarrow{j} F' \xrightarrow{q} M \to 0$$

are exact sequences where F, F' are free. Then $K \oplus F' \cong K' \oplus F$.

Proof. Construct the fibre product over F by defining

$$F \times F' = \{(x, y) \in F \oplus F'; p(x) = q(y)\}$$

and ρ to be the projection onto the first factor, so that we have the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(\rho) \arrow[r] & F \times F' \arrow[r, "\rho"] \arrow[d] & F \arrow[r] \arrow[d, "q"] & 0 \\
0 \arrow[r] & K' \arrow[r] & F' \arrow[r, "p"] & M \arrow[r] & 0
\end{tikzcd}$$

Since p, q are both surjective, so must ρ be also. Furthermore

$$\mathrm{Ker}(\rho) = \mathrm{Ker}(p) = K'$$

and we can induce a short exact sequence

$$0 \to K' \to F \times F' \to F \to 0$$

We can repeat the construction over F' to induce another exact sequence

$$0 \to K \to F \times F' \to F' \to 0$$

Since F, F' are free, both these sequences split, so that

$$K \oplus F' \cong F \times F' \cong K' \oplus F$$ □

We can now prove 2.2.1.

Proof. The proof is by induction on i, with base case true by 2.2.2. Suppose true for $i-1$ so that there exist a, b such that

$$K_{i-1} \oplus \Lambda^a \cong K'_{i-1} \oplus \Lambda^b$$

Splitting the resolutions at the i^{th} stage we obtain two exact sequences

$$0 \to K_i \to F_{i-1} \to K_{i-1} \to 0$$
$$0 \to K'_i \to F'_{i-1} \to K'_{i-1} \to 0$$

We can stabilise these by 'adding on' a number of frees, to induce exact sequences

$$0 \to K_i \to F_{i-1} \oplus \Lambda^a \to K_{i-1} \oplus \Lambda^a \to 0$$
$$0 \to K'_i \to F'_{i-1} \oplus \Lambda^b \to K'_{i-1} \oplus \Lambda^b \to 0$$

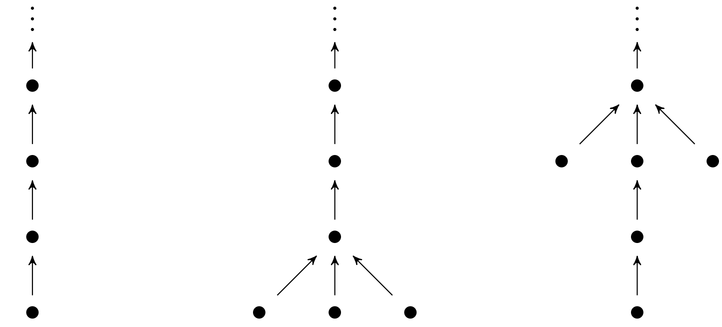
By the induction hypothesis, $K_{i-1} \oplus \Lambda^a \cong K'_{i-1} \oplus \Lambda^b$, so that applying Schanuel gives

$$K_i \oplus F'_{i-1} \oplus \Lambda^b \cong K'_i \oplus F_{i-1} \oplus \Lambda^a \qquad \square$$

Define the i^{th} syzygy of a module M as

$$\Omega_i(M) = [K_i]$$

The structure of $\Omega_i(M)$ can be represented graphically as a directed tree by taking each node to be a module in the equivalence class, and each path to represent the direct sum with one copy of Λ. If G is finite, then $\Omega_i(M)$ has three possible structures:



A B C

Defining a minimal module M_0 to be a module which does contain a summand isomorphic to Λ, we see that the three structures correspond to a unique M_0 (type A); multiple M_0 all occurring at the same (bottom-most) level of the tree (type B); and multiple M_0 occurring at two different levels of the tree (type C). We call a syzygy which can be represented by a tree of type A straight. Equivalently, a syzygy is straight when it can be written

$$\Omega_i(M) = M_0 \oplus \Lambda^{a_i}$$

for some minimal module M_0 and $a_i \in \mathbb{Z}$.

We call a resolution F_* periodic of period n when it satisfies

- $F_i = F_{i+kn}$;
- $\partial_i = \partial_{i+kn}$

for all $k \in \mathbb{N}$. This thesis considers the syzygy modules $\Omega_i(\mathbb{Z})$ over rings $\Lambda = R[G]$ where R is a commutative ring and Λ is equipped with a periodic resolution.

There also exists the notion of a projective resolution

$$\ldots \to P_m \xrightarrow{\partial_m} P_{m-1} \xrightarrow{\partial_{m-1}} \ldots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} M \to 0$$

where each P_i is a projective Λ-module. The definition of syzygies given for free resolutions also holds for projective resolutions.

2.3 Resolutions and Syzygies for Cyclic Groups

We now consider a specific choice of Λ, namely $\Lambda = \mathbb{Z}[C_n]$, where C_n is the cyclic group of order n with group presentation

$$\langle x;\ x^n = 1\rangle$$

There exists a short exact sequence

$$0 \to I_C \to \Lambda \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

where ϵ is the augmentation map defined by

$$\epsilon(\sum_{i=0}^{n-1} a_i x^i) = \sum_{i=0}^{n-1} a_i$$

and $I_C = \mathrm{Ker}(\epsilon)$ is called the augmentation ideal. We can dualise this to induce another exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\epsilon^*} \Lambda \to I_C^* \to 0$$

where we use the identifications $\Lambda^* \cong \Lambda$, $\mathbb{Z}^* \cong \mathbb{Z}$. Here $\epsilon^*(1) = \sum_{i=0}^{n-1} x^i$.

Proposition 2.3.1. $I_C \cong I_C^*$.

Proof. Define basis elements $\epsilon_i = x^i - 1$. Then the canonical basis of I_C is

$$\mathcal{E} = \{\epsilon_i\}_{1\leq i\leq n-1}$$

Defining basis elements $\phi_i = x^i - x^{i+1}$ we can define an alternative basis

$$\Phi = \{\phi_i\}_{1\leq i\leq n-1}$$

To check that Φ is a basis, construct a new basis Φ' by

$$\phi_i' = \begin{cases} \phi_i & i \neq n-2 \\ \phi_{n-2} + \phi_{n-1} & i = n-2 \end{cases}$$

From Φ', we can construct a further basis Φ'' by

$$\phi_i'' = \begin{cases} \phi_i' & i \neq n-3 \\ \phi_{n-3}' + \phi_{n-2}' & i = n-3 \end{cases}$$

Continuing this process gives a sequence of elementary basis transformations which eventually ends at $\mathcal{E}$, and so Φ is indeed a basis. Let $\rho^{\mathcal{E}}$ be the standard representation of $\mathcal{E}$, and ρ^{Φ} the standard representation of Φ. Then, using the easily verified relation

$$1 - x = \sum_{i=1}^{n-1} x^{i+1} - x^i$$

we obtain

$$\begin{aligned} \rho^{\Phi}(x)(\phi_i) &= \phi^i \cdot x \\ &= (x^i - x^{i+1})x^{n-1} \\ &= \begin{cases} -\sum\limits_{j=1}^{n-1} \phi_j & i = 1 \\ \phi_{i-1} & 2 \leq i \leq n-1 \end{cases} \end{aligned}$$

Writing c_{n-1} for the $(n-1) \times 1$ column matrix consisting entirely of $1's$, and I_{n-1} for the identity matrix of size $(n-1) \times (n-1)$, we therefore have the block matrix description

$$\rho^{\Phi}(x) = \begin{pmatrix} -c_{n-1} & I_{n-1} \\ -1 & 0 \end{pmatrix}$$

Similarly, using the fact that

$$x^j - x = x^j - 1 + 1 - x = \epsilon_j - \epsilon_1$$

one can calculate

$$\begin{aligned} \rho(x^{-1})(\epsilon_i) &= \epsilon_i \cdot x^{-1} \\ &= (x^i - 1)x \\ &= \begin{cases} \epsilon_{i+1} - \epsilon_1 & 1 \leq i \leq n-2 \\ -\epsilon_1 & i = n-1 \end{cases} \end{aligned}$$

Therefore

$$\rho^{\mathcal{E}}(x^{-1}) = \begin{pmatrix} -c_{n-1}^t & -1 \\ I_{n-1} & 0 \end{pmatrix}$$

Clearly

$$(\rho^{\mathcal{E}})^*(x) = \rho^{\mathcal{E}}(x^{-1})^t = \rho^{\Phi}(x)$$

so that Φ is the canonical basis for I_C^* and the result follows. □

Define a Λ-module

$$\mathcal{R}_n = \mathbb{Z}[C_n]/(\sum_{i=0}^{n-1} x^i)$$

with right action

$$x^a \cdot x = x^{a-1}$$

Then $I_C^* \cong \mathcal{R}_n$. Furthermore, for $n = p$ prime and $\zeta_p = \exp(2\pi i/p)$, we obtain $\mathcal{R}_p \cong \mathbb{Z}[\zeta_p]$. The module action then becomes

$$\zeta^a \cdot x = \zeta^{a-1}$$

Proposition 2.3.2. *Suppose* $\Lambda = \mathbb{Z}[C_n]$. *Then*

$$\Omega_{2j}(\mathbb{Z}) = [\mathbb{Z}]$$

$$\Omega_{2j+1}(\mathbb{Z}) = [\mathcal{R}_n]$$

Proof. Recall the two short exact sequences

$$0 \to I_C \to \Lambda \to \mathbb{Z} \to 0$$
$$0 \to \mathbb{Z} \to \Lambda \to I_C^* \to 0$$

Using 2.3.1, we can splice these two sequences together at $I_C \cong I_C^*$ to give a resolution of length 2

$$0 \to \mathbb{Z} \to \Lambda \to \Lambda \to \mathbb{Z} \to 0$$

We can now repeatedly splice at $\mathbb{Z}$ to generate a resolution of arbitrary length and period 2, and the syzygies then follow. □

We also have the following:

Proposition 2.3.3. *Suppose $\Lambda = \mathbb{Z}[C_n]$. Then $\Omega_i(\mathbb{Z})$ is straight.*

Proof. In [6] Johnson proves that if the Wedderburn decomposition of $\Lambda \otimes \mathbb{R}$ does not contain $\mathbb{H}$ as a summand, then $\Omega_1(\mathbb{Z})$ is straight. Furthermore, from 2.3.2, $\Omega_1(\mathbb{Z}) = \Omega_{2j+1}(\mathbb{Z})$. Johnson also proves that if $M_0 \cong \mathbb{Z}$, and the Wedderburn decomposition of $\Lambda \otimes \mathbb{R}$ does not contain $\mathbb{H}$ as a summand, then $\Omega_{2j}(\mathbb{Z})$ is straight. It is a standard result that $\mathbb{Q}[C_n]$ has Wedderburn decomposition

$$\mathbb{Q}[C_n] = \prod_{d|n} \mathbb{Q}[x]/c_d(x)$$

where $c_d(x)$ is the d^{th} cyclotomic polynomial. Tensoring with $\mathbb{R}$ will not generate a $\mathbb{H}$ term, hence the result. □

We briefly note that for such a resolution, we can define boundary operators as

$$\partial_{2j-1} = x - 1 \qquad \partial_{2j} = \Sigma_x$$

2.4 Resolutions and Syzygies for Metayclic Groups

For p an odd prime and q a divisor of $p-1$, define the metacyclic groups by the group presentation

$$G(p,q) = \langle x, y;\ x^p = y^q = 1,\ yx = \theta(x)y\rangle$$

where $\theta \in \mathrm{Aut}(C_p) \cong C_{p-1}$, $\mathrm{ord}(\theta) = q$. We note that there is no general expression for θ, and it must be calculated on a case by case basis. Fixing $\Lambda = \mathbb{Z}[G(p,q)]$, we outlne the treatment of Johnson and Remez in [9] who construct a special type of Λ resolution of $\mathbb{Z}$. We defer all proofs and in-depth discussion to that paper. Define a resolution of diagonal type to be a projective resolution

$$\Delta_* = (\ldots \to \Delta_{n+1} \xrightarrow{\partial_{n+1}} \Delta_n \xrightarrow{\partial_n} \Delta_{n-1} \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_2} \Delta_1 \xrightarrow{\partial_1} \Delta_0 \to M \to 0)$$

satisfying

- $\Delta_0 = \Lambda$;
- for each $k \geq 1$, $\Delta_{2k-1} = \Lambda \oplus \Lambda$ and $\Lambda_{2k} = P(k) \oplus \Lambda$ where $P(k)$ is a projective module of rank 1 over Λ;
- for each $k \geq 2$ the differential ∂_k has the diagonal form $\partial_k = \begin{pmatrix} \partial_k^+ & 0 \\ 0 & \partial_k^- \end{pmatrix}$.

Furthermore, we say Δ_* is almost free when it satisfies

$$\bigoplus_{k=1}^{q-1} P(k) \cong \Lambda^{q-1} \text{ and } P(q) \cong \Lambda$$

Remez and Johnson show that the trivial module $\mathbb{Z}$ admits an almost free diagonal resolution of period $2q$.

Recall the $\mathbb{Z}[C_p]$ module $I_C^* \cong \mathbb{Z}[\zeta_p]$. By extension of scalars, $\mathbb{Z}[\zeta_p]$ is also a Λ-module. In order to distinguish between $\mathbb{Z}[\zeta_p]$ considered as a $\mathbb{Z}[C_p]$-module and a Λ-module, we will denote it by R_C and R respectively. Since

$\theta(\Sigma_x) = \Sigma_x$, θ also induces a ring automorphism of order q on R, and we define Λ actions

$$\zeta^i \cdot x = \zeta^{i-1}$$
$$\zeta^i \cdot y = \theta^{-1}(\zeta^i)$$

Define R_0 to be the fixed ring of R under θ (or equivalently, θ^{-1})

$$R_0 = R^\theta = \{r \in R; \theta^{-1}(r) = r\}$$

Define $\pi \in R_0$ to be the unique prime in R_0 lying over p. R_0 and π then satisfy:

- $\dim_{\mathbb{Z}}(R_0) = \frac{p-1}{q}$;
- πR_0 has index p in R_0;
- $\pi^{\frac{p-1}{q}} = pu$ for some $u \in R_0$ a unit.

A diagonal resolution consists of two strands, where we can think of the lower strand corresponding to the subgroup C_q, and the upper corresponding to the subgroup C_p. To construct the lower strand we take the resolution of C_q we constructed in 2.3.2 and simply induce upwards

$$\ldots \to \Lambda \xrightarrow{y-1} \Lambda \xrightarrow{\Sigma_y} \Lambda \xrightarrow{y-1} \Lambda \xrightarrow{\Sigma_y} \ldots$$

The construction of the upper strand is not so simple. Suppose $\tilde{S}$, S_+, S_-, S are rings. We say that S is a fibre product of S_+ and S_- over S_0 when there exists a commutative square of ring homomorphisms

$$\begin{tikzcd} \tilde{S} \arrow[r] \arrow[d] & S_+ \arrow[d] \\ S_- \arrow[r] & S \end{tikzcd}$$

and if for $s_+ \in S_+$, $s_- \in S_-$ and $\phi_+(s_+) = \phi_-(s_-)$ there exists a unique

$s \in \tilde{S}$ such that $\psi_+(s) = s_+$, $\psi_-(s) = s_-$. We can decompose $\mathbb{Z}[C_p]$ as a fibre product

$$\begin{tikzcd} \mathbb{Z}[C_p] \arrow[r] \arrow[d, "\epsilon"'] & R_C \arrow[d] \\ \mathbb{Z} \arrow[r] & \mathbb{F}_p \end{tikzcd}$$

Let (S, τ) be a commutative involuted ring, i.e.

- S is a commutative ring;
- $\tau : S \to S$ an automorphism satisfying $\tau^m = Id$ for some $m \geq 2$.

and let n be a multiple of m. The cyclic ring $\mathcal{C}_n(S, \tau)$ is then the free two sided S module with basis $\{1, \mathbf{y}, \dots, \mathbf{y}^{n-1}\}$ which satisfies the relations

$$\mathbf{y}^n = 1 \qquad \mathbf{y}s = \tau(s)\mathbf{y}$$

where $s \in S$. In other words, every element $c \in \mathcal{C}_n(S, \tau)$ can be written

$$c = \sum_{i=0}^{n} s_i \mathbf{y}^i \quad s_i \in S$$

We apply this cyclic algebra construction to $\mathbb{Z}[C_p]$ using θ as our automorphism, which we take to be the identity on $\mathbb{Z}$ and $\mathbb{F}_p$, to obtain

$$\begin{tikzcd} \mathcal{C}_q(\mathbb{Z}[C_p]) \arrow[r] \arrow[d, "\epsilon"'] & \mathcal{C}_q(R_C) \arrow[d] \\ \mathcal{C}_q(\mathbb{Z}) \arrow[r] & \mathcal{C}_q(\mathbb{F}_p) \end{tikzcd}$$

We can identify

$$\mathcal{C}_q(\mathbb{Z}[C_p]) \cong \mathbb{Z}[G(p, q)]$$
$$\mathcal{C}_q(\mathbb{Z}) \cong \mathbb{Z}[C_q]$$

$$\mathcal{C}_q(\mathbb{F}_p) \cong \mathbb{F}_p[C_q]$$

to obtain

$$\begin{tikzcd}
\mathbb{Z}[G(p,q)] \arrow[r] \arrow[d, "\epsilon"'] & \mathcal{C}_q(R_C) \arrow[d] \\
\mathbb{Z}[C_q] \arrow[r] & \mathbb{F}_p[C_q]
\end{tikzcd}$$

It remains to understand the structure of $\mathcal{C}_q(R_C)$.

Suppose A is a commutative ring, and I an ideal of A. We define the upper quasi-triangular matrices over A relative to I by:

$$\mathcal{T}_n(A, I) = \{X = (x_{ij})_{1 \le i,j \le n} \in M_n(A); x_{ij} \in I \text{ if } r > s\}$$

When $I = (a)$ is principal we write $\mathcal{T}_n(A, I) = \mathcal{T}_n(A, a)$. We then obtain:

Proposition 2.4.1. $\mathcal{C}_q(R_C) = \mathcal{T}_q(R_0, \pi)$.

Λ therefore decomposes as the fibre product

$$\begin{tikzcd}
\mathbb{Z}[G(p,q)] \arrow[r] \arrow[d, "\epsilon"'] & \mathcal{T}_q(R_0,\pi) \arrow[d] \\
\mathbb{Z}[C_q] \arrow[r] & \mathbb{F}_p[C_q]
\end{tikzcd}$$

We can further decompose $\mathcal{T}_q(R_0, \pi)$ as

$$\mathcal{T}_q(A, \pi) \cong R(1) \oplus R(2) \oplus \ldots \oplus R(q)$$

where $R(k)$ is the k^{th} row of $\mathcal{T}_q(A, \pi)$ considered as a right Λ module. There exist the following relations:

- $R(q) = R$;

- $R(k)^* = R(q+1-k)$.

Define a homomorphism $T : R(k+1) \to R(k)$ by

$$T = \begin{pmatrix} 0 & I_{q-1} \\ \pi & 0 \end{pmatrix}$$

We can apply T to repeatedly to generate an infinite sequence of embeddings

$$\ldots \underset{T}{\subset} R(q) \underset{T}{\subsetneq} R(q-1) \underset{T}{\subsetneq} \ldots \underset{T}{\subsetneq} R(2) \underset{T}{\subsetneq} R(1) \underset{T}{\subsetneq} R(q) \underset{T}{\subsetneq} \ldots$$

where at each step, $R(k+1)$ embeds into $R(k)$ with index p.

Define I_G to be the augmentation ideal of Λ, i.e. the kernel of the augmentation mapping

$$\epsilon : \Lambda \to \mathbb{Z}$$

and $[y-1)$ to be the right ideal generated by $y-1$. Then (see 5.2 and 5.12 in [9]):

Proposition 2.4.2. *There exists a split exact sequence of Λ modules*

$$0 \to [y-1) \to I_G \to R(1)$$

Since this sequence splits, we can decompose the augmentation ideal of G as a direct product

$$I_G = R(1) \oplus [y-1)$$

This direct sum decomposition forms the basis of the diagonal construction. By composing the natural projections

$$\Lambda \to \mathcal{T}_q(R_0, \pi)$$
$$\mathcal{T}_q(R_0, \pi) \to R(k)$$

we define a series of projections π_k

$$\pi_k : \Lambda \to R(k)$$

and define $K(k) = \mathrm{Ker}(\pi_k)$. 2.4.2 then generates (see 6.9, 9.7, 9.8 in [9]):

Theorem 2.4.3 *There exists an exact sequence of the form*

$$
\begin{tikzcd}
 & & & & K(q) \arrow[dr] & & & \\
G(q): & 0 \arrow[r] & R(1) \arrow[r] & \Lambda \arrow[rr] \arrow[ur] & & \Lambda \arrow[r] & R(q) \arrow[r] & 0
\end{tikzcd}
$$

$G(q)$ then induces:

Theorem 2.4.4. *For $1 \leq k \leq q-1$ there exist exact sequences of the form*

$$
\begin{tikzcd}
 & & & & K(k) \arrow[dr] & & & \\
G(k): & 0 \arrow[r] & R(k+1) \arrow[r] & P(k) \arrow[rr] \arrow[ur] & & \Lambda \arrow[r] & R(k) \arrow[r] & 0
\end{tikzcd}
$$

where $P(2), \ldots, P(q)$ are projective Λ-modules of rank 1 such that

$$\bigoplus_{i=2}^{q} P(i) \cong \Lambda^{q-1}$$

Splicing these segments together generates the upper strand of our almost free diagonal resolution

$$
\begin{tikzcd}[column sep=small]
 & & & & & R(q) \arrow[dr] & & & & & R(2) \arrow[dr] & & & & & \\
0 \arrow[r] & R(1) \arrow[r] & \Lambda \arrow[rr] \arrow[dr] & & \Lambda \arrow[rr] \arrow[ur] & & P(q-1) \arrow[rr] \arrow[dr] & & \cdots \arrow[r] & \Lambda \arrow[rr] \arrow[ur] & & P(1) \arrow[rr] \arrow[dr] & & \Lambda \arrow[r] & R(1) \arrow[r] & 0 \\
 & & & K(q) \arrow[ur] & & & & K(q-1) \arrow[ur] & & & & & K(1) \arrow[ur] & & &
\end{tikzcd}
$$

Theorem 2.4.5. *Suppose $\Lambda = \mathbb{Z}[G(p,q)]$. Then $\mathbb{Z}$ admits an almost free resolution of diagonal type.*

Given the existence of such a resolution, the next question to ask is whether it can be improved upon by replacing each projective $P(k)$ with Λ. We call such a free resolution strongly diagonal. The existence of a strongly diagonal resolution of period $2q$ depends on the existence of sequences

$$\begin{tikzcd}[column sep=small] & & & & K(k) \arrow[dr] & & & \\ \tilde{G}(k): & 0 \arrow[r] & R(k+1) \arrow[r] & \Lambda \arrow[ur] \arrow[rr] & & \Lambda \arrow[r] & R(k) \arrow[r] & 0 \end{tikzcd}$$

for $1 \leq i \leq q-1$. If every such $\tilde{G}(i)$ existed, we could splice together to form an exact upper strand

$$\begin{tikzcd}[column sep=small] & & & & R(q) \arrow[dr] & & & & & R(2) \arrow[dr] & & & & \\ 0 \arrow[r] & R(1) \arrow[r] & \Lambda \arrow[r] \arrow[dr] & \Lambda \arrow[r] \arrow[ur] & \Lambda \arrow[r] \arrow[dr] & \Lambda \arrow[r] & \cdots \arrow[r] & \Lambda \arrow[r] \arrow[ur] & \Lambda \arrow[r] \arrow[dr] & \Lambda \arrow[r] & R(1) \arrow[r] & 0 \\ & & & K(q) \arrow[ur] & & K(q-1) \arrow[ur] & & & & K(1) \arrow[ur] & & \end{tikzcd}$$

The existence of a strongly diagonal resolution was shown for the dihedral groups by Johnson in [7], and for the groups $G(5,4)$ and $G(7,3)$ in [19] and [22] respectively. In an upcoming paper [10], Johnson and Remez will expand these results for certain small values of p, q:

Theorem 2.4.6. *These exists a strongly diagonal resolution for the groups:*

$$G(p,2);\ \ G(5,4);\ \ G(7,3),\ G(7,6);\ \ G(11,5),\ G(11,10);\ \ G(13,3),$$
$$G(13,4),\ G(13,6);\ \ G(17,4);\ \ G(19,3),\ G(19,6),\ G(19,9).$$

We note that for both cases, we have a complete understanding of the syzygy modules, and can write the odd syzygies as

$$\Omega_{2k+1}(\mathbb{Z}) = R(k) \oplus [y-1)$$

We also have the following (see 6.2.7 in [22]):

Theorem 2.4.7. *Suppose $\Lambda = \mathbb{Z}[G(p,q)]$. Then $\Omega_{2k+1}(\mathbb{Z})$ is straight.*

While a strongly diagonal resolution for general $G(p,q)$ might currently be out of reach, we can construct a strongly diagonal p-adic resolution. Denote the ring of p-adic integers by $\hat{\mathbb{Z}}$, and define the p-adic completion of Λ by

$$\hat{\Lambda} = \Lambda \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$$

Similarly, for M a Λ lattice we define $\hat{M}$ to be its corresponding lattice over $\hat{\Lambda}$ by

$$\hat{M} = M \otimes_{\Lambda} \hat{\Lambda}$$

Tensoring 2.4.3 with $\hat{\mathbb{Z}}$ generates an exact sequence

$$\begin{tikzcd} & & & & \hat{K}(q) \arrow[dr] & & & \\ \hat{G}(q): & 0 \arrow[r] & \hat{R}(1) \arrow[r] & \hat{\Lambda} \arrow[ur] \arrow[rr] & & \hat{\Lambda} \arrow[r] & \hat{R}(q) \arrow[r] & 0 \end{tikzcd}$$

Similar tensor products then induce exact sequences for $1 \leq i \leq q-1$

$$\begin{tikzcd} & & & & \hat{K}(k) \arrow[dr] & & & \\ \hat{G}(i): & 0 \arrow[r] & \hat{R}(k+1) \arrow[r] & \hat{\Lambda} \arrow[ur] \arrow[rr] & & \hat{\Lambda} \arrow[r] & \hat{R}(k) \arrow[r] & 0 \end{tikzcd}$$

and we can splice to generate a strongly diagonal resolution at the p-adic level. In this paper we work with the integral case; p-adic results then follow from the relevant tensor products and are left to the reader.

Finally, we note that the modules $R(k)$ have cyclotomic descriptions. Define Λ-modules

$$P^k = (\zeta - 1)^k \mathbb{Z}[\zeta_p]$$

Right actions are inherited from those defined for R. Then it is true that

$$R(k) = \begin{cases} P^i & 1 \leq k \leq q-1 \\ P^0 = R & k = q \end{cases}$$

For typesetting reasons, we write

$$\tilde{P}^k = (P^k)^*$$

Duality identifications then become

$$\tilde{P}^k = \begin{cases} P & k = 0 \\ R & k = 1 \\ P^{q+1-k} & 2 \leq k \leq q-1 \end{cases}$$

This cyclotomic formulation is extremely useful for performing calculations, and is the one we use to develop the main results of this paper.

2.5 Bilinear Forms

Let R be a commutative ring, and M a right Λ-module. A bilinear form on M is a bilinear map

$$\langle\ ,\ \rangle : M \times M \to R$$

i.e. $\forall a, b \in R$, $\langle\ ,\ \rangle$ satisfies

- $\langle xa + yb, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$;
- $\langle x, ya + zb\rangle = a\langle x, y\rangle + b\langle x, z\rangle$.

A bilinear form is non-degenerate when for all $x, y \in M$

$$\langle x, y\rangle \neq 0$$

A bilinear form is skew-symmetric when for all $x, y \in M$

$$\langle x, y\rangle = -\langle y, x\rangle$$

A bilinear form is symmetric when for all $x, y \in M$

$$\langle x, y\rangle = \langle y, x\rangle$$

We write $\langle B \rangle$ for a bilinear form $\langle\ ,\ \rangle$. If M is free with basis $\{e_1, \ldots, e_m\}$, then a bilinear form on M generates a matrix $\beta = (\beta_{ij})$ where

$$\beta_{ij} = \langle e_i, e_j \rangle$$

and this matrix uniquely determines the form. Suppose two bilinear forms $\langle A \rangle$, $\langle B \rangle$ have two associated representations α and β. We say that the forms are equivalent if

$$\alpha = U \beta U^t$$

where U is some invertible matrix. In practice, this means we can act with simultaneous elementary row and column operations upon a representation of a form $\langle B \rangle$ to reduce it to some simpler representation. For skew-symmetric forms $\langle B^- \rangle$, the existence of a symplectic basis forces the existence of a reduced form

$$\langle B^- \rangle = \begin{pmatrix} 0 & \eta_1 & 0 & 0 & 0 & 0 \\ -\eta_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \eta_n \\ 0 & 0 & 0 & 0 & -\eta_n & 0 \end{pmatrix}$$

for $\eta_i \in \mathbb{Z}$. For symmetric bilinear forms, no such general reduction exists, and the question of classifying symmetric bilinear forms is a much larger and more complicated problem. In [28], Wall developed the foundations of a theory of Hermitian forms, a general framework within which such symmetric and skew-symmetric forms sit. In a further series of papers he undertook the task of attempting to classify such hermitian forms over rings of increasing complexity, ending with a paper on integral group rings over a finite group. We note that the forms discussed later in this paper fall somewhere between his work in [29] and [30]. For further reading we also direct the reader to a paper by Fröhlich [4].

There is a strong connection between bilinear forms on a module M, and maps from M into its dual M^*. Given a bilinear form $\langle B \rangle$, it defines a pair of R-homomorphisms $F_1, F_2 : M \to M^*$ by

$$F_1(x)(y) = \langle x, y \rangle$$
$$F_2(x)(y) = \langle y, x \rangle$$

for $x, y \in M$. Reversing, given any R-homomorphism $F : M \to M^*$, F then defines a bilinear form $\langle B \rangle$ on M by

$$\langle x, y \rangle = F(x)(y)$$

Similarly a homomorphism $\tilde{F} : M^* \to M$ defines a bilinear form on M^*.

Suppose G is a finite group and $(M,\, \rho)$ is a $R[G]$-module equipped with a finite representation $\rho : G \to \mathrm{GL}_R(M)$ and a bilinear form

$$\langle B \rangle : M \times M \to R$$

$\langle B \rangle$ is $R[G]$ invariant when for all $g \in G$, $x, y \in M$ we have

$$\langle \rho(g)x, \rho(g)y \rangle = \langle x, y \rangle$$

This is equivalent to the associated dual map $F : M \to M^*$ being a $R[G]$-homomorphism.

Chapter 3

Poincaré Duality and Highly Connected Universal Covers

The treatment of the dual cell construction set out in the following chapters follows a series of lectures given by F.E.A. Johnson. However, this material is not new and similar treatments can be found in a number of souces, including [23] and [18].

3.1 Ordered Simplicial Complexes

Suppose A is a set, and $\leq$ a partial ordering on A which is reflexive, transitive, and satisfies

$$(a \leq b) \wedge (b \leq a) \Rightarrow a = b$$

We call the pair $(A, \leq)$ a poset. Suppose $(A, \leq)$, $(B, \leq)$ are posets, then an order preserving mapping $f : (A, \leq) \to (B, \leq)$ is a mapping of sets which also satisfies

$$a \leq b \Rightarrow f(a) \leq f(b)$$

$(A, \leq)$ is totally ordered when for all $a, b \in A$, either $a \leq b$ or $b \leq a$.

A simplicial complex K is a collection $K = (V_K, S_K)$ where

- V_K is a set (the vertex set);
- S_K is a set of subsets of V_K (the set of simplices) such that

- $\sigma \in S_K$ is non-empty;
- If $v \in V_K$ then $\{v\} \in S_K$;
- If $\sigma \in S_K$ and $\tau \subset \sigma$, then $\tau \in S_K$.

Suppose K, L are simplicial complexes. A simplicial mapping $F : K \to L$ is a mapping $f : V_K \to V_L$ such that for all $\sigma \in S_K$, $f(\sigma) \in S_L$.

An ordered simplicial complex is a pair $(K, \leq)$ where

- $K = (V_K, S_K)$ is a simplicial complex;
- $\leq$ is a partial ordering on V_K in such a way that for all $\sigma \in S_K$, σ is totally ordered.

Suppose $(K, \leq)$, $(L, \leq)$ are ordered simplicial complexes. Then a mapping $f : (K, \leq) \to (L, \leq)$ is a simplicial mapping which is also order preserving.

Example 3.1.1. *Consider the n-simplex $\triangle^n = [0, \ldots, n]$. Then $\triangle^n$ has canonical ordering*

$$\triangle^n = 0 < 1 < \ldots < n$$

and so can be considered as an ordered simplicial complex

We note that the set of posets, simplicial complexes and ordered simplicial complexes form categories, which we label **Pos**, **Sim** and **OS** respectively. Furthermore **Pos** and **OS** are categories with finite products.

Proposition 3.1.2. *Every finite simplicial complex K admits a partial order $\leq$ such that $(K, \leq) \in$ **OS***

Proof. Write $V_K = \{v_0, v_1, \ldots, v_n\}$. There exists an embedding

$$K \to \triangle^n$$
$$v_i \mapsto i$$

and so take as ordering the induced order. $\square$

Let $(A, \leq)$ be a poset. The nerve $N(A, \leq)$ of A is the simplicial complex $N(A, \leq) = (V_{N(A,\leq)}, S_{N(A,\leq)})$ where

$$V_{N(A,\leq)} = A$$
$$S_{N(A,\leq)} = \{\text{Totally ordered subsets of } A\}$$

We are particularly interested in one specific nerve construction. Suppose X is a simplicial complex. Then S_X has a natural partial ordering by inclusion, so we can construct the poset $(S_X, \leq)$. Define a mapping

$$\mathcal{B} : \mathbf{Simp} \to \mathbf{OS}$$
$$\mathcal{B}(X) = N(S_X, \leq)$$

We identify $\mathcal{B}$ with barycentric subdivision and call $\mathcal{B}(X)$ the derived complex. In more detail: if $S_X = \{\sigma_i\}$, we define $\hat{\sigma_i}$ to be the barycentre of σ_i. For vertex set take $V_{\mathcal{B}(X)} = \{\hat{\sigma}_i\}$. A k simplex of $\mathcal{B}(X)$ will then be a totally ordered chain

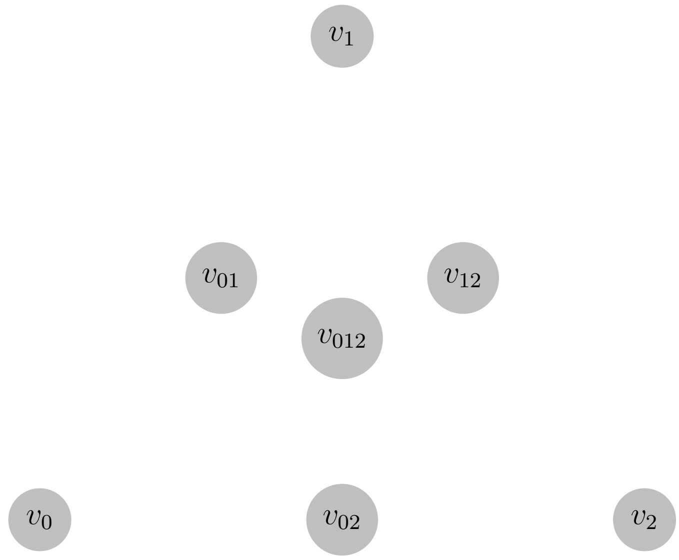
$$\hat{\sigma_0} < \hat{\sigma_1} < \ldots < \hat{\sigma_n}$$

To visualise this, it is instructive to look at a low dimensional example

Example 3.1.3. *Let $X = \triangle^2 = [0, 1, 2]$. The vertices are of the form $[i]$, and their barycentres are simply themselves, which we write as v_i. The 1-simplices take the form $[i, j]$ and we denote its barycentre by v_{ij}. Finally denote the barycentre of $[0, 1, 2]$ by v_{012}. Therefore we obtain*

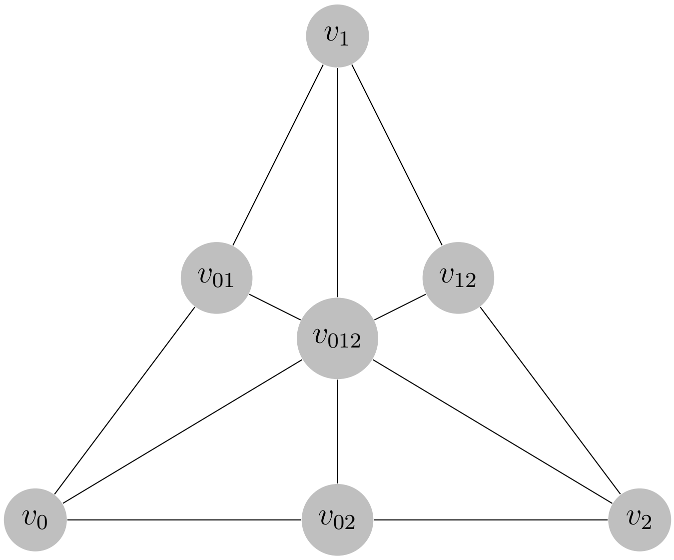
$$V_{\mathcal{B}(X)} = \{v_0, v_1, v_2, v_{01}, v_{02}, v_{12}, v_{012}\}$$

and pictorially we have



As 1*-simplices we have chains of these vertices which form a totally ordered set of size two with respect to inclusion on simplices of* X*. Our possible ordered sets look like* $\{v_i < v_{ij}\}$, $\{v_i < v_{ji}\}$, $\{v_i < v_{012}\}$, $\{v_{ij} < v_{012}\}$ *and writing out explicitly:*

$$\{v_0 < v_{01},\ v_0 < v_{02},\ v_0 < v_{012},\ v_1 < v_{01},\ v_1 < v_{12},\ v_1 < v_{012},\ v_2 < v_{02},$$
$$v_2 < v_{12},\ v_2 < v_{012},\ v_{01} < v_{012},\ v_{02} < v_{012},\ v_{12} < v_{012}\}$$



For 2*-simplices our possible ordered chains look like* $\{v_i < v_{ij} < v_{012}\}$*, and writing explicitly:*

$$\{v_0 < v_{01} < v_{012},\ v_0 < v_{02} < v_{012},\ v_1 < v_{01} < v_{012},\ v_1 < v_{12} < v_{012},$$
$$v_2 < v_{02} < v_{012},\ v_2 < v_{12} < v_{012}\}$$

Pictorially, we can just shade in all possible 2*-simplexes in the above picture.*

3.2 Simple Lefschetz Duality

We call a poset $(A, \leq)$ homogenously n-dimensional when every maximal totally ordered subset has cardinal $n+1$. Formally:

1. If $\tau \subset A$ is totally ordered then $|\tau| \leq n+1$.
2. If $\tau \subset A$ is totally ordered then there exists a totally ordered subset $\sigma \subset A$ such that $\tau \subseteq \sigma$ and $|\sigma| = n+1$.

We call a simplicial complex X homogeneously n-dimensional when every maximal simplex has dimension n. We have the following trivial results:

Lemma 3.2.1. *If* $(A, \leq)$ *is homogeneously* n*-dimensional then* $N(A, \leq)$ *is* *homogeneously n-dimensional.*

Lemma 3.2.2. *If X is a homogeneously n-dimensional simplicial complex then $\mathcal{B}(X)$ is a homogeneously n-dimensional simplicial complex.*

If $(A, \leq)$ is homogeneously n-dimensional, then we also have the notion of height. Define a function

$$h : A \to \{0, \ldots, n\}$$

by $h(a) = k$ when $a \in A$ occurs in a maximal totally ordered subset as

$$(a_0 < \ldots < a_{k-1} < a < a_{k+1} < \ldots < a_n)$$

Proposition 3.2.3. *h is well defined.*

Proof. We need to show that h is independent of the maximal totally ordered subset chosen. Suppose $h(a) = k$, and that we pick another subset containing a as

$$b_0 < \ldots < b_{l-1} < a < b_{l+1} < \ldots < b_n$$

If $k < l$, then we can construct a new totally ordered subset as

$$b_0 < \ldots < b_{l-1} < a < a_{k+1} < \ldots < a_{n+l-k}$$

But then this is a higher length than our maximal subset, which is a contradiction. We can follow the same process for $l < k$, so that we must have $k = l$ □

We can expand this definition in the obvious way to construct a height function on X a finite homogeneously n-dimensional simplicial complex,

$$h : S_X \to \{0, \ldots, n\}$$

(We note that this becomes a formalisation of dimension.) This in turn induces a height function on the derived complex

$$h : V_{\mathcal{B}(X)} \to \{0, \dots, n\}$$

Example 3.2.4. *Consider $B(\triangle^2)$. Then $h(v_{01}) = 1$, since we can construct the maximal chain*

$$v_0 < v_{01} < v_{012}$$

For $(A, \leq)$ homogeneously n dimensional, define

$$[a, +\infty) = \{b \in A;\ a \leq b\}$$
$$(a, +\infty) = \{b \in A;\ a < b\}$$

Proposition 3.2.5. *$N([a, +\infty))$ is a cone.*

Proof. From the given definitions, we have that

$$V_{N([a,+\infty))} = V_{N((a,+\infty))} + \{a\}$$
$$S_{N([a,+\infty))} = S_{N((a,+\infty))} + \{a \cup \beta;\ \beta \in S_{N((a,+\infty))}\}$$

By the definition of the join $*$, we then obtain

$$N([a, +\infty)) = \{a\} * N((a, +\infty))$$
$$N([a, +\infty)) = C(N((a, +\infty)))$$

□

Proposition 3.2.6. *If $h(a) = k$, then $N([a, +\infty))$ is homogeneously $(n-k)$-dimensional.*

Proof. Write S_i for a totally ordered subset of A of cardinal i. Then a maximal subset in $N([a, +\infty))$ must take the form

$$a = S_k < S_{k+1} < \ldots < S_n$$

and so has cardinal $n - k$ as required □

Define the formal boundary of $N([a, +\infty))$ by

$$\partial N([a, +\infty)) = N((a, +\infty)) = \bigcup_{\substack{a<b \\ h(b)=h(a)+1}} N([b, +\infty))$$

Suppose $X = (V_X, S_X)$ is an ordered simplicial complex. We can take equivalent constructions on S_X in X or V_X in $\mathcal{B}(X)$. Suppose $\sigma \in S_X$, so that $\hat{\sigma}$ is a vertex in $\mathcal{B}(X)$. We have that

$$\partial N([\sigma, +\infty)) = N((\sigma, +\infty)) = \bigcup_{\substack{\sigma<\tau \\ h(\tau)=h(\sigma)+1}} N([\tau, +\infty))$$

This allows us to define the dual cone $D(\sigma)$ as

$$D(\sigma) = N([\sigma, +\infty)) = \hat{\sigma} * \partial(N([\sigma, +\infty))$$

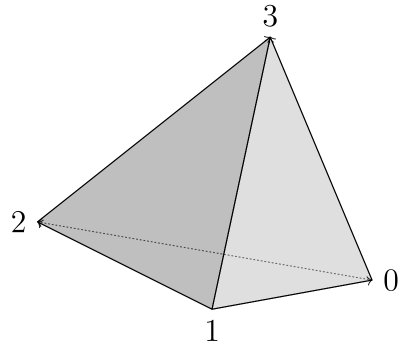
Combining with 3.2.6, we see that for each σ of dimension k in X we have constructed a dual structure in $\mathcal{B}(X)$ of dimension $n-k$. $D(\sigma)$ is clearly not a simplex in general, but with some added constraints on X it is in fact a cell.

Suppose X is a finite simplicial complex and $\rho \in S_X$. Then we say that $\tau \in S_X$ is joinable to ρ in X when

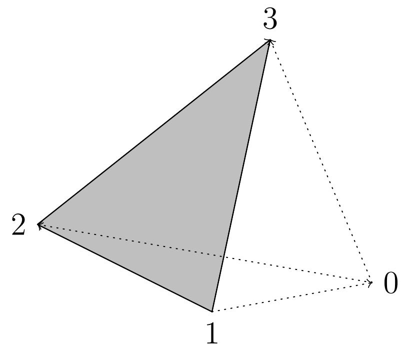
- $\tau \cap \rho = \emptyset$;
- $\tau \cup \rho \in S_X$.

Define the link of ρ in X, $\mathrm{Lk}(\rho, X)$, to be the sub-complex of X consisting of all $\tau \in S_X$ which are joinable to ρ

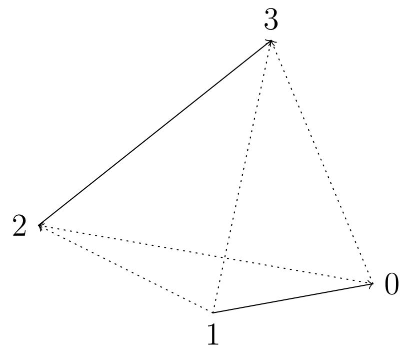
Example 3.2.7. *Take* $X = \triangle^3$*:*



First consider $\mathrm{Lk}([0], X)$*. The non intersecting* 0*-simplices are* [1]*,* [2]*,* [3]*, and since* [0, 1]*,* [0, 2]*,* [0, 3] *are all simplices, these* 0*-simplices belong in the link. Similarly for* 1*-simplices* [1, 2]*,* [1, 3]*,* [2, 3]*, and the* 2*-simplex,* [1, 2, 3]*. So we can visualise the link as*



Similarly we see that $\mathrm{Lk}([1, 2, 3], X) = [0]$*, which we visualise as above, and* $\mathrm{Lk}([01], X) = [23]$*, visualised as*



Note in the above example that for each $\rho \in S_{\triangle^3}$, $\rho * \mathrm{Lk}(\rho, \triangle^3) = \triangle^3$. This generalises to all simplicial complexes. If $\rho \in S_X$ then $\rho * \mathrm{Lk}(\rho, X)$ imbeds in X as a full subcomplex consisting of all maximal simplices which contain ρ.

Writing $\mathrm{Sd}^i(X)$ for a sequence of i subdivisions on X, we say that two simplicial complexes X and Y are combinatorially equivalent when there exist $k, l \in \mathbb{Z}$ such that

$$\mathrm{Sd}^k(X) \cong \mathrm{Sd}^l(Y)$$

We will write $X \sim Y$ to denote combinatorial equivalence. The standard n-sphere S^n has simplicial definition

$$\begin{aligned} V_{S^n} &= \{0, 1, \ldots, n+1\} \\ S_{S^n} &= \{\sigma \subset V_{S^n}; 0 < |\sigma| < n+2\} \end{aligned}$$

with degenerate case $S^{-1} = \emptyset$. Define a property of X a simplicial complex

$\mathbf{L(n, k)}$: For each k simplex $\rho \in S_X$, $\mathrm{Lk}(\rho, X) \sim S^{n-k-1}$.

We call X a combinatorial n-manifold when it satisfies $L(n, 0)$ i.e. for each $v \in V_X$, $\mathrm{Lk}(v, X) \sim S^{n-1}$.

Lemma 3.2.8. *If X is a combinatorial n-manifold then it is homogeneously n-dimensional.*

Proof. $v * \mathrm{Lk}(v, X)$ is some union of maximal simplexes. But then

$$v * \mathrm{Lk}(v, X) \sim C(S^{n-1}) \cong D^n \sim \triangle^n \qquad \square$$

Lemma 3.2.9. *S^n is a combinatorial manifold.*

Proof. Pick a $v \in V_{S^n}$. Then

$$\begin{aligned} V_{\mathrm{Lk}(v,S^n)} &= V_{S^n} - \{v\} \\ S_{\mathrm{Lk}(v,S^n)} &= \{\sigma \in S_{S^n};\ v \nsubseteq \sigma\} \\ &= \{\sigma \subset V_{\mathrm{Lk}(v,S^n)};\ 1 \leq |\sigma| \leq n+1\} \end{aligned}$$

But then this is simply the definition of S^{n-1}. $\square$

We wish to prove:

Theorem 3.2.10. *Suppose X, Y are combinatorially equivalent simplicial manifolds. Then*

$$X \textit{ is a combinatorial } n\textit{-manifold} \Leftrightarrow Y \textit{ is a combinatorial } n\textit{-manifold}$$

We proceed using a double induction argument on the following statements

$\mathbf{P(n,k)}$: If X is a simplicial complex satisfying $L(n, 0)$, then X also satisfies $L(n, k)$.

$\mathbf{\tilde{Q}(n)}$: If X, Y are simplicial complexes such that $X \sim Y$, then

$$X \text{ is a combinatorial } n\text{-manifold} \Leftrightarrow Y \text{ is a combinatorial } n\text{-manifold}$$

Clearly solving 3.2.10 is equivalent to showing that $\tilde{Q}(n)$ holds for all n.

Lemma 3.2.11. *$P(n, n)$ is true.*

Proof. By 3.2.8, any n-simplex ρ of X is necessarily maximal, so that

$$\mathrm{Lk}(\rho, X) = \emptyset = S^{-1}$$

as required. □

Proposition 3.2.12. *$P(n, k-1) \cap \tilde{Q}(n-k) \Rightarrow P(n,k)$ for $n \geq 2$ and $1 \leq k \leq n-1$.*

Proof. Let σ be a k-simplex of an X satisfying $P(n, k-1)$. Then σ can be decomposed as the disjoint union $v \sqcup \tau$ for some $v \in V_X$, and τ a $(k-1)$-simplex. Since $P(n, k-1)$ holds we have that

$$\mathrm{Lk}(\tau, X) \sim S^{n-(k-1)-1} = S^{n-k}$$

By 3.2.9, S^{n-k} is a combinatorial $(n-k)$-manifold, and by the assumption $\tilde{Q}(n-k)$ holds, so is $\mathrm{Lk}(\tau, X)$. Then since $v \in \mathrm{Lk}(\tau, X)$, $\mathrm{Lk}(v, \mathrm{Lk}(\tau, X))$ is well defined, and furthermore

$$\mathrm{Lk}(\sigma, X) = \mathrm{Lk}(v, \mathrm{Lk}(\tau, X))$$

Since $\mathrm{Lk}(\tau, X)$ is a combinatorial $(n-k)$-manifold, it satisfies $L(n-k, 0)$, therefore

$$\mathrm{Lk}(\sigma, X) \sim S^{n-k-1}$$

and $P(n,k)$ is satisfied. □

Define further conditions

$$P(n) = \bigwedge_{k=0}^{n} P(n,k) \qquad Q(n) = \bigwedge_{i=0}^{n} \tilde{Q}(i)$$

Proposition 3.2.13. *$Q(n-1) \Rightarrow P(n)$.*

Proof. First note that we can rewrite $Q(n-1)$ as

$$Q(n-1) = \bigwedge_{k=1}^{n-1} \tilde{Q}(n-k)$$

Then since $P(n, 0)$ is tautologically true, 3.2.12 gives $\bigwedge_{k=0}^{n-1} P(n,k)$ is true. But then by 3.2.11, $P(n,n)$ holds, and so does $P(n)$. □

Suppose σ a k-simplex of X, and define $\mathrm{Sd}(X, \sigma)$ to be the elementary subdivision of X formed by only subdividing X by a single extra vertex at σ.

Proposition 3.2.14. $P(n) \Rightarrow \tilde{Q}(n)$.

Proof. Suppose X is a simplicial complex satisfying $P(n)$, and let σ be a k-simplex of X. Since any subdivision is a finite number of elementary subdivisions, it is enough to set $Z = \mathrm{Sd}(X, \sigma)$ and prove

$$X \text{ a combinatorial } n\text{-manifold} \Leftrightarrow Z \text{ a combinatorial } n\text{-manifold}$$

Write

$$V_Z = V_X \sqcup \{w\}$$

where w is the unique extra vertex introduced upon subdividing at σ. Then it is easy to check that

$$\mathrm{Lk}(v, Z) \sim \begin{cases} \mathrm{Lk}(v, X) & v \in V_X \\ (\partial\sigma) * \mathrm{Lk}(\sigma, X) & v = w \end{cases}$$

Therefore for the case $v \in V_X$, we have

$$\mathrm{Lk}(v, Z) \sim S^{n-1} \;\Leftrightarrow\; \mathrm{Lk}(v, X) \sim S^{n-1}$$

for all $v \in V_X$. This leaves only $\mathrm{Lk}(w, Z)$ to consider. But then since X satisfies $P(n)$, it satisfies $P(n, k)$ in particular so that

$$\mathrm{Lk}(w, Y) \sim (\partial\sigma) * \mathrm{Lk}(\rho, X) \sim S^{k-1} * S^{n-k-1} \cong S^{n-1}$$ □

Combining 3.2.13 and 3.2.14 gives:

Corollary 3.2.15: *$P(n)$ and $Q(n)$ are true for all n.*

This proves 3.2.10, as well as the following nice result:

Theorem 3.2.16. *If X is a combinatorial n-manifold then X satisfies*

$L(n,k)$ for $0 \leq k \leq n$.

We can now prove the following:

Theorem 3.2.17. *Suppose X is an ordered simplicial n-manifold, and σ a k-simplex of X. Then $D(\hat{\sigma}) \in \mathcal{B}(X)$ is a $(n-k)$ cell.*

Proof. Note that by definition, $N((\sigma, +\infty))$ is the set of simplices joinable to $\hat{\sigma}$ formed by vertices of higher height than $\hat{\sigma}$, and so will belong in $\mathrm{Lk}(\hat{\sigma}, \mathcal{B}(X))$. Similarly, considering the construction $\mathcal{B}(\partial\sigma)$, we see that it consists of simplices joinable to $\hat{\sigma}$ formed by vertices of lower height than $\hat{\sigma}$. Therefore

$$\mathrm{Lk}(\hat{\sigma}, \mathcal{B}(X)) = N((\sigma, +\infty)) * \mathcal{B}(\partial\sigma)$$

Now $\hat{\sigma}$ is a vertex in $\mathcal{B}(X)$, which is also a combinatorial n-manifold since X is by 3.2.10. Therefore

$$\mathrm{Lk}(\hat{\sigma}, \mathcal{B}(X)) \sim S^{n-1}$$

We also have

$$\mathcal{B}(\partial\sigma) \sim \partial\sigma \sim S^{k-1}$$

so that, considering the expression

$$S^{n-1} \sim S^{k-1} * N((\sigma, +\infty))$$

we see that

$$\partial N([\sigma, +\infty)) = N((\sigma, +\infty)) \sim S^{n-k-1}$$

Finally

$$N([\sigma, +\infty)) = \hat{\sigma} * \partial N([\sigma, +\infty)) \sim \hat{\sigma} * S^{n-k-1} = D^{n-k}$$

where D^{n-k} is a disk of dimension $(n-k)$. Therefore

$$(D(\sigma), \partial D(\sigma)) \sim (D^{n-k}, S^{n-k-1})$$

the definition of a $(n-k)$-cell. $\square$

We have a constructed genuine geometric theory of duality, which takes a simplex in X to a cell in $\mathcal{B}(X)$. However it remains to formalise the algebra in terms of chain complexes

3.3 Lefschetz Complexes

We define a Lefschetz n-complex to be a collection

$$\mathfrak{X} = (X,\ \leq,\ \dim,\ [\ ,\ ])$$

where

- $(X,\ \leq)$ is a finite poset;
- $\dim : X \to \{-1, 0, 1\}$ is an order preserving mapping;
- $[\ ,\ ] : X \times X \to \{\pm 1\}$ is a mapping satisfying
  - for all $x, y \in X$, $[x, y] \neq 0 \Rightarrow \dim(y) = \dim(x) - 1$;
  - for all $x, z \in X$, $\sum_{y \in X} [z, y][y, x] = 0$.

Define $C_k(\mathfrak{X})$ to be the free abelian group on the set $\{x \in X; \dim(x) = k\}$, and define boundary operators (considering $C_k(\mathfrak{X})$ as a right module)

$$\partial_k : C_k(\mathfrak{X}) \to C_{k-1}(\mathfrak{X})$$
$$\partial_k(x) = \sum_{y \in X} y[y, x]$$

Together these define a formal chain complex $C_*(\mathfrak{X})$. $C_*(\mathfrak{X})$ is also equipped with a homology theory, which we will call Lefschetz homology, by virtue of the following:

Proposition 3.3.1. $\partial_{k-1} \circ \partial_k = 0$.

Proof. By definition for all $x \in X$

$$\begin{aligned}
\partial_{k-1} \circ \partial_k(x) &= \partial_{k-1}(\sum_{y \in X} y[y,x]) \\
&= \sum_{y \in X} \sum_{z \in X} z[z,y][y,x] \\
&= 0
\end{aligned}$$

□

We also have the notion of a cochain complex and Lefschetz cohomology. Define $C^k(\mathfrak{X})$ to be the free abelian group on $\{x \in X; \dim(x) = k\}$ considered now as a left module, and define coboundary operators by

$$\begin{aligned}
\delta_k : C^k(\mathfrak{X}) &\to C^{k+1}(\mathfrak{X}) \\
\delta(x) &= \sum_{y \in X} [x,y]y
\end{aligned}$$

Proposition 3.3.2. $\delta_{k+1} \circ \delta_k = 0$.

Proof. By definition for all $x \in X$:

$$\begin{aligned}
\delta_{k+1} \circ \delta_k(x) &= \delta_{k+1}(\sum_{y \in X} [x,y]y) \\
&= \sum_{y \in X} \sum_{z \in X} [x,y][y,z]z \\
&= 0
\end{aligned}$$

□

Example 3.3.3. *Suppose X an ordered homogenously n-dimensional simplicial complex. Define the dimension mapping* $\dim : S_X \to \{0, \dots, n\}$ *simply by the dimension of the simplex. Consider simplexes $\tau \subset \sigma$ such that* $\dim(\sigma) = \dim(\tau) + 1$. *Then for some sequence of $\alpha_j \in \mathbb{Z}$, and some $i, k \in \mathbb{Z}$ we can write*

$$\tau = [\alpha_0, \dots, \hat{\alpha}_i, \dots, \alpha_k]$$

where $\hat{\alpha_j}$ denotes the deletion of that element. We can then define the map-

ping

$$[\ ,\ ] : S_X \to S_X \to \{\pm 1\}$$
$$[\tau, \sigma] = \begin{cases} (-1)^i & \tau \subset \sigma,\ \dim(\sigma) = \dim(\tau) + 1 \\ 0 & \text{otherwise} \end{cases}$$

and we have a Lefschetz complex $\mathfrak{X}$ *associated to* X*.*

Suppose $\mathfrak{X} = (X,\ \leq,\ \dim,\ [\ ,\ ])$ is a Lefschetz n-complex. Then we define the Lefschetz cocomplex $\mathfrak{X}^*$ as

$$\mathfrak{X}^* = (X,\ \overline{\leq},\ \overline{\dim},\ \overline{[\ ,\ ]})$$

where

- $x \,\overline{\leq}\, y \Leftrightarrow y \leq x$;
- $\overline{\dim} = n - \dim$;
- $\overline{[x, y]} = [y, x]$.

For every k there then exists isomorphisms of abelian groups

$$\phi_k : C_k(\mathfrak{X}) \to C^{n-k}(\mathfrak{X}^*)$$
$$\overline{\phi}_k : C^k(\mathfrak{X}) \to C_{n-k}(\mathfrak{X}^*)$$

If, considering as modules, we make the convention that $C_*(\mathfrak{X}^*)$ is left-handed, and $C^*(\mathfrak{X}^*)$ is right-handed, we have the following:

Theorem 3.3.4. *There exist chain isomorphisms:*

$$\phi_* : C_*(\mathfrak{X}) \to C^{n-*}(\mathfrak{X}^*)$$
$$\overline{\phi}_* : C^*(\mathfrak{X}) \to C_{n-*}(\mathfrak{X}^*)$$

The proof is simply by following definitions, and since it proceeds exactly the same as for the explicit example of Poincaré duality, we defer to that later proof.

Suppose X is a finite, connected, orientable simplicial n-manifold, and $\mathcal{B}(X)$ is its barycentric subdivision. Then as per 3.3.3 we can associate a Lefschetz complex $\mathfrak{X}$ to X. Note that since any simplicial decomposition can be taken as a cellular decomposition, we can think of $C_*(\mathfrak{X})$ as a cellular chain. We also construct a dual Lefschetz complex $D(\mathfrak{X})$ by

- $C_*(D(\mathfrak{X}))$ is the free abelian group on the k-cells of $\mathcal{B}(\mathfrak{X})$ of the form $D(\sigma)$, where σ is a $(n-k)$-simplex of X;
- $\tau \subset \sigma \Rightarrow D(\sigma) \subset D(\tau)$;
- $\dim(D(\sigma)) = n - \dim(\sigma)$.

We make the convention that the incidence numbers in $D(\mathfrak{X})$ are defined by

$$[D(\sigma), D(\tau)] = [\tau, \sigma]$$

While seemingly arbitrary, we can think of this choice as ensuring that the orientability of X, and hence $\mathcal{B}(X)$ is preserved. Then $D(\mathfrak{X})$ is a Lefschetz co-complex, and our earlier geometric isomorphism

$$\begin{gathered}\{k \text{ simplices of X}\} \to \{(n-k) \text{ cells of } \mathcal{B}(X)\}\\ \sigma \mapsto D(\sigma)\end{gathered}$$

induces isomorphisms of abelian groups

$$\begin{gathered}h_k : C_k(\mathfrak{X}) \to C^{n-k}(D(\mathfrak{X}))\\ \overline{h}_k : C^k(\mathfrak{X}) \to C_{n-k}(D(\mathfrak{X}))\end{gathered}$$

Theorem 3.3.5. *There exist chain isomorphisms*

$$\begin{gathered}h_* : C_*(\mathfrak{X}) \to C^{n-*}(D(\mathfrak{X}))\\ \overline{h}_* : C^*(\mathfrak{X}) \to C_{n-*}(D(\mathfrak{X}))\end{gathered}$$

Proof. Take $h_* = \{h_k\}$. We already know that h_k is an isomorphism of abelian groups, so all that remains to check is commutativity via the diagram

$$\begin{tikzcd}
C_k(\mathfrak{X}) \arrow[r, "\partial_k"] \arrow[d, "h_k"'] & C_{k-1}(\mathfrak{X}) \arrow[d, "h_{k-1}"] \\
C^{n-k}(D(\mathfrak{X})) \arrow[r, "\delta_{n-k}"] & C^{n-k-1}(D(\mathfrak{X}))
\end{tikzcd}$$

Suppose $\sigma \in C_k(\mathfrak{X})$ a k-simplex. Then

$$\begin{aligned}
h_{k-1} \circ \partial_k(\sigma) &= h_{k-1}(\sum_{\tau \in X} \tau[\tau, \sigma]) \\
&= \sum_{\tau \in X} D(\tau)[\tau, \sigma]
\end{aligned}$$

$$\begin{aligned}
\delta_{n-k} \circ h_k(\sigma) &= \delta_{n-k}(D(\sigma)) \\
&= \sum_{D(\tau) \in \mathcal{B}(X)} D(\tau)[D(\sigma), D(\tau)] \\
&= \sum_{\tau \in X} D(\tau)[\tau, \sigma]
\end{aligned}$$

as required. □

While we have an intuitive idea of what $C_*(D(\mathfrak{X}))$ represents, currently we only have a homology theory of $C_*(D(\mathfrak{X}))$ in terms of the singular homology of $C_*(\mathfrak{X})$. We would like an independent formulation of $H_*(D(\mathfrak{X}))$, to confirm that our interpretation of $C_*(D(\mathfrak{X}))$ is correct.

Theorem 3.3.6: $H_*(C_*(D(\mathfrak{X}))) \cong H_*(X)$.

Proof. We proceed by a standard spectral sequence argument. Since $\mathcal{B}(X)$ is equipped with a cell structure, namely the dual cells to simplices in X,

$C_*(D(\mathfrak{X}))$ has a filtration

$$\ldots \subset \mathcal{F}_{p-1} \subset \mathcal{F}_p \subset \mathcal{F}_{p+1} \subset \ldots$$

where $\mathcal{F}$ is formed by dual cells of dimension p in $\mathcal{B}(X)$. Take the homology spectral sequence of $(\mathcal{F}_*)$ As entries on the first page we have

$$E^1_{p,q} = H_{p+q}(\mathcal{F}_p/\mathcal{F}_{p-1})$$

For the differentials

$$d^1_{p,q} : E^1_{p,q} \to E^1_{p-1,q}$$

consider the short exact sequence of the triple

$$0 \to \mathcal{F}_{p-1}/\mathcal{F}_{p-2} \to \mathcal{F}_p/\mathcal{F}_{p-2} \to \mathcal{F}_p/\mathcal{F}_{p-1} \to 0$$

Then the long exact sequence in homology gives rise to a boundary operator

$$\partial : H_{p+q}(\mathcal{F}_p/\mathcal{F}_{p-1}) \to H_{p+q}(\mathcal{F}_{p-1}/\mathcal{F}_{p-2})$$

and since

$$E^1_{p,q} = H_{p+q}(\mathcal{F}_p/\mathcal{F}_{p-1}) \quad E^1_{p-1,q} = H_{p+q}(\mathcal{F}_{p-1}/\mathcal{F}_{p-2})$$

we can take $d^1_{p,q} = \partial$. Defining υ_p to equal the number of p cells in $\mathcal{B}(X)$, we obtain

$$H_{p+q}(\mathcal{F}_p/\mathcal{F}_{p-1}) = \begin{cases} \mathbb{Z}^{\upsilon p} & q = 0 \\ 0 & q \neq 0 \end{cases}$$

so that

$$E^1_{p,q} = C_*(\mathcal{B}(X))$$
$$d^1_{p,q} : C_p \to C_{p-1}$$

Therefore

$$E^2_{p,q} = \begin{cases} H_p(\mathcal{B}(X)) & q = 0 \\ 0 & q \neq 0 \end{cases}$$

Looking at the differentials, since we only have one non-zero row we obtain

$$\begin{aligned} d^2_{p,q} : E^2_{p,q} \to E^2_{p-2,q-1} &\Rightarrow d^2_{p,q} = 0 \\ &\Rightarrow d^n_{p,q} = 0 \text{ for all } n \geq 2 \\ &\Rightarrow E^\infty_{p,q} = \begin{cases} H_p(\mathcal{B}(X)) & q = 0 \\ 0 & q \neq 0 \end{cases} \end{aligned}$$

Since X is finite, the filtration is bounded and

$$H_p((C_*(D(\mathfrak{X}))) \cong H_p(\mathcal{B}(X)) \cong H_p(X) \qquad \square$$

As a corollary to this we get the standard formulation of Poincaré duality

Corollary 3.3.7. $H^{n-*}(X) \cong H_*(X)$.

Finally, Lefschetz also generates a relation on the duals of these isomorphisms. Consider $h_k : C_k(\mathfrak{X}) \to C^{n-k}(D(\mathfrak{X}))$. Then

$$\begin{aligned} h_k^* : C_{n-k}(D(\mathfrak{X})) \to C^k(\mathfrak{X}) \\ (h_k^*)^{-1} : C^k(\mathfrak{X}) \to C_{n-k}(D(\mathfrak{X})) \end{aligned}$$

so we might expect that $(h_k^*)^{-1} = \pm\overline{h_k}$. Lefschetz then proves the following:

Proposition 3.3.8. $(h_k^*)^{-1} = (-1)^{\frac{n(n-1)}{2}}\overline{h_k}$.

Poincaré duality in Lefschetz's original formulation is a powerful geometric result, however one could ask how practically useful it is to construct a chain isomorphism from $\mathfrak{X}$ to $D(\mathfrak{X})$ when one is purely interested in constructing properties on $\mathfrak{X}$. It turns out that with a small shift, we can convert this formulation to an isomorphism between the chain and cochain complex of the exact same combinatorial manifold.

Theorem 3.3.9: *Suppose X is a finite, connected, orientable simplicial n-manifold with associated Lefschetz complex $\mathfrak{X}$. Then there exists a chain isomorphism*

$$h_* : C_*(\mathcal{B}(\mathfrak{X})) \to C^{n-*}(\mathcal{B}(\mathfrak{X}))$$

where $\mathcal{B}(\mathfrak{X})$ is the Lefschetz complex associated to the derived manifold $\mathcal{B}(X)$.

Proof. Since X is a simplicial manifold it admits a simplicial decomposition. Take such a decomposition, and then take the barycentric subdivision. Then a k-simplex $\triangle^k$ in $\mathfrak{X}$ defines a k-cell in $\mathcal{B}(\mathfrak{X})$ since it merely becomes a union of k-simplices contained in $\triangle^k$ with boundary $\partial\triangle^k$. So the original simplices define a cell structure on $\mathcal{B}(\mathfrak{X})$ and hence form a basis for a cellular chain complex $C_*(\mathcal{B}(\mathfrak{X}))$. Then we can construct a duality isomorphism on these simplices (now considered as cells):

$$h_* : C_*(\mathcal{B}(\mathfrak{X})) \to C^{n-k}(D(\mathfrak{X}))$$

However we then obtain

$$C^{n-k}(D(\mathfrak{X})) \cong C^{n-k}(\mathcal{B}(\mathfrak{X}))$$

Hence the result. $\square$

Proposition 3.3.10. *Suppose $h_k : C_k(\mathcal{B}(\mathfrak{X})) \to C^{n-k}(\mathcal{B}(\mathfrak{X}))$ is a duality isomorphism. Then*

$$h_k^* = (-1)^{\frac{n(n-1)}{2}} h_{n-k}$$

Proof. Recall our previous definition of $\overline{h_k}$, which here becomes

$$\overline{h_k} : C^k(\mathcal{B}(\mathfrak{X})) \to C_{n-k}(\mathcal{B}(\mathfrak{X}))$$
$$\overline{h_k}^{-1} : C_{n-k}(\mathcal{B}(\mathfrak{X})) \to C^k(\mathcal{B}(\mathfrak{X}))$$

Then $\overline{h_k}^{-1} = h_{n-k}$, and application of 3.3.8 gives the result. $\square$

3.4 Highly Connected Universal Covers

Suppose X is an orientable, closed, combinatorial-manifold of dimension n, with fundamental group $\pi_1 = G$. Writing either $n = 2m$ or $n = 2m+1$, we wish to consider those X whose universal cover $\tilde{X}$ is $(m-1)$-connected, that is

$$\pi_i(\tilde{X}) = 0 \qquad 0 \leq i \leq m-1$$

In [25], Wall labelled PL-manifolds M satisfying such conditions as highly-connected, and so we take inspiration from this nomenclature for the title of this section. In this thesis we are concerned only with the odd-dimensional cases, and from this point take X to be a orientable, closed, combinatorial $2m+1$ manifold. We will abuse notation and write X for both X and its associated Lefschetz complex, and $\tilde{X}$ for both the universal cover and the Lefschetz complex associated to it. The first question to ask is whether we can describe the homology of $\tilde{X}$. First recall that we can always split a homology group into a direct product of a free part and a torsion part, as we can for a cohomology group. The universal coefficient theorem then gives a relationship between these two decompositions (see 3.3 in [12])

Theorem 3.4.1. *Let M be a manifold, and suppose*

$$H_i(M;\mathbb{Z}) = \mathbb{Z}^{a_i} \oplus \mathrm{Tor}_i$$

where $a_i \in \mathbb{Z}$. Then

$$H^i(M;\mathbb{Z}) = \mathbb{Z}^{a_i} \oplus \mathrm{Tor}_{i-1}$$

We also have the following standard result by Hurewicz (see 4.37 in [12]).

Theorem 3.4.2. *For $n \geq 2$, if M is a $(n-1)$-connected manifold then for $i \leq n$ there exist isomorphisms*

$$h_i : \pi_i(M) \to H_i(M)$$

These results, combined with Poincaré duality, are enough to describe the homology of $\tilde{X}$.

Theorem 3.4.3. *Suppose X is a connected, orientable, closed, combinatorial $2m+1$ manifold. Suppose further that $\pi_i(\tilde{X}) = 0$ for $0 \leq i \leq j-1$. Then*

$$H_i(\tilde{X};\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, 2m+1 \\ \mathbb{Z}^a \oplus \mathrm{Tor}_j & i = m \\ \mathbb{Z}^a & i = m+1 \\ 0 & \textit{otherwise} \end{cases}$$

for $a \in \mathbb{Z}$.

Proof. Since $\tilde{X}$ is a connected space, trivially we have

$$H_0(\tilde{X};\mathbb{Z}) = \mathbb{Z}$$

Using the convention that $\mathrm{Tor}_{-1} = 0$, we also have that

$$H^0(\tilde{X};\mathbb{Z}) = \mathbb{Z}$$

Applying Poincaré duality we obtain

$$H_{2m+1}(\tilde{X};\mathbb{Z}) = \mathbb{Z}$$

Considering the conditions on $\pi_i(\tilde{X})$, application of Hurewicz gives

$$H_i(\tilde{X};\mathbb{Z}) = 0 \qquad 1 \leq i \leq m-1$$

In particular we note that

$$\mathrm{Tor}_i = 0 \qquad 0 \leq i \leq m-1$$

Therefore by the Universal Coefficient Theorem we have that

$$H^i(\tilde{X};\mathbb{Z}) = 0 \qquad 1 \leq i \leq m-1$$

Application of Poincaré duality then gives

$$H_i(\tilde{X};\mathbb{Z}) = 0 \qquad m+2 \leq i \leq 2m$$

We are left with two non-zero homology groups in the middle dimensions, m and $m+1$. Write

$$H_m(\tilde{X};\mathbb{Z}) = \mathbb{Z}^a \oplus \mathrm{Tor}_m$$

for some $a \in \mathbb{Z}$. By the Universal Coefficient Theorem

$$H^m(\tilde{X};\mathbb{Z}) = \mathbb{Z}^a \oplus \mathrm{Tor}_{m-1} = \mathbb{Z}^a$$

Poincaré duality then gives

$$H_{m+1}(\tilde{X};\mathbb{Z}) = \mathbb{Z}^a \qquad \square$$

The above result takes place entirely at the homological level, but we can use this information to reverse engineer properties at the chain level. $C_*(\tilde{X})$ is a chain complex

$$0 \to C_{2m+1}(\tilde{X}) \xrightarrow{d_{2m+1}} C_{2m}(\tilde{X}) \xrightarrow{d_{2m}} \dots \xrightarrow{d_2} C_1(\tilde{X}) \xrightarrow{d_1} C_0(\tilde{X}) \to 0$$

where each C_i is a free $\Lambda = \mathbb{Z}[G]$ module. Since $\tilde{X}$ only has non-zero homology in the top, bottom, and middle dimensions, this means that the sequence only fails to be exact at three points. However, since we assume that $\tilde{X}$ is connected, the top and bottom homology are fixed and not particularly interesting. Therefore we can augment and co-augment these complexes to remove these cases

$$0 \to \mathbb{Z} \to C_{2m+1}(\tilde{X}) \xrightarrow{d_{2m+1}} C_{2m}(\tilde{X}) \xrightarrow{d_{2m}} \dots \xrightarrow{d_2} C_1(\tilde{X}) \xrightarrow{d_1} C_0(\tilde{X}) \to \mathbb{Z} \to 0$$

Exactness will then fail at a single point, namely

$$d_{m+1} : C_{m+1} \to C_m$$

Therefore, most of the homological information of $\tilde{X}$ is encapsulated in a single boundary operator.

Now suppose we reverse the situation, and instead try to build purely algebraic chain complexes which satisfy equivalent conditions. Let C_* be a chain complex of dimension $2m+1$ where each C_i is a free Λ-module. Furthermore suppose there exists a chain isomorphism $h_* : C_* \to C^{n-*}$ satisfying $h_k^* = (-1)^{\frac{n(n-1)}{2}} h_{n-k}$ and that the homology of C_* is given by

$$H_i(C_*;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, 2j+1 \\ \mathbb{Z}^a \oplus \mathrm{Tor}_j & i = j \\ \mathbb{Z}^a & i = j+1 \\ 0 & \text{otherwise} \end{cases}$$

for some $a \in \mathbb{Z}$ We will call such a C_* a highly connected chain complex. As before we can augment and co-augment C_* to give

$$0 \to \mathbb{Z} \to C_{2m+1} \xrightarrow{d_{2m+1}} C_{2m} \xrightarrow{d_{2m}} \dots \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \to \mathbb{Z} \to 0$$

Writing the cochain complex C^* we augment and co-augment to obtain

$$0 \to \mathbb{Z} \to C^0 \xrightarrow{d_1^*} C^1 \xrightarrow{d_2^*} \dots \xrightarrow{d_{2m}^*} C^{2m} \xrightarrow{d_{2m+1}^*} C^{2m+1} \to \mathbb{Z} \to 0$$

Applying the isomorphism h_* generates the commutative chain diagram

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & C^0 & \longrightarrow & C^1 & \longrightarrow & \cdots & \longrightarrow & C^{2m} & \rightarrow & C^{2m+1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h_0} & & \downarrow{\scriptstyle h_1} & & & & \downarrow{\scriptstyle h_{2m}} & & \downarrow{\scriptstyle h_{2m+1}} & & \\ 0 & \longrightarrow & C_{2m+1} & \rightarrow & C_{2m} & \longrightarrow & \cdots & \longrightarrow & C_1 & \longrightarrow & C_0 & \longrightarrow & 0 \end{array}$$

and focusing on the middle dimension, we have the square

$$\begin{tikzcd}
C^m \arrow[r, "d^*_{m+1}"] \arrow[d, "h_m"'] & C^{m+1} \arrow[d, "h_{m+1}"] \\
C_{m+1} \arrow[r, "d_{m+1}"'] & C_m
\end{tikzcd}$$

Construct a homomorphism

$$\partial_{m+1} : C^m \to C_m$$

$$\partial_m = d_{m+1} \circ h_m = h_{m+1} \circ d^*_{m+1}$$

Then there exists a chain complex, say $\hat{C}$, which isomorphic to C_*

$$0 \to C^0 \xrightarrow{d^*_1} \dots \xrightarrow{d^*_m} C^m \xrightarrow{\partial_m} C_m \xrightarrow{d_m} \dots \xrightarrow{d_1} C_0 \to 0$$

By our earlier observations we know that $\hat{C}$ will only fail to be exact at ∂_{m+1}, and we see that all the non-trivial behaviour is described by the mapping $\partial_{m+1} : C^m \to C_m$. Considering the dual mapping $\partial^*_{m+1} : C^m \to C_m$ we obtain:

Lemma 3.4.4. $\partial^*_{m+1} = (-1)^m \partial_{m+1}$

Proof. From 3.3.10 we know that

$$h^*_m = (-1)^{\frac{2m(2m+1)}{2}} h_{2m+1-m} = (-1)^{m(2m+1)} h_{m+1}$$

Recalling that

$$\partial_{m+1} = d_{m+1} \circ h_j$$

$$d_{m+1} \circ h_m = h_{m+1} \circ d^*_{m+1}$$

we see that

$$\begin{aligned}
\partial^*_{m+1} &= h^*_m \circ d^*_{m+1} \\
&= (-1)^{m(2m+1)} h_{m+1} \circ d^*_{m+1} \\
&= (-1)^{2m^2+m} \partial_{m+1} \\
&= (-1)^m \partial_{m+1}
\end{aligned}$$

□

Returning to $\hat{C}$ we have two exact segments linked by ∂_{m+1}

$$0 \to \mathbb{Z} \xrightarrow{\epsilon^*} C^0 \xrightarrow{d_1^*} \dots \xrightarrow{d_m^*} C^m$$

$$C_m \xrightarrow{d_m} \dots \xrightarrow{d_1} C_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

Since each C_i is a free module over Λ we see that we are almost looking at a section of a Λ-resolution of $\mathbb{Z}$ and its dual, in other words, writing $F_i = C_i$, the exact sequences

$$0 \to \mathbb{Z} \xrightarrow{\epsilon^*} C^0 \xrightarrow{d_1^*} \dots \xrightarrow{d_j^*} C^m \to \Omega^*_{m+1} \to 0$$

$$0 \to \Omega_{m+1} \to C_m \xrightarrow{d_m} \dots \xrightarrow{d_1} C_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

It follows that ∂_{m+1} will induce a mapping $\tilde{\partial}_{m+1} : \Omega^*_{m+1} \to \Omega_{m+1}$ which, since $\Omega_{m+1} = \mathrm{Ker}(d_m)$, will completely describe the non-exact behaviour of $\hat{C}$. So we have the diagram

$$\begin{tikzcd}
C^m \arrow[rrr, "\partial_{m+1}"] \arrow[dr] & & & C_m \\
& \Omega^*_{m+1} \arrow[r, "\tilde{\partial}_{m+1}"'] & \Omega_{m+1} \arrow[ur] &
\end{tikzcd}$$

where $\tilde{\partial}^*_{m+1} = (-1)^m \tilde{\partial}_{m+1}$.

This diagram forms the basis for most of the material in the rest of this paper. Now suppose we wanted to calculate the non-trivial homology of C_*.

Then

$$H_m(C_*;\mathbb{Z}) = \mathrm{Ker}(d_m)/\mathrm{Im}(\partial_{m+1}) = \Omega_{m+1}/\mathrm{Im}(\partial_{m+1})$$

Take a matrix representation for ∂_{m+1}. This matrix is then equipped with a Smith Normal Form (see [20]), which takes the form

$$\mathrm{SNF}(\partial_{j+1}) = \begin{pmatrix} \eta_1 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & \eta_r & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots \end{pmatrix}$$

for some choice of $r \in \mathbb{N}$, $\eta_i \in \mathbb{Z}$ and some number $s \in \mathbb{N}$ of zero rows/columns. This form then completely determines homology (see the introduction of [3] for full details), so that:

$$H_m(C_*;\mathbb{Z}) = \mathbb{Z}^s \oplus \mathbb{Z}/\eta_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/\eta_r\mathbb{Z}$$

We also note that as discussed in Chapter 2.5, ∂_{m+1} defines a bilinear form equipped with some symmetry condition. We have therefore constructed two invariants of our chain complexes, a Λ-invariant bilinear form on the middle chain group, and homology.

Suppose that Ω_{m+1} is straight, so that we can write

$$\Omega_{m+1} = M_{m+1} \oplus \Lambda^{a_{j+1}}$$

for some minimal module M_{m+1} and choice of $a_{m+1} \in \mathbb{Z}$. For this paper we will concern ourselves only with such syzygies, and only study the minimal case i.e. $a_{m+1} = 0$.

We note that in the calculation of such $\tilde{\partial}_{m+1}$ we will not deal with any of the topology involved in constructing both genuine $\tilde{X}$ and X from these algebraic complexes. We merely produce algebraic properties any such $\tilde{X}$ must satisfy following on from the existence of free resolutions and a duality isomorphism h_* satisfying $h_k^* = (-1)^m h_{n-k}$.

Chapter 4

Cyclic Groups

4.1 Highly Connected Chain Complexes With Cyclic Fundamental Group

In this section we take our highly connected chain complexes of the previous chapter and fix $\Lambda = \mathbb{Z}[C_n]$. From our work in section 2.3 we know that a free Λ-resolution of $\mathbb{Z}$ exhibits only two syzygies which are both straight with minimal modules

$$M_{2i+1} = \mathcal{R}_n$$
$$M_{2i} = \mathbb{Z}$$

Suppose C_* is a highly-connected chain complex of dimension $4k+1$. We have the diagram

$$\begin{tikzcd}
C^{2k} \arrow[rr, "\partial_{2k+1}"] \arrow[dr] & & C_{2k} \\
& \Omega^*_{2k+1} \arrow[r, "\tilde{\partial}_{2k+1}"'] & \Omega_{2k+1} \arrow[u]
\end{tikzcd}$$

We are therefore interested in $\tilde{\partial}_{2k+1} : M^*_{2k+1} \to M_{2k+1}$, which here becomes

$\tilde{\partial}_{2k+1} : \mathcal{R}_n^* \to \mathcal{R}_n$ Furthermore from 3.4.4 we obtain

$$\tilde{\partial}_{2k+1}^* = (-1)^{2k}\tilde{\partial}_{2k+1} = \tilde{\partial}_{2k+1}$$

The study of such homomorphisms will be the main focus of this chapter, which we will cover in the next section.

Next suppose C_* is a highly-connected chain complex of dimension $4k+3$. We have the diagram

$$\begin{tikzcd} C^{2k+1} \arrow[rr, "\partial_{2k+2}"] \arrow[dr] & & C_{2k+1} \\ & \Omega_{2k+2}^* \arrow[r, "\tilde{\partial}_{2k+2}"'] & \Omega_{2k+2} \arrow[u] \end{tikzcd}$$

We are therefore interested in $\tilde{\partial}_{2k+2} : M_{2k+2}^* \to M_{2k+2}$, which here becomes $\tilde{\partial}_{2k+2} : \mathbb{Z}^* \to \mathbb{Z}$, where $\tilde{\partial}_{2k+2}^* = -\tilde{\partial}_{2k+2}$ by 3.4.4. Since $\mathbb{Z}$ is a well understood module, we can immediately give a classification in this case. Suppose A, B are right Λ-modules and define

$$\mathrm{Hom}_\Lambda^+(A,B) = \{f \in \mathrm{Hom}_\Lambda(A,B); f^* = f\}$$
$$\mathrm{Hom}_\Lambda^-(A,B) = \{f \in \mathrm{Hom}_\Lambda(A,B); f^* = -f\}$$

Proposition 4.1.1. $\mathrm{Hom}_\Lambda^-(\mathbb{Z}^*, \mathbb{Z}) = 0$.

Proof. Suppose we label the elements of $\mathbb{Z}$ as a_i. Then there exists a Λ-isomorphism

$$\upsilon : \mathbb{Z}^* \to \mathbb{Z}$$
$$\upsilon(a_i^*) = a_i$$

υ induces another Λ-isomorphism

$$\tilde{\upsilon}(\mathrm{End}_\Lambda(\mathbb{Z})) \to \mathrm{Hom}_\Lambda(\mathbb{Z}^*, \mathbb{Z})$$
$$\tilde{\upsilon}(\phi) = \phi \circ \upsilon$$

Furthermore, since $\upsilon^* : \mathbb{Z}^* \to \mathbb{Z}$ is defined by

$$\begin{aligned} \upsilon^*(a_j^*)(a_i^*) &= a_j^*(\upsilon(a_i^*)) \\ &= a_j^*(a_i) \end{aligned}$$

we obtain $\upsilon^*(a_j^*) = a_j$ i.e. $\upsilon^* = \upsilon$. Defining $\mathrm{End}_\Lambda^-(\mathbb{Z})$ in the obvious way, it is true that

$$\mathrm{End}^-(\mathbb{Z}) \cong \mathrm{Hom}_\Lambda^-(\mathbb{Z}^*, \mathbb{Z})$$

But $\mathrm{End}(\mathbb{Z}) \cong \mathbb{Z}$, so that

$$\mathrm{End}^-(\mathbb{Z}) = 0$$

Hence the result. □

4.2 Matrix Representations for $\mathbf{Hom}_{\mathbb{Z}[C_n]}(\mathcal{R}_n^*, \mathcal{R}_n)$

To classify $(4k+1)$-dimensional highly connected chain complexes, we need to classify $\mathrm{Hom}_\Lambda(\mathcal{R}_n^*, \mathcal{R}_n)$. So suppose $F \in \mathrm{Hom}_\Lambda(\mathcal{R}_n^*, \mathcal{R}_n)$. We take matrix representations, and study $F : (\mathbb{Z}^{p-1}, \rho^*) \to (\mathbb{Z}^{p-1}, \rho)$ where for $r \in \mathcal{R}_n$, $\rho(x)$, $\rho^*(x)$ are defined by the standard relations

$$\begin{aligned} \rho(x)(r) &= r \cdot x \\ \rho^*(x) &= \rho(x^{-1})^t \end{aligned}$$

Classifying such F then becomes solving the equation

$$F \circ \rho^*(x) = \rho(x) \circ F$$

for $F \in M_{p-1}(\mathbb{Z})$. Furthermore, we require $F^* = F$, which in matrix representation is equivalent to requiring

$$F^t = F$$

These equations are relatively easy to solve in small dimensions, but as n increases so does the number and length of computations required and so a brute force approach is not optimal. However a general form is obtainable by other means. Define basis elements $e_i = x^{p-i}$, and describe $\mathcal{R}_n$ by $\mathcal{R}_n = \{e_i\}_{0\leq i\leq p-2}$. (While not the most obvious description, later in the metacyclic case this will facilitate a direct comparison between two distinct methods. For the greatest continuity, we therefore adopt that description here as well.) Furthermore recall our definition of c_n to be the $n\times 1$ matrix defined by $(c_n)_{i,1} = 1$. We then have the following:

Proposition 4.2.1. $\rho(x) = \begin{pmatrix} 0 & -1 \\ I_{n-2} & -c_{n-2} \end{pmatrix}$

Proof. Since $x = -1 - \sum_{k=2}^{n-1} x^k = -\sum_{j=0}^{n-2} e_i$, we have that

$$\begin{aligned} \rho(x)(e_i) &= x^{n-i}\cdot x \\ &= x^{n-i-1} \\ &= \begin{cases} e_{i+1} & 0\leq i\leq p-3 \\ -\sum_{j=0}^{n-2} e_i & i = n-2 \end{cases} \end{aligned}$$

□

Proposition 4.2.2. $\rho^*(x) = \begin{pmatrix} -c_{n-2}^t & -1 \\ I_{n-2} & 0 \end{pmatrix}$

Proof.

$$\begin{aligned} \rho(x^{-1})(e_i) &= x^{n-i}\cdot x^{-1} \\ &= x^{n-i+1} \\ &= \begin{cases} -\sum_{j=0}^{n-2} e_i & i = 0 \\ e_{i-1} & 1\leq i\leq n-2 \end{cases} \end{aligned}$$

Therefore

$$\rho(x^{-1}) = \begin{pmatrix} -c_{n-2} & I_{n-2} \\ -1 & 0 \end{pmatrix}$$

and since $\rho^* = \rho(x^{-1})^t$, we obtain the result. $\square$

From 2.3.1 we know that $\mathcal{R}_n^* \cong \mathcal{R}_n$, so there exists an isomorphism

$$\upsilon : \mathcal{R}_n^* \to \mathcal{R}_n$$

If $\mathcal{F} \in \mathrm{End}_\Lambda(\mathcal{R}_n)$, υ will induce an isomorphism

$$\tilde{\upsilon} : \mathrm{End}_\Lambda(\mathcal{R}_n) \to \mathrm{Hom}_\Lambda(\mathcal{R}_n^*, \mathcal{R}_n)$$
$$\tilde{\upsilon}(\mathcal{F}) = \mathcal{F} \circ \upsilon$$

So we can study $\mathrm{Hom}_\Lambda(\mathcal{R}_n^*, \mathcal{R}_n)$ by studying $\mathrm{End}_\Lambda(\mathcal{R}_n)$ and one specific isomorphism υ.

Proposition 4.2.3. $\mathrm{End}_\Lambda(\mathcal{R}_n) = \mathcal{R}_n$.

Proof. Suppose $r \in \mathcal{R}_n$. Then the most general $\phi : \mathcal{R}_n \to \mathcal{R}_n$ has the form

$$\phi(r) = r \cdot \alpha \quad \alpha \in \mathbb{Z}[C_n]$$

But then writing $r = \sum_{i=0}^{n-2} a_i x^i$, we see that

$$\begin{aligned}\sum_{i=0}^{n-2} a_i x^i \cdot \sum_{j=0}^{n-1} x^i &= \sum_{i=0}^{n-2} a_i (\sum_{j=0}^{n-1} x^i) \\ &= 0\end{aligned}$$

So $\sum_{j=0}^{n-1} x^i = 0$ in $\mathrm{End}_\Lambda(\mathcal{R}_n)$, and hence the result. $\square$

In matrix representations, this means we can set

$$\rho(x)^{n-1} = -\sum_{j=0}^{n-2} \rho(x)^j$$

and express $\mathcal{F} \in \mathrm{End}_\Lambda(\mathcal{R}_n)$ as

$$\mathcal{F} = \sum_{i=0}^{n-2} a_i \rho(x)^i \quad a_i \in \mathbb{Z}$$

where $\rho(x)^0 = \rho(1) = \mathrm{Id}$. Suppose now Υ_n denotes the matrix corresponding to υ. Then we can express an $F \in \mathrm{Hom}_\Lambda(\mathcal{R}_n^*, \mathcal{R}_n)$ as

$$F = \sum_{i=0}^{n-2} a_i \rho(x)^i \Upsilon_n \quad a_i \in \mathbb{Z}$$

We have constructed a general form for F, but at the moment it remains somewhat opaque. However by considering some low dimensional examples, it becomes clear how to build up a general matrix form for firstly Υ_n and then $\rho(x)^i \circ \Upsilon_n$.

Example 4.2.4 ($\mathbf{C_5}$). *Let $\Lambda = \mathbb{Z}[C_5]$. Then*

$$\mathcal{R}_n = \mathrm{Span}_\mathbb{Z}\{e_i\}_{0 \leq i \leq 3}$$

with $e_i = x^{5-i}$. From 4.2.1 *and* 4.2.2 *we know*

$$\rho(x) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \quad \rho^*(x) = \rho(x^{-1})^t = \begin{pmatrix} -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Consider

$$\alpha = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then

$$\alpha \circ \rho^*(x) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\rho(x) \circ \alpha = \begin{pmatrix} -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Since $\alpha \circ \rho^*(x) = \rho(x) \circ \alpha$, *and* $\det(\alpha) = 1$, *we can take* $\Upsilon_5 = \alpha$. *Write* $\gamma_i = \rho(x)^i \circ \Upsilon_5$. *Then one can calculate*

$$\gamma_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\gamma_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_4 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & -1 \end{pmatrix}$$

Note that $\gamma_4 = -\sum_{i=0}^{3} \gamma_i$ *as required. An element* $F \in \mathrm{Hom}_\Lambda(\mathcal{R}_n^*, \mathcal{R}_n)$ *therefore takes the form*

$$F = a_0\gamma_0 + a_1\gamma_1 + a_2\gamma_2 + a_3\gamma_3$$

We also require a symmetry condition on F*, and calculate*

$$\begin{aligned} \gamma_3^t &= -\gamma_1 \\ \gamma_2^t &= -\gamma_2 \\ \gamma_1^t &= -\gamma_3 \\ \gamma_0^t &= \gamma_0 + \gamma_1 + \gamma_2 + \gamma_3 \end{aligned}$$

to obtain

$$\begin{aligned} F^* &= a_0(\gamma_0 + \gamma_1 + \gamma_2 + \gamma_3) - a_1\gamma_3 - a_2\gamma_2 - a_3\gamma_1 \\ &= a_0\gamma_0 + (a_0 - a_3)\gamma_1 + (a_0 - a_2)\gamma_2 + (a_0 - a_1)\gamma_3 \end{aligned}$$

Solving $F^* = F$ *and setting* $a = a_2$*,* $b = a_3$ *gives a symmetric map*

$$F^+ = a(2\gamma_0 + 2\gamma_1 + \gamma_2) + b(\gamma_3 - \gamma_1)$$

$$F^+ = \begin{pmatrix} 2a & 2a-b & a & b \\ 2a-b & 4a-2b & 3a-2b & a \\ a & 3a-2b & 4a-2b & 2a-b \\ b & a & 2a-b & 2a \end{pmatrix}$$

$$\det(F^+) = 5(a^2 + ab - b^2)^2$$

Example 4.2.5 ($\mathbf{C_7}$). *Let* $\Lambda = \mathbb{Z}[C_7]$ *and define basis elements* $e_i = x^{7-i}$ *so* *that*

$$\mathcal{R}_n = \mathrm{Span}_{\mathbb{Z}}\{e_i\}_{0\leq i\leq 5}$$

By 4.2.1 *and* 4.2.2 *we have*

$$\rho(x) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \quad \rho^*(x) = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and it is easy to check that

$$\Upsilon_7 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

satsfies $\Upsilon_7 \circ \rho^*(x) = \rho(x) \circ \Upsilon_7$ *and has unit determinant. Therefore defining* $\gamma_i = \rho(x)^i \circ \Upsilon_7$*, an element* $F \in \mathrm{Hom}_\Lambda(\mathcal{R}_n^*, \mathcal{R}_n)$ *is of the form*

$$F = a_0\gamma_0 + a_1\gamma_1 + a_2\gamma_2 + a_3\gamma_3 + a_4\gamma_4 + a_5\gamma_5$$

for $a_i \in \mathbb{Z}$ *where*

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\gamma_3 = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad \gamma_4 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 & -1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_5 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & -1 & -1 & 0 \\ 0 & -1 & -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We have the following duality relations

$$\gamma_0^t = \sum_{i=0}^{5} \gamma_i \quad \gamma_1^t = -\gamma_5$$

$$\gamma_2^t = -\gamma_4 \quad \gamma_3^t = -\gamma_3$$

$$\gamma_4^t = -\gamma_2 \quad \gamma_5^t = -\gamma_1$$

Solving $F^* = F$ and setting $a = a_3$, $b = a_4$, $c = a_5$ gives a symmetric map

$$F^+ = a(2\gamma_0 + 2\gamma_1 + 2\gamma_2 + \gamma_3) + b(\gamma_4 - \gamma_2) + c(\gamma_5 - \gamma_1)$$

$$F^+ = \begin{pmatrix} 2a & 2a-c & 2a-b & a & b & c \\ 2a-c & 4a-2c & 4a-b-2c & 3a-b-c & a+b-c & b \\ 2a-b & 4a-b-2c & 6a-2b-2c & 5a-2b-2c & 3a-b-c & a \\ a & 3a-b-c & 5a-2b-2c & 6a-2b-2c & 4a-b-2c & 2a-b \\ b & a+b-c & 3a-b-c & 4a-b-2c & 4a-2c & 2a-c \\ c & b & a & 2a-b & 2a-c & 2a \end{pmatrix}$$

$$\det(F^+) = 7(a^3 + 3a^2b - a^2c - 4ab^2 + 5abc - 2ac^2 + b^3 - b^2c - 2bc^2 + c^3)^2$$

Studying these two examples, there are obvious general forms for Υ_n and each γ_i that one could conjecture, all of which are indeed correct as we now show. Suppose $\Lambda = \mathbb{Z}[C_n]$ and $e_i = x^{n-i}$ so that

$$\mathcal{R}_n = \mathrm{Span}_{\mathbb{Z}}\{e_i\}_{0 \le i \le n-2}$$

Define $\xi_k \in GL_k(\mathbb{Z})$ to be the upper-triangular matrix consisting entirely of 1's

$$(\xi_k)_{ij} = \begin{cases} 1 & j \ge i \\ 0 & j < i \end{cases}$$

Proposition 4.2.6. $\Upsilon_n = \xi_{n-1}$.

Proof. We note that ξ_{n-1} is the matrix representation of the change of basis calculation discussed in 2.3.1, albeit over a different basis, and it follows that the result holds. For completeness we give another proof within this context however.

Proceed row by row for the calculation $\rho(x) \circ \xi_{n-1}$. Define $\mathrm{Row}(i)$ to be the i^{th} row of the product $\rho(x) \circ \xi_{n-1}$. Then by inspection, the non-zero element of the first row of $\rho(x)$ will only 'hit' the non-zero elements in the $(n-1)^{th}$ row of ξ_{n-1}, of which there is only one in position $(n-1, n-1)$. So

$$\mathrm{Row}(1) = \begin{pmatrix} 0 & \dots & 0 & -1 \end{pmatrix}$$

The non-zero elements of the second row of $\rho(x)$ will hit the first and last rows of ξ_{n-1}, and so will pick up a string of 1's ending in a $1 - 1 = 0$. So

$$\mathrm{Row}(2) = \begin{pmatrix} 1 & \dots & 1 & 0 \end{pmatrix}$$

In general, the non-zero elements of the i^{th} row of $\rho(x)$ will hit the $(i-1)^{th}$ and last rows of ξ_{n-1}, picking out in order $(i-2)$ zeroes, followed by a string of $1's$ and ending on a $1 - 1 = 0$. Combining these together gives the block matrix description

$$\rho(x) \circ \xi_{n-1} = \begin{pmatrix} 0 & -1 \\ \xi_{n-2} & 0 \end{pmatrix}$$

A similar row by row inspection gives $\xi_{n-1} \circ \rho^*(x) = \rho(x) \circ \xi_{n-1}$ $\square$

Define $\gamma_i = \rho(x)^i \circ \xi_{n-1}$, so that $F = \sum_{i=0}^{n-2} a_i \gamma_i$

Proposition 4.2.7. γ_i can be expressed in block matrix form as

$$\gamma_i = \begin{pmatrix} 0 & -\xi_i^t \\ \xi_{n-(i+1)} & 0 \end{pmatrix}$$

Proof. Proceed by induction on i. We take as base case 4.2.6, and assume that the statement is true for $i-1$, i.e.

$$\gamma_{i-1} = \begin{pmatrix} 0 & -\xi_{i-1}^t \\ \xi_{n-i} & 0 \end{pmatrix}$$

Then

$$\gamma_i = \rho(x) \circ \gamma_{i-1} = \begin{pmatrix} 0 & -1 \\ I_{n-2} & -c_{n-2} \end{pmatrix} \begin{pmatrix} 0 & -\xi_{i-1}^t \\ \xi_{n-i} & 0 \end{pmatrix}$$

We proceed on a row-by-row analysis again, defining Row(i) to be the i^{th} row of $\rho(x) \circ \gamma_{i-1}$. The non-zero element in the first row of $\rho(x)$ will only hit elements in the $(n-1)^{th}$ row of γ_{i-1} of which there is only one in position $(n-1, n-i)$. Therefore

$$\text{Row}(1) = \begin{pmatrix} 0_1 & \dots & 0_{n-i-1} & -1_{n-i} & 0_{n-i+1} & \dots & 0_{n-1} \end{pmatrix}$$

where the subscripts indicate which column an element belongs to. The second row of $\rho(x)$ has two non-zero entries which will pick out the 1^{st} and $(n-1)^{th}$ rows of γ_{i-1}. Again, we will obtain a -1 in the $(n-i)^{th}$ column, and this time we will pick up a further -1 in the $(n-i+1)^{th}$, so that

$$\text{Row}(2) = \begin{pmatrix} 0_1 & \dots & 0_{n-i-1} & -1_{n-i} & -1_{n-i+1} & 0_{n-i+2} & \dots & 0_{n-1} \end{pmatrix}$$

For $3 \leq j \leq i$ this pattern will continue, so that Row(j) will differ from Row$(j-1)$ only by the addition of an (-1) in the $(n-i+(j-1))^{th}$ column. This arises from the extra element the $(j-1)^{th}$ row of $-\xi_{i-1}^t$ has when compared to the $(j-2)^{th}$ row. Therefore in block matrix form we can write

$$\bigoplus_{j=1}^{i} \mathrm{Row}(j) = \begin{pmatrix} 0 & -\xi_i^t \end{pmatrix}$$

Next consider Row$(i+1)$. Then the $(i+1)^{th}$ row of $\rho(x)$ will pick out the i^{th} and $(n-1)^{th}$ row of γ_{i-1}, which now both correspond to the ξ_{n-i} block. Therefore it will pick out a sequence of $(n-i)$ $1's$, where the final element is cancelled out by the contribution of the -1 in the $(n-1)^{th}$ row, giving

$$\mathrm{Row}(i+1) = \begin{pmatrix} 1_1 & \dots & 1_{n-i-1} & 0_{n-i} & \dots & 0_{n-1} \end{pmatrix}$$

For $i+2 \leq j \leq n-1$, write $j = i+k$ where $2 \leq k \leq n-i-1$ the j^{th} row of γ_i differs from its $(j-1)^{th}$ by the replacement of a 1 in the $(k-1)^{th}$ column by a 0. Therefore

$$\mathrm{Row}(j) = \begin{pmatrix} 0_1 & \dots & 0_{k-1} & 1_k & \dots & 1_{n-i-1} & 0_{n-i} & \dots & 0_{n-1} \end{pmatrix}$$

and we get

$$\bigoplus_{j=i+1}^{n-1} \mathrm{Row}(j) = \begin{pmatrix} \xi_{n-i-1} & 0 \end{pmatrix}$$

By the inductive hypothesis the result then follows. □

Lemma 4.2.8. $\sum_{i=0}^{n-1} \gamma_i = 0$.

Proof.

$$\begin{aligned} \sum_{i=0}^{n-1} \gamma_i &= (\sum_{i=0}^{n-1} \rho(x)^i) \circ \xi_{n-1} \\ &= (0) \circ \xi_{n-1} \\ &= 0 \end{aligned}$$

□

It then follows that:

Theorem 4.2.9. *Suppose $F : \mathcal{R}_n^* \to \mathcal{R}_n$ is a $\mathbb{Z}[C_n]$-homomorphism. Then*

F has matrix representation

$$F = \sum_{i=0}^{n-2} a_i \gamma_i \quad a_i \in \mathbb{Z}$$

Considering symmetry conditions we obtain:

Proposition 4.2.10. $\gamma_i^t = \begin{cases} \sum\limits_{j=0}^{n-2} \gamma_j & i = 0 \\ -\gamma_{n-(i+1)} & 1 \leq i \leq n-2 \end{cases}$

Proof. Immediate from the block form of γ_i. □

We can now combine these results to give a general form for elements of $\mathrm{Hom}^+_{\mathbb{Z}[C_n]}(\mathcal{R}_n^*, \mathcal{R}_n)$.

Theorem 4.2.11. *Suppose $F : \mathcal{R}_n^* \to \mathcal{R}_n$ is a symmetric $\mathbb{Z}[C_n]$-homomorphism. Then F has matrix representation*

$$a_{\frac{n-1}{2}}\Big(\sum_{i=0}^{\frac{n-3}{2}} 2\gamma_i + \gamma_{\frac{n-1}{2}}\Big) + \sum_{i=\frac{n+1}{2}}^{n-2} a_i(\gamma_i - \gamma_{n-(i+1)})$$

where γ_i defined as above, $a_i \in \mathbb{Z}$.

Proof. The condition $F^* = F$ generates a system of equations on the coefficient terms:

$$\begin{aligned} a_0 &= a_0 \\ a_i &= a_0 - a_{n-(i+1)} \qquad (1 \leq i \leq n-2) \end{aligned}$$

The case $i = \frac{n-1}{2}$ then gives $a_0 = 2a_{\frac{n-1}{2}}$. Pick a_i for $\frac{n+1}{2} \leq i \leq n-2$ to be our representatives. Then we can write a symmetric mapping F^+ as

$$\begin{aligned} F^+ &= 2a_{\frac{n-1}{2}}\gamma_0 + \sum_{i=1}^{\frac{n-3}{2}}(2a_{\frac{n-1}{2}} - a_{n-(i+1)})\gamma_i + 2a_{\frac{n-1}{2}}\gamma_{\frac{n-1}{2}} + \sum_{i=\frac{n+1}{2}}^{n-2} a_i\gamma_i \\ &= a_{\frac{n-1}{2}}(\sum_{i=0}^{\frac{n-3}{2}} 2\gamma_i + \gamma_{\frac{n-1}{2}}) + \sum_{i=\frac{n+1}{2}}^{n-2} a_i(\gamma_i - \gamma_{n-(i+1)}) \end{aligned}$$

□

Chapter 5

Metacyclic Groups I: Further Preliminaries

5.1 Highly Connected Chain Complexes over a Metacyclic Group Ring

We now construct highly connected chain complexes fixing $\Lambda = \mathbb{Z}[G(p,q)]$. When we constructed such chain complexes in section 3.4 we assumed each C_i free, however from section 2.4 we know that the existence of a strongly diagonal resolution is currently only known for a restricted number of groups. We can still construct highly connected chain complexes using projective C_i but in doing so they lose the ability to correspond to a universal cover. However, the problems discussed in this section are still worthwhile performing in a general setting, and not just through case-by-case calculations for the groups known to admit a strongly diagonal resolution; they can accommodate any future progress made in expanding said list of groups, allow us to generate p-adic results through tensor products, and act as a first approximation to the construction of such universal covers.

As discussed in section 2.4, our diagonal resolutions (either strongly or otherwise) generate odd syzygies which are straight with minimal modules (written in the row formulation):

$$M_{2k+1} = R(k) \oplus [y-1)$$

Suppose C_* is a $4k+1$ dimensional highly connected chain complex. Then we have the diagram

$$\begin{tikzcd} C^{2k} \arrow[rr, "\partial_{2k+1}"] \arrow[dr, dash] & & C_{2k} \\ & \Omega^*_{2k+1} \arrow[r, "\tilde{\partial}_{2k+1}"'] & \Omega_{2k+1} \arrow[u, dash] \end{tikzcd}$$

We are therefore interested in $\tilde{\partial}_{2k+1} : M^*_{2k+1} \to M_{2k+1}$ where

$$\tilde{\partial}^*_{2k+1} = (-1)^{2k}\tilde{\partial}_{2k+1} = \tilde{\partial}_{2k+1}$$

We can represent $\tilde{\partial}_{2k+1}$ as a matrix

$$\tilde{\partial}_{2k+1} = \begin{pmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{pmatrix}$$

where

$$\begin{aligned} \delta_{11} :& R(k)^* \to R(k) \\ \delta_{12} :& [y-1)^* \to R(k) \\ \delta_{21} :& R(k)^* \to [y-1) \\ \delta_{22} :& [y-1)^* \to [y-1) \end{aligned}$$

with $\delta^*_{11} = \delta_{11}$, $\delta^*_{22} = \delta_{22}$, . We start by studying δ_{12}, for which we need a well known result (see B.5 in [8]), and a lemma.

Lemma 5.1.1 (Eckmann Shapiro) *Let G be a group, $H \subset G$ a subgroup, and let $i : \mathbb{Z}[H] \to \mathbb{Z}[G]$ denote the inclusion map. Suppose M is a $\mathbb{Z}[G]$-module and N a $\mathbb{Z}[H]$-module, so that $i^*(M)$ denotes restriction of* *scalars, and i_* denotes extension of scalars. Then there exist isomorphisms*

$$\mathrm{Hom}_{\mathbb{Z}[G]}(i_*(N), M) \cong \mathrm{Hom}_{\mathbb{Z}[H]}(N, i^*(M))$$
$$\mathrm{Hom}_{\mathbb{Z}[G]}(M, i_*(N)) \cong \mathrm{Hom}_{\mathbb{Z}[H]}(i^*(M), N)$$

Lemma 5.1.2. *For* $1 \leq i \leq q$, $\mathrm{Hom}_\Lambda(I_q, R(i)) = 0$.

Proof. Let $i : \mathbb{Z}[C_p] \to \Lambda$ denote the inclusion mapping. We can write

$$\bigoplus_{i=1}^{q} \mathrm{Hom}_\Lambda(I_q, R(i)) \cong \mathrm{Hom}_\Lambda(I_q, i_*(I_C))$$

Using 5.1.1, and the standard result that $\mathrm{Hom}_{\mathbb{Z}[C_p]}(\mathbb{Z}, I_C) = 0$ we calculate

$$\begin{aligned}\bigoplus_{i=1}^{q} \mathrm{Hom}_\Lambda(I_q, R(i)) &\cong \mathrm{Hom}_{\mathbb{Z}[C_p]}(i^*(I_q), I_C)\\ &= \bigoplus_{q} \mathrm{Hom}_{\mathbb{Z}[C_p]}(\mathbb{Z}, I_C)\\ &= 0\end{aligned}$$

Hence the result. $\square$

Proposition 5.1.3. $\mathrm{Hom}_\Lambda([y-1)^*, R(k)) \cong \bigoplus_{i\neq 1} \mathrm{Hom}_\Lambda(R(i)^*, R(k))$.

Proof. Let I_q be the augmentation ideal of $\mathbb{Z}[C_q]$. Then we can write $[y-1)$ as an extension

$$0 \to \bigoplus_{i\neq 1} R(i) \to [y-1) \to I_q \to 0$$

Recall from 2.3.1 that $I_q^* \cong I_q$ so dualising the above sequence, we obtain an exact sequence

$$0 \to I_q \to [y-1)^* \to \bigoplus_{j\neq 1} R(i)^* \to 0$$

Applying $\mathrm{Hom}_\Lambda(-, R(k))$ gives the long exact sequence

$$0 \to \mathrm{Hom}_\Lambda(\bigoplus_{i\neq 1} R(i)^*, R(k)) \to \mathrm{Hom}_\Lambda([y-1)^*, R(k)) \to \mathrm{Hom}_\Lambda(I_q, R(k)) \to$$

$$\to \mathrm{End}^1_\Lambda(\bigoplus_{i\neq 1} R(i)^*, R(k)) \to \dots$$

But then from 5.1.2

$$\mathrm{Hom}_\Lambda(I_q, R(k)) = 0$$

So there exists the exact sequence

$$0 \to \mathrm{Hom}_\Lambda(\bigoplus_{i\neq 1} R(i)^*, R(k)) \to \mathrm{Hom}_\Lambda([y-1)^*, R(k)) \to 0$$

and the result follows. □

It is also obvious that $\delta_{21} = \delta_{12}^*$. We therefore have three cases of Λ-homomorphisms to consider. Switching to our cyclotomic formulation, we will write these three cases as

$$\begin{aligned} F_k &: \tilde{P}^k \to P^k \\ G_{k,i} &: \tilde{P}^i \to P^k \\ H &: [y-1)^* \to [y-1) \end{aligned}$$

so that defining

$$\begin{aligned} G &= (G_{k,0}, G_{k,2}, G_{k,3}, \dots, G_{k,p-1})^t \\ G^* &= (G^*_{k,0}, G^*_{k,2}, G^*_{k,3}, \dots, G^*_{k,p-1}) \end{aligned}$$

we have

$$\tilde{\partial}_{2k+1} = \begin{pmatrix} F_k & G \\ G^* & H \end{pmatrix}$$

For the remainder of this chapter, we will produce a range of results and methods which will allow us to classify such homomorphisms, with a specific focus on F_k. In the next chapter, we will apply these and earlier results.

5.2 Homomorphisms over $\mathbb{Q}[G(p,q)]$ and the rational fixed field

As a first approximation to classifying $F_k \in \mathrm{Hom}_\Lambda(\tilde{P}^k, P^k)$ we consider what happens over $\mathbb{Q}$. Define

$$\begin{aligned}\Lambda_{\mathbb{Q}} &= \mathbb{Q}[G(p,q)]\\ K &= \mathbb{Q}[\zeta_p]\\ K_0 &= K^\theta\end{aligned}$$

where K_0 is the fixed field of K under θ, and K and K_0 are $\Lambda_{\mathbb{Q}}$-modules in the obvious way. We wish to classify $\mathrm{Hom}_{\Lambda_{\mathbb{Q}}}(\tilde{P}^k \otimes \mathbb{Q}, P^k \otimes \mathbb{Q})$. The first step towards this is to calculate $P^k \otimes \mathbb{Q}$. Recall the fibre decomposition of Λ as

$$\begin{tikzcd} \mathbb{Z}[G(p,q)] \arrow[r] \arrow[d] & \mathcal{T}_q(R_0,\pi) \arrow[d] \\ \mathbb{Z}[C_q] \arrow[r] & \mathbb{F}_p[C_q] \end{tikzcd}$$

We tensor this diagram with $\mathbb{Q}$, and calculate

$$\begin{aligned}\Lambda \otimes \mathbb{Q} &= \Lambda_{\mathbb{Q}}\\ \mathcal{T}_q(R_0,\pi) \otimes \mathbb{Q} &= M_q(R_0 \otimes \mathbb{Q}) = M_q(K_0)\\ \mathbb{Z}[C_q] \otimes \mathbb{Q} &= \mathbb{Q}[C_q]\\ \mathbb{F}_p[C_q] \otimes \mathbb{Q} &= 0\end{aligned}$$

to obtain a fibre product

$$\begin{array}{ccc} \mathbb{Q}[G(p,q)] & \longrightarrow & M_q(K_0) \\ \downarrow & & \downarrow \\ \mathbb{Q}[C_q] & \longrightarrow & 0 \end{array}$$

It follows that $\mathbb{Q}[G(p,q)] \cong \mathbb{Q}[C_q] \oplus M_q(K_0)$. Recall that

$$\begin{aligned} \mathcal{T}_q(R_0,\pi) &\cong R(1) \oplus \ldots \oplus R(q) \\ &\cong R \oplus P \oplus \ldots \oplus P^{q-1} \end{aligned}$$

Over $\mathbb{Q}$ all these modules become isomorphic, and returning to our cyclotomic interpretation, we obtain

$$\begin{aligned} P^k \otimes \mathbb{Q} &\cong K \\ \tilde{P}^k \otimes \mathbb{Q} &\cong K \end{aligned}$$

so that

$$\begin{aligned} \mathrm{Hom}_{\Lambda_{\mathbb{Q}}}(\tilde{P}^k \otimes \mathbb{Q}, P^k \otimes \mathbb{Q}) &\cong \mathrm{End}_{\Lambda_{\mathbb{Q}}}(K) \\ &\cong K_0 \end{aligned}$$

The rational homomorphism ring is completely classified by elements of the rational fixed field, which is in turn explicitly describable. To this end we recall the definition of a Galois group. Suppose F is a field, and E is some field extension that is normal and separable. The Galois group of E/F is the group of automorphisms of E that leave F fixed

$$\mathrm{Gal}(E/F) = \{\phi : E \to E; \forall f \in F, \phi(f) = f\}$$

We can therefore write

$$\theta \in \mathrm{Gal}(K/\mathbb{Q}) \cong C_{p-1}$$

Since $\mathrm{ord}(\theta) = q$, θ generates a unique subgroup of order q, $C_q \subset \mathrm{Gal}(K/\mathbb{Q})$. In fact, for any j a positive divisor of $p-1$, there exists a unique subgroup of order j, $C_j \subset \mathrm{Gal}(K/\mathbb{Q})$ and subsequent generator. Of special interest is the case $j = 2$, and we will label the generator of such a C_2 by τ. We note that τ acts as complex conjugation.

Proposition 5.2.1. *Suppose q is even. Then K_0 is totally real where*

$$K_0 \otimes_{\mathbb{Q}} \mathbb{R} = \underbrace{\mathbb{R} \times \ldots \times \mathbb{R}}_{(p-1)/q}$$

Proof. We begin with the case $q = 2$. Then $\theta = \tau$, and we can write K_0 as

$$K_0 = \mathbb{Q}[\mu_p] \quad \mu_p = \zeta_p + \zeta_p^{-1}$$

which in turn implies

$$K_0 \otimes \mathbb{R} = \underbrace{\mathbb{R} \times \ldots \times \mathbb{R}}_{(p-1)/2}$$

Now suppose $q = 2j$ where $j > 1$. Then $K_0 \subset K^{\tau}$ and

$$K_0 \otimes_{\mathbb{Q}} \mathbb{R} \subset \underbrace{\mathbb{R} \times \ldots \times \mathbb{R}}_{(p-1)/2}$$

Since $\dim_{\mathbb{Q}}(K_0) = \frac{p-1}{q}$ we must have

$$K_0 \otimes_{\mathbb{Q}} \mathbb{R} = \underbrace{\mathbb{R} \times \ldots \times \mathbb{R}}_{(p-1)/q} \qquad \square$$

For the case q odd, we need the following standard result from Galois theory (see [24])

Proposition 5.2.2. *Let $A \subset B \subset C$ be fields such that C/A and B/A are finite Galois extensions. Then there exists a short exact sequence of finite groups*

$$1 \to \mathrm{Gal}(C/B) \to \mathrm{Gal}(C/A) \to \mathrm{Gal}(B/A) \to 1$$

Proposition 5.2.3. *Suppose q is even. Then K_0 is totally complex where*

$$K_0 \otimes_{\mathbb{Q}} \mathbb{R} = \underbrace{\mathbb{C} \times \ldots \times \mathbb{C}}_{(p-1)/2q}$$

Proof. Since q is odd, $2q$ will still be a divisor of $p-1$, and so there will exist an element $\xi \in \mathrm{Gal}(K/\mathbb{Q})$ such that $\mathrm{ord}(\xi) = 2q$. Then the inclusions $K^\xi \subset K_0 \subset K$ gives rise to the exact sequence

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Gal}(K/K_0) \arrow[r] \arrow[d, "\cong"] & \mathrm{Gal}(K/K^\xi) \arrow[r] \arrow[d, "\cong"] & \mathrm{Gal}(K_0/K^\xi) \arrow[r] \arrow[d, "\cong"] & 1 \\
1 \arrow[r] & C_q \arrow[r] & C_{2q} \arrow[r] & C_2 \arrow[r] & 1
\end{tikzcd}$$

where the given isomorphisms can be checked by basic dimensional arguments. Let μ be the generator of $\mathrm{Gal}(K_0/K^\xi)$, and recall the definition of τ as the unique element of order 2 in $\mathrm{Gal}(K/\mathbb{Q})$ and hence its subgroups of even order. In this case, $\tau \in \mathrm{Gal}(K/K^\xi)$, and since q is odd, μ must be the restriction of τ to K_0. So μ will act on K_0 as complex conjugation and hence K_0 is complex

$$K_0 \otimes_{\mathbb{Q}} \mathbb{R} = \underbrace{\mathbb{C} \times \ldots \times \mathbb{C}}_{(p-1)/2q} \qquad \square$$

We are particularly interested in the elements of K_0 satisfying certain symmetry conditions. Define

$$K_0^+ = \{\alpha \in K_0; \overline{\alpha} = \alpha\}$$
$$K_0^- = \{\alpha \in K_0; \overline{\alpha} = -\alpha\}$$

It is then simple to obtain

Proposition 5.2.4. *Suppose q even. Then*

$$K_0^+ \otimes \mathbb{R} = K_0 \otimes \mathbb{R}$$
$$K_0^- \otimes \mathbb{R} = 0$$

Proposition 5.2.5: *Suppose q odd. Then*

$$K_0^+ \otimes \mathbb{R} = \underbrace{\mathbb{R} \times \ldots \times \mathbb{R}}_{(p-1)/2q}$$
$$K_0^- \otimes \mathbb{R} = \underbrace{i(\mathbb{R} \times \ldots \times \mathbb{R})}_{(p-1)/2q}$$

Note that for q odd, we have that

$$(K_0^+ \otimes \mathbb{R}) \oplus (K_0^- \otimes \mathbb{R}) \cong K_0 \otimes \mathbb{R}$$

It is also easy to see how this holds without extending scalars. Since τ is non-trivial, we can write

$$K_0 = \{a_i\gamma_i + b_i\overline{\gamma_i}\}_{1 \leq i \leq p-1}$$

where $a_i \in \mathbb{Q}$, γ_i are invariant basis elements. It then follows that

$$K_0^+ = \{a_i(\gamma_i + \overline{\gamma_i})\}_{1 \leq i \leq \frac{p-1}{2q}} \quad K_0^- = \{b_i(\gamma_i - \overline{\gamma_i})\}_{1 \leq i \leq \frac{p-1}{2q}}$$

for $a_i, b_i \in \mathbb{Q}$.

Proposition 5.2.6. $K_0^+ \oplus K_0^- \cong K_0$.

Proof. We have for each i

$$a_i(\gamma_i + \overline{\gamma_i}) + b_i(\gamma_i - \overline{\gamma_i}) = (a_i + b_i)\gamma_i + (a_i - b_i)\overline{\gamma_i}$$

Representing the coefficients as a matrix over $\mathbb{Q}$, we can reduce as

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Finally noting that there are no intersections between choices of i, we have that

$$K_0^+ \oplus K_0^- \cong \{c_i\gamma_i + d_i\overline{\gamma_i}\})_{1\leq i\leq \frac{p-1}{2q}} = K_0 \qquad \square$$

As a corollary to this proof, we note that this result will not hold in the integral case.

5.3 Characteristic Elements of P^k

We now return to $\Lambda = \mathbb{Z}[G(p,q)]$. Recall the definition of θ from the group presentation of $G(p,q)$; then θ has equivalent description

$$\theta(x^i) = x^{ia}$$

for some $a \in \mathbb{Z}$ such that $a^q = 1 \bmod p$. Define

$$b = a^{q-1} \mod p$$

so that

$$\theta^{-1}(x^i) = x^{ib}$$

Using this notation we see that

$$xy = y\theta^{-1}(x) = yx^{ib}$$

Recall $P^k = (\zeta_p - 1)^k\mathbb{Z}[\zeta_p]$ is a Λ-module via the usual right actions. For the remainder of this chapter we will write $\zeta = \zeta_p$. Writing in this form, we easily see that each P^k is monogenic. Further to this, we can completely characterise each P^k by two simple properties. Beginning with $P^0 = R$, let M be a Λ-lattice and define properties

$\mathbf{M(\Sigma)}$: $\mathrm{rk}_{\mathbb{Z}}(M) = p-1$ and $M \cdot \Sigma_x = 0$ where $\Sigma_x = \sum_{k=0}^{6} x^k$

$\mathbf{M(0)}$: There exists $\epsilon_0 \in M$ such that $\epsilon_0 \cdot y = \epsilon_0$ and $\mathrm{Span}_{\mathbb{Z}}\{\epsilon_0 \cdot x^i\}_{0 \le i \le p-2} = M$

Proposition 5.3.1. *Let M be a Λ-lattice. Then M satisfies $M(\Sigma), M(0)$ if and only if $M \cong R$ and $\mathrm{Span}_{\mathbb{Z}}\{\epsilon_0 \cdot x^i\}_{0 \le i \le p-2}$ is a $\mathbb{Z}$-basis for M.*

Proof. We start with the backwards implication. By definition R satisfies $M(\Sigma)$. Take $\epsilon_0 = 1$. Then

$$1 \cdot y = 1$$
$$R = \mathrm{Span}_{\mathbb{Z}}\{1 \cdot x^i\}_{0 \le i \le p-2}$$

so R satisfies $M(0)$ too.

Now suppose that M is a Λ-lattice satisfying $M(\Sigma)$ and $M(0)$ for some element ϵ_0. We can construct a homomorphism of abelian groups $\natural : R \to M$ defined on the basis of R by

$$\natural(1 \cdot x^i) = \epsilon_0 \cdot x^i$$

Since $M = \mathrm{Span}_{\mathbb{Z}}\{\epsilon_0 \cdot x^i\}_{0 \le i \le p-2}$, $\natural$ is surjective, and since $\mathrm{rk}_{\mathbb{Z}}(M) = \mathrm{rk}_{\mathbb{Z}}(R) = p-1$, $\natural$ is bijective. $\natural$ clearly commutes with the action of x, and via $1 \cdot y = 1$, $\epsilon_0 \cdot y = \epsilon_0$ it also commutes with the action of y, so is a Λ-isomorphism. $\square$

This result generalises to all P^k. Define the element

$$\upsilon_X = \sum_{j=1}^{p-b} x^j$$

υ_X has the following properties:

Lemma 5.3.2. *Considered as elements acting from the right*

$$(x^{b(p-1)} - 1)^k = (-1)^k (x^{p-1} - 1)^k \upsilon_X^k$$

Proof.

$$\begin{aligned}(x^{b(p-1)}-1)^k &= (x^{p-1}-1)^k(x^{(b-1)(p-1)}+x^{(b-2)(p-1)}+\ldots+1)^k \\ &= (x^{p-1}-1)(\sum_{j=0}^{b-1} x^{j(p-1)})^k\end{aligned}$$

Then

$$\begin{aligned}\sum_{j=0}^{b-1} x^{j(p-1)} &= \sum_{j=p-b+1}^{p-2} x^j + x^{p-1} + 1 \\ &= \sum_{j=p-b+1}^{p-2} x^j - \sum_{j=0}^{p-2} x^j + 1 \\ &= -\sum_{j=1}^{p-b} x^j \\ &= -v_X\end{aligned}$$

Substituting back we have

$$(x^{b(p-1)}-1)^k = (-1)^k(x^{p-1}-1)^k v_X^k$$

□

Lemma 5.3.3. *v_X is a unit with inverse $v_X^{-1} = x^{p-1}\sum_{j=0}^{p-a} x^{(p-b)j}$.*

Proof. First note that modulo p

$$ab = aa^{q-1} = a^q = 1$$

Consider the expression

$$\sum_{j=0}^{p-b-1} x^j \sum_{l=0}^{p-a-1} x^{(p-b)l} = \sum_{l=0}^{p-a-1}\sum_{j=0}^{p-b-1} x^{j+(p-b)l}$$

We order the sum by running through all values of j for a particular value of l before moving to the next value of l in the sum. Then this expression starts with a sum for the $l = 0$ terms

$$1 + x + \ldots + x^{p-b-1}$$

followed by for $l = 1$

$$x^{p-b} + x^{p-b+1} + \ldots + x^{2(p-b)-1}$$

The expression continues in this way hitting consecutive powers of x until the final chain

$$x^{(p-a-1)(p-b)} + x^{(p-a-1)(p-b)+1} + \ldots + x^{(p-a)(p-b)-1}$$

But then

$$\begin{aligned}(p-a)(p-b) - 1 &= p^2 - ap - bp + ab - 1 \\ &= 1 - 1 \qquad (p) \\ &= 0 \qquad (p)\end{aligned}$$

So, for some $n \in \mathbb{Z}$, we can write

$$\begin{aligned}\sum_{j=0}^{p-b-1} x^j \sum_{l=0}^{p-a-1} x^{(p-b)l} &= n \sum_{j=0}^{p-1} x^j + 1 \\ &= 1\end{aligned}$$

Noting that

$$\sum_{j=0}^{p-b-1} x^j = x^{p-1} v_X$$

we arrive at the result. $\square$

For $1 \leq k \leq q - 1$ define a property:

$\mathbf{M(k)}$: There exists $\epsilon_k \in M$ such that $\epsilon_k \cdot y = \epsilon_k \cdot (-1)^k v_X^k$ and $\mathrm{Span}_{\mathbb{Z}}\{\epsilon_k \cdot x^i\}_{0 \leq i \leq p-2} = M$

Proposition 5.3.4. *Let M be a Λ-lattice. Then M satisfies $M(\Sigma), M(k)$ if and only if $M \cong P^k$ and $\mathrm{Span}_{\mathbb{Z}}\{\epsilon_k \cdot x^i\}_{0\leq i\leq p-2}$ is a $\mathbb{Z}$-basis for M.*

Proof. As before we show that P^k satisfies $M(\Sigma)$, $M(k)$. $M(\Sigma)$ is obvious. Write $\epsilon_k = (\zeta-1)^k = 1\cdot(x^{p-1}-1)^k$. Then this element clearly generates, and using the fact that $1 \cdot y = 1$, we have that

$$\begin{aligned}\epsilon_k \cdot y &= 1\cdot(x^{p-1}-1)^k y\\ &= 1\cdot y(x^{b(p-1)}-1)^k\\ &= 1\cdot(-1)^k(x^{p-1}-1)^k v_X^k\\ &= \epsilon_k\cdot(-1)^k v_X^k\end{aligned}$$

So P^k satisfies $M(k)$. For the forward implication, we simply re-apply the same argument we made in 5.3.1. □

We will call such an ϵ_k a characteristic element. For the rest of this chapter we will exclusively write each generator $(\zeta-1)^k = 1\cdot(x^{p-1}-1)^k = \epsilon_k$ and defining

$$p[k,i] = \epsilon_k \cdot x^i$$

we think of P^k via the description

$$P^k = \mathrm{Span}_{\mathbb{Z}}\{p[k,i]\}_{0\leq i\leq p-2}$$

The benefit of this description is that in considering Λ-homomorphisms between such modules, we need only consider how the homomorphism acts on the characteristic element. Suppose $f : P^r \to P^s$ is some Λ-isomorphism. Then since P^r, P^s are both monogenic, f must map the generator of P^r, ϵ_r, to some multiple of the generator of P^s, ϵ_s. But then this generates f across the whole of P^r. Noting that we can also write elements of R in terms of x and acting from the right, we obtain:

Proposition 5.3.5. *$f : P^r \to P^s$ is defined by*

$$f(\epsilon_r) = \epsilon_s \cdot \lambda \quad \lambda \in R$$

if and only if

$$f(p[r,i]) = \cdot p[s,i] \cdot \lambda$$

Proof. For the forward implication, we have by definition that

$$\begin{aligned} f(p[r,i]) &= f(\epsilon_r \cdot x^i) \\ &= f(\epsilon_r) \cdot x^i \\ &= \epsilon_s \cdot \lambda x^i \\ &= p[s,i] \cdot \lambda \end{aligned}$$

where since λ is a term entirely in x, it commutes with x^i. For the backwards implication simply take $i = 0$. □

Consider the module P^q. Recalling that $R(q+i) \cong R(i)$, we see that $R(q+q) \cong R(q) \cong R$, which implies that $P^q \cong R$. Then P^q must satisfy $M(0)$ with some characteristic element ϵ_q. Unlike previously however, we cannot just take $\epsilon_q = 1 \cdot (x^{p-1} - 1)^q$, since this element does not possess the proper y action.

Recall the definition of $\pi \in R_0$ to be the unique prime in R_0 lying over p, i.e. the element satisfying

$$\pi^{\frac{p-1}{q}} = pv$$

for some $v \in R_0$ a unit

Proposition 5.3.6. *$\pi = (\zeta - 1)^q v_\pi$ for some $v_\pi \in R$ a unit.*

Proof. We have that

$$\begin{aligned} ((\zeta - 1)^q)^{\frac{p-1}{q}} &= (\zeta - 1)^{p-1} \\ &= pu \end{aligned}$$

for some $u \in R$ a unit. So clearly $(\zeta - 1)^q$ is a prime lying over p, but also $(\zeta - 1)^q \neq \pi$ since $(\zeta - 1)^q \notin R_0$. So $(\zeta - 1)^q$ must differ from π by a unit, which we will call v_π. □

Proposition 5.3.7. *$\overline{\pi} = \pi w$ for some $w \in R_0$ a unit.*

Proof. Since our involution acts as complex conjugation, it commutes with integer powers and we see that

$$\overline{\pi}^q = \overline{\pi^q} = \overline{pu} = p\overline{u}$$

where $u \in R_0$ a unit. It follows that $\overline{u}$ is a unit, and so $\overline{\pi}$ must be unit equivalent to π, say $\overline{\pi} = \pi w$ where $w \in R^\times$. Furthermore, since $\overline{\pi} \in R_0$, we obtain $w \in R_0$ since

$$\overline{\pi} w = \pi = \theta^{-1}(\pi) = \theta^{-1}(\overline{\pi} w) = \overline{\pi}\theta^{-1}(w)$$

so that $\theta^{-1}(w) = w$ as required. □

It follows from this result that it makes sense to consider π as an element acting from the right, so that $1 \cdot \pi$ will still represent the prime lying over p. Define $u_\pi = \overline{v_\pi}$, so that considering as actions on the right we write

$$\pi = (x^{p-1} - 1)^q u_\pi$$

Proposition 5.3.8. *The element $\epsilon_q = 1 \cdot \pi$ is a characteristic element for P^q.*

Proof. Firstly we have, since $\pi \in R_0$

$$\begin{aligned}
\epsilon_q \cdot y &= 1 \cdot \pi y \\
&= 1 \cdot y\theta^{-1}(\pi) \\
&= 1 \cdot \pi \\
&= \epsilon_q
\end{aligned}$$

P^q has the natural basis $P^q = \mathrm{Span}_{\mathbb{Z}}\{1 \cdot (x^{p-1} - 1)^q x^i\}_{0 \leq i \leq p-2}$. Then

$$\begin{aligned} \mathrm{Span}_{\mathbb{Z}}\{1 \cdot \pi x^i\}_{0 \leq i \leq p-2} &= \mathrm{Span}_{\mathbb{Z}}\{1 \cdot (x^{p-1} - 1)^q u_\pi x^i\}_{0 \leq i \leq p-2} \\ &= \mathrm{Span}_{\mathbb{Z}}\{\epsilon \cdot (x^{p-1} - 1)^q x^i\}_{0 \leq i \leq p-2} \end{aligned}$$

Hence the result. □

Consider $\tilde{P}^k$. Then the duality relations discussed in section 2.4 can be written

$$\tilde{P}^k \cong \begin{cases} P & k = 0 \\ R & k = 1 \\ P^{q+1-k} & 2 \leq q - 1 \end{cases}$$

It follows that each $\tilde{P}^k$ will satisfy $M(\Sigma)$ and some appropriate $M(i)$, and is therefore equipped with a characteristic element which we label $\tilde{\epsilon}_k$, so that $\tilde{P}^k = \mathrm{Span}_{\mathbb{Z}}\{\tilde{\epsilon}_k \cdot x^i\}_{0 \leq i \leq p-2}$. Define elements

$$\tilde{p}[k, i] = \tilde{\epsilon}_k \cdot x^i$$

Then we also have the following result

Proposition 5.3.9.

$$\begin{aligned} f_1 &: \tilde{P}^r \to \tilde{P}^s \\ f_2 &: \tilde{P}^r \to P^s \\ f_3 &: P^r \to \tilde{P}^s \end{aligned}$$

are Λ-homomorphisms defined by

$$\begin{aligned} f_1(\tilde{\epsilon}_r) &= \tilde{\epsilon}_s \cdot \lambda_1 \\ f_2(\tilde{\epsilon}_r) &= \epsilon_s \cdot \lambda_2 \\ f_3(\epsilon_r) &= \tilde{\epsilon}_s \cdot \lambda_3 \end{aligned}$$

for $\lambda_1, \lambda_2, \lambda_3 \in R$ *if and only if*

$$\begin{aligned} f_1(\tilde{p}[r,i]) &= \tilde{p}[s,i] \cdot \lambda_1 \\ f_2(\tilde{p}[r,i]) &= p[s,i] \cdot \lambda_2 \\ f_3(p[r,i]) &= \tilde{p}[s,i] \cdot \lambda_3 \end{aligned}$$

Proof. The same argument used in 5.3.4 applies here also and generates the result. □

For notational ease, define

$$\mathcal{P}_k = \begin{cases} P & k = 0 \\ R & k = 1 \\ P^{q+1-k} & 2 \leq k \leq q-1 \end{cases}$$

Then we can define an isomorphism $\upsilon_k : \tilde{P}^k \to \mathcal{P}_k$ by

$$\upsilon_k(\tilde{\epsilon}_k) = \begin{cases} \epsilon_1 & k = 0 \\ \epsilon_0 & k = 1 \\ \epsilon_{q+1-k} & 2 \leq k \leq q-1 \end{cases}$$

One approach to study F_k would therefore be to split it into two parts

$$F_k = \tilde{F}_k \circ \upsilon_k$$

for some $\tilde{F}_k : \mathcal{P}_k \to P^k$. However, a description of F_k^* would require us to calculate υ_k^*, and we are unable to do so with the oblique definition of υ_k given. We proceed to describe $\tilde{\epsilon}_k$ as

$$\tilde{\epsilon}_k = \sum_{j=0}^{p-2} a_j p[k,j]^* \quad a_j \in \mathbb{Z}.$$

5.4 Actions in $\tilde{P}^k$

In order to describe $\tilde{\epsilon_k}$ in the natural dual basis, we first need to understand how x and y act upon this basis. We have the following base case for x:

Lemma 5.4.1.

$$p[k,i]^* \cdot x = \begin{cases} p[k,i+1]^* - p[k,0]^* & 0 \le i \le p-3 \\ -p[k,0]^* & i = p-2 \end{cases}$$

Proof. For $a_j \in \mathbb{Z}$, and noting that $p[0,i] \cdot x = p[0,i+1]$ and setting

$$p[0,-1] = \epsilon_k \cdot x^{p-1} = -\sum_{j=0}^{p-2} p[0,j]$$

we have

$$\begin{aligned}(p[k,i]^* \cdot x)(\sum_{j=0}^{p-2} a_j p[k,j]) &= (x^{p-1} \cdot p[k,i]^*)(\sum_{j=0}^{p-2} a_j p[k,j]) \\ &= p[k,i]^*(\sum_{j=0}^{p-2} a_j p[k,j] \cdot x^{p-1}) \\ &= p[k,i]^*(\sum_{j=0}^{p-2} a_j p[k,j-1]) \\ &= p[k,i]^*(\sum_{j=1}^{p-2} a_j p[k,j-1] - a_0 \sum_{l=0}^{p-2} p[k,l]) \\ &= p[k,i]^*(\sum_{j=0}^{p-3} (a_{j+1} - a_0) p[k,j] - a_0 p[k,p-2])\end{aligned}$$

Hence the result. □

Proposition 5.4.2.

$$p[k,i]^* \cdot x^j = \begin{cases} p[k,i+j]^* - p[k,j-1]^* & 1 \leq j \leq p-2-i \\ -p[k,j-1]^* & j = p-1-i \\ p[k,i+j-p]^* - p[k,j-1]^* & p-i \leq j \leq p-1 \end{cases}$$

Proof. We proceed by (staggered) induction on j. We first consider the case $1 \leq j \leq p-2-i$.

$$p[k,i]^* \cdot x = p[k,i+1]^* - p[k,0]^*$$

so the base case holds. Suppose true for $j-1$, so that

$$p[k,i]^* \cdot x^{j-1} = p[k,i+j-1]^* - p[k,j-2]^*$$

Then

$$\begin{aligned} p[k,i]^* \cdot x^j &= (p[k,i+j-1]^* - p[k,j-2]^*) \cdot x \\ &= p[k,i+j]^* - p[k,0]^* - p[k,j-1]^* + p[k,0]^* \\ &= p[k,i+j]^* - p[k,j-1]^* \end{aligned}$$

So the statement is true for all j such that $1 \leq j \leq p-2-i$. Next suppose $j = p-1-i$. Then

$$\begin{aligned} p[k,i]^* \cdot x^j &= (p[k,i]^* \cdot x^{p-2-i}) \cdot x \\ &= (p[k,p-2]^* - p[k,p-3-i]^*) \cdot x \\ &= -p[k,0]^* - p[k,p-2-i]^* + p[k,0]^* \\ &= -p[k,p-2-i]^* \\ &= -p[k,j-1]^* \end{aligned}$$

So the statement is true here also. Finally suppose $p-i \leq j \leq p-1$. Then again we proceed by induction. The base case is now

$$\begin{aligned}
p[k,i]^* \cdot x^{p-i} &= -p[k,p-2-i]^* \cdot x \\
&= p[k,0]^* - p[k,p-1-i]^* \\
&= p[k,i+p-i-p] - p[k,p-1-i]^* \\
&= p[k,i+j-p]^* - p[k,j-1]^*
\end{aligned}$$

which holds. Suppose true for $j-1$, so that

$$p[k,i]^* \cdot x^{j-1} = p[k,i+j-1-p]^* - p[k,j-2]^*$$

Then

$$\begin{aligned}
p[k,i]^* \cdot x^{j} &= (p[k,i+j-1-p]^* - p[k,j-2]^*) \cdot x \\
&= p[k,i+j-p]^* - p[k,0]^* - p[k,j-1]^* + p[k,0]^* \\
&= p[k,i+j-p]^* - p[k,j-1]^*
\end{aligned}$$

So the statement holds for all $p-i \leq j \leq p-1$. $\square$

Having obtained a complete description of x actions in $\tilde{P}^k$, we now consider the action of y. By definition we have

$$\begin{aligned}
(p[k,i]^* \cdot y)(\sum_{j=0}^{p-2} a_j p[k,j]) &= (y^{-1} \cdot p[k,i]^*)(\sum_{j=0}^{p-2} a_j \epsilon_k \cdot x^i) \\
&= (p[k,i]^*)(\sum_{j=0}^{p-2} a_j \epsilon_k \cdot x^i y^{-1}) \\
&= (p[k,i]^*)(\sum_{j=0}^{p-2} a_j \epsilon_k \cdot y\theta(x)^i) \\
&= (p[k,i]^*)(\sum_{j=0}^{p-2} a_j \epsilon_k \cdot v_X^k x^{ai})
\end{aligned}$$

Immediately we run into a problem, namely we have no general form for a.

While a general description of the y action appears unattainable, we can at least produce a general result for the simplest possible case, the action of y on $p[0,0]^*$.

Lemma 5.4.3. *There exists a unique number η such that $1 \leq \eta \leq p-2$ and $a\eta = -1 \mod p$*

Proof. Consider elements of the form $ka \mod p$ for $1 \leq k \leq p-1$. Then since p is prime, $ka = 0 \mod p \Rightarrow k = lp$ for $l \in \mathbb{Z}$. Then $ka \mod p$ must cycle through all elements $1 \leq m \leq p-1$. Suppose otherwise, then there exist k_1, k_2 such that $1 \leq k_1 \leq k_2 \leq p-1$ and $k_1 a = k_2 a \mod p$. But then

$$\begin{aligned} k_1 a = k_2 a \mod p &\Rightarrow (k_1 - k_2)a = 0 \mod p \\ &\Rightarrow (k_1 - k_2) = 0 \mod p \\ &\Rightarrow k_1 = k_2 + lp \end{aligned}$$

a contradiction. So there exists η, $1 \leq \eta \leq p-2$ which will 'hit' $p-1$. But also noting $a \neq 1$ we obtain

$$a(p-1) = -a \neq -1 \mod p$$

Hence $1 \leq \eta \leq p-2$. □

Lemma 5.4.4. $\eta = p - b$

Proof.

$$\begin{aligned} b = a^{q-1} \mod p &\Rightarrow p - b = -a^{q-1} \mod p \\ &\Rightarrow a(p-b) = -a^q = -1 \mod p \end{aligned}$$

So by definition $p - b = \eta$. □

Proposition 5.4.5. $p[0,0]^* \cdot y = p[0,0]^* - p[0,p-b]^*$

Proof. We have that

$$\begin{aligned}
(p[0,i]^* \cdot y)(\sum_{j=0}^{p-2} a_j p[0,j]) &= p[0,i]^*(\sum_{j=0}^{p-2} a_j p[0,j] \cdot y^{-1}) \\
&= p[0,i]^*(\epsilon_0 \cdot \sum_{j=0}^{p-2} a_j x^j y^{-1}) \\
&= p[0,i]^*(\epsilon_0 \cdot y^{-1} \sum_{j=0}^{p-2} a_j x^{aj}) \\
&= p[0,i]^*(\epsilon_0 \cdot (a_0 + a_{p-b} x^{p-1} + \sum_{j=1, j\neq p-b}^{p-2} a_j x^{aj})) \\
&= p[0,i]^*((a_0 - a_{p-b}) r_0 + \ldots)
\end{aligned}$$

Hence the result. □

While limited in scope, this result is in fact enough to allow us to calculate explicit descriptions for all $\tilde{\epsilon}_k$.

5.5 Projection Homomorphisms

Recall from section 2.4 the existence of a map $T \in \mathcal{T}_q(R_0, \pi)$ which can be described in block matrix form as

$$T = \begin{pmatrix} 0 & I_{q-1} \\ \pi & 0 \end{pmatrix}$$

Then successive applications of T on any $R(k)$ generates an infinite sequence of inclusions

$$\ldots \subset R(q) \subset R(q-1) \subset \ldots \subset R(2) \subset R(1) \subset R(q) \subset \ldots$$

where each inclusion has an index of p. We wish to construct an analagous chain in our current framework. Define a projection homomorphism

$$\rho_k : P^k \to P^{k-1}$$
$$\rho_k(\epsilon_k) = \epsilon_{k-1} \cdot (x^{p-1} - 1)$$

for for $1 \leq k \leq q-1$. Then we can construct a chain

$$P^{q-1} \xrightarrow{\rho_{q-1}} P^{q-2} \xrightarrow{\rho_{q-2}} \dots \xrightarrow{\rho_2} P \xrightarrow{\rho_1} R$$

where since $(x^{p-1} - 1)^{p-1} = pu$ for some $u \in R$ a unit, we can infer that $\det(\rho_k) = p$ as needed. It remains to construct a projection $R \to P^{q-1}$. Using 5.3.1 we define an isomorphism

$$\upsilon_\pi : R \to P^q$$
$$\upsilon_\pi(\epsilon_0) = \epsilon_q$$

which allows us to define a final projection

$$\rho_q : P^q \to P^{q-1}$$
$$\rho_q(\epsilon_q) = \epsilon_{q-1} \cdot (x^{p-1} - 1)u_\pi$$

We can therefore construct an infinite chain

$$\dots \xrightarrow{\upsilon_\pi} P^q \xrightarrow{\rho_q} P^{q-1} \xrightarrow{\rho_{q-1}} \dots \xrightarrow{\rho_3} P^2 \xrightarrow{\rho_2} P \xrightarrow{\rho_1} R \xrightarrow{\upsilon_\pi} P^q \xrightarrow{\rho^q} \dots$$

Dualising this chain we obtain a new sequence of dual modules and dual maps

$$\dots \xrightarrow{\rho_q^*} \tilde{P}^q \xrightarrow{\upsilon_\pi^*} \tilde{R} \xrightarrow{\rho_1^*} \tilde{P} \xrightarrow{\rho_2^*} \tilde{P}^2 \xrightarrow{\rho_3^*} \dots \xrightarrow{\rho_{q-1}^*} \tilde{P}^{q-1} \xrightarrow{\rho_q^*} \tilde{P}^q \xrightarrow{\upsilon_\pi^*} \dots$$

Proposition 5.5.1. *$\rho_k^* : \tilde{P}^{k-1} \to \tilde{P}^k$ is defined by*

$$\rho_k^*(p[k-1,i]^*) = \begin{cases} -p[k,i]^* \cdot (x^{p-1}-1)x & 1 \leq k \leq q-1 \\ -p[k,i]^* \cdot (x^{p-1}-1)x\overline{u_\pi} & k = q \end{cases}$$

Proof. We consider the case $k = q$, as it contains the other case by simply deleting u_π. Then by definition we have that

$$\begin{aligned} \rho_k^*(p[k-1,i]^*)(\sum_{j=0}^{p-2} a_j p[k,j]) &= p[k-1,i]^*(\rho_k((\sum_{j=0}^{p-2} a_j p[k,j])) \\ &= p[k-1,i]^*(\sum_{j=0}^{p-2} a_j p[k-1,j] \cdot (x^{p-1}-1)u_\pi) \\ &= (p[k-1,i]^* \cdot (x-1)\overline{u_\pi})(\sum_{j=0}^{p-2} a_j p[k-1,j]) \end{aligned}$$

Since the action of x is independent of k, we can infer

$$(p[k-1,i]^* \cdot (x-1)\overline{u_\pi})(\sum_{j=0}^{p-2} a_j p[k-1,j]) = (p[k,i]^* \cdot (x-1)\overline{u_\pi})(\sum_{j=0}^{p-2} a_j p[k,j])$$

which implies

$$\begin{aligned} \rho_k^*(p[k-1,i]^*) &= p[k,i]^* \cdot (x-1)\overline{u_\pi} \\ &= -p[k,i]^* \cdot (x^{p-1}-1)x\overline{u_\pi} \end{aligned}$$

□

Proposition 5.5.2. *$\upsilon_\pi^* : \tilde{P}^q \to \tilde{R}$ is defined by*

$$\upsilon_\pi^*(p[q,i]^*) = p[0,i]^*$$

Proof. By definition we have

$$\upsilon_\pi^*(p[q,i]^*)(\sum_{j=0}^{p-2} a_j p[0,i]) = p[q,i]^*(\upsilon_\pi(\sum_{j=0}^{p-2} a_j p[0,i]))$$

$$= p[q,i]^*(\sum_{j=0}^{p-2} a_j p[q,i]))$$
$$= p[0,i]^*(\sum_{j=0}^{p-2} a_j p[0,i]))$$

Hence the result. □

Our goal is to be able to project calculations in a general $\tilde{P}^k$ up to $\tilde{R}$, perform them there, and then move back down to give the result in terms of our original module. To this end we define the inverse map

$$(\upsilon_\pi^*)^{-1} : \tilde{R} \to \tilde{P}^q$$
$$(\upsilon_\pi^*)^{-1}(p[0,i]^*) = p[q,i]^*$$

For the projections, while it is evident we cannot define an inverse over the entirety of $\tilde{P}^k$, we can define it over a certain subset. Define semi-inverses

$$(\rho_k^*)^{-1} : \tilde{P}^k \to \tilde{P}^{k-1}$$
$$(\rho_k^*)^{-1}(p[k,i]^* \cdot (x^{p-1}-1)x) = -p[k-1,i]^* \quad 1 \le k \le q-1$$
$$(\rho_k^*)^{-1}(p[k,i]^* \cdot (x^{p-1}-1)x\overline{u_\pi}) = -p[k-1,i]^* \quad k = q$$

For $s > r$ we will write

$$\sum_{j=s}^{r} \rho_j^* = \rho_s^* \circ \ldots \circ \rho_{r+1}^* \circ \rho_r^*$$
$$\sum_{j=r}^{s} (\rho_j^*)^{-1} = (\rho_r^*)^{-1} \circ \ldots \circ (\rho_{s+1}^*)^{-1} \circ (\rho_s^*)^{-1}$$

We wish to prove

$$p[k,i]^* \cdot y = (\sum_{j=k+1}^{q} (\rho_j^*)^{-1} \circ \upsilon_\pi^{-1})((\upsilon_\pi \circ \sum_{j=q}^{k+1} \rho_j^*)(p[k,i]^*) \cdot y)$$

for which we need to check that $(\upsilon_\pi \circ \sum_{j=q}^{k+1} \rho_j^*)(p[k,i]^*) \cdot y)$ belongs to the domain of $(\rho_j^*)^{-1}$ for $k+1 \leq j \leq q$. To achieve this, recall the definition of u_π. We obtain the following:

Lemma 5.5.3. $\theta^{-1}(u_\pi) = (-1)^q u_\pi \upsilon_X^{-q}$.

Proof. Since $\pi \in R_0$ we have

$$\begin{aligned} \pi &= \theta^{-1}(\pi) \\ &= (x^{b(p-1)} - 1)^q \theta^{-1}(u_\pi) \end{aligned}$$

which implies

$$\begin{aligned} (x^{p-1} - 1)^q u_\pi &= (-1)^q (x^{p-1} - 1)^q \upsilon_X^q \theta^{-1}(u_\pi) \\ \Rightarrow \ \theta^{-1}(u_\pi) &= (-1)^q u_\pi \upsilon_X^{-q} \end{aligned}$$

□

Corollary 5.5.4.

$$\theta^{-1}(\overline{u_\pi}) = (-1)^q \overline{u_\pi} \overline{\upsilon_X^{-q}}$$

Proof. Immediate from the relation

$$\overline{\theta^{-1}(u_\pi)} = \theta^{-1}(\overline{u_\pi})$$

□

We are now able to prove:

Proposition 5.5.5.

$$p[k,i]^* \cdot y = (\sum_{j=k+1}^{q} (\rho_j^*)^{-1} \circ \upsilon_\pi^{-1})((\upsilon_\pi \circ \sum_{j=q}^{k+1} \rho_j^*)(p[k,i]^*) \cdot y)$$

Proof. We know that $p[0,i]^* \cdot y = p[0,i]^* \cdot \lambda$ for some $\lambda \in R$. Using 5.5.4 we obtain

$$\begin{aligned}
&(\upsilon_\pi \circ \sum_{j=q}^{k+1} \rho_j^*)(p[k,i]^*) \cdot y \\
&= p[0,i]^* \cdot (-1)^{q-k}(x^{p-1}-1)^{q-k}x^{q-k}\overline{u_\pi}y \\
&= p[0,i]^* \cdot y(-1)^{q-k}(x^{b(p-1)}-1)^{q-k}x^{b(q-k)}\theta^{-1}(\overline{u_\pi}) \\
&= p[0,i]^* \cdot \lambda(-1)^{3q-2k}(x^{(p-1)}-1)^{q-k}\upsilon_X^{q-k}x^{b(q-k)}\overline{u_\pi}\overline{\upsilon_X^{-q}} \\
&= p[0,i]^* \cdot ((-1)^{q-k}(x^{p-1}-1)^{q-k}x^{q-k}\overline{u_\pi})(-1)^{2q-k}x^{(b-1)(q-k)}\upsilon_X^{q-k}\overline{\upsilon_X^{-q}}\lambda
\end{aligned}$$

While a messy expression, the important thing to note is that even after y acts, we still remain in the domain over which our sequence of inverse projections are defined. So

$$(\sum_{j=k+1}^{q}(\rho_j^*)^{-1} \circ \upsilon_\pi^{-1})((\upsilon_\pi \circ \sum_{j=q}^{k+1}\rho_j^*)(p[k,i]^*) \cdot y)$$

is a well-defined expression, and since we have Λ-homomorphisms

$$\begin{aligned}
&(\sum_{j=k+1}^{q}(\rho_j^*)^{-1} \circ \upsilon_\pi^{-1})((\upsilon_\pi \circ \sum_{j=q}^{k+1}\rho_j^*)(p[k,i]^*) \cdot y) \\
&= (\sum_{j=k+1}^{q}(\rho_j^*)^{-1} \circ \upsilon_\pi^{-1} \circ \upsilon_\pi \circ \sum_{j=q}^{k+1}\rho_j^*)(p[k,i]^*) \cdot y \\
&= p[k,i]^* \cdot y \qquad \square
\end{aligned}$$

5.6 Characteristic elements in $\tilde{P}^k$

We wish to calculate a natural dual basis description for $\tilde{\epsilon}_k$

$$\tilde{\epsilon_k} = \sum_{i=0}^{p-2} a_i p[k,i]^*$$

We start in $\tilde{R}$:

Proposition 5.6.1. $\tilde{\epsilon}_0 = p[0,0]^*$.

Proof. Since $\tilde{R} \cong P$, we want to show that

$$p[0,0]^* \cdot y = -p[0,0]^* \cdot v_X$$

From 5.4.5 we know

$$p[0,0]^* \cdot y = p[0,0]^* - p[0,p-b]^*$$

and using 5.4.2 we calculate

$$\begin{aligned}
-p[0,0]^* \cdot v_X &= -p[0,0]^* \cdot (\sum_{j=1}^{p-b} x^j) \\
&= -\sum_{j=1}^{p-b}(p[0,j]^* - p[0,j-1]^*) \\
&= -(p[0,p-b]^* - p[0,0]^*) \\
&= p[0,0]^* - p[0,p-b]^* \\
&= p[0,0]^* \cdot y
\end{aligned}$$

We also have

$$\begin{aligned}
\mathrm{Span}_{\mathbb{Z}}\{p[0,0]^* \cdot x^i\}_{0 \le i \le p-2} &= \mathrm{Span}_{\mathbb{Z}}\{p[0,i]^* - p[0,i-1]^*, p[0,0]^*\}_{1 \le i \le p-2} \\
&= \mathrm{Span}_{\mathbb{Z}}\{p[0,i]^*\}_{0 \le i \le p-2}
\end{aligned}$$

□

Corollary 5.6.2. $\tilde{\epsilon}_q = p[q,0]^*$.

For $\tilde{P}$, first recall 5.3.7 and the relation $\overline{\pi} = \pi w$ where $w \in R_0^\times$.

Lemma 5.6.3. $x^q \overline{u_\pi} = (-1)^q u_\pi w$

Proof. We calculate

$$\begin{aligned}(x^{p-1}-1)^q u_\pi w &= \pi w \\ &= \overline{\pi} \\ &= \overline{(x^{p-1}-1)^q u_\pi} \\ &= (x-1)^q \overline{u_\pi} \\ &= (-1)^q x^q (x^{p-1}-1)^q \overline{u_\pi}\end{aligned}$$

Comparing terms we obtain the result. □

Proposition 5.6.4. $\tilde{\epsilon}_1 = p[1,0]^* \cdot x$

Proof. To show $p[1,0]^* \cdot xy = p[1,0]^* \cdot x$ we project $p[1,0]^* \cdot xy$ into $\tilde{R}$, have y act, and then project back up. We will abuse notation and omit explicitly writing these projections. Using 5.6.3 we then have

$$\begin{aligned}p[1,0]^* \cdot xy &= p[0,0]^* \cdot ((-1)^{q-1}(x^{p-1}-1)^{q-1}x^{q-1}\overline{u_\pi})xy \\ &= p[0,0]^* \cdot (-1)^{q-1}(x^{p-1}-1)^{q-1}x^q\overline{u_\pi}y \\ &= p[0,0]^* \cdot (-1)^{2q-1}(x^{p-1}-1)^{q-1}u_\pi wy \\ &= p[0,0]^* \cdot y(-1)^{2q-1}(x^{b(p-1)}-1)^{q-1}\theta^{-1}(u_\pi)w \\ &= p[0,0]^* \cdot (-1)^{4q-1}v_X(x^{p-1}-1)^{q-1}v_X^{q-1}u_\pi v_X^{-q}w \\ &= p[0,0]^* \cdot (-1)^{2q-1}(x^{p-1}-1)^{q-1}u_\pi w \\ &= p[0,0]^* \cdot (-1)^{q-1}(x^{p-1}-1)^{q-1}x^q\overline{u_\pi} \\ &= p[1,0]^* \cdot x\end{aligned}$$

We also have

$$\begin{aligned}\mathrm{Span}_{\mathbb{Z}}\{p[1,0]^* \cdot x^{i+1}\}_{0\leq i\leq p-2} &= \mathrm{Span}_{\mathbb{Z}}\{p[1,i]^* - p[1,i-1]^*, -p[1,p-2]^*\}_{1\leq i\leq p-2} \\ &= \mathrm{Span}_{\mathbb{Z}}\{p[1,i]^*\}_{0\leq i\leq p-2}\end{aligned}$$

Hence the result. □

Proposition 5.6.5. *For $2 \leq k \leq q-1$*

$$\tilde{\epsilon}_k = p[k,0]^* \cdot x^k u_\pi^{-1}$$

Proof. We want to show $p[k,0]^* \cdot x^k u_\pi^{-1} y = p[k,0]^* \cdot (-1)^{q+1-k} x^k u_\pi^{-1} v_X^{q+1-k}$. Then

$$\begin{aligned}
p[k,0]^* \cdot x^k u_\pi^{-1} y &= p[0,0]^* \cdot ((-1)^{q-k}(x^{p-1}-1)^{q-k} x^{q-k} \overline{u_\pi}) x^k u_\pi^{-1} y \\
&= p[0,0]^* \cdot (-1)^{q-k}(x^{p-1}-1)^{q-k} x^q \overline{u_\pi} u_\pi^{-1} y \\
&= p[0,0]^* \cdot (-1)^{2q-k}(x^{p-1}-1)^{q-k} u_\pi u_\pi^{-1} w y \\
&= p[0,0]^* \cdot (-1)^{2q-k}(x^{p-1}-1)^{q-k} w y \\
&= p[0,0]^* \cdot y(-1)^{2q-k}(x^{b(p-1)}-1)^{q-k} w \\
&= p[0,0]^* \cdot (-1)^{3q-2k+1} v_X (x^{p-1}-1)^{q-k} v_X^{q-k} w \\
&= p[0,0]^* \cdot (-1)^{2(q-k)+1} (x^{p-1}-1)^{q-k} x^q \overline{u_\pi} u_\pi^{-1} v_X^{q-k+1} \\
&= p[k,0]^* \cdot (-1)^{q-k+1} x^k u_\pi^{-1} v_X^{q-k+1}
\end{aligned}$$

We also have

$$\begin{aligned}
&\mathrm{Span}_{\mathbb{Z}}\{p[k,0]^* \cdot x^{i+k} u_\pi^{-1}\}_{0 \leq i \leq p-2} \\
&= \mathrm{Span}_{\mathbb{Z}}\{p[k,0]^* \cdot x^{i+k}\}_{0 \leq i \leq p-2} \\
&= \mathrm{Span}_{\mathbb{Z}}\{p[k,k+i]^* - p[k,k+i-1]^*, -p[k,p-2], p[k,0]\}_{0 \leq i \leq p-k-2,} \\
&\oplus \mathrm{Span}_{\mathbb{Z}}\{p[k,k+i-p]^* - p[k,k+i-p-1]^*\}_{p-k-1 \leq i \leq p-2} \\
&= \mathrm{Span}_{\mathbb{Z}}\{p[k,i]^*\}_{0 \leq i \leq p-2}
\end{aligned}$$

□

Given these dual characteristic elements, we can define how ρ_k^* acts on them

Proposition 5.6.6. *Suppose $\rho_k : P^k \to P^{k-1}$ is defined as above. Then*

$\rho_k^* : \tilde{P}^{k-1} \to \tilde{P}^k$ is defined by

$$\rho_k^*(\tilde{\epsilon}_{k-1}) = \begin{cases} \tilde{\epsilon}_k \cdot (-1)(x^{p-1} - 1) & k = 1, 3 \leq k \leq q \\ \tilde{\epsilon}_k \cdot (-1)(x^{p-1} - 1)u_\pi & k = 2 \end{cases}$$

Proof. We proceed case by case, beginning with $k = 1$ and $\rho_1^* : \tilde{R} \to \tilde{P}$. Then

$$\begin{aligned} \rho_1^*(\tilde{\epsilon}_0) &= \rho_1^*(p[0,0]^*) \\ &= p[1,0]^* \cdot (-1)(x^{p-1} - 1)x \\ &= \tilde{\epsilon}_1 \cdot (-1)(x^{p-1} - 1) \end{aligned}$$

For $k = 2$ and $\rho_2^* : \tilde{P} \to \tilde{P}^2$ we obtain

$$\begin{aligned} \rho_2^*(\tilde{\epsilon}_1) &= \rho_2^*(p[1,0]^* \cdot x) \\ &= p[2,0]^* \cdot (-1)(x^{p-1} - 1)x^2 \\ &= p[2,0]^* \cdot (-1)(x^{p-1} - 1)x^2 u_\pi^{-1} u_\pi \\ &= \tilde{\epsilon}_2 \cdot (-1)(x^{p-1} - 1)u_\pi \end{aligned}$$

For $3 \leq k \leq q - 1$ and $\rho_k^* : \tilde{P}^{k-1} \to \tilde{P}^k$ we have

$$\begin{aligned} \rho_k^*(\tilde{\epsilon}_{k-1}) &= \rho_k^*(p[k-1,0]^* \cdot x^k u_\pi^{-1}) \\ &= p[k,0]^* \cdot (-1)(x^{p-1} - 1)x^{k+1} u_\pi^{-1} \\ &= \tilde{\epsilon}_k \cdot (-1)(x^{p-1} - 1) \end{aligned}$$

Finally for $k = q$ and $\rho_q^* : \tilde{P}^{q-1} \to \tilde{P}^q$ we have

$$\begin{aligned} \rho_q^*(\tilde{\epsilon}_{q-1}) &= \rho_q^*(p[q-1,0]^* \cdot x^{q-1} u_\pi^{-1}) \\ &= p[q,0]^* \cdot (-1)(x^{p-1} - 1)x^q \overline{u_\pi} u_\pi^{-1} \\ &= p[q,0]^* \cdot (-1)(x^{p-1} - 1)u_\pi u_\pi^{-1} \\ &= p[q,0]^* \cdot (-1)(x^{p-1} - 1) \end{aligned}$$

$$= \tilde{\epsilon}_q \cdot (-1)(x^{p-1} - 1) \qquad \square$$

Finally we can also now write 5.5.2 in terms of dual characteristic elements.

Corollary 5.6.7. *$\upsilon_\pi^* : \tilde{P}^q \to \tilde{R}$ is defined by*

$$\upsilon_\pi^*(\tilde{\epsilon}_q) = \tilde{\epsilon}_0$$

5.7 Duality Isomorphisms

We are now equipped to study the duals of duality isomorphisms. We previously defined

$$\upsilon_k : \tilde{P}^k \to \mathcal{P}_k$$

for $0 \leq k \leq q-1$ by

$$\upsilon_k(\tilde{\epsilon}_k) = \begin{cases} \epsilon_1 & k = 0 \\ \epsilon_0 & k = 1 \\ \epsilon_{q+1-k} & 2 \leq k \leq q-1 \end{cases}$$

Lemma 5.7.1.

$$\upsilon_k(p[k,i]^*) = \begin{cases} \epsilon_1 \cdot \sum\limits_{j=0}^{i} x^j & k = 0 \\ \epsilon_0 \cdot \sum\limits_{j=0}^{i} x^{j-1} & k = 1 \\ \epsilon_{q+1-k} \cdot \sum\limits_{j=0}^{i} x^{j-k} u_\pi & 2 \leq k \leq q-1 \end{cases}$$

Proof. We have that

$$p[k,0]^* \cdot \sum_{j=0}^{i} x^i = \sum_{j=0}^{i} (p[k,i]^* - p[k,i-1]^*) + p[k,0]^*$$

$$= p[k,i]^*$$

So that

$$\begin{aligned} \upsilon_k(p[k,i]^*) &= \upsilon_k(p[k,0]^* \cdot \sum_{j=0}^{i} x^j) \\ &= \upsilon_k(p[k,0]^*) \cdot \sum_{j=0}^{i} x^j \end{aligned}$$

Using 5.6.1, 5.6.4, 5.6.5 it follows that

$$\upsilon_k(p[k,0]^*) = \begin{cases} \epsilon_1 & k = 0 \\ \epsilon_0 \cdot x^{p-1} & k = 1 \\ \epsilon_{q+1-k} \cdot x^{-k} u_\pi & 2 \le k \le q-1 \end{cases}$$

Hence the result. □

Again, the simplest case to consider is $k = 0$.

Proposition 5.7.2. *$\upsilon_0^* : \tilde{P} \to R$ is defined by*

$$\upsilon_0^*(\tilde{\epsilon}_1) = -\epsilon_0$$

Proof. υ_0^* is defined on $p[1,0]^*$ by

$$\begin{aligned} \upsilon_0^*(p[1,0]^*)(\sum_{j=0}^{p-2} a_j p[0,j]^*) &= p[1,0]^*(\upsilon_0(\sum_{j=0}^{p-2} a_j p[0,j]^*)) \\ &= p[1,0]^*(\sum_{j=0}^{p-2} a_j (p[1,0] \cdot \sum_{l=0}^{j} x^l)) \\ &= p[1,0]^*(\sum_{j=0}^{p-2} a_j p[1,0] + \sum_{j=1}^{p-2} a_j p[1,1] + \ldots) \end{aligned}$$

$$= \sum_{j=0}^{p-2} a_j$$

which implies

$$\upsilon_0^*(p[1,0]^*) = \sum_{j=0}^{p-2} p[0,j]$$

Then

$$\begin{aligned}
\upsilon_0^*(\tilde{\epsilon}_1) &= \upsilon_0^*(p[1,0]^* \cdot x) \\
&= \sum_{j=0}^{p-2} p[0,j] \cdot x \\
&= \sum_{j=1}^{p-2} p[0,j] + p[0,p-1] \\
&= -p[0,0] \\
&= -\epsilon_0
\end{aligned}$$

□

We next turn to the case $k = 1$:

Proposition 5.7.3. *$\upsilon_1^* : \tilde{P} \to R$ is defined by*

$$\upsilon_1^*(\tilde{\epsilon}_0) = -\epsilon_1$$

Proof. υ_1^* is defined on $p[0,0]^*$ by

$$\begin{aligned}
\upsilon_1^*(p[0,0]^*)(\sum_{j=0}^{p-2} a_j p[1,0]^*) &= p[0,0]^*(\upsilon_1(\sum_{j=0}^{p-2} a_j p[1,0]^*)) \\
&= p[0,0]^*(\sum_{j=0}^{p-2} a_j (p[0,0] \cdot \sum_{l=0}^{j} x^{l-1})) \\
&= p[0,0]^*(a_0 p[0,0] \cdot x^{p-1} + a_1 p[0,0] \cdot (x^{p-1} + 1) + \ldots) \\
&= p[0,0]^*(-\sum_{j=0}^{p-2} \sum_{l=0}^{j} a_l p[0,j])
\end{aligned}$$

which implies that

$$\upsilon_1^*(p[0,0]^*) = -p[1,0]$$

Hence the result. $\square$

Finally we have:

Proposition 5.7.4. *Suppose* $2 \le k \le q-1$. *Then* $\upsilon_k^* : \tilde{P}^{q+1-k} \to P^k$ *is defined by*

$$\upsilon_k^*(\tilde{\epsilon}_{q+1-k}) = \epsilon_k \cdot (-1)^{q+1}\overline{w}^{-1}$$

Proof. For notation ease define

$$\mu = (-1)^q \overline{w} u_\pi^{-1}$$

We note that by rearranging 5.x we obtain

$$\overline{\mu} u_\pi = x^q$$

Then υ_k^* is defined on $p[q+1-k,0]^* \cdot x^{q+1-k}\mu$ by

$$\begin{aligned}
&\upsilon_k^*(p[q+1-k,0]^* \cdot x^{q+1-k}\mu)(\sum_{j=0}^{p-2} a_j p[k,j]^*) \\
&= (p[q+1-k,0]^* \cdot x^{q+1-k}\mu)(\upsilon_k(\sum_{j=0}^{p-2} a_j p[k,j]^*)) \\
&= (p[q+1-k,0]^* \cdot x^{q+1-k}\mu)(\sum_{j=0}^{p-2} a_j p[q+1-k,0] \cdot \sum_{l=0}^{j} x^{l-k} u_\pi) \\
&= p[q+1-k,0]^*(\sum_{j=0}^{p-2} a_j p[q+1-k,0] \cdot \sum_{l=0}^{j} x^{l-k} u_\pi \overline{\mu} x^{p-q-1+k}) \\
&= p[q+1-k,0]^*(\sum_{j=0}^{p-2} a_j p[q+1-k,0] \cdot \sum_{l=0}^{j} x^{l-q-1} x^q)
\end{aligned}$$

$$= p[q+1-k,0]^*(\sum_{j=0}^{p-2} a_j p[q+1-k,0] \cdot \sum_{l=0}^{j} x^{l-1})$$

$$= p[q+1-k,0]^*(-\sum_{j=0}^{p-2}\sum_{l=0}^{j} a_l p[q+1-k,j])$$

which implies

$$\upsilon_k^*(p[q+1-k,0]^* \cdot x^{q+1-k}\mu) = -p[k,0]$$

Hence the result. $\square$

Given this result, we would like to have an explicit description for w, and we introduce the following property of G a metacyclic group:

G(π): $\overline{\pi} = (-1)^q \pi$

We will shortly show that $G(\pi)$ is satisfied for a large range of groups. We also have the following:

Theorem 5.7.5. *Suppose G satisfies $G(\pi)$. Then $\upsilon_k^* : \tilde{P}^k \to \mathcal{P}_k$ is given by*

$$\upsilon_k^* = \begin{cases} -\upsilon_1 & k=0 \\ -\upsilon_0 & k=1 \\ -\upsilon_{q+1-k} & 2 \leq k \leq q-1 \end{cases}$$

Proof. For $k = 0,1$, the result follows immediately from 5.7.2 and 5.7.3 respectively. For $2 \leq k \leq q-1$, since

$$G \text{ satisfies } G(\pi) \Rightarrow w = (-1)^q$$

5.7.4 becomes

$$\upsilon_k^*(\tilde{\epsilon}_{q+1-k}) = \epsilon_k \cdot (-1)^{2q+1} = -\epsilon_k$$

and the result follows. $\square$

Define $\upsilon_q : \tilde{P}^q \to P$ by

$$\upsilon_q(\tilde{\epsilon}_q) = \epsilon_1$$

Then it is immediately obvious that:

Lemma 5.7.6. *υ_q can be decomposed as*

$$\upsilon_q = \upsilon_0 \circ \upsilon_\pi^*$$

This allows us to rewrite 5.5.1 as:

Corollary 5.7.7.

$$\rho_k^* = \begin{cases} -(\upsilon_k^{-1} \circ \rho_k \circ \upsilon_{k-1}) & k = 1 \\ -(\upsilon_k^{-1} \circ \rho_q \circ \upsilon_\pi \circ \upsilon_{k-1}) & k = 2 \\ -(\upsilon_k^{-1} \circ \rho_{q+2-k} \circ \upsilon_{k-1}) & 3 \leq k \leq q \end{cases}$$

5.8 Endomorphisms of P^k

The final type of mappings for us to consider are Λ-endomorphisms of P^k.

Proposition 5.8.1. *Suppose $\phi_k : P^k \to P^k$ is a Λ-endomorphism. Then*

$$\phi_k(\epsilon_k) = \epsilon_k \cdot \alpha \quad \alpha \in R_0$$

Proof. The most general ϕ_k is given by

$$\phi_k(\epsilon_k) = \epsilon_k \cdot \alpha \quad \alpha \in R$$

But then, recalling that $\epsilon_0 \cdot y = \epsilon_0$, and noting that all x terms commute, ϕ_k must also satisfy

$$\begin{aligned} \epsilon_k \cdot \alpha &= \phi_k(\epsilon_k) \\ &= \phi_k(\epsilon_0 \cdot (x^{p-1} - 1)^k) \end{aligned}$$

$$
\begin{aligned}
&= \phi_k(\epsilon_0 \cdot y(x^{p-1}-1)^k) \\
&= \phi_k(\epsilon_0 \cdot (x^{a(p-1)-1})^k y) \\
&= \phi_k(\epsilon_0 \cdot (x^{p-1}-1)^k)(\sum_{j=1}^{a-1} x^{j(p-1)})^k y \\
&= \epsilon_0 \cdot (x^{p-1}-1)^k \alpha(\sum_{j=1}^{a-1} x^{j(p-1)})^k y \\
&= \epsilon_0 \cdot (x^{a(p-1)}-1)^k \alpha y \\
&= \epsilon_0 \cdot y(x^{p-1}-1)^k \theta^{-1}(\alpha) \\
&= \epsilon_k \cdot \theta^{-1}(\alpha)
\end{aligned}
$$

Hence the result. □

Proposition 5.8.2: *$\phi_k^* : \tilde{P}^k \to \tilde{P}^k$ is defined by*

$$\phi_k^*(\tilde{\epsilon}_k) = \tilde{\epsilon}_k \cdot \overline{\alpha}$$

Proof. By definition we have

$$
\begin{aligned}
\phi_k^*(p[k,i]^*)(p[k,j]) &= p[k,i]^*(\phi_k(p[k,j])) \\
&= p[k,i]^*(p[k,j] \cdot \alpha) \\
&= (\alpha \cdot p[k,i]^*)(p[k,j]) \\
&= (p[k,i]^* \cdot \overline{\alpha})(p[k,i])
\end{aligned}
$$

which implies

$$\phi_k^*(p[k,i]^*) = p[k,i]^* \cdot \overline{\alpha}$$

Hence the result. □

Note that we can define an endomorphism on the dual modules in the exact same way

Proposition 5.8.3: Suppose $\tilde{\phi}_k : \tilde{P}^k \to \tilde{P}^k$ is a Λ-homomorphism. Then

$$\tilde{\phi}_k(\tilde{\epsilon}_k) = \tilde{\epsilon}_k \cdot \alpha \quad \alpha \in R_0$$

Proof. Analogous to 5.8.2 □

5.9 R_0 for Metacyclic Groups of Even Order

In section 5.2, we saw the fixed field K_0 classified rational homomorphisms. Similarly, the presence of R_0 in the result of 5.8.1 and the condition $G(\pi)$ show the important role of R_0 in the integral case, and we devote time to the study of this fixed ring. Define symmetric and skew-symmetric subsets of R_0 by

$$R_0^+ = \{\alpha \in R_0; \overline{\alpha} = \alpha\}$$
$$R_0^- = \{\alpha \in R_0; \overline{\alpha} = -\alpha\}$$

Then $G(\pi)$ is equivalent to stating that $\pi \in R_0^+$ for q even and $\pi \in R_0^-$ for q odd. For q even, it turns out this is simple to prove. Suppose $\Lambda = \mathbb{Z}[G(p, 2r)]$ for some $r \in \mathbb{Z}$.

Proposition 5.9.1. *Let $G = G(p, 2r)$. Then $R_0 = R_0^+$*

Proof. Let $\alpha \in R_0$. Then the most general form of α can be written

$$\alpha = \sum_{i=0}^{p-2} a_i \sum_{j=0}^{2r-1} \theta^j(\zeta^i)$$

(Note that in practice, many of the a_i's would be set to zero). Then since θ has order $2r$, θ^r has order 2, hence $\theta^r(\zeta^k) = \zeta^{-k}$. So we can write

$$\alpha = \sum_{i=0}^{p-1} a_i \sum_{j=0}^{r-1} (\theta^j(\zeta^i) + \theta^r\theta^j(\zeta^i))$$

$$= \sum_{i=0}^{p-1} a_i \sum_{j=0}^{r-1} (\theta^j(\zeta^i) + \theta^j(\zeta^i)^{-1})$$

Then

$$\begin{aligned}
\overline{\alpha} &= \sum_{i=0}^{p-1} a_i \sum_{j=0}^{r-1} (\overline{\theta^j(\zeta^i)} + \overline{\theta^j(\zeta^i)^{-1}}) \\
&= \sum_{i=0}^{p-1} a_i \sum_{j=0}^{r-1} (\theta^j(\zeta^i)^{-1} + \theta^j(\zeta^i)) \\
&= \alpha
\end{aligned}$$ □

It then easily follows that

Corollary 5.9.3. *Suppose $G = G(p, 2r)$. Then G satisfies $G(\pi)$.*

Corollary 5.9.2. *If $G = G(p, 2r)$, then $R_0^- = 0$.*

We compare this with our rational calculations, where K_0 is totally symmetric.

Example 5.9.4 ($G(5, 2)$). *Writing $\gamma_1 = \zeta + \zeta^4$, $\gamma_2 = \zeta^2 + \zeta^3$; $G(5, 2) = D_{10}$ has group presentation and fixed ring*

$$G(5, 2) = \langle x, y;\ x^5 = y^2 = 1,\ yx = x^4 y\rangle$$
$$R_0 = \{a\gamma_1 + b\gamma_2;\ a, b \in \mathbb{Z}\}$$

It can be calculated

$$\gamma_1\gamma_2 = -1$$
$$(2\gamma_1 + 3\gamma_2)^2 = 5\gamma_2^2$$

so that

$$\pi = 2\gamma_1 + 3\gamma_2$$

Furthermore

$$\begin{aligned} v_\pi &= -\zeta - \zeta^2 + \zeta 4 \\ \Rightarrow u_\pi &= 1 + 2x + x^2 \end{aligned}$$

5.10 R_0 of Quadratic Type for Metacyclic Groups of Odd Order

Unfortunately, when q is odd, things are not so simple. The presence of a genuine automorphism of order 2 means that both R_0^+, R_0^- are non-zero. Furthermore, from the proof of 5.2.6 we see that there exist elements in R_0 which cannot be split into symmetric and skew-symmetric parts. This complexity makes trying to ascertain symmetry conditions on π highly non-trivial. However we can achieve a completely general result for one subclass of $G(p, 2r+1)$.

Suppose $G = G(p, 2r+1)$ where $2r+1 = \frac{p-1}{2}$. For these values of q, R_0 has dimension two and writing $R_0 = \mathbb{Z}[\alpha]/m(\alpha)$ where $m(\alpha)$ is the minimal polynomial of R_0, $m(\alpha)$ is quadratic and we call R_0 of quadratic type. For some choice of $\gamma \in R_0$, $\gamma_i \neq 1$, we can write

$$R_0 = \mathbb{Z}[\gamma] = \{a + b\gamma; a, b \in \mathbb{Z}\}$$

The existence of the involution on R_0 means that $\overline{\gamma} \neq \gamma$, and it follows that $1 + \gamma + \overline{\gamma} = 0$. Therefore

$$R_0 = \{a\gamma + b\overline{\gamma};\ a, b, \in \mathbb{Z}\}$$

We will sometimes find it useful to switch between these two definitions.

Example 5.10.1 ($G(7,3)$). *$G(7,3)$ has group presentation*

$$G(7,3) = \langle x, y;\ x^7 = y^3 = 1,\ yx = x^2y\rangle$$

Then we can take $\gamma = \zeta + \zeta^2 + \zeta^4$, so that $\overline{\gamma} = \zeta^3 + \zeta^5 + \zeta^6$, and we get the description

$$R_0 = \{a(\zeta + \zeta^2 + \zeta^4) + b(\zeta^3 + \zeta^5 + \zeta^6);\ a, b \in \mathbb{Z}\}$$

The structure of these cyclotomic quadratic fields was considered by Hasse in [11]. Recall the definition of the Legendre symbol as

$$(\frac{a}{p}) = \begin{cases} 1 & a \text{ is a square modulo } p, a \neq 0 \text{ modulo } p \\ -1 & a \text{ is not a square modulo } p, a \neq 0 \text{ modulo } p \\ 0 & a = 0 \text{ modulo } p \end{cases}$$

Then Hasse proves (page 529 in [11]):

Proposition 5.10.2. *Let p be a prime number, and let $\hat{p} = (-1)^{\frac{p-1}{2}}p$. Then*

$$\sqrt{\hat{p}} = \sum_{a \not\equiv 0 \mathrm{mod} p} (\frac{a}{p})\zeta_p^a$$

Lemma 5.10.3. $\pi = \sqrt{\hat{p}}$

Proof. In the case $q = \frac{p-1}{2}$, π is the unique element satisfying

$$\pi^2 = pu \quad u \in R_0^\times$$

Then

$$(\sqrt{\hat{p}})^2 = \hat{p} = (-1)^{\frac{p-1}{2}}p$$

Since π is unique, we must then have

$$\sqrt{\hat{p}} = \pi$$

□

We can further connect Hasse's result to our formulation of R_0 via the following

Proposition 5.10.4. *Let p be a prime, $q = 2r + 1 = \frac{p-1}{2}$. Then $z \in \mathbb{F}_p^x$ is a square if and only if there exists an automorphism $\theta : \mathbb{F}_p^x \to \mathbb{F}_p^x$ such that* $\mathrm{ord}(\theta) = q$ *and $z = \theta^k(1)$ for some $1 \leq k \leq q - 1$.*

Proof. Suppose z is a square. Let $y = a^2$ be any square (for instance, for $p \geq 3$ you can always pick $y = 4$). Then define a θ by $\theta^k(i) = y^k i$ for $i \in \mathbb{F}_p^x$. Then it is obvious that any power of y will also be a square, and also

$$y^q = (a^2)^q = a^{p-1} = 1 \ \ (p)$$

by Fermat's Little Theorem. Therefore $\mathrm{ord}(\theta) = q$, and θ applied successively to 1 simply cycles through all $\frac{p-1}{2}$ squares in $\mathbb{F}_p$ before returning back to 1. Therefore $z = \theta^k(1)$ for some k.

For the converse suppose we are given an automorphism $\theta : \mathbb{F}_p^x \to \mathbb{F}_p^x$ such that $\mathrm{ord}(\theta) = q$. Let $m = \theta(1)$. Then $\theta^q(1) = m^q = 1$. Since q odd means $q + 1$ is even this means we can write

$$\theta(1) = m = m^{q+1} = (m^{\frac{q+1}{2}})^2$$

Clearly $\theta^2(1) = m^2$ is a square, as is $\theta^{2j}(1)$ for $1 \leq j \leq \frac{q-1}{2}$. But then we can write

$$\theta^{2j+1}(1) = m^{2j+1} = m^{2j+1+q} = (m^{j+\frac{q+1}{2}})^2$$

and hence the result. $\square$

Proposition 5.10.5. $\pi = \gamma - \overline{\gamma}$

Proof. We can express γ by

$$\gamma = \sum_{k=0}^{q-1} \theta^k(\zeta_p)$$

Using 5.10.4, this becomes

$$\gamma = \sum_{a \text{ is a square mod} p} \zeta_p^a$$

Then rewriting 5.10.2 using this and 5.10.3 gives

$$\pi = \sqrt{\hat{p}} = \gamma - \overline{\gamma} \qquad \square$$

This complete description of π gives us the following

Proposition 5.10.6. *Suppose $G = G(p, 2r+1)$ where $2r+1 = \frac{p-1}{2}$. Then*

$$R_0^+ = \{a; a \in \mathbb{Z}\}$$
$$R_0^- = \{a\pi; a \in \mathbb{Z}\}$$

Proof. Any $r \in R_0$ can be expressed as

$$r = a\gamma + b\overline{\gamma} \quad a, b \in \mathbb{Z}$$

Then

$$\overline{r} = a\overline{\gamma} + b\gamma$$

Solving $\overline{r} = r$ implies $b = a$, so that

$$r = a(\gamma + \overline{\gamma}) = -a$$

where we can absorb the sign into the constant. Solving $\overline{r} = -r$ implies $b = -a$, so that

$$r = a(\gamma - \overline{\gamma}) = a\pi \qquad \square$$

We see that in this case, the element π generates R_0^-. We also note the formal corollary

Corollary 5.10.7. *Suppose $G = G(p, 2r+1)$ where $2r+1 = \frac{p-1}{2}$. Then G satisfies $G(\pi)$.*

Finally, we note that in this case, the unit group $R_0^\times$ is trivial, with $R_0^\times = \{\pm 1\}$. As we shall see, this simplicity does not survive as we increase the order of R_0.

5.11 R_0 of Quartic Type for Metacyclic Groups of Odd Order

Suppose $\Lambda = \mathbb{Z}[G(p, 2r+1)]$ where $2r+1 = \frac{p-1}{4}$. For these values of q, R_0 has dimension four with quartic minimal polynomial so we call it of quartic type. We can express R_0 in four variables, say $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in R_0$, such that

$$R_0 = \{a\gamma_1 + b\gamma_2 + c\gamma_3 + d\gamma_4;\ a, b, c, d \in \mathbb{Z}\}$$

Once again, since q is odd, there exists a genuine involution on γ_i ($\gamma_i \neq 1$). We can then write R_0 as

$$R_0 = \{a\gamma_1 + b\overline{\gamma_1} + c\gamma_2 + d\overline{\gamma_2};\ a, b, c, d \in \mathbb{Z}\}$$

As a first approximation to R_0^+, R_0^- we have the following standard descriptions

$$\begin{aligned} R_0^- &= \{a(\gamma_1 - \overline{\gamma_1}) + b(\gamma_2 - \overline{\gamma_2});\ a, b \in \mathbb{Z}\} \\ R_0^+ &= \{a(\gamma_1 + \overline{\gamma_1}) + b(\gamma_2 + \overline{\gamma_2});\ a, b \in \mathbb{Z}\} \end{aligned}$$

From our work in the case $2r+1 = \frac{p-1}{2}$, we might expect a further description of R_0^- involving a factor of the element π. Towards this end we consider the lowest dimensional case. We note however that even here the calculations are unfeasibly onerous to do by hand. To deal with this, computational methods can be employed and a series of Python scripts written to automate the calculation of:

- a for a given p, q;

- the basis elements of R_0;
- products in R_0 (this takes as input the coefficients of γ_i, $\overline{\gamma_i}$ and outputs the coefficients of these terms in the final product).

These building blocks then facilitate the calculation of units and π for a greater range of groups. However, the time taken to perform the most general search algorithm for units increases exponentially in $\frac{p-1}{q}$, and so this computational approach is infeasible in general. Furthermore, calculation time also increases with p, albeit in a more manageable way.

Example 5.11.1($G(13,3)$). *$G(13,3)$ has group presentation*

$$G(13,3) = \langle x, y;\ x^{13} = y^3 = 1,\ yx = x^3 y\rangle$$

and we pick representatives for R_0 as

$$\begin{aligned}\gamma_1 &= \zeta + \zeta^3 + \zeta^9\\ \gamma_2 &= \zeta^2 + \zeta^5 + \zeta^6\end{aligned}$$

Consider the element

$$(\gamma_1 + \overline{\gamma_1}) + 2(\gamma_2 + \overline{\gamma_2})$$

Then it can be calculated that

$$((\gamma_1 + \overline{\gamma_1}) + 2(\gamma_2 + \overline{\gamma_2}))(2(\gamma_1 + \overline{\gamma_1}) + (\gamma_2 + \overline{\gamma_2})) = -1$$

Therefore $u = (\gamma_1 + \overline{\gamma_1}) + 2(\gamma_2 + \overline{\gamma_2})$ is a unit. Suppose $u_1, u_2, \ldots, u_k$ are units. Then we define

$$< u_1, u_2, \ldots, u_k >= \{\pm u_1^{n_1} u_2^{n_2} \ldots u_k^{n_k}; n_k \in \mathbb{Z}\}$$

a generator representation for a unit group. In $G(13,3)$ we obtain

$$\begin{aligned}&< u, u^{-1} > = \{\pm 1, \pm u, \pm u^{-1}, \pm u^2, \pm u^{-2}, \ldots\}\\ &< u, u^{-1} > \subset R_0^{\times}\end{aligned}$$

We note that $< u, u^{-1} >$*, and hence* $R_0^\times$*, is infinite, providing a potentially infinite number of conditions to check against to find* π*, namely*

$$\pi^4 = 13v \quad v \in R_0^\times$$

As it happens, one can calculate

$$(\gamma_1 - \overline{\gamma_1})^4 = 13u^{-2}$$
$$(\gamma_2 - \overline{\gamma_2})^4 = 13u^2$$

providing two options for the unique π*. But then it is also easily checked that*

$$(\gamma_2 - \overline{\gamma_2})u = (\gamma_1 - \overline{\gamma_1})$$

so that uniqueness is preserved. Since u, u^{-1} *are clearly symmetric, either choice of representative gives that* π *is skew-symmetric. Making a choice of representative:*

$$\pi = (\gamma_2 - \overline{\gamma_2})$$

then R_0^- *has description*

$$R_0^- = \{\pi(a + bu); a, b \in \mathbb{Z}\}$$

In this example, as in those R_0 considered in section 5.10, all elements of R_0^- contain a factor of π. However this is not the case in general.

Example 5.11.2 ($G(37,9)$**).** $G(37,9)$ *has group presentation*

$$G(37,9) = \langle x, y;\ x^{37} = y^9 = 1,\ yx = x^7 y\rangle$$

and we pick representatives for R_0 *as*

$$\gamma_1 = \zeta + \zeta^7 + \zeta^9 + \zeta^{10} + \zeta^{12} + \zeta^{16} + \zeta^{26} + \zeta^{33} + \zeta^{34}$$
$$\gamma_2 = \zeta^2 + \zeta^{14} + \zeta^{15} + \zeta^{18} + \zeta^{20} + \zeta^{24} + \zeta^{29} + \zeta^{31} + \zeta^{32}$$

Then we can find a unit u with inverse u^{-1} as

$$\begin{aligned} u &= 7(\gamma_1 + \overline{\gamma_1}) + 5(\gamma_2 + \overline{\gamma_2}) \\ u^{-1} &= -5(\gamma_1 + \overline{\gamma_1}) - 7(\gamma_2 + \overline{\gamma_2}) \end{aligned}$$

One can calculate

$$((\gamma_1 - \overline{\gamma_1}) + (\gamma_2 - \overline{\gamma_2}))^4 = 37u^2 \Rightarrow \pi = (\gamma_1 - \overline{\gamma_1}) + (\gamma_2 - \overline{\gamma_2}))$$

Furthermore

$$\pi u^{-1} = ((\gamma_1 - \overline{\gamma_1}) - (\gamma_2 - \overline{\gamma_2}))$$

Define a subset $R_0^\pi \subset R_0$ by

$$R_0^\pi = \{\pi(a + bu^{-1}); a, b \in \mathbb{Z}\}$$

We see that $R_0^\pi \ncong R_0^-$ since

$$\begin{aligned} &a((\gamma_1 - \overline{\gamma_1}) + (\gamma_2 - \overline{\gamma_2})) + b((\gamma_1 - \overline{\gamma_1}) - (\gamma_2 - \overline{\gamma_2})) \\ &= (a+b)(\gamma_1 - \overline{\gamma_1}) + (a-b)(\gamma_2 - \overline{\gamma_2}) \\ &= 2c(\gamma_1 + \overline{\gamma_1}) + d(\gamma_2 - \overline{\gamma_2}) \end{aligned}$$

For $G(37, 9)$ we have obtained our first genuinely distinct structure in the integral case. We note however, that extending scalars to $\mathbb{Q}$ gives

$$R_0^- \otimes \mathbb{Q} \cong R_0^\pi \otimes \mathbb{Q}$$

Furthermore, extending to $\mathbb{R}$, we see that $\pi \otimes \mathbb{R}$ must be a skew-symmetric unit, in other words i, again matching up with our work in Section 5.2.

Similar calculations can be performed for a selection of p, q. In each case we make the choice $\gamma_1 = \sum_{k=0}^{q-1} \theta^k(\zeta)$, which then fixes γ_2. Define $u_1, u_2, v_1, v_2, \pi_1, \pi_2$ by

- $u = u_1(\gamma_1 + \overline{\gamma_1}) + u_2(\gamma_2 + \overline{\gamma_2})$ is a unit with inverse $u^{-1} = v_1(\gamma_1 + \overline{\gamma_1}) + v_2(\gamma_2 + \overline{\gamma_2})$;
- $\pi = \pi_1(\gamma_1 - \overline{\gamma_1}) + \pi_2(\gamma_2 - \overline{\gamma_2})$ satisfies $\pi^4 = pu^2$.

We obtain the following in tabulated form

p	q	u_1	u_2	v_1	v_2	π_1	π_2
13	3	1	2	-2	-1	0	1
29	7	2	3	-3	-2	1	0
37	9	7	5	-5	-7	1	1
53	13	4	3	-3	-4	1	0
61	15	22	17	-17	-22	1	2
101	25	9	11	-11	-9	1	1
109	27	118	143	-143	-118	3	4

Corollary 5.11.3. *$G(\pi)$ is satsified for the groups $G(13,3)$, $G(29,7)$, $G(37,9)$, $G(53,13)$, $G(61,15)$, $G(101,25)$, $G(109,27)$.*

Chapter 6

Metacyclic Groups II

6.1 Generator Representations for $\mathbf{Hom}_{\mathbb{Z}[G(p,q)]}(\tilde{P}^k, P^k)$

In the previous chapter we constructed duality isomorphisms, projection homomorphisms, and endomorphisms for each P^k defined on the generators of each. We now use these constituent parts to construct general homomorphisms $F_k : \tilde{P}^k \to P^k$. Broadly, the problem splits into five distinct classes dependent on k

1. $k = 0$
2. $k = 1$
3. $1 < k < \dfrac{q+1}{2}$
4. $\dfrac{q+1}{2} < k < q - 1$
5. $k = \dfrac{q+1}{2}$

We consider each in turn.

6.1.1 k=0

We want to classify $F_0 \in \mathrm{Hom}_\Lambda(\tilde{R}, R)$. We could construct such an F_0 as

$$\tilde{R} \xrightarrow{\upsilon_0} P \xrightarrow{\rho_1} R \xrightarrow{\phi_0} R$$

This construction is not unique, but we have the following:

Proposition 6.1.1.1. *F_0 is invariant under choice of construction.*

Proof. We want to classify all the different paths one can take from $\tilde{R}$ into R. Then we have two general cases, which will look like

$$\tilde{R} \to P \to R$$
$$\tilde{R} \to \tilde{P} \to R$$

albeit with an arbitrary number of endomorphisms interspersed along these paths. But then since $\phi_k : P^k \to P^k$, $\tilde{\phi}_k^* : \tilde{P}^k \to \tilde{P}^k$ are independent of k, with each contributing an element of R_0, we can collect them together into one single endomorphism at any point along the path. Up to sign and elements of R_0, which can also be absorbed into any endomorphism, ρ_1 is the only Λ-homomorphism from P into R, and ρ_1^* is the only Λ-homomorphism from $\tilde{R}$ into $\tilde{P}$. But then, from 5.7.7

$$\rho_1^* = -\upsilon_1^{-1} \circ \rho_1 \circ \upsilon_0$$

So they both contribute the same term to any F_0. Finally consider what happens if we choose to include more than one duality isomorphism, such as

$$\tilde{R} \to P \to P \to \tilde{R} \to \tilde{P} \to R$$

While these isomorphisms do not contribute any terms to F_0, upon taking duals, each contributes a minus sign, and so it is important to consider them. However any path from $\tilde{R}$ into R must contain an odd number of these isomorphisms, and mod 2 all paths are equivalent. □

Proposition 6.1.1.2. *Suppose* $F_0 \in \mathrm{Hom}_\Lambda(\tilde{R}, R)$. *Then*

$$F_0(\tilde{\epsilon}_0) = \epsilon_0 \cdot (x^{p-1} - 1)\alpha \quad \alpha \in R_0$$

Proof. Using 6.1.1.1 , it is enough to consider the construction

$$\tilde{R} \xrightarrow{\upsilon_0} P \xrightarrow{\rho_1} R \xrightarrow{\phi_0} R$$

Then, for $\alpha \in R_0$ we get

$$\begin{aligned}(\phi_0 \circ \rho_1 \circ \upsilon_0)(\tilde{\epsilon}_0) &= (\phi_0 \circ \rho_1)(\epsilon_1) \\ &= \phi_0(\epsilon_0 \cdot (x^{p-1} - 1)) \\ &= \epsilon_0 \cdot (x^{p-1} - 1)\alpha\end{aligned}$$

□

Proposition 6.1.1.3. *Suppose* $F_0^+ \in \mathrm{Hom}_\Lambda^+(\tilde{R}, R)$. *Then*

$$F_0^+(\tilde{\epsilon}_0) = \epsilon_0 \cdot (x^{p-1} - 1)\alpha_+ \quad \alpha_+ \in R_0^+$$

Proof. From 5.7.6 and 5.7.7 we have that

$$\begin{aligned}\upsilon_0^* \circ \rho_1^* &= (-\upsilon_1) \circ (-\upsilon_1^{-1} \circ \rho_1 \circ \upsilon_0) \\ &= \rho_1 \circ \upsilon_0\end{aligned}$$

Then $F_0^* : \tilde{R} \to R$ is defined by

$$\begin{aligned}F_0^*(\tilde{\epsilon}_0) &= (\phi_0 \circ \rho_1 \circ \upsilon_0)^*(\tilde{\epsilon}_0) \\ &= (\upsilon_0^* \circ \rho_1^* \circ \phi_0^*)(\tilde{\epsilon}_0) \\ &= (\rho_1 \circ \upsilon_0 \circ \phi_0^*)(\tilde{\epsilon}_0) \\ &= \epsilon_0 \cdot (x^{p-1} - 1)\overline{\alpha}\end{aligned}$$

Solving $F_0^* = F_0$ then reduces to solving $\overline{\alpha} = \alpha$. Hence the result. □

6.1.2 k=1

Proposition 6.1.2.1. *Suppose G satisfies $G(\pi)$ and $F_1 \in \mathrm{Hom}_\Lambda(\tilde{P}, P)$. Then*

$$F_1(\tilde{\epsilon}_1) = \epsilon_1 \cdot (x^{p-1} - 1)^{q-1} u_\pi \alpha \quad \alpha \in R_0$$

Proof. We again construct F_1 by composing projections, endomorphisms and duality isomorphisms, and writing

$$\sum_{j=2}^{q} \rho_j = \rho_2 \circ \rho_3 \circ \dots \rho_q$$

we could construct such an F_1 by

$$\tilde{P} \xrightarrow{\upsilon_1} R \xrightarrow{\upsilon_\pi} P^q \xrightarrow{\sum_{j=2}^{q} \rho_j} P \xrightarrow{\phi_1} P$$

Considering all possible paths from $\tilde{P}$ into P, any path will contain

- An odd number of duality isomorphisms;
- A total of $q - 1$ projections and dual projections, containing one of either ρ_2^* or $\rho_q \circ \upsilon_\pi$;
- An arbitrary number of endomorphisms which we can combine into one endomorphism.

and again F_1 is invariant of choice of construction. We therefore obtain

$$\begin{aligned}
F_1(\tilde{\epsilon}_1) &= (\phi_1 \circ \sum_{j=2}^{q} \rho_j \circ \upsilon_\pi \circ \upsilon_1)(\tilde{\epsilon}_1) \\
&= (\phi_1 \circ \sum_{j=2}^{q} \rho_j \circ \upsilon_\pi)(\epsilon_0) \\
&= (\phi_1 \circ \sum_{j=2}^{q} \rho_j)(\epsilon_q) \\
&= \phi_1(\epsilon_1 \cdot (x^{p-1} - 1)^{q-1} u_\pi)
\end{aligned}$$

$$= \epsilon_1 \cdot (x^{p-1} - 1)^{q-1} u_\pi \alpha$$

for $\alpha \in R_0$. $\square$

Proposition 6.1.2.2: $F_1^*(\tilde{\epsilon}_1) = \epsilon_1 \cdot (-1)^q (x^{p-1} - 1)^{q-1} u_\pi \overline{\alpha}$

Proof. We begin by noting that

$$\begin{aligned}
(\sum_{j=2}^{q} \rho_j)^* &= \sum_{j=q}^{2} (\rho^j)^* \\
&= (-1)^{q-1} \sum_{j=q}^{3} (\upsilon_j^{-1} \circ \rho_{q+2-j} \circ \upsilon_{j-1}) \circ (\upsilon_2^{-1} \circ \rho_q \circ \upsilon_\pi \circ \upsilon_1) \\
&= (-1)^{q-1} \upsilon_q^{-1} \circ \sum_{j=2}^{q} \rho_j \circ \upsilon_\pi \circ \upsilon_1
\end{aligned}$$

So that

$$\begin{aligned}
F_1^* &= \upsilon_1^* \circ \upsilon_\pi^* \circ (\sum_{j=2}^{q} \rho_j)^* \circ \phi_1^* \\
&= (-1)^q (\upsilon_0 \circ \upsilon_\pi^* \circ \upsilon_q^{-1} \circ \sum_{j=2}^{q} \rho_j \circ \upsilon_\pi \circ \upsilon_1 \circ \phi_1^*) \\
&= (-1)^q (\upsilon_q \circ \upsilon_q^{-1} \circ \sum_{j=2}^{q} \rho_j \circ \upsilon_\pi \circ \upsilon_1 \circ \phi_1^*) \\
&= (-1)^q (\sum_{j=2}^{q} \rho_j \circ \upsilon_\pi \circ \upsilon_1 \circ \phi_1^*)
\end{aligned}$$

Then by inspection

$$F_1^*(\tilde{\epsilon}_1) = \epsilon_1 \cdot (-1)^q (x^{p-1} - 1)^{q-1} u_\pi \overline{\alpha}$$ $\square$

Corollary 6.1.2.3: *Suppose G satsfies $G(\pi)$ and $F_1^+ \in \mathrm{Hom}_\Lambda^+(\tilde{P}, P)$. Then*

$$F_1^+(\tilde{\epsilon}_1) = \begin{cases} \epsilon_1 \cdot (x^{p-1}-1)^{q-1} u_\pi \alpha_+ & q = 2r \\ \epsilon_1 \cdot (x^{p-1}-1)^{q-1} u_\pi \alpha_- & q = 2r+1 \end{cases}$$

where $\alpha_+ \in R_0^+$, $\alpha_- \in R_0^+$.

6.1.3 $1 < k < \frac{q+1}{2}$

Proposition 6.1.3.1: *Suppose G satsfies $G(\pi)$, $1 < k < \frac{q+1}{2}$ and $F_k \in \mathrm{Hom}_\Lambda(\tilde{P}^k, P^k)$. Then*

$$F_k(\tilde{\epsilon}_k) = \epsilon_k \cdot (x^{p-1}-1)^{q+1-2k}\alpha \quad \alpha \in R_0$$

Proof. We take the construction

$$\tilde{P}^k \xrightarrow{\upsilon_k} P^{q+1-k} \xrightarrow{\sum_{j=k+1}^{q+1-k} \rho_j} P^k \xrightarrow{\phi_k} P^k$$

By the same argument used in 6.1.1.1, F_k is invariant under the construction chosen. Then for $\alpha \in R_0$

$$\begin{aligned} F_k(\tilde{\epsilon}_k) &= (\phi_k \circ \sum_{j=k+1}^{q+1-k} \rho_j \circ \upsilon_k)(\tilde{\epsilon}_k) \\ &= ((\phi_k \circ \sum_{j=k+1}^{q+1-k} \rho_j)(\epsilon_{q+1-k}) \\ &= \phi_k(\epsilon_k \cdot (x^{p-1}-1)^{q+1-2k}) \\ &= \epsilon_k \cdot (x^{p-1}-1)^{q+1-2k})\alpha \end{aligned}$$

□

Proposition 6.1.3.2. $F_k^*(\tilde{\epsilon}_k) = \epsilon_k \cdot (-1)^q (x^{p-1}-1)^{q+1-2k}\overline{\alpha}$

Proof. We have that

$$(\sum_{j=k+1}^{q+1-k} \rho_j)^* = \sum_{q+1-k}^{j=k+1} \rho_j^*$$

$$= (-1)^{q+1-2k}(\sum_{q+1-k}^{j=k+1} \upsilon_j^{-1} \circ \rho_{q+2-j} \circ \upsilon_{j-1})$$

$$= (-1)^{q+1}(\upsilon_{q+1-k}^{-1} \circ \sum_{j=k+1}^{q+1-k} \rho_j \circ \upsilon_k)$$

Then

$$\begin{aligned} F_k^* &= \upsilon_k^* \circ (\sum_{j=k+1}^{q+1-k} \rho_j)^* \circ \phi_k^* \\ &= (-1)^{q+2}(\upsilon_{q+1-k} \circ \upsilon_{q+1-k}^{-1} \circ \sum_{j=k+1}^{q+1-k} \rho_j \circ \upsilon_k \circ \phi_k^*) \\ &= (-1)^q(\sum_{j=k+1}^{q+1-k} \rho_j \circ \upsilon_k \circ \phi_k^*) \end{aligned}$$

which implies

$$F_k^*(\tilde{\epsilon}_k) = \epsilon_k \cdot (-1)^q (x^{p-1} - 1)^{q+1-2k}\overline{\alpha}$$

□

Corollary 6.1.3.3. *Suppose G satisfies $G(\pi)$, $1 < k < \frac{q+1}{2}$ and $F_k^+ \in \mathrm{Hom}_\Lambda^+(\tilde{P}^k, P^k)$. Then*

$$F_k^+(\tilde{\epsilon}_k) = \begin{cases} \epsilon_k \cdot (x^{p-1} - 1)^{q+1-2k}\alpha_+ & q = 2r \\ \epsilon_k \cdot (x^{p-1} - 1)^{q+1-2k}\alpha_- & q = 2r+1 \end{cases}$$

where $\alpha_+ \in R_0^+$, $\alpha_- \in R_0^-$.

6.1.4 $\frac{q+1}{2} < k < q - 1$

Proposition 6.1.4.1. *Suppose G satsfies $G(\pi)$, $\frac{q+1}{2} < k < q - 1$ and $F_k \in \mathrm{Hom}_\Lambda(\tilde{P}^k, P^k)$. Then*

$$F_k(\tilde{\epsilon}_k) = \epsilon_k \cdot (x^{p-1} - 1)^{2(q-k)+1} u_\pi \alpha \quad \alpha \in R_0$$

Proof. We take the construction

$$\tilde{P}^k \xrightarrow{\upsilon_k} P^{q+1-k} \xrightarrow{\sum_{j=1}^{q+1-k} \rho_j} R \xrightarrow{\upsilon_\pi} P^q \xrightarrow{\sum_{j=k+1}^{q} \rho_j} P^k \xrightarrow{\phi_k} P^k$$

By the usual argument, F_k is independent of the construction chosen. Then noting that

$$\sum_{j=1}^{q+1-k} \rho_j(\epsilon_{q+1-k}) = \epsilon_0 \cdot (x^{p-1}-1)^{q+1-k}$$

$$\sum_{j=k+1}^{q} \rho_j(\epsilon_q) = \epsilon_k \cdot (x^{p-1}-1)^{q-k} u_\pi$$

We obtain

$$F_k(\tilde{\epsilon}_k) = \epsilon_k \cdot (x^{p-1}-1)^{2(q-k)+1} u_\pi \alpha$$

for $\alpha \in R_0$. □

Proposition 6.1.4.2. $F_k^*(\tilde{\epsilon}_k) = \epsilon_k \cdot (x^{p-1}-1)^{2(q-k)+1} u_\pi \overline{\alpha}$

Proof. By the usual argument, we have that

$$(\sum_{j=k+1}^{q} \rho_j)^* = (-1)^{q-k}(\upsilon_q^{-1} \circ \sum_{j=2}^{q+1-k} \rho_j \circ \upsilon_k)$$

We can also calculate

$$\begin{aligned}
&(\sum_{j=1}^{q+1-k} \rho_j)^* \\
&= (-1)^{q+1-k}(\upsilon_{q+1-k}^{-1} \circ \sum_{k+1}^{q-1} \rho_j \circ \upsilon_2) \circ (\upsilon_2^{-1} \circ \rho_q \circ \upsilon_\pi \upsilon_1) \circ (\upsilon_1^{-1} \circ \rho_1 \circ \upsilon_0) \\
&= (-1)^{q+1-k}(\upsilon_{q+1-k}^{-1} \circ \sum_{k+1}^{q} \rho_j \circ \upsilon_\pi \circ \rho_1 \circ \upsilon_0)
\end{aligned}$$

Then

$$\upsilon_k^* \circ (\sum_{j=1}^{q+1-k} \rho_j)^* \circ \upsilon_\pi^* = (-1)^{q+2-k}(\upsilon_{q+1-k} \circ \upsilon_{q+1-k}^{-1} \circ \sum_{k+1}^{q} \rho_j \circ \upsilon_\pi \circ \rho_1 \circ \upsilon_0 \circ \upsilon_\pi^*)$$
$$= (-1)^{q-k}(\sum_{k+1}^{q} \rho_j \circ \upsilon_\pi \circ \rho_1 \circ \upsilon_q)$$

So that

$$\begin{aligned} F_k^* &= \upsilon_k^* \circ (\sum_{j=1}^{q+1-k} \rho_j)^* \circ \upsilon_\pi^* \circ (\sum_{j=k+1}^{q} \rho_j)^* \circ \phi_k^* \\ &= (-1)^{2(q-k)}(\sum_{k+1}^{q} \rho_j \circ \upsilon_\pi \circ \rho_1 \circ \upsilon_q \circ \upsilon_q^{-1} \circ \sum_{j=2}^{q+1-k} \rho_j \circ \upsilon_k \circ \phi_k^*) \\ &= \sum_{k+1}^{q} \rho_j \circ \upsilon_\pi \circ \sum_{j=1}^{q+1-k} \rho_j \circ \upsilon_k \circ \phi_k^* \end{aligned}$$

which implies

$$F_k^*(\tilde{\epsilon}_k) = \epsilon_k \cdot (x^{p-1} - 1)^{2(q-k)+1} u_\pi \overline{\alpha}$$ □

Corollary 6.1.4.3. *Suppose G satisfies $G(\pi)$, $\frac{q+1}{2} < k < q-1$, and $F_k^+ \in \mathrm{Hom}_\Lambda^+(\tilde{P}^k, P^k)$. Then*

$$F_k^+(\tilde{\epsilon}_k) = \epsilon_k \cdot (x^{p-1} - 1)^{2(q-k)+1} u_\pi \alpha_+ \quad \alpha_+ \in R_0^+$$

6.1.5 $k = \frac{q+1}{2}$

Proposition 6.1.5.1. *Suppose $F_{\frac{q+1}{2}} \in \mathrm{Hom}_\Lambda(\tilde{P}^{\frac{q+1}{2}}, P^{\frac{q+1}{2}})$. Then*

$$F_{\frac{q+1}{2}}(\tilde{\epsilon}_{\frac{q+1}{2}}) = \epsilon_{\frac{q+1}{2}} \cdot \alpha \quad \alpha \in R_0$$

Proof. Since $\tilde{P}^{\frac{q+1}{2}} \cong P^{\frac{q+1}{2}}$ there is only one possible path to take, namely

$$\tilde{P}^{\frac{q+1}{2}} \xrightarrow{\upsilon_{\frac{q+1}{2}}} P^{\frac{q+1}{2}} \xrightarrow{\phi_{\frac{q+1}{2}}} P^{\frac{q+1}{2}}$$

Then, for $\alpha \in R_0$

$$\begin{aligned} F_{\frac{q+1}{2}}(\tilde{\epsilon}_{\frac{q+1}{2}}) &= \phi_{\frac{q+1}{2}} \circ \upsilon_{\frac{q+1}{2}}(\tilde{\epsilon}) \\ &= \phi_{\frac{q+1}{2}}(\epsilon_{\frac{q+1}{2}}) \\ &= \epsilon_{\frac{q+1}{2}} \cdot \alpha \end{aligned}$$

□

Proposition 6.1.5.2. *Suppose G satisfies $G(\pi)$, and $F^+_{\frac{q+1}{2}} \in \mathrm{Hom}^+_\Lambda(\tilde{P}^{\frac{q+1}{2}}, P^{\frac{q+1}{2}})$. Then*

$$F_{\frac{q+1}{2}}(\tilde{\epsilon}_{\frac{q+1}{2}}) = \epsilon_{\frac{q+1}{2}} \cdot \alpha_-$$

where $\alpha \in R_0^-$.

Proof. $F^*_{\frac{q+1}{2}} : \tilde{P}^{\frac{q+1}{2}} \to P^{\frac{q+1}{2}}$ is defined by

$$\begin{aligned} F^*_{\frac{q+1}{2}} &= \upsilon^*_{\frac{q+1}{2}} \circ \phi_{\frac{q+1}{2}} \\ &= -\upsilon_{\frac{q+1}{2}} \circ \phi_{\frac{q+1}{2}} \end{aligned}$$

which implies that

$$F^*_{\frac{q+1}{2}}(\tilde{\epsilon}_{\frac{q+1}{2}}) = \epsilon_{\frac{q+1}{2}} \cdot -\overline{\alpha}$$

So solving $F^*_{\frac{q+1}{2}} = F_{\frac{q+1}{2}}$ reduces to $\overline{\alpha} = -\alpha$. Hence the result. □

Collating these results together, and using 5.9.3, we obtain:

Theorem 6.1.5.3. *Suppose $G = (p, 2r)$ and $F^+_k \in \mathrm{Hom}^+_\Lambda(\tilde{P}^k, P^k)$. Then*

$$F_k(\tilde{\epsilon}_k) = \begin{cases} \epsilon_k \cdot (x^{p-1} - 1)\alpha_+ & k = 0 \\ \epsilon_k \cdot (x^{p-1} - 1)^{q+1-2k} u_\pi \alpha_+ & k = 1 \\ \epsilon_k \cdot (x^{p-1} - 1)^{q+1-2k} \alpha_+ & 1 < k \leq \frac{q}{2} \\ \epsilon_k \cdot (x^{p-1} - 1)^{2(q-k)+1} u_\pi \alpha_+ & \frac{q+2}{2} \leq k < q \end{cases}$$

where $\alpha_+ \in R_0^+$, $\alpha_- \in R_0^-$.

Theorem 6.1.5.4. *Suppose $G = G(p, 2r+1)$, G satisfies $G(\pi)$, and that $F_k^+ \in \mathrm{Hom}_\Lambda^+(\tilde{P}^k, P^k)$. Then*

$$F_k^+(\tilde{\epsilon}_k) = \begin{cases} \epsilon_k \cdot (x^{p-1}-1)\alpha_+ & k = 0 \\ \epsilon_k \cdot (x^{p-1}-1)^{q+1-2k} u_\pi \alpha_- & k = 1 \\ \epsilon_k \cdot (x^{p-1}-1)^{q+1-2k} \alpha_- & 1 < k < \frac{q+1}{2} \\ \epsilon_k \cdot \alpha_- & k = \frac{q+1}{2} \\ \epsilon_k \cdot (x^{p-1}-1)^{2(q-k)+1} u_\pi \alpha_+ & \frac{q+1}{2} < k < q \end{cases}$$

where $\alpha_+ \in R_0^+$, $\alpha_- \in R_0^-$.

6.2 Matrix Representations for $\mathbf{Hom}_{\mathbb{Z}[G(p,q)]}(\tilde{P}^k, P^k)$

In section 3.4 we discussed how given a matrix representation for the middle boundary map of a highly connected chain complex X^{4k+1}, we can compute its Smith Normal Form to compute the only homology group of note, namely $H_{2j}(X^{4k+1}; \mathbb{Z})$. Seeking matrix representations for F_k is therefore a worthwhile cause. We have two approaches. The first is a brute force approach which uses the tools set out in Chapter 4.

Suppose $F_k \in \mathrm{Hom}_\Lambda(\tilde{P}^k, P^k)$, and $\rho : G \to GL_{\mathbb{Z}}(P^k)$ an integral representation. Then F_k can be expressed as $F_k : (\mathbb{Z}^{p-1}, \rho^*) \to (\mathbb{Z}^{p-1}, \rho)$ by first taking a general $F_k = (a_{ij})_{1 \le i,j \le p-1}$, $a_{ij} \in \mathbb{Z}$ and then solving

$$\begin{aligned} F_k \circ \rho^*(x) &= \rho(x) \circ F_k \\ F_k \circ \rho^*(y) &= \rho(y) \circ F_k \\ F_k^t &= F^k \end{aligned}$$

This method has the benefit of providing solutions without any calculations within R_0, but quickly becomes unfeasible since the size and number of calculations increases with p.

The second approach utilises our work with so called generator representations for F_k. Recall we took basis elements $p[k, i] = \epsilon_k \cdot x^i$ so that

$P^k = \mathrm{Span}_{\mathbb{Z}}\{p[k,i]\}$. Then we have two definitions depending on which basis for $\tilde{P}^k$ we pick. If we choose $\tilde{p}[k,i]$, then we have that

$$F_k(\tilde{p}[k,i]) = F_k(\tilde{\epsilon_k}) \cdot x^i$$

One can then simply expand out the right hand side and read off coefficients of basis elements to generate the matrix for F_k. In half of the cases, this method has the benefit of only requiring simple expansion and cancellation of terms even for large p. The difficulty for the other half is not knowing a priori an explicit form for u_π, and as the calculations in sections 5.9 through 5.11 show, it is far from trivial to compute π and hence u_π. Furthermore, this does not generate a symmetric matrix, even though it corresponds to a symmetric function, because of the choice of basis. To generate a symmetric matrix as our brute force method does, we need to define F_k over the natural dual basis

$$\begin{aligned} F_k(p[k,i]^*) &= F_k(p[k,0]^*) \cdot (1 + \ldots x^i) \\ &= F_k(\tilde{\epsilon_k}) \cdot w_k(1 + \ldots x^i) \end{aligned}$$

where w_k is some unit dependent on k. To generate these symmetric matrices is again computationally difficult because for all $k \neq 0, 1$ it requires knowledge of u_π. We proceed to work through both methods for the example $G = G(5,2)$ to verify that they do in fact coincide for this group.

Example 6.2.1 **($G = G(5,2)$).** *$G(5,2) = D_{10}$ has group presentation*

$$G(5,2) = \langle x, y;\ x^5 = y^2 = 1,\ yx = x^4 y\rangle$$

and two homomorphisms to consider

$$\begin{aligned} F_0 &: \tilde{R} \to R \\ F_1 &: \tilde{P} \to P \end{aligned}$$

We begin with our brute force approach, and take our standard basis. We will write $\rho_k(g)$ for the representation of g in P^k. Then using 4.2.1 *and* 4.2.2

we obtain

$$\rho_0(x) = \rho_1(x) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\rho_0^*(x) = \rho_1^*(x) = \begin{pmatrix} -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The calculations $\rho(y)(p[k,i]) = p[k,i] \cdot y$ *give*

$$\rho_0(y) = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{pmatrix} \quad \rho_1(y) = \begin{pmatrix} 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\rho_0^*(y) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \rho_1^*(y) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0-1 & \end{pmatrix}$$

Solving the requisite equations for $F_k = (a_{ij})_{ij}$ *and setting* $a_{13} = a$, $a_{12} = b$ *gives*

$$F_0 = F_1 = \begin{pmatrix} 2a & b & a & 2a-b \\ b & 2b & 2b-a & a \\ a & 2b-a & 2b & b \\ 2a-b & a & b & 2a \end{pmatrix}$$

Now consider our second method, beginning with $k = 0$. *Then* $w_0 = 1$ *and* $\alpha \in R_0$ *can be written*

$$\alpha = a(x + x^4) + b(x^2 + x^3) \quad a, b \in \mathbb{Z}$$

Therefore, writing $e_i = p[0, i]$

$$\begin{aligned} F_0(e_i^*) &= \epsilon_0 \cdot (x^4 - 1)(1 + \ldots + x^i)(a(x + x^4) + b(x^2 + x^3)) \\ &= \epsilon_0 \cdot (x^4 - x^i)(a(x + x^4) + b(x^2 + x^3)) \\ &= \epsilon_0 \cdot (a(1 + x^3 - x^{i+1} - x^{i-1}) + b(x + x^2 - x^{i+2} - x^{i-2})) \end{aligned}$$

and we obtain

$$\begin{aligned} F_0(e_0^*) &= \epsilon_0 \cdot (a(1 + x^3 - x - x^4) + b(x + x^2 - x^2 - x^3)) \\ &= \epsilon_0 \cdot (a(2 + x^2 + 2x^3) + b(x - x^3)) \\ &= 2ae_0 + be_1 + ae_2 + (2a - b)e_3 \\ F_0(e_1^*) &= \epsilon_0 \cdot (a(1 + x^3 - x^2 - 1) + b(x + x^2 - x^3 - x^4)) \\ &= \epsilon_0 \cdot (a(x^3 - x^2) + b(1 + 2x + 2x^2)) \\ &= be_0 + 2be_1 + (2b - a)e_2 + ae_3 \\ F_0(e_2^*) &= \epsilon_0 \cdot (a(1 + x^3 - x^3 - x) + b(x + x^2 - x^4 - 1)) \\ &= \epsilon_0 \cdot (a(1 - x) + b(2x + 2x^2 + x^3)) \\ &= ae_0 + (2b - a)e_1 + 2be_2 + be_3 \\ F_0(e_3^*) &= \epsilon_0 \cdot (a(1 + x^3 - x^4 - x^2) + b(x + x^2 - 1 - x)) \\ &= \epsilon_0 \cdot (a(2 + x + 2x^3) + b(-1 + x^2)) \\ &= (2a - b)e_0 + ae_1 + be_2 + 2ae_3 \end{aligned}$$

Writing as a matrix

$$F_0 = \begin{pmatrix} 2a & b & a & 2a - b \\ b & 2b & 2b - a & a \\ a & 2b - a & 2b & b \\ 2a - b & a & b & 2a \end{pmatrix}$$

and we have verified that we obtain the same result. For F_1 *recall from* 5.6.3*,* 5.9.1 *that*

$$u_\pi = 1 + 2x + x^2$$
$$w_k = x^4$$

Therefore, writing $e_i = p[1, i]$

$$\begin{aligned} F_1(e_i^*) &= \epsilon_1 \cdot x^4(x^4-1)(1+2x+x^2)(1+\ldots+x^i)(a(x+x^4)+b(x^2+x^3)) \\ &= \epsilon_0 \cdot (2+x+x^4)(a(1+x^3-x^{i+1}-x^{i-1})+b(x+x^2-x^{i+2}-x^{i-2})) \end{aligned}$$

so that

$$\begin{aligned} F_1(e_0^*) &= 2be_0 + (3b-a)e_1 + be_2 + (a-b)e_3 \\ F_1(e_1^*) &= (3b-a)e_0 + (6b-2a)e_1 + (5b-2a)e_2 + be_3 \\ F_1(e_2^*) &= be_0 + (5b-2a)e_1 + (6b-2a)e_2 + (3b-a)e_3 \\ F_1(e_3^*) &= (a-b)e_0 + be_1 + (3b-a)e_2 + 2be_3 \end{aligned}$$

and writing as a matrix

$$F_1 = \begin{pmatrix} 2b & 3b-a & b & a-b \\ 3b-a & 6b-2a & 5b-2a & b \\ b & 5b-2a & 6b-2a & 3b-a \\ a-b & b & 3b-a & 2b \end{pmatrix}$$

At first glance, this does not match up with our previous result. However, by making the basis change $a = 3b - c$ $c \in \mathbb{Z}$*, we find that*

$$F_1 = \begin{pmatrix} 2b & c & b & 2b-c \\ c & 2c & 2c-b & b \\ b & 2c-b & 2c & c \\ 2b-c & b & c & 2b \end{pmatrix} = F_0$$

In this example, $F_0 = F_1$ are defined over two separate variables, and so the calculation of the Smith Normal Form is non-trivial. However, considering G satisfying $\dim_{\mathbb{Z}}(R_0) = 1$, these groups will generate matrices more amenable to general calculations.

Suppose $G = G(p, p-1)$. Then $R_0 = \{a; a \in \mathbb{Z}\}$. This simplicity of the fixed ring allows a particularly nice result for F_0. Define a $n \times n$ matrix T_n by

$$(T)_{ij} = \begin{cases} 2 & i = j \\ 1 & i \neq j \end{cases}$$

We then obtain:

Proposition 6.2.2. *Suppose $G = G(p, p-1)$. Then F_0 has matrix representation $F_0 = aT_{p-1}$, $a \in \mathbb{Z}$.*

Proof. Simply rewriting 6.1.1.3 over the natural dual basis gives

$$\begin{aligned} F_0(p[0,i]^*) &= \epsilon_0 \cdot a(x^{p-1} - 1)(1 + \ldots + x^i) \\ &= \epsilon_0 \cdot a(x^{p-1} - x^i) \\ &= \epsilon_0 \cdot -a(\sum_{j=0}^{p-2} x^j + x^i) \\ &= -a(\sum_{j=0}^{p-2} p[0,j] + p[0,i]) \end{aligned}$$

and absorbing the sign into the constant term gives the result. □

T_n is a particularly nice matrix for our calculations by virtue of the following result

Proposition 6.2.3: $\det(T_n) = n + 1$

Proof. We proceed by induction on n. As base case take $n = 1$, which is clearly true. Suppose true for all $i \leq n - 1$, and consider T_n. Perform a simultaneous row and column operation on T_n by subtracting row 2 from row 1, and column 2 from column 1 to give an equivalent matrix

$$\tilde{T}_n = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots \\ -1 & 2 & 1 & 1 & \dots \\ 0 & 1 & & & \\ 0 & 1 & & T_{n-2} & \\ \vdots & \vdots & & & \end{pmatrix}$$

Then, recalling the earlier of definition of c_n as a column consisting entirely of 1's:

$$\begin{aligned} \det(T_n) &= 2\det(T_{n-1}) + \det(\begin{pmatrix} -1 & c_{n-2}^t \\ 0 & T_{n-2} \end{pmatrix}) \\ &= 2\det(T_{n-1}) - \det(T_{n-2}) \\ &= 2n - (n-1) \\ &= n+1 \end{aligned}$$

Hence the result. $\square$

Corollary 6.2.4. *F_0 has Smith Normal Form*

$$a \begin{pmatrix} I_{p-2} & 0 \\ 0 & p \end{pmatrix}$$

We proceed to give symmetric matrix representations and Smith Normal Forms for F_k, $k \geq 1$, for the example $G(5,4)$.

Example 6.2.5 ($G(5,4)$). *For $a \in \mathbb{Z}$, F_1, F_2, F_3 can be written:*

$$F_1 = \begin{pmatrix} 4a & 3a & 2a & a \\ 3a & 6a & 4a & 2a \\ 2a & 4a & 6a & 3a \\ a & 2a & 3a & 4a \end{pmatrix} \quad \mathrm{SNF}(F_1) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 5a & 0 & 0 \\ 0 & 0 & 5a & 0 \\ 0 & 0 & 0 & 5a \end{pmatrix}$$

$$F_2 = \begin{pmatrix} 2a & 2a & a & 0 \\ 2a & 4a & 3a & a \\ a & 3a & 4a & 2a \\ 0 & a & 2a & 2a \end{pmatrix} \quad \mathrm{SNF}(F_2) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 5a \end{pmatrix}$$

$$F_3 = \begin{pmatrix} 6a & 7a & 3a & -a \\ 7a & 14a & 11a & 3a \\ 3a & 11a & 14a & 7a \\ -a & 3a & 7a & 6a \end{pmatrix} \quad \mathrm{SNF}(F_3) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 5a & 0 & 0 \\ 0 & 0 & 5a & 0 \\ 0 & 0 & 0 & 5a \end{pmatrix}$$

Suppose that $G = G(p, \frac{p-1}{2})$ where $\frac{p-1}{2} = 2r+1$ for some r. Then from 5.10.6 we know that

$$R_0^+ = \{a; a \in \mathbb{Z}\}$$
$$R_0^- = \{a\pi; a \in \mathbb{Z}\}$$

Considering F_0, we note that we have the same situation as in 6.2.2, and therefore obtain:

Proposition 6.2.6. *Suppose $G = G(p, \frac{p-1}{2})$ where $\frac{p-1}{2}$ is odd. Then F_0 has matrix representation $F_0 = aT_{p-1}$, $a \in \mathbb{Z}$.*

Example 6.2.7 $G(7,3)$**.** *Calculating symmetric matrices for $G = G(7,3)$ gives*

$$F_1^+ = \begin{pmatrix} 6a & 5a & 4a & 3a & 2a & a \\ 5a & 10a & 8a & 6a & 4a & 2a \\ 4a & 8a & 12a & 9a & 6a & 3a \\ 3a & 6a & 9a & 12a & 8a & 4a \\ 2a & 4a & 6a & 8a & 10a & 5a \\ a & 2a & 3a & 4a & 5a & 6a \end{pmatrix} \quad \mathrm{SNF}(F_1) \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7 \end{pmatrix}$$

$$F_2^+ = \begin{pmatrix} 4a & 5a & 4a & 2a & 0 & -a \\ 5a & 10a & 10a & 7a & 3a & 0 \\ 4a & 10a & 14a & 12a & 7a & 2a \\ 2a & 7a & 12a & 14a & 10a & 4a \\ 0 & 3a & 7a & 10a & 10a & 5a \\ -a & 0 & 2a & 4a & 5a & 4a \end{pmatrix} \quad \mathrm{SNF}(F_2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7 \end{pmatrix}$$

6.3 Representations for $\mathbf{Hom}_\Lambda(\tilde{P}^i, P^k)$

In section 5.1 we saw that writing our boundary homomorphism in block matrix form, the off-diagonal entries were generated by homomorphisms

$$G_{k,i} : \tilde{P}^i \to P^k$$

where $i \neq 1$. We can then use the same reasoning as for the results in Section 6.1 to obtain

Proposition 6.3.1. *Suppose* $G_{k,0} \in \mathrm{Hom}_\Lambda(\tilde{R}, P^k)$. *Then*

$$G_{k,0}(\tilde{\epsilon}_0) = \begin{cases} \epsilon_k \cdot (x^{p-1} - 1)\alpha & k = 0 \\ \epsilon_k \cdot \alpha & k = 1 \\ \epsilon_k \cdot (x^{p-1} - 1)^{q+1-k} u_\pi \alpha & 2 \leq k \leq q-1 \end{cases}$$

for $\alpha \in R_0$.

Proposition 6.3.2. *Suppose* $2 \leq i \leq q-1$, *and* $G_{k,i} \in \mathrm{Hom}_\Lambda(\tilde{P}^i, P^k)$. *Then*

$$G_{k,i}(\tilde{\epsilon}_i) = \begin{cases} \epsilon_k \cdot (x^{p-1} - 1)^{q+1-i-k} \alpha & k < q+1-i \\ \epsilon_k \cdot (x^{p-1} - 1)^{2q+1-i-k} u_\pi \alpha & k > q+1-i \\ \epsilon_k \cdot \alpha & k = q+1-i \end{cases}$$

for $\alpha \in R_0$.

Matrix representations for these maps can then be generated in the usual way.

6.4 Representations for $\mathbf{Hom}_\Lambda([y-1)^*,[y-1))$

Recall the definition of $[y-1)$ as the right ideal generated by $y-1$ over Λ. Define basis elements

$$e[i,j] = (y-1)y^{q-i}x^j$$

Then since

$$\begin{aligned}
(y-1)y &= y^2 - y \\
&= -y + 1 - 1 + y^{q-1} - y^{q-1} + \ldots - y^3 + y^2 \\
&= \sum_{k=2}^{q}(y^k - y^{k+1}) \\
&= -\sum_{k=2}^{q}(y-1)y^k \\
&= -\sum_{l=0}^{q-2}(y-1)y^{q-l}
\end{aligned}$$

we can write

$$[y-1) = \mathrm{Span}_{\mathbb{Z}}\{e[i,j]\}_{\substack{0\le i\le q-2\\ 0\le j\le p-1}}$$

Define a right Λ action on $[y-1)$ by

$$e[i,j]\cdot g = e[i,j]g^{-1}$$

The following are then trivial:

Lemma 6.4.1.

$$e[i,j]\cdot x = \begin{cases} e[i,p-1] & j=0 \\ e[i,j-1] & 1\le j\le p-1 \end{cases}$$

Lemma 6.4.2.

$$e[i,j]\cdot y = \begin{cases} e[i+1,aj] & 0 \leq i \leq q-3 \\ -\sum\limits_{k=0}^{q-2} e[k,aj] & i = q-2 \end{cases}$$

where a is determined from the group presentation.

From this description we see that $[y-1)$ looks like a product

$$[y-1) = \mathcal{R}_q \otimes_\Lambda \mathbb{Z}[C_p]$$

where we recall the definition of $\mathcal{R}$ from section 2.3. We therefore expect to be able to express a homomorphism

$$H : [y-1)^* \to [y-1)$$

as a product of univariate functions

$$H = h_1 \otimes h_2$$
$$h_1 : \mathcal{R}_q^* \to \mathcal{R}_q$$
$$h_2 : \mathbb{Z}[C_p]^* \to \mathbb{Z}[C_p]$$

In fact this is easily verified by calculating actions in the dual space.

Proposition 6.4.3.

$$e[i,j]^* \cdot x = \begin{cases} e[i,p-1]^* & j = 0 \\ e[i,j-1] & 1 \leq j \leq p-1 \end{cases}$$

Proof. Labelling constants $a[i,j] \in \mathbb{Z}$ we have

$$(e[i,j]^* \cdot x)(\sum_{i=0}^{q-2}\sum_{j=0}^{p-1} a[i,j]e[i,j])$$

$$= e[i,j]^*(\sum_{i=0}^{q-2}\sum_{j=0}^{p-1} a[i,j]e[i,j] \cdot x^{-1})$$

$$= e[i,j]^*(\sum_{i=0}^{q-2}\sum_{j=0}^{p-1} a[i,j](y-1)y^{q-i}x^{j+1})$$

$$= e[i,j]^*(\sum_{i=0}^{q-2}(\sum_{j=0}^{p-2} a[i,j]e[i,j+1] + a[i,p-1]e[i,0]))$$

$$= e[i,j]^*(\sum_{i=0}^{q-2}(\sum_{j=1}^{p-1} a[i,j-1]e[i,j] + a[i,p-1]e[i,0]))$$

Hence the result. □

Proposition 6.4.4.

$$e[i,j]^* \cdot y = \begin{cases} e[i+1,aj]^* - e[0,aj]^* & 0 \leq i \leq q-3 \\ -e[0,aj]^* & i = q-2 \end{cases}$$

Proof.

$$(e[i,j]^* \cdot y)(\sum_{i=0}^{q-2}\sum_{j=0}^{p-1} a[i,j]e[i,j])$$

$$= e[i,j]^*(\sum_{i=0}^{q-2}\sum_{j=0}^{p-1} a[i,j](y-1)y^{q-i}x^j y$$

$$= e[i,j]^*(\sum_{i=0}^{q-2}\sum_{j=0}^{p-1} a[i,j](y-1)y^{q-(i-1)}x^{bj}$$

$$= e[i,j]^*(\sum_{i=0}^{q-2}\sum_{j=0}^{p-1} a[i,aj](y-1)y^{q-(i-1)}x^{j}$$

$$= e[i,j]^*(\sum_{j=0}^{p-1}(-a[0,aj]\sum_{k=0}^{q-2} e[k,aj] + \sum_{i=1}^{q-2} a[i,aj]e[i-1,j]))$$

$$= e[i,j]^*(\sum_{j=0}^{p-1}(-a[0,aj]\sum_{k=0}^{q-2} e[k,aj] + \sum_{i=0}^{q-3} a[i+1,aj]e[i,j]))$$

$$= e[i,j]^*(\sum_{j=0}^{p-1}(\sum_{i=0}^{q-3}(a[i+1,aj] - a[0,aj])e[i,j] - a[0,aj]e[q-2,j]))$$

Hence the result. $\square$

Considering H, we see that the requirement $H(e[i,j]^* \cdot x) = H(e[i,j]^*) \cdot x$ becomes a relation purely in j, i.e. x, and that $H(e[i,j]^* \cdot y) = H(e[i,j]^*) \cdot y$ is a relation purely in y, and so H can be split into two univariate functions. But then we have already classified matrix representations for $h_1 : \mathcal{R}_q \to \mathcal{R}_q$ in Chapter 4, so that only h_2 remains. Define a matrix

$$\alpha_n = \begin{pmatrix} 0 & I_{n-1} \\ 1 & 0 \end{pmatrix}$$

Lemma 6.4.5. *Suppose $\Lambda = \mathbb{Z}[C_p]$. Then*

$$\rho(x) = \alpha_p$$

Proof. Immediate from 6.4.1 by disregarding the i indices. $\square$

Proposition 6.4.6. *Suppose $h_2 : \mathbb{Z}[C_p]^* \to \mathbb{Z}[C_p]$ is a $\mathbb{Z}[C_p]$-homomorphism. Then h_2 has matrix representation*

$$h_2 = \sum_{i=0}^{p-1} a_i \alpha_p^i$$

where $a_i \in \mathbb{Z}$.

Proof. We can decompose h_2 as

$$h_2 = \tilde{h} \circ \upsilon$$

where $\upsilon : \mathbb{Z}[C_p]^* \to \mathbb{Z}[C_p]$ is an isomorphism, and $\tilde{h} \in \mathrm{End}_{\mathbb{Z}[C_p]}$. But then from 6.4.1 and 6.4.3 we see that we can simply take $\upsilon = I_p$. Furthermore, since $\mathrm{End}_{\mathbb{Z}[C_p]} \cong \mathbb{Z}[C_p]$, $\tilde{h}$ can be expressed as

$$h_2 = \sum_{i=0}^{p-1} a_i \rho(x)^i$$

where $a_i \in \mathbb{Z}$. Hence the result. $\square$

Summarising, we obtain:

Theorem 6.4.7: *Suppose $H : [y-1)^* \to [y-1)$ is a Λ-homomorphism. Then H can be expressed as a product*

$$(h_1 \otimes h_2) : (\mathcal{R}_q^* \otimes \mathbb{Z}[C_p]^*) \to (\mathcal{R}_q \otimes \mathbb{Z}[C_p])$$

where

$$h_1 = \sum_{i=0}^{q-2} a_i \gamma_i$$
$$h_2 = \sum_{j=0}^{p-1} b_i \alpha_i$$

and $a_i, b_i \in \mathbb{Z}$, γ_i as defined in Chapter 4.

We also require $H^* = H$, giving the following

Theorem 6.4.8: *Suppose $H : [y-1)^* \to [y-1)$ is a symmetric Λ-homomorphism. Then H can be expressed as a product*

$$(h_1 \otimes h_2) : (\mathcal{R}_q^* \otimes \mathbb{Z}[C_p]^*) \to (\mathcal{R}_q \otimes \mathbb{Z}[C_p])$$

where

$$h_1 = a_{\frac{n-1}{2}}\left(\sum_{i=0}^{\frac{n-3}{2}} 2\gamma_i + \gamma_{\frac{n-1}{2}}\right) + \sum_{i=\frac{n+1}{2}}^{n-2} a_i(\gamma_i - \gamma_{n-(i+1)})$$

$$h_2 = b_0\alpha_0 + \sum_{j=1}^{\frac{p-1}{2}} b_i(\alpha_i + \alpha_{p-i})$$

and $a_i, b_i \in \mathbb{Z}$.

Bibliography

[1] D. Barden, *Simply connected five-manifolds,* Annals of Mathematics 82 (1965) 365-385.

[2] C. Curtis and I. Reiner, *Methods of Representation Theory with Applications to Finite Groups and Orders, Volume 1* Wiley, (1981).

[3] J.G. Dumas, F. Heckenbach, B.D. Saunders, and V. Welker, *Computing Simplicial homology based on efficient Smith normal form algorithms* Algebra, Geometry, and Software Systems (2003) 177-207.

[4] A. Fröhlich, *Orthogonal and symplectic representations of groups,* Proc. London Mat Soc (3) 24 (1972) 470-505.

[5] F.E.A. Johnson, *Flat algebraic manifolds,* (in "Geometry of low-dimensional manifolds", Proceedings of the 1989 Durham Conference) LMS Lecture Notes vol. 150 73-91 CUP (1991).

[6] F.E.A. Johnson, *Stable modules and the D(2) problem,* LMS Lecture Notes in Mathematics, vol. 301, Cambridge University Press, Cambridge (2003)

[7] F.E.A. Johnson, *Syzygies and diagonal resolutions for dihedral groups*, Communications in Algebra 44 (2015) 2034-2047.

[8] F.E.A. Johnson, *Syzygies and Homotopy Theory*, Algebra and Applications, vol. 17 (2011), Springer.

[9] F.E.A. Johnson and J.J. Remez, *Diagonal resolutions for metacyclic groups,* J. Algebra 474 (2017) 329-360.

[10] F.E.A. Johnson and J.J. Remez, *Strongly diagonal resolutions for metacyclic groups,* (in preparation).

[11] H. Hasse, *Number Theory,* Springer-Verlag (1980).

[12] A. Hatcher, *Algebraic Topology* Cambridge University Press (2002).

[13] D. Hilbert, *Über die Theorie der Algebraischen Formen,* Cambridge Mathematical Library, Cambridge Mathematical Press, 6 (1993).

[14] I. Kaplansky *Fields and Rings* Chicago Lectures in Mathematics (2nd ed.), University Of Chicago Press (1972)

[15] S. Lefschetz, *Algebraic Topology* American Mathematical Society, Colloquium Publications vol. 27 (1942).

[16] I. Madsen, C.B. Thomas, C.T.C. Wall, *The topological spherical space form problem - II existence of free actions,* Topology 15 (1976) 375-382.

[17] J.W. Milnor, *Groups which operate on S^n without fixed points,* Am. J. Math. 79 (1957) 612-623.

[18] J. Munkres, *Elements of Algebriac Toplogy* Addison-Wesley (1984).

[19] A.J. Nadim, *A periodic monogenic resolution,* PhD Thesis, University College London (2015).

[20] M. Newman, *The Smith normal form* Linear Algebra and its Applications 254 (1997) 367-381.

[21] T. Petrie, *Free metacyclic group actions on homotopy spheres,* Annals of Mathematics 94 (1971) 108-124.

[22] J.J. Remez *Diagonal representations for the metacylic groups $G(pq)$,* PhD Thesis, University College London (2012).

[23] H. Seifert and W. Threllfall (trans. M. Goldman, Joan Birman), *A Textbook of Topology* Pure and Applied Mathematics, vol. 89, London: Academic Press Inc. (1980).

[24] The Stacks Project Authors, *The Stacks Project* https://stacks.math.columbia.edu (2018).

[25] C.T.C. Wall, *Classification of* $(n-1)$*-connected* $2n$*-manifolds,* Annals of Mathematics 75 (1962) 163-189.

[26] C.T.C. Wall, *Classification problems in differential topology VI: classification of* $(s-1)$*-connected* $(2s+1)$*-manifolds,* Topology 6 (1967) 271-296.

[27] C.T.C. Wall, *Poincaré complexes I,* Annals of Mathematics 86 (1967) 213-245.

[28] C.T.C. Wall, *On the axiomatic foundations of the theory of hermitian forms,* Proc. Camb. Phil. Soc. 67 (1970) 243-250.

[29] C.T.C. Wall, *On the classification of hermitian forms II semisimple rings,* Inventiones Math. 18 (1972) 119-141.

[30] C.T.C. Wall, *On the classification of hermitian forms VI group rings,* Annals of Mathematics 103 (1976) 1-80.