Localised modes due to defects in high contrast periodic media via two-scale homogenization

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Abstract

The spectral problem for an infinite periodic medium perturbed by a compact defect is considered. For a high contrast small ε -size periodicity and a finite size defect we consider the critical ε^2 -scaling for the contrast. We employ (high contrast) two-scale homogenization for deriving asymptotically explicit limit equations for the localised modes (exponentially decaying eigenfunctions) and associated eigenvalues. Those are expressed in terms of the eigenvalues and eigenfunctions of a perturbed version of a two-scale limit operator introduced by V.V. Zhikov with an emergent explicit nonlinear dependence on the spectral parameter for the spectral problem at the macroscale. Using the method of asymptotic expansions supplemented by a high contrast boundary layer analysis we establish the existence of the actual eigenvalues near the eigenvalues of the limit operator, with “ ε square root” error bounds. An example for circular or spherical defects in a periodic medium with isotropic homogenized properties is given and displays explicit limit eigenvalues and eigenfunctions. Further results on improved error bounds for the eigenfunctions are discussed, by combining our results with those of M. Cherdantsev (*Mathematika*. 2009;55:29–57) based on the technique of strong two-scale resolvent convergence and associated two-scale compactness properties.

Keywords: localised modes, defects in periodic media, high-contrast homogenization, eigenvalue problem, boundary layer analysis, error bounds

1 Introduction

We dedicate this work to the memory of Professor V.V. Zhikov. Among many important contributions of V.V. Zhikov are developments of two-scale homogenization techniques for high contrast spectral problems, e.g. [30, 31, 34], and of error estimates in homogenization theory, e.g. [19, 32, 34]. The present work, whose preliminary version was completed in 2006, relates to the both topics and was greatly influenced by Zhikov then and has had a continued influence by his ideas since.

Studying the spectral properties of operators with periodic coefficients, with and without defects, has received considerable attention in the mathematical literature, see e.g. review [20]. Some additional recent interest is motivated by problems in physics associated with photonic or phononic crystals and photonic crystal fibers, see e.g. [18], [24]. A photonic crystal fiber (PCF) for example represents geometrically a periodic medium (whose

physical properties vary across the fiber but not along it), with the defect being its “core”, which is a propagating channel or a waveguide: electromagnetic waves of certain frequencies (the band gap frequencies) fail to propagate in the surrounding periodic medium and hence remain localised inside the PCF, which allows for them to propagate along the core for long distances with little loss [24]. Mathematically, the problem reduces to an appropriate spectral problem at the cross-section of the PCF, cf. Figure 1. This is that of characterisation of localised modes or eigenstates (whenever such exist) in the band gaps in the Floquet-Bloch spectrum for the Maxwell’s operator in the surrounding periodic medium with a fixed “propagation constant” (the wave vector along the fiber). The latter cross-sectional geometry is a periodic medium perturbed by a finite size heterogeneity (domain Ω_2 in Fig. 1). The problem is hence first in detecting the band gaps in the periodic medium without defects and then in finding, in the presence of a defect, the extra point spectrum in the gaps as well as the associated eigenfunctions, the localised states. In the present work we aim at detecting such localised modes in an asymptotically explicit form due to defects in a high contrast periodic medium using the tools of (high contrast) homogenization theory. In physical terms, this corresponds to a simplified model with scalar rather than the Maxwell equations and with in effect zero propagation constant. We expect that this nevertheless captures the essence of the underlying effects, making thereby the proposed methodology more transparent and avoiding at the same time additional technical complications. For an asymptotic analysis of a full three-dimensional Maxwell PCF (although with a moderate contrast) see [13]. In any case, we expect the problem and the methods we develop here to be of mathematical interest, in particular in part of obtaining error bounds for high-contrast homogenization problems in the presence of a boundary layer due to the defect.

Considerable literature is devoted to problems from the above described general class. Apart from numerous computational approaches (e.g., [22]), most of the mathematical treatments have been qualitative, establishing the existence of the band gaps, of the point spectrum in the gaps in the presence of defects, some bounds on the number of the eigenvalues in the gap, on the pattern of the (exponential) decay of the eigenmodes, etc, see [20] and further references therein. If however the problem contains one or more small parameters, e.g. *high contrast*, often in the presence of other small parameters, e.g. thickness of thin periodic (high contrast) structures, the asymptotic methods become potentially applicable for more explicit answers to the above questions. In mathematical literature, various results on the existence and the asymptotic description of the band gaps in high contrast periodic media have been obtained on this way, e.g. [15, 17]. In particular, methods of homogenization theory, including those of high contrast homogenization, have proven to be particularly fruitful for problems from the above general class, see e.g. [30, 31, 11, 16].

Hempel and Lienau [17] studied the spectral problem for a matrix-inclusion high contrast periodic medium and established asymptotically explicit band gaps by using min-max variational methods. Zhikov [30, 31] has independently used for this case techniques of high-contrast homogenization of “double-porosity” type. The related periodic medium has periodicity cell size ε and the contrast between the “inclusion” and the “matrix” of order ε^2 (ε is small), which is equivalent to the scaling of [17]. As a result the spectrum converges in the sense of Hausdorff to an explicitly described limit spectrum which contains gaps. Zhikov, using the techniques of two-scale convergence [21, 3], has made significant further advances having additionally described an associated (two-scale) limit operator and clarified further the convergence of the spectra in terms of the strong two-scale resolvent convergence, the associated convergence of the spectral projectors and certain additional

compactness properties. Zhikov has also shown that at the “macroscale” the spectral problem displays an emergent explicit nonlinear dependence on the spectral parameter, see (3.16), (3.20) below.

In this paper, using the (high contrast) homogenization theory methods, we show that if such a rapidly oscillating high-contrast medium is perturbed by a compact defect of size of order one, asymptotically explicit eigenvalues and eigenfunctions can emerge in the gap when $\varepsilon \rightarrow 0$. The essential spectrum is known to remain unchanged under such a perturbation [9, 2, 14], and the existence of the point spectrum in the gaps of the unperturbed operator has also been established on some accounts together with some estimates on the number of eigenvalues in the gap, see [2, 14] and further references in review [20]. We argue that the homogenization techniques allow to substantially refine this information, providing an explicit asymptotic description and tight bounds on the (convergent) eigenvalues and eigenfunctions, and ultimately on their number via some kind of “asymptotic completeness” of the spectrum in the gap as described by an explicit limit operator.

We employ in this work the method of asymptotic expansions supplemented by its rigorous justification, in the case of regular boundaries for both the periodic inclusions and the defect. This allows to obtain not only an explicit description of the limit equations and the convergence results, but also to establish *the rate* of convergence: the main technical result of this work is the error estimate (4.1), with the “ ε -square-root” bound being typical for classical homogenization with boundary-layer effects, see e.g. [19] and further references therein. The method of asymptotic expansions in the “moderate contrast” classical homogenization as well as its rigorous justification are well developed, see e.g. [8, 25, 6, 19]. Applications of asymptotic methods for high contrast homogenization can be found e.g. in [26]. On the other hand, error estimates in homogenization can be obtained by other methods, see e.g. recent review [35], including the so called spectral method, e.g. [7, 29, 4, 10, 33], as well as by modifications of the asymptotic expansions method with no assumptions on the regularity of the coefficients, e.g. [32]. One novel technical ingredient in the present work is perhaps the execution of a delicate boundary layer asymptotic analysis in the high contrast case (Section 5).

Cherdantsev [12] considered essentially the same problem by an alternative method of two-scale convergence, following the general ideas of Zhikov [30, 31]. With an additional restriction on the boundary-layer inclusions, he established not only strong two-scale resolvent convergence but also associated compactness properties via some novel technical analysis of an ε -uniform exponential decay of the eigenfunctions in the gap. As a result he obtained stronger results on the “full” Hausdorff spectral convergence, although without the error estimates. Since our approach and that of [12] are essentially independent and the results complement each other, it is natural to combine them as we also discuss in the present paper.

The structure of the paper is the following. We first formulate the problem (Section 2), then give an explicit description of the limit problem in Section 3. Section 4 establishes the convergence and error bounds for the eigenvalues and the eigenfunctions (Theorem 4.1). We give full proof of Theorem 4.1, whose most technical part (Theorem 4.2, including a number of accompanying technical lemmas and propositions) is given in the Section 5. An explicit example illustrating the existence of the point spectrum for the limit operator for spherical defects in a surrounding periodic medium with isotropic effective properties is given in Section 6. Finally we discuss the improved error estimates on the eigenfunctions by combining our results with those of Cherdantsev [12] in Section 7. Appendix A gives

formal derivation of the limit problem, as well as of the correctors required for the rigorous justification.

2 Formulation of the problem

We consider a high contrast two-phase periodic medium with a small periodicity size and with a “finite size” defect filled by a third phase. The geometric configuration is displayed on Figure 1.

Figure 1: Geometric configuration: a defect in a rapidly oscillating high contrast medium

The precise mathematical formulation is the following. Let $Q := [0, 1]^n$ be the reference periodicity cell in $\mathbb{R}^n$, $n \geq 2$, and let $Q_0 \subset Q$ be a domain (a “reference inclusion”) in Q , with infinitely smooth boundary ∂Q_0 , $\overline{Q_0} \subset Q$ (the overbar denotes the closure of the set). Denote by Q_1 the complement of Q_0 in Q , $Q_1 := Q \setminus \overline{Q_0}$. Let $\hat{Q}_0^\varepsilon$ be the corresponding contracted set, i.e. $\hat{Q}_0^\varepsilon := \{x : x/\varepsilon \in Q_0\}$, where $\varepsilon > 0$ is a small positive parameter. We denote by Q_0^ε the ε -periodic cloning of $\hat{Q}_0^\varepsilon$, i.e. $Q_0^\varepsilon := \hat{Q}_0^\varepsilon + \varepsilon\mathbb{Z}^n$. Let the “defect domain” Ω_2 be an ε -independent bounded domain with infinitely smooth boundary. We denote by $\tilde{Q}_0^\varepsilon$ the set of all the inclusions in Q_0^ε which intersect with the boundary $\partial\Omega_2$ of Ω_2 , and by $\tilde{\Omega}_0^\varepsilon$ the union of all the parts from $\tilde{Q}_0^\varepsilon$ outside Ω_2 , i.e. $\tilde{\Omega}_0^\varepsilon := \tilde{Q}_0^\varepsilon \setminus \overline{\Omega_2}$, see Figure 1.

One phase, the “inclusions phase” of the resulting composite medium, denoted Ω_0^ε , is the collection of all the small inclusions lying entirely outside the defect Ω_2 , i.e. $\Omega_0^\varepsilon := Q_0^\varepsilon \setminus (\Omega_2 \cup \tilde{Q}_0^\varepsilon)$. The “matrix phase”, denoted Ω_1^ε , is the complement to the inclusions outside the defect, i.e. $\Omega_1^\varepsilon := \mathbb{R}^n \setminus \overline{(Q_0^\varepsilon \cup \tilde{Q}_0^\varepsilon \cup \Omega_2)}$.

We assume that the matrix and the defect are filled with materials with ε -independent uniform properties a_1 and a_2 respectively, and the inclusions are filled with materials with

ε -dependent (uniform) properties: $a_0(\varepsilon)$ and $\tilde{a}_0(\varepsilon)$ for the “full” (domain Ω_0^ε) and the “cut” (domain $\tilde{\Omega}_0^\varepsilon$) inclusions, respectively. Mathematically, for every positive (small enough) ε we consider the spectral problem

$$A_\varepsilon u^\varepsilon = \lambda(\varepsilon) u^\varepsilon \quad (2.1)$$

for operator A_ε , self-adjoint in $L^2(\mathbb{R}^n)$ with a domain $D(A_\varepsilon)$ dense in $H^1(\mathbb{R}^n)$,

$$A_\varepsilon u^\varepsilon := -\nabla \cdot \left(a(x, \varepsilon) \nabla u^\varepsilon(x) \right), \quad x \in \mathbb{R}^n, \quad (2.2)$$

where

$$a(x, \varepsilon) = \begin{cases} a_0(\varepsilon), & x \in \Omega_0^\varepsilon, \\ \tilde{a}_0(\varepsilon), & x \in \tilde{\Omega}_0^\varepsilon, \\ a_1, & x \in \Omega_1^\varepsilon, \\ a_2, & x \in \Omega_2. \end{cases} \quad (2.3)$$

We assume that

$$a_0(\varepsilon) = a_0 \varepsilon^2 \quad (2.4)$$

(which is sometimes called a double porosity-type scaling), with a_0 , as well as a_1 and a_2 being arbitrary positive constants. There is a degree of freedom in this work for selecting the scaling for $\tilde{a}_0(\varepsilon)$ in the “boundary layer”, so we only require that

$$0 < \tilde{a}_0(\varepsilon) \leq \tilde{A}_0 \quad (2.5)$$

with some ε -independent positive $\tilde{A}_0$. Hence the results of the present paper remain valid, for example, both for the double porosity scaling where $\tilde{a}_0(\varepsilon) = \tilde{a}_0 \varepsilon^2$ with some ε -independent positive $\tilde{a}_0$, in particular for $\tilde{a}_0 = a_0$ (physically, the “cut” inclusions outside the defect being kept), as well for $\tilde{a}_0(\varepsilon)$ being “of order one”, e.g. $\tilde{a}_0(\varepsilon) = A_0$, in particular for $A_0 = a_1$ or $A_0 = a_2$ (the cut inclusions being replaced by the matrix or defect material). We also remark that the results presented in this paper will remain equally valid for the boundary inclusions $\tilde{Q}_0^\varepsilon$ also cutting out of the “defect” part Ω_2 (the maximal generality has not been pursued to avoid unnecessary further technical complications). Emphasize however that for subsequent stronger results on the Hausdorff convergence of the spectra and the convergence of the eigenfunctions (see [12] and Section 7 below) a more restrictive (essentially “order one”) choice for $\tilde{a}_0(\varepsilon)$ becomes necessary, to exclude additional modes which might otherwise exist near the defect’s boundary.

The equation (2.1) is understood in the usual weak sense, implying the continuity of u^ε and of the conormal derivatives at the boundaries of $\Omega_0^\varepsilon, \Omega_1^\varepsilon, \Omega_2$ and $\tilde{\Omega}_0^\varepsilon$.

Hence for every fixed positive ε (2.1)–(2.2) represents a spectral problem for an operator with periodic coefficients “perturbed” by a localised defect. We are mostly interested in the existence and asymptotics for the eigenvalues $\lambda(\varepsilon)$ and the associated localised solutions (eigenfunctions) $u^\varepsilon(x)$ of (2.1) when $\varepsilon \rightarrow 0$.

3 Homogenization and the limit problem

We describe in this section the formal asymptotic procedure for solving (2.1)–(2.2) when $\varepsilon \rightarrow 0$. It is rigorously justified in the subsequent sections.

One can seek a formal solution to the spectral problem (2.1)–(2.2) in the form of a standard two-scale ansatz:

$$u^\varepsilon(x) = u^{(0)}(x, x/\varepsilon) + \varepsilon u^{(1)}(x, x/\varepsilon) + \varepsilon^2 u^{(2)}(x, x/\varepsilon) + r^\varepsilon(x), \quad (3.1)$$

$$\lambda(\varepsilon) = \lambda_0 + o(1), \quad (3.2)$$

where $u^{(0)}(x, y)$, $u^{(1)}(x, y)$ and $u^{(2)}(x, y)$ are functions to be determined which are Q -periodic in y , the remainders $r^\varepsilon(x)$ and $o(1)$ are expected to be small when $\varepsilon \rightarrow 0$, with λ_0 and $u^{(0)}(x, y)$ subsequently having the meaning of the eigenvalues and eigenfunctions of a “limit problem”. [We subsequently show that the remainder $o(1)$ in (3.2) is in fact “of order $\varepsilon^{1/2}$ ”, i.e. $O(\varepsilon^{1/2})$, see (4.1).]

A formal substitution of (3.1)–(3.2) into (2.1) results upon a straightforward calculation in the following structure of the main-order term $u^{(0)}(x, y)$, see Appendix A:

$$u^{(0)}(x, y) = \begin{cases} u_0(x), & x \in \Omega_2, \text{ or } x \in \mathbb{R}^n \setminus \Omega_2 \text{ and } y \notin Q_0, \\ u_0(x) + v(x, y), & x \in \mathbb{R}^n \setminus \Omega_2, y \in Q_0, \end{cases} \quad (3.3)$$

which highlights the fact that $u^{(0)}$ varies only at the “slow” scale x (as is the case in the classical, i.e. not high-contrast, homogenization) everywhere outside the domain of “soft” inclusions Ω_0^ε , however may depend on the fast variable $y = x/\varepsilon$ in Ω_0^ε . Further, the pair of functions $(u_0(x), v(x, y))$ must solve the following limit coupled spectral problem:

$$-\nabla \cdot a_2 \nabla u_0(x) = \lambda_0 u_0(x), \quad x \in \Omega_2, \quad (3.4)$$

$$-\nabla \cdot A^{\text{hom}} \nabla u_0(x) = \lambda_0 (u_0 + \langle v \rangle_y), \quad x \in \mathbb{R}^n \setminus \Omega_2, \quad (3.5)$$

$$-a_0 \Delta_y v = \lambda_0 (u_0 + v), \quad y \in Q_0; \quad v = 0, \quad y \in \partial Q_0 \quad (x \in \mathbb{R}^n \setminus \Omega_2), \quad (3.6)$$

$$(u_0)_- = (u_0)_+, \quad a_2 \left(\frac{\partial u_0}{\partial n} \right)_- = \left(A_{ij}^{\text{hom}} \frac{\partial u_0}{\partial x_j} n_i \right)_+, \quad x \in \partial \Omega_2. \quad (3.7)$$

Here

$$\langle v \rangle_y(x) := \int_Q v(x, y) dy \quad (3.8)$$

denotes the averaging with respect to y over the periodicity cell Q (extending v by zero outside Q_0); Δ_y is the Laplace operator, $(\cdot)_-$ and $(\cdot)_+$ denote respectively the interior and exterior limit values of the appropriate entities at the boundary $\partial \Omega_2$ of Ω_2 , n is the interior unit normal to $\partial \Omega_2$, $\partial/\partial n := n \cdot \nabla$ is the normal derivative, summation is henceforth implied with respect to repeated indices. In (3.5) $A^{\text{hom}} = (A_{ij}^{\text{hom}})$ is the standard “soft inclusions” (or perforated domain) homogenized matrix for the above described periodic medium with $a_0 = 0$, see e.g. [19] §3.1:

$$A_{ij}^{\text{hom}} \xi_i \xi_j = \inf_{w \in C_{\text{per}}^\infty(Q)} \int_{Q \setminus Q_0} a_1 |\xi + \nabla w|^2 dy \quad (\xi \in \mathbb{R}^n), \quad (3.9)$$

where $C_{\text{per}}^\infty(Q)$ denotes the class of infinitely smooth Q -periodic functions.

Notice that, following the pattern of Zhikov [31], the above limit problem (3.4)–(3.7) can be interpreted¹ as a spectral problem for a two-scale limit non-negative self-adjoint

¹See some further discussion in Section 7 on the limit operator.

operator (which we will denote A_0) acting in the following Hilbert space $\mathcal{H}_0$:

$$\mathcal{H}_0 = \left\{ u(x, y) \in L^2(\mathbb{R}^n \times Q) \mid \begin{array}{l} u(x, y) = u_0(x) + v(x, y), \quad u_0 \in L^2(\mathbb{R}^n), \\ v \in L^2(\mathbb{R}^n \times Q); v(x, y) = 0 \text{ if } y \in Q_1 \text{ or } x \in \Omega_2 \end{array} \right\}. \quad (3.10)$$

The operator A_0 is generated by the (closed) symmetric and non-negative quadratic form B_0 defined on dense in $\mathcal{H}_0$ domain

$$\mathcal{V} = \left\{ u(x, y) \in \mathcal{H}_0 \mid \begin{array}{l} u(x, y) = u_0(x) + v(x, y), \quad u_0 \in H^1(\mathbb{R}^n), \\ v \in L^2(\mathbb{R}^n; H_{per}^1(Q)); v(x, y) = 0 \text{ if } y \in Q_1 \text{ or } x \in \Omega_2 \end{array} \right\}, \quad (3.11)$$

where $H_{per}^1(Q)$ stands for the closure in $H^1(Q)$ of $C_{per}^\infty(Q)$. The limit two-scale form B_0 is then defined as follows: for $u(x, y) = u_0(x) + v(x, y) \in \mathcal{V}$ and $w(x, y) = w_0(x) + z(x, y) \in \mathcal{V}$,

$$B_0(u, w) = \int_{\Omega_2} a_2 \nabla u_0 \cdot \nabla w_0 dx + \int_{\mathbb{R}^n \setminus \Omega_2} A^{\text{hom}} \nabla u_0 \cdot \nabla w_0 dx + \int_{\mathbb{R}^n \setminus \Omega_2} \int_{Q_0} a_0 \nabla_y v \cdot \nabla_y z dy dx. \quad (3.12)$$

The resulting (self-adjoint) operator A_0 is defined in standard way on a dense domain $D(A_0) \subset \mathcal{V}$. The limit spectral problem (3.4)–(3.7) can then be equivalently stated as follows: find $u = u_0 + v \in \mathcal{V}$, $u \neq 0$, and $\lambda_0 \geq 0$ such that

$$B_0(u, w) = \lambda_0(u, w)_{\mathcal{H}_0}, \quad \text{for all } w \in \mathcal{V},$$

where $(u, w)_{\mathcal{H}_0}$ is the inner product in $\mathcal{H}_0$ taken as the standard inner product in $L^2(\mathbb{R}^n \times Q)$.

On the other hand, the limit problem (3.4)–(3.7) can be re-arranged as follows, cf. [30, 31]. Let λ_j and $\varphi_j(y)$ be the eigenvalues and the (ortho-normalized) eigenfunctions respectively of $-a_0 \Delta_y$ in Q_0 with Dirichlet boundary conditions on ∂Q_0 . Assume $\lambda_0 \neq \lambda_j$ for all $j \geq 1$. Boundary value problem (3.6) implies that $v(x, y)$ can be uniquely presented as

$$v(x, y) = u_0(x)V(y), \quad (3.13)$$

where V is a solution of the problem

$$-a_0 \Delta_y V = \lambda_0 V + \lambda_0, \quad y \in Q_0; \quad V = 0, \quad y \in \partial Q_0. \quad (3.14)$$

It is further assumed that V is extended by zero to Q and is then periodically extended on the whole $\mathbb{R}^n$. Then, substituting (3.13) back into (3.5) we arrive at the following spectral problem for u_0 with a nonlinear dependence on the spectral parameter λ :

$$-\nabla \cdot a_2 \nabla u_0(x) = \lambda_0 u_0(x), \quad x \in \Omega_2, \quad (3.15)$$

$$-\nabla \cdot A^{\text{hom}} \nabla u_0(x) = \beta(\lambda_0) u_0(x), \quad x \in \mathbb{R}^n \setminus \Omega_2, \quad (3.16)$$

$$(u_0)_- = (u_0)_+, \quad a_2 \left(\frac{\partial u_0}{\partial n} \right)_- = \left(A_{ij}^{\text{hom}} \frac{\partial u_0}{\partial x_j} n_i \right)_+, \quad x \in \partial \Omega_2. \quad (3.17)$$

Here

$$\beta(\lambda) := \lambda + \lambda \langle V \rangle \quad (3.18)$$

is the function introduced by Zhikov [30, 31]. Applying the spectral decomposition to (3.14), cf. [30, 31],

$$V(y) = \lambda_0 \sum_{j=1}^{\infty} \frac{\langle \varphi_j \rangle_y}{\lambda_j - \lambda_0} \varphi_j(y), \quad (3.19)$$

and substituting the latter into (3.18) results in

$$\beta(\lambda) = \lambda + \lambda^2 \sum_{j: \langle \varphi_j \rangle_y \neq 0} \frac{\langle \varphi_j \rangle_y^2}{\lambda_j - \lambda}, \quad (3.20)$$

which is a sign-changing function, also derived in [30, 31], singular near the inclusions' eigenvalues λ_j , $j = 1, 2, \dots$, see Figure 2. (Strictly speaking, $\lambda_1, \lambda_2, \lambda_3, \dots$ on Fig. 2 denote only the eigenvalues corresponding to φ_j with non-zero mean, ordered by increasing values.)

Figure 2: Function $\beta(\lambda)$

The spectrum of the “unperturbed” limit operator (i.e. with no defects) consists of the union of λ where $\beta(\lambda) \geq 0$ and of λ_j , $j = 1, 2, \dots$, [31]. If $\langle \varphi_j \rangle_y = 0$ the spectrum of the limit operator also contains (infinite multiplicity) point spectrum at $\lambda = \lambda_j$. Hence the limit operator has gaps when $\beta(\lambda) < 0$, $\lambda \neq \lambda_j$, $j = 1, 2, \dots$.

In this work, for the perturbed linear operator A_0 , restricting ourselves only to values of λ in the gaps of the unperturbed limit operator, we define λ_0 to be an eigenvalue of A_0 if $\beta(\lambda_0) < 0$, $\lambda_0 \neq \lambda_j$, $j \geq 1$, and the system (3.15)–(3.20) admits a solution $u_0(x)$ decaying at infinity (hence, decaying exponentially since $\beta(\lambda_0) < 0$). We define as an eigenfunction (or an eigenvector) of A_0 the pair of functions (u_0, v) , where $u_0(x)$ is the above solution and $v(x, y)$ is related to $u_0(x)$ via (3.6) (equivalently via (3.13) and (3.14) or (3.19)) for $y \in Q_0$ and $x \in \mathbb{R}^n \setminus \Omega_2$ and extended by zero for the remaining values of x and $y \in Q$.

One can show from (3.15)–(3.17) that the perturbed limit operator A_0 inside the gaps of the unperturbed operator can only develop isolated eigenvalues of finite multiplicity, see [12] §7. As we show in Section 6, the problem (3.15)–(3.17) admits in some cases an explicit calculation of its eigenvalues and eigenfunctions.

4 Convergence and error bounds for eigenvalues and eigenmodes

The main result of the present work is the following theorem establishing the closeness of the spectrum of the original operator A_ε to the spectrum of the above described limit operator A_0 as $\varepsilon \rightarrow 0$:

Theorem 4.1. *Let λ_0 be an eigenvalue of limit operator A_0 , $\beta(\lambda_0) < 0$, $\lambda_0 \neq \lambda_j, j \geq 1$. Then there exists $\varepsilon_0 > 0$ and a constant $C_1 > 0$ independent of ε such that for any $0 < \varepsilon \leq \varepsilon_0$ there exists an isolated eigenvalue $\lambda(\varepsilon)$ of operator A_ε of finite multiplicity, such that*

$$|\lambda(\varepsilon) - \lambda_0| \leq C_1 \varepsilon^{1/2}. \quad (4.1)$$

Moreover if (u_0, v) is an eigenfunction of A_0 which corresponds to λ_0 then the function

$$u^{\text{appr}}(x, \varepsilon) := \begin{cases} u_0(x) + v(x, x/\varepsilon), & x \in \Omega_0^\varepsilon, \\ u_0(x), & x \in \Omega_1^\varepsilon \cup \Omega_2 \cup \tilde{\Omega}_0^\varepsilon, \end{cases} \quad (4.2)$$

is an approximate eigenfunction for A_ε at least in the following sense²: for any small $\delta > 0$, there exist constants $c_j(\varepsilon)$ such that

$$\left\| u^{\text{appr}} - \sum_{j \in J_\varepsilon} c_j(\varepsilon) u_j^\varepsilon \right\|_{L_2(\mathbb{R}^n)} < C_2 \varepsilon^{1/2}, \quad (4.3)$$

where $J_\varepsilon = \{j : |\lambda^{(j)}(\varepsilon) - \lambda_0| < \delta\}$ is a finite set of indices (for each ε), and $\lambda^{(j)}(\varepsilon)$, $u_j^\varepsilon(x)$ are eigenvalues and L_2 -normalized eigenfunctions of A_ε , and the constant C_2 is independent of ε .

For proving the theorem, we first establish estimates somewhat related to the “strong resolvent convergence” of operators A_ε when $\varepsilon \rightarrow 0$. Strictly speaking, those are related to some generalization of the latter: the usual resolvent convergence is not suitable for our purposes because the space where the limit operator A_0 acts differs from the space natural for operators A_ε . One therefore has to refer to the so-called *two-scale* strong resolvent convergence, see Zhikov [30, 31].

Let $u^0(x, y) = (u_0(x), v(x, y)) \in D(A_0)$ be an eigenfunction of the operator A_0 corresponding to an eigenvalue λ_0 , $\beta(\lambda_0) < 0$, $\lambda_0 \neq \lambda_j, j \geq 1$. Denote by U_ε the “transfer” operator, constructing from u^0 the approximate eigenfunction u^{appr} via (4.2), i.e.

$$(U_\varepsilon u^0)(x) := \begin{cases} u_0(x) + v(x, x/\varepsilon), & x \in \Omega_0^\varepsilon, \\ u_0(x), & x \in \Omega_1^\varepsilon \cup \Omega_2 \cup \tilde{\Omega}_0^\varepsilon. \end{cases} \quad (4.4)$$

²See [12] and the discussion in Section 7 on strengthening, under an additional restriction, of this result on convergence of the eigenfunctions.

Notice that due to the regularity of u_0 and v (which solve (3.15)–(3.17)), decomposition (3.19) and the exponential decay of u_0 at infinity when $\beta(\lambda_0) < 0$, $U_\varepsilon u^0 \in L_2(\mathbb{R}^n)$.

Denoting by I the unity operator, we formulate next the main technical statement of this work, close to that of the two-scale resolvent convergence, cf. [30, 31]:

Theorem 4.2. *Let $u^0(x, y) = (u_0(x), v(x, y)) \in D(A_0)$ be an eigenfunction of the operator A_0 :*

$$A_0 u^0 = \lambda_0 u^0, \quad \lambda_0 \neq \lambda_j, \quad j \geq 1; \quad \beta(\lambda_0) < 0. \quad (4.5)$$

Consider the following (resolvent) problem for the original operator A_ε :

$$(A_\varepsilon + I) \tilde{u}^\varepsilon = (\lambda_0 + 1) U_\varepsilon u^0. \quad (4.6)$$

Then

$$\|U_\varepsilon u^0 - \tilde{u}^\varepsilon\|_{L_2(\mathbb{R}^n)} \leq C \varepsilon^{1/2}, \quad (4.7)$$

with a constant C independent of ε .

Remark 4.1. The above estimate can be equivalently rewritten in a form which somewhat clarifies the role of operator U_ε :

$$\|U_\varepsilon (A_0 + I)^{-1} u^0 - (A_\varepsilon + I)^{-1} U_\varepsilon u^0\| \leq C \varepsilon^{1/2}, \quad (4.8)$$

where u^0 is an eigenfunction of operator A_0 , cf. [32].

The proof of Theorem 4.2 constitutes the key technical component for establishing the main result (4.1). The proof of the central error bound (4.7) requires, among other ingredients, the development of a high contrast version of the asymptotic analysis of the boundary layer near the boundary of the defect Ω_2 , conceptually somewhat similar to e.g. [19] §1.4. The complete proof of Theorem 4.2 is given in Section 5. We remark here that the need of executing a series of technical error estimates in Section 5 is caused by the fact that “globally” we can explicitly construct only a main order term in the asymptotic expansion, with the major obstacle being the need to control the effect of the boundary layer near the defect’s border $\partial\Omega_2$. Constructing or analyzing the boundary layer in homogenization is still an open problem in general, even in the classical (“moderate contrast”) case, see e.g. [23] for some developments. Nevertheless, the boundary layer’s effect can instead be somewhat controlled via an order-optimal error bound of order $\varepsilon^{1/2}$ in the H^1 norm, see e.g. [19] §1.4 for the case of boundaries with Dirichlet or Neumann conditions. In the present work we face however the *high contrast* version of the boundary layer problem and in effect show that even in this case (for the interface conditions) the boundary layer accounts for the “ ε -square root” error, but in the L_2 norm, see (4.7). The series of technical lemmas and propositions in Section 5 ensure that the approximation $U_\varepsilon u^0$ is close to the exact solution $\tilde{u}^\varepsilon$ in the sense of appropriate quadratic forms, see (5.2). Namely, the main-order contributions of the quadratic forms corresponding to $U_\varepsilon u^0$ and $\tilde{u}^\varepsilon$ coincide and the errors are controllably small, including those due to the boundary layer.

Denote by $\sigma_{\text{ess}}(A_\varepsilon)$ the essential spectrum of operator A_ε . The next step is, partly, a specialization of a more general methodology, see e.g. [19] §11.1, to the present context:

Lemma 4.3. *Let λ_0 be an eigenvalue of operator A_0 , $\beta(\lambda_0) < 0$, $\lambda_0 \neq \lambda_j$, $j \geq 1$, and let $u^0 = (u_0, v)$ be associated eigenfunction. Then:*

(i) For sufficiently small ε there exists $c > 0$ independent of ε , such that

$$(\lambda_0 - c, \lambda_0 + c) \cap \sigma_{\text{ess}}(A_\varepsilon) = \emptyset. \quad (4.9)$$

(ii) There exists, for sufficiently small ε , an isolated eigenvalue (hence of finite multiplicity) $\lambda(\varepsilon)$ of operator A_ε , such that

$$|\lambda(\varepsilon) - \lambda_0| < C_1 \varepsilon^{1/2}, \quad (4.10)$$

with constant C_1 independent of ε .

(iii) For any small $\delta > 0$, there exist constants $c_j(\varepsilon)$ such that

$$\|U_\varepsilon u^0 - \sum_{j \in J_\varepsilon} c_j(\varepsilon) u_j^\varepsilon\|_{L_2} < C_2 \varepsilon^{1/2}, \quad (4.11)$$

where $J_\varepsilon = \{j : |\lambda^{(j)}(\varepsilon) - \lambda_0| < \delta\}$; $\lambda^{(j)}(\varepsilon), u_j^\varepsilon(x)$ are eigenvalues and (L_2 -normalized) eigenfunctions of A_ε , and the constant C_2 is independent of ε .

Proof. (i) The assertion (4.9) follows from the Hausdorff convergence of the spectra of the “unperturbed” (i.e. with no defects) operators with periodic coefficients [17, 31], and the stability of the essential spectrum due to localized defects, e.g. [14].

(ii) To prove (4.10) recall that the distance from a point μ to the spectrum $\sigma(B)$ of a linear self-adjoint operator B in $L_2(\mathbb{R}^n)$ can be bounded from above as follows

$$\text{dist}(\mu, \sigma(B)) \leq \frac{\|Bu - \mu u\|_{L_2(\mathbb{R}^n)}}{\|u\|_{L_2(\mathbb{R}^n)}}, \quad (4.12)$$

with any $u \in D(B)$, $u \neq 0$. Let us take as B and μ , respectively, $B = B_\varepsilon := (A_\varepsilon + I)^{-1}$ and $\mu = (\lambda_0 + 1)^{-1}$. Then B is bounded with $D(B) = L^2(\mathbb{R}^n)$, and obviously $\lambda \in \sigma(A_\varepsilon)$ if and only if $(\lambda + 1)^{-1} \in \sigma(B_\varepsilon)$. Now select $u = U_\varepsilon u^0$. Then according to Theorem 4.2, see (4.8), the numerator in (4.12) can be estimated as follows

$$\begin{aligned} & \|(A_\varepsilon + I)^{-1} U_\varepsilon u^0 - (\lambda_0 + 1)^{-1} U_\varepsilon u^0\|_{L_2(\mathbb{R}^n)} = \\ & = \|(A_\varepsilon + I)^{-1} U_\varepsilon u^0 - U_\varepsilon (A_0 + I)^{-1} u^0\|_{L_2(\mathbb{R}^n)} \leq C \varepsilon^{1/2}, \end{aligned} \quad (4.13)$$

(where we have also used that $(1 + \lambda_0)^{-1} u^0 = (A_0 + I)^{-1} u^0$). Obviously, the denominator in (4.12) is bounded from below (e.g. $\|U_\varepsilon u^0\|_{L_2(\mathbb{R}^n)} \geq \|u_0\|_{L_2(\Omega_2)} > 0$). As a result, $\text{dist}(\mu, \sigma(B_\varepsilon)) \leq c_1 \varepsilon^{1/2}$, with some ε -independent c_1 . Using then, for example, for small enough ε obvious inequality, $\text{dist}(\lambda_0, \sigma(A_\varepsilon)) \leq L(\varepsilon) \text{dist}(\mu, \sigma(B))$ with $L(\varepsilon) = (\mu - c_1 \varepsilon^{1/2})^{-2}$ we arrive at $\text{dist}(\lambda_0, \sigma(A_\varepsilon)) \leq c \varepsilon^{1/2}$, with some ε -independent positive constant c . Finally notice that, for sufficiently small ε , the interval $(\lambda_0 - c \varepsilon^{1/2}, \lambda_0 + c \varepsilon^{1/2})$ may contain only isolated eigenvalues of A_ε by (4.9). This proves the existence of eigenvalues $\lambda(\varepsilon)$ satisfying (4.10).

(iii) The assertion (4.11) is a consequence of (4.13) and general results, see e.g. [28] or [19] §11.1, and follows by applying spectral decomposition of $U_\varepsilon u^0$ with respect to A_ε , and using (i) and (ii). Namely, if P_λ^ε are spectral projectors of A_ε , then

$$(A_\varepsilon + I)^{-1} U_\varepsilon u^0 - (\lambda_0 + 1)^{-1} U_\varepsilon u^0 = \int_0^\infty \left(\frac{1}{\lambda + 1} - \frac{1}{\lambda_0 + 1} \right) dP_\lambda^\varepsilon(U_\varepsilon u^0),$$

and (4.11) follows from (4.8) and the orthogonality properties of the spectral projectors. $\square$

Now the Theorem 4.1 directly follows from Theorem 4.2 and Lemma 4.3. $\square$

Remark 4.2. It is relatively straightforward to slightly modify the statement of Theorem 4.1 for limit eigenvalues λ_0 with “multiplicities”. Namely, let for a given λ_0 , $\beta(\lambda_0) < 0$, $\lambda_0 \neq \lambda_j$, $j = 1, 2, \dots$, there exist m , $2 \leq m < \infty$, linearly independent eigenfunctions $u_j^{(0)}(x, y)$, $j = 1, \dots, m$ of the two-scale limit operator A_0 . Then we claim that, for sufficiently small ε , there exist *not less than* m eigenvalues of A_ε (counted with their own multiplicities) such that the error bound (4.7) is valid for each of them. The above proof could be modified for this case by for example first “orthogonalizing” $u_j^{(0)}(x, y)$, $j = 1, \dots, m$ (with respect to the inner product induced by the quadratic form B_0 , see (3.12)), and then showing that the associated approximations $U_\varepsilon u_j^0$, $j = 1, \dots, m$ are “approximately” mutually orthogonal for small ε in the “original” space $L_2(\mathbb{R}^n)$. The latter would then allow to modify the above existence and error bounds argument for the case of multiplicities. We do not elaborate on this in detail to avoid further technical complications, but also since (under an additional restriction on the boundary inclusions) a further strengthening of this result is possible (via a further advance of the theory, see [12]), to the effect that there are not only “at least” but also “at most” as many eigenvalues as m near λ_0 : see [12] and the discussion in Section 7.

5 Proof of Theorem 4.2

We give in this section a full proof of the main technical Theorem 4.2, whose key error bound (4.7) establishes the closeness of the exact solution $\tilde{u}^\varepsilon$ of (4.6) to the approximate solution $U_\varepsilon u^0$ constructed via (4.4) in terms of the solution $u^0 := (u_0(x), v(x, y))$ of the limit problem (3.4)–(3.7).

The proof of the Theorem 4.2 will be divided into a number of stages. The plan is roughly as follows. The closeness of $U_\varepsilon u^0$ and $\tilde{u}$ is established employing the associated quadratic form b_ε , see (5.2) below, where $U_\varepsilon u^0$ is replaced by its modification $U_1^\varepsilon(x)$, incorporating some higher-order terms in the asymptotic expansion (3.1), see (5.1). Namely, we show that it is sufficient for our purposes to establish the closeness in the sense of (5.5), Lemma 5.1. The technical proof of Lemma 5.1 itself then follows by first splitting the quadratic form in the left hand side of (5.5) into those corresponding to $U_1^\varepsilon(x)$ and $\tilde{u}$, and then splitting the former further into a number of components (corresponding to the various domains of the integration in (5.2)) and examining those separately, see Propositions 5.2–5.5. For each component the main-order parts are explicitly evaluated and the errors are bounded. Eventually everything is assembled together and the main-order terms cancel each other as anticipated, whereas the errors are shown to be “at worst” of order $\varepsilon^{1/2}$. (The latter $\varepsilon^{1/2}$ -errors correspond in a sense to the effect of the boundary-layer near the defect’s border, and those of order ε or higher to the truncation of the asymptotic ansatz away from it.) An essential specific technical ingredient used in the course of implementing the above strategy is the employment of the so-called “extension lemma” in Proposition 5.3. Extension lemmas have been intensively used for homogenization problems in perforated domains before, see e.g. [19] and further references therein, as is briefly reviewed by us below too, see (5.22) and accompanying discussion.

To proceed, notice first that the above approximation $U_\varepsilon u^0$ as defined by (4.4) lies in appropriate functional spaces, in particular $U_\varepsilon u^0 \in H^1(\mathbb{R}^n)$. Observe to this end that u_0 is infinitely smooth in Ω_2 and $\mathbb{R}^n \setminus \Omega_2$ as a solution of elliptic equations with constant coefficients (3.15) and (3.16) respectively. Next u_0 decays exponentially at infinity, as a

decaying solution of equation with constant coefficients (3.16) outside Ω_2 , since $\beta(\lambda_0) < 0$ and hence the fundamental solution of (3.16) in the whole $\mathbb{R}^n$ is exponentially decaying. Further since, by (3.13), $v(x, x/\varepsilon) = u_0(x)V(x/\varepsilon)$, where $V(y)$ specified by (3.14) is an H^1 periodic function and its restrictions to Q_0 and Q_1 are infinitely smooth, we conclude that $v(x, x/\varepsilon)$ is an exponentially decaying function belonging to $H^1(\mathbb{R}^n \setminus \Omega_2)$.

We further aim at establishing error bounds in the energy norms, i.e. with respect to the quadratic forms (5.2) below associated with the equation (4.6). For this, we slightly alter the approximation $U_\varepsilon u^0$ by adding to it in Ω_1^ε the first-order corrector from the asymptotic expansion (3.1), hence introducing the following corrected approximation:

$$U_1^\varepsilon(x) = \begin{cases} U_\varepsilon u^0(x) = u_0(x), & x \in \Omega_2 \cup \tilde{\Omega}_0^\varepsilon, \\ U_\varepsilon u^0(x) = u_0(x) + v(x, x/\varepsilon) = u_0(x)(1 + V(x/\varepsilon)), & x \in \Omega_0^\varepsilon, \\ U_\varepsilon u^0(x) + \varepsilon N_j(x/\varepsilon)u_{0,j}(x) = u_0(x) + \varepsilon N_j(x/\varepsilon)u_{0,j}(x), & x \in \Omega_1^\varepsilon. \end{cases} \quad (5.1)$$

Here $\varepsilon N_j(x/\varepsilon)u_{0,j}(x)$ is the first order corrector, see (A.4)-(A.5).

Consider now for any $\varepsilon > 0$ the quadratic form b_ε corresponding to the operator $A_\varepsilon + I$:

$$b_\varepsilon(w, u) := \sum_{j=0}^2 \int_{\Omega_j^\varepsilon} a_j(\varepsilon) \nabla w \cdot \nabla u \, dx + \int_{\tilde{\Omega}_0^\varepsilon} \tilde{a}_0(\varepsilon) \nabla w \cdot \nabla u \, dx + \int_{\mathbb{R}^n} w u \, dx, \quad (5.2)$$

(for brevity of notation, $\Omega_2^\varepsilon := \Omega_2$, $a_j(\varepsilon) := a_j$, $j = 1, 2$). In particular, for the actual solution $\tilde{u}^\varepsilon$ of (4.6),

$$b_\varepsilon(\tilde{u}^\varepsilon, w) = (f^\varepsilon, w)_{\mathbb{R}^n}, \quad \forall w \in H^1(\mathbb{R}^n), \quad (5.3)$$

where

$$(f^\varepsilon, w)_{\mathbb{R}^n} := \int_{\mathbb{R}^n} f^\varepsilon w \, dx,$$

and

$$f^\varepsilon := (\lambda_0 + 1)U_\varepsilon u^0 = \begin{cases} (\lambda_0 + 1)u_0(x) \left(1 + V(x/\varepsilon)\right), & x \in \Omega_0^\varepsilon \\ (\lambda_0 + 1)u_0(x), & x \notin \Omega_0^\varepsilon, \end{cases} \quad (5.4)$$

via (4.6), (4.4) and (3.13).

The domain of the form (5.2) is $H^1(\mathbb{R}^n)$, however we extend it to all “piecewise H^1 ” functions w , i.e. such that $w \in H^1(\Omega_j^\varepsilon)$, $j = 0, 1, 2$, $w \in H^1(\tilde{\Omega}_0^\varepsilon)$, for which $b_\varepsilon(w, w)$, as directly defined by the right hand side of (5.2), is bounded. In particular, U_1^ε is in this “extended” domain.

The proof of Theorem 4.2 will be based on the following key technical lemma. We first state the lemma, then prove the theorem assuming it is valid, and then prove the lemma itself (which will in turn consist of several technical steps).

Lemma 5.1. *There exists an ε and w -independent $C > 0$ such that for any sufficiently small $\varepsilon > 0$ and for any $w \in H^1(\mathbb{R}^n)$*

$$|b_\varepsilon(w, U_1^\varepsilon - \tilde{u}^\varepsilon)| \leq C\varepsilon^{1/2}b_\varepsilon(w, w)^{1/2}. \quad (5.5)$$

Proof of Theorem 4.2:

Assume Lemma 5.1 is valid. For any $\varepsilon > 0$, select $w = w^\varepsilon(x) := U_2^\varepsilon(x) - \tilde{u}^\varepsilon(x)$, where U_2^ε is another “corrected” approximation constructed as follows:

$$U_2^\varepsilon(x) = \begin{cases} U_\varepsilon u^0(x) = u_0(x), & x \in \Omega_2, \\ U_\varepsilon u^0(x) + \varepsilon \chi_\varepsilon(x) N_j(x/\varepsilon) u_{0,j}(x), & x \in \Omega_1^\varepsilon \cup \Omega_0^\varepsilon \cup \tilde{\Omega}_0^\varepsilon. \end{cases} \quad (5.6)$$

Here $\varepsilon N_j(x/\varepsilon)u_{0,j}(x)$ is the first order corrector, constructed everywhere outside the defect, with (for example) a harmonic extension of $N(y)$ onto Q_0 ,

$$\chi_\varepsilon(x) := \chi(\text{dist}(x, \partial\Omega_2)\varepsilon^{-1}) \quad (5.7)$$

and $\chi(t)$ is a cut-off function: $\chi \in C^\infty(\mathbb{R})$; $\chi(t) = 0, t < 1/2$ and $\chi(t) = 1, t > 1$. Notice that both U_2^ε and $\tilde{u}^\varepsilon$ are in $H^1(\mathbb{R}^n)$. Hence by Lemma 5.1,

$$|b_\varepsilon(U_2^\varepsilon - \tilde{u}^\varepsilon, U_1^\varepsilon - \tilde{u}^\varepsilon)| \leq C\varepsilon^{1/2}b_\varepsilon(U_2^\varepsilon - \tilde{u}^\varepsilon, U_2^\varepsilon - \tilde{u}^\varepsilon)^{1/2}, \quad (5.8)$$

with C denoting henceforth constants independent of ε whose precise value is insignificant and can change from line to line.

On the other hand, by the non-negativity of the “extended” quadratic form, obviously,

$$b_\varepsilon(U_2^\varepsilon - \tilde{u}^\varepsilon, U_2^\varepsilon - \tilde{u}^\varepsilon) \leq 2|b_\varepsilon(U_2^\varepsilon - \tilde{u}^\varepsilon, U_1^\varepsilon - \tilde{u}^\varepsilon)| + b_\varepsilon(U_1^\varepsilon - U_2^\varepsilon, U_1^\varepsilon - U_2^\varepsilon). \quad (5.9)$$

Notice next that from (5.1) and (5.6)

$$U_1^\varepsilon(x) - U_2^\varepsilon(x) = \begin{cases} 0, & x \in \Omega_2, \\ -\chi_\varepsilon(x)\varepsilon N_j(x/\varepsilon)u_{0,j}(x), & x \in \Omega_0^\varepsilon \cup \tilde{\Omega}_0^\varepsilon, \\ (1 - \chi_\varepsilon(x))\varepsilon N_j(x/\varepsilon)u_{0,j}(x), & x \in \Omega_1^\varepsilon. \end{cases} \quad (5.10)$$

Then, due to the small size (of order ε near $\partial\Omega_2$) of the support of $1 - \chi(\text{dist}(x, \partial\Omega_2)\varepsilon^{-1})$ as well as of $\tilde{\Omega}_0^\varepsilon$, and to the regularity and exponential decay of $u_0(x)$, we conclude that

$$b_\varepsilon(U_1^\varepsilon - U_2^\varepsilon, U_1^\varepsilon - U_2^\varepsilon) \leq C\varepsilon. \quad (5.11)$$

Combining (5.9) with (5.8) and (5.11) implies:

$$\begin{aligned} b_\varepsilon(U_2^\varepsilon - \tilde{u}^\varepsilon, U_2^\varepsilon - \tilde{u}^\varepsilon) &\leq 2C\varepsilon^{1/2}b_\varepsilon(U_2^\varepsilon - \tilde{u}^\varepsilon, U_2^\varepsilon - \tilde{u}^\varepsilon)^{1/2} + C\varepsilon \leq \\ &\frac{1}{2}b_\varepsilon(U_2^\varepsilon - \tilde{u}^\varepsilon, U_2^\varepsilon - \tilde{u}^\varepsilon) + (2C^2 + C)\varepsilon, \end{aligned}$$

which yields, via (5.2),

$$\|U_2^\varepsilon - \tilde{u}^\varepsilon\|_{L^2(\mathbb{R}^n)}^2 \leq b_\varepsilon(U_2^\varepsilon - \tilde{u}^\varepsilon, U_2^\varepsilon - \tilde{u}^\varepsilon) \leq C\varepsilon. \quad (5.12)$$

Notice finally that from (5.6), the boundedness of N_j and χ_ε as well as boundedness and exponential decay of $u_{0,j}$,

$$\|U_\varepsilon u^0 - U_2^\varepsilon\|_{L^2(\mathbb{R}^n)}^2 \leq C\varepsilon.$$

This together with (5.12) implies via the triangle inequality that

$$\|U_\varepsilon u^0 - \tilde{u}^\varepsilon\|_{L^2(\mathbb{R}^n)} \leq C\varepsilon^{1/2},$$

with appropriate constant C . This establishes (4.7) and hence proves the theorem. $\square$

Proof of Lemma 5.1:

First, using (5.3) the entity in the left hand side of (5.5) can be evaluated as follows:

$$b_\varepsilon(w, U_1^\varepsilon - \tilde{u}^\varepsilon) = b_\varepsilon(w, U_1^\varepsilon) - (w, f^\varepsilon)_{\mathbb{R}^n} = I_1(\varepsilon) + I_2(\varepsilon), \quad (5.13)$$

where

$$I_1(\varepsilon) := b_\varepsilon(w, U_1^\varepsilon) = \sum_{j=0}^2 \int_{\Omega_j^\varepsilon} a_j(\varepsilon) \nabla w \cdot \nabla U_1^\varepsilon dx + \int_{\tilde{\Omega}_0^\varepsilon} \tilde{a}_0(\varepsilon) \nabla w \cdot \nabla U_1^\varepsilon dx + \int_{\mathbb{R}^n} w U_1^\varepsilon dx, \quad (5.14)$$

and, via (5.4),

$$I_2(\varepsilon) := -(w, f^\varepsilon)_{\mathbb{R}^n} = -(\lambda_0 + 1) \left(\int_{\Omega_0^\varepsilon} w u_0 (1 + V) dx + \int_{\tilde{\Omega}_0^\varepsilon} w u_0 dx + \int_{\Omega_2} w u_0 dx + \int_{\Omega_1^\varepsilon} w u_0 dx \right). \quad (5.15)$$

It is further convenient to break $I_1(\varepsilon)$ into four separate terms for the four integration domains:

$$I_1(\varepsilon) = \tilde{A}_0(\varepsilon) + \sum_{j=0}^2 A_j(\varepsilon), \quad (5.16)$$

where

$$\tilde{A}_0(\varepsilon) := \int_{\tilde{\Omega}_0^\varepsilon} \tilde{a}_0(\varepsilon) \nabla w \cdot \nabla U_1^\varepsilon dx + \int_{\tilde{\Omega}_0^\varepsilon} w U_1^\varepsilon dx, \quad (5.17)$$

and

$$A_0(\varepsilon) := \int_{\Omega_0^\varepsilon} \varepsilon^2 a_0 \nabla w \cdot \nabla U_1^\varepsilon dx + \int_{\Omega_0^\varepsilon} w U_1^\varepsilon dx, \quad (5.18)$$

$$A_j(\varepsilon) := \int_{\Omega_j^\varepsilon} a_j \nabla w \cdot \nabla U_1^\varepsilon dx + \int_{\Omega_j^\varepsilon} w U_1^\varepsilon dx, \quad j = 1, 2. \quad (5.19)$$

We will be separately estimating $\tilde{A}_0(\varepsilon)$, $A_j(\varepsilon)$, $j = 0, 1, 2$, and then $I_2(\varepsilon)$ in the series of the following propositions, and will subsequently derive (5.5) by combining all these estimates.

Proposition 5.2.

$$\left| \tilde{A}_0(\varepsilon) \right| \leq C \varepsilon^{1/2} b_\varepsilon(w, w)^{1/2}. \quad (5.20)$$

Proof. Notice that the measure of $\tilde{\Omega}_0^\varepsilon$ is bounded by $C\varepsilon$. As a result, using Cauchy-Schwartz inequality,

$$\left| \int_{\tilde{\Omega}_0^\varepsilon} \tilde{a}_0(\varepsilon) \nabla w \cdot \nabla U_1^\varepsilon dx \right| \leq \left(\int_{\tilde{\Omega}_0^\varepsilon} \tilde{a}_0(\varepsilon) \nabla w \cdot \nabla w dx \right)^{1/2} \left(\int_{\tilde{\Omega}_0^\varepsilon} \tilde{a}_0(\varepsilon) |\nabla U_1^\varepsilon|^2 dx \right)^{1/2} \leq C \varepsilon^{1/2} b_\varepsilon(w, w)^{1/2},$$

where we have used (2.5) and the L^∞ -boundedness of ∇U_1^ε in $\tilde{\Omega}_0^\varepsilon$ via (5.1) and the boundedness of u_0 . The second integral in (5.17) is bounded similarly, which leads to (5.20). $\square$

Consider next the integrals over Ω_1^ε in (5.19):

$$A_1(\varepsilon) = \int_{\Omega_1^\varepsilon} a_1 \nabla w \cdot \nabla U_1^\varepsilon dx + \int_{\Omega_1^\varepsilon} w U_1^\varepsilon dx. \quad (5.21)$$

Before formulating the corresponding result for $A_1(\varepsilon)$, we need to use the following technical construction. One can extend any function w from $H^1\left(\mathbb{R}^n \setminus (\Omega_0^\varepsilon \cup \tilde{Q}_0^\varepsilon)\right)$ (let us remind that $\tilde{Q}_0^\varepsilon$ is the set of all the inclusions in Q_0^ε which intersect with the boundary $\partial\Omega_2$ of Ω_2 , see Section 2) into the whole of $H^1(\mathbb{R}^n)$, controlling its norm “uniformly” with respect to ε . More precisely, for any ε and any $w \in H^1\left(\mathbb{R}^n \setminus (\Omega_0^\varepsilon \cup \tilde{Q}_0^\varepsilon)\right)$ there exists a function $\hat{w} \in H^1(\mathbb{R}^n)$ such that

$$\hat{w}(x) = w(x), \quad x \in \mathbb{R}^n \setminus (\Omega_0^\varepsilon \cup \tilde{Q}_0^\varepsilon) \quad \text{and} \quad \|\hat{w}\|_{H^1(\mathbb{R}^n)} \leq C\|w\|_{H^1(\mathbb{R}^n \setminus (\Omega_0^\varepsilon \cup \tilde{Q}_0^\varepsilon))}, \quad (5.22)$$

where C does not depend on ε and w . The above follows e.g. via a straightforward modification of the so called “extension lemma”, see e.g. [19] §3.1 Lemma 3.2 which uses the extension construction, see for the latter e.g. [27] §6.3.1, p.181, Theorem 5.

Proposition 5.3.

$$A_1(\varepsilon) = (\beta(\lambda_0) + |Q_1|) \int_{\mathbb{R}^n \setminus \Omega_2} \hat{w} u_0 dx + \int_{\partial\Omega_2} \hat{w} n_i A_{ij}^{\text{hom}} u_{0,j} dS + \hat{A}_1(\varepsilon), \quad (5.23)$$

where

$$|\hat{A}_1(\varepsilon)| \leq C\varepsilon^{1/2} b_\varepsilon(w, w)^{1/2}. \quad (5.24)$$

Proof. 1. Let μ^ε be characteristic function of Ω_1^ε . We can rewrite the first integral in (5.21) via (5.1) as follows

$$\int_{\Omega_1^\varepsilon} a_1 \nabla w \cdot \nabla U_1^\varepsilon dx = \int_{\mathbb{R}^n \setminus \Omega_2} \nabla \hat{w} \cdot p_1^\varepsilon dx, \quad (5.25)$$

where

$$(p_1^\varepsilon(x))_i := \mu^\varepsilon(x) a_1 \left(u_{0,i}(x) + N_{j,i}(y) u_{0,j}(x) + \varepsilon N_j(y) u_{0,ji}(x) \right), \quad x \in \mathbb{R}^n \setminus \Omega_2, \quad y = x/\varepsilon. \quad (5.26)$$

(Henceforth p_i denotes the i -th component of the appropriate vector field $p(x)$.) The above flow $(p_1^\varepsilon(x))_i$ can be re-written in the following form

$$(p_1^\varepsilon(x))_i = A_{ij}^{\text{hom}} u_{0,j} + g_i^j(x/\varepsilon) u_{0,j} + \varepsilon \mu^\varepsilon a_1 N_j(x/\varepsilon) u_{0,ji}(x). \quad (5.27)$$

Here

$$g_i^j(y) := \mu(y) a_1 (\delta_{ij} + N_{j,i}(y)) - A_{ij}^{\text{hom}}, \quad y \in Q, \quad (5.28)$$

μ is the characteristic function of Q_1 , and A_{ij}^{hom} are the entries of the homogenized matrix A^{hom} , see (A.9), and $N_j(y)$ are here assumed extended by zero on Q_0 . It follows then from (A.5) that vector field $g_i^j \in [L^2(Q)]^n$ (with fixed j) is divergence free in the whole of the periodicity cell Q in the following (weak) sense:

$$\int_Q \partial_i \psi g_i^j dy = 0, \quad \forall \psi \in H^1(\square) \quad (5.29)$$

($H^p(\square)$ stands for the closure in $H^p(Q)$ of all Q -periodic C^∞ functions and $\partial_i := \partial/\partial y_i$). Since (5.28) and (A.9) also imply that g_i^j have zero mean value over Q , they can be rewritten

as “divergences” of skew-symmetric field $G_{ik}^j(y) \in H^1(\square)$ (which is in fact a generalization of the curl operation to arbitrary n , see e.g. [19] §1.1 pp. 6-7):

$$g_i^j(y) = \partial_k G_{ik}^j(y), \quad G_{ik}^j(y) = -G_{ki}^j(y). \quad (5.30)$$

Consequently (5.27) can be rewritten as

$$(p_1^\varepsilon(x))_i = A_{ij}^{\text{hom}} u_{0,j}(x) + \varepsilon \frac{\partial}{\partial x_k} \left(G_{ik}^j(x/\varepsilon) u_{0,j}(x) \right) + \varepsilon \mu^\varepsilon a_1 N_j(x/\varepsilon) u_{0,ji}(x) - \quad (5.31)$$

$$\varepsilon G_{ik}^j(x/\varepsilon) u_{0,jk}(x), \quad x \in \mathbb{R}^n \setminus \Omega_2.$$

Function $\frac{\partial}{\partial x_k} (G_{ik}^j u_{0,j})$ from (5.31) is not divergence free in $\mathbb{R}^n \setminus \Omega_2$. We introduce its “divergence free modification” $\frac{\partial}{\partial x_k} (\chi_\varepsilon G_{ik}^j u_{0,j})$ which differs from $\frac{\partial}{\partial x_k} (G_{ik}^j u_{0,j})$ insignificantly by employing again the cut-off function $\chi_\varepsilon(x)$, see (5.7). As a result,

$$(p_1^\varepsilon(x))_i = A_{ij}^{\text{hom}} u_{0,j}(x) + \varepsilon \frac{\partial}{\partial x_k} \left(\chi_\varepsilon G_{ik}^j(x/\varepsilon) u_{0,j}(x) \right) + \varepsilon \mu^\varepsilon a_1 N_j(x/\varepsilon) u_{0,ji}(x) + \quad (5.32)$$

$$(r_1^\varepsilon(x))_i, \quad x \in \mathbb{R}^n \setminus \Omega_2,$$

where

$$(r_1^\varepsilon(x))_i = \varepsilon \frac{\partial}{\partial x_k} \left((1 - \chi_\varepsilon(x)) G_{ik}^j(x/\varepsilon) u_{0,j}(x) \right) - \varepsilon G_{ik}^j(x/\varepsilon) u_{0,jk}(x). \quad (5.33)$$

2. Integrating by parts in (5.25) and using (5.32), (5.30) and (5.29) we obtain

$$\int_{\Omega_1^\varepsilon} a_1 \nabla w \cdot \nabla U_1^\varepsilon dx = \int_{\mathbb{R}^n \setminus \Omega_2} \nabla \hat{w} \cdot p_1^\varepsilon dx = - \int_{\mathbb{R}^n \setminus \Omega_2} \hat{w} A_{ij}^{\text{hom}} u_{0,ij} dx \quad (5.34)$$

$$+ \int_{\partial \Omega_2} \hat{w} n_i A_{ij}^{\text{hom}} u_{0,j} dS + R_1^\varepsilon + R_2^\varepsilon,$$

with the “remainders”

$$R_1^\varepsilon := \int_{\mathbb{R}^n \setminus \Omega_2} \nabla \hat{w} \cdot r_1^\varepsilon dx, \quad (5.35)$$

$$R_2^\varepsilon := \int_{\Omega_1^\varepsilon} w_{,i} \varepsilon a_1 N_j(x/\varepsilon) u_{0,ji}(x) dx. \quad (5.36)$$

We argue that both remainders are “small”. Let us consider R_1^ε . The only term of “order one” in R_1^ε is that corresponding to $\varepsilon \frac{\partial}{\partial x_k} \left((1 - \chi^\varepsilon) G_{ik}^j \right)$, see (5.33), however the size of the support of $(1 - \chi^\varepsilon)$ is of order ε . So we can apply the Cauchy-Schwartz inequality to each term of (5.33) in (5.35) and use the fact that G_{ik}^j is ε -periodic and hence its H^1 norm over the support of $(1 - \chi^\varepsilon)$ is bounded by $C\varepsilon^{-1/2}$. As a result,

$$|R_1^\varepsilon| \leq \varepsilon^{1/2} \|\hat{w}\|_{H^1(\mathbb{R}^n)}. \quad (5.37)$$

(Having also noticed that the remaining, “order ε ”, terms in R_1^ε contribute only order ε terms into the right hand side of (5.37), upon straightforward application of the Cauchy-Schwartz inequality and the exponential decay of u_0 .) Finally, combining (5.37) with the extension bound (5.22) we conclude that

$$|R_1^\varepsilon| \leq C\varepsilon^{1/2}b_\varepsilon(w, w)^{1/2}. \quad (5.38)$$

Notice next that R_2^ε is of order ε with the exponentially decaying u_0 , which implies $|R_2^\varepsilon| \leq \varepsilon b_\varepsilon(w, w)^{1/2}$.

3. Summarising, we obtain following estimate on the combined smallness of both remainders:

$$|R_1^\varepsilon + R_2^\varepsilon| \leq C\varepsilon^{1/2}b_\varepsilon(w, w)^{1/2}. \quad (5.39)$$

As a result we can evaluate $A_1(\varepsilon)$ (see (5.21) and (5.1)) as follows

$$\begin{aligned} A_1(\varepsilon) = & - \int_{\mathbb{R}^n \setminus \Omega_2} \hat{w} A_{ij}^{\text{hom}} u_{0,ij} dx + \int_{\Omega_1^\varepsilon} w u_0 dx + \\ & + \int_{\partial\Omega_2} \hat{w} n_i A_{ij}^{\text{hom}} u_{0,j} dS + \tilde{A}_1(\varepsilon), \end{aligned} \quad (5.40)$$

where

$$\tilde{A}_1(\varepsilon) := R_1^\varepsilon + R_2^\varepsilon + \int_{\Omega_1^\varepsilon} w \varepsilon N_j u_{0,j} dx,$$

and consequently, via (5.39) and the straightforward estimate for the last integral,

$$|\tilde{A}_1(\varepsilon)| \leq C\varepsilon^{1/2}b_\varepsilon(w, w)^{1/2}. \quad (5.41)$$

4. Let us consider the integral over Ω_1^ε in (5.40):

$$\int_{\Omega_1^\varepsilon} w u_0 dx = \int_{\mathbb{R}^n \setminus \Omega_2} \mu^\varepsilon w u_0 dx, \quad (5.42)$$

regarding here $\mu^\varepsilon(x) = \mu(x/\varepsilon)$ as the characteristic function of $\mathbb{R}^n \setminus Q_0^\varepsilon$, see Section 2. In a standard way, the associated Q -periodic characteristic function $\mu(y)$ can be presented as follows:

$$\mu(y) = |Q_1| + \Delta_y M(y), \quad y \in Q, \quad (5.43)$$

where $M \in H^2(\square)$. Then (5.42) can be evaluated in the following way:

$$\begin{aligned} \int_{\mathbb{R}^n \setminus \Omega_2} \mu^\varepsilon w u_0 dx &= \int_{\mathbb{R}^n \setminus \Omega_2} \mu^\varepsilon \hat{w} u_0 dx = \int_{\mathbb{R}^n \setminus \Omega_2} (|Q_1| + \varepsilon^2 \Delta_x M(x/\varepsilon)) \hat{w} u_0 dx = \\ &= |Q_1| \int_{\mathbb{R}^n \setminus \Omega_2} \hat{w} u_0 dx - \int_{\mathbb{R}^n \setminus \Omega_2} \varepsilon^2 \nabla_x M(x/\varepsilon) \cdot \nabla_x (\hat{w} u_0) dx + \int_{\partial\Omega_2} \varepsilon^2 n_i \left(\frac{\partial}{\partial x_i} M(x/\varepsilon) \right) \hat{w} u_0 dS. \end{aligned} \quad (5.44)$$

Following the pattern of the previous estimates (i.e. again using the boundedness and the exponential decay of u_0) the second integral on the right hand side of (5.44) can be readily bounded by $C\varepsilon b_\varepsilon(w, w)^{1/2}$. For the last integral,

$$\begin{aligned} \left| \int_{\partial\Omega_2} \varepsilon^2 n_i \left(\frac{\partial}{\partial x_i} M(x/\varepsilon) \right) \hat{w} u_0 dS \right| &\leq C \int_{\partial\Omega_2} \varepsilon^2 |\nabla_x M(x/\varepsilon)| |\hat{w}| dS = \\ C\varepsilon \int_{\partial\Omega_2} |\nabla_y M(x/\varepsilon)| |\hat{w}| dS &\leq C\varepsilon \|\nabla_y M\|_{L^\infty(Q)} \|\hat{w}\|_{L^2(\partial\Omega_2)} \leq C\varepsilon b_\varepsilon(w, w)^{1/2}. \end{aligned}$$

Here we have used the classical property of the L^∞ -boundedness of $\nabla_y M$ for $M(y)$ solving the Laplace equation (5.43), the boundedness of $|\partial\Omega_2|$, continuity of the trace operator from $H^1(\Omega_2)$ into $L^2(\partial\Omega_2)$, and (5.22).

Hence the last two terms in (5.44) can be estimated as follows:

$$\left| \int_{\partial\Omega_2} \varepsilon^2 n_i \left(\frac{\partial}{\partial x_i} M(x/\varepsilon) \right) \hat{w} u_0 dS - \int_{\mathbb{R}^n \setminus \Omega_2} \varepsilon^2 \nabla_x M(x/\varepsilon) \cdot \nabla_x (\hat{w} u_0) dx \right| \leq C\varepsilon b_\varepsilon(w, w)^{1/2}. \quad (5.45)$$

5. As a result we have

$$\begin{aligned} A_1(\varepsilon) = & - \int_{\mathbb{R}^n \setminus \Omega_2} \hat{w} A_{ij}^{\text{hom}} u_{0,ij} dx + |Q_1| \int_{\mathbb{R}^n \setminus \Omega_2} \hat{w} u_0 dx + \\ & + \int_{\partial\Omega_2} \hat{w} n_i A_{ij}^{\text{hom}} u_{0,j} dS + \hat{A}_1(\varepsilon), \end{aligned} \quad (5.46)$$

with $\hat{A}_1(\varepsilon)$ satisfying (5.24). Finally, using equation (3.16) for u_0 we obtain (5.23). $\square$

Proposition 5.4.

$$A_0(\varepsilon) = \int_{\Omega_0^\varepsilon} w f^\varepsilon dx - (\beta(\lambda_0) - |Q_1| \lambda_0) \int_{\mathbb{R}^n \setminus \Omega_2} \hat{w} u_0 dx + \hat{A}_0(\varepsilon), \quad |\hat{A}_0(\varepsilon)| \leq C\varepsilon b_\varepsilon(w, w)^{1/2}. \quad (5.47)$$

Proof. Consider the flows associated with U_1^ε in (5.18), i.e. let $p_0^\varepsilon(x) := a_0 \varepsilon^2 \nabla U_1^\varepsilon(x)$, $x \in \Omega_0^\varepsilon$. Using (5.1), cf. (4.4) and (3.13),

$$(p_0^\varepsilon(x))_i = a_0 \varepsilon V_{,i}(x/\varepsilon) u_0(x) + a_0 \varepsilon^2 (r_0^\varepsilon(x, x/\varepsilon))_i, \quad x \in \Omega_0^\varepsilon, \quad (5.48)$$

where

$$(r_0^\varepsilon(x, y))_i := u_{0,i}(x)(1 + V(y)). \quad (5.49)$$

Then $A_0(\varepsilon)$, see (5.18), can be evaluated as follows:

$$A_0(\varepsilon) = a_0 \int_{\Omega_0^\varepsilon} w_{,i} \varepsilon V_{,i} u_0 dx + a_0 \int_{\Omega_0^\varepsilon} \varepsilon^2 \nabla w \cdot r_0^\varepsilon dx + \int_{\Omega_0^\varepsilon} w u_0 (1 + V) dx. \quad (5.50)$$

For the first integral in (5.50) decompose w into its extension $\hat{w}$, see (5.22), and $z^\varepsilon := w - \hat{w}$ and notice that $z^\varepsilon \in H_0^1(\Omega_0^\varepsilon)$. As a result,

$$a_0 \int_{\Omega_0^\varepsilon} w_{,i} \varepsilon V_{,i} u_0 dx = a_0 \int_{\Omega_0^\varepsilon} \hat{w}_{,i} \varepsilon V_{,i} u_0 dx + a_0 \int_{\Omega_0^\varepsilon} z_{,i}^\varepsilon \varepsilon V_{,i} u_0 dx. \quad (5.51)$$

Applying Cauchy-Schwartz inequality to the first integral in (5.51) and using (5.22) we conclude that it is small:

$$\left| a_0 \int_{\Omega_0^\varepsilon} \hat{w}_{,i} \varepsilon V_{,i} u_0 dx \right| \leq C \varepsilon \|\hat{w}\|_{H^1(\mathbb{R}^n)} \leq C \varepsilon b_\varepsilon(w, w)^{1/2} \quad (5.52)$$

(having also used the boundedness of u_0 and V and the exponential decay of u_0).

For the latter integral in (5.51), upon integration by parts and using (3.14),

$$a_0 \int_{\Omega_0^\varepsilon} z_{,i}^\varepsilon \varepsilon V_{,i} u_0 dx = \lambda_0 \int_{\Omega_0^\varepsilon} z^\varepsilon u_0 (1 + V) - a_0 \int_{\Omega_0^\varepsilon} z^\varepsilon \varepsilon V_{,i} u_{0,i} dx.$$

The last integral on the right hand side is bounded by $C \varepsilon b_\varepsilon(w, w)^{1/2}$ by splitting $z^\varepsilon = w - \hat{w}$ and using again the Cauchy-Schwartz inequality and (5.51). Splitting in turn the other integral,

$$\begin{aligned} \lambda_0 \int_{\Omega_0^\varepsilon} z^\varepsilon u_0 (1 + V) dx &= \lambda_0 \int_{\Omega_0^\varepsilon} w u_0 (1 + V) dx - \lambda_0 \int_{\Omega_0^\varepsilon} \hat{w} u_0 (1 + V) dx = \\ &= \lambda_0 \int_{\Omega_0^\varepsilon} w u_0 (1 + V) dx - \lambda_0 (|Q_0| + \langle V \rangle) \int_{\mathbb{R}^n \setminus \Omega_2} \hat{w} u_0 dx + R_0^\varepsilon(x). \end{aligned}$$

Here we have used the argument as in (5.44): denoting by $\mu_0(y)$ the characteristic function of the inclusion Q_0 , similarly to (5.43),

$$\mu_0(y)(1 + V(y)) = (|Q_0| + \langle V \rangle) + \Delta_y W(y), \quad y \in Q,$$

for some $W \in H^2(\square)$, and then proceeding with evaluation of $\int_{\Omega_0^\varepsilon} \hat{w} u_0 (1 + V) dx$ as in (5.44). As a result, $|R_0^\varepsilon(x)| \leq C \varepsilon b_\varepsilon(w, w)^{1/2}$.

Notice finally that the remainder term in (5.50) is also small:

$$\left| \int_{\Omega_0^\varepsilon} \varepsilon^2 \nabla w \cdot r_0^\varepsilon dx \right| \leq C \varepsilon b_\varepsilon(w, w)^{1/2},$$

(having again used the boundedness of u_0 and V and the exponential decay of u_0).

As a result of the above estimates, (5.50) yields

$$A_0(\varepsilon) = \lambda_0 \int_{\Omega_0^\varepsilon} w u_0 (1 + V) dx - \lambda_0 (|Q_0| + \langle V \rangle) \int_{\mathbb{R}^n \setminus \Omega_2} \hat{w} u_0 dx + \int_{\Omega_0^\varepsilon} w u_0 (1 + V) dx + \hat{A}_0(\varepsilon), \quad (5.53)$$

with $\hat{A}_0(\varepsilon) \leq \varepsilon^{1/2} b_\varepsilon(w, w)^{1/2}$.

Finally, using (5.4) and (3.18), we observe that (5.53) yields (5.47). $\square$

Consider now the integrals $A_2(\varepsilon)$ over the defect domain Ω_2 , see (5.19):

Proposition 5.5.

$$A_2(\varepsilon) = \int_{\Omega_2} w f^\varepsilon dx - \int_{\partial\Omega_2} \hat{w} n_i a_2 u_{0,i} dS + \hat{A}_2(\varepsilon), \quad (5.54)$$

where f^ε is given by (5.4), and

$$|\hat{A}_2(\varepsilon)| \leq C \varepsilon^{1/2} b_\varepsilon(w, w)^{1/2}. \quad (5.55)$$

Proof.

$$\begin{aligned} A_2(\varepsilon) &= \int_{\Omega_2} a_2 \nabla w \cdot \nabla U_1^\varepsilon dx + \int_{\Omega_2} w U_1^\varepsilon dx = \\ &\int_{\Omega_2} a_2 \nabla \hat{w} \cdot \nabla U_1^\varepsilon dx + \int_{\Omega_2} \hat{w} U_1^\varepsilon dx + \tilde{A}_2(\varepsilon), \end{aligned} \quad (5.56)$$

where

$$\tilde{A}_2(\varepsilon) = \int_{\Omega_2 \cap \tilde{Q}_0^\varepsilon} a_2 \nabla (w - \hat{w}) \cdot \nabla U_1^\varepsilon dx + \int_{\Omega_2 \cap \tilde{Q}_0^\varepsilon} (w - \hat{w}) U_1^\varepsilon dx.$$

Integrating by parts and recalling (5.1), we obtain (with the consistent choice of the normal n to $\partial\Omega_2$ being inward for Ω_2)

$$A_2(\varepsilon) = - \int_{\Omega_2} \hat{w} a_2 \Delta u_0 dx + \int_{\Omega_2} \hat{w} u_0 dx - \int_{\partial\Omega_2} \hat{w} n_i a_2 u_{0,i} dS + \tilde{A}_2(\varepsilon) = \quad (5.57)$$

$$(\lambda_0 + 1) \int_{\Omega_2} \hat{w} u_0 dx - \int_{\partial\Omega_2} \hat{w} n_i a_2 u_{0,i} dS + \tilde{A}_2(\varepsilon) = \int_{\Omega_2} \hat{w} f^\varepsilon dx - \int_{\partial\Omega_2} \hat{w} n_i a_2 u_{0,i} dS + \tilde{A}_2(\varepsilon),$$

having used (3.15) and (5.4). The above expression can be rewritten as

$$A_2(\varepsilon) = \int_{\Omega_2} w f^\varepsilon dx - \int_{\partial\Omega_2} \hat{w} n_i a_2 u_{0,i} dS + \hat{A}_2(\varepsilon),$$

where

$$\hat{A}_2(\varepsilon) := \int_{\Omega_2 \cap \tilde{Q}_0^\varepsilon} a_2 \nabla (w - \hat{w}) \cdot \nabla U_1^\varepsilon dx + \int_{\Omega_2 \cap \tilde{Q}_0^\varepsilon} (w - \hat{w}) U_1^\varepsilon dx + \int_{\Omega_2 \cap \tilde{Q}_0^\varepsilon} (\hat{w} - w) f^\varepsilon dx.$$

Arguing further as in Proposition 5.2, we obtain (5.55). $\square$

We can now complete the proof of the Lemma 5.1 with the aid of the established Propositions 5.2–5.5 as follows.

Combine (5.54) with (5.20), (5.23), (5.47) which are all substituted into (5.16), and then employ (3.7). As a result, $I_1(\varepsilon)$, see(5.14), is evaluated as follows:

$$I_1(\varepsilon) = b_\varepsilon(w, U_1^\varepsilon) = \int_{\Omega_2 \cup \Omega_0^\varepsilon} w f^\varepsilon dx + (\lambda_0 + 1) |Q_1| \int_{\mathbb{R}^n \setminus \Omega_2} \hat{w} u_0 dx + \hat{I}_1(\varepsilon), \quad (5.58)$$

where

$$|\hat{I}_1(\varepsilon)| \leq C \varepsilon^{1/2} b_\varepsilon(w, w)^{1/2}. \quad (5.59)$$

Let us now consider $I_2(\varepsilon)$, see (5.15). Noticing that the last term in the right hand side of (5.15) is a constant times (5.42), we can employ again (5.44), (5.45) which results in:

$$\int_{\Omega_1^\varepsilon} w u_0 dx = |Q_1| \int_{\mathbb{R}^n \setminus \Omega_2} \hat{w} u_0 dx + R^\varepsilon, \quad (5.60)$$

where

$$|R^\varepsilon| \leq C \varepsilon b_\varepsilon(w, w)^{1/2}. \quad (5.61)$$

Notice next that the integral over $\tilde{\Omega}_0^\varepsilon$ in (5.15) is small due to the smallness of the measure of $\tilde{\Omega}_0^\varepsilon$:

$$\left| \int_{\tilde{\Omega}_0^\varepsilon} w u_0 dx \right| \leq C \varepsilon^{1/2} b_\varepsilon(w, w)^{1/2}. \quad (5.62)$$

As a result of employing (5.60)–(5.62) in (5.15) it can be rewritten in the following form:

$$I_2(\varepsilon) = -(\lambda_0 + 1) \left(\int_{\Omega_0^\varepsilon} w u_0 (1 + V) dx + \int_{\Omega_2} w u_0 dx + |Q_1| \int_{\mathbb{R}^n \setminus \Omega_2} \hat{w} u_0 dx \right) + \hat{I}_2(\varepsilon), \quad (5.63)$$

where

$$|\hat{I}_2(\varepsilon)| \leq C \varepsilon^{1/2} b_\varepsilon(w, w)^{1/2}. \quad (5.64)$$

Adding finally (5.63) and (5.58) and using (5.4), we conclude that (5.13) can be bounded as follows:

$$|b_\varepsilon(w, U_1^\varepsilon - \tilde{u}^\varepsilon)| \leq C \varepsilon^{1/2} b_\varepsilon(w, w)^{1/2}, \quad (5.65)$$

which is identical to (5.5) and hence proves Lemma 5.1. $\square$

6 An example

A straightforward analysis of the limit problem (3.15)–(3.17) shows that one can sometimes explicitly calculate isolated eigenvalues of operator A_0 and associated eigenfunctions, at least in the case when $\Omega_2 = \{x : |x| < R\}$ (i.e. the defect is a ball of some radius $R > 0$) and $A^{\text{hom}} = a^{\text{hom}} I$, i.e. the perforated homogenized matrix is isotropic. The latter isotropy is the case e.g. when the microscopic periodic inclusions Q_0 have appropriate symmetries, in particular being themselves balls in which case $\beta(\lambda)$ is in fact explicitly found in terms of Bessel functions and is in particular a simple trigonometric function for $n = 3$, see [5]. We sketch the details below.

Under the above assumptions a solution to the spectral problem (3.15)–(3.17) is sought by separation of variables in the spherical coordinates $x = (r, \omega)$, $r := |x|$, $\omega := x/|x| \in S^{n-1}$, the unit sphere in $\mathbb{R}^n$ ($n \geq 2$):

$$u_0(x) = \begin{cases} r^{-(n-2)/2} J_m((\lambda/a_2)^{1/2} r) P_m(\omega), & |x| \leq R, \\ \alpha r^{-(n-2)/2} I_m(|\beta(\lambda)/a^{\text{hom}}|^{1/2} r) P_m(\omega), & |x| \geq R, \end{cases} \quad (6.1)$$

assuming $\lambda > 0$, $\beta(\lambda) < 0$. Parameters m , λ and α are to be found. Here $J_m(z)$ and $I_m(z)$ are the Bessel and the modified Bessel functions respectively, see e.g. [1], $P_m(\omega)$ are the spherical functions which are the eigenfunctions of the Laplace-Beltrami operator Δ_ω on S^{n-1} :

$$(\Delta_\omega + m^2 - (n-2)^2/4) P_m(\omega) = 0. \quad (6.2)$$

The spherical spectral problem (6.2) is a classical one and defines explicit eigenvalues m and eigenfunctions P_m (e.g. for $n = 3$ the spectral parameter m is half-integer and P_m , in spherical coordinates (θ, φ) , are products of the Legendre polynomials of $\cos \theta$ and trigonometric functions of φ ; for $n = 2$ those are trigonometric functions and m is integer). Having selected m and associated $P_m(\omega)$ the function $u_0(x)$ determined by (6.1) automatically satisfies the equations (3.15) and (3.16), and exponentially decays at infinity ($r \rightarrow \infty$) with the rate $\exp(-|\beta(\lambda)/a^{\text{hom}}|^{1/2} r)$, due to the exponential decay of $I_m(z)$. The remaining

parameters α and λ (the eigenvalue) are determined from (3.17) at $r = R$, which specializes to:

$$J_m \left((\lambda/a_2)^{1/2} R \right) = \alpha I_m \left(|\beta(\lambda)/a^{\text{hom}}|^{1/2} R \right) \quad (6.3)$$

$$\begin{aligned} & (\lambda a_2)^{1/2} J'_m \left((\lambda/a_2)^{1/2} R \right) - a_2 \frac{n-2}{2R} J_m \left((\lambda/a_2)^{1/2} R \right) = \\ & = \alpha \left| \beta(\lambda) a^{\text{hom}} \right|^{1/2} I'_m \left(|\beta(\lambda)/a^{\text{hom}}|^{1/2} R \right) - \alpha a^{\text{hom}} \frac{n-2}{2R} I_m \left(|\beta(\lambda)/a^{\text{hom}}|^{1/2} R \right), \end{aligned} \quad (6.4)$$

where J'_m and I'_m denote derivatives of the relevant (modified) Bessel functions.

All λ with $\beta(\lambda) < 0$ for which there exists a solution to (6.2), (6.3)–(6.4) describe the point spectrum of the operator A_0 in the “gaps”. One can see that it is generally non-empty. Say, for $n = 3$ and $m = 1/2$, $P_{1/2}(\omega) \equiv 1$ is an eigenfunction of (6.2) (which determines the spherically symmetric solutions via (6.1)), and $J_{1/2}$ and $I_{1/2}$ are represented by explicit trigonometric and exponential functions respectively, e.g. [1]. This allows replacing (up to insignificant multiplicative constants) the radial parts in the right hand sides of (6.1) by $r^{-1} \sin((\lambda/a_2)^{1/2} r)$ and $r^{-1} \exp(-|\beta(\lambda)/a^{\text{hom}}|^{1/2} r)$, transforming (6.3)–(6.4) into:

$$\sin \left((\lambda/a_2)^{1/2} R \right) = \alpha \exp \left(-|\beta(\lambda)/a^{\text{hom}}|^{1/2} R \right) \quad (6.5)$$

$$\begin{aligned} & (\lambda a_2)^{1/2} \cos \left((\lambda/a_2)^{1/2} R \right) - \frac{a_2}{R} \sin \left((\lambda/a_2)^{1/2} R \right) = \\ & -\alpha a^{\text{hom}} \left[|\beta(\lambda)/a^{\text{hom}}|^{1/2} + \frac{1}{R} \right] \exp \left(-|\beta(\lambda)/a^{\text{hom}}|^{1/2} R \right). \end{aligned} \quad (6.6)$$

The condition of solvability of (6.5)–(6.6) obviously reads:

$$\cotan \left((\lambda/a_2)^{1/2} R \right) + \frac{a^{\text{hom}} - a_2}{(\lambda a_2)^{1/2} R} = - \left(\frac{a^{\text{hom}} |\beta(\lambda)|}{\lambda a_2} \right)^{1/2}, \quad \beta(\lambda) < 0. \quad (6.7)$$

Noticing that the left hand side of (6.7) is a function of $\lambda^{1/2} R$ one can easily see that by varying $R > 0$ one can obtain infinitely many solutions of (6.7) at any λ in any gap ($\beta(\lambda) < 0$) of the unperturbed linear operator.

Remark finally that, when $Q_0 = \{y \in Q : |y - y_0| \leq \rho\}$ i.e. is a ball of some radius $\rho < 1/2$, $\beta(\lambda)$ is itself explicitly found in terms of a simple radially symmetric solution of (3.14) and then (3.18), see [5] p.419 Remark 1. In particular, for $n = 3$,

$$\beta(\lambda) = \left(1 - \frac{4}{3} \pi \rho^3 \right) \lambda + 4\pi \rho \left(1 - \rho \lambda^{1/2} \cotan \left(\lambda^{1/2} \rho \right) \right). \quad (6.8)$$

7 Discussion: further refinement of the results.

In this section we describe a further refinement of the results formulated in Theorem 4.1 using the results of Cherdantsev [12], who studied a similar problem applying an alternative technique of the two-scale convergence [21, 3, 30, 31] and who followed in turn a general strategy of Zhikov [30, 31]. Remark that, apart from the following important strengthening of the results (subject to an additional restriction on the boundary inclusions as below), the approach of [12] potentially allows less regular boundaries for both the periodic inclusions Q_0 and for the defect Ω_2 . It is also, in principle, extendable for (periodically) variable

$a_j(y)$, $j = 0, 1$, as well as variable $a_2(x)$ in the defect with some minimal assumptions on the regularity. Its disadvantage however is in the intrinsic inability of the method of two-scale convergence to provide error bounds, i.e. the rate of convergence as we do in Theorem 4.1, see (4.1) and (4.3). It may therefore be advantageous to combine the results based on the two methods, as discussed below.

For the results of [12] to be applicable to our setting, the only but essential additional assumption which has to be made is restricting further our assumption (2.5) on the boundary inclusions. Namely, again not pursuing here the maximal generality, following formula (2.5) of [12] we now require

$$\hat{A}_0 \varepsilon^{2-\theta} \leq \tilde{a}_0(\varepsilon) \leq \tilde{A}_0, \quad (7.1)$$

where $\hat{A}_0$ and $\tilde{A}_0$ are positive constants and $0 < \theta \leq 2$.

The restriction (7.1) plays an important role, which may be interpreted as that of excluding additional modes which may in principle be localized near the boundary inclusions for small ε and which cannot be accounted for by the limit two-scale problem. The latter possibility was essentially excluded by the analysis of Cherdantsev in [12], which has not only established a version of the strong two-scale resolvent convergence but also established a key property of two-scale spectral compactness (Theorem 6.1 of [12]). Namely, if u^ε is a sequence of normalized eigenfunctions of the original operator A_ε , see (2.2), with associated eigenvalues $\lambda(\varepsilon) \rightarrow \lambda_0$, then up to a subsequence u^ε strongly two-scale converges to a (non-zero) $u^{(0)}(x, y)$ which is an eigenfunction of the limit two-scale operator A_0 for which λ_0 is the eigenvalue. (We emphasize that for establishing this key result Cherdantsev followed again the general strategy of Zhikov, which however required for him to develop a key novel technical ingredient of a uniform exponential decay of the eigenfunctions of A_ε , see Theorem 3.1 of [12].)

The above result of [12], when applied to our setting, immediately implies the following. Given λ_0 an isolated eigenvalue of A_0 of a finite multiplicity m with $\beta(\lambda_0) < 0$ and $\lambda_0 \neq \lambda_j$, $j \geq 0$, for small enough ε there will be exactly m eigenvalues of A_ε (counted with their multiplicities) near λ_0 . (Notice that this is formally valid for $m = 0$ as well: if λ_0 is *not* an eigenvalue then, for small ε , there are no eigenvalues of A_ε near λ_0 .) This in turn implies that in (4.3), for small enough ε , $|J_\varepsilon| = m$, which can be interpreted as an $\varepsilon^{1/2}$ -error bound not only on the eigenvalues but also on the eigenfunctions. We formulate this refined result as the following theorem.

Theorem 7.1. *Let all the assumptions of Theorem 4.1 be satisfied, and let additionally the assumption (2.5) be strengthened to (7.1), and let λ_0 be an isolated eigenvalue of the limit operator A_0 of a finite multiplicity $m \geq 0$. Then there exists $\varepsilon_0 > 0$, and constants $\delta > 0$ and $C_1 > 0$ independent of ε such that for any $0 < \varepsilon \leq \varepsilon_0$ there exist, within the δ -neighbourhood of λ_0 ($\{\lambda : |\lambda - \lambda_0| < \delta\}$), exactly m eigenvalues $\lambda^{(j)}(\varepsilon)$ of operator A_ε counted with their multiplicities, such that*

$$|\lambda^{(j)}(\varepsilon) - \lambda_0| \leq C_1 \varepsilon^{1/2}, \quad j = 1, \dots, m. \quad (7.2)$$

Moreover if $u^0(x, y) = (u_0(x), v(x, y))$ is an eigenfunction of A_0 which corresponds to λ_0 then there exist constants c_j , $j = 1, \dots, m$, such that

$$\left\| u^0(x, x/\varepsilon) - \sum_{j=1}^m c_j u_j^\varepsilon \right\|_{L_2(\mathbb{R}^n)} \leq C_2 \varepsilon^{1/2}, \quad (7.3)$$

where $u_j^\varepsilon(x)$ are L_2 -normalized eigenfunctions associated with $\lambda^{(j)}(\varepsilon)$, and the constant C_2 is independent of ε .

In particular, if $m = 1$ i.e. λ_0 is a simple isolated eigenvalue of A_0 , then

$$\|u^\varepsilon(x) - u^0(x, x/\varepsilon)\|_{L^2(\mathbb{R}^n)} \leq C \varepsilon^{1/2}, \quad |\lambda(\varepsilon) - \lambda_0| \leq C_1 \varepsilon^{1/2}, \quad (7.4)$$

i.e. there holds an $\varepsilon^{1/2}$ -order L^2 -error bound between the exact and the approximate eigenfunctions, as well as the eigenvalues.

Finally, let us remark that we expect the conditions on the regularity of the boundaries of both the periodic inclusion Q_0 and the defect Ω_2 could also be relaxed. Indeed, while assuming the C^∞ -smoothness of the above domain, the presented argument has only used up to C^2 -regularity. Furthermore, in principle, the argument could be refined to relax the regularity restrictions further, cf. e.g. [32], [19] §8.3.

A Formal derivation of the limit problem (3.4)–(3.7).

This appendix formally derives the limit equations (3.4)–(3.7) via two-scale asymptotic expansions in the form (3.1). It also establishes the structure of a corrector term in (3.1) as required for subsequent rigorous justification.

The asymptotic solution to (3.4)–(3.7) is sought in the form of two-scale ansatz (3.1)–(3.2), where $u^{(0)}(x, y)$, $u^{(1)}(x, y)$ and $u^{(2)}(x, y)$ are assumed to depend periodically on the “fast” variable y only outside the defect Ω_2 . The exact solution $u^\varepsilon(x)$ satisfies the standard continuity conditions at the boundary $\partial\Omega_0^\varepsilon$ of the small inclusions away from the defect:

$$(u^\varepsilon(x))_0 = (u^\varepsilon(x))_1, \quad x \in \partial\Omega_0^\varepsilon, \quad (A.1)$$

and

$$\left(a_0 \varepsilon^2 \frac{\partial u^\varepsilon}{\partial n}(x)\right)_0 = \left(a_1 \frac{\partial u^\varepsilon}{\partial n}(x)\right)_1, \quad x \in \partial\Omega_0^\varepsilon, \quad (A.2)$$

where the subscripts “0” and “1” denote the limit values at the boundary evaluated in Ω_0^ε and Ω_1^ε , respectively; n stands for unit normal to $\partial\Omega_0^\varepsilon$ which we select as outward for the matrix phase Ω_1^ε (and hence inward for the inclusions Ω_0^ε). Similar boundary conditions are also satisfied at the boundaries of the defect $\partial\Omega_2$ and of the “boundary layer” inclusions $\tilde{\Omega}_0^\varepsilon$.

The ansatz (3.1)–(3.2) is first substituted into the equation (2.1)–(2.2) and the interface conditions (A.1)–(A.2), away from the defect.

By equating first the terms of order ε^{-2} in (2.1) and of order ε^{-1} in (A.2),

$$-\nabla_y \cdot a_1 \nabla_y u^{(0)}(x, y) = 0, \quad y \in Q_1; \quad a_1 \frac{\partial}{\partial n_y} u^{(0)}(x, y) = 0, \quad y \in \partial Q_0,$$

with n being the outward unit normal for Q_1 , and $\partial/\partial n_y := n \cdot \nabla_y$ is the normal derivative “in y ”.

This is a homogeneous Neumann problem in Q_1 with periodic boundary conditions in y , whose solution is an arbitrary constant in Q_1 (i.e. is independent of y) which implies (3.3). The balance of the terms of order ε^0 in (A.1) implies (cf. (3.6))

$$v(x, y) = 0, \quad y \in \partial Q_0. \quad (A.3)$$

Equating next the terms of order ε^{-1} in (2.1) and of order ε^0 in (A.2), we arrive at:

$$-\nabla_y \cdot a_1 \nabla_y u^{(1)}(x, y) = 0, \quad y \in Q_1; \quad a_1 \frac{\partial}{\partial n_y} u^{(1)}(x, y) = -a_1 \frac{\partial}{\partial n_x} u_0(x), \quad y \in \partial Q_0,$$

$\partial/\partial n_x := n \cdot \nabla_x$, together with the periodicity conditions in y . This is a standard corrector problem for “soft” inclusions (or perforated) periodic domains. As a result,

$$u^{(1)}(x, y) = N_j(y) \frac{\partial u_0(x)}{\partial x_j}, \quad (\text{A.4})$$

where N_j is the solution of the “soft” unit cell problem with periodic boundary conditions (e.g. [19] §3.1):

$$a_1 \Delta N_j = 0, \quad y \in Q_1; \quad a_1 \frac{\partial}{\partial n_y} N_j = -a_1 n_j, \quad y \in \partial Q_0. \quad (\text{A.5})$$

It is convenient for some subsequent analysis to view the functions $N_j(y)$ as defined in the whole of the periodicity cell Q : with this aim they are extended into the inclusion domain Q_0 by, for example, harmonic continuation: $\Delta N_j = 0$, in Q_0 , $N_j \in C(Q)$. The above procedure determines N_j up to a constant which can be specified by the condition that the average of N_j over Q is zero: $\langle N_j(y) \rangle_y = 0$.

Equate now the terms of order ε^0 in (2.1). As a result, in Q_0 (outside Ω_2),

$$-a_0 \Delta_y v(x, y) = \lambda_0 (u_0(x) + v(x, y)), \quad (\text{A.6})$$

which together with (A.3) fully recovers (3.6).

In turn, in Q_1 , taking into account (3.3) and (A.4), the balance of order ε^0 terms in (2.1) yields:

$$-a_1 \Delta_y u^{(2)}(x, y) = 2a_1 N_{j,k}(y) u_{0,jk}(x) + a_1 \Delta_x u_0(x) + \lambda_0 u_0(x), \quad y \in Q_1. \quad (\text{A.7})$$

(Henceforth comma in subscript denotes differentiation with respect to variables with following indices.)

This equation has to be supplemented by boundary conditions which result from equating in (A.2) the terms of order ε^1 , yielding

$$a_1 \frac{\partial}{\partial n_y} u^{(2)} = -a_1 N_j \frac{\partial}{\partial n_x} u_{0,j} + a_0 \frac{\partial}{\partial n_y} v, \quad y \in \partial Q_0. \quad (\text{A.8})$$

Treating (A.7)–(A.8) as boundary value problem for $u^{(2)}$ in y for any fixed x , the Green’s formula together with the periodicity boundary conditions in y imply:

$$a_1 \int_{Q_1} \Delta_y u^{(2)} = a_1 \int_{\partial Q_0} \frac{\partial}{\partial n_y} u^{(2)},$$

which yields:

$$\begin{aligned} & (-a_1 \Delta_x u_0(x) - \lambda_0 u_0(x)) |Q_1| - 2a_1 u_{0,jk}(x) \int_{Q_1} N_{j,k}(y) dy = \\ & \int_{\partial Q_0} \left(-a_1 N_j(y) u_{0,jk}(x) n_k(y) + a_0 \frac{\partial}{\partial n_y} v(x, y) \right) dy. \end{aligned}$$

Applying the integration by parts to the right hand side surface integrals and using (A.6) we arrive at (3.5) where

$$A_{ij}^{\text{hom}} := \left\langle \mu(y) a_1 \left(\delta_{ij} + N_{j,i}(y) \right) \right\rangle_y, \quad (\text{A.9})$$

and $\mu(y)$ is the characteristic function of Q_1 ($\mu(y) = 1$, $y \in Q_1$; $\mu(y) = 0$, $y \in Q_0$). This is a well-known representation of the entries of the homogenized matrix A^{hom} in a perforated domain (with “holes” in soft inclusions Q_0), equivalent to (3.9), see e.g. [19] §3.1. Hence the limit equation (3.5) is recovered.

To complete the formal derivation of the limit problem (3.4)–(3.7) the natural equation (3.4) is simply postulated within the (homogeneous) defect Ω_2 . The limit interface conditions (3.7) at the boundary of the defect are also postulated: they have the meaning of the continuity of the fields and the flows “to main order” and the proof of the Theorem 4.2 ensures a posteriori that those produce a controllably small boundary layer, ultimately ensuring the main result (4.1) of the paper.

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