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APPROXIMATE SOLUTIONS TO A MODEL OF TWO-COMPONENT REACTIVE FLOW

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Abstract. We consider a model of motion of binary mixture, based on the compressible Navier-Stokes system. The mass balances of chemically reacting species are described by the reaction-diffusion equations with generalized form of multicomponent diffusion flux. Under a special relation between the two density dependent viscosity coefficients and for singular cold pressure we construct the weak solutions passing through several levels of approximation.

1. **Introduction.** The present article concerns with the proof of existence of weak solutions to a model of motion of two-component reactive mixture. To describe such a model we use the Navier-Stokes system supplemented by two reaction-diffusion equations for the species, which express the conservation of mass, the balance of momentum and the species masses conservation, respectively:

$$\left.\begin{array}{r}\partial_t\varrho+\operatorname{div}(\varrho\mathbf{u})=0\\ \partial_t(\varrho\mathbf{u})+\operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u})-\operatorname{div}(2\varrho\mathbf{D}(\mathbf{u}))+\nabla p=\mathbf{0}\\ \partial_t\varrho_A+\operatorname{div}(\varrho_A\mathbf{u})+\operatorname{div}(\mathcal{F}_A)=\varrho\omega\\ \partial_t\varrho_B+\operatorname{div}(\varrho_B\mathbf{u})+\operatorname{div}(\mathcal{F}_B)=-\varrho\omega\end{array}\right\}\text{in}\quad(0,T)\times\Omega. \tag{1}$$

In the above system $\varrho=\varrho(t,x)$ denotes the total mass density being the sum of the species densities $\varrho=\varrho_A+\varrho_B$ and $\mathbf{u}=\mathbf{u}(t,x)$ is the velocity vector field, $\mathbf{D}(\mathbf{u})$ stands for the symmetric part of the velocity gradient, $\mathbf{D}(\mathbf{u})=\frac{1}{2}\left(\nabla\mathbf{u}+\nabla^T\mathbf{u}\right)$, p is the internal pressure, ω is the species A production rate, $\mathcal{F}_k$ denotes the diffusion flux of the k-th species, $k\in S=\{A,B\}$; for simplicity we take $\Omega=\mathbb{T}^3$. We consider

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only the three dimensional, physically reasonable case. We prescribe the following initial data:

$$\begin{aligned} \varrho(0,x) = \varrho^0(x), \quad \varrho^0(x) \geq 0, \quad \varrho\mathbf{u}(0,x) = (\varrho\mathbf{u})^0(x), \\ \varrho_k(0,x) = \varrho_k^0(x), \quad k \in S, \quad \varrho^0(x) = \varrho_A^0(x) + \varrho_B^0(x) \quad \text{for all } x \in \Omega. \end{aligned} \tag{2}$$

We assume that the pressure $p = p(\varrho, \varrho_A, \varrho_B)$ obeys the following state equation

$$p(\varrho, \varrho_A, \varrho_B) = p_E(\varrho) + p_M(\varrho, \varrho_A, \varrho_B), \tag{3}$$

where p_E is the modification of the standard barotropic pressure ϱ^γ in the region of small densities; it is a continuous function such that

$$p_E'(\varrho) \sim \begin{cases} c\varrho^{-4k-1} & \text{for } \varrho \leq 1, \quad k > 1, \\ \varrho^{\gamma-1} & \text{for } \varrho > 1, \quad \gamma > 1. \end{cases} \tag{4}$$

By p_M we denote the classical molecular pressure for isothermal process given, in accordance with the Boyle law, by the constitutive equation

$$p_M = \sum_{k\in S} p_k = \sum_{k \in S} \frac{\varrho_k}{m_k}, \tag{5}$$

where m_k is the molar mass of k-th species (we take the perfect gas constant equal to 1) and we assume that $m_A \neq m_B$.

The species mass fluxes $\mathcal{F}_A, \mathcal{F}_B$ are given in a general form

$$\mathcal{F}_k = -\sum_{l \in S} C_{kl}\mathbf{d}_l, \quad k \in S, \tag{6}$$

where C_{kl}, $k, l \in S$ are the multicomponent flux diffusion coefficients, $\mathbf{d}_k$ is the diffusion force for k-th species which depends on the gradients of the partial pressures in the following way

$$\mathbf{d}_k = \nabla\left(\frac{p_k}{p_M}\right) + \left(\frac{p_k}{p_M} - \frac{\varrho_k}{\varrho}\right)\nabla \log p_M.$$

Supposing the following form of the matrix C (see Giovangigli [12], Chapter 7)

$$C = \frac{C_0(\varrho, \varrho_A, \varrho_B)}{\varrho}\begin{pmatrix} \varrho_B & -\varrho_A \\ -\varrho_B & \varrho_A \end{pmatrix}, \tag{7}$$

we verify, by use of (6), that

$$\begin{aligned} \mathcal{F}_A &= -\frac{C_0}{p_M}\left(\left(\frac{\varrho_B}{\varrho m_A} + \frac{\varrho_A}{\varrho m_B}\right)\nabla\varrho_A - \frac{\varrho_A}{\varrho m_B}\nabla\varrho\right), \\ \mathcal{F}_B &= -\frac{C_0}{p_M}\left(\left(\frac{\varrho_B}{\varrho m_A} + \frac{\varrho_A}{\varrho m_B}\right)\nabla\varrho_B - \frac{\varrho_B}{\varrho m_A}\nabla\varrho\right). \end{aligned}$$

In what follows we assume that the diffusion coefficient C_0 is equal to the Boyle pressure, thus $\frac{C_0}{p_M} = 1$.

An important consequence of (7) is that $\mathcal{F}_B + \mathcal{F}_A = 0$, therefore system (1) is a priori linearly dependent. For this reason, while solving both species equations separately, we have to verify that the obtained solution is compatible with the constraint $\varrho = \varrho_A + \varrho_B$.

The molar production rate $\omega = \omega(\varrho_A)$ is a Lipschitz continuous function. We additionally assume existence of constants $\underline{\omega}$ and $\overline{\omega}$ such that

$$-\underline{\omega} \leq \omega(\varrho_A) \leq \overline{\omega}, \qquad \text{for all } 0 \leq \varrho_A \leq \varrho, \tag{8}$$

and we suppose

$$\omega(\varrho_A) \geq 0 \quad \text{whenever} \quad \varrho_A = 0. \tag{9}$$

In majority of studies devoted to systems modeling the multicomponent reactive flows, the diffusion fluxes are described by the Fick law [10, 6, 7, 20, 13]. This approximation does not take into account the cross-effects that are well-known to play an important role in many phenomena. Furthermore, such an assumption leads to inconsistency with the second law of thermodynamics, when the pressure depends on the chemical composition of the mixture. In that case, the sign of the entropy production may fail to be nonnegative, which contradicts physical admissibility of the process. It is also a serious obstacle in the proof of fundamental a-priori estimates. In fact, we are aware of only one result concerning the global in time existence of solutions to system (1) equipped with the relevant constitutive relations for heat conducting mixtures [12]. This was, however, established only for the initial data sufficiently close to an equilibrium state. A relevant result on the local in time well posedness of the Maxwell-Stefan multicomponent diffusion system in the isobaric, isothermal case is presented in [1].

Our aim is to construct a suitable approximate system in order to complement the considerations from [21], where the issue of weak sequential stability of solutions was addressed. An important feature of the system studied there was the form of the viscosity coefficients $\mu = \mu(\varrho)$ and $\nu = \nu(\varrho)$ in the momentum equation

$$\partial_t(\varrho\mathbf{u}) + \operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) - \operatorname{div}(2\mu\mathbf{D}(\mathbf{u})) - \nabla(\nu\operatorname{div}\mathbf{u}) + \nabla p = 0.$$

They were assumed to satisfy the relation proposed for the first time in [2]

$$\nu(\varrho) = 2\varrho\mu'(\varrho) - 2\mu(\varrho). \tag{10}$$

In this article we restrict ourselves to the particular case when $\mu(\varrho) = \varrho$ and $\nu(\varrho) = 0$. This condition is necessary to obtain better integrability of density what compensates the lack of an information about the velocity. Indeed, in comparison to the Navier-Stokes system with the constant viscosity coefficients[15, 8], the vanishing viscosity coefficients lead to a problem with defining $\mathbf{u}$ itself, as it cannot be controlled independently of ϱ any more. This obstacle was solved on the level of weak sequential stability of solutions by Mellet and Vasseur in [17], where the Bresch-Desjardins inequality was coupled with an additional estimate for the norm of $\varrho|\mathbf{u}|^2$ in $L^\infty(0,T;L\log L(\Omega))$. However, construction of an approximate system which preserves both: the Bresch-Desjardins structure and better integrability of $\varrho|\mathbf{u}|^2$ seems to be still an open problem. The modification of the pressure in the regions of small densities (4), as it was suggested in [4], is one of possible ways to overcome this difficulty. Nevertheless, the existence of solutions for such a problem has never been carefully checked even for the Navier-Stokes system, some hints are given in [3].

Here, application of the concept of Bresch-Desjardins has an essential advantage. Namely, it enables to regain a part of the regularity of the degenerated parabolic system of an arbitrary large number of species reaction-diffusion equations

$$\partial_t\varrho_k + \operatorname{div}(\varrho_k\mathbf{u}) + \operatorname{div}(\mathcal{F}_k) = \varrho\omega_k, \quad k = 1,...n, \tag{11}$$

which sum up to the hyperbolic continuity equation. It was observed in [18] that provided the additional regularity of the density is available, system (11) with the multicomponent diffusion (6) admits global in time weak solution.

The goal of the present note is to prove existence to the approximation of system (1), more precisely to system (12), as stated in Theorem 1 below. It is the first step in the scheme of proving existence of weak solutions to the original problem. Our approach requires to use the pressure of the type (4), which guarantees that vacuum states are not admissible. Thanks to that we are able to control the density from below and hence also the velocity. From the mathematical viewpoint our result achieves two goals:

- construction of the approximative solutions; this part is very important by itself and it actually completes the results of paper [21];
- presentation of the procedure/technique of proving the existence of weak solutions to the systems describing complex flows; it may be applicable in many others problems which are not only connected to (1), as e.g. the general class of models with degenerate parabolic equations.

The main result of this paper reads as follows

Theorem 1.1. *Let $\varepsilon, \eta, \delta$ be fixed positive parameters. Assume that the initial data $\varrho^0, (\varrho\mathbf{u})^0, \varrho_A^0, \varrho_B^0$ satisfy (2) together with the following bounds*

$$\int_\Omega \left(\frac{1}{2} \frac{\left|(\varrho\mathbf{u})^0\right|^2}{\varrho^0} + \varrho^0 \pi(\varrho^0) \right) \,\mathrm{d}x < \infty, \qquad \int_\Omega \frac{\left|\nabla \varrho^0\right|^2}{\varrho^0} \,\mathrm{d}x < \infty,$$

where $\pi'(y) = p_E(y)/y^2$. Then there exist functions $\varrho, \mathbf{u}, \varrho_A, \varrho_B$, solutions to

$$\begin{aligned}
&\partial_t \varrho + \operatorname{div}(\varrho\mathbf{u}) - \varepsilon\Delta\varrho = 0,\\
&\partial_t(\varrho\mathbf{u}) + \operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) - \operatorname{div}(2\varrho\mathbf{D}(\mathbf{u}))\\
&\qquad\qquad + \nabla p + \eta\Delta^2\mathbf{u} - \delta\varrho\nabla\Delta^{2s+1}\varrho + \varepsilon(\nabla\varrho\cdot\nabla)\mathbf{u} = \mathbf{0},\\
&\partial_t\varrho_A - \varepsilon\Delta\varrho_A + \operatorname{div}(\varrho_A\mathbf{u})\\
&\qquad\qquad - \operatorname{div}\left(\left(\frac{\varrho_B}{\varrho m_A} + \frac{\varrho_A}{\varrho m_B}\right)\nabla\varrho_A - \left(\frac{\varrho_A}{\varrho m_B}\right)\nabla\varrho\right) = \varrho\omega,\\
&\partial_t\varrho_B - \varepsilon\Delta\varrho_B + \operatorname{div}(\varrho_B\mathbf{u})\\
&\qquad\qquad - \operatorname{div}\left(\left(\frac{\varrho_A}{\varrho m_B} + \frac{\varrho_B}{\varrho m_A}\right)\nabla\varrho_B - \left(\frac{\varrho_B}{\varrho m_A}\right)\nabla\varrho\right) = -\varrho\omega,
\end{aligned} \tag{12}$$

where the first equation holds a.e. on $(0,T)\times\Omega$ together with the initial condition $\varrho(0,x) = \varrho^0(x)$, $x\in\Omega$ and the remaining ones are satisfied in the sense of distributions on $(0,T)\times\Omega$ with the initial conditions satisfied in the sense of distributions on Ω. Moreover, we have

$$\begin{gathered}
\varrho \in L^2(0,T;W^{2s+2,2}(\Omega)) \cap L^\infty(0,T;W^{2s+1,2}(\Omega)), \quad \partial_t\nabla\varrho \in L^2((0,T)\times\Omega),\\
\|\varrho^{-1}\|_{L^\infty((0,T)\times\Omega)} \le c(\delta),\\
\mathbf{u} \in L^2(0,T;W^{2,2}(\Omega)) \cap L^\infty(0,T;L^2(\Omega)),\\
\varrho_A, \varrho_B \in L^\infty(0,T;L^2(\Omega)) \quad \varrho_A, \varrho_B \in L^2(0,T;W^{1,2}(\Omega)).
\end{gathered}$$

In addition,

$$0 \le \varrho_A, \varrho_B \le \varrho, \quad \text{and} \quad \varrho_A + \varrho_B = \varrho \quad \text{a.e. on } (0,T)\times\Omega.$$

Let us now outline the strategy of the proof and thus the structure of the paper. At the beginning of Section 2 we introduce the n-dimensional Faedo-Galerkin approximation for the momentum equation, truncations of coefficients in the equations of species and additional five parameters $\kappa_1, \kappa_2, \varepsilon, \eta$ and δ which indicate the level of the approximation. The parameters κ_1, κ_2 are responsible for smoothing coefficients of the species mass balance equations and the molecular pressure, ε is the rate of dissipation in the continuity equation, η regularizes the velocity field, while by δ we insert to the momentum equation the artificial smoothing operator

$\delta\varrho\nabla\Delta^{2s+1}\varrho$ with s sufficiently large, inspired by the capillarity forces [5]. The main result achieved in this section is the existence of solutions to the approximation of (12) for all the para meters being fixed and positive. It is formulated in Theorem 2.1 and it provides the starting point for Section 3, where the passages to the limit $\kappa_1, \kappa_2 \to 0$ and $n \to \infty$ are performed which completes the proof of Theorem 1.1.

It should be emphasized that establishing Theorem 1.1 is really the corner stone of the proof of the existence of solutions to the original problem (1). Indeed, at this level of approximation, it is relatively easy to derive the Bresch-Desjardins inequality which results in the following estimate

$$\begin{aligned}
&\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}+\nabla\phi(\varrho)|^2 + \frac{\delta}{2}|\nabla\Delta^s\varrho|^2 + \varrho\pi(\varrho)\right)(T)\ \mathrm{d}x \\
&\quad + \int_0^T\int_\Omega \frac{p_E'(\varrho)}{\varrho}|\nabla\varrho|^2\ \mathrm{d}x\ \mathrm{d}t + \frac{1}{2}\int_\Omega \varrho|\nabla\mathbf{u}-\nabla^T\mathbf{u}|^2\ \mathrm{d}x \\
&\quad + 2\delta\int_0^T\int_\Omega |\Delta^{s+1}\varrho|^2\ \mathrm{d}x\ \mathrm{d}t + \delta\varepsilon\int_\Omega |\Delta^{s+1}\varrho|^2\ \mathrm{d}x + \eta\int_\Omega |\Delta\mathbf{u}|^2\ \mathrm{d}x \le c
\end{aligned} \tag{13}$$

with $\pi(\varrho) = \int_0^\varrho y^{-2}p_E(y)\ \mathrm{d}y \ge 0$, $\nabla\phi(\varrho) = 2\frac{\nabla\varrho}{\varrho}$ and a constant c which depends only on the initial data. The proof of (13) is presented in Section 4. This estimate can be used to perform the passage to the limit with remaining approximation parameters $\varepsilon, \eta, \delta$ exactly as in [21].

2. **First level of approximation – construction of solution.** For the constant parameters $\varepsilon, \eta, \kappa_1, \kappa_2, \delta > 0$ (we skip all the indexes when no confusion can arise) we will be looking for a set of four functions $(\varrho, \mathbf{u}, \varrho_A, \varrho_B)$ satisfying the following regularization of the original system.

1. Approximate continuity equation:

$$\partial_t\varrho + \operatorname{div}(\varrho\mathbf{u}) - \varepsilon\Delta\varrho = 0, \tag{14}$$

with the initial condition

$$\varrho(0,x) = \varrho_\delta^0(x), \tag{15}$$

where

$$\varrho_\delta^0 \in C^{2+\nu}(\Omega), \quad \inf_{x\in\Omega}\varrho_\delta^0(x) > 0. \tag{16}$$

2. The Faedo-Galerkin approximation for the weak formulation of the momentum balance:

$$\begin{aligned}
&\int_\Omega \varrho\mathbf{u}(T)\boldsymbol{\phi}\ \mathrm{d}x + \eta\int_0^T\int_\Omega \Delta\mathbf{u}\cdot\Delta\boldsymbol{\phi}\ \mathrm{d}x\ \mathrm{d}t - \int_0^T\int_\Omega (\varrho\mathbf{u}\otimes\mathbf{u}) : \nabla\boldsymbol{\phi}\ \mathrm{d}x\ \mathrm{d}t \\
&\qquad + \int_0^T\int_\Omega 2\varrho\mathbf{D}(\mathbf{u}) : \nabla\boldsymbol{\phi}\ \mathrm{d}x\ \mathrm{d}t - \int_0^T\int_\Omega p_{\kappa_2}(\varrho,\varrho_A,\varrho_B)\operatorname{div}\boldsymbol{\phi}\ \mathrm{d}x\ \mathrm{d}t \\
&\qquad - \delta\int_0^T\int_\Omega \varrho\boldsymbol{\phi}\cdot\nabla\Delta^{2s+1}\varrho\ \mathrm{d}x\ \mathrm{d}t + \varepsilon\int_0^T\int_\Omega (\nabla\varrho\cdot\nabla)\mathbf{u}\cdot\boldsymbol{\phi}\ \mathrm{d}x\ \mathrm{d}t \\
&\quad = \int_\Omega (\varrho\mathbf{u})^0\boldsymbol{\phi}\ \mathrm{d}x
\end{aligned} \tag{17}$$

satisfied for any test function $\boldsymbol{\phi} \in X_n$, where $X_n = \operatorname{span}\{\boldsymbol{\phi}_i\}_{i=1}^n$ where $\{\boldsymbol{\phi}_i\}_{i=1}^\infty$ is an orthonormal basis in $L^2(\Omega)$, such that $\boldsymbol{\phi}_i \in W^{2,2}(\Omega)$ for all $i \in \mathbb{N}$.

The regularized internal pressure is equal to

$$p(\varrho, \varrho_A, \varrho_B) = p_E(\varrho) + \left(\frac{\varrho_A}{\sqrt{\varrho} m_A} + \frac{\varrho_B}{\sqrt{\varrho} m_B}\right)_{\kappa_2} \sqrt{\varrho}.$$

3. The species mass balance equations with truncated and regularized coefficients:

$$\begin{aligned}
&\partial_t \varrho_A - \varepsilon \Delta \varrho_A + \operatorname{div}(\varrho_A \mathbf{u}) \\
&\qquad - \operatorname{div}\left(\left(\frac{\varrho_B^+}{\varrho m_A} + \frac{\varrho_A^+}{\varrho m_B}\right)_{\kappa_1} \nabla \varrho_A - \left(\frac{\varrho_A^+}{\varrho m_B}\right)_{\kappa_1} \nabla \varrho\right) = \varrho \omega_{\kappa_1}, \\
&\partial_t \varrho_B - \varepsilon \Delta \varrho_B + \operatorname{div}(\varrho_B \mathbf{u}) \\
&\qquad - \operatorname{div}\left(\left(\frac{\varrho_A^+}{\varrho m_B} + \frac{\varrho_B^+}{\varrho m_A}\right)_{\kappa_1} \nabla \varrho_B - \left(\frac{\varrho_B^+}{\varrho m_A}\right)_{\kappa_1} \nabla \varrho\right) = -\varrho \omega_{\kappa_1},
\end{aligned} \tag{18}$$

where we set

$$\varrho_i^+ = \begin{cases} 0 & \text{if} \quad \varrho_i < 0, \\ \varrho_i & \text{if} \quad 0 \le \varrho_i < \varrho, \\ \varrho & \text{if} \quad \varrho \le \varrho_i, \end{cases} \qquad \text{for } i \in S. \tag{19}$$

The initial conditions are

$$\begin{aligned}
&\varrho_A(0,x) = \varrho^0_{A,\delta}(x), \quad \varrho_B(0,x) = \varrho^0_{B,\delta}(x), \\
&\varrho^0_{A,\delta}, \varrho^0_{B,\delta} \in C^{2+\nu}(\Omega), \quad \varrho^0_{A,\delta} + \varrho^0_{B,\delta} = \varrho^0_\delta.
\end{aligned} \tag{20}$$

Moreover, the constraint $\varrho_A(t,x)+\varrho_B(t,x) = \varrho(t,x)$ is satisfied for $(t,x) \in [0,T]\times\Omega$.

The operators $f \to f_{\kappa_i}$, $\kappa_i = (\kappa^i_t, \kappa^i_x)$, $i = 1,2$ are the standard smoothing operators that apply to the variables x and t in the case of functions $\varrho, \varrho_A, \varrho_B$. However, the regularization over time in (18) means that instead of $\varrho, \varrho_A, \varrho_B$ we consider their continuous extensions respectively in the class $V_{\mathbb{R}}$ that will be specified later on. We also assume that the supports of these extensions are contained in the time-space cylinder $(-2T, 2T) \times \Omega$.

Theorem 2.1. *Let $\varepsilon, \kappa_1, \kappa_2, \eta, \delta$ be fixed positive parameters. Approximate problem (14-20) admits a strong solution $\{\varrho, \mathbf{u}, \varrho_A, \varrho_B\}$ belonging to the regularity class*

$$\varrho \in C([0,T]; C^{2+\nu}(\Omega)), \quad \partial_t \varrho, \in C([0,T]; C^{0,\nu}(\Omega)), \quad \inf_{[0,T]\times\Omega} \varrho > 0,$$

$$\mathbf{u} \in C^1([0,T], X_n),$$

$$\varrho_i \in L^\infty(0,T; W^{1,2}(\Omega)), \quad \partial_t \varrho_i, \Delta \varrho_i \in L^2((0,T)\times\Omega), \quad i \in \{A,B\},$$

$$\varrho_A + \varrho_B = \varrho.$$

Proof. The strategy of the proof is following:

1. We linearize system (18).
2. We set $\mathbf{u} \in C([0,T]; X_n)$ for which we find the mappings

   $$\mathbf{u} \mapsto \varrho(\mathbf{u}) \quad \text{and} \quad \mathbf{u} \mapsto (\varrho_A(\mathbf{u}), \varrho_B(\mathbf{u}))$$

   determining the unique solution to the continuity equation and the species mass balance equations.
3. For sufficiently small time interval $[0, \tau^0]$ we find the unique solution to the momentum equation applying the Banach fixed point theorem. Then we extend the existence result for the maximal time interval.
4. We recover the semi-linear system (18) using a version of the Leray-Schauder fixed point theorem.

□

The proof will be given in the following subsections.

2.1. **Continuity equation.** Here we present the argument for existence of smooth, unique solution to problem (14-16) in the situation when the vector field $\mathbf{u}(x,t)$ is given and belongs to $C([0,T];X_n)$.
The following result can be proven by the Galerkin approximation and the well known statements about the regularity of linear parabolic systems (for the details of the proof see [9], Lemma 3.1).

Lemma 2.2. *Let $\mathbf{u} \in C([0,T];X_n)$ for n fixed and let $\varrho^0_\delta \in C^{2+\nu}(\Omega)$, $\nu \in (0,1)$ be such that*

$$0 < \underline{\varrho^0} \leq \varrho^0 \leq \overline{\varrho^0} < \infty.$$

Then there exists the unique classical solution to (14-16), i.e. $\varrho \in V_{[0,T]}$, where

$$V_{[0,T]} = \left\{ \begin{array}{cc} \varrho & \in C\left([0,T];C^{2+\nu}(\Omega)\right), \\ \partial_t \varrho & \in C\left([0,T];C^{0,\nu}(\Omega)\right). \end{array} \right\} \tag{21}$$

Moreover, the mapping $\mathbf{u} \mapsto \varrho(\mathbf{u})$ maps bounded sets in $C([0,T];X_n)$ into bounded sets in $V_{[0,T]}$ and is continuous with values in $C\big([0,T];C^{2+\nu'}(\Omega)\big)$, $0 < \nu' < \nu < 1$. Finally,

$$\underline{\varrho^0} e^{-\int_0^\tau \|\operatorname{div}\mathbf{u}\|_\infty \mathrm{d}t} \leq \varrho(\tau,x) \leq \overline{\varrho^0} e^{\int_0^\tau \|\operatorname{div}\mathbf{u}\|_\infty \mathrm{d}t} \quad \text{for all } \tau \in [0,T],\ x \in \Omega. \tag{22}$$

2.2. **Linearized species mass balance equations.** In this subsection we shall prove the existence of solutions to the linearization of system (18). For $\tilde{\varrho}_A, \tilde{\varrho}_B \in L^\infty\left(0,T;W^{1,2}(\Omega)\right)$ fixed, $\mathbf{u}$ and $\varrho(\mathbf{u})$ satisfying the assumptions and assertion of Lemma 2.2, we investigate the following system of linear parabolic equations with smooth coefficients

$$\begin{aligned} &\partial_t \varrho_A - \varepsilon \Delta \varrho_A + \operatorname{div}(\varrho_A \mathbf{u}) \\ &\qquad - \operatorname{div}\left(\left(\frac{\widetilde{\varrho_B^+}}{\varrho m_A} + \frac{\widetilde{\varrho_A^+}}{\varrho m_B} \right)_{\kappa_1} \nabla \varrho_A - \left(\frac{\widetilde{\varrho_A^+}}{\varrho m_B} \right)_{\kappa_1} \nabla \varrho \right) = \varrho \left(\omega\left(\widetilde{\varrho_A}\right)\right)_{\kappa_1}, \\ &\partial_t \varrho_B - \varepsilon \Delta \varrho_B + \operatorname{div}(\varrho_B \mathbf{u}) \\ &\qquad - \operatorname{div}\left(\left(\frac{\widetilde{\varrho_A^+}}{\varrho m_B} + \frac{\widetilde{\varrho_B^+}}{\varrho m_A} \right)_{\kappa_1} \nabla \varrho_B - \left(\frac{\widetilde{\varrho_B^+}}{\varrho m_A} \right)_{\kappa_1} \nabla \varrho \right) = -\varrho \left(\omega\left(\widetilde{\varrho_A}\right)\right)_{\kappa_1}. \end{aligned} \tag{23}$$

The existence of unique solution to system (23) with the initial conditions (20) is stated in the following lemma.

Lemma 2.3. *Let $\kappa_1 > 0$ and assumptions of Lemma 2.2 be satisfied. Suppose that $\varrho^0_{A,\delta}, \varrho^0_{B,\delta} \in C^{2+\nu}(\Omega)$, then problem (23) with the initial data (20) possesses the unique strong solution (ϱ_A, ϱ_B) belonging to the regularity class $\left(V_{[0,T]}\right)^2$.*

Moreover, the mapping $\mathbf{u} \mapsto (\varrho_A(\mathbf{u}), \varrho_B(\mathbf{u}))$ maps bounded sets in $C([0,T];X_n)$ into bounded sets in $\left(V_{[0,T]}\right)^2$ and is continuous with values in $\Big(C([0,T];C^{2+\nu'}(\Omega))\Big)^2$.

In addition

$$\varrho_A + \varrho_B = \varrho. \tag{24}$$

Proof. Existence of unique classical solutions can be shown using the classical result about solvability of the linear parabolic Cauchy problem with variable coefficients:

$$\mathcal{L}(t,x,\frac{\partial}{\partial t},\frac{\partial}{\partial x})u = f(t,x) \quad \text{in } (0,T)\times\mathbb{R}^3,$$
$$u(0,\cdot) = u^0 \qquad \text{in } \mathbb{R}^3,$$

where

$$\mathcal{L}(t,x,\frac{\partial}{\partial t},\frac{\partial}{\partial x})u := \partial_t u - \sum_{i,j=1}^n a_{i,j}(t,x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n a_i(t,x)\frac{\partial u}{\partial x_i} + a(t,x)u.$$

A relevant existence theory for such systems, not only within the framework of continuously differentiable functions but also for the Sobolev spaces can be found in the book of Ladyženskaja, Solonnikov and Uralceva [14]. Here, however, it is more convenient to apply the result from the analytic semigroup theory taken over from the book of Lunardi [16], which requires merely continuity of coefficients with respect to time.

Theorem 2.4 (Theorem 5.1.9 in [16])**.** *Let all the coefficients of operator $\mathcal{L}$ and f be uniformly continuous functions belonging to $C^{0,\nu}([0,T]\times\mathbb{R}^3)$, with $0<\nu<1$, and let $u^0 \in C^{2+\nu}(\mathbb{R}^3)$. Then the above problem has a unique solution from the class $u \in C^{1,2+\nu}([0,T]\times\mathbb{R}^3)$ which satisfies the inequality*

$$\|u\|_{C^{1,2+\nu}([0,T]\times\mathbb{R}^3)} \le c\left(\|f\|_{C^{0,\nu}([0,T]\times\mathbb{R}^3)} + \|u^0\|_{C^{2+\nu}(\mathbb{R}^3)}\right). \tag{25}$$

Note, in particular, that the assertion of Lemma 2.2 guaranties uniform continuity in the time interval [0,T] of the "worst" term proportional to $\Delta\varrho$ which plays the role of force in the system (23). Thus the existence of regular, unique solution belonging to the class $\left(V_{[0,T]}\right)^2$ is straightforward.
The continuity of the mapping $\mathbf{u} \mapsto (\varrho_A(\mathbf{u}), \varrho_B(\mathbf{u}))$ follows from uniqueness of solution in the class $\left(V_{[0,T]}\right)^2$, compact embeddings in the spaces of Hölder continuous functions and the Arzelà-Ascoli theorem.

The proof of (24) follows by subtracting both equations of (18) from the approximate continuity equation. We obtain

$$\begin{aligned}\partial_t\xi - \varepsilon\Delta\xi + \operatorname{div}(\xi\mathbf{u}) - \operatorname{div}\left(\left(\frac{\widetilde{\varrho_B^+}}{\varrho m_A} + \frac{\widetilde{\varrho_A^+}}{\varrho m_B}\right)_{\kappa_1}\xi\right) &= 0,\\ \xi(0,x) &= 0,\end{aligned} \tag{26}$$

where we denoted $\xi = \varrho - \varrho_A - \varrho_B$. The unique solution of the resulting system must be, due to the initial condition, equal to 0 for (t,x) in $[0,T]\times\Omega$.
By this remark, the proof of Lemma 2.3 is complete. □

2.3. **Momentum equation.** Now we prove that there exists $T = T(n)$ and $\mathbf{u} \in C([0,T];X_n)$ satisfying (17). To this purpose we apply the fixed point argument to the mapping

$$\mathcal{T} : C([0,T];X_n) \to C([0,T];X_n),$$
$$\mathcal{T}[\mathbf{u}](t) = \mathcal{M}_{\varrho(t)}\left[P_n(\varrho\mathbf{u})^0 + \int_0^t P_n\mathcal{N}(\mathbf{u})(s)\mathrm{d}s\right], \tag{27}$$

where P_n is the orthogonal projection of $L^2(\Omega)$ onto X_n,

$$\begin{aligned}\mathcal{N}(\mathbf{u}) = &-\operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) + \operatorname{div}(2\varrho\mathbf{D}(\mathbf{u})) + \nabla p_{\kappa_2}(\varrho,\varrho_A,\varrho_B)\\ &-\delta\varrho\nabla\Delta^{2s+1}\varrho + \eta\Delta^2\mathbf{u} + \varepsilon(\nabla\varrho\cdot\nabla)\mathbf{u}\end{aligned}$$

and

$$\mathcal{M}_{\varrho}\left[\cdot\right]: X_n \to X_n, \int_{\Omega} \varrho \mathcal{M}_{\varrho}\left[\mathbf{w}\right] \phi \,\mathrm{d}x = < \mathbf{w}, \phi >, \quad \mathbf{w}, \phi \in X_n.$$

First, observe that $P_n \mathcal{N}(\mathbf{u})(t)$ is bounded in X_n for $t \in [0, T]$. Using the equivalence of norms on the finite dimensional space X_n we can easily check that

$$\begin{aligned} \|P_n \mathcal{N}(\mathbf{u})\|_{X_n} \leq & c \left[\|\mathbf{u}\|_{X_n} + \|\varrho\|_{L^\infty(\Omega)} \left(\|\mathbf{u}\|^2_{X_n} + \|\mathbf{u}\|_{X_n}\right) \right. \\ & \left. + \|\varrho\|^{\gamma}_{L^\infty(\Omega)} + \|\varrho\|_{L^\infty(\Omega)} + \|\varrho\|_{L^\infty(\Omega)} \|\varrho\|_{W^{4s+3,\infty}(\Omega)}\right]. \end{aligned} \tag{28}$$

To justify that the last term on the r.h.s. is bounded, one needs to know that the unique solution ϱ to the approximate continuity equation (14) is more regular than it was indicated in Lemma 2.2. More precisely, using the fact that $\mathbf{u}$ is actually smooth with respect to space, we can put the term $\mathrm{div}(\varrho \mathbf{u})$ to the r.h.s. of (14) and then bootstrap the procedure leading to regularity (21), see e.g. [14], Chapter IV. By this argument, the term $P_n \varrho \nabla \Delta^{2s+1} \varrho$ in the approximate momentum equation makes sense, i.e. it is bounded in $L^1(0, T; X_n)$.

Concerning the operator $\mathcal{M}_{\varrho}$, it is easy to see that provided $\varrho(t, x) \geq \underline{\varrho} > 0$, one has

$$\|\mathcal{M}_{\varrho}\|_{\mathcal{L}(X_n, X_n)} \leq \underline{\varrho}^{-1}.$$

Moreover, since $\mathcal{M}_{\varrho} - \mathcal{M}_{\varrho'} = \mathcal{M}_{\varrho'} \left(\mathcal{M}^{-1}_{\varrho'} - \mathcal{M}^{-1}_{\varrho}\right) \mathcal{M}_{\varrho}$ we verify that

$$\|\mathcal{M}_{\varrho(t)} - \mathcal{M}_{\varrho'(t)}\|_{\mathcal{L}(xX_n, X_n)} \leq c \underline{\varrho}^{-2} \|(\varrho - \varrho')(t)\|_{L^1(\Omega)}$$

for $t \in [0, T]$. Thus, by virtue of continuity of mappings $\mathbf{u} \to \varrho(\mathbf{u})$, $\mathbf{u} \to \varrho_i(\mathbf{u})$, $i = A, B$ and the estimates established in Lemmas 2.2 and 2.3 one can verify that $\mathcal{T}[\mathbf{u}]$ maps the ball

$$B_{R, \tau^0} = \left\{ \mathbf{u} \in C([0, \tau^0], X_n) : \|\mathbf{u}\|_{C([0,\tau^0], X_n)} \leq R, \mathbf{u}(0, x) = P_n \left(\frac{(\varrho \mathbf{u})^0}{\varrho^0_\delta} \right) \right\}$$

into itself and it is a contraction, for sufficiently small $\tau^0 > 0$. It therefore possesses the unique fixed point satisfying (17) on the time interval $[0, \tau^0]$. In view of previous remarks, the proof of this step can be done by a minor modification of the procedure described in [19], Section 7.7, so we skip this part.

Additionally, the time regularity of $\mathbf{u}$ may be improved by differentiating (27) with respect to time and estimating the norm of the resulting r.h.s. in X_n, so we get

$$\mathbf{u} \in C^1([0, \tau^0], X_n).$$

This is the crucial information that enables to extend this solution to the maximal time interval $[0, T]$. Indeed, provided the system enjoys the estimates independent of τ^0, we can iterate the local construction of solution described above to get the solution for any $T > 0$. The existence of such a bound is based on the energy estimate and a bound from below for the density (22). Both of them can be again derived analogously to [19], so for the sake of consistency, we recall here only the idea of the proof.

We first differentiate (17) with respect to t, then we observe that it is possible to use $\mathbf{u}$ as a test function. We obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega\left(\frac{1}{2}\varrho|\mathbf{u}|^2+\frac{\delta}{2}|\nabla\Delta^s\varrho|^2+\varrho\pi(\varrho)\right)\,\mathrm{d}x+\int_\Omega 2\varrho|\mathbf{D}(\mathbf{u})|^2\,\mathrm{d}x\\+\eta\int_\Omega|\Delta\mathbf{u}|^2\,\mathrm{d}x+\delta\varepsilon\int_\Omega|\Delta^{s+1}\varrho|^2\,\mathrm{d}x\\\le\int_\Omega\left(\frac{\varrho_A}{\sqrt{\varrho}m_A}+\frac{\varrho_B}{\sqrt{\varrho}m_B}\right)_{\kappa_2}\sqrt{\varrho}\,\mathrm{div}\,\mathbf{u}\,\mathrm{d}x,\end{aligned}\tag{29}$$

where $\pi'(y)=p_E(y)/y^2$.

Applying the Cauchy inequality (with ϵ) we see that the r.h.s. may be bounded as follows

$$\begin{aligned}\left|\int_\Omega\left(\frac{\varrho_A}{\sqrt{\varrho}m_A}+\frac{\varrho_B}{\sqrt{\varrho}m_B}\right)_{\kappa_2}\sqrt{\varrho}\,\mathrm{div}\,\mathbf{u}\,\mathrm{d}x\right|&\le\epsilon\int_\Omega\varrho|\,\mathrm{div}\,\mathbf{u}|^2\,\mathrm{d}x+c(\epsilon,\kappa_2)\\&\le3\epsilon\int_\Omega\varrho|\mathbf{D}(\mathbf{u})|^2\,\mathrm{d}x+c(\epsilon,\kappa_2),\end{aligned}\tag{30}$$

where the last inequality in (30) follows by the following observation

$$(\mathrm{div}\,\mathbf{u})^2=\sum_{i,j=1}^3\partial_iu_i\partial_ju_j\le\sum_{i,j=1}^3\frac{1}{2}\left((\partial_iu_i)^2+(\partial_ju_j)^2\right)\le3|\mathbf{D}(\mathbf{u})|^2.$$

Hence, for ϵ sufficiently small, the r.h.s. of (29) can be absorbed by the l.h.s. and we get several, uniform in time estimates, in particular

$$\sqrt{\varrho}\mathbf{u}\in L^\infty(0,T;L^2(\Omega)),\quad\sqrt{\eta}\Delta\mathbf{u}\in L^2(0,T;L^2(\Omega)).\tag{31}$$

From these bounds, using the estimate for $\varrho\pi(\varrho)$ and the Korn-Poincaré inequality we deduce boundedness of the $L^2(0,T;W^{2,2}(\Omega))$ norm of $\mathbf{u}$. Next, by the equivalence of norms of $\mathbf{u}$ we actually have that $\mathbf{u}\in L^2(0,T;X_n)$ also $\mathbf{u}\in L^2(0,T;W^{1,\infty}(\Omega))$. Therefore the bounds from below and above for ϱ can be derived exactly as in the proof of estimate (22) from Lemma 2.2. This in turn allows us to explore the uniform estimate on $\varrho|\mathbf{u}|^2$ following from (29) to show the boundedness of $\mathbf{u}$ in $C([0,T];L^2(\Omega))$. Having this, we can again take advantage of equivalence of norms, to deduce that we have uniformly in time

$$\|\mathbf{u}\|_{C([0,T];X_n)}\le c.$$

At this point we can return to the procedure of construction of local in time solution and repeat it until we reach an approximate solution defined on $[0,T]$ for arbitrary large, but finite $T>0$, exactly as in [19], Section 7.7.

2.4. **Nonlinear equations of species mass conservation.** Completing the proof of Theorem 2.1 requires to check that the original system (18) can be recovered. To this purpose we will need the following version of the fixed point theorem (for the proof see e.g. [11], Theorem 11.3).

Theorem 2.5. *Let $\mathcal{T}:X\to X$ be a continuous, compact mapping, X a Banach space. Let for any $\lambda\in[0,1]$ the fixed points $\lambda\mathcal{T}u=u$, $u\in X$ be bounded. Then $\mathcal{T}$ possesses at least one fixed point in X.*

We will apply the theorem above to the mapping

$$\mathcal{T}: W_{[0,T]} \times W_{[0,T]} \to W_{[0,T]} \times W_{[0,T]},$$
$$\mathcal{T}(\widetilde{\varrho_A}, \widetilde{\varrho_B}) = (\varrho_A, \varrho_B),$$

where (ϱ_A, ϱ_B) is a unique, global in time solution to system (23) and $W_{[0,T]}$ denotes the following class of functions

$$W_{[0,T]} = \{L^2(0,T; W^{2,2}(\Omega)) \cap L^\infty(0,T; W^{1,2}(\Omega))\}. \tag{32}$$

For κ fixed we can show the boundedness of $\mathcal{T}$ in the class $\left(V_{[0,T]}\right)^2$ using Theorem 2.4, moreover, the obtained solution is unique. Therefore, proving compactness and continuity of this mapping in $C([0,T]; C^{2+\nu'}(\Omega))$ follows exactly as in the proof of Lemma 2.3.

The only assumption of the theorem above that needs to be checked is whether any solution to

$$\lambda \mathcal{T}(\varrho_A, \varrho_B) = (\varrho_A, \varrho_B)$$

is bounded for $\lambda \in [0,1]$. This identity rewrites as

$$\begin{aligned} \partial_t \varrho_A - \left(\varepsilon + \left(\frac{\varrho_B^+}{\varrho m_A} + \frac{\varrho_A^+}{\varrho m_B}\right)_{\kappa_1}\right) \Delta \varrho_A + \left(\mathbf{u} - \nabla\left(\frac{\varrho_B^+}{\varrho m_A} + \frac{\varrho_A^+}{\varrho m_B}\right)_{\kappa_1}\right) \nabla \varrho_A \\ + \operatorname{div} \mathbf{u} \varrho_A = \lambda \varrho \left(\omega\left(\varrho_A\right)\right)_{\kappa_1} - \lambda \operatorname{div}\left(\left(\frac{\varrho_A^+}{\varrho m_B}\right)_{\kappa_1} \nabla \varrho\right), \end{aligned} \tag{33}$$

and similarly for the species B. So, we first multiply the above equation by ϱ_A and we get:

$$\begin{aligned} &\frac{d}{dt} \int_\Omega \frac{\varrho_A^2}{2} \,\mathrm{d}x + \int_\Omega \left(\varepsilon + \left(\frac{\varrho_B^+}{\varrho m_A} + \frac{\varrho_A^+}{\varrho m_B}\right)_{\kappa_1}\right) |\nabla \varrho_A|^2 \,\mathrm{d}x \\ &= \int_\Omega \varrho_A \mathbf{u} \cdot \nabla \varrho_A \,\mathrm{d}x + \lambda \int_\Omega \left(\frac{\varrho_A^+}{\varrho m_B}\right)_{\kappa_1} \nabla \varrho \cdot \nabla \varrho_A \,\mathrm{d}x + \lambda \int_\Omega \varrho \omega_{\kappa_1} \varrho_A \,\mathrm{d}x. \end{aligned} \tag{34}$$

The r.h.s. is estimated due to assumed regularity of $\varrho, \mathbf{u}$ and by the definition of $\omega\left(\varrho_A\right)$. We obtain

$$\begin{aligned} &\left| \int_\Omega \varrho_A \mathbf{u} \cdot \nabla \varrho_A \,\mathrm{d}x + \lambda \int_\Omega \left(\frac{\varrho_A^+}{\varrho m_B}\right)_{\kappa_1} \nabla \varrho \cdot \nabla \varrho_A \,\mathrm{d}x + \lambda \int_\Omega \varrho \omega_{\kappa_1} \varrho_A \,\mathrm{d}x \right| \\ &\quad \le \|\mathbf{u}\|_{L^\infty(\Omega)} \|\varrho_A\|_{L^2(\Omega)} \|\nabla \varrho_A\|_{L^2(\Omega)} + c \|\nabla \varrho_A\|_{L^2(\Omega)} \|\nabla \varrho\|_{L^2(\Omega)} \\ &\quad\quad + c\overline{\omega} \|\varrho\|_{L^\infty(\Omega)} \|\varrho_A\|_{L^1(\Omega)}, \end{aligned}$$

where the r.h.s. is absorbed by the l.h.s. after application of the Cauchy inequality. The same holds for ϱ_B. Next, multiplying (33) by $\partial_t \varrho_A$ we get

$$\begin{aligned} &\int_\Omega |\partial_t \varrho_A|^2 \,\mathrm{d}x + \frac{\varepsilon}{2} \frac{d}{dt} \int_\Omega |\nabla \varrho_A|^2 \,\mathrm{d}x + \frac{1}{2} \frac{d}{dt} \int_\Omega \left(\frac{\varrho_B^+}{\varrho m_A} + \frac{\varrho_A^+}{\varrho m_B}\right)_{\kappa_1} |\nabla \varrho_A|^2 \,\mathrm{d}x \\ &= -\int_\Omega \left(\nabla \varrho_A \cdot \mathbf{u} \partial_t \varrho_A + \varrho_A \operatorname{div} \mathbf{u} \partial_t \varrho_A\right) \,\mathrm{d}x - \lambda \int_\Omega \operatorname{div}\left(\left(\frac{\varrho_A^+}{\varrho m_A}\right)_{\kappa_1} \nabla \varrho\right) \partial_t \varrho_A \,\mathrm{d}x \\ &\quad + \frac{1}{2} \int_\Omega \partial_t \left(\frac{\varrho_B^+}{\varrho m_A} + \frac{\varrho_A^+}{\varrho m_B}\right)_{\kappa_1} |\nabla \varrho_A|^2 \,\mathrm{d}x + \lambda \int_\Omega \varrho \omega_{\kappa_1} \partial_t \varrho_A \,\mathrm{d}x. \end{aligned}$$

By the properties of mollifiers, regularity of ϱ and $\mathbf{u}$ we can estimate the r.h.s., note, however, that this cannot be done independently of κ_1.

Resuming, we have shown that

$$\text{ess}\sup_{t\in(0,T)} \|\varrho_A\|^2_{W^{1,2}(\Omega)} + \int_0^T \|\partial_t\varrho_A\|_{L^2(\Omega)}\, \mathrm{d}t \le c(\kappa_1) \tag{35}$$

and from this we may deduce that also

$$\|\nabla^2\varrho_A\|_{L^2((0,T)\times\Omega)} \le c(\kappa_1). \tag{36}$$

Moreover, the fixed point satisfies $\varrho_A + \varrho_B = \varrho$, so the proof of Theorem 2.1 is now complete.

□

3. **Second level of approximation.** The aim of this section is to recover system (12). To this purpose we first derive the estimates uniform with respect to κ_1 and then extract subsequences in order to let $\kappa_1 \to 0$ in the approximate system. Having this, we prove that the species densities ϱ_A, ϱ_B are nonnegative, which is necessary to remove truncations from the coefficients of system (18). The last part of this section is devoted to the limit passage with the dimension of the Faedo-Galerkin approximation. Observe that the final regularity of solutions does not allow to test the momentum equation by $\mathbf{u}$, it is, however, sufficient to use $\nabla\log\varrho$ instead and hence we end up with the Bresch-Desjardins estimate, as announced in the introduction.

3.1. **Estimates independent of κ_1.** From the previous section we deduce that the first energy estimate holds independently of κ_1, thus

$$\begin{aligned}
&\|\sqrt{\varrho_{\kappa_1}}\mathbf{u}_{\kappa_1}\|_{L^\infty(0,T;L^2(\Omega))} + \|\sqrt{\varrho_{\kappa_1}}\nabla\mathbf{u}_{\kappa_1}\|_{L^2(0,T;L^2(\Omega))}\\
&+\|\sqrt{\eta}\Delta\mathbf{u}_{\kappa_1}\|_{L^2(0,T;L^2(\Omega))} + \|\sqrt{\varepsilon\delta}\Delta^{s+1}\varrho_{\kappa_1}\|_{L^2(0,T;L^2(\Omega))}\\
&+\|\sqrt{\delta}\nabla\Delta^s\varrho_{\kappa_1}\|_{L^\infty(0,T;L^2(\Omega))} + \|p_E(\varrho_{\kappa_1})\|_{L^\infty(0,T;L^1(\Omega))} \le c.
\end{aligned} \tag{37}$$

By this we see that the construction of $\varrho_{\kappa_1}(\mathbf{u}_{\kappa_1})$ performed in Lemma 2.2 can be repeated. In particular, the sequence ϱ_{κ_1} is uniformly separated from 0 as long as n is fixed.

In addition, repeating estimate (34) we also verify that

$$\|\varrho_{A,\kappa_1}, \varrho_{B,\kappa_1}\|_{L^\infty(0,T;L^2(\Omega))} + \|\varrho_{A,\kappa_1}, \varrho_{B,\kappa_1}\|_{L^2(0,T;W^{1,2}(\Omega))} \le c. \tag{38}$$

Thus, the time derivatives of $\varrho_{A,\kappa_1}, \varrho_{B,\kappa_1}$ can be estimated in $L^2(0,T;W^{-1,2}(\Omega))$ directly from (23).

3.2. **Passage to the limit $\kappa_1, \kappa_2 \to 0$.** Having n fixed, all the norms of $\mathbf{u}_{\kappa_1}$ are equivalent and the limit function $\mathbf{u} \in C([0,T];X_n)$, thus the passage to the limit in the continuity equation is trivial and the limit $\varrho \in V_{[0,T]}$ on account of Lemma 2.2. Concerning the species mass balance equations, the Aubin-Lions argument can be applied and we get compactness of ϱ_{A,κ_1} in $L^2(0,T;L^q(\Omega))$ for $q<6$, in particular $\varrho_{A,\kappa_1} \to \varrho_A$ a.e. on $(0,T)\times\Omega$. By this and the bounds from (37) we easily check that the limit equation of mass conservation of species A

$$\partial_t\varrho_A + \operatorname{div}(\varrho_A\mathbf{u}) - \varepsilon\Delta\varrho_A - \operatorname{div}\left(\left(\frac{\varrho_B^+}{\varrho m_A} + \frac{\varrho_A^+}{\varrho m_B}\right)\nabla\varrho_A - \left(\frac{\varrho_A^+}{\varrho m_B}\right)\nabla\varrho\right) = \varrho\omega \tag{39}$$

is satisfied in the sense of distributions on $(0,T)\times\Omega$, but the standard density argument enables to extend the class of test functions to $L^2(0,T;W^{1,2}(\Omega))$. Moreover, due to (38) the initial condition is satisfied in the sense of distributions on Ω. Similarly for ϱ_B.

The passage to the limit in the momentum equation is straightforward.

Our next goal is to deduce from the form of system (14-20) that for $\kappa_1=0$ the limit functions ϱ_A,ϱ_B satisfy not only the mass constraint (24) but they are also nonnegative a.e. in $(0,T)\times\Omega$.

We have

Lemma 3.1. *For $\delta,\varepsilon,\eta>0$ and a positive integer n fixed, let $(\varrho,\mathbf{u},\varrho_A,\varrho_B)$ be a solution to (14-20) with $\kappa_1=0$ as specified above. Then*

$$\varrho_A=\varrho_A^+,\quad \varrho_B=\varrho_B^+\quad a.e.\ in\ (0,T)\times\Omega.$$

Proof. In what follows, we focus only on the proof of nonnegativity of ϱ_A, the case of ϱ_B can be shown analogously.

By virtue of (38), we are allowed to test (39) with a function $(\varrho_{A-}+l)^{q-1}$, $l>0$, $q\in(1,2]$, where

$$\varrho_{A-}=\begin{cases}-\varrho_A & \text{if}\quad \varrho_A<0,\\ 0 & \text{if}\quad 0\le\varrho_A,\end{cases}$$

and then pass to the limit $l\to0^+$. Observe that $\varrho_A^+\varrho_{A-}=0$ and $\varrho_B^+\varrho_{A-}=\varrho\varrho_{A-}$ in case when $\varrho_A<0$ or $\varrho_B^+\varrho_{A-}=0$ for $\varrho_A\ge0$, thus

$$\begin{aligned}-\frac{1}{q}\frac{d}{dt}\int_\Omega\varrho_{A-}^q\ \mathrm{d}x-\frac{4\varepsilon(q-1)}{q^2}\int_\Omega\left|\nabla\varrho_{A-}^{q/2}\right|^2\ \mathrm{d}x-\frac{4(q-1)}{m_Aq^2}\int_\Omega\left|\nabla\varrho_{A-}^{q/2}\right|^2\ \mathrm{d}x\\ =(1-q)\int_\Omega\mathbf{u}\cdot\nabla\varrho_{A-}\varrho_{A-}^{q-1}\ \mathrm{d}x+\int_\Omega\varrho\omega(\varrho_A)\varrho_{A-}^{q-1}\ \mathrm{d}x.\end{aligned}$$

Since $\varrho_{A-}>0$ enforces $\omega\left(\varrho_A\right)\ge0$, we put the last term from the r.h.s. to the l.h.s., so multiplying the above expression by -1 we get

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\int_\Omega\varrho_{A-}^q\ \mathrm{d}x+\frac{4\varepsilon(q-1)}{q^2}\int_\Omega\left|\nabla\varrho_{A-}^{q/2}\right|^2\ \mathrm{d}x+\frac{4(q-1)}{m_Aq^2}\int_\Omega\left|\nabla\varrho_{A-}^{q/2}\right|^2\ \mathrm{d}x\\ +\int_\Omega\varrho\omega(\varrho_A)\varrho_{A-}^{q-1}\ \mathrm{d}x=\frac{2(q-1)}{q}\int_\Omega\mathbf{u}\cdot\nabla\varrho_{A-}^{q/2}\varrho_{A-}^{q/2}\ \mathrm{d}x.\end{aligned}\tag{40}$$

Now, the r.h.s. may be bounded by use of the Cauchy inequality

$$\left|\int_\Omega\mathbf{u}\cdot\nabla\varrho_{A-}^{q/2}\varrho_{A-}^{q/2}\ \mathrm{d}x\right|\le\|\mathbf{u}\|_{L^\infty(\Omega)}\left(\epsilon\|\nabla\varrho_{A-}^{q/2}\|^2_{L^2(\Omega)}+c(\epsilon)\|\varrho_{A-}^{q/2}\|^2_{L^2(\Omega)}\right)$$

and the first of the resulting terms is absorbed by the l.h.s. of (40) provided $\frac{4}{m_Aq^2}>\frac{2\epsilon\|\mathbf{u}\|_\infty}{q}$, while the other is bounded since $\varrho_A\in L^\infty(0,T;L^2(\Omega))$.

Further, as the three last terms from the l.h.s. of (40) are nonnegative we get that

$$\frac{d}{dt}\int_\Omega\varrho_{A-}^q\ \mathrm{d}x\le c(q-1),$$

thus, passing to the limit $q\to1^+$ and integrating by time we conclude that

$$\int_\Omega\varrho_{A-}(t)\ \mathrm{d}x\le\int_\Omega\varrho_{A-}(0)\ \mathrm{d}x.$$

Since the integrant from the r.h.s. is equal to 0 a.e. in Ω, there must be $\varrho_{A-}(t,x)=0$ a.e. in $(0,T)\times\Omega$. □

Obviously, positiveness of species masses coupled with (24) leads to the following inequality

$$0 \le \varrho_A, \varrho_B \le \varrho, \quad \text{a.e. in } (0,T)\times\Omega.$$

This fact allows us to verify that the estimates uniform with respect to κ_1 are in fact uniform with respect to κ_2. Therefore passage to the limit $\kappa_2 \to 0$ can be performed identically as the previous one.

3.3. **Estimates independent of the dimension of the Galerkin approximation.** Observe that the estimates derived in the previous section are independent of n. In particular, due to bounds from (37) we deduce that

$$\mathbf{u}_n \to \mathbf{u} \quad \text{weakly in } L^2(0,T;W^{2,2}(\Omega)), \tag{41}$$

and

$$\varrho_n \to \varrho \quad \text{weakly in } L^2(0,T;W^{2s+2,2}(\Omega)), \tag{42}$$

at least for a suitable subsequence. In addition the r.h.s. of the linear parabolic problem

$$\partial_t\varrho_n - \varepsilon\Delta\varrho_n = \operatorname{div}(\varrho_n\mathbf{u}_n),$$
$$\varrho_n(0,x) = \varrho_\delta^0,$$

is uniformly bounded in $L^2(0,T;L^6(\Omega))$ and the initial condition is sufficiently smooth, thus, applying the $L^p - L^q$ theory to this problem we conclude that $\{\partial_t\varrho_n\}_{n=1}^\infty$ is uniformly bounded in $L^2(0,T;L^6(\Omega))$. Therefore, the standard compact embeddings imply

$$\varrho_n \to \varrho \quad \text{a.e. in } (0,T)\times\Omega;$$

whence the limit passage in the approximate continuity equation is straightforward. Having that, we can also identify the limit for $n\to\infty$ in all terms of the momentum equation, except for the convective term, the additional capillarity force and the pressure. To handle the first one observe that

$$\varrho_n\mathbf{u}_n \to \varrho\mathbf{u} \quad \text{weakly}^* \ \text{ in } L^\infty(0,T;L^2(\Omega)),$$

due to the uniform estimates (37) and the strong convergence of the density. Next, one can show that for any $\phi \in \cup_{n=1}^\infty X_n$ the family of functions $\int_\Omega \varrho_n\mathbf{u}_n(t)\phi \,\mathrm{d}x$ is bounded and equi-continuous in $C([0,T])$, thus via the Arzelà-Ascoli theorem and density of smooth functions in $L^2(\Omega)$ we get that

$$\varrho_n\mathbf{u}_n \to \varrho\mathbf{u} \quad \text{in } C([0,T];L^2_{weak}(\Omega)).$$

Finally, by the compact embedding $L^2(\Omega)\subset W^{-1,2}(\Omega)$ and the weak convergence of $\mathbf{u}_n$ (41) we verify that

$$\varrho_n\mathbf{u}_n\otimes\mathbf{u}_n \to \varrho\mathbf{u}\otimes\mathbf{u} \quad \text{weakly in } L^2((0,T)\times\Omega).$$

Concerning the capillarity term, we first rewrite it in the form

$$\int_\Omega \varrho_n\nabla\Delta^{2s+1}\varrho_n\cdot\phi \,\mathrm{d}x = \int_\Omega \Delta^s\operatorname{div}(\varrho_n\phi)\,\Delta^{s+1}\varrho_n \,\mathrm{d}x.$$

Due to (42) and boundedness of the time derivative of ϱ_n, we infer that

$$\varrho_n \to \varrho \quad \text{strongly in } L^2(0,T;W^{2s+1,2}(\Omega)), \tag{43}$$

thus

$$\int_\Omega \Delta^s\operatorname{div}(\varrho_n\phi)\,\Delta^{s+1}\varrho_n \,\mathrm{d}x \to \int_\Omega \Delta^s\operatorname{div}(\varrho\phi)\,\Delta^{s+1}\varrho \,\mathrm{d}x,$$

for every $\phi \in \cup_{n=1}^{\infty} X_n$. Moreover, by the penultimate estimate of (37) and since the set $\cup_{n=1}^{\infty} X_n$ is dense in $W^{2s+2}(\Omega)$, this convergence holds for all $\phi \in L^2(0,T;W^{2s+2}(\Omega))$.

Passage to the limit in the molecular part of the pressure is an easy task, since due to (38) there exist the subsequences such that

$$\varrho_{k,n} \to \varrho_k, \quad \text{weakly in } L^2(0,T;W^{1,2}(\Omega)),\ k \in S.$$

So, the only uncertain part is the nonlinear barotropic pressure. Its strong convergence is a consequence of pointwise convergence of the density, and the bounds from (37). Taking s sufficiently large we can show that the density is separated from 0 uniformly with respect to all approximation parameters except for δ. Indeed, since by the Sobolev embedding $\|\varrho^{-1}\|_{L^\infty(\Omega)} \le c\|\varrho^{-1}\|_{W^{3,k}(\Omega)}$ for $k > 1$ and

$$\|\nabla^3 \varrho^{-1}\|_{L^k(\Omega)} \le (1 + \|\nabla^3\varrho\|_{L^{2k}(\Omega)})^3(1 + \|\varrho^{-1}\|_{L^{4k}(\Omega)})^4,$$

is bounded on account of (37), provided that $2s+1 \ge 4$, we have

$$\|\varrho^{-1}\|_{L^\infty((0,T)\times\Omega)} \le c(\delta) \quad \text{a.e. in } (0,T)\times\Omega. \tag{44}$$

By this observation, passage to the limit $n \to \infty$ in the species mass balance equations may be performed identically as the passage $\kappa_1 \to 0$ from the previous subsection. Theorem 1.1 is proved.

□

4. **Estimates independent of ε, η and δ.** At this level of approximation, it is relatively easy to derive the Bresch-Desjardins inequality. Indeed, as we know that $\varrho \in L^2(0,T;W^{2s+2,2}(\Omega)) \cap L^\infty((0,T)\times\Omega)$ and $\mathbf{u} \in L^2(0,T;W^{2,2}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$, we can differentiate the approximate continuity equation with respect to x to observe that $\nabla\varrho$ satisfies the following system

$$\begin{aligned} \partial_t(\nabla\varrho) - \varepsilon\Delta(\nabla\varrho) &= -\nabla \operatorname{div}(\varrho\mathbf{u}),\\ \nabla\varrho(0,x) &= \nabla\varrho_\delta^0. \end{aligned} \tag{45}$$

Since the r.h.s. of (45) is bounded in $L^2((0,T)\times\Omega)$, we can again apply the maximal $L^p - L^q$ theory for such problems to deduce that $\partial_t\nabla\varrho \in L^2((0,T)\times\Omega)$. Hence, the function $\nabla\phi = 2\frac{\nabla\varrho}{\varrho}$ belongs to $W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;W^{2s+1,2}(\Omega))$, thus it is an admissible test function for the approximate momentum equation $(12)_2$.

Our next aim is to prove the following inequality:

Lemma 4.1. *We have*

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u} + \nabla\phi(\varrho)|^2 + \frac{\delta}{2}|\nabla\Delta^s\varrho|^2 + \varrho\pi(\varrho)\right)\,\mathrm{d}x\\ &\quad + \int_\Omega \nabla\phi(\varrho)\cdot\nabla p\,\mathrm{d}x + \frac{1}{2}\int_\Omega \varrho|\nabla\mathbf{u} - \nabla^T\mathbf{u}|^2\,\mathrm{d}x\\ &\quad + \int_\Omega \left(2\delta|\Delta^{s+1}\varrho|^2 + \delta\varepsilon|\Delta^{s+1}\varrho|^2\right)\,\mathrm{d}x + \eta\int_\Omega |\Delta\mathbf{u}|^2\,\mathrm{d}x\\ &\le -\varepsilon\int_\Omega (\nabla\varrho\cdot\nabla)\mathbf{u}\cdot\nabla\phi\,\mathrm{d}x + \varepsilon\int_\Omega \Delta\varrho\frac{|\nabla\phi|^2}{2}\,\mathrm{d}x\\ &\quad + \varepsilon\int_\Omega \varrho\nabla\phi(\varrho)\cdot\nabla\left(\phi'(\varrho)\Delta\varrho\right)\,\mathrm{d}x - \varepsilon\int_\Omega \operatorname{div}(\varrho\mathbf{u})\phi'(\varrho)\Delta\varrho\,\mathrm{d}x\\ &\quad - \eta\int_\Omega \Delta\mathbf{u}\cdot\nabla\Delta\phi(\varrho)\,\mathrm{d}x + \int_\Omega\left(\frac{\varrho_A}{m_A} + \frac{\varrho_B}{m_B}\right)\operatorname{div}\mathbf{u}\,\mathrm{d}x \end{aligned} \tag{46}$$

in $\mathcal{D}'(0,T)$, *where* $\pi(\varrho) = \int_0^\varrho y^{-2} p_E(y)\ dy \geq 0$.

Proof. The basic idea of the proof is to find the explicit form of the term:

$$\frac{d}{dt}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho\mathbf{u}\cdot\nabla\phi(\varrho) + \varrho|\nabla\phi(\varrho)|^2\right)\ \mathrm{d}x. \tag{47}$$

For this purpose we first multiply the approximate continuity equation by $\frac{|\nabla\phi(\varrho)|^2}{2}$ and we obtain the following sequence of equalities

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \frac{1}{2}\varrho|\nabla\phi(\varrho)|^2\ \mathrm{d}x\\
&= \int_\Omega \left(\varrho\partial_t\frac{|\nabla\phi(\varrho)|^2}{2} - \frac{|\nabla\phi(\varrho)|^2}{2}\operatorname{div}(\varrho\mathbf{u}) + \varepsilon\frac{|\nabla\phi(\varrho)|^2}{2}\Delta\varrho\right)\ \mathrm{d}x\\
&= \int_\Omega \left(\varrho\nabla\phi(\varrho)\cdot\nabla\left(\phi'(\varrho)\partial_t\varrho\right) - \frac{|\nabla\phi(\varrho)|^2}{2}\operatorname{div}(\varrho\mathbf{u}) + \varepsilon\frac{|\nabla\phi(\varrho)|^2}{2}\Delta\varrho\right)\ \mathrm{d}x\\
&= \int_\Omega \left(\varepsilon\varrho\nabla\phi(\varrho)\cdot\nabla\left(\phi'(\varrho)\Delta\varrho\right) - \varrho\nabla\mathbf{u} : \nabla\phi(\varrho)\otimes\nabla\phi(\varrho)\right)\ \mathrm{d}x\\
&\quad + \int_\Omega \left(\varepsilon\frac{|\nabla\phi(\varrho)|^2}{2}\Delta\varrho - \frac{|\nabla\phi(\varrho)|^2}{2}\operatorname{div}(\varrho\mathbf{u})\right)\ \mathrm{d}x\\
&\quad - \int_\Omega \left(\varrho\mathbf{u}\otimes\nabla\phi(\varrho) : \nabla^2\phi(\varrho) + \varrho\nabla\phi(\varrho)\cdot\nabla\left(\phi'(\varrho)\varrho\operatorname{div}\mathbf{u}\right)\right)\ \mathrm{d}x\\
&= \int_\Omega \left(\varepsilon\varrho\nabla\phi(\varrho)\cdot\nabla\left(\phi'(\varrho)\Delta\varrho\right) - \varrho\nabla\mathbf{u} : \nabla\phi(\varrho)\otimes\nabla\phi(\varrho)\right)\ \mathrm{d}x\\
&\quad + \int_\Omega \left(\varrho\mathbf{u}\Delta\phi(\varrho)\cdot\nabla\phi(\varrho) + \frac{|\nabla\phi(\varrho)|^2}{2}\operatorname{div}(\varrho\mathbf{u})\right)\ \mathrm{d}x\\
&\quad - \int_\Omega \operatorname{div}(\varrho\mathbf{u}\otimes\nabla\phi(\varrho)\nabla\phi(\varrho))\ \mathrm{d}x\\
&\quad + \int_\Omega \left(\varrho^2\phi'(\varrho)\Delta\phi(\varrho)\operatorname{div}\mathbf{u} + \varrho\left|\nabla\phi(\varrho)\right|^2\operatorname{div}\mathbf{u}\right)\ \mathrm{d}x\\
&= \int_\Omega \left(\varepsilon\varrho\nabla\phi(\varrho)\cdot\nabla\left(\phi'(\varrho)\Delta\varrho\right) - \varrho\nabla\mathbf{u} : \nabla\phi(\varrho)\otimes\nabla\phi(\varrho) + \varepsilon\frac{|\nabla\phi(\varrho)|^2}{2}\Delta\varrho\right)\ \mathrm{d}x\\
&\quad + \int_\Omega \left(\varrho^2\phi'(\varrho)\Delta\phi(\varrho)\operatorname{div}\mathbf{u} + \varrho\left|\nabla\phi(\varrho)\right|^2\operatorname{div}\mathbf{u}\right)\ \mathrm{d}x.
\end{aligned} \tag{48}$$

In the above series of equalities, each one holds pointwisely with respect to time due to the regularity of ϱ and $\nabla\phi$. This is not the case of the middle integrant of (47), for which one should really think of weak in time formulation. Denote

$$V = W^{2s+1,2}(\Omega), \quad H = L^2(\Omega) \quad \text{and} \quad \mathbf{v} = \varrho\mathbf{u}, \quad \mathbf{h} = \nabla\phi.$$

We know that $\mathbf{v} \in L^2(0,T;H)$ and its weak derivative with respect to time variable $\mathbf{v}' \in L^2(0,T;V^*)$, where V^* denotes the dual space to V. Moreover, $\mathbf{h} \in L^2(0,T;V)$, $\mathbf{h}' \in L^2(0,T;H^*)$. Now, let $\mathbf{v}_m$, $\mathbf{h}_m$ denote the standard mollifications in time of $\mathbf{v}$ and $\mathbf{h}$ respectively. By the properties of mollifiers we know that

$$\mathbf{v}_m, \mathbf{v}_m' \in C^\infty(0,T;H), \quad \mathbf{h}_m, \mathbf{h}_m' \in C^\infty(0,T;V),$$

and

$$\begin{aligned}
&\mathbf{v}_m \to \mathbf{v} \quad \text{in } L^2(0,T;H), \qquad &&\mathbf{h}_m \to \mathbf{h} \quad \text{in } L^2(0,T;V),\\
&\mathbf{v}_m' \to \mathbf{v}' \quad \text{in } L^2(0,T;V^*), \qquad &&\mathbf{h}_m' \to \mathbf{h}' \quad \text{in } L^2(0,T;H^*).
\end{aligned} \tag{49}$$

For these regularized sequences we may write

$$\frac{d}{dt}\int_\Omega \mathbf{v}_m \cdot \mathbf{h}_m \, \mathrm{d}x = \frac{d}{dt}(\mathbf{v}_m, \mathbf{h}_m)_H = (\mathbf{v}'_m, \mathbf{h}_m)_H + (\mathbf{v}_m, \mathbf{h}'_m)_H. \tag{50}$$

Using the Riesz representation theorem we verify that $\mathbf{v}'_m \in C^\infty(0,T;H)$ uniquely determines the functional $\Phi_{\mathbf{v}'_m} \in H^*$ such that $(\mathbf{v}'_m, \psi)_H = \langle \Phi_{\mathbf{v}'_m}, \psi \rangle_{H^*,H}$, $\forall \psi \in H$. Since $H^* \subset V^*$ densely, this functional belongs to V^* in the sense $\langle \Phi_{\mathbf{v}'_m}, \psi \rangle_{V^*,V} = \int_\Omega \mathbf{v}'_m \cdot \psi \, \mathrm{d}x$, $\forall \psi \in V$. Therefore, by identification of H and H^*, the first term on the r.h.s. of (50) can be understood as $(\mathbf{v}'_m, \mathbf{h}_m)_H = \langle \mathbf{v}'_m, \mathbf{h}_m \rangle_{H^*,H} = \langle \mathbf{v}'_m, \mathbf{h}_m \rangle_{V^*,V}$. For the second term from the r.h.s. of (50), we use the Riesz representation theorem to write $(\mathbf{v}_m, \mathbf{h}'_m)_H = \langle \mathbf{v}_m, \mathbf{h}'_m \rangle_{H,H^*}$ and thus we obtain

$$-\int_0^T (\mathbf{v}_m, \mathbf{h}_m)_H \psi' \, \mathrm{d}t = \int_0^T \langle \mathbf{v}'_m, \mathbf{h}_m \rangle_{V^*,V} \psi \, \mathrm{d}t + \int_0^T \langle \mathbf{v}_m, \mathbf{h}'_m \rangle_{H,H^*} \psi \, \mathrm{d}t,$$

for all $\psi \in \mathcal{D}(0,T)$. Observe that both integrants from the r.h.s. are uniformly bounded in $L^1(0,T)$, thus, using (49), we let $m \to \infty$ to obtain

$$\frac{d}{dt}(\mathbf{v}, \mathbf{h})_H = \langle \mathbf{v}', \mathbf{h} \rangle_{V^*,V} + \langle \mathbf{v} \cdot \mathbf{h}' \rangle_{H,H^*} \quad \text{in } \mathcal{D}'(0,T).$$

Coming back to our original notation, this means that the operation

$$\frac{d}{dt}\int_\Omega \varrho \mathbf{u} \cdot \nabla \phi(\varrho) \, \mathrm{d}x = \langle \partial_t(\varrho \mathbf{u}), \nabla \phi \rangle_{V^*,V} + \int_\Omega \varrho \mathbf{u} \cdot \partial_t \nabla \phi \, \mathrm{d}x \tag{51}$$

is well defined and is nothing but equality between two scalar distributions. By the fact that $\partial_t \nabla \phi$ exists a.e. in $(0,T) \times \Omega$ we may use approximate continuity equation to write

$$\int_\Omega \varrho \mathbf{u} \cdot \partial_t \nabla \phi \, \mathrm{d}x = \int_\Omega (\mathrm{div}(\varrho \mathbf{u}))^2 \phi'(\varrho) \, \mathrm{d}x - \varepsilon \int_\Omega \mathrm{div}(\varrho \mathbf{u}) \phi'(\varrho) \Delta \varrho \, \mathrm{d}x, \tag{52}$$

whence the first term on the r.h.s. of (51) may be evaluated by testing the approximate momentum equation by $\nabla \phi(\varrho)$

$$\begin{aligned}
&\langle \partial_t(\varrho \mathbf{u}), \nabla \phi \rangle_{V^*,V} \\
&\quad = -\int_\Omega 2\varrho \Delta \phi(\varrho) \,\mathrm{div}\, \mathbf{u} \, \mathrm{d}x + 2\int_\Omega \nabla \mathbf{u} : \nabla \phi(\varrho) \otimes \nabla \varrho \, \mathrm{d}x \\
&\qquad - 2\int_\Omega \nabla \phi(\varrho) \cdot \nabla \varrho \,\mathrm{div}\, \mathbf{u} \, \mathrm{d}x - \int_\Omega \nabla \phi(\varrho) \cdot \nabla p \, \mathrm{d}x \\
&\qquad + \delta \int_\Omega \varrho \nabla \Delta^{2s+1} \varrho \cdot \nabla \phi(\varrho) \, \mathrm{d}x - \int_\Omega \nabla \phi(\varrho) \cdot \mathrm{div}(\varrho \mathbf{u} \otimes \mathbf{u}) \, \mathrm{d}x \\
&\qquad - \eta \int_\Omega \Delta^2 \mathbf{u} \cdot \nabla \phi(\varrho) \, \mathrm{d}x - \varepsilon \int_\Omega (\nabla \varrho \cdot \nabla) \mathbf{u} \cdot \nabla \phi(\varrho) \, \mathrm{d}x.
\end{aligned} \tag{53}$$

Recalling the form of $\phi(\varrho)$ it can be deduced that the combination of (48) with (51-53) yields

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega\left(\varrho\mathbf{u}\cdot\nabla\phi(\varrho)+\frac{1}{2}\varrho|\nabla\phi(\varrho)|^2\right)\,\mathrm{d}x+\int_\Omega\nabla p\cdot\nabla\phi(\varrho)+2\delta|\Delta^{s+1}\varrho|^2\,\mathrm{d}x\\
&\quad=-\int_\Omega\nabla\phi(\varrho)\operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u})\,\mathrm{d}x+\int_\Omega(\operatorname{div}(\varrho\mathbf{u}))^2\phi'(\varrho)\,\mathrm{d}x\\
&\qquad-\varepsilon\int_\Omega\operatorname{div}(\varrho\mathbf{u})\phi'(\varrho)\Delta\varrho\,\mathrm{d}x-\eta\int_\Omega\Delta\mathbf{u}\cdot\nabla\Delta\phi(\varrho)\,\mathrm{d}x\\
&\qquad+\varepsilon\int_\Omega\frac{|\nabla\phi(\varrho)|^2}{2}\Delta\varrho\,\mathrm{d}x-\varepsilon\int_\Omega(\nabla\varrho\cdot\nabla)\mathbf{u}\cdot\nabla\phi(\varrho)\,\mathrm{d}x\\
&\qquad+\varepsilon\int_\Omega\varrho\nabla\phi(\varrho)\cdot\nabla\left(\phi'(\varrho)\Delta\varrho\right)\,\mathrm{d}x.
\end{aligned}
\tag{54}
$$

It is then easy to check that the first two terms from the r.h.s of (54) can be transformed into

$$
\begin{aligned}
&\int_\Omega\left[(\operatorname{div}(\varrho\mathbf{u}))^2\phi'(\varrho)-\nabla\phi(\varrho)\operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u})\right]\,\mathrm{d}x\\
&\qquad\qquad=\int_\Omega\left[2\varrho|\mathbf{D}(\mathbf{u})|^2-\frac{1}{2}\varrho|\nabla\mathbf{u}-\nabla^T\mathbf{u}|^2\right]\,\mathrm{d}x,
\end{aligned}
$$

and thus, the assertion of Lemma 4.1 follows by adding (29) to (54). □

Our next aim is to derive uniform estimates from inequality (46). To this purpose we will integrate it with respect to time. For any $\psi_m\in\mathcal{D}(0,T)$, the first term from the l.h.s. of (46) equals

$$
\begin{aligned}
&\int_0^T\frac{d}{dt}\int_\Omega\left(\frac{1}{2}\varrho|\mathbf{u}+\nabla\phi(\varrho)|^2+\frac{\delta}{2}|\nabla\Delta^s\varrho|^2+\varrho\pi(\varrho)\right)\,\mathrm{d}x\;\psi_m\,\mathrm{d}t\\
&\qquad=-\int_0^T\!\!\int_\Omega\left(\frac{1}{2}\varrho|\mathbf{u}+\nabla\phi(\varrho)|^2+\frac{\delta}{2}|\nabla\Delta^s\varrho|^2+\varrho\pi(\varrho)\right)\psi_m'\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{55}
$$

Now, choosing a sequence of $\psi_m\in\mathcal{D}(0,\tau)$ such that $\psi_m\to1$ pointwisely in $(0,\tau)$, $\psi_m\to0$ pointwisely in $[\tau,T)$, $0<\tau<T$ we see that ψ_m' approximates the inner normal vector to the boundary of the time interval $[0,\tau]$. In other words, it generates two Dirac distributions at the ends of $[0,\tau]$. Thus, using the fact that

$$
t\mapsto\int_\Omega\left(\frac{1}{2}\varrho|\mathbf{u}+\nabla\phi(\varrho)|^2+\frac{\delta}{2}|\nabla\Delta^s\varrho|^2+\varrho\pi(\varrho)\right)\,\mathrm{d}x\in C([0,\tau]),
$$

we let $m \to \infty$ in (55) and from (46) we get

$$\begin{aligned}
&\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}+\nabla\phi(\varrho)|^2 + \frac{\delta}{2}|\nabla\Delta^s\varrho|^2 + \varrho\pi(\varrho)\right)(\tau)\ \mathrm{d}x \\
&\quad + \int_0^\tau\int_\Omega \nabla\phi(\varrho)\cdot\nabla p(\varrho,\varrho_A)\ \mathrm{d}x\ \mathrm{d}t \\
&\quad + \int_0^\tau\int_\Omega \left(\frac{1}{2}\varrho|\nabla\mathbf{u}-\nabla^T\mathbf{u}|^2 + 2\delta|\Delta^{s+1}\varrho|^2 + \delta\varepsilon|\Delta^{s+1}\varrho|^2 + \eta|\Delta\mathbf{u}|^2\right)\ \mathrm{d}x\ \mathrm{d}t \\
&\le \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}+\nabla\phi(\varrho)|^2 + \frac{\delta}{2}|\nabla\Delta^s\varrho|^2 + \varrho\pi(\varrho)\right)(0)\ \mathrm{d}x \qquad (56)\\
&\quad + \varepsilon\int_0^\tau\int_\Omega \nabla\varrho\cdot\nabla\mathbf{u}\cdot\nabla\phi\ \mathrm{d}x\ \mathrm{d}t \\
&\quad + \varepsilon\int_0^\tau\int_\Omega \Delta\varrho\frac{|\nabla\phi|^2}{2}\ \mathrm{d}x\ \mathrm{d}t + \varepsilon\int_0^\tau\int_\Omega \varrho\nabla\phi(\varrho)\cdot\nabla\left(\phi'(\varrho)\Delta\varrho\right)\ \mathrm{d}x\ \mathrm{d}t \\
&\quad - \varepsilon\int_0^\tau\int_\Omega \operatorname{div}(\varrho\mathbf{u})\phi'(\varrho)\Delta\varrho\ \mathrm{d}x\ \mathrm{d}t - \eta\int_0^\tau\int_\Omega \Delta\mathbf{u}\cdot\nabla\Delta\phi(\varrho)\ \mathrm{d}x\ \mathrm{d}t.
\end{aligned}$$

The only non-positive contribution to the l.h.s. of (56) is contained in the second integral, as we can not determine the sign of the part corresponding to molecular pressure. However, we have

$$\int_\Omega \nabla\phi\cdot\nabla p_M(\varrho,\varrho_A)\ \mathrm{d}x = \int_\Omega\left(\frac{2|\nabla\varrho|^2}{\varrho m_B} + \left(\frac{1}{m_A}-\frac{1}{m_B}\right)\frac{2\nabla\varrho\cdot\nabla\varrho_A}{\varrho}\right)\ \mathrm{d}x$$

moreover,

$$\left(\frac{1}{m_A}-\frac{1}{m_B}\right)\int_\Omega \frac{\nabla\varrho\cdot\nabla\varrho_A}{\varrho}\ \mathrm{d}x = \left(\frac{1}{m_A}-\frac{1}{m_B}\right)\int_\Omega\left(Y_A\frac{|\nabla\varrho|^2}{\varrho} + \nabla\varrho\cdot\nabla Y_A\right)\ \mathrm{d}x$$

and

$$\left|\int_\Omega \nabla\varrho\cdot\nabla Y_A\ \mathrm{d}x\right| \le c(\varepsilon)\int_\Omega \frac{|\nabla\varrho|^2}{\varrho}\ \mathrm{d}x + \varepsilon\int_\Omega \varrho|\nabla Y_A|^2\ \mathrm{d}x. \qquad (57)$$

To control the second term we, we use $Y_A = \frac{\varrho_A}{\varrho} \in L^2(0,T;W^{1,2}(\Omega))$ as a test function in (39), we obtain

$$\begin{aligned}
&\int_\Omega \frac{1}{2}\varrho Y_A^2(T)\ \mathrm{d}x + \left(\varepsilon + \frac{1}{\max\{m_A,m_B\}}\right)\int_0^T\int_\Omega \varrho|\nabla Y_A|^2\ \mathrm{d}x\ \mathrm{d}t \\
&\quad \le \int_\Omega \frac{1}{2}\varrho Y_A^2(0)\ \mathrm{d}x + \int_0^T\int_\Omega \varrho|\omega(Y)|Y_A\ \mathrm{d}x\ \mathrm{d}t + c\int_0^T\int_\Omega |\nabla\varrho\cdot\nabla Y_A|\ \mathrm{d}x\ \mathrm{d}t.
\end{aligned} \qquad (58)$$

Hence, by the Cauchy inequality, we can justify that the $L^1(\Omega)$ norm of $\varrho|\nabla Y_A|^2$ is controlled by the $L^1(\Omega)$ norm of $\frac{|\nabla\varrho|^2}{\varrho}$ independently of the approximation parameters, so we end up with

$$\int_\Omega |\nabla\phi\cdot\nabla p_M(\varrho,\varrho_A)|\ \mathrm{d}x \le c(m_A,m_B)\int_\Omega \frac{|\nabla\varrho|^2}{\varrho}\ \mathrm{d}x.$$

Finally, the Gronwall-type argument can be applied to absorb this term by the l.h.s. of (46).

Concerning the ε and η-dependent terms from the r.h.s of (56), they can be handled similarly. For example, the first one is estimated by

$$\left|\varepsilon\int_\Omega \nabla\varrho\cdot\nabla\mathbf{u}\cdot\nabla\phi\ \mathrm{d}x\right| \le 2\varepsilon\|\nabla\mathbf{u}\|_{L^6(\Omega)}\|\varrho^{-1}\|_\infty\|\varrho\|^2_{W^{1,6/5}(\Omega)}.$$

The Sobolev imbedding implies that for ε sufficiently smaller than η and s sufficiently large we have

$$\left|\varepsilon\int_\Omega \nabla\varrho\cdot\nabla\mathbf{u}\cdot\nabla\phi\;\mathrm{d}x\right| \le \frac{\eta}{3}\|\Delta\mathbf{u}\|^2_{L^2(\Omega)} + c(\varepsilon)\|\varrho^{-1}\|^2_{L^\infty(\Omega)}\|\varrho\|^4_{H^{2s+1}(\Omega)}$$

and the last term is bounded uniformly in time due to (29) provided ε is sufficiently smaller than δ. The last term on the r.h.s. of (56) can be estimated exactly as in (30), thus the estimate (13) is valid.

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