Singular Limit of Navier-Stokes System Leading to a Free/Congested Zones Two–Phase model

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Abstract. The aim of this work is to justify mathematically the derivation of a viscous free/congested zones two–phase model from the isentropic compressible Navier-Stokes equations with a singular pressure playing the role of a barrier.

Titre et résumé en Français. *Modèle bi-phasique gérant zones libres/zones congestionnées comme limite singulière d'un système de Navier-Stokes compressible.* Le but de ce papier est de justifier mathématiquement l'obtention d'un modèle biphasique visqueux gérant zones libres/zones congestionnées comme limite singulière des équations de Navier-Stokes compressible barotrope à l'aide d'une pression singulière jouant le rôle d'une barrière. Ce type de systèmes macroscopiques pour modéliser le mouvement de foule a été proposé dans de nombreux papiers. Le lecteur interessé est renvoyé par exemple au papier de review [B. Maury, Actes des Colloques Caen 2012-Rouen 2011].

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1. Introduction

Macroscopic approaches for modelling the motion of crowd have been recently proposed in various papers where the swarm is identified through a density $\rho = \rho(t,x)$, see for instance a review paper by Maury [12]. The density is transported through a vector field $u(t,x)$ which itself solves an equation

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expressing the variation of velocity for each individual under some factors. The following system is obtained

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) = F(\rho, u), \end{cases} \tag{1.1}$$

where F is an appropriate differential operator that has to be defined depending on the applications; for instance, repulsive/attractive terms may be included to model congestion.

For modelling the traffic jams, some systems that mix free/congested regions have been also proposed, namely

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla \pi = 0 \\ 0 \le \rho \le \rho^*, \qquad (\rho - \rho^*)\pi = 0 \end{cases} \tag{1.2}$$

for given function ρ^*. The interested reader is referred to paper by Berthelin [1] in which the existence of solutions to system (1.2) was proven for $\rho^* = const.$ using the convergence of some special solutions, called the sticky blocks. For various extensions of this work (when ρ^* depends on the velocity or on the number of lanes in the portion of the road) we refer to a recent work of Berthelin and Broizat [2] and the references therein.

Formal justification of system (1.2) from (1.1) with $F(\rho, u)$ being a gradient of a specific singular pressure term has been given by Degond *et al.* in [7] (see also the proposed numerical scheme for $\rho^* = 1$). Note that a more complex model than (1.2) has been also formally derived by these authors for collective motion (namely with the extra constraint on the velocity $|u| = 1$).

The main objective of this note is to justify mathematically the viscous version of (1.2) as a limit of the isentropic compressible Navier-Stokes equations. This limit will be obtained by introducing a small parameter ε in front of a singular pressure and by letting $\varepsilon \to 0$. The important feature of such system is that it preserves the constraint $0 \le \rho^\varepsilon \le 1$ for any $\varepsilon > 0$ fixed.

2. Singular compressible Navier-Stokes model and the associated free boundary system

We consider the system of compressible barotropic Navier-Stokes equations

$$\begin{cases} \partial_t \rho^\varepsilon + \operatorname{div}(\rho^\varepsilon u^\varepsilon) = 0 \\ \partial_t(\rho^\varepsilon u^\varepsilon) + \operatorname{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) - 2\operatorname{div}(\mu(\rho^\varepsilon) D(u^\varepsilon)) \\ \qquad -\nabla(\lambda(\rho^\varepsilon)\operatorname{div}(u^\varepsilon)) + \nabla p_1(\rho^\varepsilon) + \nabla p_2^\varepsilon(\rho^\varepsilon) = 0 \end{cases} \tag{2.1}$$

in a fixed bounded domain Ω.

In the above system p_1 is the barotropic pressure

$$p_1(\rho^\varepsilon) = a(\rho^\varepsilon)^\alpha \quad a \ge 0, \quad \alpha > 1, \tag{2.2}$$

while p_2^ε is the singular pressure in the spirit of [5, 8]

$$p_2^\varepsilon(\rho^\varepsilon) = \varepsilon(\rho^\varepsilon)^\gamma P(\rho^\varepsilon), \quad \gamma > 1, \quad \varepsilon > 0. \tag{2.3}$$

The singular pressure $P(\cdot) \in \mathcal{C}^1(0,1)$ is strictly increasing function, such that

$$\lim_{\rho^\varepsilon \to \rho_*^-} P(\rho^\varepsilon) = +\infty \tag{2.4}$$

and $\rho_* = 1$ stands for the upper threshold of the density.
We supplement system (2.1) with the following initial conditions

$$\rho^\varepsilon(x)|_{t=0} = \rho_0^\varepsilon(x), \quad u^\varepsilon(0)|_{t=0} = u_0^\varepsilon(x) \quad x \in \Omega, \tag{2.5}$$

where

$$0 \leq \rho_0^\varepsilon \leq 1, \quad \int_\Omega \rho_0^\varepsilon = M \tag{2.6}$$

and the Dirichlet boundary conditions

$$u^\varepsilon|_{\partial\Omega} = 0.$$

Our concern is to investigate the limit when ε tends to zero and justify that $(\rho^\varepsilon,\ u^\varepsilon,\ p_2^\varepsilon(\rho^\varepsilon))$ tends (in some sense) to (ρ, u, π) which satisfies the following free boundary problem

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0 \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) \\ \qquad -2\mathrm{div}(\mu(\rho)D(u)) - \nabla(\lambda(\rho)\mathrm{div}(u)) + \nabla p_1(\rho) + \nabla\pi = 0 \end{cases} \tag{2.7}$$

with

$$\begin{cases} 0 \leq \rho \leq 1, \\ \pi \geq 0, \\ (1-\rho)\pi = 0. \end{cases} \tag{2.8}$$

Such free boundary system has been derived by Lions and Masmoudi [11] where they considered $p_\gamma(\rho) = a\rho^\gamma$ with γ tending to $+\infty$. The same limit has been studied in [9] with viscosities depending on the density when some surface tension is included. However, such form of pressure does not guarantee the congestion constraint $0 \leq \rho^\gamma \leq 1$ for fixed γ, which is a problem for numerical investigation, as mentioned in the recent paper by Maury [12]. We will see that the pressure P defined in (2.4) plays a role of a barrier and implies that the constraint $0 \leq \rho^\varepsilon \leq 1$ is automatically satisfied for any $\varepsilon > 0$. This, however, asks for a special behavior of $P(\cdot)$ close to 1. An important example of such barrier used for instance in Self-Organized Hydrodynamics [6], [7] is of the form

$$p^\varepsilon(\rho^\varepsilon) = \varepsilon\left(\frac{1}{\frac{1}{\rho^\varepsilon} - 1}\right)^\gamma = \varepsilon\left(\frac{\rho^\varepsilon}{1-\rho^\varepsilon}\right)^\gamma.$$

3. One-dimensional case

The aim of this section is to prove the global-in-time existence of regular solutions to system (2.1) when $\Omega = [0, L]$ and μ, λ are positive constants. We will also perform the limit passage leading to the free boundary system (2.7–2.8). More precisely, we prove the following results

Theorem 3.1. *Let ε, μ, λ be fixed positive constants and let $(u^0,\rho^0) \in H^1_0(0,L) \times H^1(0,L)$ with $0 < \rho^0 < 1$. Assume that the singular pressure satisfies*

$$P(\rho) = (1-\rho)^{-\beta} \tag{3.1}$$

with $\beta, \gamma > 1$. Then there exists a regular solution $(u^\varepsilon, \rho^\varepsilon)$ to (2.1–2.4) such that

$$\|\rho^\varepsilon\|_{L^\infty(0,T;H^1(0,L))} + \|\rho^\varepsilon\|_{H^1(0,T;L^2(0,L))} \le c,$$
$$\|u^\varepsilon\|_{L^2(0,T;H^1_0(0,L))} + \|u^\varepsilon\|_{L^\infty(0,T;L^2(0,L))} \le c$$

uniformly with respect to ε and there exist constants c and $C(\varepsilon)$ s.t.

$$0 < c \le \rho^\varepsilon \le C(\varepsilon) < 1. \tag{3.2}$$

Remark 3.2. The full regularity and uniqueness of this solution for ε fixed can also be proved, see Theorem 3.4 below. However the proof relies on the estimates which strongly depend on ε.

Theorem 3.3. *Under assumptions of the previous theorem, there exists a subsequence already denoted $(\rho^\varepsilon, u^\varepsilon, \pi^\varepsilon)$ s.t.*

$$\begin{aligned} \rho^\varepsilon &\to \rho \quad \text{in } \mathcal{C}([0,T]\times[0,L]),\\ u^\varepsilon &\to u \quad \text{in } L^2(0,T;\mathcal{C}[0,L]),\\ \pi^\varepsilon = p_2^\varepsilon &\rightharpoonup \pi \quad \text{in } \mathcal{M}^+((0,T)\times(0,L)), \end{aligned} \tag{3.3}$$

where (u,ρ,π) satisfies (2.7–2.8).

3.1. Proof of Theorem 3.1

As mentioned before, Theorem 3.1 may be obtained as a corollary of a stronger result formulated below in Theorem 3.4 by use of Lagrangian coordinates.

We drop the index ε when no confusion can arise and we define

$$\mathrm{x} = \int_0^x \rho(\tau,s)ds, \qquad \tau = t. \tag{3.4}$$

Using (3.4) and denoting $\nu = 2\mu + \lambda$, system (2.1) may be transformed into the following one

$$\begin{cases} \rho_\tau + \rho^2 u_\mathrm{x} = 0 \\ u_\tau - \nu(\rho u_\mathrm{x})_\mathrm{x} + (p_1(\rho))_\mathrm{x} + (p_2^\varepsilon(\rho))_\mathrm{x} = 0 \end{cases} \tag{3.5}$$

with the Dirichlet boundary conditions

$$u|_{\mathrm{x}=0} = u|_{\mathrm{x}=M} = 0$$

and the initial data

$$\rho|_{\tau=0} = \rho_0, \quad u|_{\tau=0} = u_0, \quad \text{in} \quad [0,M], \tag{3.6}$$

such that

$$0 < \rho_0 < 1. \tag{3.7}$$

For the above system we will prove the following theorem.

Theorem 3.4. *Assume that* $(u^0, \rho^0) \in H_0^1(0,M) \times H^1(0,M)$ *and that* (3.7) *is satisfied. Then system (*3.5-3.6*) possesses a global unique solution* (ρ, u) *such that*

$$\begin{aligned} \rho \in L^\infty(0,T;H^1(0,M)),\ \ \rho_\tau \in L^2((0,T)\times(0,M)), \\ u_{\mathrm{x}} \in L^\infty(0,T;L^2(0,M)) \cap L^2(0,T;H^1(0,M)). \end{aligned} \tag{3.8}$$

Moreover there exist positive constants c_ρ, C_ρ *such that*

$$0 < c_\rho \leq \rho^\varepsilon \leq C_\rho(\varepsilon) < 1. \tag{3.9}$$

The local in time solvability of system (2.1-2.6) with monotone pressure, is a classical result, see f.i. [14]. Therefore, in order to show global in time existence it is enough to prove uniform in time estimates. This will be a purpose of the following paragraphs.

To deduce bounds on the density we first test $(3.5)_2$ by u and then by $\frac{\rho_{\mathrm{x}}}{\rho}$ and we sum the obtained expressions. This leads to

$$\sup_{\tau\in(0,T)} \int_0^M ((\log\rho)_{\mathrm{x}})^2(\tau)d\mathrm{x} + \int_0^T\int_0^M \left|(p_2^\varepsilon)'(\rho)\frac{(\rho_{\mathrm{x}})^2}{\rho}\right| d\mathrm{x}\ d\tau \leq c. \tag{3.10}$$

The lower bound is deduced from the control of the first integral while boundedness of the second integral clearly forces the upper bound, recall that $\beta > 1$.

It is then natural to expect that u is more regular than it follows from the basic energy estimate. Regularity (3.8) can be shown in a standard way, by testing $(3.5)_2$ by $-u_{\mathrm{xx}}$. The proof of uniqueness is then straightforward. □

Note that (3.8) allows to back to Eulerian coordinates, since $\partial_t h(t,x) = \partial_\tau h(\tau,\mathrm{x}) - u(\tau,\mathrm{x})\rho(\tau,\mathrm{x})\partial_{\mathrm{x}} h(\tau,\mathrm{x})$ and $\partial_x h(t,x) = \rho(\tau,\mathrm{x})\partial_{\mathrm{x}} h(\tau,\mathrm{x})$ which finishes the proof of Theorem 3.1. □

3.2. Recovering the two-phase system

In this subsection we prove Theorem 3.3. Let us first focus on establishing the estimates which are uniform with respect to ε. The basic energy equality for system (2.1) in the Eulerian coordinates reads

$$\frac{d}{dt}\int_0^L \left(\frac{1}{2}\rho^\varepsilon|u^\varepsilon|^2 + \rho^\varepsilon\big(e_1(\rho^\varepsilon) + e_2^\varepsilon(\rho^\varepsilon)\big)\right) + \nu\int_0^L |\partial_x u^\varepsilon|^2 = 0 \tag{3.11}$$

with $e_1(\rho^\varepsilon) = \frac{a}{\alpha-1}(\rho^\varepsilon)^{\alpha-1}$ and $e_2^\varepsilon(\rho^\varepsilon) = \int_0^{\rho^\varepsilon}\frac{p_2^\varepsilon(s)}{s^2}ds$. As in [11], the bound on $\rho e_2^\varepsilon(\rho^\varepsilon)$ does not provide bound for p_2^ε uniform with respect to ε. To solve this problem we perform a Bogovskii-type of estimate. Note that the arguments to conclude will be different than in [11].

Uniform estimate of the pressure. We test the momentum equation in (2.1) by $\phi(t,x) = \psi(t)\left(\int_0^x \rho^\varepsilon(t,y)dy - \overline{\rho^\varepsilon}\right)$, where $\overline{\rho^\varepsilon} = \frac{1}{L}\int_0^L \rho^\varepsilon(x,t)\ dx$ and $\psi(t) \in \mathcal{C}_0^\infty((0,L))$, we obtain

$$\int_0^T \psi \int_0^L (p_1 + p_2^\varepsilon)\,(\rho^\varepsilon - \overline{\rho^\varepsilon})\,dx\,dt = -\int_0^T \psi' \int_0^L \rho^\varepsilon u^\varepsilon \left(\int_0^x \rho^\varepsilon dy - \overline{\rho^\varepsilon}\right) dx\,dt$$
$$+ \int_0^T \psi \overline{\rho^\varepsilon} \int_0^L \rho^\varepsilon (u^\varepsilon)^2 dx\,dt + \nu \int_0^T \psi \int_0^L u_x (\rho^\varepsilon - \overline{\rho^\varepsilon}) dx\,dt$$

The r.h.s. is controlled thanks to (3.11) and (3.9), thus the l.h.s. is bounded uniformly with respect to ε. We then split the l.h.s. into two terms

$$I_1 + I_2 = \int_{\{\rho^\varepsilon < \frac{\overline{\rho}_0+1}{2}\}} p_2^\varepsilon\,(\rho^\varepsilon - \overline{\rho^\varepsilon})\,dx\,dt + \int_{\{\rho^\varepsilon \geq \frac{\overline{\rho}_0+1}{2}\}} p_2^\varepsilon\,(\rho^\varepsilon - \overline{\rho^\varepsilon})\,dx\,dt \leq c.$$

The integrant in I_1 is far away from singularity, thus it is bounded, whence the integrant in I_2 is larger than $\frac{1-\overline{\rho}_0}{2} p_2^\varepsilon$ which implies that $p_2^\varepsilon = \varepsilon p_2(\rho^\varepsilon)$ is bounded in $L^1((0,T)\times(0,L))$ uniformly with respect to ε. The same conclusion can be drawn for $p_2^\varepsilon \rho^\varepsilon$.

Passage to the limit $\varepsilon \to 0$. Using the Arzelà-Ascoli theorem we prove that

$$\rho^\varepsilon \to \rho \quad \text{in} \quad \mathcal{C}([0,T]\times[0,L]), \tag{3.12}$$

and (3.9) implies that $p_1(\rho^\varepsilon) \to p_1(\rho)$ strongly in $\mathcal{C}([0,T]\times[0,L])$.
Thanks to the uniform bounds on the pressure, up to a subsequence, we have

$$p_2^\varepsilon(\rho^\varepsilon) \rightharpoonup \pi, \quad \rho^\varepsilon p_2^\varepsilon(\rho^\varepsilon) \rightharpoonup \pi_1 \quad \text{in} \quad \mathcal{M}^+((0,T)\times(0,L)), \tag{3.13}$$

but thanks to (3.12) we may identify the second limit as

$$\rho^\varepsilon p_2^\varepsilon(\rho^\varepsilon) \rightharpoonup \rho\pi \quad \text{in} \quad \mathcal{M}^+((0,T)\times(0,L)). \tag{3.14}$$

Concerning the convergence of the velocity, by (3.11) we deduce that

$$u^\varepsilon \rightharpoonup u \quad \text{in } L^2(0,T;H_0^1(0,L)), \quad u^\varepsilon \rightharpoonup^* u \quad \text{in } L^\infty(0,T;L^2(0,L))$$

up to a subsequence. Therefore $\rho^\varepsilon u^\varepsilon \rightharpoonup \rho u$ in $L^4((0,T)\times(0,L))$. In addition, $(\rho^\varepsilon u^\varepsilon)_x$ is uniformly bounded in $L^2((0,T)\times(0,L))$. From the momentum equation and the L^1 bound on the pressure we can assert that $(\rho^\varepsilon u^\varepsilon)_t \in L^1(0,T;W^{-1,1}(0,L))$. Thus, an application of the generalized Aubin-Lions lemma [13]) yields

$$\rho^\varepsilon u^\varepsilon \to \rho u \quad \text{in } L^2(0,T;\mathcal{C}[0,L]).$$

Hence, (3.9) and (3.12) imply strong convergence of u^ε, as stated in (3.3).

In order to conclude it remains to prove that (ρ,π) satisfies constraint $(2.8)_3$. Due to singularity of the pressure, we cannot use the same argument as in [11]. Nevertheless, using (3.1) we may write

$$\varepsilon\rho^\varepsilon p_2^\varepsilon(\rho^\varepsilon) = -\varepsilon\frac{(\rho^\varepsilon)^\gamma}{(1-\rho^\varepsilon)^{\beta-1}} + \varepsilon p_2^\varepsilon(\rho^\varepsilon). \tag{3.15}$$

Letting $\varepsilon \to 0$, we see that the l.h.s. converges to $\rho\pi$ and the second term on the r.h.s. converges to π, on account of (3.14) and (3.13) respectively, while the middle term vanishes due to the uniform bound on p_2^ε. □

4. Multi-dimensional case

Let us now comment what are main differences in the proof for the multi-dimensional case, we refer to [?] for more details.

- In general, the global-in-time regular solutions are not known to exist, thus one needs to work with the weak solutions.
- The constraint $0 \leq \rho^\varepsilon \leq 1$ can be obtained for sufficiently strong singularity in the pressure (i.e. $\beta > 3$), otherwise it holds only for the limit.
- The strong convergence of density is not an automatic consequence of the a-priori estimates. For this reason, verification of (3.14) requires some compactness of the so-called *effective pressure.*

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References

[1] F. Berthelin. Existence and weak stability for a pressureless model with unilateral constraint. *Math. Model. Methods in Applied Sciences,* 249–272, (2002).

[2] F. Berthelin, D. Broizat. A model for the evolution of traffic jams in multilane. *Kinetic and Related Models,* Vol. 5, no. 4, (2012), 697–728.

[3] L. Boudin. A solution with bounded expansion rate to the model of viscous pressureless gases. *SIAM J. Math. Anal.,* 172–193, (2000).

[4] N.F. Carnahan, K.E. Starling. Equation of state for nonreacting rigid spheres. *J. Chem. Phys.,* 51:635–638, (1980).

[5] P. Degond, J. Hua. Self-Organized Hydrodynamics with congestion and path formation in crowds. *J. Comput. Phys.,* 237:299–319, (2013).

[6] P. Degond, J. Hua, L. Navoret. Numerical simulations of the Euler system with congestion constraint. *J. Comput. Phys.,* 230:8057–8088, (2011).

[7] E. Feireisl, H. Petzeltová, E. Rocca, G. Schimperna. Analysis of a phase-field model for two-phase compressible fluids. *Math. Models Methods Appl. Sci.* 20, no. 7, 1129–1160, (2010).

[8] S. Labbé, E. Maitre. A free boundary model for Korteweg fluids as a limit of barotropic compressible Navier-Stokes equations. *Methods Appl. Anal.,* 20, no. 2, 165–177, (2013).

[9] P.-L. Lions. *Mathematical Topics in Fluid Mechanics, Vol 2: Compressible Models.* Oxford University Press, New York, (1998).

[10] P.–L. Lions, N. Masmoudi. On a free boundary barotropic model. *Annales I.H.P.,* 373–410, (1999).

[11] B. Maury. Prise en compte de la congestion dans les modèles de mouvements de foules. Actes des Colloques Caen 2012-Rouen 2011.

[12] C. Perrin, E. Zatorska. A free/congested two-phase model from weak solutions to compressible Navier-Stokes equations. Forthcoming paper (2014).

[13] J. Simon. *Compact sets in the space $L^p(0, T; B)$.* Ann. Mat. Pura Appl. (4), 146:65–96, 1987.

[14] V.A. Solonnikov. *The solvability of the initial-boundary value problem for the equations of motion of a viscous compressible fluid.* Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), 56:128–142, 197, 1976.

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