Hill *Res. Number Theory* (2019) 5:2
https://doi.org/10.1007/s40993-018-0140-z

Research in Number Theory

RESEARCH

Non-residually finite extensions of arithmetic groups

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Abstract

The aim of the article is to show that there are many finite extensions of arithmetic groups which are not residually finite. Suppose G is a simple algebraic group over the rational numbers satisfying both strong approximation, and the congruence subgroup problem. We show that every arithmetic subgroup of G has finite extensions which are not residually finite. More precisely, we investigate the group

$$\bar{H}^2(\mathbb{Z}/n) = \lim_{\substack{\rightarrow \\ \Gamma}} H^2(\Gamma, \mathbb{Z}/n),$$

where Γ runs through the arithmetic subgroups of G. Elements of $\bar{H}^2(\mathbb{Z}/n)$ correspond to (equivalence classes of) central extensions of arithmetic groups by $\mathbb{Z}/n$; non-zero elements of $\bar{H}^2(\mathbb{Z}/n)$ correspond to extensions which are not residually finite. We prove that $\bar{H}^2(\mathbb{Z}/n)$ contains infinitely many elements of order n, some of which are invariant for the action of the arithmetic completion $\widehat{G(\mathbb{Q})}$ of $G(\mathbb{Q})$. We also investigate which of these (equivalence classes of) extensions lift to characteristic zero, by determining the invariant elements in the group

$$\bar{H}^2(\mathbb{Z}_l) = \lim_{\substack{\leftarrow \\ t}} \bar{H}^2(\mathbb{Z}/l^t).$$

We show that $\bar{H}^2(\mathbb{Z}_l)^{\widehat{G(\mathbb{Q})}}$ is isomorphic to $\mathbb{Z}_l{}^c$ for some positive integer c. When $G(\mathbb{R})$ has no simple components of complex type, we prove that $c = b + m$, where b is the number of simple components of $G(\mathbb{R})$ and m is the dimension of the centre of a maximal compact subgroup of $G(\mathbb{R})$. In all other cases, we prove upper and lower bounds on c; our lower bound (which we believe is the correct number) is $b + m$.

Keywords: Cohomology of arithmetic groups, Congruence subgroup property, Residually finite group

Mathematics Subject Classification: 11F77, 11F06

1 Introduction

An abstract group G is said to be residually finite if, for every non-trivial element g, there is a subgroup H of finite index in the group, which does not contain g. The content of this statement is not changed if we insist that H is a normal subgroup of G. This is equivalent to the statement that the canonical map from the group to its profinite completion is injective.

Arithmetic groups are residually finite. Indeed, if Γ is an arithmetic group and $1 \neq \gamma \in \Gamma$, then there is even a congruence subgroup which does not contain γ. On the other hand, Deligne wrote down a central extension $\tilde{\Gamma}$ of $\mathrm{Sp}_{2n}(\mathbb{Z})$ ($n \geq 2$) by $\mathbb{Z}$, such that $\tilde{\Gamma}$ is not residually finite. More precisely, the group $\tilde{\Gamma}$ fits into an exact sequence:

$$1 \to \mathbb{Z} \to \tilde{\Gamma} \to \mathrm{Sp}_{2n}(\mathbb{Z}) \to 1,$$

and any subgroup of finite index in $\tilde{\Gamma}$ contains $2\mathbb{Z}$.

In this note, we show that a weaker version of Deligne's result holds for a large class of arithmetic groups.

We briefly recall Deligne's construction. The Lie group $\mathrm{Sp}_{2n}(\mathbb{R})$, is not simply connected. In fact, its fundamental group is isomorphic to $\mathbb{Z}$. We shall write $\tilde{\mathrm{Sp}}_{2n}(\mathbb{R})$ for the universal cover of $\mathrm{Sp}_{2n}(\mathbb{R})$, so we have an exact sequence:

$$1 \to \mathbb{Z} \to \tilde{\mathrm{Sp}}_{2n}(\mathbb{R}) \to \mathrm{Sp}_{2n}(\mathbb{R}) \to 1.$$

One defines $\tilde{\Gamma}$ to be the preimage of $\mathrm{Sp}_{2n}(\mathbb{Z})$ in $\tilde{\mathrm{Sp}}_{2n}(\mathbb{R})$. Note that $\tilde{\mathrm{Sp}}_{2n}(\mathbb{R})$ is a Lie group, but is not the group of real points of an algebraic group; in fact Sp_{2n} is simply connected as an algebraic group. Thus $\tilde{\Gamma}$ is not an arithmetic group.

There are some cases for which Deligne's argument generalizes easily. Suppose G is an algebraic group over $\mathbb{Q}$, which is simple and simply connected. As we have seen above, the group $G(\mathbb{R})$ may fail to be simply connected with the archimedean topology; this happens whenever a maximal compact subgroup of $G(\mathbb{R})$ has infinite centre. We shall assume that the fundamental group $\pi_1(G(\mathbb{R}))$ has more than 2 elements. We can define just as before an extension

$$1 \to \pi_1(G(\mathbb{R})) \to \tilde{\Gamma} \to \Gamma \to 1,$$

where Γ is an arithmetic subgroup of $G(\mathbb{Q})$. There is also a canonical double cover $\tilde{G}(\mathbb{R})^{\mathrm{met}}$ of $G(\mathbb{R})$, called the metaplectic cover:

$$1 \to \mu_2 \to \tilde{G}(\mathbb{R})^{\mathrm{met}} \to G(\mathbb{R}) \to 1, \quad \mu_2 = \{1, -1\}.$$

By the universal property of the universal cover, there is a canonical map

$$\pi_1(G(\mathbb{R})) \to \mu_2.$$

Deligne's argument shows that if G has the congruence subgroup property, then every subgroup of finite index in $\tilde{\Gamma}$ contains $\ker\left(\pi_1(G(\mathbb{R})) \to \mu_2\right)$.

To show that this generalization is not vacuous, we remark that $\pi_1(G(\mathbb{R}))$ is infinite whenever there is a Shimura variety associated to G, and the congruence subgroup property is known to hold for simple, simply connected groups of rational rank at least 2.

In this paper, we shall deal also with groups G, for which Deligne's construction cannot be used. The most easily stated consequence of our results is the following.

Theorem 1 *Let G be a simple algebraic group over $\mathbb{Q}$, which is algebraically simply connected, and has positive real rank. Assume also that G and has finite congruence kernel. Let Γ be an arithmetic subgroup of $G(\mathbb{Q})$. Then there is a finite abelian group A and an extension of groups*

$$1 \to A \to \tilde{\Gamma} \to \Gamma \to 1,$$

such that $\tilde{\Gamma}$ is not residually finite.

1.1 Residual finiteness of hyperbolic groups

It is an important open question in geometric group theory whether every Gromov-hyperbolic group is residually finite (see for example [1,9,11,15,26]). This question turns out to be related to the following conjecture of Serre [20].

Conjecture 1 Let $G/\mathbb{Q}$ be a simple, simply connected algebraic group of real rank 1. Then the congruence kernel of G is infinite.

As a consequence of Theorem 1, we obtain the following.

Corollary 1 *If every Gromov–hyperbolic group is residually finite then Conjecture 1 is true.*

Proof Let Γ be an arithmetic subgroup of a Lie group with real rank 1. It is known that Γ is Gromov–hyperbolic (see chapter 7 of [10]). Since hyperbolicity is invariant under quasi-isometry, every finite extension of Γ is also hyperbolic, and hence by assumption residually finite. If the congruence kernel were finite, then the groups $\tilde{\Gamma}$ from Theorem 1 would provide a counterexample to this.

In fact one can show as a consequence of the results proved here the following slightly more precise result.

Corollary 2 *Assume that every Gromov–hyperbolic group is residually finite. If $G/\mathbb{Q}$ is a simple, simply connected group of real rank 1 then for every positive integer n, the congruence kernel of G has a subquotient isomorphic to $\mathbb{Z}/n$.*

2 Statement of results

Throughout this section, we fix a simple algebraic group $G/\mathbb{Q}$, such that

1. G is (algebraically) simply connected;
2. G has positive real rank (i.e. $G(\mathbb{R})$ is not compact, and arithmetic subgroups of G are infinite);
3. The congruence kernel of $G/\mathbb{Q}$ is finite (and hence conjecturally the real rank of G is at least 2).

We do not assume that G is absolutely simple.

We'll show that Theorem 1 is a consequence of the following result.

Theorem 2 *Let $G/\mathbb{Q}$ be as described above and let Γ be an arithmetic subgroup of $G(\mathbb{Q})$. For every positive integer n there is a subgroup Δ of finite index in Γ and a central extension*

$$1 \to \mathbb{Z}/n \to \tilde{\Delta} \to \Delta \to 1,$$

such that $\tilde{\Delta}$ is not residually finite. More precisely, every subgroup of finite index in $\tilde{\Delta}$ contains the subgroup $\mathbb{Z}/n$.

Proof of Theorem 1 We'll now show that Theorem 1 is a consequence of Theorem 2. Let Γ be an arithmetic group with finite congruence kernel. By Theorem 2, there is a subgroup Δ of finite index in Γ and a central extension $\tilde{\Delta}$ of Γ by $\mathbb{Z}/n$, such that every subgroup of finite index in $\tilde{\Delta}$ contains $\mathbb{Z}/n$. Let $\sigma \in H^2(\Delta, \mathbb{Z}/n)$ be the cohomology class corresponding to this extension. By Shapiro's lemma, there is an isomorphism $H^2(\Delta, \mathbb{Z}/n) \cong H^2(\Gamma, A)$, where A is the induced representation $A = \mathrm{ind}_\Delta^\Gamma(\mathbb{Z}/n)$. We'll write Σ for the image of σ in $H^2(\Gamma, A)$. Corresponding to the cohomology class Σ, there is a (non-central) extension $\tilde{\Gamma}$ of Γ by A. These two group extensions are related by the following commutative diagram:

$$\begin{array}{ccccccccccc}
1 & \longrightarrow & A & \longrightarrow & \tilde{\Gamma} & \xrightarrow{\mathrm{pr}} & \Gamma & \longrightarrow & 1 & & (\Sigma) \\
 & & \Vert & & \uparrow & & \uparrow & & & & \\
1 & \longrightarrow & \mathrm{ind}_\Delta^\Gamma(\mathbb{Z}/n) & \longrightarrow & \mathrm{pr}^{-1}(\Delta) & \longrightarrow & \Delta & \longrightarrow & 1 & & \\
 & & \downarrow & & \downarrow & & \Vert & & & & \\
1 & \longrightarrow & \mathbb{Z}/n & \longrightarrow & \tilde{\Delta} & \longrightarrow & \Delta & \longrightarrow & 1 & & (\sigma)
\end{array}$$

Suppose for the sake of argument that $\tilde{\Gamma}$ is residually finite. Hence the subgroup $\mathrm{pr}^{-1}\Delta$ is residually finite. There is therefore a subgroup $\Phi \subset \mathrm{pr}^{-1}(\Delta)$ of finite index, such that $\Phi \cap A$ is trivial. The image of Φ in $\tilde{\Delta}$ is then a subgroup of finite index in $\tilde{\Delta}$, whose intersection with $\mathbb{Z}/n$ is trivial. This is a contradiction. □

2.1 Some refinements of Theorem 2

Let $G/\mathbb{Q}$ be simple, simply connected, and have real rank at least 1. Furthermore assume that the congruence kernel of G is finite (and hence, conjecturally at least, the real rank of G is at least 2). Fix an arithmetic subgroup Γ of G.

Suppose that we have a central extension of Γ by $\mathbb{Z}/n$ as follows:

$$1 \to \mathbb{Z}/n \to \tilde{\Gamma} \to \Gamma \to 1.$$

We shall write $\sigma \in H^2(\Gamma, \mathbb{Z}/n)$ for the cohomology class of this extension. Suppose for a moment that $\tilde{\Gamma}$ is residually finite. We can then find a subgroup $\Delta \subset \tilde{\Gamma}$ of finite index, such that the intersection of Δ with $\mathbb{Z}/n$ is trivial. Hence Δ projects bijectively onto a subgroup of Γ, which we shall also call Δ. The preimage of Δ in $\tilde{\Gamma}$ is the direct sum $\mathbb{Z}/n \oplus \Delta$. As a result of this, we know that the restriction of σ to Δ is trivial.

This means that in order to construct a non-residually finite extension of Γ, we need a non-zero element of the direct limit

$$\bar{H}^2(\mathbb{Z}/n) = \lim_{\substack{\rightarrow \\ \Delta}} H^2(\Delta, \mathbb{Z}/n),$$

where Δ runs over subgroups of finite index in Γ. The argument above shows that Theorem 2 is implied by the following result.

Theorem 3 *For every positive integer n, there are infinitely many elements of order n in $\bar{H}^2(\mathbb{Z}/n)$.*

We shall actually prove a stronger result, which needs a little more notation to state. We shall write $\widehat{G(\mathbb{Q})}$ for the *arithmetic completion* of the group $G(\mathbb{Q})$, i.e.

$$\widehat{G(\mathbb{Q})} = \lim_{\substack{\leftarrow \\ \Delta}} G(\mathbb{Q})/\Delta,$$

where Δ runs through the subgroups of finite index in Γ. There is a natural projection $\widehat{G(\mathbb{Q})} \to G(\mathbb{A}_f)$, and the congruence kernel $\mathrm{Cong}(G)$ is, by definition, the kernel of this map. This means that we have an extension of topological groups

$$1 \to \mathrm{Cong}(G) \to \widehat{G(\mathbb{Q})} \stackrel{\mathrm{pr}}{\to} G(\mathbb{A}_f) \to 1.$$

The group $\widehat{G(\mathbb{Q})}$ acts smoothly on $\bar{H}^2(\mathbb{Z}/n)$.

Let S be a finite set of prime numbers. By an *S-arithmetic level*, we shall mean an open subgroup L of $\widehat{G(\mathbb{Q})}$ of the form

$$L = \mathrm{pr}^{-1}\left(\prod_{p\in S} G(\mathbb{Q}_p) \times L^S\right), \qquad L^S = \prod_{p\notin S} K_p,$$

where each K_p is a compact open subgroup of $G(\mathbb{Q}_p)$, chosen so that L is open in $\widehat{G(\mathbb{Q})}$.

Theorem 4 *Let L be an S-arithmetic level in $\widehat{G(\mathbb{Q})}$ for some finite set of primes S. For every positive integer n, there are infinitely many elements of order n in $\bar{H}^2(\mathbb{Z}/n)^L$.*

Theorem 4 will be proved in Sect. 4. The proof requires a technical result on the cohomology of finite groups of Lie type, which is proved in Sect. 5. By modifying the argument slightly, one can also prove the following result.

Theorem 5 *Let n be a positive integer. Then there are infinitely many elements σ of order n in $\bar{H}^2(\mathbb{Z}/n)$ with the following property. There is a prime number p depending on σ, such that for all primes $q \neq p$ the element σ is fixed by* $\mathrm{pr}^{-1}\left(G(\mathbb{Q}_q)\right)$.

2.2 Virtual lifting to characteristic zero

Let l be a prime number. Any central extension of Γ by $\mathbb{Z}/l^{t+1}$ gives rise to a central extension by $\mathbb{Z}/l^t$. We'll say that the extension of Γ by $\mathbb{Z}/l^r$ *virtually lifts to characteristic zero* if for every $t > r$ there is a arithmetic subgroup Δ_t of Γ and a central extension of Δ_t by $\mathbb{Z}/l^t$, such that the extensions fit into a commutative diagram of the following form.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z}/l^t & \longrightarrow & \tilde{\Delta}_t & \longrightarrow & \Delta_t & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbb{Z}/l^r & \longrightarrow & \tilde{\Gamma} & \longrightarrow & \Gamma & \longrightarrow & 1
\end{array}$$

Here the map $\mathbb{Z}/l^t \to \mathbb{Z}/l^r$ is the usual reduction map, and the map $\Delta_t \to \Gamma$ is the inclusion.

Equivalently, an element of $\bar{H}^2(\mathbb{Z}/l^r)$ virtually lifts to characteristic zero if it is in the image of the following group.

$$\bar{H}^2(\mathbb{Z}_l) = \lim_{\substack{\leftarrow \\ t}} \bar{H}^2(\mathbb{Z}/l^t).$$

There is a continuous action of $\widehat{G(\mathbb{Q})}$ on the cohomology group $\bar{H}^2(\mathbb{Z}_l)$. Our next result will show that there are indeed families of non-residually finite central extensions, which virtually lift to characteristic zero. Before stating the result we'll need a little notation. The group $G(\mathbb{R})$ is semi-simple over $\mathbb{R}$, and decomposes as a product of finitely many simple groups $G_i(\mathbb{R})$. We'll say that a simple group G_i over $\mathbb{R}$ is of *complex type* if G_i is the restriction of scalars of a group defined over $\mathbb{C}$, or equivalently if $G_i(\mathbb{C})$ is a product of two simple groups; otherwise we say that G_i is of real type. We'll write $b_{\mathbb{R}}$ for the number of simple factors of $G(\mathbb{R})$ of real type and $b_{\mathbb{C}}$ for the number of simple factors of $G(\mathbb{R})$ of complex type. We'll also write m for the dimension of the centre of a maximal compact subgroup $K_\infty \subset G(\mathbb{R})$.

Theorem 6 *The group $\bar{H}^2(\mathbb{Z}_l)^{\widehat{G(\mathbb{Q})}}$ is isomorphic to ${\mathbb{Z}_l}^c$ for some positive integer c. More precisely, c is in the range*

$$b_{\mathbb{R}} + b_{\mathbb{C}} + m \leq c \leq b_{\mathbb{R}} + 2b_{\mathbb{C}} + m,$$

where $b_{\mathbb{R}}$, $b_{\mathbb{C}}$ and m are the integers defined above. In particular $\bar{H}^2(\mathbb{Z}_l)$ is non-zero.

For comparison, we note that the construction of Deligne implies the bound $c \geq m$; this is because $\pi_1(G(\mathbb{R}))$ has a finite index subgroup isomorphic to $\mathbb{Z}^m$.

As an easy consequence of the theorem, we obtain the following:

Corollary 3 *Let $G/\mathbb{Q}$ be simple and simply connected with finite congruence kernel. There is a subgroup of $\bar{H}^2(\mathbb{Z}/l^t)^{\widehat{G(\mathbb{Q})}}$ isomorphic to $(\mathbb{Z}/l^t)^c$, all of whose elements virtually lift to characteristic zero, where c is the positive integer in Theorem 6.*

Theorem 6 and its corollary will be proved in section 6. The proof requires a result on the cohomology of compact symmetric spaces, which is proved in the appendix.

Remark 1 We stress that Theorem 6 implies $\bar{H}^2(\mathbb{Z}_l)^{\widehat{G(\mathbb{Q})}}$ is non-zero even in cases where $H^2(\Gamma, \mathbb{C}) = 0$ for all arithmetic subgroups Γ of $G(\mathbb{Q})$. This happens when G has large real rank and the symmetric space associated to G has no complex structure, for example when $G = \mathrm{SL}_5/\mathbb{Q}$. The extensions constructed by the method of Deligne exist only in the case $m > 0$; our result shows that $\bar{H}^2(\mathbb{Z}_l)^{\widehat{G(\mathbb{Q})}}$ is non-zero even in cases where $m = 0$.

Remark 2 The author suspects that $\mathrm{rank}_{\mathbb{Z}_l}\left(\bar{H}^2(\mathbb{Z}_l)^{\widehat{G(\mathbb{Q})}}\right) = b_{\mathbb{R}} + b_{\mathbb{C}} + m$. Proving this would amount to showing that the restriction map $H^3_{\mathrm{cts}}(G(\mathbb{Q}_l), \mathbb{Q}_l) \to H^3(G(\mathbb{Q}), \mathbb{Q}_l)$ is surjective. The evidence for this is very slight, but we note that $\dim H^3_{\mathrm{cts}}(G(\mathbb{Q}_l), \mathbb{Q}_l)$ is at least twice as big as $\dim H^3(G(\mathbb{Q}), \mathbb{Q}_l)$.

As an example, consider the case $G = \mathrm{Res}^k_{\mathbb{Q}}(SL_{\geq 3}/k)$, where k is an imaginary quadratic field. In this case $m = 0$, $b_{\mathbb{R}} = 0$ and $b_{\mathbb{C}} = 1$, so our result implies that the rank c is either 1 or 2. In this case $H^3_{\mathrm{cts}}(G(\mathbb{Q}_l), \mathbb{Q}_l)$ is 2-dimensional and $H^3(G(\mathbb{Q}), \mathbb{Q}_l)$ is 1-dimensional (see Sect. 6.3), so the restriction map is either surjective or zero. If the restriction map is non-zero, then the rank is 1. One might expect to prove that the rank is 1 by evaluating an appropriate l-adic Borel regulator; however the author has not done this in any case.

As long as $G(\mathbb{R})$ has no simple factors of complex type, Theorem 6 tells us precisely the rank of $\bar{H}^2(\mathbb{Z}_l)^{\widehat{G(\mathbb{Q})}}$. Some examples are given in Table 1. In this table, $\mathrm{Spin}(r, s)$ denotes the Spin group of an arbitrary quadratic form over $\mathbb{Q}$ of signature (r, s). The congruence subgroup property for such groups was established by Kneser [12].

The case $\mathrm{SL}_2/\mathbb{Q}$ and its forms of rank 0 are not included in the table. This is because these groups have infinite congruence kernel, and indeed for these groups we have $\bar{H}^2(\mathbb{Z}/n) = 0$ and $\bar{H}^2(\mathbb{Z}_l) = 0$.

3 Background material

3.1 Continuous cohomology

We shall make use of the continuous cohomology groups $H^\bullet_{\mathrm{cts}}(G, A)$, where G is a topological group and A is an abelian topological group, which is a G-module via a continuous action $G \times A \to A$.

Table 1 Values of the rank of $\bar{H}^2(\mathbb{Z}_l)^{\widehat{G(\mathbb{Q})}}$

G		m	$b_{\mathbb{R}}$	$c = \mathrm{rank}_{\mathbb{Z}_l}\left(\bar{H}^2(\mathbb{Z}_l)^{\widehat{G(\mathbb{Q})}}\right)$
$\mathrm{SL}_n/\mathbb{Q}$	$(n \geq 3)$	0	1	1
$\mathrm{Sp}_{2n}/\mathbb{Q}$	$(n \geq 2)$	1	1	2
$\mathrm{Spin}(r,s)$	$(r \geq s \geq 3)$	0	1	1
$\mathrm{Spin}(r,2)$	$(r \geq 3)$	1	1	2
$\mathrm{Spin}(2,2)$		2	2	4
$\mathrm{Res}^k_{\mathbb{Q}}(\mathrm{SL}_n/k)$	$(n \geq 3, k$ totally real$)$	0	$[k:\mathbb{Q}]$	$[k:\mathbb{Q}]$
$\mathrm{Res}^k_{\mathbb{Q}}(\mathrm{SL}_2/k)$	$(k$ totally real, $k \neq \mathbb{Q})$	$[k:\mathbb{Q}]$	$[k:\mathbb{Q}]$	$2[k:\mathbb{Q}]$
$\mathrm{Res}^k_{\mathbb{Q}}(\mathrm{Sp}_{2n}/k)$	$(k$ totally real$)$	$[k:\mathbb{Q}]$	$[k:\mathbb{Q}]$	$2[k:\mathbb{Q}]$

In all cases under consideration here, the group G will be metrizable, locally compact, totally disconnected, separable and σ-compact. The coefficient group A will always be Polonais (a topological group is Polonais if its topology admits a separable complete metric; see page 3 of [18]). Under these restriction, the continuous cohomology groups defined in [7] (based on continuous cocycles) are the same as those defined in [16–18] based on Borel measurable cocycles. This is proved in Theorem 1 of [25].

If A is a continuous H-module for some closed subgroup H of G, then we shall write $\mathrm{ind}_H^G(A)$ for the induced module, consisting of all continuous functions $f : G \to A$ satisfying $f(hg) = h \cdot f(g)$ for all $g \in G$ and $h \in H$. This agrees with the notation of [7] but not [16–18]. The following version of Shapiro's lemma holds for these induced representations.

Theorem 7 (Shapiro's Lemma) *Let H be a closed subgroup of G, where G satisfies the conditions above. For any continuous H-module A, there is a canonical isomorphism of topological groups:*

$$H^\bullet_{\mathrm{cts}}\left(G, \mathrm{ind}_H^G A\right) = H^\bullet_{\mathrm{cts}}(H, A).$$

Proof This follows Propositions 3 and 4 of [7] in view of the remark following Proposition 4. □

We shall also make frequent use of the following.

Theorem 8 (The Hochschild–Serre spectral sequence) *Let H be a closed normal subgroup of a group G, where G satisfies the conditions above, and let A be a continuous Polonais representation of G. If the groups $H^\bullet(H, A)$ are all Hausdorff, then there is a first quadrant spectral sequence converging to $H^\bullet(G, A)$, with E_2 sheet given by*

$$E_2^{r,s} = H^r_{\mathrm{cts}}\left(G, H^s_{\mathrm{cts}}(H, A)\right).$$

Proof This follows from Theorem 9 of [18] in all cases under consideration. □

3.2 The derived functor of projective limit

By a *projective system*, we shall mean a sequence of abelian groups A_t, indexed by $t \in \mathbb{N}$, and connected by group homomorphisms as follows:

$$A_1 \leftarrow A_2 \leftarrow A_3 \leftarrow \cdots.$$

We shall write $\lim\limits_{\stackrel{\leftarrow}{t}} A_t$ for the projective limit of the system. The functor $\lim\limits_{\stackrel{\leftarrow}{t}}$ is left-exact from the category of projective systems of abelian groups to the category of abelian groups. As such, it has derived functors $\left(\lim\limits_{\stackrel{\leftarrow}{t}}\right)^{\bullet} A_t$. It is known (Corollary 3.5.4 of [24]) that the higher derived functors $\left(\lim\limits_{\stackrel{\leftarrow}{t}}\right)^{n} A_t$ for $n \geq 2$ are all zero.

The projective system (A_t) is said to satisfy the *Mittag–Leffler property* if for every $t \in \mathbb{N}$, there is a $j \in \mathbb{N}$ with the property that for all $k > j$ the image of A_k in A_t is equal to the image of A_j in A_t. For example, if the Abelian groups A_t are all finite then the projective system has the Mittag–Leffler property. Similarly, if the groups A_t are all finite dimensional vector spaces connected by linear maps, then the projective system satisfies the Mittag–Leffler condition.

Proposition 1 (Proposition 3.5.7 of [24]) *If the projective system* (A_t) *satisfies the Mittag–Leffler condition then* $\left(\lim\limits_{\stackrel{\leftarrow}{t}}\right)^{1} A_t = 0$.

Theorem 9 (Theorem 3.5.8 of [24]) *Let* $\cdots \to C_2^r \to C_1^r$ *be a projective system of cochain complexes of abelian groups, each indexed by* $r \geq 0$. *Assume that this projective system has the Mittag–Leffler property, and let* $C^r = \lim\limits_{\stackrel{\leftarrow}{t}} C_t^r$ *be the projective limit of the complexes. Then we have* $H^0(C^\bullet) = \lim\limits_{\stackrel{\leftarrow}{t}} H^0(C_t^\bullet)$. *Furthermore, for every* $r \geq 0$ *there is a short exact sequence*

$$0 \to \left(\lim\limits_{\stackrel{\leftarrow}{i}}\right)^{1} H^r(C_i^\bullet) \to H^{r+1}(C^\bullet) \to \lim\limits_{\stackrel{\leftarrow}{i}} H^{r+1}(C_i^\bullet) \to 0.$$

As a simple example, we show how to express the cohomology of $G(\mathbb{Q})$ in terms of the cohomology of its S-arithmetic subgroups. As before, we let $G/\mathbb{Q}$ be a simple, simply connected algebraic group, and $K_f = \prod_p K_p$ a compact open subgroup of $G(\mathbb{A}_f)$. We shall write Γ for the arithmetic group $G(\mathbb{Q}) \cap K_f$. More generally, if S is a finite set of prime numbers, then we use the notation Γ^S for the corresponding S-arithmetic group, i.e.

$$\Gamma^S = G(\mathbb{Q}) \cap K^S, \quad K^S = \left(\prod_{p \in S} G(\mathbb{Q}_p)\right) \times K_f.$$

Proposition 2 *For any field* $\mathbb{F}$, *we have* $H^\bullet(G(\mathbb{Q}), \mathbb{F}) = \lim\limits_{\stackrel{\leftarrow}{S}} H^\bullet(\Gamma^S, \mathbb{F})$. *In the case* $\mathbb{F} = \mathbb{C}$ *we have* $H^\bullet(G(\mathbb{Q}), \mathbb{C}) = H^\bullet(\mathfrak{g}, \mathfrak{k}, \mathbb{C})$, *where* $H^\bullet(\mathfrak{g}, \mathfrak{k}, \mathbb{C})$ *are the relative Lie algebra cohomology groups studied in [5].*

Proof For each $r \geq 0$ we shall write $C^r(\Gamma^S, \mathbb{F})$ for the usual (inhomogeneous) cochain complex, consisting of all functions $f : \left(\Gamma^S\right)^r \to \mathbb{F}$. Since $G(\mathbb{Q})$ is the union of the groups Γ^S, it follows that

$$C^r(G(\mathbb{Q}), \mathbb{F}) = \lim\limits_{\stackrel{\leftarrow}{S}} C^r(\Gamma^S, \mathbb{F}).$$

The maps in this projective system are restrictions of functions, and they are obviously surjective. Therefore the projective system satisfies the Mittag–Leffler condition. As a consequence, we have short exact sequences

$$0 \to \left(\lim_{\leftarrow S}\right)^1 H^r(\Gamma^S, \mathbb{F}) \to H^{r+1}(G(\mathbb{Q}), \mathbb{F}) \to \lim_{\leftarrow S} H^{r+1}(\Gamma^S, \mathbb{F}) \to 0.$$

By the theory of the Borel–Serre compactification (see [4]), the cohomology groups $H^r(\Gamma^S, \mathbb{F})$ are finite dimensional vector spaces. Therefore the system $\left(H^r(\Gamma^S, \mathbb{F})\right)_S$ satisfies the Mittag–Leffler condition, so $\left(\lim_{\leftarrow S}\right)^1 H^r(\Gamma^S, \mathbb{F}) = 0$.

In the case $\mathbb{F} = \mathbb{C}$, the theorem of [2] implies that $H^r(\Gamma^S, \mathbb{C}) = H^r(\mathfrak{g}, \mathfrak{k}, \mathbb{C})$ whenever S contains more than r primes. Hence the projective limit (over S) is in this case $H^r(\mathfrak{g}, \mathfrak{k}, \mathbb{C})$. □

3.3 The congruence kernel

Let $G/\mathbb{Q}$ be a simple, simply connected group with real rank at least 1. By Kneser's strong approximation theorem (see [13]) the group $G(\mathbb{Q})$ is dense is $G(\mathbb{A}_f)$, where $\mathbb{A}_f$ is the ring of finite adèles of $\mathbb{Q}$. It follows that there is an isomorphism of topological groups:

$$G(\mathbb{A}_f) = \lim_{\substack{\leftarrow \\ \text{congruence subgroups } \Gamma}} G(\mathbb{Q})/\Gamma,$$

where Γ runs over the congruence subgroups of $G(\mathbb{Q})$. Recall that an *arithmetic subgroup* of G is any subgroup of $G(\mathbb{Q})$, which is commensurable with a congruence subgroup. The *arithmetic completion* $\widehat{G(\mathbb{Q})}$ is defined to be the completion of $G(\mathbb{Q})$ with respect to the arithmetic subgroups of G, i.e.

$$\widehat{G(\mathbb{Q})} = \lim_{\substack{\leftarrow \\ \text{arithmetic subgroups } \Gamma}} G(\mathbb{Q})/\Gamma.$$

There is a canonical surjective homomorphism $\widehat{G(\mathbb{Q})} \to G(\mathbb{A}_f)$. The *congruence kernel* $\mathrm{Cong}(G)$ is defined to be the kernel of this map, so we have a short exact sequence:

$$1 \to \mathrm{Cong}(G) \to \widehat{G(\mathbb{Q})} \to G(\mathbb{A}_f) \to 1.$$

The congruence kernel is trivial if and only if every arithmetic subgroup of G is a congruence subgroup. If $G(\mathbb{R})$ is simply connected as an analytic group, then the congruence kernel is never trivial, but may still be finite. It has been conjectured by Serre [20], that the congruence kernel is finite if and only if each simple factor of G over $\mathbb{Q}$ has real rank at least 2. In the case that $\mathrm{Cong}(G)$ is finite, it is known that $\mathrm{Cong}(G)$ is contained in the centre of $\widehat{G(\mathbb{Q})}$, and is a cyclic group.

4 Proof of Theorem 4

In this section, we assume that the group $G/\mathbb{Q}$ is a simple, simply connected algebraic group with positive real rank. We shall also assume that the congruence kernel $\mathrm{Cong}(G)$ is finite. Hence, conjecturally that the real rank of G is at least 2.

4.1 The groups $\mathcal{C}(L, \mathbb{Z}/n)$

Let L be an open subgroup of the arithmetic completion $\widehat{G(\mathbb{Q})}$. We shall write $\Gamma(L)$ for the group $G(\mathbb{Q}) \cap L$. Since $G(\mathbb{Q})$ is dense in $\widehat{G(\mathbb{Q})}$, it follows that $\Gamma(L)$ is dense in L. If L is compact and open then $\Gamma(L)$ is an arithmetic group and L is its profinite completion. If L is an S-arithmetic level, then $\Gamma(L)$ is an S-arithmetic group.

We shall write $\mathcal{C}(L, \mathbb{Z}/n)$ for the group of continuous functions $f : L \to \mathbb{Z}/n$. We regard $\mathcal{C}(L, \mathbb{Z}/n)$ as a $\Gamma(L) \times L$-module, in which (for the sake of argument) $\Gamma(L)$ acts by left-translation and L acts by right-translation. We regard $\Gamma(L)$ as a discrete topological group, and L as a topological group with the subspace topology from $\widehat{G(\mathbb{Q})}$. We do not assume that elements of $\mathcal{C}(L, \mathbb{Z}/n)$ are uniformly continuous, and so the action of L is not smooth unless L is compact. The action is continuous, where $\mathcal{C}(L, \mathbb{Z}/n)$ is equipped with the compact–open topology.

We shall also use the following notation, which was introduced earlier:

$$\bar{H}^\bullet(\mathbb{Z}/n) = \lim_{\substack{\rightarrow \\ \Delta}} H^\bullet(\Delta, \mathbb{Z}/n),$$

where Δ ranges of the arithmetic subgroups.

Proposition 3 *For each open subgroup L of $\widehat{G(\mathbb{Q})}$, there is a canonical isomorphism of L-modules:*

$$\bar{H}^\bullet(\mathbb{Z}/n) = H^\bullet(\Gamma(L), \mathcal{C}(L, \mathbb{Z}/n)).$$

The cohomology groups $\bar{H}^r(\mathbb{Z}/n) = H^r(\Gamma(L), \mathcal{C}(L, \mathbb{Z}/n))$ are discrete (and hence Hausdorff).

Proof As a first step, we'll show that the groups $H^\bullet(\Gamma(L), \mathcal{C}(L, \mathbb{Z}/n))$ do not depend on the level L. Let K be an open subgroup of L. As a $\Gamma(L)$-module, we have

$$\mathcal{C}(L, \mathbb{Z}/n) \cong_{\Gamma(L)} \mathrm{ind}_{\Gamma(K)}^{\Gamma(L)} \mathcal{C}(K, \mathbb{Z}/n).$$

By Shapiro's Lemma (Theorem 7), we have an isomorphism of topological groups:

$$H^\bullet(\Gamma(L), \mathcal{C}(L, \mathbb{Z}/n)) = H^\bullet(\Gamma(K), \mathcal{C}(K, \mathbb{Z}/n)).$$

It's therefore sufficient to consider the case that the level L is compact and open. Under this assumption, we have (as $\Gamma(L)$-modules):

$$\mathcal{C}(L, \mathbb{Z}/n) = \lim_{\substack{\rightarrow \\ \Delta}} \mathrm{ind}_{\Delta}^{\Gamma(L)} (\mathbb{Z}/n),$$

where Δ ranges over the arithmetic subgroups of $\Gamma(L)$. Since direct limits commute with cohomology, this implies

$$H^\bullet(\Gamma(L), \mathcal{C}(L, \mathbb{Z}/n)) = \lim_{\substack{\rightarrow \\ \Delta}} H^\bullet\left(\Gamma(L), \mathrm{ind}_{\Delta}^{\Gamma(L)} \mathbb{Z}/n\right).$$

Applying Shapiro's Lemma again, we have

$$H^\bullet(\Gamma(L), \mathcal{C}(L, \mathbb{Z}/n)) = \lim_{\substack{\rightarrow \\ \Delta}} H^\bullet(\Delta, \mathbb{Z}/n).$$

If we choose L to be compact, then $\mathcal{C}(L, \mathbb{Z}/n)$ is discrete, and therefore the group $H^\bullet(\Gamma(L), \mathcal{C}(L, \mathbb{Z}/n))$ is discrete. □

Lemma 1 *We have $H^0_{\mathrm{cts}}(L, \mathcal{C}(L, \mathbb{Z}/n)) = \mathbb{Z}/n$ and $H^s_{\mathrm{cts}}(L, \mathcal{C}(L, \mathbb{Z}/n)) = 0$ for $s > 0$. In particular the groups $H^s_{\mathrm{cts}}(L, \mathcal{C}(L, \mathbb{Z}/n))$ are Hausdorff.*

Proof As a continuous L-module, we have $\mathcal{C}(L, \mathbb{Z}/n) = \mathrm{ind}_1^L(\mathbb{Z}/n)$. The result follows from this using Shapiro's Lemma. □

Proposition 4 *Let L be any open subgroup of $\widehat{G(\mathbb{Q})}$. Then there is a first quadrant spectral sequence with $E_2^{r,s} = H^r_{\mathrm{cts}}(L, \tilde{H}^s(\mathbb{Z}/n))$ which converges to $H^{r+s}(\Gamma(L), \mathbb{Z}/n)$.*

Proof We've seen that $H^\bullet_{\mathrm{cts}}(L, \mathcal{C}(L, \mathbb{Z}/n))$ and $H^\bullet(\Gamma(L), \mathcal{C}(L, \mathbb{Z}/n))$ are both Hausdorff. We therefore have two Hochschild–Serre spectral sequences, both of which converge to $H^{r+s}_{\mathrm{cts}}(\Gamma(L) \times L, \mathcal{C}(L, \mathbb{Z}/n))$:

$$H^r_{\mathrm{cts}}(L, H^s(\Gamma(L), \mathcal{C}(L, \mathbb{Z}/n))), \qquad H^r(\Gamma(L), H^s_{\mathrm{cts}}(L, \mathcal{C}(L, \mathbb{Z}/n))).$$

By Lemma 1, the second of these two spectral sequence collapses and we have $H^\bullet_{\mathrm{cts}}(\Gamma(L) \times L, \mathcal{C}(L, \mathbb{Z}/n)) = H^\bullet(\Gamma(L), \mathbb{Z}/n)$. The result now follows from Proposition 3. □

4.2 Low degree terms

We shall now describe some of the low degree terms of the spectral sequence of Proposition 4.

Lemma 2 *With the notation introduced above,*

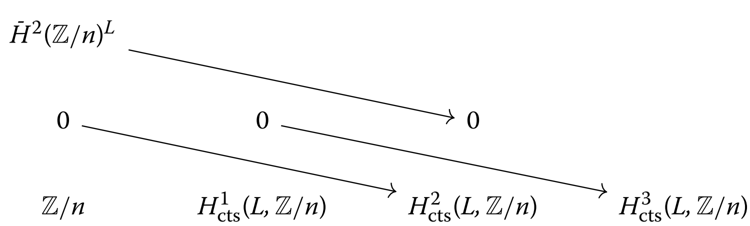
$$\tilde{H}^0(\mathbb{Z}/n) = \mathbb{Z}/n, \qquad \tilde{H}^1(\mathbb{Z}/n) = 0.$$

Proof For $\tilde{H}^0$, note that for any arithmetic group Δ,

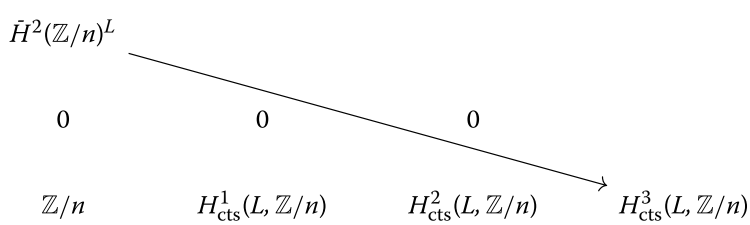
$$H^0(\Delta, \mathbb{Z}/p^r) = \mathbb{Z}/n.$$

Furthermore the restriction maps from one of these groups to another, are all the identity map. For $\tilde{H}^1$, we must show that for every element $\sigma \in H^1(\Delta, \mathbb{Z}/n)$, there is an arithmetic subgroup $\Delta' \subset \Delta$, such that the restriction of σ to Δ' is zero. Any such σ is a homomorphism $\Delta \to \mathbb{Z}/n$, so we may simply set $\Delta' = \ker \sigma$. □

By Lemma 2, we know that $E_2^{r,0} = H^r_{\mathrm{cts}}(L, \mathbb{Z}/n)$ and $E_2^{r,1} = 0$. Therefore the bottom left corner of the E_2 sheet of the spectral sequence looks like this:



These groups all remain the same in the E_3 sheet, where we have a map $\tilde{H}^2(\mathbb{Z}/n)^L \to H^3_{\mathrm{cts}}(L, \mathbb{Z}/n)$.



(1)

This map is part of the exact sequence:

$$H^2(\Gamma(L), \mathbb{Z}/n) \to \left(\bar{H}^2(\mathbb{Z}/n)\right)^L \to H^3_{\text{cts}}(L, \mathbb{Z}/n) \to H^3(\Gamma(L), \mathbb{Z}/n). \tag{2}$$

We now recall the theorem which we are proving:

Theorem *Let S be a finite set of prime numbers and let L be an S-arithmetic level. Then the group $\bar{H}^2(\mathbb{Z}/n)^L$ contains infinitely many elements of order n.*

Proof Let L be an S-arithmetic level. In this case the group $\Gamma(L)$ is an S-arithmetic group. By the theory of the Borel–Serre compactification, there is a resolution of $\mathbb{Z}$ as a $\Gamma(L)$-module consisting of finitely generated $\mathbb{Z}[\Gamma(L)]$-modules. This implies that the cohomology groups $H^r(\Gamma(L), \mathbb{Z}/n)$ are all finite. In view of this, the sequence in Equation 2 has the form

$$\text{finite} \to \left(\bar{H}^2(\mathbb{Z}/n)\right)^L \to H^3_{\text{cts}}(L, \mathbb{Z}/n) \to \text{finite}.$$

To prove the theorem, it is therefore sufficient to show that $H^3_{\text{cts}}(L, \mathbb{Z}/n)$ contains infinitely many elements of order n.

It will be useful to have the following notation. A prime number p will be called a *tame* prime if it satisfies all of the following conditions:

1. p is not in the finite set S;
2. p is not a factor of $|\text{Cong}(G)|$;
3. p is not a factor of n;
4. G is unramified over $\mathbb{Q}_p$.
5. The group K_p is a maximal hyperspecial compact open subgroup of $G(\mathbb{Q}_p)$ (see [23]). This implies that if we let K_p^0 be the maximal pro-p normal subgroup of K_p, then the quotient $G(\mathbb{F}_p) = K_p/K_p^0$ is a product of some of the simply connected finite Lie groups described in [22].
6. $H^r(G(\mathbb{F}_p), \mathbb{Q}/\mathbb{Z}) = 0$ for $r = 1, 2$. We recall from [22] that this condition is satisfied for all but finitely many of the groups $G(\mathbb{F}_p)$.

We note that all but finitely many primes are tame. For each tame prime p, we shall write K_p^* for a lift of K_p to $\widehat{G(\mathbb{Q})}$; note that such a lift exists and is unique by conditions (2) and (6). The group L contains the following subgroup

$$K_{\text{tame}} = \prod_{p \text{ tame}} K_p^*,$$

Evidently, $\text{pr}(K_{\text{tame}})$ is a direct summand of $\text{pr}(L)$; since $K_{\text{tame}} \cap \text{Cong}(G)$ is trivial, it follows that K_{tame} is a direct summand of L. Hence by the Künneth formula, $H^3_{\text{cts}}(K_{\text{tame}}, \mathbb{Z}/n)$ is a direct summand of $H^3_{\text{cts}}(L, \mathbb{Z}/n)$. It is therefore sufficient to prove that $H^3_{\text{cts}}(K_{\text{tame}}, \mathbb{Z}/n)$ contains infinitely many elements of order n.

Since the coefficient ring $\mathbb{Z}/n$ is finite, we have (by Proposition 8, section 2.2 of [21]) a decomposition

$$H^3_{\text{cts}}(K_{\text{tame}}, \mathbb{Z}/n) = \lim_{\substack{\to \\ U \text{ finite}}} H^3_{\text{cts}}\left(\prod_{p \in U} K_p, \mathbb{Z}/n\right).$$

By the Künneth formula, the group on the right contains a subgroup of the form

$$\bigoplus_{p \text{ tame}} H^3_{\text{cts}}(K_p, \mathbb{Z}/n).$$

By conditions (3) and (5) for tame primes p, we may identify $H^\bullet_{\mathrm{cts}}(K_p, \mathbb{Z}/n)$ with $H^\bullet(G(\mathbb{F}_p), \mathbb{Z}/n)$. To prove the theorem, it is therefore sufficient so show that there are infinitely many tame primes p, such that $H^3(G(\mathbb{F}_p), \mathbb{Z}/n)$ contains an element of order n. This follows from Theorem 10, which will be proved in the next section. □

5 A lemma on the cohomology of finite Lie groups

In this section we shall prove Theorem 10, which completes the proof of Theorem 4.

Before stating the theorem, we note that if G is an algebraic group over $\mathbb{Q}$, then we may write G in the form $\mathscr{G} \times_{\mathbb{Z}} \mathbb{Q}$, for some group scheme $\mathscr{G}$ over $\mathbb{Z}$. The group $\mathscr{G}(\mathbb{F}_p)$ depends on the choice of $\mathscr{G}$, not just on G. Nevertheless if we alter the group scheme $\mathscr{G}$ then only finitely many of the groups $\mathscr{G}(\mathbb{F}_p)$ will change. Because of this, the following statement makes sense, where we are writing $G(\mathbb{F}_p)$ in place of $\mathscr{G}(\mathbb{F}_p)$ for some fixed choice of $\mathscr{G}$.

Theorem 10 *Let $G/\mathbb{Q}$ be a simple, simply connected algebraic group. For every positive integer n there are infinitely many prime numbers p, such that $H^3(G(\mathbb{F}_p), \mathbb{Z}/n)$ contains an element of order n.*

I assume this sort of result is known to experts, and many special cases are consequences of results in algebraic K-theory (for example the results of [19] imply the case SL_r).

In the proof we shall use the Cartan–Eilenberg theory of invariant cohomology classes, which we recall now. Let T be a subgroup of a finite group G, and let A be a G-module. We shall write Rest^G_T and CoRest^T_G for the restriction and corestriction maps between $H^\bullet(G, A)$ and $H^\bullet(T, A)$. A cohomology class $\sigma \in H^r(T, A)$ is called *invariant* if for every $g \in G$,

$$\mathrm{Rest}^T_{T\cap T^g}(\sigma) = \mathrm{Rest}^{T^g}_{T\cap T^g}(\sigma^g).$$

We'll use the following result.

Proposition 5 (Chapter XII, Proposition 9.4 of [6]) *Let T be a subgroup of a finite group G. If $\sigma \in H^\bullet(T, A)$ is an invariant cohomology class. Then*

$$\mathrm{Rest}^G_T\left(\mathrm{CoRest}^T_G(\sigma)\right) = [G : T] \cdot \sigma.$$

As a corollary to this, we note the following.

Corollary 4 *Let T be a subgroup of a finite group G. Let d be a positive integer and l a prime number, such that $\big|[G : T]\big|_l = \big|d\big|_l$. If $H^r(T, \mathbb{Z})$ contains an invariant class of order dl^t then $H^r(G, \mathbb{Z})$ contains an element of order l^t.*

Proof Let $\tau = \mathrm{CoRest}^T_G(\sigma)$, where σ is the invariant class on T of order dl^t. By Proposition 5, the restriction of τ to T has order $\frac{dl^t}{\gcd(dl^t,[G:T])}$. The condition on d implies that the order of $\mathrm{Rest}^G_T(\tau)$ is a multiple of l^t. Hence the order of τ is a multiple of l^t, so some multiple of τ has order l^t. □

In order to apply the corollary, it will be useful to note the following.

Lemma 3 *Let l be a prime number, and let x be an integer such that $x \equiv 1 \bmod 2l$. Then for every integer d we have*

$$|x^d - 1|_l = |d(x-1)|_l.$$

Proof We recall that the l-adic logarithm function $\log_l$ converges on the multiplicative group $1 + 2l\mathbb{Z}_l$. If $\log_l$ converges at an element x, then we have $|\log(x)|_l = |x-1|_l$. Our congruence condition implies that $\log_l(x)$ and $\log_l(x^d)$ both converge, so we have

$$|x^d - 1|_l = \left|\log_l(x^d)\right|_l = |d \cdot \log_l(x)|_l = |d \cdot (x-1)|_l.$$

□

Proof of Theorem 10 By the Chinese remainder theorem, it is sufficient to prove the theorem in the case $n = l^t$, where l is a prime number.

We shall introduce some notation. We fix a semi-simple model $\mathscr{G}$ of G over $\mathbb{Z}$, and let k be a number field such that $\mathscr{G}$ splits over $\mathscr{O}_k$. Let $\mathscr{T}$ be a maximal torus in $\mathscr{G}$, defined and split over $\mathscr{O}_k$. Let P be the lattice of algebraic characters $\mathscr{T} \to \mathrm{GL}_1/\mathscr{O}_k$. The roots of $\mathscr{G}$ with respect to $\mathscr{T}$ are elements of the lattice P. Consider the element

$$Q = \sum_{\alpha \in \Phi} \alpha \otimes \alpha \ \in \ \mathrm{Sym}^2(P),$$

where Φ is the set of roots. If we identify elements of P with a group of characters of the Lie algebra $\mathfrak{t}$ of $\mathscr{T}$, then we may similarly identify elements of $\mathrm{Sym}^2(P)$ with quadratic forms on $\mathfrak{t}$. The element Q corresponds to the restriction of the Killing form to $\mathfrak{t}$. Therefore Q is non-zero.

Let e be the largest positive integer, such that Q is a multiple of e in the lattice $\mathrm{Sym}^2(P)$. Also let $d_1, \ldots, d_r$ be the degrees of the basic polynomial invariants of the Weyl group of G/k (where r is the rank of G/k). The smallest of these degrees is $d_1 = 2$, and the others depend on the root system (see [22]). By extending the number field k if necessary, we may assume that k contains a primitive root of unity of order $d_1 \cdots d_r \cdot e \cdot n$. By the Chebotarev density theorem, there are infinitely many prime numbers which split in k; we'll show that each of these prime numbers has the desired property.

From now on we fix a prime number p which splits in k, and we are attempting to show that $H^3(\mathscr{G}(\mathbb{F}_p), \mathbb{Z}/n)$ contains an element of order n. By abusing notation slightly we shall write $G(\mathbb{F}_p)$ for the group $\mathscr{G}(\mathbb{F}_p)$. We may identify $G(\mathbb{F}_p)$ with $\mathscr{G}(\mathscr{O}_k/\mathfrak{p})$ for some prime ideal $\mathfrak{p}$ above p). We shall also write $T(\mathbb{F}_p)$ for the subgroup $\mathscr{T}(\mathscr{O}_k/\mathfrak{p})$.

Identifying $H^3(G(\mathbb{F}_p), \mathbb{Z}/n)$ with the n-torsion in $H^4(G(\mathbb{F}_p), \mathbb{Z})$, we see that it's sufficient to prove there is an element of order n in $H^4(G(\mathbb{F}_p), \mathbb{Z})$.

We shall use the following formula for the order of the the group $G(\mathbb{F}_p)$ (see Theorem 25, in Chapter 9 of [22])

$$|G(\mathbb{F}_p)| = p^N (p^{d_1} - 1) \cdots (p^{d_r} - 1).$$

In this formula, N is the number of positive roots; r is the rank and $d_1, \ldots, d_r$ are the degrees of the fundamental invariants of the Weyl group. Note also that since T is a split torus of rank r, we have

$$|T(\mathbb{F}_p)| = (p-1)^r.$$

Since p splits in k and k contains an primitive $2l$-th root of unity (because $d_1 = 2$), we have $p \equiv 1 \bmod 2l$. Hence by Lemma 3,

$$|p^{d_i} - 1|_l = |d_i(p-1)|_l.$$

We therefore have

$$\left|[G(\mathbb{F}_p) : T(\mathbb{F}_p)]\right|_l = |d_1 \cdots d_r|_l.$$

By Corollary 4, it is sufficient to show that $H^4(T(\mathbb{F}_p), \mathbb{Z})$ has an invariant element of order $d_1 \cdots d_r \cdot n$. It will actually be more convenient to find an invariant element of $H^4(T(\mathbb{F}_p), \mathbb{Z}) \otimes (\mathbb{F}_p^\times)^{\otimes 2}$; this is because such an element is canonical, whereas the invariant element of $H^4(T(\mathbb{F}_p), \mathbb{Z})$ would depend on a choice of primitive root modulo p.

We shall construct our invariant element of H^4 from elements of H^2. Since $T(\mathbb{F}_p)$ is a finite group, we have canonical isomorphisms:

$$H^2(T(\mathbb{F}_p), \mathbb{Z}) \cong H^1(T(\mathbb{F}_p), \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(T(\mathbb{F}_p), \mathbb{Q}/\mathbb{Z}).$$

Tensoring with $\mathbb{F}_p^\times$, we get

$$H^2(T(\mathbb{F}_p), \mathbb{Z}) \otimes \mathbb{F}_p^\times \cong \mathrm{Hom}(T(\mathbb{F}_p), \mathbb{F}_p^\times) \cong P/(p-1)P,$$

where as before, P is the lattice of algebraic characters of T.

Recall that the cohomology ring of the cyclic group $\mathbb{F}_p^\times$ is the symmetric algebra on $H^2(\mathbb{F}_p^\times, \mathbb{Z})$. The group $T(\mathbb{F}_p)$ is a product of copies of $\mathbb{F}_p^\times$, and so by the Künneth formula, $H^\bullet(T(\mathbb{F}_p), \mathbb{Z})$ contains as a subring the algebra $H^\bullet(\mathbb{F}_p^\times, \mathbb{Z})^{\otimes r}$, which is isomorphic to the symmetric algebra on $H^2(\mathbb{F}_p^\times, \mathbb{Z})^r$. More canonically, this subring is the symmetric algebra on $H^2(T(\mathbb{F}_p), \mathbb{Z})$. In particular, $H^4(T(\mathbb{F}_p), \mathbb{Z})$ contains $\mathrm{Sym}^2(H^2(T(\mathbb{F}_p), \mathbb{Z}))$ as a subgroup; this is the subgroup generated by cup products of elements of $H^2(T(\mathbb{F}_p), \mathbb{Z})$[1]. From this, we see that $H^4(T(\mathbb{F}_p), \mathbb{Z}) \otimes (\mathbb{F}_p^\times)^{\otimes 2}$ contains as a subgroup the group

$$\mathrm{Sym}^2\left(H^2(T(\mathbb{F}_p), \mathbb{Z}) \otimes \mathbb{F}_p^\times\right) \cong \mathrm{Sym}^2(P/(p-1)P) \cong \mathrm{Sym}^2(P)/(p-1).$$

We claim that the following element of $H^4(T(\mathbb{F}_p), \mathbb{Z})$ is an invariant cohomology class:

$$q = \sum_{\alpha \in \Phi} \alpha \cup \alpha,$$

where Φ is the root system of G with respect to T. The element q is evidently in the subgroup $\mathrm{Sym}^2(P)/(p-1)$. Equivalently, we can regard q as the quadratic function $q : T(\mathbb{F}_p) \to (\mathbb{F}_p^\times)^{\otimes 2}$ defined by

$$q(t) = \sum \alpha(t) \otimes \alpha(t).$$

Here we are writing the group $(\mathbb{F}_p^\times)^{\otimes 2}$ additively.

Suppose g is an element of $G(\mathbb{F}_p)$ and suppose that both t and $g^{-1}tg$ are in $T(\mathbb{F}_p)$. To show that q is an invariant class, we must show that $q(t) = q(g^{-1}tg)$. Evidently we have

$$q(g^{-1}tg) = \sum_{\alpha \in \Phi} \alpha(g^{-1}tg) \otimes \alpha(g^{-1}tg).$$

The numbers $\alpha(t)$ are the non-zero eigenvalues in the action of t on the Lie algebra $\mathfrak{g} \otimes \mathbb{F}_p$. These eigenvalues are the same as those of $g^{-1}tg$, and so the numbers $\alpha(t)$ are the same (possibly in a different order) as the numbers $\alpha(g^{-1}tg)$. From this it follows that $q(t^g) = q(t)$, so q is an invariant class in $H^4(T(\mathbb{F}_p), \mathbb{Z}) \otimes (\mathbb{F}_p^\times)^{\otimes 2}$.

It remains to determine the order of q in $H^4(T(\mathbb{F}_p), \mathbb{Z}) \otimes (\mathbb{F}_p^\times)^{\otimes 2}$, or equivalently the order of q in the subgroup $\mathrm{Sym}^2(P)/(p-1)$. By definition, q is the the reduction modulo $p-1$ of the element $Q \in \mathrm{Sym}^2(P)$. We defined e to be the largest integer such that Q is a multiple of e. Since we are assuming that $p \equiv 1 \bmod e$, the order of q in $\mathrm{Sym}^2(P)/(p-1)$ is precisely $\frac{p-1}{e}$.

To summarize, we have shown that $H^4(T(\mathbb{F}_p), \mathbb{Z})$ has an invariant element of order $\frac{p-1}{e}$. Therefore $H^4(G(\mathbb{F}_p), \mathbb{Z})$ has an element of order $\frac{p-1}{d_1 \cdots d_r \cdot e}$. Since $p \equiv 1 \bmod (d_1 \cdots d_r \cdot e \cdot n)$ it follows that $H^4(G(\mathbb{F}_p), \mathbb{Z})$ has an element of order n. □

[1] This is a proper subgroup if and only if $r \geq 3$.

6 Proof of Theorem 6

6.1 The groups $\bar{H}^2(\mathbb{Z}_l)$

As before, we let L be an open subgroup of the arithmetic completion $\widehat{G(\mathbb{Q})}$, and we shall now fix a prime number l. We introduce a new module

$$\mathcal{C}(L, \mathbb{Z}_l) = \{\text{continuous functions } f : L \to \mathbb{Z}_l\}.$$

Again, we regard the group $\mathcal{C}(L, \mathbb{Z}_l)$ as a $\Gamma(L) \times L$-module. We have

$$\mathcal{C}(L, \mathbb{Z}_l) = \lim_{\substack{\leftarrow \\ t}} \mathcal{C}(L, \mathbb{Z}/l^t).$$

We define, analogously to the notation $\bar{H}^\bullet(\mathbb{Z}/l^t)$,

$$\bar{H}^\bullet(\mathbb{Z}_l) = H^\bullet(\Gamma(L), \mathcal{C}(L, \mathbb{Z}_l)). \tag{3}$$

The main focus of this section is to determine the group $\bar{H}^2(\mathbb{Z}_l)^{\widehat{G(\mathbb{Q})}}$. We begin by establishing some easy properties of the modules $\bar{H}^\bullet(\mathbb{Z}_l)$.

Proposition 6 *The cohomology groups $\bar{H}^\bullet(\mathbb{Z}_l)$ have the following properties:*

1. *The groups $\bar{H}^\bullet(\mathbb{Z}_l)$ do not depend on the open subgroup L in their definition (Equation 3).*
2. $\bar{H}^0(\mathbb{Z}_l) = \mathbb{Z}_l$,
3. $\bar{H}^1(\mathbb{Z}_l) = 0$,
4. $\bar{H}^2(\mathbb{Z}_l) = \lim_{\substack{\leftarrow \\ t}} \bar{H}^2(\mathbb{Z}/l^t)$.
5. *The group $\bar{H}^2(\mathbb{Z}_l)$ is torsion-free and contains no non-zero divisible elements.*
6. *For any open subgroup L of $\widehat{G(\mathbb{Q})}$ we have $\bar{H}^2(\mathbb{Z}_l)^L = \lim_{\substack{\leftarrow \\ t}} \left(\bar{H}^2(\mathbb{Z}/l^t)^L\right)$.*

Proof
1. Suppose M is an open subgroup of L. Then we have an isomorphism of $\Gamma(L)$-modules $\mathcal{C}(L, \mathbb{Z}_l) = \mathrm{ind}_M^L \mathcal{C}(M, \mathbb{Z}_l)$. The result follows from this by Shapiro's Lemma (Theorem 7).
2. Since $\Gamma(L)$ is dense in L, it follows that the $\Gamma(L)$-invariant continuous functions on L are constant. This shows that $\bar{H}^0(\mathbb{Z}_l) = \mathbb{Z}_l$.

(3,4) For each $r > 0$ we have by Theorem 9 a short exact sequence

$$0 \to \left(\lim_{\substack{\leftarrow \\ t}}\right)^1 H^{r-1}(\Gamma(L), \mathcal{C}(L, \mathbb{Z}/l^t)) \to \bar{H}^r(\mathbb{Z}_l) \\ \to \lim_{\substack{\leftarrow \\ t}} H^r(\Gamma(L), \mathcal{C}(L, \mathbb{Z}/l^t)) \to 0.$$

In the notation of the previous section, we have

$$0 \to \left(\lim_{\substack{\leftarrow \\ t}}\right)^1 \bar{H}^{r-1}(\mathbb{Z}/l^t) \to \bar{H}^r(\mathbb{Z}_l) \to \lim_{\substack{\leftarrow \\ t}} \bar{H}^r(\mathbb{Z}/l^t) \to 0.$$

By Lemma 2, $\bar{H}^0(\mathbb{Z}/l^t) = \mathbb{Z}/l^t$ and $\bar{H}^1(\mathbb{Z}/l^t) = 0$. Both of these projective systems consist of finite groups, so satisfy the Mittag–Leffler condition. Therefore $\left(\lim_{\substack{\leftarrow \\ t}}\right)^1$

vanishes on both of them. As a result of this we have for $r = 1, 2$:

$$\bar{H}^r(\mathbb{Z}_l) = \varprojlim_t \bar{H}^r(\mathbb{Z}/l^t).$$

In particular $\bar{H}^1(\mathbb{Z}_l) = 0$.

(5) Consider the short exact sequence of modules:

$$0 \to \mathcal{C}(L, \mathbb{Z}_l) \stackrel{\times l^t}{\to} \mathcal{C}(L, \mathbb{Z}_l) \to \mathcal{C}(L, \mathbb{Z}/l^t) \to 0.$$

This gives the exact sequence in cohomology

$$\cdots \to \bar{H}^1(\mathbb{Z}/l^t) \to \bar{H}^2(\mathbb{Z}_l) \stackrel{\times l^t}{\to} \bar{H}^2(\mathbb{Z}_l) \to \cdots.$$

We already saw in Lemma 2 that $\bar{H}^1(\mathbb{Z}/l^t) = 0$. This shows that $\bar{H}^2(\mathbb{Z}_l)$ is torsion-free. Suppose σ is a divisible element in $\bar{H}^2(\mathbb{Z}_l)$. Then the image of σ in $\bar{H}^2(\mathbb{Z}/l^t)$ is a divisible element for each t. Since $\bar{H}^2(\mathbb{Z}/l^t)$ is a $\mathbb{Z}/l^t$-module, the image of σ in $\bar{H}^2(\mathbb{Z}/l^t)$ must be zero. By (4) it follows that $\sigma = 0$.

(6) This follows because the functor $\varprojlim_t$ commutes with the functor $-^L$ of L-invariant elements. □

Proposition 7 *For any S-arithmetic level $L \subset \widehat{G(\mathbb{Q})}$, there is an exact sequence as follows:*

$$0 \to H^2_{\text{cts}}(L, \mathbb{Z}_l) \to H^2(\Gamma(L), \mathbb{Z}_l) \to \bar{H}^2(\mathbb{Z}_l)^L \to H^3_{\text{cts}}(L, \mathbb{Z}_l) \to H^3(\Gamma(L), \mathbb{Z}_l).$$

Remark 3 It is tempting to suggest that the exact sequence of the proposition follows from a spectral sequence of the form $H^r_{\text{cts}}(L, \bar{H}^s(\mathbb{Z}_l)) \implies H^{r+s}(\Gamma(L), \mathbb{Z}_l)$, which would be proved in the same way as in the finite coefficient case (Proposition 4). Unfortunately this is not quite so simple. The problem is that the groups $\bar{H}^r(\mathbb{Z}_l)$ will probably not be Hausdorff for $r \geq 3$, and so there is no off-the-shelf spectral sequence for us to use. Admittedly we could truncate at $\bar{H}^2(\mathbb{Z}_l)$ to obtain a spectral sequence with three rows, or we could try to work with the more general spectral sequence constructed in [8]. Instead we've gone for a more elementary approach, and we prove the exact sequence of the proposition by taking the projective limit of such exact sequences in the finite coefficient cases.

Proof For any $t \geq 0$ the spectral sequence in Equation 1 gives rise to an exact sequence:

$$\begin{aligned} 0 \to H^2_{\text{cts}}(L, \mathbb{Z}/l^t) \to H^2(\Gamma(L), \mathbb{Z}/l^t) \to \bar{H}^2(\mathbb{Z}/l^t)^L \\ \to H^3_{\text{cts}}(L, \mathbb{Z}/l^t) \to H^3(\Gamma(L), \mathbb{Z}/l^t). \end{aligned}$$

We shall write A_t for the image of the map $H^2(\Gamma(L), \mathbb{Z}/l^t) \to \bar{H}^2(\mathbb{Z}/l^t)^L$ and B_t for the image the map $\bar{H}^2(\mathbb{Z}/l^t)^L \to H^3_{\text{cts}}(L, \mathbb{Z}/l^t)$. We therefore have three exact sequences:

$$0 \to H^2_{\text{cts}}(L, \mathbb{Z}/l^t) \to H^2(\Gamma(L), \mathbb{Z}/l^t) \to A_t \to 0, \quad (4)$$

$$0 \to A_t \to \bar{H}^2(\mathbb{Z}/l^t)^L \to B_t \to 0, \quad (5)$$

$$0 \to B_t \to H^3_{\text{cts}}(L, \mathbb{Z}/l^t) \to H^3(\Gamma(L), \mathbb{Z}/l^t). \quad (6)$$

As $\Gamma(L)$ is an S-arithmetic group, the cohomology groups $H^\bullet(\Gamma(L), \mathbb{Z}/l^t)$ are all finite. From the Equation 4 it follows that $H^2_{\mathrm{cts}}(L, \mathbb{Z}/l^t)$ and A_t are both finite, and hence

$$\left(\lim_{\substack{\leftarrow \\ t}}\right)^1 H^2_{\mathrm{cts}}(L, \mathbb{Z}/l^t) = 0, \qquad \left(\lim_{\substack{\leftarrow \\ t}}\right)^1 A_t = 0.$$

From this it follows that we have exact sequences

$$0 \to \lim_{\substack{\leftarrow \\ t}} H^2_{\mathrm{cts}}(L, \mathbb{Z}/l^t) \to \lim_{\substack{\leftarrow \\ t}} H^2(\Gamma(L), \mathbb{Z}/l^t) \to \lim_{\substack{\leftarrow \\ t}} A_t \to 0,$$

$$0 \to \lim_{\substack{\leftarrow \\ t}} A_t \to \tilde{H}^2(\mathbb{Z}_l)^L \to \lim_{\substack{\leftarrow \\ t}} B_t \to 0,$$

$$0 \to \lim_{\substack{\leftarrow \\ t}} B_t \to \lim_{\substack{\leftarrow \\ t}} H^3_{\mathrm{cts}}(L, \mathbb{Z}/l^t) \to \lim_{\substack{\leftarrow \\ t}} H^3(\Gamma(L), \mathbb{Z}/l^t).$$

In the second of these we have used part (6) of Proposition 6. Splicing the exact sequences together again, we obtain the following exact sequence:

$$0 \to \lim_{\substack{\leftarrow \\ t}} H^2_{\mathrm{cts}}(L, \mathbb{Z}/l^t) \to \lim_{\substack{\leftarrow \\ t}} H^2(\Gamma(L), \mathbb{Z}/l^t) \to \tilde{H}^2(\mathbb{Z}_l)^L$$
$$\to \lim_{\substack{\leftarrow \\ t}} H^3_{\mathrm{cts}}(L, \mathbb{Z}/l^t) \to \lim_{\substack{\leftarrow \\ t}} H^3(\Gamma(L), \mathbb{Z}/l^t).$$

Recall again that as $\Gamma(L)$ is an S-arithmetic group, the groups $H^\bullet(\Gamma(L), \mathbb{Z}/l^t)$ must be finite. Furthermore, the spectral sequence in Eq. 1 shows that the groups $H^1_{\mathrm{cts}}(L, \mathbb{Z}/l^t)$ and $H^2_{\mathrm{cts}}(L, \mathbb{Z}/l^t)$ are also finite. As a result, all of these projective systems satisfy the Mittag–Leffler condition, so we have:

$$\left(\lim_{\substack{\leftarrow \\ t}}\right)^1 H^1_{\mathrm{cts}}(L, \mathbb{Z}/l^t), \quad \left(\lim_{\substack{\leftarrow \\ t}}\right)^1 H^1(\Gamma(L), \mathbb{Z}/l^t),$$
$$\left(\lim_{\substack{\leftarrow \\ t}}\right)^1 H^2_{\mathrm{cts}}(L, \mathbb{Z}/l^t), \quad \left(\lim_{\substack{\leftarrow \\ t}}\right)^1 H^2(\Gamma(L), \mathbb{Z}/l^t).$$

As a result of this, we have

$$\lim_{\substack{\leftarrow \\ t}} H^2_{\mathrm{cts}}(L, \mathbb{Z}/l^t) = H^2_{\mathrm{cts}}(L, \mathbb{Z}_l),$$

$$\lim_{\substack{\leftarrow \\ t}} H^2(\Gamma(L), \mathbb{Z}/l^t) = H^2(\Gamma(L), \mathbb{Z}_l),$$

$$\lim_{\substack{\leftarrow \\ t}} H^3_{\mathrm{cts}}(L, \mathbb{Z}/l^t) = H^3_{\mathrm{cts}}(L, \mathbb{Z}_l),$$

$$\lim_{\substack{\leftarrow \\ t}} H^3(\Gamma(L), \mathbb{Z}/l^t) = H^3(\Gamma(L), \mathbb{Z}_l).$$

Substituting these into our previous exact sequence we get

$$0 \to H^2_{\mathrm{cts}}(L, \mathbb{Z}_l) \to H^2(\Gamma(L), \mathbb{Z}_l) \to \tilde{H}^2(\mathbb{Z}_l)^L \to H^3_{\mathrm{cts}}(L, \mathbb{Z}_l) \to H^3(\Gamma(L), \mathbb{Z}_l).$$

□

6.2 The groups $H^\bullet_{\text{cts}}(L, \mathbb{Z}_l)$

We shall next concentrate on the continuous cohomology groups in the exact sequence of Proposition 7.

From now on, we assume that L is an S-arithmetic level for some finite set of primes S. Recall that this means L is the pre-image in $\widehat{G(\mathbb{Q})}$ of an open subgroup of $G(\mathbb{A}_f)$ of the form

$$\prod_{p\in S} G(\mathbb{Q}_p) \times \prod_{p\notin S} K_p, \qquad \text{where } K_p \text{ is compact and open in } G(\mathbb{Q}_p).$$

It will be convenient to call a prime number p a *tame prime* if it satisfies all of the following conditions:

1. $p \notin S$.
2. $p \neq l$.
3. G is unramified over $\mathbb{Q}_p$.
4. K_p is a maximal hyperspecial compact open subgroup of $G(\mathbb{Q}_p)$. This implies that if we let K_p^0 be the maximal pro-p normal subgroup of K_p, then the group $G(\mathbb{F}_p) = K_p/K_p^0$ is a product of some of the simply connected finite groups of Lie type described in detail in [22].
5. $H^r(G(\mathbb{F}_p), \mathbb{Q}/\mathbb{Z}) = 0$ for $r = 1, 2$. We recall from [22] that this condition is satisfied for all but finitely many of the groups $G(\mathbb{F}_p)$.

We note that for tame primes p we have $H^\bullet_{\text{cts}}(K_p, \mathbb{Z}/l^t) = H^\bullet(G(\mathbb{F}_p), \mathbb{Z}/l^t)$ by condition (2).

All but finitely many of the prime numbers are tame. We shall write W for the set of primes not in S which are not tame. The group $L/\text{Cong}(G)$ decomposes in the form

$$L/\text{Cong}(G) = L_S \times K_W \times K_{\text{tame}},$$

where we are using the notation:

$$L_S = \prod_{p\in S} G(\mathbb{Q}_p), \quad K_W = \prod_{p\in W} K_p, \quad K_{\text{tame}} = \prod_{p \text{ tame}} K_p.$$

Lemma 4 *With the notation introduced above, $H^r_{\text{cts}}(K_{\text{tame}}, \mathbb{Z}_l) = 0$ for $r = 1, 2, 3$.*

Proof For each $r > 0$ we have by Theorem 9 a short exact sequence

$$0 \to \left(\lim_{\substack{\leftarrow \\ t}}\right)^1 H^{r-1}_{\text{cts}}(K_{\text{tame}}, \mathbb{Z}/l^t) \to H^r_{\text{cts}}(K_{\text{tame}}, \mathbb{Z}_l) \to \lim_{\substack{\leftarrow \\ t}} H^r_{\text{cts}}(K_{\text{tame}}, \mathbb{Z}/l^t) \to 0.$$

Furthermore, for any r we have

$$H^r_{\text{cts}}(K_{\text{tame}}, \mathbb{Z}/l^t) = \lim_{\substack{\rightarrow \\ U}} H^r_{\text{cts}}\left(\prod_{p\in U} K_p, \mathbb{Z}/l^t\right),$$

where U runs through the finite sets of tame primes. For such primes p we have $H^r_{\text{cts}}(K_p, \mathbb{Q}_l/\mathbb{Z}_l) = 0$ for $r = 1, 2$. Hence by an obvious long exact sequence we have $H^r_{\text{cts}}(K_p, \mathbb{Z}/l^t) = 0$ for $r = 1, 2$. By the Künneth formula we have

$$H^r_{\text{cts}}\left(\prod_{p\in U} K_p, \mathbb{Z}/l^t\right) = 0 \quad \text{for } r = 1, 2.$$

Hence $H^r_{\mathrm{cts}}(K_{\mathrm{tame}}, \mathbb{Z}/l^t) = 0$ for $r = 1, 2$. Since the projective system $H^0_{\mathrm{cts}}(K_{\mathrm{tame}}, \mathbb{Z}/l^t) = \mathbb{Z}/l^t$ satisfies the Mittag–Leffler condition, we have $H^r_{\mathrm{cts}}(K_{\mathrm{tame}}, \mathbb{Z}_l) = 0$ for $r = 1, 2$.

We'll now concentrate on the group $H^3_{\mathrm{cts}}(K_{\mathrm{tame}}, \mathbb{Z}_l)$. By the short exact sequence above, together with the fact that $H^2_{\mathrm{cts}}(K_{\mathrm{tame}}, \mathbb{Z}/l^t) = 0$, we have

$$H^3_{\mathrm{cts}}(K_{\mathrm{tame}}, \mathbb{Z}_l) = \varprojlim_t H^3_{\mathrm{cts}}(K_{\mathrm{tame}}, \mathbb{Z}/l^t).$$

We also have (using the Künneth formula and the fact that $H^\bullet(K_p, \mathbb{Z}/l^t)$ is isomorphic to $H^\bullet(G(\mathbb{F}_p), \mathbb{Z}/l^t)$):

$$H^3_{\mathrm{cts}}(K_{\mathrm{tame}}, \mathbb{Z}_l) = \varprojlim_t \left(\bigoplus_{p \text{ tame}} H^3(G(\mathbb{F}_p), \mathbb{Z}/l^t) \right) \subseteq \prod_{p \text{ tame}} \left(\varprojlim_t H^3(G(\mathbb{F}_p), \mathbb{Z}/l^t) \right). \tag{7}$$

Consider any tame prime number p. Since $G(\mathbb{F}_p)$ is finite, we have $H^3(G(\mathbb{F}_p), \mathbb{Q}_l) = 0$, and therefore $H^3(G(\mathbb{F}_p), \mathbb{Z}_l) = H^2(G(\mathbb{F}_p), \mathbb{Q}_l/\mathbb{Z}_l) = 0$. From this it follows that

$$\varprojlim_t H^3(G(\mathbb{F}_p), \mathbb{Z}/l^t) = 0.$$

By Eq. 7 we have $H^3_{\mathrm{cts}}(K_{\mathrm{tame}}, \mathbb{Z}_l) = 0$. □

(It might be tempting to imagine that the result above can be extended further in a simple way. However, we note that the projective system in Eq. 7 does not satisfy the Mittag–Leffler condition, so we do not expect $H^4_{\mathrm{cts}}(K_{\mathrm{tame}}, \mathbb{Z}_l)$ to be finitely generated as a $\mathbb{Z}_l$-module).

Lemma 5 *Let L be an S-arithmetic level in $\widehat{G(\mathbb{Q})}$. For $r = 0, 1, 2, 3$ we have*

$$H^r_{\mathrm{cts}}(L/\mathrm{Cong}(G), \mathbb{Z}_l) = H^r_{\mathrm{cts}}(L_S \times K_W, \mathbb{Z}_l),$$
$$H^r_{\mathrm{cts}}(L, \mathbb{Q}_l) = H^r_{\mathrm{cts}}(G(\mathbb{Q}_l), \mathbb{Q}_l) = H^r_{\mathrm{Lie}}(\mathfrak{g} \otimes \mathbb{Q}_l, \mathbb{Q}_l).$$

Here $\mathfrak{g}$ is the Lie algebra of G over $\mathbb{Q}$.

Proof Recall that we have a decomposition of the group $L/\mathrm{Cong}(G)$ in the form $L_S \times K_W \times K_{\mathrm{tame}}$. This gives rise to the following spectral sequence

$$H^r_{\mathrm{cts}}(L_S \times K_T, H^s(K_{\mathrm{tame}}, \mathbb{Z}_l)) \implies H^{r+s}_{\mathrm{cts}}(L/\mathrm{Cong}(G), \mathbb{Z}_l).$$

From Lemma 4, we see that the bottom left corner of the E_2-sheet is as follows:

$$\begin{array}{cccc}
0 & \cdots & & \\
0 & 0 & \cdots & \\
0 & 0 & 0 & \cdots \\
H^0_{\mathrm{cts}}(L_S \times K_T, \mathbb{Z}_l) & H^1_{\mathrm{cts}}(L_S \times K_T, \mathbb{Z}_l) & H^2_{\mathrm{cts}}(L_S \times K_T, \mathbb{Z}_l) & H^3_{\mathrm{cts}}(L_S \times K_T, \mathbb{Z}_l)
\end{array}$$

This shows that $H^r_{\mathrm{cts}}(L/\mathrm{Cong}(G), \mathbb{Z}_l) = H^r(L_S \times K_T, \mathbb{Z}_l)$ for $r \leq 3$.

Since $\mathrm{Cong}(G)$ is a finite group we have

$$H^s(\mathrm{Cong}(G), \mathbb{Q}_l) = \begin{cases} \mathbb{Q}_l & \text{if } s = 0, \\ 0 & \text{if } s > 0. \end{cases}$$

Therefore the spectral sequence $H^r_{\mathrm{cts}}(L/\mathrm{Cong}(G), H^s(\mathrm{Cong}(G), \mathbb{Q}_l))$ collapses to the bottom row, so we have

$$H^r_{\mathrm{cts}}(L, \mathbb{Q}_l) = H^r_{\mathrm{cts}}(L/\mathrm{Cong}(G), \mathbb{Q}_l).$$

In particular for $r \le 3$ we have

$$H^r_{\mathrm{cts}}(L, \mathbb{Q}_l) = H^r_{\mathrm{cts}}\left(\prod_{p\in S} G(\mathbb{Q}_p) \times \prod_{p\in W} K_p, \mathbb{Q}_l\right).$$

We'll calculate these cohomology groups above using the Künneth formula.

Suppose first that p is a prime in W, which is not equal to l. The group K_p contains a normal pro-p subgroup of finite index. From this it follows that

$$H^s_{\mathrm{cts}}(K_p, \mathbb{Q}_l) = \begin{cases} \mathbb{Q}_l & s = 0, \\ 0 & s > 0. \end{cases}$$

Next, suppose that p is a prime in S which is not equal to l. We recall from [7] that there is a spectral sequence which calculates the cohomology of $G(\mathbb{Q}_p)$ in terms of the cohomology of its compact open subgroups. Let K^0_p be a maximal pro-p subgroup of $G(\mathbb{Q}_p)$. The subgroup K^0_p is compact and open in $G(\mathbb{Q}_p)$. There are finitely many maximal compact subgroups of $G(\mathbb{Q}_p)$, which contain K^0_p; we call these subgroups $K_1, \ldots, K_n$. In the spectral sequence, the E_1-sheet is given by

$$E_1^{r,s} = \bigoplus_{i_0<\ldots<i_r} H^s_{\mathrm{cts}}(K_{i_0,\ldots,i_s}, \mathbb{Q}_l).$$

Here we are using the notation

$$K_{i_1,\ldots,i_s} = K_{i_1} \cap \ldots \cap K_{i_s}.$$

The map $E_1^{r-1,s} \to E_1^{r,s}$ is an alternating sum of restriction maps; in other words, its $(i_0, \ldots, i_r)$-component is equal to

$$\sum_{j=0}^{r} (-1)^j \mathrm{Rest}^{K_{i_0,\ldots,\hat{i}_j,\ldots,i_r}}_{K_{i_0,\ldots,i_r}} \left(\sigma_{i_0,\ldots,\hat{i}_j,\ldots,i_r}\right).$$

As we are assuming here that $p \neq l$, the cohomology groups $H^s_{\mathrm{cts}}(K_{i_0,\ldots,i_s}, \mathbb{Q}_l)$ are zero for $s > 0$. Therefore the spectral sequence consists of a single row in E_1; this row is the simplicial cochain complex of a simplex with n vertices. As this simplex is contractable, we have

$$H^s_{\mathrm{cts}}(G(\mathbb{Q}_p), \mathbb{Q}_l) = \begin{cases} \mathbb{Q}_l & s = 0, \\ 0 & s > 0. \end{cases}$$

From the Künneth formula we have for $r \le 3$:

$$H^r_{\mathrm{cts}}(L, \mathbb{Q}_l) = H^r_{\mathrm{cts}}(L_l, \mathbb{Q}_l),$$

where L_l is either K_l or $G(\mathbb{Q}_l)$, depending on whether the prime l is in S or not. In either case we have $H^\bullet_{\mathrm{cts}}(L_l, \mathbb{Q}_l) = H^\bullet(\mathfrak{g} \otimes \mathbb{Q}_l, \mathbb{Q}_l)$; this is proved in [14] for K_l and in [7] for $G(\mathbb{Q}_l)$. As a result of this we have $H^r_{\mathrm{cts}}(L, \mathbb{Q}_l) = H^r(\mathfrak{g} \otimes \mathbb{Q}_l, \mathbb{Q}_l)$ for $r \le 3$. □

Lemma 6 *We have* $H^0(\mathfrak{g} \otimes \mathbb{Q}_l, \mathbb{Q}_l) = \mathbb{Q}_l$, $H^1(\mathfrak{g} \otimes \mathbb{Q}_l, \mathbb{Q}_l) = 0$, $H^2(\mathfrak{g} \otimes \mathbb{Q}_l, \mathbb{Q}_l) = 0$ *and* $H^3(\mathfrak{g} \otimes \mathbb{Q}_l, \mathbb{Q}_l) = \mathbb{Q}_l^b$, *where* b *is the number of simple components of* $G \times_{\mathbb{Q}} \mathbb{C}$. *(Note that in the notation of the introduction we have* $b = b_{\mathbb{R}} + 2b_{\mathbb{C}}$*).*

Proof The dimension of the Lie algebra cohomology does not depend on the base field, so we may instead calculate the cohomology of $\mathfrak{g} \otimes \mathbb{C}$. There is a decomposition:

$$\mathfrak{g} \otimes \mathbb{C} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_b,$$

where each $\mathfrak{g}_i$ is a complex simple Lie algebra. By Whitehead's first and second lemmas we have $H^r(\mathfrak{g}_i, \mathbb{C}) = 0$ for $r = 1, 2$, and $H^0(\mathfrak{g}_i, \mathbb{C}) = \mathbb{C}$. It is well known (see for example section 1.6 of [5]) that $H^\bullet(\mathfrak{g}_i, \mathbb{C})$ is isomorphic to the singular cohomology of a compact connected Lie group with Lie algebra $\mathfrak{g}_i$. Hence by Proposition 8 each group $H^3(\mathfrak{g}_i, \mathbb{C})$ is 1-dimensional. The lemma follows from the Künneth formula. □

6.3 The end of the proof

Recall from Proposition 7 that we have an exact sequence:

$$0 \to H^2_{\mathrm{cts}}(L, \mathbb{Z}_l) \to H^2(\Gamma(L), \mathbb{Z}_l) \to \bar{H}^2(\mathbb{Z}_l)^L \to H^3_{\mathrm{cts}}(L, \mathbb{Z}_l) \to H^3(\Gamma(L), \mathbb{Z}_l).$$

Tensoring with $\mathbb{Q}_l$ and using Lemma 5 and Lemma 6, we have an exact sequence of $\mathbb{Q}_l$-vector spaces. We've seen in Proposition 6 that $\bar{H}^2(\mathbb{Z}_l)$ is torsion-free. This implies that we may regard $\bar{H}^2(\mathbb{Z}_l)$ as a subgroup of $\bar{H}^2(\mathbb{Q}_l)$, where we are using the notation $\bar{H}^2(\mathbb{Q}_l) = \bar{H}^2(\mathbb{Z}_l) \otimes \mathbb{Q}_l$. It follows that $\bar{H}^2(\mathbb{Z}_l)^L \otimes \mathbb{Q}_l = \bar{H}^2(\mathbb{Q}_l)^L$. In view of this, we have an exact sequence

$$0 \to H^2(\Gamma(L), \mathbb{Q}_l) \to \bar{H}^2(\mathbb{Q}_l)^L \to H^3(\mathfrak{g} \otimes \mathbb{Q}_l, \mathbb{Q}_l) \to H^3(\Gamma(L), \mathbb{Q}_l). \tag{8}$$

The vector spaces in Eq. 8 are all finite dimensional; this follows for $\bar{H}^2(\mathbb{Q}_l)^L$, because it is between the finite dimensional spaces $H^2(\Gamma(L), \mathbb{Q}_l)$ and $H^3(\mathfrak{g} \otimes \mathbb{Q}_l, \mathbb{Q}_l)$.

We shall next take the projective limit over S of the sequences in Eq. 8. Since the sequences consist of finite dimensional vector spaces, the derived functors $\left(\varprojlim_S\right)^1$ all vanish, so we have the following exact sequence:

$$0 \to H^2(G(\mathbb{Q}), \mathbb{Q}_l) \to \bar{H}^2(\mathbb{Q}_l)^{\widehat{G(\mathbb{Q})}} \to H^3(\mathfrak{g} \otimes \mathbb{Q}_l, \mathbb{Q}_l) \to H^3(G(\mathbb{Q}), \mathbb{Q}_l).$$

Here we have used Proposition 2, which shows that the projective limit (over S) of the groups $H^r(\Gamma(L), \mathbb{Q}_l)$ is $H^r(G(\mathbb{Q}), \mathbb{Q}_l)$.

The dimensions of the groups $H^r(G(\mathbb{Q}), \mathbb{Q}_l)$ are the same as those of $H^r(G(\mathbb{Q}), \mathbb{C})$, and by Proposition 2 these are the same as the the relative Lie algebra cohomology groups $H^r(\mathfrak{g}, \mathfrak{k}, \mathbb{C})$. Here $\mathfrak{k}$ is the Lie algebra of a maximal compact subgroup K_∞ of $G(\mathbb{R})$.

Recall from section 1.6 of [5] that the relative Lie algebra cohomology groups are isomorphic to the singular cohomology groups $H^r(X^*, \mathbb{C})$, where X^* is the compact dual of the symmetric space $X = G(\mathbb{R})/K_\infty$. We calculate the dimensions of these spaces in the appendix. The results (see Corollary 5) are:

$$\dim H^2(\mathfrak{g}, \mathfrak{k}, \mathbb{C}) = \dim(Z(K_\infty)).$$

$$\dim H^3(\mathfrak{g}, \mathfrak{k}, \mathbb{C}) = \#\text{simple components of } G \times \mathbb{R} \text{ of complex type.}$$

If we write $b_{\mathbb{C}}$ for the number of simple components of $G \times \mathbb{R}$ of complex type, and $b_{\mathbb{R}}$ for the number of simple components of real type, then the number of simple components of $G \times \mathbb{C}$ is $b_{\mathbb{R}} + 2b_{\mathbb{C}}$. Substituting these dimensions, we have an exact sequence of the form

$$0 \to \mathbb{Q}_l^{\dim(Z(K_\infty))} \to \bar{H}^2(\mathbb{Q}_l)^{\widehat{G(\mathbb{Q})}} \to \mathbb{Q}_l^{b_{\mathbb{R}}+2b_{\mathbb{C}}} \to \mathbb{Q}_l^{b_{\mathbb{C}}}.$$

It follows that the dimension of $\bar{H}^2(\mathbb{Q}_l)^{\widehat{G(\mathbb{Q})}}$ is between $\dim(Z(K_\infty)) + b_{\mathbb{R}} + b_{\mathbb{C}}$ and $\dim(Z(K_\infty)) + b_{\mathbb{R}} + 2b_{\mathbb{C}}$.

Since $\bar{H}^2(\mathbb{Z}_l)$ is torsion-free, it follows that $\bar{H}^2(\mathbb{Z}_l)^{\widehat{G(\mathbb{Q})}}$ is a torsion-free $\mathbb{Z}_l$-module, which spans $\bar{H}^2(\mathbb{Q}_l)^{\widehat{G(\mathbb{Q})}}$. On the other hand, $\bar{H}^2(\mathbb{Z}_l)$ has no non-zero divisible elements. This implies that $\bar{H}^2(\mathbb{Z}_l)^{\widehat{G(\mathbb{Q})}} \cong {\mathbb{Z}_l}^c$, where $c = \dim \bar{H}^2(\mathbb{Q}_l)^{\widehat{G(\mathbb{Q})}}$. This finishes the proof of the Theorem 6.

Corollary *There is a subgroup of $\bar{H}^2(\mathbb{Z}/l^t)^{\widehat{G(\mathbb{Q})}}$ isomorphic to $(\mathbb{Z}/l^t)^c$, all of whose elements virtually lift to characteristic zero, where $c = \mathrm{rank}_{\mathbb{Z}_l}\left(\bar{H}^2(\mathbb{Z}_l)^{\widehat{G(\mathbb{Q})}}\right)$.*

Proof We have a short exact sequence

$$0 \to \mathcal{C}(L, \mathbb{Z}_l) \stackrel{\times l^t}{\to} \mathcal{C}(L, \mathbb{Z}_l) \to \mathcal{C}(L, \mathbb{Z}/l^t) \to 0.$$

This gives a long exact sequence containing the following terms

$$\bar{H}^1(\mathbb{Z}/l^t) \to \bar{H}^2(\mathbb{Z}_l) \stackrel{\times l^t}{\to} \bar{H}^2(\mathbb{Z}_l) \to \bar{H}^2(\mathbb{Z}/l^t).$$

We've shown that $\bar{H}^1(\mathbb{Z}/l^t) = 0$, so we have

$$0 \to \bar{H}^2(\mathbb{Z}_l) \stackrel{\times l^t}{\to} \bar{H}^2(\mathbb{Z}_l) \to \bar{H}^2(\mathbb{Z}/l^t).$$

We'll write A for the subgroup of elements in $\bar{H}^2(\mathbb{Z}/l^t)$ which virtually lift to characteristic zero. By definition, A is the image of $\bar{H}^2(\mathbb{Z}_l)$ in $H^2(\mathbb{Z}/l^t)$. We therefore have a short exact sequence

$$0 \to \bar{H}^2(\mathbb{Z}_l) \stackrel{\times l^t}{\to} \bar{H}^2(\mathbb{Z}_l) \to A \to 0.$$

Taking $\widehat{G(\mathbb{Q})}$-invariants, we have an exact sequence

$$0 \to \bar{H}^2(\mathbb{Z}_l)^{\widehat{G(\mathbb{Q})}} \stackrel{\times l^t}{\to} \bar{H}^2(\mathbb{Z}_l)^{\widehat{G(\mathbb{Q})}} \to A^{\widehat{G(\mathbb{Q})}}.$$

The result follows because $\bar{H}^2(\mathbb{Z}_l)^{\widehat{G(\mathbb{Q})}}$ is isomorphic to ${\mathbb{Z}_l}^c$. □

Acknowlegements
I'd like to thank Lars Louder for many useful discussions. I'd also like to thank the anonymous referee for suggesting several improvements.

Appendix: A result on compact symmetric spaces

In this appendix, we shall calculate the low dimensional cohomology of the compact simple symmetric spaces. Such spaces have the form G/K, where G is a compact, connected, simple Lie group and K is a closed, connected subgroup. In the following result, we shall use the shorthand $H^\bullet(X)$ for the singular cohomology on the topological space X with coefficients in $\mathbb{R}$.

Proposition 8 *Let G be a compact, connected simple Lie group and K a closed, connected subgroup. There are isomorphisms:*

$$H^1(G/K) = 0,$$
$$H^2(G/K) = \mathfrak{z}(\mathfrak{k})^*,$$
$$H^3(G/K) = \begin{cases} \mathbb{R} & \text{if } K \text{ is trivial,} \\ 0 & \text{otherwise.} \end{cases}$$

Here $\mathfrak{z}(\mathfrak{k})^*$ denotes the dual space of the centre of the Lie algebra $\mathfrak{k}$ of K.

Remark 4 One might expect to be able to look this result up in tables; however I didn't manage to find such tables, so I am including a proof. The result is a straightforward consequence of results in Borel's thesis [3].

Proof We shall first review some results from [3] on the cohomology of compact Lie groups and their classifying spaces. Suppose that G is compact connected Lie group. We shall write BG for the classifying space of G. This is a space with a fibre bundle

$$\begin{array}{ccc} G & \to & EG \\ & & \downarrow \\ & & BG \end{array}$$

such that the cohomology of EG is the cohomology of a point.

The real singular cohomology of G is (as a ring) an exterior algebra whose generators are cohomology classes $x_1, \ldots, x_n$ in odd dimensions $s_1, \ldots, s_n$. For each generator $x_i \in H^{s_i}(G)$, there is an element $y_i \in H^{s_i+1}(BG)$ called the transgression of x_i. Furthermore the cohomology ring $H^\bullet(BG)$ is equal to the polynomial ring $\mathbb{R}[y_1, \ldots, y_n]$. In particular BG only has non-zero real cohomology in even dimensions.

As an example of this, let T be an n-dimensional compact torus, and let $\mathfrak{t}$ be its Lie algebra. Recall that the cohomology of T is exactly the exterior algebra of $H^1(T)$. Furthermore, we may identify $H^1(T)$ with the dual space of $\mathfrak{t}$. As a result of this, we know that $H^\bullet(BT)$ is the algebra $\mathbb{R}[\mathfrak{t}]$ of polynomial functions on $\mathfrak{t}$. Furthermore, $H^{2n}(BT)$ is the space of homogeneous polynomials on $\mathfrak{t}$ of degree n.

Suppose now that G is a compact, connected Lie group and T is a maximal torus in G. We shall write W for the Weyl group of G with respect to T. We have a fibre bundle

$$\begin{array}{ccc} G/T & \to & BT \\ & & \downarrow \\ & & BG \end{array} \qquad (BT = EG/T).$$

Corresponding to this there is a spectral sequence

$$H^r(BG, H^s(G/T)) \Rightarrow H^{r+s}(BT).$$

As G is connected, it follows easily that BG is simply connected. This implies that $H^s(G/T)$ is a trivial bundle on BG, and so the spectral sequence takes the form

$$H^r(BG) \otimes H^s(G/T) \Rightarrow H^{r+s}(BT).$$

In particular we have an edge map $H^\bullet(BG) \to H^\bullet(BT)$. We'll use the following result, which describes this edge map.

Proposition 9 (Proposition 27.1 of [3]) *Let G be a compact, connected Lie group and T a maximal torus in G. The edge map $H^\bullet(BG) \to H^\bullet(BT)$ is injective. Its image is the subspace $H^\bullet(BT)^W$ of W-invariant polynomial functions on* $\mathfrak{t}$.

As a result of this proposition, we know that for semi-simple G we have $H^2(BG) = 0$. This is because there are no W-invariant linear forms on $\mathfrak{t}$. For simple G we have

$$H^4(BG) = \mathbb{R}.$$

Recall that $H^4(BG)$ is the space of W-invariant quadratic forms on $\mathfrak{t}$. The restriction of the Killing form is one such form, and any other is a constant multiple of this. As a consequence, we see that $H^1(G) = H^2(G) = 0$ and $H^3(G) = \mathbb{R}$. This proves Proposition 8 in the case that K is trivial.

Assume now that K is a non-trivial closed, connected subgroup of G. We shall use the spectral sequence of the following fibration:

$$\begin{array}{ccc} G/K & \to & BK \\ & & \downarrow \\ & & BG. \end{array}$$

That is:

$$H^r(BG) \otimes H^s(G/K) \Rightarrow H^{r+s}(BK). \qquad (9)$$

Let S be a maximal torus in K and $T \supset S$ be a maximal torus in G, and let W_G and W_K be the corresponding Weyl groups. From the spectral sequence in Eq. 9 we have an edge map

$$H^\bullet(BG) \to H^\bullet(BK).$$

By Proposition 9, we may interpret this as a map

$$\mathbb{R}[\mathfrak{t}]^{W_G} \to \mathbb{R}[\mathfrak{s}]^{W_K},$$

where $\mathfrak{s}$ is the Lie algebra of S. This map has been determined by Borel:

Proposition 10 (Proposition 28.2 of [3]) *The above edge map is given by restricting a polynomial function on* $\mathfrak{t}$ *to the subspace* $\mathfrak{s}$.

We can now finish proving our proposition. Suppose that G is simple and K is non-trivial. Then $H^4(BG)$ is generated by the Killing form. Since the Killing form is negative definite, its restriction to $\mathfrak{s}$ is non-zero. This shows that the edge map $H^4(BG) \to H^4(BK)$ is injective. The E_2-sheet of the spectral sequence in Eq. 9 looks like this:

$$\begin{array}{ccccc} H^3(G/K) & 0 & 0 & 0 & \\ & & & & \\ H^2(G/K) & 0 & 0 & 0 & \\ & & & & \\ H^1(G/K) & 0 & 0 & 0 & \\ & & & & \\ \mathbb{R} & 0 & 0 & 0 & H^4(BG) \end{array}$$

These groups all remain unchanged until the E_4 sheet, where we have a map $H^3(G/K) \to H^4(BG)$. From this we see that

$$H^1(G/K) = H^1(BK) = 0 \quad \text{because 1 is odd},$$
$$H^2(G/K) = H^2(BK) = H^1(K) = \mathfrak{z}(\mathfrak{k})^*.$$

Furthermore there is an exact sequence:

$$0 \to H^3(BK) \to H^3(G/K) \to H^4(BG) \to H^4(BK).$$

We've seen that $H^3(BK) = 0$ (because 3 is odd) and the edge map $H^4(BG) \to H^4(BK)$ is injective, so it follows that $H^3(G/K) = 0$. □

We finally translate the result above into a more usable form. In the following corollary, G is a semi-simple, simply connected algebraic group over $\mathbb{R}$ and K_∞ is a maximal compact subgroup of $G(\mathbb{R})$. We shall write $b_\mathbb{R}$ and $b_\mathbb{C}$ for the number of simple components of G of real and of complex type. We write $\mathfrak{g}$ and $\mathfrak{k}$ for the Lie algebras of $G(\mathbb{R})$ and K_∞ respectively, and we write $\mathfrak{z}(\mathfrak{k})$ for the centre of $\mathfrak{k}$.

Corollary 5 *With the notation introduced above, we have*

$$H^2(\mathfrak{g}, \mathfrak{k}, \mathbb{R}) = \mathfrak{z}(\mathfrak{k})^*, \quad \dim H^3(\mathfrak{g}, \mathfrak{k}, \mathbb{R}) = b_\mathbb{C}.$$

Proof Recall from section 1.6 of [5], that the $\mathfrak{g}, \mathfrak{k}$-cohomology is the same as the singular cohomology of the compact symmetric space $X = H(\mathbb{R})/K_\infty$, where H is the compact form of G over $\mathbb{R}$; in other words, $H(\mathbb{R})$ is a maximal compact subgroup of $G(\mathbb{C})$ containing K_∞.

Let $G = \prod_{i=1}^{b_\mathbb{R}+b_\mathbb{C}} G_i$, where each G_i is a simple, simply connected group over $\mathbb{R}$; the groups $G_1, \ldots, G_{b_\mathbb{R}}$ are assumed to be of real type and the others are of complex type. The subgroup K_∞ also decomposes as $\prod K_i$, where each K_i is a maximal compact subgroup of $G_i(\mathbb{R})$. Similarly the compact form H decomposes as $\prod H_i$, where each H_i is the compact form of G_i. This gives us a decomposition

$$X = \prod X_i, \quad \text{where } X_i = H_i(\mathbb{R})/K_i.$$

Our aim is to calculate the cohomology of X using the Künneth formula.

Case 1. In the case that G_i is of real type, The group $G_i \times \mathbb{C}$ is simple over $\mathbb{C}$. Therefore $H_i(\mathbb{R})$, being a maximal compact subgroup of $G_i(\mathbb{C})$, is a simple, simply connected Lie group. By Proposition 8 we have

$$H^1(X_i) = 0, \qquad H^2(X_i) = \mathfrak{z}(\mathfrak{k}_i)^*, \qquad H^3(X_i) = 0.$$

Case 2. Suppose instead that G_i is of complex type. In this case $G_i \times \mathbb{C}$ splits as a direct sum of two simple groups and we have $G_i(\mathbb{C}) \cong G_i(\mathbb{R}) \times G_i(\mathbb{R})$; the subgroups $G_i(\mathbb{R})$ and K_i are diagonally embedded in $G_i(\mathbb{C})$. As $H_i(\mathbb{R})$ is a maximal compact subgroup of $G_i(\mathbb{C})$, we have $H_i(\mathbb{R}) = K_i \times K_i$. In this case, the compact symmetric space X_i is the quotient $(K_i \times K_i)/K_i$, which is homeomorphic to K_i. By Proposition 8 we have

$$H^1(X_i) = 0, \qquad H^2(X_i) = \mathfrak{z}(\mathfrak{k}_i)^*, \qquad H^3(X_i) = \mathbb{R}.$$

By the Künneth formula we have

$$H^2(\mathfrak{g}, \mathfrak{k}, \mathbb{R}) = \bigoplus_i \mathfrak{z}(\mathfrak{k}_i)^* = \mathfrak{z}(\mathfrak{k})^*,$$

$$H^3(\mathfrak{g}, \mathfrak{k}, \mathbb{R}) = \bigoplus_{i \text{ of complex type}} \mathbb{R} = \mathbb{R}^{b_\mathbb{C}}.$$

□

Received: 30 July 2018 Accepted: 20 October 2018
Published online: 02 November 2018

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