Syntomic regulators of Asai–Flach classes

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Abstract.

In this paper, we derive a formula for the p-adic syntomic regulators of Asai–Flach classes. These are cohomology classes forming an Euler system associated to a Hilbert modular form over a quadratic field, introduced in an earlier paper [LLZ16] by Antonio Lei and the first and third authors. The formula we develop here is expressed in terms of differential operators acting on overconvergent Hilbert modular forms; it is analogous to existing formulae for the regulators of Beilinson–Flach classes, but a novel feature is the appearance of a projection operator associated to a critical-slope Eisenstein series. We conclude the paper with numerical calculations giving strong evidence for the non-vanishing of these regulators in an explicit example.

§1. Introduction

1.1. Aims of the paper

Let F be a real quadratic field, and $\mathcal{F}$ a Hilbert modular newform over F, of level coprime to p and weights $\geqslant 2$. Associated to $\mathcal{F}$ is a 4-dimensional p-adic representation of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ (the *Asai Galois representation* of $\mathcal{F}$), which is the tensor induction of the (perhaps more familiar) 2-dimensional representation of $\mathrm{Gal}(\overline{\mathbf{Q}}/F)$ associated to $\mathcal{F}$. The preceding paper [LLZ16], by Antonio Lei and the first and third authors, defines a collection of Galois cohomology classes (*étale Asai–Flach classes*) for the Asai Galois representation, and proves that these form an Euler system; however, the question of whether this Euler system is non-trivial remains open in general.

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It is certainly expected that the étale Asai–Flach classes should be non-zero, for the following reason. These classes are the étale realisations of classes in motivic cohomology, whose images in Deligne–Beilinson cohomology are known to be related to leading terms of complex L-functions as predicted by Beilinson's conjecture (as shown in [Kin98] and [LLZ16, Theorem 5.4.8]). Perrin-Riou's p-adic extension of Beilinson's conjecture [PR95, §4.2] thus predicts that the images of these classes under the p-adic syntomic regulator should be related to p-adic L-functions; and since there is a direct comparison between the syntomic and étale regulators via the Bloch–Kato exponential map, this should give non-vanishing of the étale classes.

In this work, we give some evidence towards a non-vanishing result of this form, assuming that p splits in F. We express the pairing between the Bloch–Kato logarithm of the étale Asai–Flach class $\mathrm{AF}^{[\mathcal{F},j]}_{\text{ét}}$ and the differential associated to $\mathcal{F}$ using the theory of overconvergent modular forms. Our result is somewhat analogous to the formulae of [BDR15, KLZ15] in the setting of Rankin–Selberg convolutions, although there are important differences, such as the lack of any immediate connection to p-adic L-functions. We use this to give very strong numerical evidence (although a little less than a fully rigorous proof) for the non-vanishing of the 3-adic Asai–Flach classes for an explicit example of a Hilbert modular eigenform over $\mathbf{Q}(\sqrt{13})$.

1.2. Statement of results

We now state our results slightly more formally. Our first main result does not involve Hilbert modular forms at all, but is a result about the Eisenstein classes for $\mathrm{GL}_2/\mathbf{Q}$. Let $N \geqslant 1$ be coprime to p, and let L be a p-adic field containing the N-th roots of unity. For $k \in \mathbf{Z}$, denote by $S^\dagger_k(N, L)$ the space of overconvergent cusp forms of weight k with q-expansion coefficients in L.

Theorem A. *Let $k \geqslant 0$, and let $\chi : (\mathbf{Z}/N\mathbf{Z})^\times \to L^\times$ be a Dirichlet character modulo N with $\chi(-1) = (-1)^k$. If $k = 0$, assume χ is not trivial. Let Θ be Coleman's differential operator $S^\dagger_{-k}(N, L) \to S^\dagger_{k+2}(N, L)$ (acting as $\left(q\frac{\mathrm{d}}{\mathrm{d}q}\right)^{k+1}$ on q-expansions).*

Define the critical-slope Eisenstein quotient *as the unique 1-dimensional quotient of the space*

$$S^\dagger_{k+2}(N, L)/\Theta\left(S^\dagger_{-k}(N, L)\right)$$

on which the Hecke operators $T(\ell) - 1 - \ell^{k+1}\chi(\ell)$ (for $\ell \nmid Np$), $U(\ell) - 1$ (for $\ell \mid N$), and $U(p) - p^{k+1}\chi(p)$ act as 0.

If we identify $S^\dagger_{k+2}(N,L)/\Theta\left(S^\dagger_{-k}(N,L)\right)$ *with a rigid cohomology group as in Equation* (1c) *below, then the linear functional given by pairing with the* χ*-isotypical part of the weight* $k+2$ *Eisenstein class factors through this quotient, and maps it isomorphically to* L*. Moreover, this linear functional maps the critical-slope Eisenstein eigenform* $E^{(k+2)}_{\mathrm{crit},\chi} \in S^\dagger_{k+2}(N,L)$ *to an explicit product of* p*-adic Dirichlet* L*-values.*

Now let F be a real quadratic field, with $p = \mathfrak{p}_1\mathfrak{p}_2$ split in F as before; let σ_i be the embedding $F \hookrightarrow L$ corresponding to the prime $\mathfrak{p}_i$. Let $\mathcal{F}$ be a Hilbert modular eigenform, of level $U_1(\mathfrak{N})$ for some $\mathfrak{N}$ coprime to p. Choose an embedding of the coefficient field of $\mathcal{F}$ into L, and suppose that $\mathcal{F}$ has weights (k_1+2, k_2+2) at the embeddings σ_1, σ_2 respectively, where $k_i \geqslant 0$. We write $\mathcal{F}^{[\mathfrak{p}_1,\mathfrak{p}_2]}$ for the form obtained from $\mathcal{F}$ by setting to 0 all Fourier–Whittaker coefficients $c(\mathfrak{m},\mathcal{F})$ with $\mathfrak{m}$ divisible by one or both of the $\mathfrak{p}_i$. This is an element of the space $S^\dagger_{(k_1+2,k_2+2)}(\mathfrak{N},L)$ of overconvergent Hilbert modular forms of tame level $\mathfrak{N}$ and weight (k_1+2,k_2+2).

Theorem B. *The form* $\mathcal{F}^{[\mathfrak{p}_1,\mathfrak{p}_2]}$ *is in the image of the differential operator*

$$\Theta_1 : S^\dagger_{(-k_1,k_2+2)}(\mathfrak{N},L) \hookrightarrow S^\dagger_{(k_1+2,k_2+2)}(\mathfrak{N},L),$$

acting on q*-expansions as* $\left(q_1 \frac{\partial}{\partial q_1}\right)^{k_1+1}$*; and for any integer* $0 \leqslant j \leqslant \min(k_1,k_2)$*, we have the formula*

$$\left\langle \log\left(\mathrm{AF}^{[\mathcal{F},j]}_{\text{ét}}\right), \omega_{\mathcal{F}} \right\rangle = (*) \cdot \lambda_{\mathrm{Eis}}\left(\left[\Theta_1^{-1}\left(\mathcal{F}^{[\mathfrak{p}_1,\mathfrak{p}_2]}\right)\right]_{k_1-j}\right),$$

where log *is the Bloch–Kato logarithm,* $(*)$ *is an explicit non-zero constant,*

$$\left[\Theta_1^{-1}\left(\mathcal{F}^{[\mathfrak{p}_1,\mathfrak{p}_2]}\right)\right]_{k_1-j} \in S^\dagger_{k_1+k_2-2j+2}(N,L)$$

is the Rankin–Cohen bracket, and λ_{Eis} *denotes the linear functional defined by pairing with the level* N *Eisenstein class. In particular, if the projection of* $\left[\Theta_1^{-1}\left(\mathcal{F}^{[\mathfrak{p}_1,\mathfrak{p}_2]}\right)\right]_{k_1-j}$ *to the critical-slope Eisenstein quotient is non-zero, then the class* $\mathrm{AF}^{[\mathcal{F},j]}_{\text{ét}}$ *does not vanish.*

See Theorem 5.3.9 below for a precise statement, and an explicit formula for the constant $(*)$. Assuming a certain hypothesis regarding the rate of convergence of various power series, we have computed explicitly this projection for an example with $p=3$ and $F = \mathbf{Q}(\sqrt{13})$, and verified that the critical-slope projection is indeed non-zero.

1.3. Relations to other work

The Asai–Flach classes in the cohomology of a Hilbert modular surface can be regarded as a "degenerate case" of diagonal cycles on the product of a Hilbert surface and an elliptic curve. Since the initial release of this paper in preprint form, analogues of our regulator formula in this diagonal-cycle case have been announced by Blanco-Chacón and Sols [BCS17] and by Fornea [For17]; there is a substantial overlap between their computations and ours.

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§2. Preliminaries on elliptic modular forms

We start by recalling some facts about elliptic modular forms and their p-adic analogues.

2.1. Nearly holomorphic modular forms

Let $\mathcal{H}$ be the upper half-plane. Recall (cf. [Urb14, §2.1.1]) that a C^∞ function $f : \mathcal{H} \to \mathbf{C}$ is said to be a *nearly-holomorphic modular form* of level N, weight r and degree $\leqslant n$ if:

- The function f transforms like a modular form of weight r under $\Gamma_1(N)$.
- The absolute value $|f(\gamma\tau)|$ is bounded as $\operatorname{Im}\tau \to \infty$, for every $\gamma \in \mathrm{GL}_2^+(\mathbf{Q})$.
- The function f can be written in the form

$$\sum_{j=0}^{n} f_j(\tau)(\operatorname{Im}\tau)^{-j}$$

where f_j are holomorphic functions.

We write $M_r^{\leqslant n}(N, \mathbf{C})$ for the space of such functions.

Definition 2.1.1. *We say $f \in M_r^{\leqslant n}(N, \mathbf{C})$ is* strongly cuspidal *if all the f_j vanish at ∞, and the same holds with f replaced by $f \mid_r \gamma$ for* *any $\gamma \in \mathrm{SL}_2(\mathbf{Z})$. We write $\mathbf{S}_r^{\leqslant n}(N, \mathbf{C})$ for the space of strongly cuspidal forms.*

The Maass–Shimura differential operator $\delta := \frac{1}{2\pi i}\left(\frac{\mathrm{d}}{\mathrm{d}\tau} + \frac{r}{2i\operatorname{Im}(\tau)}\right)$ gives maps

$$M_r^{\leqslant n}(N, \mathbf{C}) \to M_{r+2}^{\leqslant n+1}(N, \mathbf{C}), \qquad \mathbf{S}_r^{\leqslant n}(N, \mathbf{C}) \to \mathbf{S}_{r+2}^{\leqslant n+1}(N, \mathbf{C}).$$

Shimura has shown that if $r > 2n$ the inclusion $M_r(N, \mathbf{C}) \hookrightarrow M_r^{\leqslant n}(N, \mathbf{C})$ has a left inverse, the "holomorphic projection" map Π^{hol}, characterised by the condition that $\Pi^{\mathrm{hol}}(\delta^j f) = 0$ for all $j \in \{1, \dots, n\}$ and all holomorphic modular forms $f \in M_{r-2j}(N)$. This map clearly sends $\mathbf{S}_r^{\leqslant n}(N)$ to $S_r(N)$.

2.2. Geometric interpretation

Let $\mathscr{H}^{(r)}$ denote[1] the r-th symmetric power of the first relative de Rham cohomology sheaf of the universal elliptic curve $\mathcal{E}/Y_1(N)$, extended to a vector bundle on $X_1(N)$ as in [Urb14, §2.2.1]. The n-th power of the Hodge line bundle ω^r embeds naturally in $\mathscr{H}^{(r)}$, and one has

$$\begin{aligned} M_r^{\leqslant n}(N, \mathbf{C}) &:= H^0\left(X_1(N)_{\mathbf{C}}, \mathscr{H}^{(n)} \otimes \omega^{r-n}\right), \\ \mathbf{S}_r^{\leqslant n}(N, \mathbf{C}) &:= H^0\left(X_1(N)_{\mathbf{C}}, \mathscr{H}^{(n)} \otimes \omega^{r-n}(-C)\right), \end{aligned}$$

where C is the divisor of cusps. (The first formula is [Urb14, Proposition 1], and the second is proved similarly.) We can use this to define $M_r^{\leqslant n}(N, L)$ and $\mathbf{S}_r^{\leqslant n}(N, L)$ for any coefficient field L of characteristic 0 containing the N-th roots of unity.[2]

Remark 2.2.1. Note that the space $S_r^{\mathrm{nh}}(N, \mathbf{C})$ of nearly-holomorphic cusp forms defined in [DR14, §2.3], for $r \geqslant 2$, is a subspace of our space $M_r^{\leqslant (r-2)}(N, \mathbf{C})$, but it is not the same as our space $\mathbf{S}_r^{\leqslant (r-2)}(N, \mathbf{C})$ of strongly cuspidal forms. Darmon and Rotger work with a certain "parabolic" sheaf $(\mathscr{H}^{(k)} \otimes \Omega^1_{X_1(N)})_{\mathrm{par}}$, intermediate between $\mathscr{H}^{(r-2)} \otimes \omega^2$ and $\mathscr{H}^{(r-2)} \otimes \Omega^1 = \mathscr{H}^{(r-2)} \otimes \omega^2(-C)$. In terms of functions on $\mathcal{H}$, this

[1] For consistency with our previous works, we have used square brackets for "homological" objects, and round brackets for "cohomological" ones. Thus $\mathscr{H}^{[r]}$ is the dual of $\mathscr{H}^{(r)}$. The sheaves $\mathscr{H}^{(r)}$ and $\mathscr{H}^{[r]}$ are actually isomorphic to each other, but their filtrations and their natural Hecke actions are not the same, so we shall not treat this isomorphism as an identification.

[2] This is in order to avoid problems with the non-rationality of the cusp ∞ in the standard $\mathbf{Q}$-model of $Y_1(N)$.

corresponds to requiring that $f_0(\infty) = 0$ but with no condition on the higher f_j (and similarly at the other cusps of $\Gamma_1(N)$).

2.3. Nearly overconvergent forms

Let $p \nmid N$ be prime, and let L be a finite extension of $\mathbf{Q}_p$, again containing the N-th roots of unity. Following [Urb14, §3.2.1], we make the following definitions:

Definition 2.3.1. *Let $r \in \mathbf{Z}$ and $n \in \mathbf{Z}_{\geqslant 0}$. We define the space of nearly-overconvergent p-adic modular forms of degree $\leqslant n$, and its subspace of strongly cuspidal forms, by*

$$M_r^{\dagger,\leqslant n}(N, L) \coloneqq H^0\left(X_1(N)^{\mathrm{rig}}, j^\dagger\left(\mathscr{H}^{(n)} \otimes \omega^{r-n}\right)\right),$$
$$\mathbf{S}_r^{\dagger,\leqslant n}(N, L) \coloneqq H^0\left(X_1(N)^{\mathrm{rig}}, j^\dagger\left(\mathscr{H}^{(n)} \otimes \omega^{r-n}(-C)\right)\right)$$

where $X_1(N)^{\mathrm{rig}}$ is the rigid space over L associated to $X_1(N)$, C is the divisor of cusps, and j is the inclusion of the ordinary locus $X_1(N)^{\mathrm{ord}}$ into $X_1(N)^{\mathrm{rig}}$.

For $n = 0$ these are the familiar spaces $M_r^\dagger(N, L)$ and $S_r^\dagger(N, L)$ of overconvergent modular (resp. cusp) forms with q-expansion coefficients in L. We shall often omit the coefficient field L or the tame level N (or both) from the notation if these are clear from context.

Over the ordinary locus $X_1(N)^{\mathrm{ord}}$, the inclusion $\omega \hookrightarrow \mathscr{H}$ admits a left inverse, the "unit-root splitting", giving an isomorphism $\mathscr{H} \cong \omega \oplus \omega^{-1}$. By [Urb14, Proposition 6], composing the unit-root splitting with restriction to the ordinary locus gives an *injective* map $M_r^{\dagger,\leqslant n}(N, L) \hookrightarrow H^0\left(X_1(N)^{\mathrm{ord}}, \omega^r\right)$ (the space of p-adic modular forms); in particular, we may describe nearly-overconvergent p-adic modular forms by q-expansions in $L[[q]]$.

The differential operator δ is defined on the spaces $M_r^{\dagger,\leqslant n}$ and $\mathbf{S}_r^{\dagger,\leqslant n}$, and we have the following crucial fact (cf. [Urb14, §3]):

Proposition 2.3.2. *For any integer $k \geqslant 0$ the operator*

$$\delta^{k+1} : M_{-k}^\dagger \to M_{k+2}^{\dagger,\leqslant(k+1)}$$

has image contained in $S_{k+2}^\dagger$, and it coincides with Coleman's differential operator Θ, acting as $\left(q\frac{\mathrm{d}}{\mathrm{d}q}\right)^{k+1}$ on q-expansions. *Q.E.D.*

2.4. Rigid cohomology

Let $k \geqslant 0$ be an integer, and K any field of characteristic 0. We have the following general sheaf-theoretic fact:

Proposition 2.4.1. *Define complexes of sheaves on $X_1(N)_K$ by*

$$\begin{aligned}
\mathrm{DR}^\bullet\left(\mathscr{H}^{(k)}\right) &= \left[\mathscr{H}^{(k)} \xrightarrow{\nabla} \mathscr{H}^{(k)} \otimes \Omega^1(C)\right], \\
\mathrm{DR}^\bullet_c\left(\mathscr{H}^{(k)}\right) &= \left[\mathscr{H}^{(k)}(-C) \xrightarrow{\nabla} \mathscr{H}^{(k)} \otimes \Omega^1,\right], \\
\mathrm{DR}^\bullet_{\mathrm{par}}\left(\mathscr{H}^{(k)}\right) &= \left[\mathscr{H}^{(k)} \xrightarrow{\nabla} (\mathscr{H}^{(k)} \otimes \Omega^1)_{\mathrm{par}}\right], \\
\mathrm{BGG}^\bullet\left(\mathscr{H}^{(k)}\right) &= \left[\omega^{-k} \xrightarrow{\Theta} \omega^{k+2}\right], \\
\mathrm{BGG}^\bullet_c\left(\mathscr{H}^{(k)}\right) &= \left[\omega^{-k}(-C) \xrightarrow{\Theta} \omega^{k+2}(-C)\right], \\
\mathrm{BGG}^\bullet_{\mathrm{par}}\left(\mathscr{H}^{(k)}\right) &= \left[\omega^{-k} \xrightarrow{\Theta} \omega^{k+2}(-C)\right].
\end{aligned}$$

For $? \in \{\varnothing, c, \mathrm{par}\}$*, there are maps of complexes* $\mathrm{BGG}^\bullet_? \to \mathrm{DR}^\bullet_?$ *which are quasi-isomorphisms.* *Q.E.D.*

Here $(\mathscr{H}^{(k)} \otimes \Omega^1)_{\mathrm{par}}$ is the subsheaf of $\mathscr{H}^{(k)} \otimes \Omega^1(C)$ mentioned in Remark 2.2.1 above, ∇ denotes the Gauss–Manin connection, and Θ the differential operator of Proposition 2.3.2. The map $\mathrm{BGG}^\bullet\left(\mathscr{H}^{(k)}\right) \to \mathrm{DR}^\bullet(\mathscr{H}^{(k)})$ is the natural inclusion in degree 1, and in degree 0 it is characterised by the fact that its composite with the natural map $\mathscr{H}^{(k)} \to \omega^{-k}$ is multiplication by $(-1)^k k!$; the other maps are characterised similarly.

Remark 2.4.2. Note that this construction is purely algebraic, and does not use the splitting $\mathscr{H} \cong \omega \oplus \omega^{-1}$ of rigid-analytic sheaves over $X_1(N)^{\mathrm{ord}}$.

If we let $K = \mathbf{Q}_p$, where $p \nmid N$, and let j denote the inclusion of the ordinary locus $X_1(N)^{\mathrm{ord}}$ in $X_1(N)^{\mathrm{rig}}$ as above, then the hypercohomology groups $\mathbb{H}^*\left(X_1(N)^{\mathrm{rig}}, j^\dagger \mathrm{DR}^\bullet_?\left(\mathscr{H}^{(k)}\right)\right)$ compute various flavours of rigid cohomology of the special fibre (with coefficients in $\mathscr{H}^{(k)}$). The hypercohomology groups of $j^\dagger\mathrm{DR}^\bullet\left(\mathscr{H}^{(k)}\right)$ and $j^\dagger\mathrm{DR}^\bullet_{\mathrm{par}}\left(\mathscr{H}^{(k)}\right)$ compute the rigid cohomology of the mod p varieties $\bar{Y}_1(N)^{\mathrm{ord}}$ and $\bar{X}_1(N)^{\mathrm{ord}}$ respectively. As in [HLTT16, §6.5], we interpret the hypercohomology of $j^\dagger\mathrm{DR}^\bullet_c$ as "rigid cohomology of $\bar{Y}_1(N)^{\mathrm{ord}}$ with compact supports towards the cusps" (but not towards the supersingular locus), and we denote it by $H^1_{\mathrm{rig},c-\partial}\left(\bar{Y}_1(N)^{\mathrm{ord}}, \mathscr{H}^{(k)}\right)$.

Combining this with the quasi-isomorphisms of Proposition 2.4.1, and the fact that $X_1(N)^{\mathrm{ord}}$ is affinoid (so all higher sheaf cohomology groups vanish), we obtain presentations in terms of overconvergent modular forms for these three rigid cohomology groups. More precisely, for

L a finite extension of $\mathbf{Q}_p$ containing the N-th roots of unity, this construction gives isomorphisms

$$H^1_{\mathrm{rig}}\left(\bar{Y}_1(N)^{\mathrm{ord}}, \mathscr{H}^{(k)}\right) \otimes_{\mathbf{Q}_p} L \cong \frac{M^\dagger_{k+2}(N,L)}{\Theta\left(M^\dagger_{-k}(N,L)\right)}, \tag{1a}$$

$$H^1_{\mathrm{rig}}\left(\bar{X}_1(N)^{\mathrm{ord}}, \mathscr{H}^{(k)}\right) \otimes_{\mathbf{Q}_p} L \cong \frac{S^\dagger_{k+2}(N,L)}{\Theta\left(M^\dagger_{-k}(N,L)\right)}, \tag{1b}$$

$$H^1_{\mathrm{rig},c-\partial}\left(\bar{Y}_1(N)^{\mathrm{ord}}, \mathscr{H}^{(k)}\right) \otimes_{\mathbf{Q}_p} L \cong \frac{S^\dagger_{k+2}(N,L)}{\Theta\left(S^\dagger_{-k}(N,L)\right)}. \tag{1c}$$

All three isomorphisms are clearly compatible with Hecke operators away from p, and the action of the p-power Frobenius map φ on the rigid cohomology corresponds to the operator $p^{k+1}\langle p\rangle V_p$ on $M^\dagger_{k+2}(N,L)$, where V_p acts on q-expansions as $q \mapsto q^p$. (The operator $\langle p\rangle$ appears because the cusp ∞ is not rational in our model of $Y_1(N)$; see [KLZ15, §6.1].)

We can also consider the cohomology with compact supports towards the supersingular points (but not the cusps), which we denote $H^\bullet_{\mathrm{rig},c-ss}\left(\bar{Y}_1(N)^{\mathrm{ord}}, -\right)$. This is dual to the preceding theory; although such dualities are presumably well-known to the experts, we have not been able to find a reference for rigid cohomology with partial compact support, so we shall briefly sketch the proof. For brevity, in this discussion we write X, Y for $X_1(N)$ and $Y_1(N)$, and omit the subscript "rig" from cohomology groups. Let $\mathscr{H}^{[k]}$ denote the dual of $\mathscr{H}^{(k)}$ (so that $\mathscr{H}^{[k]} \cong \mathscr{H}^{(k)}$ as isocrystals, but the filtration and Frobenius actions are shifted). If $\Omega^\bullet_c = \left[\mathcal{O}(-C) \to \Omega^1\right]$ is the de Rham complex of X^{rig}, with trivial coefficients, relative to the cuspidal divisor C, then we have pairings of complexes

$$\mathrm{DR}^\bullet_c(\mathscr{H}^{(k)}) \otimes \mathrm{DR}^\bullet(\mathscr{H}^{[k]}) \to \Omega^\bullet_c.$$

Let $R\underline{\Gamma}_{\mathrm{ord}}$ denote the derived functor of sections with support in X^{ord}, as defined in [LS07, §5.2]. By Corollary 5.3.6 of *op.cit.* we obtain a pairing in the derived category of abelian sheaves on X^{rig},

$$j^\dagger\left(\mathrm{DR}^\bullet_c(\mathscr{H}^{(k)})\right) \otimes R\underline{\Gamma}_{\mathrm{ord}}\left(\mathrm{DR}^\bullet(\mathscr{H}^{[k]})\right) \to R\underline{\Gamma}_{\mathrm{ord}}\left(\Omega^\bullet_c\right).$$

Since $\mathbb{H}^2\left(X^{\mathrm{rig}}, R\underline{\Gamma}_{\mathrm{ord}}\left(\Omega^\bullet_c\right)\right) = H^2_c\left(\bar{Y}^{\mathrm{ord}}\right) = \mathbf{Q}_p(-1)$, this gives a bilinear pairing

$$\langle -,-\rangle : H^1_{c-ss}\left(\bar{Y}^{\mathrm{ord}}, \mathscr{H}^{[k]}(1)\right) \times H^1_{c-\partial}\left(\bar{Y}^{\mathrm{ord}}, \mathscr{H}^{(k)}\right) \to \mathbf{Q}_p, \tag{2}$$

compatible with the action of Frobenius. Moreover, if $\bar{Z} = \bar{Y}^{\mathrm{ord}} - \bar{Y}$ is the 0-dimensional scheme of supersingular points in characteristic p, there are long exact excision sequences (cf. [LS07, Proposition 8.2.18]):

$$\begin{array}{ccccccccc} H^0(\bar{Y}, \mathscr{H}^{[k]}(1)) & \longrightarrow & H^0(\bar{Z}) & \longrightarrow & H^1_{c-ss}(\bar{Y}^{\mathrm{ord}}) & \longrightarrow & H^1(\bar{Y}) & \longrightarrow & 0, \\ H^2_c(\bar{Y}, \mathscr{H}^{(k)}) & \longleftarrow & H^2_{\bar{Z}}(\bar{Y}) & \longleftarrow & H^1_{c-\partial}(\bar{Y}^{\mathrm{ord}}) & \longleftarrow & H^1_c(\bar{Y}) & \longleftarrow & 0. \end{array}$$

(omitting coefficients except in the first column, for reasons of space). There are pairings between each term in the top sequence and the corresponding term in the bottom sequence, compatible with the horizontal maps (compare [Ked06, eq. 9.3.2.1]). By Poincaré duality for rigid cohomology with coefficients [Ked06, Theorem 1.2.3], all of these pairings except the middle one are known to be perfect. Hence the middle pairing, which is (2), must be a perfect duality also.

Remark 2.4.3. (a) Presentations of rigid cohomology similar to (1) are fundamental in the p-adic regulator computations of [DR14] and [KLZ15]. However, unlike these previous works, in the present paper we shall project to an *Eisenstein* eigenspace in the rigid cohomology, rather than a cuspidal one; so it is important to distinguish carefully between the three slightly different cohomology spaces (1a)–(1c). Our account is based on the description of the theory for Hilbert modular forms given in [TX16]. (There is a minor error in [DR14] at this point – it is claimed in equation (2.30) of *op.cit.* that the quotient $\frac{S^\dagger_{k+2}(N,L)}{\Theta\left(S^\dagger_{-k}(N,L)\right)}$ computes parabolic cohomology.)

(b) The cohomology groups $H^\bullet_{\mathrm{rig},c-ss}$ do not seem to have a direct description in terms of overconvergent modular forms, unlike the groups $H^\bullet_{\mathrm{rig},c-\partial}$. They can be interpreted as the cohomology of the mapping fibre of restriction to the "infinitesimal boundary" of the supersingular residue discs.

2.5. Overconvergent projection operators

There exist two slightly different generalisations of the holomorphic projection operator to nearly-overconvergent modular forms.

Urban's overconvergent projector: In [Urb14, §3.3.4], Urban shows that whenever $r \notin \{2, 3, \dots, 2n\}$ there is an isomorphism

$$M_r^{\dagger, \leqslant n} = \bigoplus_{j=0}^{n} \delta^j \left(M^\dagger_{r-2j}\right)$$

and hence there is a unique projection map (denoted by $\mathcal{H}^\dagger$ in *op.cit.*) onto $M_r^\dagger$, characterised by vanishing on the subspace $\bigoplus_{j=1}^n \delta^j(M_{r-2j}^\dagger)$. This map evidently sends $\mathbf{S}_r^{\dagger,\leqslant n}$ to $S_r^\dagger$.

Darmon and Rotger's overconvergent projector: For $r = k+2 \geqslant 2$, Darmon and Rotger have defined a space $S_{k+2}^{\mathrm{n-oc}}$ intermediate between our spaces $M_{k+2}^{\dagger,\leqslant k}$ and $\mathbf{S}_{k+2}^{\dagger,\leqslant k}$ (see [DR14, Definition 2.4]), and a map (denoted by Π^{oc} in *op.cit.*)

$$S_{k+2}^{\mathrm{n-oc}} \to S_{k+2}^\dagger/\Theta\left(M_{-k}^\dagger\right).$$

This map is defined as follows: $S_{k+2}^{\mathrm{n-oc}}$ is the overconvergent sections of $(\mathscr{H}^{(k)} \otimes \Omega^1)_{\mathrm{par}}$, and Π^{oc} sends such a section f to the element of $S_{k+2}^\dagger/\Theta\left(M_{-k}^\dagger\right)$ representing the class of f in $H^1_{\mathrm{rig}}\left(\bar{X}_1(N)^{\mathrm{ord}}, \mathscr{H}^{(k)}\right)$. Note that this is only well-defined modulo $\Theta\left(M_{-k}^\dagger\right)$, rather than modulo $\Theta\left(S_{-k}^\dagger\right)$ as claimed in *op.cit.*, because of the error in (2.30) of *op.cit.* mentioned above. However, the same construction with $\mathscr{H}^{(k)} \otimes \Omega^1$ in place of $(\mathscr{H}^{(k)} \otimes \Omega^1)_{\mathrm{par}}$ does give a well-defined map

$$\mathbf{S}_{k+2}^{\dagger,\leqslant k} \to S_{k+2}^\dagger/\Theta\left(S_{-k}^\dagger\right),$$

which we denote by the same symbol Π^{oc}. This map is characterised by vanishing on $\delta(\mathbf{S}_k^{\leqslant k})$; in particular, if n is small enough that Urban's projector is defined on $\mathbf{S}_{k+2}^{\dagger,\leqslant n}$, then the restriction of Π^{oc} to this subspace coincides with the image of Urban's projector in the quotient.

§3. A "compactification" of the GL_2 Eisenstein class

We begin with some computations relating to the Eisenstein classes for $\mathrm{GL}_2/\mathbf{Q}$; our goal is to understand the linear functional defined by pairing with the Eisenstein class in terms of the presentations of rigid cohomology given in Equation (1). We fix an integer $N \geqslant 4$, and abbreviate $Y_1(N)$ simply by Y.

3.1. The Eisenstein class of level N

Let $k \geqslant 0$. We write $\mathrm{Eis}^k_{\mathrm{rig},N} \in H^1_{\mathrm{rig}}\left(\bar{Y}, \mathscr{H}^{[k]}(1)\right)^{\varphi=1}$ for the rigid realisation of the level N Eisenstein class, as in [KLZ15, §4.2].

Definition 3.1.1. *The* Eisenstein subspace $H^1_{\mathrm{rig}}\left(\bar{Y}, \mathscr{H}^{[k]}(1)\right)^{\mathrm{Eis}}$ *of* $H^1_{\mathrm{rig}}\left(\bar{Y}, \mathscr{H}^{[k]}(1)\right)$ *is defined as the maximal subspace on which all of the*

Hecke operators $U'(\ell) - 1$ (for $\ell \mid N$) and $T'(\ell) - 1 - \ell^{k+1}\langle \ell^{-1}\rangle$ (for $\ell \nmid Np$) act as zero.

We have $\mathrm{Eis}^k_{\mathrm{rig},N} \in H^1_{\mathrm{rig}}\left(\bar{Y}, \mathscr{H}^{[k]}(1)\right)^{\mathrm{Eis}}$, by [KLZ17, Remark 4.3.5], hence our choice of terminology.

Proposition 3.1.2. *Let $L/\mathbf{Q}_p$ be a finite extension and let $\chi : (\mathbf{Z}/N\mathbf{Z})^\times \to L^\times$ be a Dirichlet character such that $\chi(-1) = (-1)^k$. Then the intersection of $H^1_{\mathrm{rig}}\left(\bar{Y}, \mathscr{H}^{[k]}(1)\right)^{\mathrm{Eis}} \otimes L$ with the χ-eigenspace for the diamond operators is 1-dimensional over L, unless $k = 0$ and χ is the trivial character, in which case it is zero.*

Proof. By the standard comparison theorems, it suffices to prove the corresponding statement with rigid cohomology over L replaced by de Rham cohomology over $\mathbf{C}$. Moreover, by applying the Atkin–Lehner involution W_N we may interchange the Hecke operators $T'(\ell)$ and $U'(\ell)$ with their more familiar cousins $U(\ell)$ and $T(\ell)$. Since the Eichler–Shimura isomorphism furnishes an isomorphism of Hecke modules $H^1_{\mathrm{dR}}\left(Y_1(N)_{\mathbf{C}}, \mathscr{H}^{[k]}\right) \cong M_{k+2}(N, \mathbf{C}) \oplus \overline{S_{k+2}(N, \mathbf{C})}$, we are reduced to checking that the eigenspace in $M_{k+2}(N, \mathbf{C})$ on which the diamond operators act by χ, $U(\ell) = 1$ for $\ell \mid N$, and $T(\ell) = 1 + \ell^{k+1}\chi(\ell)$ for $\ell \nmid N$ has dimension 1 (or 0 if $k = 2$ and χ is trivial). This follows easily from standard results on the Hecke module structure of Eisenstein series, cf. [DS05, Theorems 4.5.2 and 4.6.2]. Q.E.D.

Remark 3.1.3. From the explicit formula given in [KLZ15, §4.3] for the de Rham realisation of the Eisenstein class, one sees that if χ is a Dirichlet character mod N of the appropriate sign, then the projection of $\mathrm{Eis}^k_{\mathrm{rig},N}$ to the χ-eigenspace in $H^1_{\mathrm{rig}}\left(\bar{Y}, \mathscr{H}^{[k]}(1)\right)^{\mathrm{Eis}} \otimes \overline{\mathbf{Q}}_p$ is non-zero.[3] Hence it must be a basis vector of this space, by the preceding proposition.

3.2. Lifting to compact supports

We define the Eisenstein subspace of $H^1_{\mathrm{rig},c-ss}\left(\bar{Y}^{\mathrm{ord}}, \mathscr{H}^{[k]}(1)\right)$ in the same way as for $H^1_{\mathrm{rig}}\left(\bar{Y}, \mathscr{H}^{[k]}(1)\right)$ above (requiring that the same Hecke operators act as zero).

[3] In more detail: this projection is represented by a modular form F such that $a_n(F)$ is given, up to an explicit non-zero constant, by $\sum_{d \mid n} (n/d)^{k+1} G(\chi, \zeta_N^d)$, where $G(\chi, \zeta_N^d) := \sum_{b \in (\mathbf{Z}/N\mathbf{Z})^\times} \chi(b)\zeta_N^{bd}$. Since $\mathbf{Q}(\zeta_N)$ is isomorphic as a Galois module to the regular representation of $(\mathbf{Z}/N\mathbf{Z})^\times$, the sum $G(\chi, \zeta_N^d)$ must be non-zero for some d; and if n is equal to the least $d \geqslant 1$ for which this holds, then $a_n(F)$ is non-zero.

Lemma 3.2.1. *The natural map*

$$H^1_{\mathrm{rig},c-ss}\left(\bar{Y}^{\mathrm{ord}}, \mathscr{H}^{[k]}(1)\right) \to H^1_{\mathrm{rig}}\left(\bar{Y}, \mathscr{H}^{[k]}(1)\right)$$

restricts to an isomorphism between the Eisenstein subspaces.

Proof. Let $\bar{Z}$ be the subscheme of supersingular points in $\bar{Y}$. The Gysin sequence for rigid cohomology gives us a long exact sequence

$$0 \to \frac{H^0_{\mathrm{rig}}\left(\bar{Z}, \mathscr{H}^{[k]}(1)\right)}{\operatorname{image} H^0_{\mathrm{rig}}\left(\bar{Y}, \mathscr{H}^{[k]}(1)\right)} \to H^1_{\mathrm{rig},c-ss}\left(\bar{Y}^{\mathrm{ord}}, \mathscr{H}^{[k]}(1)\right) \\ \to H^1_{\mathrm{rig}}\left(\bar{Y}, \mathscr{H}^{[k]}(1)\right) \to 0$$

where the exactness at the right-hand end is a consequence of the fact that $\bar{Z}$ is zero-dimensional, so its H^1_{rig} vanishes. So it suffices to show that the given system of eigenvalues cannot occur in the the first group in the above sequence.

The systems of Hecke eigenvalues appearing in $H^0_{\mathrm{rig}}\left(\bar{Z}, \mathscr{H}^{[k]}(1)\right)$ are well-understood. By a theorem of Deuring and Serre [Ser96], there is a bijection of finite sets

$$\bar{Z}(\overline{\mathbf{F}}_p) \cong B^\times \backslash (B \otimes \mathbf{A}_f)^\times / U_{B,1}(N),$$

where B is the quaternion algebra ramified at $\{p, \infty\}$, and $U_{B,1}(N) \subset (B \otimes \mathbf{A}_f)^\times$ is a suitable level group. This bijection is compatible with Hecke operators away from p, and induces an isomorphism of prime-to-p Hecke modules between $H^0_{\mathrm{rig}}\left(\bar{Z}, \mathscr{H}^{[k]}(1)\right) \otimes_{\mathbf{Q}_p} L$ and the space of automorphic forms for $B^\times$ of level $U_{B,1}(N)$ and weight the representation $\mathrm{Sym}^k(L^2)$ of $(B \otimes L)^\times \cong \mathrm{GL}_2(L)$, for any extension $L/\mathbf{Q}_p$ splitting B. (This last statement is not proved in *op.cit.*, but a proof can be extracted as a special case of the much more general results on Hilbert modular varieties proved in [TX16].)

In turn, this space of automorphic forms for $B^\times$ is related to classical modular forms via the Jacquet–Langlands correspondence. This gives the following concrete description of the systems of prime-to-p Hecke eigenspaces appearing in $H^0_{\mathrm{rig}}\left(\bar{Z}, \mathscr{H}^{[k]}(1)\right)$:

- If $k = 0$ there is a 1-dimensional subspace generated by the constant function, which is exactly the image of $H^0_{\mathrm{rig}}(\bar{Y}, \mathbf{Q}_p(1))$; if $k \geqslant 1$, this image is zero.
- In either case, the remaining eigenspaces correspond to the Hecke eigenvalues of cusp forms of level $\Gamma_1(N) \cap \Gamma_0(p)$ which are new at p.

Hence the Hecke operators $T'(\ell)$ cannot act via $T'(\ell) = 1 + \ell^{k+1}\langle \ell^{-1}\rangle$ on any non-zero element in the first group of the above short exact sequence. Q.E.D.

Corollary 3.2.2. *The rigid Eisenstein class* $\mathrm{Eis}^k_{\mathrm{rig},N}$ *has a* unique *lift* $\widetilde{\mathrm{Eis}}^k_{\mathrm{rig},N} \in H^1_{\mathrm{rig},c-ss}\left(\bar{Y}^{\mathrm{ord}}, \mathscr{H}^{[k]}(1)\right)^{\mathrm{Eis}}$. *Moreover, this lift lies in the* $\varphi = 1$ *eigenspace.*

Proof. It remains only to check that the lift lies in the $\varphi = 1$ eigenspace; but this is immediate from the fact that the Frobenius commutes with the prime-to-p Hecke action and with the forgetful map from $H^1_{\mathrm{rig},c-ss}(\bar{Y}^{\mathrm{ord}})$ to $H^1_{\mathrm{rig}}(\bar{Y})$. Q.E.D.

Via the Poincaré duality pairing (2), we can regard this "compactified" Eisenstein class $\widetilde{\mathrm{Eis}}^k_{\mathrm{rig},N}$ as a linear functional on the space $H^1_{\mathrm{rig},c-\partial}\left(\bar{Y}^{\mathrm{ord}}, \mathscr{H}^{(k)}\right)$, extending the functional on $H^1_{\mathrm{rig},c}(\bar{Y}, \mathscr{H}^{(k)})$ given by $\mathrm{Eis}^k_{\mathrm{rig},N}$. As we have seen above, the space $H^1_{\mathrm{rig},c-\partial}\left(\bar{Y}^{\mathrm{ord}}, \mathscr{H}^{(k)}\right)$ can be computed in terms of overconvergent modular forms.

Definition 3.2.3. *We define the* critical-slope Eisenstein quotient *of* $S^\dagger_{k+2}(N, L)$ *to be the maximal* L*-vector space quotient of the space* $S^\dagger_{k+2}(N, L)/\Theta\left(S^\dagger_{-k}(N, L)\right)$ *on which the following Hecke operators are zero:*

- *the Hecke operators* $T(\ell) - 1 - \ell^{k+1}\langle \ell\rangle$ *for primes* $q \nmid Np$*;*
- *the operators* $U(\ell) - 1$ *for* $\ell \mid N$*;*
- *the operator* $U(p) - p^{k+1}\langle p\rangle$*.*

Proposition 3.2.4. *Under the isomorphism*

$$H^1_{\mathrm{rig},c-\partial}\left(\bar{Y}^{\mathrm{ord}}, \mathscr{H}^{(k)}\right) \otimes_{\mathbf{Q}_p} L \cong \frac{S^\dagger_{k+2}(N, L)}{\Theta\left(S^\dagger_{-k}(N, L)\right)},$$

the linear functional on $S^\dagger_{k+2}(N, L)$ *given by pairing with* $\widetilde{\mathrm{Eis}}^k_{\mathrm{rig},N}$ *factors through projection to the critical-slope Eisenstein quotient.*

Proof. For $\ell \ne p$, the operators $T'(\ell) - 1 - \ell^{k+1}\langle \ell^{-1}\rangle$ and $U'(\ell) - 1$ annihilate the Eisenstein class, so the linear functional given by pairing with this class must factor through the quotient where the adjoints of these operators act as 0.

To see how the linear functional interacts with $U(p)$, we use the fact that the Eisenstein class is invariant under φ, whose adjoint is $\varphi^{-1} = p^{-1-k}\langle p\rangle^{-1}U(p)$. So the linear functional factors through the cokernel of the map $1 - \varphi^{-1} = 1 - p^{-1-k}\langle p^{-1}\rangle U(p)$. Q.E.D.

It follows from Proposition 3.1.2 that, for any $\chi : (\mathbf{Z}/N\mathbf{Z})^\times \to \overline{\mathbf{Q}}_p^\times$ such that $\chi(-1) = (-1)^k$, the χ-eigenspace for the diamond operators in the critical-slope Eisenstein quotient is 1-dimensional. Moreover, it follows from Remark 3.1.3 that the Eisenstein class defines a non-zero linear functional on each such eigenspace. Hence Proposition 3.2.4 characterises the class $\widetilde{\mathrm{Eis}}^k_{\mathrm{rig},N}$ up to a unit in $\overline{\mathbf{Q}}_p^\times$ for each such χ.

3.3. The "Eisenstein period"

As well as the critical-slope Eisenstein quotient described above, there is also a critical-slope Eisenstein *subspace* of $S^\dagger_{k+2}(N, L)$, defined as the largest subspace where the above operators act as zero. This is of course spanned by classical modular forms: for each χ of the appropriate sign (non-trivial if $k = 0$) we can consider the level N Eisenstein series

$$E_\chi^{(k+2)} = \frac{L(\chi, -1-k)}{2} + \sum_{n \geqslant 1} q^n \left(\sum_{\substack{d \mid n \\ (d,N)=1}} \chi(d) d^{k+1} \right).$$

Then $E^{(k+2)}_{\mathrm{crit},\chi} := E^{(k+2)}_\chi(\tau) - E^{(k+2)}_\chi(p\tau)$ is a classical form of level $\Gamma_1(N) \cap \Gamma_0(p)$, which vanishes at all the cusps contained in the ordinary locus, and is therefore in $S^\dagger_{k+2}(N, \overline{\mathbf{Q}}_p)$. This form spans the χ-isotypical part of the critical-slope Eisenstein subspace.

It follows from the computations of Bellaïche [Bel12] that the map from the critical-slope subspace to the critical-slope quotient is an isomorphism if and only if a certain value of the p-adic Dirichlet L-function of χ is non-zero; otherwise, this map is the zero map. In this section, we shall give an alternative proof of this result, by computing explicitly the Poincaré duality pairing of the two classes involved. Together with the results of the previous section, this completes the proof of Theorem A from the introduction.

Proposition 3.3.1. *Suppose χ is primitive modulo N. Then we have*

$$\left\langle \widetilde{\mathrm{Eis}}^k_{\mathrm{rig},N}, E^{(k+2)}_{\mathrm{crit},\chi} \right\rangle = \frac{(-1)^{k+1} k! N^k}{4 G(\chi^{-1})} L_p(\chi^{-1}, 1+k) L(\chi, -1-k),$$

where $\langle -, - \rangle$ is the Poincaré duality pairing (2).

Here $G(\chi^{-1})$ is the Gauss sum $\sum_{u \in (\mathbf{Z}/N\mathbf{Z})^\times} \chi(u)^{-1} \zeta_N^u$, and $L_p(\chi, s)$ denotes the p-adic L-function of χ, so that for all integers $s \leqslant 0$ such that $(-1)^s = -\chi(-1) = (-1)^{k+1}$, we have $L_p(\chi, s) = (1 - p^{-s}\chi(p)) L(\chi, s)$. Thus the right-hand side of the above formula is the product of a critical L-value and a non-critical p-adic L-value.

Remark 3.3.2. This p-adic L-value could potentially be zero; this is exactly the pathological case identified by Bellaïche in which the critical-slope Eisenstein generalised eigenspace is non-semisimple of dimension > 1. If $k = 0$ then this does not occur, since $L_p(\chi^{-1}, 1)$ is the p-adic logarithm of a cyclotomic unit, and therefore non-zero.

We shall give only an outline of the proof. Firstly, we note that the Poincaré duality pairing (2) can be expressed in terms of residues. Classes in $H^1_{c-\partial}(\bar{Y}^{\mathrm{ord}}, \mathscr{H}^{(k)})$ are represented by $\mathscr{H}^{(k)}$-valued overconvergent 1-differentials ω on X^{ord}; while classes in $H^1_{c-ss}(\bar{Y}^{\mathrm{ord}}, \mathscr{H}^{[k]}(1))$ are represented by pairs $[\mu, G]$, where μ is an $\mathscr{H}^{[k]}$-valued 1-differential on some strict neighbourhood V of X^{ord} in X^{rig}, with simple poles at the cusps, and G is a section of $\mathscr{H}^{[k]}$ over the "boundary" $V \cap X^{\mathrm{ss}}$ satisfying $\nabla G = \mu$. In terms of these representatives, the Poincaré duality pairing is given by the formula

$$\Big\langle [\mu, G], [\omega] \Big\rangle = \sum_{x \in \bar{X}^{\mathrm{ss}}} \mathrm{Res}_x(G \cdot \omega),$$

where $\cdot$ denotes the pairing $\mathscr{H}^{[k]} \times \mathscr{H}^{(k)} \to \mathbf{Q}_p$, and Res_x denotes the residue map at x. (This is a consequence of a more general formula in which ω and μ are taken to be differentials defined on some strict neighbourhood of Y^{ord} in X^{rig}, in which case one needs to take into account residues at cusps as well as supersingular points; see e.g. [Col89, Proposition 4.5] for the case of trivial coefficients, or in a more general setting [LS07, Proposition 6.4.18]. We are taking advantage of the fact that the cuspidal locus of $\bar{X}$ lifts canonically to characteristic 0 and we can choose ω and μ to have logarithmic poles at the cusps, from which it follows that the residue terms at the cusps are all zero.)

In our case, ω is the class of the differential associated to the form $E^{(k+2)}_{\mathrm{crit},\chi}$, which is exact; we may write it as $\nabla(A)$ for some overconvergent $\mathscr{H}^{(k)}$-valued analytic function on X^{ord}. Since $\mathrm{Res}_x(G \cdot \nabla(A)) = -\mathrm{Res}_x(\nabla(G) \cdot A) = -\mathrm{Res}_x(\mu \cdot A)$, and the sum of the residues of the 1-differential $\mu \cdot A$ at all points of $\bar{X} - \bar{Y}^{\mathrm{ord}}$ must be zero, we can write this as

$$\Big\langle [\mu, G], [\nabla A] \Big\rangle = \sum_{c \in C} \mathrm{Res}_c(\mu \cdot A),$$

where C is the set of cusps. We may take for A the ordinary Eisenstein series of weight $-k$,

$$F^{(-k)}_{\mathrm{ord},\chi} = (?) + \sum_{n \geqslant 1} \left(\sum_{\substack{dd' = n \\ (d,N)=(d',p)=1}} \chi(d)(d')^{-1-k} \right) q^n,$$

where $? = \frac{1}{2}\zeta_p(1+k)$ if $\chi = 1$ and $? = 0$ otherwise. In terms of the BGG complex, the pairing between $\mathscr{H}^{[k]}$ and $\mathscr{H}^{(k)} \otimes \Omega^1$ is induced by the pairing

$$\{-,-\} : \omega^{-k} \times \omega^{k+2}(-C) \to \omega^2(-C) = \Omega^1, \quad \{f, g\} = (-1)^k k! fg,$$

so we are reduced to computing

$$(-1)^k k! \sum_{c \in C} F^{(-k)}_{\mathrm{ord},\chi}(c) \cdot F^{(k+2)}_{0,1/N}(c),$$

where $F^{(k+2)}_{0,1/N}$ is the classical level N Eisenstein series that is the de Rham realisation of the weight k Eisenstein class.

It is well-known that $X_1(N)$ has exactly $\frac{1}{2}\phi(d)\phi(N/d)$ cusps of width d for each integer $d \mid N$. Proposition 3.10 of [Kat04] gives formulae for the constant terms of the Eisenstein series $F^{(k+2)}_{0,1/N}$ at each of these cusps, and one can obtain corresponding formulae for $F^{(-k)}_{\mathrm{ord},\chi}$ by p-adic interpolation. One concludes that both Eisenstein series vanish at all cusps of width $< N$. The cusps of width N are exactly those lying above the cusp 0 of $X_0(N)$, and they biject with $(\mathbf{Z}/N\mathbf{Z})^\times/(\pm 1)$. If c is an element of this quotient, then the constant terms of the two Eisenstein series are respectively $\frac{N^{k+1}\chi(c)}{2G(\chi^{-1})} L_p(\chi^{-1}, 1+k)$ and $-N^{-1} \left(\sum_{n = c \bmod N} n^{-s}\right)\Big|_{s=-1-k}$. Thus the sum over $c \in C$ is given by

$$\begin{aligned}
&\tfrac{1}{2} \sum_{c \in (\mathbf{Z}/N\mathbf{Z})^\times} \left(\tfrac{N^{k+1}\chi(c)}{2G(\chi^{-1})} L_p(\chi^{-1}, 1+k)\right) \left(-N^{-1} \sum_{\substack{n=c \\ \bmod N}} n^{-s}\right)\Bigg|_{s=-1-k} \\
&\quad = \tfrac{-N^k L_p(\chi^{-1},1+k)}{4G(\chi^{-1})} \left(\sum_{c \in (\mathbf{Z}/N\mathbf{Z})^\times} \chi(c) \sum_{\substack{n=c \\ \bmod N}} n^{-s}\right)\Bigg|_{s=-1-k} \\
&\quad = \frac{-N^k}{4G(\chi^{-1})} L_p(\chi^{-1}, 1+k) L(\chi, -1-k).
\end{aligned}$$

Remark 3.3.3. The above formula has a complex-analytic counterpart. The Eisenstein series $E^{(k+2)}_{\mathrm{crit},\chi}$ vanishes at every p-ordinary cusp of the modular curve of level $\Gamma_1(N) \cap \Gamma_0(p)$, i.e. every cusp above the cusp ∞ of $\Gamma_0(p)$. The Atkin–Lehner involution W_{Np} interchanges these with the cusps above 0, so the product $E^{(k+2)}_{\mathrm{crit},\chi} \cdot W_{Np}\left(E^{(k+2)}_{\mathrm{crit},\chi}\right)$ vanishes at every cusp and the Petersson product

$$\left\langle E^{(k+2)}_{\mathrm{crit},\chi}, W_N\left(E^{(k+2)}_{\mathrm{crit},\chi}\right)\right\rangle$$

is well-defined. Using the well-known fact that $\langle f, g\rangle$ is the residue at $s = k+2$ of the Rankin–Selberg L-function $L(\overline{f}, g, s)$ (up to an explicit non-zero constant factor), one can compute the above pairing as a product of various explicit constants and the quantity $L(\chi^{-1}, 1+k)L(\chi, -1-k)$.

3.4. Small levels

In order to compute explicit examples, it will be convenient to relax the assumption that $N \geqslant 4$. Of course the modular curve $Y_1(N)$ does not exist as a fine moduli space for $N \leqslant 3$, so we shall use the following workaround: we choose an auxiliary prime q not dividing Np, and form the modular curve of level $\Gamma_1(N) \cap \Gamma(q)$. The cohomology groups of this curve (and its compactifications, special fibres, etc) will all carry an action of $\mathrm{GL}_2(\mathbf{F}_q)$, and we simply project to the invariants under the action of this group. With these conventions, we can define $\mathrm{Eis}^k_{\mathrm{rig},N}$ and $\widetilde{\mathrm{Eis}}^k_{\mathrm{rig},N}$ for any $N \geqslant 1$ and $k \geqslant 0$. (Note that if $N \leqslant 2$ and k is odd, or if $N = 1$ and $k = 0$, then these classes are both 0.)

§4. Preliminaries on Hilbert modular forms

Virtually all of the theory of overconvergent modular forms and rigid cohomology described in §2 can be generalised to the setting of Hilbert modular forms (although in the present paper we shall not need to consider holomorphic projection operators in the Hilbert setting). We shall let F denote a real quadratic field, and σ_1, σ_2 the embeddings $F \hookrightarrow \mathbf{R}$ (in some fixed order). We write $\mathcal{H}_F$ for the product $\mathcal{H} \times \mathcal{H}$, with $\mathrm{GL}_2^+(F)$ acting on the first factor via σ_1 and the second via σ_2. A *weight* will be a quadruple of integers $\mu = (r_1, r_2, t_1, t_2)$ such that $r_1 + 2t_1 = r_2 + 2t_2$.

4.1. Nearly-holomorphic Hilbert modular forms

As in [LLZ16], we interpret Hilbert modular forms of level $U \subset \mathrm{GL}_2(\mathbf{A}_{F,f})$ as functions on the quotient $(\mathrm{GL}_2(\mathbf{A}_{F,f}) \times \mathcal{H}_F)/U$ which are holomorphic on each coset of $\mathcal{H}_F$, and transform appropriately under left translation by $\mathrm{GL}_2^+(F)$. The restriction of any such function to $\mathcal{H}_F$ has a *Fourier–Whittaker expansion*

$$\mathcal{F}(\tau_1, \tau_2) = \sum_{\lambda \gg 0} \sigma_1(\lambda)^{-t_1}\sigma_2(\lambda)^{-t_2} c(\lambda, \mathcal{F}) \exp\left(2\pi i\left[\tau_1\sigma_1(\lambda) + \tau_2\sigma_2(\lambda)\right]\right),$$

where the Fourier–Whittaker coefficients $c(-, \mathcal{F})$ are smooth functions on $\mathbf{A}_F^\times$. We are most interested in the case where $U = U_1(\mathfrak{N}) \coloneqq \{g \in \mathrm{GL}_2(\widehat{\mathcal{O}}_F) : g = \left(\begin{smallmatrix} * & * \\ 0 & 1\end{smallmatrix}\right) \bmod \mathfrak{N}\}$, for some ideal $\mathfrak{N} \lhd \mathcal{O}_F$.

Definition 4.1.1. *Define a* nearly-holomorphic Hilbert modular form *over F, of weight μ, degree $\leqslant (n_1, n_2)$ and level $\mathfrak{N}$, to be a C^∞ function*

$$f : (\mathrm{GL}_2(\mathbf{A}_{F,f}) \times \mathcal{H}_F)/U_1(\mathfrak{N}) \to \mathbf{C}$$

which transforms appropriately under left translation by $\mathrm{GL}_2^+(F)$, and whose restriction to every coset of $\mathcal{H}_F$ can be written in the form

$$\sum_{i_1=0}^{n_1} \sum_{i_2=0}^{n_2} f_{i_1 i_2}(\tau_1, \tau_2)(\operatorname{Im} \tau_1)^{-i_1}(\operatorname{Im} \tau_2)^{-i_2}$$

with the $f_{i_1 i_2}$ holomorphic and bounded at ∞. We write $M_\mu^{\leqslant(n_1,n_2)}(\mathfrak{N}, \mathbf{C})$ for the space of such forms.

As in the case $F = \mathbf{Q}$, we have two notions of cuspidality: one can require that all the $f_{i_1 i_2}$ vanish at ∞ (strong cuspidality) or only that f_{00} does so (weak cuspidality). We write $\mathbf{S}_\mu^{\leqslant(n_1,n_2)}(\mathfrak{N}, \mathbf{C})$ for the subspace of strongly cuspidal forms.

Note 4.1.2. There are two Shimura–Maass derivative operators δ_1 and δ_2, one for each real place. The operator δ_1 is a map

$$M_\mu^{\leqslant(n_1,n_2)}(\mathfrak{N}, \mathbf{C}) \to M_{\mu+(2,0,-1,0)}^{\leqslant(n_1+1,n_2)}(\mathfrak{N}, \mathbf{C})$$

given by $\frac{1}{2\pi i}\left(\frac{\partial}{\partial \tau_1} + \frac{r_1}{2i \operatorname{Im}(\tau_1)}\right)$ on each coset of $\mathcal{H}_F$; and similarly for δ_2.

These spaces also have a geometric interpretation. Let L be any subfield of $\mathbf{C}$ containing the images of the embeddings σ_1, σ_2 of F. We write $Y_1(\mathfrak{N})$ for the base-extension to L of the Hilbert modular surface, and $Y_1^*(\mathfrak{N})$ the finite covering of a subset of the components of $Y_1(\mathfrak{N})$ described in [LLZ16]. On $Y_1^*(\mathfrak{N})$ the line bundle $\omega^{(r_1,r_2)}$ is defined for any $r_i \in \mathbf{Z}$, and if $r_i \geqslant 0$ this embeds in the vector bundle $\mathscr{H}^{(r_1,r_2)}$.

If $\mu = (r_1, r_2, t_1, t_2)$ is a weight (so, in particular, $r_1 = r_2 \bmod 2$), then the tensor product

$$\mathscr{H}^{(n_1,n_2)} \otimes \omega^{(r_1-n_1, r_2-n_2)} \otimes \det{}^{(t_1,t_2)}$$

(where "det" denotes the trivial line bundle with its Hecke action twisted by the determinant map) descends to $Y_1(\mathfrak{N})$; we denote the descent by $\mathscr{H}^{(\mu,n_1,n_2)}$. If we assume our field L contains the N-th roots of 1, where N is the integer such that $\mathfrak{N} \cap \mathbf{Z} = N\mathbf{Z}$, then we may define

$$\begin{aligned} M_\mu^{\leqslant(n_1,n_2)}(\mathfrak{N}, L) &\coloneqq H^0\left(X_1(\mathfrak{N}), \mathscr{H}^{(\mu,n_1,n_2)}\right), \\ \mathbf{S}_\mu^{\leqslant(n_1,n_2)}(\mathfrak{N}, L) &\coloneqq H^0\left(X_1(\mathfrak{N}), \mathscr{H}^{(\mu,n_1,n_2)}(-C)\right), \end{aligned}$$

where $X_1(\mathfrak{N})$ denotes any smooth toroidal compactification of $Y_1(\mathfrak{N})$, and C denotes its boundary divisor. These spaces are independent of the choice of toroidal compactification [TX16, §2.12], they have a natural Hecke action, and they are consistent with Definition 4.1.1 when $L = \mathbf{C}$.

Finally, if N is as above, the map $\iota : \mathcal{H} \to \mathcal{H}_F$ defined by $\tau \mapsto (\tau, \tau)$ gives a pullback map

$$\iota^* : M_\mu^{\leqslant (n_1, n_2)}(\mathfrak{N}, L) \to M_{r_1+r_2}^{\leqslant (n_1+n_2)}(N, L). \tag{3}$$

4.2. Overconvergent and nearly-overconvergent p-adic Hilbert modular forms

We choose a prime $p \nmid \mathfrak{N}$ unramified in F, a finite extension $L/\mathbf{Q}_p$ containing μ_N, and an embedding $\sigma_1 : F \hookrightarrow L$.

Overconvergent forms If $\mu = (r_1, r_2, t_1, t_2)$ is a weight, then we define spaces $M_\mu^\dagger(\mathfrak{N}, L) \supseteq S_\mu^\dagger(\mathfrak{N}, L)$ of overconvergent p-adic Hilbert modular (resp. cusp) forms of level $U_1(\mathfrak{N})$ and weight μ with coefficients in L as in [TX16, §3]:

$$\begin{aligned} M_\mu^\dagger(\mathfrak{N}, L) &\coloneqq H^0\left(X_1(\mathfrak{N})_L^{\mathrm{rig}}, j^\dagger \omega^{(\mu)}\right), \\ S_\mu^\dagger(\mathfrak{N}, L) &\coloneqq H^0\left(X_1(\mathfrak{N})_L^{\mathrm{rig}}, j^\dagger \omega^{(\mu)}(-C)\right). \end{aligned}$$

Here j is the inclusion of the ordinary locus $X_1(\mathfrak{N})_L^{\mathrm{ord}}$ into $X_1(\mathfrak{N})_L^{\mathrm{rig}}$ (see Notation 4.3.3 below). These spaces have the following properties:

- These spaces are independent of the choice of toroidal compactification $X_1(\mathfrak{N})$ [TX16, §3.3].
- Overconvergent forms $\mathcal{F} \in M_\mu^\dagger(\mathfrak{N}, L)$ have Fourier–Whittaker coefficients $c(\mathfrak{m}, \mathcal{F}) \in L$, for every fractional ideal $\mathfrak{m} \subseteq \mathfrak{d}^{-1}$; and if $\mathcal{F}$ is cuspidal (or if $(r_1, r_2) \neq (0, 0)$) then $\mathcal{F}$ is uniquely determined by its Fourier–Whittaker coefficients. (This follows from the corresponding statement for not-necessarily-overconvergent p-adic Hilbert modular forms; see e.g. [Hid04, Corollary 4.23].)
- The space $M_\mu^\dagger(\mathfrak{N}, L)$ has an action of the normalised Hecke operators $\mathcal{T}(\mathfrak{q})$ for each prime $\mathfrak{q} \nmid p\mathfrak{N}$ [TX16, §3.7], and $\mathcal{U}(\mathfrak{q})$ for $\mathfrak{q} \mid p\mathfrak{N}$ [TX16, §3.18], having the expected effect on Fourier–Whittaker coefficients; in particular, we have $c(\mathfrak{m}, \mathcal{U}(\mathfrak{q})\mathcal{F}) = c(\mathfrak{m}\mathfrak{q}, \mathcal{F})$. These operators preserve the subspaces of cusp forms.
- The operators $\mathcal{U}(\mathfrak{p})$ for $\mathfrak{p} \mid p$ have right inverses $\mathcal{V}(\mathfrak{p})$ satisfying

$$c(\mathfrak{m}, \mathcal{V}(\mathfrak{p})\mathcal{F}) = \begin{cases} 0 & \text{if } \mathfrak{p} \nmid \mathfrak{m}, \\ c(\mathfrak{m}/\mathfrak{p}, \mathcal{F}) & \text{if } \mathfrak{p} \mid \mathfrak{m}. \end{cases}$$

(See [TX16, Lemma 3.20]; note that the operator $S_{\mathfrak{p}}$ in the notation of *op.cit.* is obviously invertible.)

- If $r_1, r_2 \geqslant 2$, then $M^\dagger_\mu(\mathfrak{N}, L)$ contains the space of classical Hilbert modular forms of weight μ and level $U_1(\mathfrak{N}) \cap U_0(p)$ with coefficients in L as a Hecke-invariant subspace [TX16, §3.3], and the Fourier–Whittaker coefficients and Hecke operators coincide with the classical ones on this subspace.

We shall frequently omit the field L and/or the level $\mathfrak{N}$ from the notation if these are clear from context. As with classical Hilbert modular forms, the spaces $S^\dagger_\mu$ are independent of (t_1, t_2) up to a canonical isomorphism, so we shall also occasionally omit (t_1, t_2) from the notation and just write $S^\dagger_{(r_1,r_2)}(\mathfrak{N}, L)$; this identification twists the Fourier–Whittaker coefficients and the actions of the operators $\mathcal{T}(\mathfrak{q})$ and $\mathcal{U}(\mathfrak{q})$ by a power of $\mathrm{Nm}(\mathfrak{q})$.

Nearly-overconvergent forms Exactly as in the elliptic modular case, we can define nearly-overconvergent spaces by replacing the line bundles $\omega^{(\mu)}$ with the larger vector bundles $\mathscr{H}^{(\mu,n_1,n_2)}$, for $n_1, n_2 \geqslant 0$. The resulting spaces $M^{\dagger,\leqslant(n_1,n_2)}_\mu(\mathfrak{N}, L) \supset \mathbf{S}^{\dagger,\leqslant(n_1,n_2)}_\mu(\mathfrak{N}, L)$ have actions of the operators δ_1 and δ_2.

4.3. Theta operators and rigid cohomology

We now assume $p = \mathfrak{p}_1\mathfrak{p}_2$ is split in F, and (without loss of generality) $\mathfrak{p}_1$ is the prime corresponding to the embedding $\sigma_1 : F \hookrightarrow L \subseteq \overline{\mathbf{Q}}_p$.

Proposition 4.3.1 (Tian–Xiao)**.**

(i) [TX16, Proposition 3.24] *Every slope of the operator $\mathcal{U}(\mathfrak{p}_i)$ acting on $S^\dagger_\mu(\mathfrak{N})$ is $\geqslant t_i$.*

(ii) [TX16, Remark 2.17(1)] *If $r_1 \geqslant 2$, the operator $\delta_1^{(r_1-1)}$ is an injective map*
$$\Theta_1 : S^\dagger_{w_1(\mu)} \hookrightarrow S^\dagger_\mu,$$
where $w_1(\mu) = (2 - r_1, r_2, t_1 + r_1 - 1, t_2)$, which preserves Fourier–Whittaker coefficients[4], and commutes with the action of the normalised Hecke operators $\mathcal{T}(\mathfrak{q})$ and $\mathcal{U}(\mathfrak{q})$. In particular, the image of Θ_1 is a Hecke-invariant subspace of $S^\dagger_\mu$ on which every slope of $\mathcal{U}(\mathfrak{p}_1)$ is $\geqslant t_1 + r_1 - 1$. The same holds mutatis mutandis for Θ_2 if $r_2 \geqslant 2$.

[4]Given our conventions for Fourier–Whittaker expansions, the fact that Θ_1 preserves Fourier–Whittaker coefficients, while decreasing t_1 by $r_1 - 1$, amounts to stating that it acts on Fourier–Whittaker expansions in the same way as the operator $\left(\frac{1}{2\pi i}\frac{\partial}{\partial\tau_1}\right)^{r_1-1}$ on $\mathcal{H}_F$.

Notation 4.3.2. *Somewhat abusively, we shall write $U(\mathfrak{p}_i)$ for the operator $p^{-t_i}\mathcal{U}(\mathfrak{p}_i)$ (for $i = 1, 2$), and $U(p) = U(\mathfrak{p}_1)U(\mathfrak{p}_2)$. Similarly, we write $V(\mathfrak{p}_i) = p^{t_i}\mathcal{V}(\mathfrak{p}_i)$, and $V(p) = V(\mathfrak{p}_1)V(\mathfrak{p}_2)$.*

These operators are then well-defined on $S^\dagger_{(r_1,r_2)}$ (independent of the choice of the t_i). In this language, part (i) of the theorem states that these operators have all slopes $\geqslant 0$, and part (ii) states that Θ_1 increases the slopes of $U(\mathfrak{p}_1)$ by $r_1 - 1$ (while leaving the slopes of $U(\mathfrak{p}_2)$ unchanged).

The differential operators Θ_i have a geometric interpretation via rigid cohomology. To state this, we shall need to introduce some notation. Suppose $(r_1, r_2) = (k_1 + 2, k_2 + 2)$ with $k_i \geqslant 0$.

Notation 4.3.3.

- *Let $\mathcal{Y}$ be the smooth model over $\mathbf{Z}_p$ of the Hilbert modular variety $Y_1(\mathfrak{N})$.*
- *Let $\mathcal{X}$ be a smooth toroidal compactification of $\mathcal{Y}$, and X its generic fibre.*
- *Let $\mathcal{C} = \mathcal{X} - \mathcal{Y}$ be the boundary, which is a relative simple normal crossing divisor over $\mathbf{Z}_p$.*
- *Let $\bar{X}, \bar{Y}, \bar{C}$ be the special fibres of $\mathcal{X}, \mathcal{Y}, \mathcal{C}$.*
- *We let $\bar{Z}$ denote the closed subvariety of $\bar{Y}$ parametrising Hilbert–Blumenthal abelian surfaces which are non-ordinary at one or both of $\{\mathfrak{p}_1, \mathfrak{p}_2\}$ (the vanishing locus of the total Hasse invariant).*
- *We write $\bar{Y}^{\mathrm{ord}} = \bar{Y} - \bar{Z}$, and similarly $\bar{X}^{\mathrm{ord}}$.*
- *We write $\mathscr{H}^{(\mu-2)}$ for the F-isocrystal on $\mathcal{Y}$ corresponding to the algebraic representation of $\mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_2$ of weight (k_1, k_2, t_1, t_2).*

Proposition 4.3.4 (Tian–Xiao, [TX16, Theorem 3.5 & Lemma 4.11])**.** *The complex $\mathrm{BGG}^\bullet_c\left(\mathscr{H}^{(\mu-2)}\right)$ of sheaves on X given by*

$$\left[\omega^{w_1w_2(\mu)} \xrightarrow{(\Theta_2,-\Theta_1)} \omega^{w_1(\mu)} \oplus \omega^{w_2(\mu)} \xrightarrow{(\Theta_1,\Theta_2)} \omega^{\mu}\right](-C)$$

maps quasi-isomorphically to the de Rham complex $\mathrm{DR}^\bullet_c\left(\mathscr{H}^{(\mu-2)}\right)$ relative to the cuspidal divisor C; and taking overconvergent sections over the tube of $\bar{X}^{\mathrm{ord}}$ induces an isomorphism

$$\frac{S^\dagger_\mu(\mathfrak{N}, L)}{\Theta_1\left(S^\dagger_{w_1(\mu)}(\mathfrak{N}, L)\right) + \Theta_2\left(S^\dagger_{w_2(\mu)}(\mathfrak{N}, L)\right)} \cong H^2_{\mathrm{rig},c-\partial}\left(\bar{Y}^{\mathrm{ord}}, \mathscr{H}^{(\mu-2)}\right) \otimes_{\mathbf{Q}_p} L$$

for any p-adic field L containing the N-th roots of unity.

(As in the case of modular curves, one can define similarly groups $H^2_{\mathrm{rig},c-ss}\left(\bar{Y}^{\mathrm{ord}}, \mathscr{H}^{(\mu-2)}\right)$ with compact support towards the supersingular locus, but we will not use them here.)

4.4. Rankin–Cohen brackets

Let $\mathcal{F}$ be a holomorphic Hilbert cusp form of weight $(\underline{r}, \underline{t})$, where $\underline{r} = (r_1, r_2)$, and level $U_1(\mathfrak{N})$.

Proposition-Definition 4.4.1 (Rankin–Cohen, cf. [Zag94])**.** *In the above setting, for any $n \geqslant 0$, the function on $\mathcal{H}$ defined by*

$$[\mathcal{F}]_n \coloneqq \sum_{a_1+a_2=n} (-1)^{a_1} \binom{r_1+n-1}{a_2} \binom{r_2+n-1}{a_1} \iota^* \left(\delta_1^{a_1} \delta_2^{a_2} \mathcal{F}\right)$$
$$= \sum_{a_1+a_2=n} (-1)^{a_1} \binom{r_1+n-1}{a_2} \binom{r_2+n-1}{a_1} \iota^* \left(\theta_1^{a_1} \theta_2^{a_2} \mathcal{F}\right)$$

(where θ_j is the differential operator $\frac{1}{2\pi i}\frac{\partial}{\partial z_j} = q_j \frac{\partial}{\partial q_j}$ on $\mathcal{H}_F$, and ι^* is as in (3)*) is a holomorphic modular form of weight $r_1 + r_2 + 2n$ and level $U_1(N)$, where $\mathfrak{N} \cap \mathbf{Z} = N\mathbf{Z}$. We call this the* n-th Rankin–Cohen bracket *of $\mathcal{F}$.*

The equality of the two expressions for $[\mathcal{F}]_n$ is part of [Lan08, Theorem 1]. From the first expression one sees that $[\mathcal{F}]_n$ is a nearly-holomorphic modular form of level N, weight $r_1 + r_2 + 2n$ and degree $\leqslant n$, and from the second expression one sees that it is actually holomorphic. Note that $[\mathcal{F}^\sigma]_n = (-1)^n [\mathcal{F}]_n$, and that $[\mathcal{F}]_0$ is just $\iota^*(\mathcal{F})$. Moreover, the brackets $[\mathcal{F}]_n$ are unchanged if one twists $\mathcal{F}$ by a power of the adèle norm character.

Proposition 4.4.2 (Lanphier)**.** *We have*

$$\iota^* \left(\delta_1^n \mathcal{F}\right) = \sum_{j=0}^{n} \frac{(-1)^j \binom{n}{j} \binom{r_1+n-1}{n-j}}{\binom{r_1+r_2+2j-2}{j} \binom{r_1+r_2+n+j-1}{n-j}} \delta^{n-j} \left([\mathcal{F}]_j\right).$$

Proof. See [Lan08, Theorem 1]. (Lanphier's result is stated for a product of two elliptic modular forms, but the same identity is valid in the Hilbert setting also.) Q.E.D.

Corollary 4.4.3. *If $\mathcal{F}$ is a (holomorphic) Hilbert cusp form, then*

$$[\mathcal{F}]_n = (-1)^n \binom{r_1+r_2+2n-2}{n} \left(\Pi_{\mathrm{hol}} \circ \iota^* \circ \delta_1^n\right)(\mathcal{F}).$$

Proof. We note firstly that Shimura's projection operator is well-defined, since $\iota^*(\delta_1^n \mathcal{F})$ has weight $r_1 + r_2 + 2n$ and degree $\leqslant n$, and $r_1 + r_2 > 0$. Applying Π_{hol} to Lanphier's formula, all the terms go to 0 except the $j = n$ term, which gives thte stated formula. Q.E.D.

The theory of Rankin–Cohen brackets extends to overconvergent forms:

Proposition 4.4.4. *If $\mathcal{F}$ is an overconvergent Hilbert cusp form of weight (r_1, r_2), then there are overconvergent elliptic cusp forms $[\mathcal{F}]_n$ (the Rankin–Cohen brackets of $\mathcal{F}$) of weight $r_1 + r_2 + 2n$, for all $n \geqslant 0$, given by the same formulae as in Proposition-Definition 4.4.1.*

Proof. The case $n = 0$ is obvious from the definition of overconvergent Hilbert modular forms, as sections of the automorphic line bundles $\omega^{(r_1,r_2)}$ over a strict neighbourhood of the ordinary locus in the Hilbert modular variety. Pulling such a section back to the image of $Y_1(N)$ gives a section of $\iota^*\left(\omega^{(r_1,r_2)}\right) = \omega^{r_1+r_2}$ over the ordinary locus of $Y_1(N)$, and (by considering Fourier–Whittaker coefficients) this section must vanish at the cusps of $Y_1(N)$, and therefore defines an overconvergent modular form.

For $n \geqslant 1$, we use the theory of nearly-overconvergent p-adic modular forms. We may interpret $\theta_1^{a_1}\theta_2^{a_2}\mathcal{F}$, for $\mathcal{F} \in S^{\dagger}_{(r_1,r_2)}$ and any a_1, a_2 with $a_1 + a_2 = n$, as the degree 0 part of the nearly-overconvergent Hilbert modular form $\delta_1^{a_1}\delta_2^{a_2}\mathcal{F}$, which has a polynomial Fourier–Whittaker expansion in which the Fourier–Whittaker coefficients are polynomials in two variables X_1, X_2 of degree $\leqslant n$. Pulling this back via ι gives a nearly-overconvergent elliptic cusp form of weight $r_1 + r_2 + 2(a_1 + a_2)$; and exactly the same computation as in the classical case shows that in the linear combination defining $[\mathcal{F}]_n$, all the positive-degree terms cancel to 0, and the result is an overconvergent form. Q.E.D.

We now relate Rankin–Cohen brackets to overconvergent projection operators. We let r_1, r_2, n be integers, with $n \geqslant 1$ (the case $n = 0$ being trivial), and write $t = r_1 + r_2 + 2n$.

Proposition 4.4.5. *If $\mathcal{F} \in S^{\dagger}_{(r_1,r_2)}(\mathfrak{N}, L)$, then in $\mathbf{S}^{\dagger,\leqslant n}_t$ we have the equality*

$$[\mathcal{F}]_n = (-1)^n \binom{t-2}{n} \iota^*\left(\delta_1^n \mathcal{F}\right) \bmod \delta\left(\mathbf{S}^{\dagger,\leqslant n-1}_{t-2}\right).$$

In particular,

(i) if $r_1 + r_2 \geqslant 1$, then Urban's overconvergent projector is defined on $\iota^*\left(\delta_1^n \mathcal{F}\right)$, and maps it to $(-1)^n \binom{t-2}{n}^{-1} [\mathcal{F}]_n$;

(ii) if $r_1 + r_2 \geqslant 2 - n$, then Darmon and Rotger's overconvergent projector is defined on $\iota^*\left(\delta_1^n \mathcal{F}\right)$, and maps it to the image of $(-1)^n \binom{t-2}{n}^{-1} [\mathcal{F}]_n$ modulo $\Theta\left(S^{\dagger}_{2-t}\right)$.

Proof. If $r_1 + r_2 \geqslant 1$ then Lanphier's identity is valid for $\mathcal{F} \in S^{\dagger}_{(r_1,r_2)}(\mathfrak{N}, L)$, so we may argue as in the case of classical forms. However, when $r_1 + r_2 \leqslant 0$, we cannot argue in this fashion, since some of the binomial coefficients in the denominator are 0. Hence we use a slightly different argument.

We can write $[\mathcal{F}]_n$ as $\iota^*\left(P(\delta_1, \delta_2)\mathcal{F}\right)$, where $P(X, Y)$ is the polynomial $\sum_{a=0}^{n}(-1)^a\binom{r_1+n-1}{n-a}\binom{r_2+n-1}{a}X^aY^{n-a}$. Since we have $P(X, -X) = (-1)^n\binom{t-2}{n}X^n$, we can write

$$P(X, Y) = (-1)^n\binom{t-2}{n}X^n + (X + Y)Q(X, Y)$$

for some homogenous polynomial $Q \in \mathbf{Z}[X, Y]$ of degree $n - 1$. Since $\iota^*((\delta_1 + \delta_2)\mathcal{G}) = \delta(\iota^*\mathcal{G})$ for any nearly-overconvergent $\mathcal{G}$, we have that

$$[\mathcal{F}]_n - (-1)^n\binom{t-2}{n}\iota^*\left(\delta_1^n\mathcal{F}\right) = \delta\left(\iota^*Q(\delta_1, \delta_2)\mathcal{F}\right) \in \delta\left(\mathbf{S}^{\dagger, \leqslant n-1}_{t-2}\right).$$

Q.E.D.

Remark 4.4.6. Lanphier's formula is also valid if $r_1 + r_2 < 2 - 2n$. In the intermediate cases $2 - 2n \leqslant r_1 + r_2 \leqslant 0$, we do not know whether $\iota^*\left(\delta_1^n\mathcal{F}\right)$ lies in $\sum_{a=0}^{n}\delta^a\left(S^{\dagger}_{t-2a}\right)$.

4.5. P-depletion

Definition 4.5.1. *If $\mathcal{F} \in S^{\dagger}_{\mu}$, and $\mathfrak{a}$ is a square-free product of primes dividing p, we define the $\mathfrak{a}$-depletion of $\mathcal{F}$ by*

$$\mathcal{F}^{[\mathfrak{a}]} = \Big(1 - \mathcal{V}(\mathfrak{a})\mathcal{U}(\mathfrak{a})\Big)\mathcal{F},$$

so that $c(\mathfrak{m}, \mathcal{F}^{[\mathfrak{a}]}) = 0$ if $\mathfrak{a} \mid \mathfrak{m}$, and $c(\mathfrak{m}, \mathcal{F}^{[\mathfrak{a}]}) = c(\mathfrak{m}, \mathcal{F})$ otherwise.

We advance here a conjecture relating these depletion operators to the differential operators Θ_i in the case of p a split prime.

Conjecture 4.5.2. *Suppose $r_1, r_2 \geqslant 2$, and assume $\mathcal{F}$ is a classical eigenform of level $\mathfrak{N}$. Then $\mathcal{F}^{[\mathfrak{p}_1]}$ is in the image of the map $\Theta_1 : S^{\dagger}_{w_1(\mu)} \hookrightarrow S^{\dagger}_{\mu}$.*

Although simple to state, this conjecture appears to be surprisingly difficult. Notice that it is automatic that $\Theta_1^{-1}(\mathcal{F}^{[\mathfrak{p}_1]})$ exists as a p-adic Hilbert modular form, since we can write it as a uniform limit of the forms $\theta_1^{p^n(p-1)-r_1+1}(\mathcal{F}^{[\mathfrak{p}_1]})$ as $n \to \infty$; the difficulty is ensuring that it is overconvergent. We can only prove the conjecture under an irritating additional assumption:

Proposition 4.5.3. *Assume that $\mathcal{F}$ is non-ordinary at $\mathfrak{p}_2$. Then Conjecture 4.5.2 holds.*

Proof. The operators $\mathcal{U}(\mathfrak{p}_1)$ and $\mathcal{U}(\mathfrak{p}_2)$ are both invertible on H^2_{rig}. Hence, since the form $\mathcal{F}^{[\mathfrak{p}_1]}$ is in the kernel of $\mathcal{U}(\mathfrak{p}_1)$, it must lie in the sum $\operatorname{image}\Theta_1 + \operatorname{image}\Theta_2$.

We consider the projection of $\mathcal{F}^{[\mathfrak{p}_1]}$ to the quotient

$$\frac{\operatorname{image}(\Theta_2)}{\operatorname{image}(\Theta_1) \cap \operatorname{image}(\Theta_2)}.$$

Since all the maps are Hecke-equivariant, this quotient has an action of $\mathcal{U}(\mathfrak{p}_2)$, and all slopes of $\mathcal{U}(\mathfrak{p}_2)$ are $\geqslant t_2 + k_2 - 1$. However, by assumption, $\mathcal{F}$ is non-ordinary at $\mathfrak{p}_2$; thus $\mathcal{F}^{[\mathfrak{p}_1]}$ lies in a sum of finite-dimensional generalised eigenspaces for $\mathcal{U}(\mathfrak{p}_2)$ whose slopes σ satisfy $t_2 < \sigma < t_2 + k_2 - 1$, and hence its image in this quotient is zero. Q.E.D.

We shall not in fact use this result directly; instead, we shall use the following proposition due to Fornea. However, we leave Proposition 4.5.3 in situ, since Fornea's argument is partially based on the proof of Proposition 4.5.3 from an earlier preprint version of this paper.

Proposition 4.5.4 (Fornea; [For17, Corollary 4.7])**.** *For any $\mathcal{F}$ as in Conjecture 4.5.2, the form $\mathcal{F}^{[\mathfrak{p}_1,\mathfrak{p}_2]} = (\mathcal{F}^{[\mathfrak{p}_1]})^{[\mathfrak{p}_2]}$ is in the image of Θ_1 (and also of Θ_2).*

(Fornea's argument is formulated for forms that are ordinary at $\mathfrak{p}_2$, but it in fact suffices to have at least one $\mathfrak{p}_2$-stabilisation of non-critical slope, as is clear from the proof, and this holds for all such $\mathcal{F}$.)

§5. Evaluation of the regulator

We now begin the computation of the regulators of the Asai–Flach classes attached to Hilbert modular eigenforms. We assume $p = \mathfrak{p}_1\mathfrak{p}_2$ is split in F, and σ_1 is the embedding of F into our coefficient field L corresponding to $\mathfrak{p}_1$, as above.

We fix, throughout this section, a level $\mathfrak{N}$ coprime to p, a weight μ of the form (k_1+2, k_2+2, t_1, t_2) with $k_1, k_2 \geqslant 0$, and a Hilbert modular eigenform $\mathcal{F}$ of level $U_1(\mathfrak{N})$ and weight μ with coefficients in L. We assume (for simplicity) that $\mathfrak{N}$ is sufficiently large; the case where $\mathfrak{N}$ is not sufficiently large can be handled by introducing full level q structure for an auxiliary prime q and then passing to invariants, but we shall not spell out the details explicitly.

5.1. Cohomology classes from Hilbert eigenforms

Let $Y = Y_1(\mathfrak{N})$ be the Hilbert modular surface considered as a $\mathbf{Q}_p$-variety, and $\mathcal{Y}$ its smooth $\mathbf{Z}_p$-model. Then the $\mathcal{F}$-eigenspace for the Hecke operators acting on $H^2_{\mathrm{dR}}\left(Y, \mathscr{H}^{(\mu-2)}(t_1+t_2)\right) \otimes_{\mathbf{Q}_p} L$ is 4-dimensional, and lifts isomorphically to the compactly-supported cohomology of Y. We denote this space by $M_{\mathrm{dR}}(\mathcal{F})$. By comparison with the rigid cohomology of the special fibre $\bar{Y}$ of $\mathcal{Y}$, the space $M_{\mathrm{dR}}(\mathcal{F})$ has an L-linear action of the Frobenius map. Moreover, for any $i \ne 2$ the $\mathcal{F}$-eigenspaces in H^i_{dR} and $H^i_{\mathrm{dR},c}$ vanish (and likewise in rigid cohomology).

Remark 5.1.1. The overconvergent filtered F-isocrystal $\mathscr{H}^{(\mu-2)}(t_1+t_2)$ is independent of the choice of the t_i, up to a canonical isomorphism, and we denote it by $\mathscr{H}^{(k_1,k_2)}$. These isomorphisms twist the Hecke operators $\mathcal{T}(\mathfrak{q})$ by a power of $\mathrm{Norm}(\mathfrak{q})$; so we obtain an identification between the spaces $M_{\mathrm{dR}}(\mathcal{F})$ and $M_{\mathrm{dR}}(\mathcal{F}[R])$ for any $R \in \mathbf{Z}$, where $\mathcal{F}[R]$ is the form of weight $(k_1+2, k_2+2, t_1+R, t_2+R)$ obtained by twisting $\mathcal{F}$ by the R-th power of the finite adèle norm. We can thus regard $M_{\mathrm{dR}}(\mathcal{F})$ as a subspace of $H^2_{\mathrm{dR}}(Y, \mathscr{H}^{(k_1,k_2)}) \otimes L$ canonically associated to the twisting-equivalence class of forms $\{\mathcal{F}[R] : R \in \mathbf{Z}\}$.

If α_i, β_i are the roots of the polynomial

$$P_{\mathfrak{p}_i}(\mathcal{F}, X) = X^2 - p^{-t_i} c(\mathfrak{p}_i, \mathcal{F}) X + p^{k_i+1} \varepsilon_{\mathcal{F}}(\mathfrak{p}_i),$$

for $i = 1, 2$, then the eigenvalues of φ on $M_{\mathrm{dR}}(\mathcal{F})$ are the pairwise products $\{\alpha_1\alpha_2, \alpha_1\beta_2, \beta_1\alpha_2, \beta_1\beta_2\}$. (Note that α_i and β_i are the eigenvalues of the operator $U(\mathfrak{p}_i)$ on the $\mathcal{F}$-eigenspace at level $p\mathfrak{N}$.) Thus the polynomial

$$P_p(\mathcal{F}, X) = \det(1 - X\varphi : M_{\mathrm{dR}}(\mathcal{F})) = (1 - \alpha_1\alpha_2 X) \dots (1 - \beta_1\beta_2 X)$$

is the local Asai Euler factor of $\mathcal{F}$ at p.

Notation 5.1.2. *Let $P \in 1 + XL[X]$ be a monic polynomial. We write $H^*_{\mathrm{fp}}(\mathcal{Y}, \mathscr{H}^{(k_1,k_2)}, P)$ for Besser's finite-polynomial cohomology of $\mathcal{Y}$, with coefficients in $\mathscr{H}^{(k_1,k_2)}$, relative to the polynomial P; and similarly $H^*_{\mathrm{fp},c}(\mathcal{Y}, \mathscr{H}^{(k_1,k_2)}, P)$ for its compactly supported analogue.*

By construction, we have a long exact sequence of L-vector spaces

$$\dots \to H^i_{\mathrm{fp}}(\mathcal{Y}, \mathscr{H}^{(k_1,k_2)}, P) \to \mathrm{Fil}^0 H^i_{\mathrm{dR}}(Y, \mathscr{H}^{(k_1,k_2)})_L \xrightarrow{P(\varphi)} H^i_{\mathrm{rig}}(\bar{Y}, \mathscr{H}^{(k_1,k_2)})_L \to \dots,$$

and similarly for compactly supported cohomology. This sequence is compatible with the action of the Hecke operators away from p, so from the vanishing statements above, we see that the natural map

(4)
$$H^2_{\mathrm{fp},c}\left(\mathcal{Y}, \mathscr{H}^{(k_1,k_2)}(n), P\right)[\mathcal{F}] \to \left(\mathrm{Fil}^n M_{\mathrm{dR}}(\mathcal{F})\right) \cap \left(M_{\mathrm{rig}}(\mathcal{F})^{P(p^{-n}\varphi)=0}\right)$$

is an isomorphism for all $n \in \mathbf{Z}$. If η is a class in $\mathrm{Fil}^n M_{\mathrm{dR}}(\mathcal{F})$, and P is any polynomial such that $P(p^{-n}\varphi)$ annihilates η, then we write $\tilde{\eta}$ for the preimage of η in $H^2_{\mathrm{fp},c}$.

Notation 5.1.3. *We let j be an integer with $0 \leqslant j \leqslant \min(k_1, k_2)$, and we write $m = k_1 + k_2 - 2j \geqslant 0$.*

We write $\mathcal{Y}_{\mathbf{Q}}$ for the smooth $\mathbf{Z}_p$-model of $Y_1(N)$, where $N = \mathfrak{N} \cap \mathbf{Z}$ as usual. There is a morphism of filtered F-isocrystals on $\mathcal{Y}_{\mathbf{Q}}$, the *Clebsch–Gordan map*,

$$CG^{(k_1,k_2,j)} : \iota^* \mathscr{H}^{(k_1,k_2)}(j) \to \mathscr{H}^{(m)}.$$

Proposition 5.1.4. *Suppose $P(p^{-1}) \neq 0$, and let*

$$\tilde{\eta} \in H^2_{\mathrm{fp},c}\left(\mathcal{Y}, \mathscr{H}^{(k_1,k_2)}(1+j), P\right)[\mathcal{F}]$$

be the unique preimage of $\eta \in \mathrm{Fil}^{1+j} M_{\mathrm{dR}}(\mathcal{F})^{P(p^{-1-j}\varphi)=0}$ under the isomorphism (4). Then

$$\left\langle \log_p \left(\mathrm{AF}^{[\mathcal{F},j]}_{\text{ét}}\right), \eta \right\rangle_{\mathrm{dR},Y} = \left\langle \mathrm{Eis}^m_{\mathrm{syn},N}, CG^{(k_1,k_2,j)}\left(\iota^* \tilde{\eta}\right) \right\rangle_{\mathrm{fp},P,\mathcal{Y}_{\mathbf{Q}}},$$

where $\mathrm{AF}^{[\mathcal{F},j]}_{\text{ét}} \in H^1(\mathbf{Q}, M_{\text{ét}}(\mathcal{F})^*(-j))$ is the Asai–Flach class defined in [LLZ16]*, and we have written $\log_p$ for the map*

$$H^1_{\mathrm{e}}(\mathbf{Q}_p, M_{\text{ét}}(\mathcal{F})^*(-j)) \to M_{\mathrm{dR}}(\mathcal{F})^*_{\mathbf{Q}_p} / \mathrm{Fil}^{-j}$$

induced by the Bloch–Kato logarithm and the de Rham comparison isomorphism $M_{\mathrm{dR}}(\mathcal{F}) \cong \mathbf{D}_{\mathrm{dR}}(M_{\text{ét}}(\mathcal{F}))$.

Proof. Let $CG^{[k_1,k_2,j]} : \mathscr{H}^{[m]} \to \iota^* \mathscr{H}^{[k_1,k_2]}(-j)$ be the dual of the map $CG^{(k_1,k_2,j)}$. This map also makes sense in étale cohomology, and the étale Asai–Flach class was defined in [LLZ16] as the image of the class

$$\mathrm{AF}^{[k_1,k_2,j]}_{\text{ét},\mathfrak{N}} := (\iota_* \circ CG^{[k_1,k_2,j]})(\mathrm{Eis}^m_{\text{ét},N})$$

under projection to the $\mathcal{F}$-eigenspace.

We recall from [LLZ16] that the syntomic and étale Asai–Flach classes are both defined as realisations of the same class in motivic cohomology. The compatibility of syntomic and étale cohomology via the Bloch–Kato exponential map (cf. [KLZ15, Proposition 5.4.1]) therefore gives the equality

$$\log_p\left(\mathrm{AF}_{\text{ét}}^{[\mathcal{F},j]}\right) = AJ_{\mathcal{F},\mathrm{syn}}\left(\mathrm{AF}_{\mathrm{syn}}^{[k_1,k_2,j]}\right),$$

where $\mathrm{AF}_{\mathrm{syn}}^{[k_1,k_2,j]} = (\iota_* \circ CG^{[k_1,k_2,j]})(\mathrm{Eis}^m_{\mathrm{syn},N})$ and $AJ_{\mathcal{F},\mathrm{syn}}$ is the projection map

$$\begin{aligned} H^3_{\mathrm{syn}}(\mathcal{Y}, \mathscr{H}^{(k_1,k_2)}(2-j)) &\to H^1_{\mathrm{syn}}(\operatorname{Spec}\mathbf{Z}_p, M_{\mathrm{rig}}(\mathcal{F})^*(-j)) \\ &\cong \frac{M_{\mathrm{dR}}(\mathcal{F})^*(-j)}{(1-\varphi)\operatorname{Fil}^0 M_{\mathrm{dR}}(\mathcal{F})^*(-j)} \xrightarrow{\cong} M_{\mathrm{dR}}(\mathcal{F})^*/\operatorname{Fil}^{-j}, \end{aligned}$$

where the last map is given by $(1-\varphi)^{-1}$ (which is well-defined, since all eigenvalues of Frobenius on $M_{\mathrm{rig}}(\mathcal{F})^*(-j)$ are Weil numbers of weight $(-m-2) < 0$).

Via the compatibility of the cup-products in rigid and finite-polynomial cohomology,

$$\left\langle AJ_{\mathcal{F},\mathrm{syn}}\left(\mathrm{AF}_{\mathrm{syn}}^{[k_1,k_2,j]}\right), \eta\right\rangle_{\mathrm{dR},Y} = \left\langle \mathrm{AF}_{\mathrm{syn}}^{[k_1,k_2,j]}, \tilde{\eta}\right\rangle_{\mathrm{fp},P,\mathcal{Y}};$$

we use here the fact that $P(p^{-1}) \neq 0$ in order to define the right-hand side, since this is required to define the trace map $H^5_{\mathrm{fp},c}(\mathcal{Y}, \mathbf{Q}_p(3), P) \to L$. Since the cup-product in FP-cohomology satisfies the adjunction formula, we obtain the statement above. Q.E.D.

Proposition 5.1.5. *Suppose that at least one of the following hypotheses is satisfied:*

- $m \geqslant 1$ *(that is, we do not have $k_1 = k_2 = j$);*
- $\mathcal{F}$ *is not CM, and not twist-equivalent to its internal conjugate.*

Then the map

$$CG^{(k_1,k_2,j)} \circ \iota^* : H^2_{\mathrm{dR},c}\left(Y, \mathscr{H}_{\mathrm{dR}}^{(k_1,k_2)}(1+j)\right)_L \to H^2_{\mathrm{dR},c}\left(Y_{\mathbf{Q}}, \mathscr{H}_{\mathrm{dR}}^{(m)}(1)\right)_L$$

is zero on the direct summand $M_{\mathrm{dR}}(\mathcal{F})(1+j)$ of the domain.

Proof. There is nothing to prove unless $k_1 + k_2 = 2j$ (i.e. unless $m = 0$), since otherwise the target of this map is zero. To settle the remaining case we can use the comparison with étale cohomology: we have $M_{\mathrm{dR}}(\mathcal{F}) = \mathbf{D}_{\mathrm{dR}}\left(M_{\text{ét}}(\mathcal{F})\right)$, so it is sufficient to show that the map

$$CG^{(k_1,k_2,j)} \circ \iota^* : H^2_{\text{ét},c}\left(Y, \mathscr{H}^{(k_1,k_2)}_{\text{ét}}(1+j)\right)_L \to H^2_{\text{ét},c}\left(Y_{\mathbf{Q}}, \mathbf{Q}_p(1)\right)_L$$

is zero on the direct summand $M_{\text{ét}}(\mathcal{F})(1+j)$. But $H^2_{\text{ét},c}\left(Y_{\mathbf{Q}}, \mathbf{Q}_p(1)\right)_L$ is 1-dimensional since it is dual to $H^0_{\text{ét}}\left(Y_{\mathbf{Q}}, \mathbf{Q}_p\right)_L$, and $M_{\text{ét}}(\mathcal{F})$ is an irreducible 4-dimensional Galois representation (cf. Proposition 9.4.1 in [LLZ16]). Since the map $CG^{(k_1,k_2,j)} \circ \iota^*$ is Galois-equivariant, its restriction to $M_{\text{ét}}(\mathcal{F})$ must be zero. Q.E.D.

Recall that $m = k_1 + k_2 - 2j \geqslant 0$. It follows from the above that $(CG^{(k_1,k_2,j)} \circ \iota^*)(\tilde{\eta})$ lies in the image of the natural map

$$\frac{H^1_{\mathrm{rig},c}(\overline{Y}_{\mathbf{Q}}, \mathscr{H}^{(m)}(1))}{P(\varphi)\,\mathrm{Fil}^0} \to H^2_{\mathrm{fp},c}\left(\mathcal{Y}_{\mathbf{Q}}, \mathscr{H}^{(m)}(1), P\right).$$

(cf. [KLZ15, §2.5]). Moreover, this natural map is compatible with cup-product up to a factor of $\frac{1}{P(p^{-1})}$. We thus deduce the following:

Proposition 5.1.6. *If ξ is any element of $\in H^1_{\mathrm{rig},c}(\overline{Y}_{\mathbf{Q}}, \mathscr{H}^{(m)}(1))$ mapping to $(CG^{(k_1,k_2,j)} \circ \iota^*)(\tilde{\eta})$, then we have*

$$\begin{aligned}\left\langle \log_p\left(\mathrm{AF}^{[\mathcal{F},j]}_{\text{ét}}\right), \eta \right\rangle_{\mathrm{dR},Y} &= \frac{1}{P(p^{-1})}\left\langle \mathrm{Eis}^m_{\mathrm{rig},N}, \xi\right\rangle_{\mathrm{rig},\bar{Y}_{\mathbf{Q}}} \\ &= \frac{1}{P(p^{-1})}\left\langle \widetilde{\mathrm{Eis}}^m_{\mathrm{rig},N}, \xi|_{\bar{Y}^{\mathrm{ord}}_{\mathbf{Q}}}\right\rangle_{\mathrm{rig},\bar{Y}^{\mathrm{ord}}_{\mathbf{Q}}}.\end{aligned}$$

The restriction $\xi|_{\bar{Y}^{\mathrm{ord}}_{\mathbf{Q}}}$ is a class in $H^1_{\mathrm{rig},c-\partial}(\bar{Y}^{\mathrm{ord}}_{\mathbf{Q}}, \mathscr{H}^{(m)}(1))$. We have already seen that this rigid cohomology group has a simple presentation in terms of overconvergent cusp forms, and that the linear functional given by product with the rigid Eisenstein class corresponds to projection to the critical-slope Eisenstein quotient of this space. So we would like to compute a representative of $(CG^{(k_1,k_2,j)} \circ \iota^*)(\tilde{\eta})$ as an overconvergent modular form.

5.2. Representatives over the ordinary locus

We now recall how classes in finite-polynomial cohomology may be constructed. We shall not work with $\mathcal{Y}$ itself, but rather with a slightly modified version of the toroidal compactification $\mathcal{X}$: we let $\mathcal{X}^{\mathrm{ord}}$ be the complement, in $\mathcal{X}$, of the subscheme $\bar{Z}$ of supersingular points in the special fibre $\bar{X}$. Thus $\mathcal{X}^{\mathrm{ord}}$ is a smooth, but non-proper, $\mathbf{Z}_p$-scheme, whose generic fibre is the proper $\mathbf{Q}_p$-variety X, and whose special fibre is the non-proper $\mathbf{F}_p$-variety $\bar{X}^{\mathrm{ord}} = \bar{X} - \bar{Z}$ defined above.

As before, we fix k_1, k_2, j with $0 \leqslant j \leqslant \min(k_1, k_2)$, and we set $m = k_1 + k_2 - 2j$. We define a crude, but explicit, space which is an "approximation" to the finite-polynomial cohomology of $\mathcal{X}^{\mathrm{ord}}$:

Definition 5.2.1. *We let $B = B(\mathfrak{N}, k_1, k_2, P)$ denote the L-vector space consisting of pairs (f, g), where $f \in S_\mu(\mathfrak{N}, L)$ is a Hilbert modular form with coefficients in L, and $g = (g_1, g_2) \in S^\dagger_{w_1(\mu)}(\mathfrak{N}, L) \oplus S^\dagger_{w_2(\mu)}(\mathfrak{N}, L)$ is a pair of overconvergent forms satisfying*

$$P\left(p^{-1-j}\varphi\right)(f) = \Theta_1(g_1) + \Theta_2(g_2),$$

modulo the equivalence relation $(f, g_1, g_2) \cong (f, g_1 + \Theta_2(h), g_2 - \Theta_1(h))$ for $h \in S^\dagger_{w_1 w_2(\mu)}$.

Here φ acts on $S^\dagger_{(k_1+2,k_2+2)}$ as the Hecke operator $p^{k_1+k_2+2}\langle p\rangle V(p)$. Note that B is finite-dimensional: the natural map $(f, g) \mapsto f$ has finite-dimensional target, and its kernel is a presentation of the rigid cohomology group $H^1_{\mathrm{rig},c-\partial}\left(\bar{Y}^{\mathrm{ord}}, \mathscr{H}^{(\mu-2)}\right)$ and is therefore also finite-dimensional.

As in [Bes00b, Ban02], we attach to $\mathcal{X}^{\mathrm{ord}}$ and the filtered isocrystal $\mathscr{H}^{(k_1,k_2)}$ various complexes of L-vector spaces: de Rham cohomology complexes $\mathrm{Fil}^r C^\bullet_{\mathrm{dR}}$ for every $r \geqslant 0$; a rigid cohomology complex $C^\bullet_{\mathrm{rig}}(\mathcal{X}^{\mathrm{ord}})$ equipped with an action of Frobenius; and a specialisation map relating the two. The actual complexes $C^\bullet$ are rather hard to describe explicitly (they depend on various choices of injective resolutions), but once they are chosen, we can write $H^2_{\mathrm{fp},c-\partial}\left(\mathcal{Y}^{\mathrm{ord}}, \mathscr{H}^{(k_1,k_2)}(1+j), P\right)$ as a mapping fibre:

$$\frac{\left\{(x, y) : x \in \mathrm{Fil}^{1+j} C^2_{\mathrm{dR}}, y \in C^1_{\mathrm{rig}} \,\middle|\, dx = 0, dy = P(p^{-1-j}\varphi)\mathrm{sp}(x)\right\}}{\left\{(dx', P(\varphi)\mathrm{sp}(x') - dy') : x' \in \mathrm{Fil}^{1+j} C^1_{\mathrm{dR}}, y' \in C^0_{\mathrm{rig}}\right\}}. \tag{5}$$

Proposition 5.2.2. *There exist maps*

$$S_{(k_1,k_2)}(\mathfrak{N}) \to \mathrm{Fil}^{1+j} C^2_{\mathrm{dR}},$$

and

$$S^\dagger_{(-k_1,k_2+2)} \oplus S^\dagger_{(k_1+2,-k_2)} \to C^1_{\mathrm{rig}},$$

which assemble into a map $B \to H^2_{\mathrm{fp},c-\partial}\left(\mathcal{Y}^{\mathrm{ord}}, \mathscr{H}^{(k_1,k_2)}(1+j), P\right)$.

Proof. The complex $\mathrm{Fil}^m C^\bullet_{\mathrm{dR}}$ is given by the global sections of a suitable injective resolution of the algebraic de Rham complex of sheaves on X (cf. [TX16, §2.14]),

$$\mathrm{Fil}^m \mathrm{DR}^\bullet_c(\mathscr{H}^{(k_1,k_2)}) := \left(\mathrm{Fil}^{m-\bullet} \mathscr{H}^{(k_1,k_2)}(-C) \otimes \Omega^\bullet_{X/\mathbf{Q}_p}(\log C)\right).$$

Hence, for any m, there is a natural map

$$H^0\left(X, \mathrm{Fil}^{m-2} \mathscr{H}^{(k_1,k_2)}(-C) \otimes \Omega^2(\log C)\right) \to C^2_{\mathrm{dR}}.$$

However, the sheaf $\Omega^2(\log C)$ is just the automorphic line bundle $\omega^{(2,2)}$, and $\mathrm{Fil}^{k_1+k_2} \mathscr{H}^{(k_1,k_2)}$ is $\omega^{(k_1,k_2)}$. So the source of the above map is $H^0(X, \omega^{(k_1+2,k_2+2)}(-C)) = S_\mu(\mathfrak{N})$.

The rigid cohomology is handled similarly, replacing the variety X with the rigid-analytic space X^{rig}, and the algebraic de Rham complex with its overconvergent analogue $j^\dagger \mathrm{DR}^\bullet_c(\mathscr{H}^{(k_1,k_2)})$, where j is the inclusion of X^{ord} into X. This gives a natural map

$$H^0\left(X^{\mathrm{rig}}, j^\dagger \mathrm{DR}^\bullet_1(\mathscr{H}^{(k_1,k_2)})\right) \to C^\bullet_{\mathrm{rig}}.$$

However, the complex $\mathrm{DR}^\bullet_c(\mathscr{H}^{(k_1,k_2)})$ is quasi-isomorphic to its subcomplex $\mathrm{BGG}^\bullet_c(\mathscr{H}^{(k_1,k_2)})$ by Proposition 4.3.4, and $H^0(X^{\mathrm{rig}}, j^\dagger \mathrm{BGG}^1_c)$ is precisely the direct sum $S^\dagger_{(-k_1,k_2+2)} \oplus S^\dagger_{(k_1+2,-k_2)}$.

Finally, the specialisation map $C^\bullet_{\mathrm{dR}} \to C^\bullet_{\mathrm{rig}}$ is chosen to be compatible with the differentials and with the natural inclusion $H^0(X, \mathrm{DR}^\bullet_c) \hookrightarrow H^0(X^{\mathrm{rig}}, j^\dagger \mathrm{DR}^\bullet_c)$; so we do indeed obtain a map from B into the quotient (5). Q.E.D.

Remark 5.2.3. The map from B to H^2_{fp} is neither injective nor surjective in general. We do not know at present how to give a convenient presentation for the space H^2_{fp} in terms of p-adic modular forms.

Proposition 5.2.4. *If $b = (f, g_1, g_2) \in B$, and ρ is the image of b in H^2_{fp}, then $(CG^{(k_1,k_2,j)} \circ \iota^*)(\rho)$ is represented by the nearly-overconvergent elliptic modular form*

$$\frac{k_1!k_2!}{(k_1-j)!(k_2-j)!}\iota^*\left((-1)^j \delta_1^{k_1-j}(g_1) - \delta_2^{k_2-j}(g_2)\right) \in \mathbf{S}^{\dagger,\leqslant m}_{m+2}(N).$$

Proof. Via the map of complexes $\mathrm{BGG}^\bullet_c \hookrightarrow \mathrm{DR}^\bullet_c$ described above, the pair (g_1, g_2) determines an overconvergent rigid 1-differential on the ordinary locus of $X_1(\mathfrak{N})$ with values in $\mathscr{H}^{(k_1,k_2)}$, and clearly $\iota^*(\rho)$ is represented by the restriction to $X_{\mathbf{Q}}$ of this 1-differential (since the restriction map on 2-differentials is 0).

Let us recall how this map of complexes is defined in degree 1, where it is given by a map of sheaves

$$\begin{aligned}\left(\omega^{(-k_1,k_2+2)} \oplus \omega^{(k_1+2,-k_2)}\right)(-C) &\to \mathscr{H}^{(k_1,k_2)}(-C) \otimes \Omega^1(\log C)\\ &= \mathscr{H}^{(k_1,k_2)}(-C) \otimes (\omega^{(2,0)} \oplus \omega^{(0,2)}).\end{aligned}$$

This map is characterised by sending a section α of $\omega^{(-k_1,k_2+2)}(-C)$ to the unique section ξ of $\mathscr{H}^{(k_1,0)} \otimes \omega^{(0,k_2+2)}(-C)$ such that $\nabla(\xi)$ lies in $\omega^{(k_1+2,k_2+2)}(-C)$, and the image of ξ via the quotient map $\mathscr{H}^{(k_1,0)} \to \omega^{(-k_1,0)}$ is $(-1)^k k!\alpha$.

This can be given a more explicit form after restriction to the ordinary locus. We consider the *Hilbert modular Igusa tower* $\widetilde{\mathcal{Y}}$, which is the p-adic formal scheme classifying Hilbert–Blumenthal abelian surfaces A with polarisations, points of order $\mathfrak{N}$, and embeddings of $\mathcal{O}_F$-module schemes $\mu_{p^\infty} \otimes \mathcal{O}_F \hookrightarrow A$ (cf. [Hid04, §4.1]). Then $\widetilde{\mathcal{Y}}$ is an inverse limit of finite Galois coverings of the identity component of $\mathcal{Y}^{\mathrm{ord}}$, and as in the case of elliptic modular forms studied in [KLZ15, §4.5], the pullback of $\mathscr{H}^{(k_1,k_2)}$ to $\widetilde{\mathcal{Y}}$ has a canonical basis of sections $\{v^{(k_1-a_1,a_1)} \otimes v^{(k_2-a_2,a_2)} : 0 \leqslant a_i \leqslant k_i\}$. Given $g_1 \in S^\dagger_{(-k_1,k_2+2)}$ as above, we can consider the section of $\mathscr{H}^{(k_1,k_2)}(-C) \otimes \Omega^1(\log C)$ given by the formula

$$\xi_1 = \sum_{a=0}^{k_1} \frac{(-1)^a k_1!}{(k_1-a)!} \theta_1^{k_1-a}(g_1) \cdot (v^{(k_1-a,a)} \otimes v^{(k_2,0)}) \frac{\mathrm{d}q_2}{q_2},$$

where θ_1 is the differential operator acting on q-expansions as $q_1 \frac{\partial}{\partial q_1}$. One checks that this form lies in the subsheaf $\mathscr{H}^{(k_1,0)} \otimes \omega^{(0,k_2+2)}(-C)$, and maps to $(-1)^{k_1} k_1! g_1$ in the quotient $\omega^{(-k_1,k_2+2)}(-C)$. Moreover, computing $\nabla(\xi)$ using the formula for the Gauss–Manin connection in our basis given in [KLZ15, §4.5] (which carry over unchanged to the Hilbert case), the series telescopes and one finds that $\nabla(\xi) = \theta_1^{k+1}(g_1)$, which in particular lies in the subsheaf $\omega^{(k_1+2,k_2+2)}(-C)$. Thus ξ_1 must be the pullback to $\widetilde{\mathcal{Y}}$ of the image of g_1 in DR^1_c. Similarly, $g_2 \in S^\dagger_{w_2(\mu)}$ is mapped to

$$\xi_2 = -\sum_{a=0}^{k_2} \frac{(-1)^a k_2!}{(k_2-a)!} \theta_2^{k_2-a}(g_2) \cdot (v^{(k_1,0)} \otimes v^{(k_2-a,a)}) \frac{\mathrm{d}q_1}{q_1}.$$

If $\xi = \xi_1 + \xi_2$, then one verifies easily that

$$\nabla(\xi) = \left(\theta_1^{k_1+1}(g_1) + \theta_2^{k_2+1}(g_2)\right) \cdot \left(\tfrac{\mathrm{d}q_1}{q_1} \wedge \tfrac{\mathrm{d}q_2}{q_2}\right) = f \cdot \left(\tfrac{\mathrm{d}q_1}{q_1} \wedge \tfrac{\mathrm{d}q_2}{q_2}\right).$$

We now consider the images of these forms under the composition of pullback to $]\bar{X}^{\mathrm{ord}}_{\mathbf{Q}}[\subset X^{\mathrm{rig}}_{\mathbf{Q}}$ and the Clebsch–Gordan map. Considering the characters by which the diagonal torus acts, we see that Clebsch–Gordan must send $v^{(k_1-a,a)} \otimes v^{(k_2,0)}$ to zero if $a < j$, and to a scalar multiple of $v^{(m-a+j,a-j)}$ otherwise; and in the boundary case $a = j$, one computes (using the formulae in [KLZ15, §5.1]) that its image is $\frac{k_2!}{(k_2-j)!} v^{(m,0)}$. Similarly, $v^{(k_1,0)} \otimes v^{(k_2-j,j)}$ maps to $(-1)^j \frac{k_1!}{(k_1-j)!} v^{(m,0)}$ plus higher-order terms. Consequently, we have

$$(CG^{(k_1,k_2,j)} \circ \iota^*)(\xi) = \frac{k_1!k_2!}{(k_1-j)!(k_2-j)!} \iota^* \left((-1)^j \theta_1^{k_1-j}(g_1) - \theta_2^{k_2-j}(g_2) \right) v^{(m,0)} + \dots,$$

where the dots indicate terms involving $v^{(m-b,b)}$ with $b \geqslant 1$. Hence $(CG^{(k_1,k_2,j)} \circ \iota^*)(\xi)$ and the form in the statement of the proposition are both overconvergent sections of $\mathscr{H}^{(m)} \otimes \Omega^1_{X^{\mathrm{rig}}}$ whose images under the unit-root splitting coincide as p-adic modular forms, and hence they must be equal. Q.E.D.

5.3. Choice of the g_i

To make the above formulae completely explicit, we explain how to choose the polynomial P and the forms g_i giving a lifting of $\mathcal{F}$ to the quotient B. Recall that we are using the notation $P_p(\mathcal{F}, T)$ for the polynomial $\det(1 - T\varphi : M_{\mathrm{rig}}(\mathcal{F}))$, whose roots are the eigenvalues of φ^{-1} on $M_{\mathrm{rig}}(\mathcal{F})$. We consider the Hecke operator $P_p(\mathcal{F}, V(p))$ obtained by evaluating this polynomial at the operator $V(p)$ of Notation 4.3.2.

Lemma 5.3.1. *The overconvergent form*

$$P_p(\mathcal{F}, V(p)) \cdot \mathcal{F} \in S^\dagger_\mu(\mathfrak{N}, L)$$

is in the kernel of the operator $U(p)^3$.

Proof. It follows easily from the recurrences satisfied by the Fourier–Whittaker coefficients of $\mathcal{F}$ that

$$P_p(\mathcal{F}, V(p)) \cdot \mathcal{F} = (1 - \alpha_1\alpha_2\beta_1\beta_2 V(p^2))\mathcal{F}^{[p]},$$

where $\mathcal{F}^{[p]}$ is the p-depletion of $\mathcal{F}$. This is clearly in the kernel of $U(p)^3$. Q.E.D.

Since $U(p)$ acts invertibly on $H^2_{\mathrm{rig},c-\partial}(Y_1(\mathfrak{N})^{\mathrm{ord}}, \mathscr{H}^{(k_1,k_2)})$, it follows from Proposition 4.3.4 that $P_p(\mathcal{F}, V(p)) \cdot \mathcal{F}$ lies in the sum of the images of the two Θ operators. As the Frobenius map on $\mathscr{H}^{(k_1,k_2)} \otimes \Omega^2$ is given by $p^{k_1+k_2+2}\langle p\rangle V(p)$, this gives a lifting of $\mathcal{F}$ to B, taking the polynomial P to be

$$P(T) = P_p\left(\mathcal{F}, \frac{p^j T}{p^{k_1+k_2+1}\varepsilon_{\mathcal{F}}(p)}\right).$$

We can build a specific choice of lifting to B by considering Hecke operators at the primes $\mathfrak{p}_1, \mathfrak{p}_2$ above p. Recall that we have defined $U(\mathfrak{p}_i) = p^{-t_i}\mathcal{U}(\mathfrak{p}_i)$, and similarly $V(\mathfrak{p}_i)$.

Notation 5.3.2. *Write $P_{\mathfrak{p}_i}(\mathcal{F}, T)$ for the polynomial $(1-\alpha_{\mathfrak{p}_i}T)(1-\beta_{\mathfrak{p}_i}T)$.*

Thus $P_p(\mathcal{F}, T)$ is the "star product" of $P_{\mathfrak{p}_1}(\mathcal{F}, T)$ and $P_{\mathfrak{p}_2}(\mathcal{F}, T)$ in the notation of [Bes00a, §2] – the polynomial whose roots are the pairwise products of the roots of the two quadratics. One sees easily that for each $i \in \{1, 2\}$ we have $P_{\mathfrak{p}_i}(\mathcal{F}, V(\mathfrak{p}_i)) \cdot \mathcal{F} = \mathcal{F}^{[\mathfrak{p}_i]}$, the $\mathfrak{p}_i$-depletion of $\mathcal{F}$; this is in the kernel of $U(\mathfrak{p}_i)$, and hence defines the trivial class in H^2_{rig}, as before.

Proposition 5.3.3. *For each $i \in \{1, 2\}$, we can find a pair of forms $\left(g_1^{(i)}, g_2^{(i)}\right) \in S^\dagger_{w_1(\mu)} \oplus S^\dagger_{w_2(\mu)}$ such that:*

- *We have $\Theta_1\left(g_1^{(i)}\right) + \Theta_2\left(g_2^{(i)}\right) = \mathcal{F}^{[\mathfrak{p}_i]}$.*
- *For every prime $\mathfrak{q} \nmid p\mathfrak{N}$, the pair*

$$\left((\mathcal{T}(\mathfrak{q}) - \mu(\mathfrak{q}))\cdot g_1^{(i)}, (\mathcal{T}(\mathfrak{q}) - \mu(\mathfrak{q}))\cdot g_2^{(i)}\right)$$

  defines the zero class in $H^1_{\mathrm{rig},c-\partial}(Y^{\mathrm{ord}}, \mathscr{H}^{(k_1,k_2)})$, where $\mu(\mathfrak{q})$ is the $\mathcal{T}(\mathfrak{q})$-eigenvalue of $\mathcal{F}$.
- *Both $g_1^{(i)}$ and $g_2^{(i)}$ are in the kernel of $U(\mathfrak{p}_i)$.*

Proof. Since $U(\mathfrak{p}_i)$ acts invertibly on the rigid H^2, the existence of a pair $(g_1^{(i)}, g_2^{(i)})$ satisfying the first condition is immediate. Since the system of Hecke eigenvalues associated to $\mathcal{F}$ does not appear in $H^1_{\mathrm{rig}}(Y^{\mathrm{ord}}, \mathscr{H}^{(k_1,k_2)})$, we can arrange that the second condition is satisfied.

Finally, since the $\mathfrak{p}_1$-depletion operator $1 - V(\mathfrak{p}_1)U(\mathfrak{p}_1)$ acts on $S^\dagger_{w_1(\mu)} \oplus S^\dagger_{w_2(\mu)}$ compatibly with its Hecke action and with the map to $S^\dagger_\mu$, and it sends $\mathcal{F}^{[\mathfrak{p}_1]}$ to itself, applying this operator to an arbitrary pair $(g_1^{(i)}, g_2^{(i)})$ satisfying the first two conditions will give a pair satisfying all three. Q.E.D.

Remark 5.3.4.

(1) The third condition implies the second, since $U(\mathfrak{p}_i)$ acts invertibly on $H^1_{\mathrm{rig},c-\partial}\left(\bar{Y}^{\mathrm{ord}}, \mathscr{H}^{(k_1,k_2)}\right)$. In fact, one can check that this group actually vanishes unless $k_1 = k_2 = 0$; via various exact sequences this ultimately follows from the fact that the group $\mathrm{GL}_2(\mathcal{O}_F)$ has the congruence subgroup property, which forces $H^1(Y_1(\mathfrak{N}), -)$ to vanish for any coefficient sheaf.

(2) If Conjecture 4.5.2 holds for $\mathcal{F}$, then we can take $g_1^{(1)} = \Theta_1^{-1}(\mathcal{F}^{[\mathfrak{p}_1]})$, and $g_2^{(1)} = 0$. Similarly, we can choose $g_1^{(2)} = 0$ if Conjecture 4.5.2 holds for the internal conjugate $\mathcal{F}^\sigma$. However, we are not assuming this at present.

Since $P_p(\mathcal{F}, X)$ is the star product of $P_{\mathfrak{p}_1}$ and $P_{\mathfrak{p}_2}$, we can construct a preimage of $P_p(\mathcal{F}, V(p)) \cdot \mathcal{F}$ out of the four forms $g_r^{(s)}$, following the construction of cup-products in [Bes00a]. We choose polynomials $a(T_1, T_2)$ and $b(T_1, T_2)$ such that

$$a(T_1,T_2)P_{\mathfrak{p}_1}(\mathcal{F},T_1) + b(T_1,T_2)P_{\mathfrak{p}_2}(\mathcal{F},T_2) = P_p(\mathcal{F},T_1T_2). \tag{6}$$

Then the forms (h_1, h_2) defined by

$$h_1 = a\left(p^{-(k_1+1)}V(\mathfrak{p}_1), V(\mathfrak{p}_2)\right) g_1^{(1)} + b(p^{-(k_1+1)}V(\mathfrak{p}_1), V(\mathfrak{p}_2))g_1^{(2)}$$

and

$$h_2 = a(V(\mathfrak{p}_1), p^{-(k_2+1)}V(\mathfrak{p}_2))g_2^{(1)} + b(V(\mathfrak{p}_1), p^{-(k_2+1)}V(\mathfrak{p}_2))g_2^{(2)}$$

satisfy $\Theta_1(h_1) + \Theta_2(h_2) = P_p(\mathcal{F}, V(p)) \cdot \mathcal{F}$, and they define the unique lift of $\mathcal{F}$ to B which lies in the $\mathcal{F}$-eigenspace for the Hecke operators outside p.

Proposition 5.3.5. *The identity* (6) *is satisfied by the polynomials*

$$\begin{aligned} a(T_1,T_2) = \alpha_1\beta_1\alpha_2\beta_2(\alpha_2+\beta_2)T_1^2T_2^3 \\ - \alpha_1\beta_1\alpha_2\beta_2T_1^2T_2^2 - \alpha_2\beta_2(\alpha_1+\beta_1)T_1T_2^2 + 1 \end{aligned}$$

and

$$b(T_1,T_2) = \alpha_1^2\beta_1^2\alpha_2\beta_2T_1^4T_2^2 - \alpha_1\beta_1(\alpha_2+\beta_2)T_1^2T_2 - \alpha_1\beta_1T_1^2 + (\alpha_1+\beta_1)T_1.$$

These polynomials are carefully chosen so that almost all of their terms will contribute nothing to the final formula, because of the following lemma (which is an analogue for Hilbert modular forms of [DR14, Lemma 2.17] and [KLZ15, Lemma 6.4.6] in the Rankin–Selberg setting).

Lemma 5.3.6. *Suppose x, y are non-negative integers with $x > y$, and let $\mathcal{G}$ be a p-adic Hilbert modular form (not necessarily overconvergent) whose Fourier–Whittaker coefficients $c(\mathfrak{m}, \mathcal{G})$ are zero unless $v_{\mathfrak{p}_1}(\mathfrak{m}) \geqslant x$ and $v_{\mathfrak{p}_2}(\mathfrak{m}) = y$. Then the p-adic elliptic modular form $\iota^*(\mathcal{G})$ is in the kernel of $U(p)^{1+y}$.*

Proof. If $\lambda \in (\mathfrak{d}^{-1})^+$ satisfies $v_{\mathfrak{p}_1}(\lambda) \geqslant x$ and $v_{\mathfrak{p}_2}(\lambda) = y$, then we must have $v_p(\operatorname{Tr}\lambda) = y$. Since the coefficient of q^n in the Fourier expansion of $\iota^*(\mathcal{G})$ is given by $\sum_{\lambda : \operatorname{Tr}(\lambda) = n} \lambda^{-\underline{t}} c(\lambda, \mathcal{G})$, this implies that the Fourier expansion of $\iota^*(\mathcal{G})$ is supported on coefficients of p-adic valuation y, and is therefore in the kernel of $U(p)^{1+y}$. Q.E.D.

Since all the monomials in $b(T_1, T_2)$ are of the form $T_1^x T_2^y$ with $x > y$, and we are applying the operators $b(p^{-(k_1+1)}V(\mathfrak{p}_1), V(\mathfrak{p}_2))$ and $b(V(\mathfrak{p}_1), p^{-(k_2+1)}V(\mathfrak{p}_2))$ to forms which are $\mathfrak{p}_2$-depleted, the result will pull back to a differential which lies in the kernel of $U(p)$ and is therefore exact. Similarly, the terms involving $T_1^2 T_2^3$ and $T_1 T_2^2$ in $a(T_1, T_2)$ can be neglected.

So if $\eta = \omega_{\mathcal{F}}$ is the de Rham cohomology class associated to the eigenform $\mathcal{F}$, and $\tilde{\eta}$ is the unique Hecke-equivariant lifting of $\omega_{\mathcal{F}}$ to FP-cohomology, we conclude that the restriction of $CG^{(k_1,k_2,j)}(\iota^*(\tilde{\eta}))$ to the ordinary locus is represented by the nearly-overconvergent cusp form

$$(1 - \alpha_1\beta_1\alpha_2\beta_2 p^{-2-2j} V(p^2)) \cdot \left[(-1)^j \delta_1^{k_1-j} g_1^{(1)} - \delta_2^{k_2-j} g_2^{(1)}\right],$$

and the forms $g_1^{(2)}, g_2^{(2)}$ do not enter the formula. Combining this formula for $CG^{(k_1,k_2,j)}(\iota^*(\tilde{\eta}))$ with Proposition 5.1.6, we have:

Proposition 5.3.7. *Let $\omega_{\mathcal{F}}$ be the class in $\operatorname{Fil}^{k_1+k_2+2} M_{\mathrm{dR}}(\mathcal{F})$ corresponding to the eigenform $\mathcal{F}$, and let $g_1^{(1)}, g_2^{(1)}$ be overconvergent Hilbert modular forms satisfying the conditions of Proposition 5.3.3. Then*

$$\left\langle \log\left(\mathrm{AF}_{\text{ét}}^{[\mathcal{F},j]}\right), \omega_{\mathcal{F}} \right\rangle = \frac{\left(1 - \frac{p^{2j}}{\alpha_1\beta_1\alpha_2\beta_2}\right)}{\left(1 - \frac{p^j}{\alpha_1\alpha_2}\right)\left(1 - \frac{p^j}{\alpha_1\beta_2}\right)\left(1 - \frac{p^j}{\beta_1\alpha_2}\right)\left(1 - \frac{p^j}{\beta_1\beta_2}\right)}$$
$$\times \tfrac{k_1!k_2!}{(k_1-j)!(k_2-j)!} \left\langle \widetilde{\mathrm{Eis}}^m_{\mathrm{rig},N}, \Pi^{\mathrm{oc}} \iota^* \left((-1)^j \delta_1^{k_1-j}(g_1^{(1)}) - \delta_2^{k_2-j} g_2^{(1)}\right) \right\rangle_{\mathrm{rig}}.$$

Remark 5.3.8. Notice that all the products $\{\alpha_1\alpha_2, \dots, \beta_1\beta_2\}$ have complex absolute value $p^{(k_1+k_2+2)/2}$, which is strictly larger than p^j, so the Euler factors are all non-zero.

This formula is not convenient in practice, since we do not have an explicit description of the overconvergent forms $(g_1^{(1)}, g_2^{(1)})$. If Conjecture 4.5.2 holds (e.g. if $\mathcal{F}$ is non-ordinary at $\mathfrak{p}_2$), we can take $g_1^{(1)} =$

$\Theta_1^{-1}\left(\mathcal{F}^{[\mathfrak{p}_1]}\right)$ and $g_2^{(1)} = 0$; then we can write the above formula in terms of Rankin–Cohen brackets, since

$$\left(\Pi^{\mathrm{oc}} \circ \iota^* \circ \delta_1^{k_1-j}\right)\left(\Theta_1^{-1}\mathcal{F}^{[\mathfrak{p}_1]}\right) = \tfrac{(-1)^{k_1-j}(k_1-j)!(k_2-j)!}{(k_1+k_2-2j)!}\left[\Theta_1^{-1}\mathcal{F}^{[\mathfrak{p}_1]}\right]_{k_1-j}$$

modulo $\theta^{m+1}(S^{\dagger}_{-m}(N,L))$, by Proposition 4.4.5(ii). However, following an idea of Fornea [For17], we can modify the above argument slightly so we still obtain a canonically-defined answer without needing to impose additional hypotheses, using the fact that although we do not have uniquely-determined antiderivatives of $\mathcal{F}^{[\mathfrak{p}_1]}$ or $\mathcal{F}^{[\mathfrak{p}_2]}$, by Proposition 4.5.4 we do have such an antiderivative for $\mathcal{F}^{[\mathfrak{p}_1,\mathfrak{p}_2]}$. This gives our main theorem:

Theorem 5.3.9 (Theorem B)**.** *We have the formula*

$$\left\langle \log\left(\mathrm{AF}^{[\mathcal{F},j]}_{\text{ét}}\right), \omega_{\mathcal{F}}\right\rangle = \frac{\left(1-\frac{p^{2j}}{\alpha_1\beta_1\alpha_2\beta_2}\right)}{\left(1-\frac{p^j}{\alpha_1\alpha_2}\right)\left(1-\frac{p^j}{\alpha_1\beta_2}\right)\left(1-\frac{p^j}{\beta_1\alpha_2}\right)\left(1-\frac{p^j}{\beta_1\beta_2}\right)}$$
$$\times \frac{(-1)^{k_1}k_1!k_2!}{(k_1+k_2-2j)!}\left\langle \widetilde{\mathrm{Eis}}^{k_1+k_2-2j}_{\mathrm{rig},N}, \left[\Theta_1^{-1}\mathcal{F}^{[\mathfrak{p}_1,\mathfrak{p}_2]}\right]_{k_1-j}\right\rangle_{\mathrm{rig}}.$$

In particular, if the projection of $\left[\Theta_1^{-1}\mathcal{F}^{[\mathfrak{p}_1,\mathfrak{p}_2]}\right]_{k_1-j}$ *to the critical-slope Eisenstein quotient is non-zero, then the étale Asai–Flach class* $\mathrm{AF}^{[\mathcal{F},j]}_{\text{ét}}$ *is also non-zero.*

Proof. We replace the identity $P_p(T_1T_2) = a(T_1,T_2)P_{\mathfrak{p}_1}(\mathcal{F},T_1) + b(T_1,T_2)P_{\mathfrak{p}_2}(\mathcal{F},T_2)$ with the slightly different identity

$$\begin{aligned}P_p(T_1T_2) = (1 - \alpha_1\beta_1\alpha_2\beta_2 T_1^2T_2^2)&P_{\mathfrak{p}_1}(T_1)P_{\mathfrak{p}_2}(T_2)\\ &+ b(T_1,T_2)P_{\mathfrak{p}_2}(T_2) + b'(T_1,T_2)P_{\mathfrak{p}_1}(T_1),\end{aligned}$$

where b' is the polynomial obtained from b by interchanging the indices 1 and 2 throughout. Substituting in $V(\mathfrak{p}_i)$ for T_i, this gives us

$$\begin{aligned}P_p(\mathcal{F},V(p))\cdot\mathcal{F} = (1 - \alpha_1\beta_1\alpha_2\beta_2 &V(p)^2)\mathcal{F}^{[\mathfrak{p}_1,\mathfrak{p}_2]}\\ &+ b(V(\mathfrak{p}_1),V(\mathfrak{p}_2))\mathcal{F}^{[\mathfrak{p}_2]} + b'(V(\mathfrak{p}_1),V(\mathfrak{p}_2))\mathcal{F}^{[\mathfrak{p}_1]}.\end{aligned}$$

We use this to construct an integral of $P_p(\mathcal{F},V(p))\cdot\mathcal{F}$, as before. Using the fact that $b(T_1,T_2)$ contains only monomials with higher powers of T_1 than T_2, and vice versa for $b'(T_1,T_2)$, the integrals of the second and third terms become exact after pulling back to $\mathcal{Y}_{\mathbf{Q}}$. This gives the formula stated. Q.E.D.

§6. An example for $D = 13$

6.1. The newform $\mathcal{F}$

Let F be the field $\mathbf{Q}(\sqrt{13})$. Note that this field has narrow class number 1. We let $\sigma_1 : F \hookrightarrow \mathbf{R}$ be the embedding corresponding to the positive square root, and σ_2 its conjugate.

Using Dembélé's algorithms for computing Hilbert modular forms via Brandt matrices (cf. [Dem07]), which are implemented in Magma [BCP97], we find that there is a unique Hilbert modular form $\mathcal{F}$ over F of weight $(2, 8, 3, 0)$ and level 1, up to scalars. If we write $\mu(\mathfrak{m})$ for the $\mathcal{T}(\mathfrak{m})$-eigenvalue of $\mathcal{F}$, then the quantities $\mu(\mathfrak{m})$ all lie in the field F itself. For the first few prime values of $\mathfrak{m}$ the values of $\mu(\mathfrak{m})$ are given by the following table:

prime $\mathfrak{p}$	$\mathrm{Nm}(\mathfrak{p})$	$\mathcal{T}(\mathfrak{p})$-eigenvalue $\mu(\mathfrak{p})$
2	4	-104
$(\sqrt{13}+5)/2$	3	$-3\sqrt{13}-60$
$(-\sqrt{13}+5)/2$	3	$3\sqrt{13}-60$
5	25	-11375
7	49	-1368913
11	121	-2664662
$(3\sqrt{13}+13)/2$	13	-3380
$(\sqrt{13}+9)/2$	17	$-3744\sqrt{13}-15795$
$(-\sqrt{13}+9)/2$	17	$3744\sqrt{13}-15795$
19	361	556580414
$\sqrt{13}+6$	23	$9438\sqrt{13}+35100$
$-\sqrt{13}+6$	23	$-9438\sqrt{13}+35100$
$2\sqrt{13}+9$	29	$19860\sqrt{13}-84456$
$-2\sqrt{13}+9$	29	$-19860\sqrt{13}-84456$

(The left-hand column gives, for each ideal, the totally-positive generator having the smallest possible trace.) Notice that $\lambda(\mathfrak{p})$ is always divisible by $\mathfrak{p}^3$, since $t_1 = 3$. Moreover, $\lambda(\sigma(\mathfrak{m})) = \sigma(\lambda(\mathfrak{m}))$, where σ is the Galois automorphism of F. (This can be used to speed up the computations somewhat, since it is not necessary to compute $\lambda(\mathfrak{p})$ and $\lambda(\sigma(\mathfrak{p}))$ separately.) We normalise $\mathcal{F}$ by setting $c(\mathfrak{d}^{-1}, \mathcal{F}) = 1$ (this is different from the normalisation used in [LLZ16], but it makes the computations simpler). Then we have $c(\lambda, \mathcal{F}) = \mu(\mathfrak{d}\lambda)$, and the values $\mu(\mathfrak{m})$ for all $\mathfrak{m}$ are easily computed once one knows $\lambda(\mathfrak{p})$ for each prime $\mathfrak{p}$.

We set $p = 3$, and we embed F in $\mathbf{Q}_3$ using the embedding corresponding to the prime $\mathfrak{p}_1 = (\sqrt{13}+5)/2$. Then $\mathcal{F}$ is ordinary at $\mathfrak{p}_1$, but non-ordinary at $\mathfrak{p}_2$, since its $\mathcal{T}(\mathfrak{p}_2)$-eigenvalue maps to $2\cdot 3+3^2+2\cdot 3^3+\dots$. Hence $\mathcal{F}^{[\mathfrak{p}_1]}$ is in the image of Θ_1.

For any $n \in \mathbf{N}$ the set $\{\lambda \in \left(\mathfrak{d}^{-1}\right)^+ : \mathrm{Tr}(\lambda) = n\}$ is finite (and easy to compute), so one can evaluate the q-expansion of the overconvergent elliptic modular form $\iota^*\left(\Theta_1^{-1}\mathcal{F}^{[\mathfrak{p}_1]}\right)$ up to degree N via the formula

$$\iota^*\left(\Theta_1^{-1}\mathcal{F}^{[\mathfrak{p}_1]}\right) = \sum_{n\geqslant 1}\left(\sum_{\substack{\lambda\in(\mathfrak{d}^{-1})^+\\ \mathrm{Tr}(\lambda)=n,\ \mathfrak{p}_1\nmid\lambda}} \frac{c(\lambda,\mathcal{F})}{\sigma_1(\lambda)^4}\right) q^n.$$

(Evaluating the q-expansion up to degree N requires the computation of the $\lambda(\mathfrak{p})$ for primes $\mathfrak{p}$ of norm up to $\frac{13}{4}N^2$.) Since $\mathcal{F}$ has level 1, this must be an overconvergent 3-adic modular form of tame level 1 and weight 8. The theory does not seem to give any immediate bound for its radius of overconvergence; since $\mathcal{F}^{[\mathfrak{p}_1]}$ is r-overconvergent for every $r < 1/4$, it seems likely that $\iota^*\left(\Theta_1^{-1}\mathcal{F}^{[\mathfrak{p}_1]}\right)$ should also have this property, but we have not proved this.

Remark 6.1.1. A similar result is sketched for elliptic modular forms in [Lau14, §2.3.2], but the argument does not seem to generalise to this 2-dimensional setting.

6.2. A basis for overconvergent modular forms

Since $X_0(3)$ has genus 0, one has a convenient explicit presentation for the space $S_k^\dagger(1,r)$ of r-overconvergent 3-adic cusp forms, for any even integer $k \geqslant 2$. For $k = 8$ and any $r < \frac{p}{p+1} = \frac{3}{4}$, a Banach basis is given by the forms

$$\left(3^{\lfloor 6rn\rfloor} g^n \cdot E_8^{\mathrm{ord}}\right)_{n\geqslant 1}$$

where $E_8^{\mathrm{ord}} = 1 - \frac{240}{1093}\sum_{n\geqslant 1}\left(\sum_{d\mid n, 3\nmid d} d^7\right) q^n$ is the ordinary weight 8 Eisenstein series, and g is the meromorphic modular function $\left(\frac{\Delta(3z)}{\Delta(z)}\right)^{1/2}$, which gives an isomorphism $X_0(3) \cong \mathbf{P}^1$. See [Loe07] for further details. For the purposes of our example we will take $r = \frac{1}{6}$.

The matrix A of the Hecke operator $U(3)$ on $S_k^\dagger(1,r)$ in the above basis has been extensively studied by many authors (going back to work of Kolberg in the 1960s), and the entries satisfy a wealth of congruences and recurrence relations. Using these relations, one can verify that for

$k = 8$ and $r = \frac{1}{6}$, the matrix entries $(a_{ij})_{1 \leqslant i,j \leqslant \infty}$ have the following two properties: [5]

- If $j > 3i$ then $a_{ij} = 0$.
- For all i, j we have $v_3(a_{ij}) \geqslant 2i$.

It follows that all entries of the matrix lie $3^2\mathbf{Z}_3$, and for any $N \geqslant 1$, we have $a_{ij} = 0 \bmod 3^{2N}$ if $i \geqslant N$ or if $j \geqslant 3N - 2$. For instance, modulo 3^{10} the only non-zero entries of the matrix are

$$\begin{pmatrix} 48087 & 21195 & 9 & & & & & & & & & \\ 4374 & 52488 & 51030 & 8019 & 14580 & 81 & & & & & & \\ & & 39366 & & 6561 & 6561 & 15309 & 21870 & 729 & & & \\ & & & & & & & & & 39366 & 39366 & 6561 \end{pmatrix}.$$

6.3. Numerical linear algebra

Definition 6.3.1. *Let us say that an infinite matrix $A = (a_{ij})_{i,j \geqslant 1}$ over $\mathbf{Z}_p$ is* computable *if, for every $N \geqslant 1$, there exists $R = R(N) \geqslant 0$ such that $v_p(a_{ij}) \geqslant N$ whenever $i > R$ or $j > R$, and there is an algorithm which, given an integer N, computes such a bound $R(N)$ and the values $a_{ij} \bmod p^N$ for all $1 \leqslant i, j \leqslant R(N)$.*

Via the theory of Newton polygons, one sees that if A is computable, then the dimension of the slope $\leqslant n$ subspace of A (the sum of the generalised eigenspaces for all eigenvalues of valuation $\leqslant n$) is a computable function of $n \in \mathbf{Z}_{\geqslant 0}$.

Remark 6.3.2. More precisely, for each integer $r \geqslant 0$ let us define $c_r \in \mathbf{Z}_p$ to be $(-1)^r$ times the sum of the determinants of the $r \times r$

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[5] The first property is obvious. Let us sketch the proof of the second. It is convenient to extend the definition of a_{ij} to allow $i = 0$ or $j = 0$ (which gives the matrix of $U(3)$ on the full space of overconvergent forms $M_8^\dagger(1, r)$). Then the operator $U(3)$ is an "operator of rational generation" in the sense of [Smi04]: the generating function $\sum_{i,j \geqslant 0} a_{ij} X^i Y^j$ is a rational function. Explicitly, it is given by

$$\frac{\sum_{i,j} m_{ij} X^i Y^j}{(1093+2106X-2187X^2)(1-270XY-8748X^2Y-108XY^2-59049X^3Y-729X^2Y^2-9XY^3)},$$

where m_{ij} are the entries of the matrix

$$\begin{pmatrix} 1093 & 2106 & -2187 & 0 & 0 & 0 \\ 0 & -230580 & -34222176 & -2449943010 & -48920206932 & -282300396318 \\ 0 & -40068 & -5959575 & -304338546 & -1742595039 & 0 \end{pmatrix}.$$

Substituting $3^{-2}X$ and Y in place of X and Y gives the rational function whose coefficients are $3^{-2i}a_{ij}$; and this function is easily seen to be a ratio of polynomials over $\mathbf{Z}$ whose constant terms are 3-adic units, so its power-series coefficients are in $\mathbf{Z}_3$. One can prove in the same way the slightly stronger bound $v_3(a_{i,3i-t}) \geqslant 2i + \frac{1}{2}t$, which is the optimal linear bound on $v_3(a_{ij})$.

diagonal minors of A (the trace of $\bigwedge^r A$), so that formally $\sum_{r \geqslant 0} c_r t^r =$ "$\det(1 - tA)$". Then the dimension of the slope n subspace is equal to the total length of the edges of slope n in the Newton polygon of A, which is the convex hull of the points $\{(r, c_r) : r \geqslant 0\}$. Any vertex (r, c_r) such that $c_r > rn$ will not affect the slope n edges. However, it is easily seen that $v_p(c_r) \geqslant N(r - R(N))$ for any $N \geqslant 1$, and if $N > n$ then this is eventually larger than rn; so the set of r such that $c_r \leqslant rn$ is finite, and computable as a function of n.

We now define a "condition number" for non-zero eigenvalues of computable matrices. For simplicity, we suppose that the eigenvalue λ is known exactly as an element of $\mathbf{Z}_p \cap \mathbf{Q}^\times$, and that the λ-eigenspace is one-dimensional, as this is the case in all the examples we shall consider.

Definition 6.3.3. *We define the* condition number *of λ to be the largest non-zero power of p appearing as an elementary divisor of the $R(N) \times R(N)$ truncation of $(A - \lambda) \bmod p^N$, where $N > v_p(\lambda)$ is any integer sufficiently large that this truncation has exactly one elementary divisor which is zero (in $\mathbf{Z}/p^N\mathbf{Z}$).*

Note that the condition number is always at least $v_p(\lambda)$ (but it may be much larger). If c is the condition number of λ, then the image modulo p^N of the kernel of $(A - \lambda) \bmod p^{N+c}$ is free of rank 1 over $\mathbf{Z}/p^N\mathbf{Z}$. Since it must contain the mod p^N reduction of the kernel of $A - \lambda$, these spaces must be equal. Thus we may calculate the mod p^N reduction of the λ-eigenspace of A by performing our calculations modulo p^{N+c}.

We now apply this to our $U(3)$ example. As we saw in the previous section, the matrix of $U(3)$ in the Kolberg basis of $S_8^\dagger(1, \frac{1}{6})$ is computable (and it suffices to take $R(N) = 3\lceil \frac{N}{2} \rceil - 3$). We find that the slope $\leqslant 7$ subspace is 2-dimensional, and hence must be spanned by the classical level 3 newform $q + 6q^2 - 27q^3 + \dots$ (whose $U(3)$-eigenvalue is -27, of slope 3) and the critical-slope Eisenstein series

$$E_{\mathrm{crit}}^{(8)} = \sum_{n \geqslant 1} \left(\sum_{d \mid n, 3 \nmid d} (n/d)^7 \right) q^n = q + 129q^2 + 2187q^3 + 16513q^4 + \dots$$

(whose $U(3)$-eigenvalue is 3^7). In particular, both of these $U(3)$ eigenspaces are 1-dimensional, so the critical-slope Eisenstein series is *not* a critical eigenform in the sense of [Bel12, Definition 2.12].

We computed the matrix A modulo 3^{20} (which is zero outside the top left 27×27 submatrix) and computed the Smith normal form of $A - 3^7$. The smallest non-zero elementary divisor of this matrix was

3^9, so exactly 9 digits of 3-adic precision were lost, and this computation determines the kernel of $A - 3^7$ modulo 3^{11}: it is spanned by $(42041, 1, 54513, 21870, 0, 0, \dots)^t$. Up to a normalisation factor this is (of course) just the expansion of $E^{(8)}_{\mathrm{crit}}$ in the Kolberg basis. Much more interestingly, this computation also determines the kernel of $(A - 3^7)^t$, so we can use it to write down a non-zero linear functional factoring through projection to the critical-slope Eisenstein subspace.

6.4. The result

We computed the Hecke eigenvalues of $\mathcal{F}$ for all primes of norm up to $10,000$, which was sufficient to determine the first 55 coefficients of the form $h = \iota^*\left(\Theta_1^{-1}\mathcal{F}^{[\mathfrak{p}_1]}\right)$ in the Kolberg basis of $S_8^\dagger(1, \frac{1}{6})$. These coefficients appeared to be tending rapidly to zero 3-adically; in fact the coefficient b_n of the n-th basis vector appeared to have p-adic valuation growing approximately as $\frac{1}{2}n$ (supporting our conjecture that this form is r-overconvergent for all $r < \frac{1}{4}$). However, since we have no precise bounds on the b_n, we have been forced to assume such a bound:

Conjecture 6.4.1. *We have $v_3(b_n) \geqslant 10$ for all $n > 55$.*

Under this conjecture, we find that the coefficient of q in the critical-slope Eisenstein projection of h is $3^{-2} + 3^{-1} + 2 + 3^1 + 2 \cdot 3^2 + 3^5 + 2 \cdot 3^6 + O(3^7)$. In particular, it is non-zero.

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