Research Article

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On the steady flow of reactive gaseous mixture

Abstract: We present the study of systems of equations governing a steady flow of polyatomic, heat-conducting reactive gas mixture. It is shown that the corresponding system of PDEs admits a weak solution and renormalized solution to the continuity equation, provided the adiabatic exponent for the mixture γ is greater than $\frac{5}{3}$.

Keywords: Multicomponent flow, chemically reacting gas, steady compressible Navier–Stokes–Fourier system, weak solution

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1 Introduction

In this paper we consider a system of equations modelling the steady flow of compressible, heat-conducting, n-component mixtures, undergoing reversible chemical reactions in a bounded domain $\Omega \subset \mathbb{R}^3$. These are n reaction-diffusion equations coupled to the Navier–Stokes–Fourier system describing the fluid motion. The applications of such systems are numerous, especially in engineering, like spatial vehicle reentry, crystal and polymers growth, combustion or atmospheric pollution [12], but they may also be used to describe population or the chemotaxis models [4, 11]. This is a strong motivation for investigating the mathematical structure and properties of the corresponding PDEs. Our goal is to extend the existence theory [20, 25] to the case of heat-conducting mixtures with strong cross-diffusion.

The motion of gaseous mixture can be described by a system of equations governing the total mass density $\varrho = \varrho(x)$, the velocity vector field $\mathbf{u} = \mathbf{u}(x)$, the absolute temperature $\vartheta = \vartheta(x)$ and the species concentrations $Y_k(x)$, $k \in \{1, \ldots, n\}$. These equations express the conservation of the total mass, momentum, total energy and species masses. They may be written as

$$\begin{cases} \operatorname{div}(\varrho\mathbf{u}) = 0, \\ \operatorname{div}(\varrho\mathbf{u} \otimes \mathbf{u}) + \operatorname{div}\mathbf{S} + \nabla\pi = \varrho\mathbf{f}, \\ \operatorname{div}(\varrho E\mathbf{u}) + \operatorname{div}(\pi\mathbf{u}) + \operatorname{div}\mathbf{Q} + \operatorname{div}(\mathbf{S}\mathbf{u}) = \varrho\mathbf{f} \cdot \mathbf{u}, \\ \operatorname{div}(\varrho Y_k\mathbf{u}) + \operatorname{div}\mathbf{F}_k = m_k\omega_k, \quad k \in \{1, \ldots, n\}. \end{cases} \tag{1.1}$$

In the above equations, $\mathbf{S}$ denotes the viscous tensor (it has the opposite sign to the viscous stress tensor also considered in continuum mechanics), π the internal pressure of the fluid, $\mathbf{f}$ the external force, E the specific total energy, $\mathbf{Q}$ the heat flux, ω_k the molar production rate of the k-th species, $\mathbf{F}_k$ the diffusion flux of the k-th species, and m_k the molar mass of the k-th species.

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The fundamental difficulty in such type of systems is the coupling between the fluid mechanics part, governed by the Navier–Stokes–Fourier system, with the species reaction-diffusion equations. The coupling appears through the form of the pressure, which may depend on the species concentrations, and through the heat-flux which, in contrast to the single-component flows, includes a term representing the transfer of energy due to species molecular diffusion.

Here we focus on the second type of coupling, while we assume that the molar masses are comparable, i.e.

$$m_1 = \cdots = m_n = 1, \tag{1.2}$$

which leads to the pressure independent of the species concentrations. Assumption (1.2) is fulfilled for the mixtures of *isomers* or in the state-to-state fluid models where each quantum state is a separate pseudo species [2]. A similar assumption was made in [10], where the existence of variational weak entropy solutions was shown, yet for the diagonal Fick diffusion. This approximation, however, does not take into account the cross-effects that are well known to play an important role in the flows of multicomponent fluids. In the recent papers [25, 27] the model of isothermal flow with the Fick diffusion and pressure depending on the species concentration was considered. However, it turns out that this assumption leads to inconsistency with the *Second Law of Thermodynamics* when the heat-conductivity is taken into account.

Reaction-diffusion equations with more complex diffusion and even their coupling to the incompressible Navier–Stokes equations were investigated using many different approaches [1, 3, 5, 12, 13, 15]. Our goal in this paper is to provide analogue existence results with no restriction on the size of data but in the compressible setting. So far, such results were available only for systems with viscosity coefficients satisfying the so-called *BD relation* and some further restrictions on the form of the pressure [19, 26].

In our case, due to relatively weak coupling between the Navier–Stokes equations and the rest of system (1.1), a part of the techniques is a combination of methods developed first by Lions [14] and Feireisl [8] for evolutionary barotropic flows and by Mucha, Pokorný [16] and Novotný, Pokorný [20] for the stationary Navier–Stokes–Fourier system. The main difference and the biggest difficulty in the present paper concerns much more complex forms of the entropy inequality, which is a source of majority of a-priori estimates. Therefore, the approximation scheme had to be considerably modified in comparison to the single-fluid case considered in [20]. Some ideas for construction of the solution to the subsystem of species reaction-diffusion equations can be already found in our previous work [17]. Here, however, we had to extend them by adding further regularizations necessary to handle the general form of the diffusion matrix.

The outline of the paper is the following. In the next section we introduce the main assumptions on the parameters of our model. In Section 3 we present the definition of the weak solution and formulate the main result: existence of a weak solution under certain restriction on the parameters of our model. Sections 4 and 5 contain the description of the approximate scheme. In Section 6 we prove existence of a solution to the full approximation. In Sections 7 and 8 we pass to the limit with all regularizing parameters to get existence of a solution to the original problem.

2 Formulation of the problem

We consider system (1.1) supplemented by the no-slip boundary conditions

$$\mathbf{u}|_{\partial\Omega} = \mathbf{0}, \tag{2.1}$$

together with

$$\mathbf{F}_k \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{2.2}$$

and the Robin boundary condition for the heat flux:

$$-\mathbf{Q} \cdot \mathbf{n} + L(\vartheta - \vartheta_0) = 0. \tag{2.3}$$

The last condition means that the heat flux through the boundary is proportional to the difference of the temperature inside Ω and the known external temperature ϑ_0.

We assume that the total mass of the mixture is given by

$$\int_\Omega \varrho \, dx = M > 0.$$

The mass fractions Y_k, $k \in \{1, \dots, n\}$ are defined by $Y_k = \frac{\varrho_k}{\varrho}$, where ϱ_k are the species densities and $\sum_{k=1}^n \varrho_k = \varrho$. Thus, by definition, the mass fractions satisfy

$$\sum_{k=1}^n Y_k = 1.$$

The consistency with the principle of mass conservation requires that the diffusion fluxes and the species production rates satisfy

$$\sum_{k=1}^n \mathbf{F}_k = \mathbf{0}, \quad \sum_{k=1}^n \omega_k = 0, \tag{2.4}$$

keeping in mind that the species molar masses are equal (see (1.2)).

2.1 Fundamental thermodynamic relations

We consider a pressure $\pi = \pi(\varrho, \vartheta)$ of the form

$$\pi(\varrho, \vartheta) = \pi_c(\varrho) + \pi_m(\varrho, \vartheta), \tag{2.5}$$

where π_m obeys the Boyle law

$$\pi_m = \sum_{k=1}^n \varrho Y_k \vartheta = \varrho \vartheta, \tag{2.6}$$

i.e., it represents the pressure for an ideal mixture of n species, whose molar masses are equal to 1 and we take, without loss of generality, the gaseous constant $R = 1$.

The first component of (2.5), π_c, is the so-called *cold pressure* or the *barotropic correction*:

$$\pi_c = \varrho^\gamma, \quad \gamma > 1.$$

The specific total energy E is a sum of the specific kinetic and specific internal energies:

$$E(\varrho, \mathbf{u}, \vartheta, \varrho_1, \dots, \varrho_n) = \frac{1}{2}|\mathbf{u}|^2 + e(\varrho, \vartheta, Y_1, \dots, Y_n),$$

where the latter, similarly to the pressure, consists of two components

$$e = e_c(\varrho) + e_m(\vartheta, Y_1, \dots, Y_n).$$

The cold energy e_c and the ideal gas mixture energy e_m are given by

$$e_c = \frac{1}{\gamma - 1}\varrho^{\gamma-1}, \quad e_m = \sum_{k=1}^n Y_k e_k = \vartheta \sum_{k=1}^n c_{vk} Y_k,$$

where c_{vk} is the mass constant-volume specific heat. The constant-pressure specific heat, denoted by c_{pk}, is related (under assumption (1.2)) to c_{vk} in the following way:

$$c_{pk} = c_{vk} + 1, \tag{2.7}$$

and both c_{vk} and c_{pk} are assumed to be constant (but possibly different for each constituent).

In agreement with the second law of thermodynamics, there exists a differentiable function called the *specific entropy of the mixture* $s(\varrho, \vartheta, Y_1, \dots, Y_n)$ that can be expressed in terms of the partial specific entropies $s_k = s_k(\varrho, \vartheta, Y_k)$ of the k-th species:

$$s = \sum_{k=1}^n Y_k s_k. \tag{2.8}$$

The differential of entropy is related to the differential of energy, total density and mass fractions by the Gibbs formula

$$\vartheta \mathbf{D}s = \mathbf{D}e + \pi \mathbf{D}\left(\frac{1}{\varrho}\right) - \sum_{k=1}^{n} g_k \mathbf{D}Y_k, \tag{2.9}$$

with the Gibbs functions

$$g_k = h_k - \vartheta s_k, \tag{2.10}$$

where $h_k = h_k(\vartheta)$ and $s_k = s_k(\varrho, \vartheta, Y_k)$ denote the specific enthalpy and the specific entropy of the k-th species, respectively, with the following exact forms:

$$h_k = c_{pk}\vartheta, \quad s_k = c_{vk} \log \vartheta - \log \varrho - \log Y_k.$$

The cold pressure and the cold energy correspond to isentropic processes, therefore using (2.9) one can derive an equation for the specific entropy s:

$$\text{div}(\varrho s \mathbf{u}) + \text{div}\left(\frac{\mathbf{Q}}{\vartheta} - \sum_{k=1}^{n} \frac{g_k}{\vartheta} \mathbf{F}_k\right) = \sigma, \tag{2.11}$$

where σ is the entropy production rate

$$\sigma = -\mathbf{S} : \frac{\nabla \mathbf{u}}{\vartheta} - \frac{\mathbf{Q} \cdot \nabla \vartheta}{\vartheta^2} - \sum_{k=1}^{n} \mathbf{F}_k \cdot \nabla\left(\frac{g_k}{\vartheta}\right) - \frac{\sum_{k=1}^{n} g_k \omega_k}{\vartheta}. \tag{2.12}$$

2.2 The form of transport fluxes

The viscous tensor $\mathbf{S}$ is determined by the Newton rheological law as

$$\mathbf{S} = \mathbf{S}(\vartheta, \nabla \mathbf{u}) = -\mu\left[\nabla \mathbf{u} + (\nabla \mathbf{u})^t - \frac{2}{3} \text{div } \mathbf{u}\mathbf{I}\right] - \nu(\text{div } \mathbf{u})\mathbf{I}, \tag{2.13}$$

where $\mu = \mu(\vartheta) > 0$ and $\nu = \nu(\vartheta) \geq 0$ are the shear and bulk viscosity coefficients, respectively, and $\mathbf{I}$ is the identity matrix.

The heat flux $\mathbf{Q}$ consists of two terms (we already neglect the *Soret* and the *Dufour* effects) representing the transfer of energy due to the species molecular diffusion and the Fourier law, respectively:

$$\mathbf{Q} = \sum_{k=1}^{n} h_k \mathbf{F}_k + \mathbf{q}, \quad \mathbf{q} = -\kappa \nabla \vartheta, \tag{2.14}$$

where $\kappa = \kappa(\vartheta) > 0$ is the thermal conductivity coefficient.

The diffusion flux of the k-th species $\mathbf{F}_k$ is given by

$$\mathbf{F}_k = -\varrho Y_k \sum_{l=1}^{n} D_{kl}^{\varrho} \nabla Y_l,$$

where $D_{kl}^{\varrho} = D_{kl}^{\varrho}(\varrho, \vartheta, Y_1, \ldots, Y_n)$, $k, l = 1, \ldots, n$ are the multicomponent diffusion coefficients.

The diffusion matrix. The coefficients ϱD_{kl}^{ϱ} depend only on ϑ and $Y_1, \ldots, Y_n$, see for instance [12], therefore we introduce another matrix

$$(D_{kl})_{k,l=1}^{n} = \varrho(D_{kl}^{\varrho})_{k,l=1}^{n} = (D_{kl}(\vartheta, Y_1, \ldots, Y_n))_{k,l=1}^{n}.$$

The main properties of the diffusion matrix D, discussed in [12, Chapter 7], are

$$D = D^t, \quad N(D) = \mathbb{R}\vec{Y}, \quad R(D) = \vec{Y}^{\perp}, \quad D \text{ is positive semidefinite over } \mathbb{R}^n, \tag{2.15}$$

where we assumed that $\vec{Y} = (Y_1, \ldots, Y_n)^t > 0$. Here, $N(D)$ denotes the nullspace of matrix D, $R(D)$ denotes its range, $U = (1, \ldots, 1)^t$, and $U^{\perp}$ denotes the orthogonal complement of $\mathbb{R}U$.

Furthermore, we assume that the matrix D is homogeneous of a non-negative order with respect to $Y_1, \ldots, Y_n$ and that D_{ij} are differentiable functions of $\vartheta, Y_1, \ldots, Y_n$ for any $i, j \in \{1, \ldots, n\}$ such that

$$|D_{ij}(\vartheta, \vec{Y})| \leq C(\vec{Y})(1 + \vartheta^a)$$

for some $a \geq 0$.

Remark 2.1. As a consequence of (2.15) the matrix D is positive definite over $U^\perp$. This property corresponds to the positivity of the entropy production rate associated with the diffusive process, see [24]. Indeed, according to above definitions, σ may be rewritten in the form

$$\sigma = -\mathbf{S}(\vartheta, \nabla \mathbf{u}) : \frac{\nabla \mathbf{u}}{\vartheta} + \frac{\kappa |\nabla \vartheta|^2}{\vartheta^2} - \sum_{k=1}^{n} \mathbf{F}_k \cdot \nabla(\log p_k) - \frac{\sum_{k=1}^{n} g_k \omega_k}{\vartheta},$$

where we denote $p_k = \varrho Y_k \vartheta$. Let us investigate the structure of the third term, we have

$$
\begin{aligned}
-\sum_{k=1}^{n} \mathbf{F}_k \cdot \nabla(\log p_k) &= -\sum_{k=1}^{n} \frac{\mathbf{F}_k}{p_k} \cdot \nabla p_k \\
&= -\sum_{k=1}^{n} \mathbf{F}_k \cdot \left(\frac{\nabla Y_k}{Y_k} + \frac{\nabla(\varrho \vartheta)}{\varrho \vartheta} \right) \quad \text{(due to (2.4))} \\
&= \sum_{l,k=1}^{n} D_{kl} \nabla Y_l \cdot \nabla Y_k \geq c \sum_{k=1}^{n} |\nabla Y_k|^2.
\end{aligned}
\tag{2.16}
$$

The transport coefficients. The coefficients μ, ν, κ are continuous functions of temperature and the following growth conditions are imposed:

$$\underline{\mu}(1 + \vartheta) \leq \mu(\vartheta) \leq \overline{\mu}(1 + \vartheta), \quad 0 \leq \nu(\vartheta) \leq \overline{\nu}(1 + \vartheta), \quad \underline{\kappa}(1 + \vartheta^m) \leq \kappa(\vartheta) \leq \overline{\kappa}(1 + \vartheta^m),$$

with $m > 0$ and the positive constants $\underline{\mu}, \overline{\mu}, \overline{\nu}, \underline{\kappa}, \overline{\kappa}$.

The species production rates. We assume that the species production rates are smooth bounded functions of $(\varrho, \vartheta, Y_1, \ldots, Y_n)$ such that

$$\omega_k(\varrho, \vartheta, Y_1, \ldots, Y_n) \geq 0 \quad \text{whenever } Y_k = 0.
\tag{2.17}$$

Next, in accordance with the second law of thermodynamics we assume that

$$-\sum_{k=1}^{n} g_k \omega_k \geq 0,
\tag{2.18}$$

where g_k are specified in (2.10). Note that thanks to this inequality, (2.16), together with (2.13) and (2.14), yields that the entropy production rate defined in (2.12) is non-negative.

2.3 Notation

The matrices from $\mathbb{R}^{n \times n}$ are denoted by capital letters, the vectors from $\mathbb{R}^n$ are denoted by $\vec{}$. The vectors from $\mathbb{R}^3$ and the tensors from $\mathbb{R}^{3 \times 3}$ are denoted by small and capital bold letters, respectively. We use a generic constant denoted by C which may change from line to line. When it is important, its dependence of parameters will be indicated in parentheses.

We work in the framework of Sobolev and Lebesgue spaces denoted by $W^{m,p}(\Omega)$, $m \in \mathbb{N}$, and $L^p(\Omega)$, $p \geq 1$, respectively, endowed with the standard norms. For brevity we will write

$$\|u\|_{W^{m,p}(\Omega)} = \|u\|_{m,p}, \quad \|u\|_{L^p(\Omega)} = \|u\|_p, \quad \|u\|_{L^p(\partial\Omega)} = \|u\|_{p,\partial\Omega}$$

independently whether u is a vector or scalar. By $C_0^\infty(\Omega)$ we denote the space of C^∞ functions on Ω with zero value at the boundary $\partial\Omega$.

3 Weak solutions, main result

We are now in a position to formulate the definition of weak solutions for our system.

Definition 3.1. We say the set of functions $(\varrho, \mathbf{u}, \vartheta, \vec{Y})$ is a weak solution to problem (1.1)–(2.3), (2.5)–(2.14), (2.17), (2.18) provided $\varrho \geq 0$ a.e. in Ω, $\varrho \in L^\gamma(\Omega)$, $\int_\Omega \varrho \, dx = M$, $\mathbf{u} \in W_0^{1,2}(\Omega)$, $\varrho|\mathbf{u}|^3 \in L^1(\Omega)$, $Y_k \geq 0$ a.e. in Ω, $\vec{Y} \in W^{1,2}(\Omega)$, $\mathbf{F}_k \cdot \mathbf{n}|_{\partial\Omega} = 0$, and $\sum_{k=1}^n Y_k = 1$ a.e. in Ω, $\vartheta > 0$, a.e. in Ω, $\vartheta^m \nabla \vartheta \in L^1(\Omega)$, $\vartheta \in L^1(\partial\Omega)$, and the following integral equalities hold:

- the weak formulation of the continuity equation

$$\int_\Omega \varrho \mathbf{u} \cdot \nabla \psi \, dx = 0$$

  for any test function $\psi \in C^\infty(\overline{\Omega})$;

- the weak formulation of the momentum equation

$$-\int_\Omega (\varrho(\mathbf{u} \otimes \mathbf{u}) : \nabla \boldsymbol{\phi} + \mathbf{S} : \nabla \boldsymbol{\phi}) \, dx - \int_\Omega \pi \operatorname{div} \boldsymbol{\phi} \, dx = \int_\Omega \varrho \mathbf{f} \cdot \boldsymbol{\phi} \, dx$$

  for any test function $\boldsymbol{\phi} \in C_0^\infty(\Omega)$;

- the weak formulation of the species equations

$$-\int_\Omega Y_k \varrho \mathbf{u} \cdot \nabla \psi \, dx - \int_\Omega \mathbf{F}_k \cdot \nabla \psi \, dx = \int_\Omega \omega_k \psi \, dx$$

  for any test function $\psi \in C^\infty(\overline{\Omega})$ and for all $k = 1, \ldots, n$;

- the weak formulation of the total energy balance

$$-\int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e \right) \mathbf{u} \cdot \nabla \psi \, dx + \int_\Omega \kappa \nabla \vartheta \cdot \nabla \psi \, dx - \int_\Omega \left(\sum_{k=1}^n h_k \mathbf{F}_k \right) \cdot \nabla \psi \, dx$$

$$= \int_\Omega \varrho \mathbf{f} \cdot \mathbf{u} \psi \, dx + \int_\Omega (\mathbf{S}\mathbf{u}) \cdot \nabla \psi \, dx + \int_\Omega \pi \mathbf{u} \cdot \nabla \psi \, dx - \int_{\partial\Omega} L(\vartheta - \vartheta_0) \psi \, dS$$

  for any test function $\psi \in C^\infty(\overline{\Omega})$.

Remark 3.2. Indeed, there exists a more general definition of a solution to our problem which is based on replacing the weak formulation of the total energy balance by the entropy inequality and the global total energy balance; see [20] or [21] for more details. The reason for this is that it allows to prove the existence of those variational entropy solutions for a larger range of parameters. Nonetheless, we prefer to stay here with a (more direct) definition of weak solutions to our problem, even for the prize of more restrictive conditions on m and γ.

Remark 3.3. The kinetic energy balance is (at least formally) nothing but the momentum equation multiplied by $\mathbf{u}$ and integrated over Ω, thus we may write the balance of the internal energy in the form

$$\operatorname{div}(\varrho e \mathbf{u}) + \operatorname{div}\left(\sum_{k=1}^n h_k \mathbf{F}_k - \kappa \nabla \vartheta \right) + \pi \operatorname{div} \mathbf{u} + \mathbf{S} : \nabla \mathbf{u} = 0.$$

However, the balance of the total energy is equivalent to the balance of the internal energy and the momentum equation only for sufficiently regular solutions, but it might be not true for weak solutions introduced above.

We will also use the notion of the renormalized solution to the continuity equation.

Definition 3.4. Let $\mathbf{u} \in W_{\mathrm{loc}}^{1,2}(\Omega)$ and $\varrho \in L_{\mathrm{loc}}^{6/5}(\Omega)$ solve

$$\operatorname{div}(\varrho \mathbf{u}) = 0$$

in the sense of distributions on Ω, then the pair $(\varrho, \mathbf{u})$ is called a renormalized solution to the continuity equation, if

$$\text{div}(b(\varrho)\mathbf{u}) + (\varrho b'(\varrho) - b(\varrho))\,\text{div}\,\mathbf{u}\,dx = 0,$$

in the sense of distributions on Ω, for all $b \in W^{1,\infty}(0, \infty) \cap C^1([0, \infty))$, such that $sb'(s) \in L^\infty(0, \infty)$.

The main theorem of this paper reads as follows.

Theorem 3.5. *Let $\gamma > \frac{5}{3}$, $M > 0$, $m > 1$, $a < \frac{3m-2}{2}$. Let $\Omega \in C^2$. Then there exists at least one weak solution to our problem above in the sense of Definition 3.1. Moreover, $(\varrho, \mathbf{u})$ is the renormalized solution to the continuity equation in the sense of Definition 3.4.*

4 First level of approximation

This section is devoted to the main level of approximation on which all essential approximation parameters appear: $\varepsilon > 0$ indicating additional dissipation and relaxation in the continuity equation as well as in the species mass balance equations, $\lambda > 0$ providing elliptic regularization in the species equations while written in terms of entropic variables, and $\delta > 0$ improving the integrability of the density and providing the bound from below for the temperature. Combining ideas from [10, 20, 25] we consider the following approximative system. We look for $\varrho \in W^{2,q}(\Omega) \cap L^\beta(\Omega)$ for some $q \geq \frac{6}{5}$, $\mathbf{u} \in W_0^{1,2}(\Omega)$, $\vec{Y} \in W^{1,2}(\Omega)$, $\vartheta \in W^{1,2}(\Omega) \cap L^{3B}(\Omega) \cap L^{2B}(\partial\Omega)$ with $\varrho \geq 0$, $Y_k \geq 0$ and $\vartheta > 0$ a.e. in Ω satisfying the following system:

- The approximate continuity equation

$$\varepsilon\varrho + \text{div}(\varrho\mathbf{u}) = \varepsilon\Delta\varrho + \varepsilon\overline{\varrho}, \quad \nabla\varrho \cdot \mathbf{n}|_{\partial\Omega} = 0 \tag{4.1}$$

is satisfied pointwisely and we require $\overline{\varrho} > 0$, $\int_\Omega \overline{\varrho}\,dx = M$, thus we may take $\overline{\varrho} = \frac{M}{|\Omega|}$.

- The weak formulation of the approximate momentum equation

$$\int_\Omega \left(\frac{1}{2}\varrho\mathbf{u} \cdot \nabla\mathbf{u} \cdot \boldsymbol{\phi} - \frac{1}{2}\varrho(\mathbf{u} \otimes \mathbf{u}) : \nabla\boldsymbol{\phi} - \mathbf{S} : \nabla\boldsymbol{\phi}\right)dx - \int_\Omega (\pi + \delta\varrho^\beta + \delta\varrho^2)\,\text{div}\,\boldsymbol{\phi}\,dx = \int_\Omega \varrho\mathbf{f} \cdot \boldsymbol{\phi}\,dx \tag{4.2}$$

is satisfied for each $\boldsymbol{\phi} \in C_0^\infty(\Omega)$.

- The weak formulation of the approximate species balance equations $(k = 1, 2, \ldots, n)$

$$\int_\Omega \varepsilon Y_k\varrho\psi\,dx - \int_\Omega Y_k\varrho\mathbf{u} \cdot \nabla\psi\,dx - \int_\Omega \widehat{\mathbf{F}}_k \cdot \nabla\psi\,dx + \lambda\int_\Omega (\nabla\log Y_k \cdot \nabla\psi + \log Y_k\psi)\,dx$$

$$= \int_\Omega \left[\omega_k\psi - \varepsilon\varrho\nabla Y_k \cdot \nabla\psi + \varepsilon\,\text{div}(Y_k\nabla\varrho)\psi - \varepsilon\nabla Y_k \cdot \nabla\psi + \varepsilon\overline{\varrho}_k\psi\right]dx \tag{4.3}$$

is satisfied for any $\psi \in C^\infty(\overline{\Omega})$, where $\overline{\varrho}_k > 0$, $k = 1, \ldots, n$ satisfy $\sum_{k=1}^n \overline{\varrho}_k = \overline{\varrho}$; for instance, we take $\overline{\varrho}_k = \frac{\overline{\varrho}}{n} = \frac{M}{n|\Omega|}$. Moreover,

$$\widehat{\mathbf{F}}_k = -Y_k\sum_{l=1}^n \widehat{D}_{kl}(\vartheta, \vec{Y})\nabla Y_l, \quad \widehat{D}_{kl}(\vartheta, \vec{Y}) = \frac{1}{(\sigma_Y + \varepsilon)^r}D_{kl}(\vartheta, \vec{Y}) \tag{4.4}$$

for suitable $r \geq 1$ (connected with the order of the homogeneity of $D(\cdot, \vec{Y})$) and $\sigma_Y = \sum_{k=1}^n Y_k$.

- The weak formulation of the approximate total energy equation

$$-\int_\Omega \left[\varrho e + \frac{1}{2}\varrho|\mathbf{u}|^2 + (\pi + \delta\varrho^\beta + \delta\varrho^2)\right]\mathbf{u} \cdot \nabla\psi\,dx - \int_\Omega (\mathbf{S}\mathbf{u} \cdot \nabla\psi + \delta\vartheta^{-1}\psi)\,dx$$

$$+ \int_\Omega \kappa_\delta\frac{\varepsilon + \vartheta}{\vartheta}\nabla\vartheta \cdot \nabla\psi\,dx + \int_{\partial\Omega} \left[(L + \delta\vartheta^{B-1})(\vartheta - \vartheta_0) + \varepsilon\log\vartheta + \lambda\vartheta^{\frac{B}{2}}\log\vartheta\right]\psi\,dS$$

$$+ \sum_{k=1}^n c_{vk}\int_\Omega \left[-\widehat{\mathbf{F}}_k \cdot \nabla\psi + \vartheta(\varepsilon(\varrho + 1)Y_k + \lambda)\frac{\nabla Y_k}{Y_k} \cdot \nabla\psi\right]dx$$

$$= \int_\Omega \varrho \mathbf{f} \cdot \mathbf{u} \psi \, \mathrm{d}x + \frac{\delta}{\beta - 1} \int_\Omega (\varepsilon \beta \overline{\varrho} \varrho^{\beta-1} \psi + \varrho^\beta \mathbf{u} \cdot \nabla \psi - \varepsilon \beta \varrho^\beta \psi) \, \mathrm{d}x$$

$$+ \delta \int_\Omega (2\varepsilon \overline{\varrho} \varrho \psi + \varrho^2 \mathbf{u} \cdot \nabla \psi - \varepsilon \beta \varrho^2 \psi) \, \mathrm{d}x \tag{4.5}$$

is satisfied for any $\psi \in C^\infty(\overline{\Omega})$, where $\kappa_\delta = \kappa + \delta \vartheta^B + \delta \vartheta^{-1}$.

Remark 4.1. Above β and B are some positive, large enough numbers that will be determined in the course of the proof.

For this system we will prove the following result.

Theorem 4.2. *Let $0 < \lambda \ll \varepsilon \ll \delta$, β and B be sufficiently large positive numbers. Let $\Omega \in C^2$. Then there exists a solution to the approximate system in the sense specified above.*

The existence of solutions to the above system will be proven below, by introducing another artificial level of approximation.

5 Full approximation

Now, our task is to construct a regular solution defined in the previous section, i.e. to prove Theorem 4.2. For this purpose we introduce two new parameters: the dimension $N \in \mathbb{N}$ of the Galerkin approximation in the momentum equation, and the regularization η of the coefficients in the temperature and the momentum equations. Note that since the species molar masses are assumed to be equal, the species concentrations do not appear in the momentum equation. This allows to solve more or less separately the Navier–Stokes–Fourier system and the reaction-diffusion system and to combine them via a suitable fixed-point theorem. In order to consider the former, we used the slightly modified strategy from [20] by Novotný and Pokorný. The existence of weak solutions to a similar system of reaction-diffusion equations in the evolutionary case is due to Mucha, Pokorný and Zatorska [17]. In [18, 19] the same authors investigated also the coupling between the two systems, note however, that the assumptions on the form of the fluxes and pressure are now quite different. Nevertheless, we use some of their arguments here.

In the basic level of approximation we look for the set of functions $\{\varrho, \mathbf{u}, \vec{Y}, \vartheta\}$ satisfying the following system:

- The approximate continuity equation

$$\varepsilon \varrho + \mathrm{div}(\varrho \mathbf{u}) = \varepsilon \Delta \varrho + \varepsilon \overline{\varrho}, \quad \nabla \varrho \cdot \mathbf{n}|_{\partial\Omega} = 0 \tag{5.1}$$

  is satisfied pointwisely.
- The Galerkin approximation for the momentum equation

$$\int_\Omega \left(\frac{1}{2} \varrho \mathbf{u} \cdot \nabla \mathbf{u} \cdot \mathbf{w} - \frac{1}{2} \varrho (\mathbf{u} \otimes \mathbf{u}) : \nabla \mathbf{w} - \mathbf{S}_\eta : \nabla \mathbf{w} \right) \mathrm{d}x - \int_\Omega (\pi + \delta \varrho^\beta + \delta \varrho^2) \, \mathrm{div} \, \mathbf{w} \, \mathrm{d}x = \int_\Omega \varrho \mathbf{f} \cdot \mathbf{w} \, \mathrm{d}x \tag{5.2}$$

  is satisfied for each test function $\mathbf{w} \in X_N$, $X_N = \mathrm{span}\{\mathbf{w}_i\}_{i=1}^N \subset W_0^{1,2}(\Omega)$, where $\{\mathbf{w}_i\}_{i=1}^N$ are the first N eigenfunctions of the Laplace operator with Dirichlet boundary condition; indeed, $\mathbf{u} \in X_N$. In (5.2) we denoted

$$\mathbf{S}_\eta = -\frac{\mu_\eta(\vartheta)}{1 + \eta \vartheta} \left[\nabla \mathbf{u} + (\nabla \mathbf{u})^t - \frac{2}{3} \, \mathrm{div} \, \mathbf{u} \mathbf{I} \right] - \frac{\nu_\eta(\vartheta)}{1 + \eta \vartheta} (\mathrm{div} \, \mathbf{u}) \mathbf{I}. \tag{5.3}$$

- The approximate species mass balance equations

$$\mathrm{div} \, \mathbf{J}_k = R_k, \quad \mathbf{J}_k \cdot \mathbf{n}|_{\partial\Omega} = 0 \tag{5.4}$$

are satisfied pointwisely. Here, we denoted

$$\mathbf{J}_k = -\sum_{l=1}^{n} Y_k Y_l \widehat{D}_{kl}(\vartheta, \vec{Y}) \nabla Y_l / Y_l - (\varepsilon(\varrho + 1)Y_k + \lambda)\nabla Y_k / Y_k, \tag{5.5}$$

$$R_k = \omega_k(\varrho, \vartheta, \vec{Y}) + \varepsilon \overline{\varrho}_k - \varepsilon Y_k \varrho - \text{div}(Y_k \varrho \mathbf{u}) + \varepsilon \, \text{div}(Y_k \nabla \varrho) - \lambda \log Y_k, \tag{5.6}$$

and $\widehat{D}_{kl}(\vartheta, \vec{Y})$ was defined in (4.4).

- The approximate internal energy balance

$$-\text{div}\left(\kappa_{\delta,\eta} \frac{\varepsilon + \vartheta}{\vartheta} \nabla \vartheta\right) = -\text{div}(\varrho e \mathbf{u}) - \pi \, \text{div}\, \mathbf{u} + \frac{\delta}{\vartheta} - \mathbf{S}_\eta : \nabla \mathbf{u}$$
$$+ \delta\varepsilon(\beta\varrho^{\beta-2} + 2)|\nabla\varrho|^2 - \text{div}\left(\vartheta \sum_{k=1}^{n} c_{vk} J_k\right), \tag{5.7}$$

with the boundary condition

$$\kappa_{\delta,\eta} \frac{\varepsilon + \vartheta}{\vartheta} \nabla \vartheta \cdot \mathbf{n}|_{\partial\Omega} + (L + \delta\vartheta^{B-1})(\vartheta - \vartheta_0^\eta) + \varepsilon \log \vartheta + \lambda\vartheta^{\frac{B}{2}} \log \vartheta = 0, \tag{5.8}$$

is satisfied pointwisely and ϑ_0^η stands for a smooth, strictly positive approximation of ϑ_0.

Remark 5.1. Note that in fact we expect to have in (5.7) in the last term c_{pk} instead of c_{vk}, which can be easily observed by setting $\lambda = \varepsilon = \delta = 0$. However, since $\sum_{k=1}^{n} \mathbf{F}_k = \sum_{k=1}^{n} \widehat{\mathbf{F}}_k = \mathbf{0}$, both c_{vk} and c_{pk} lead after the limit passages to the same. Hence we prefer to keep c_{vk} in the approximate scheme.

In the above system $\kappa_{\delta,\eta}$, μ_η, ν_η are regularizations of functions κ_δ, μ, and ν extended by constants $\kappa_\delta(0)$, $\mu(0)$ and $\nu(0)$ to the negative half-line.

The existence of solutions is formulated in the following theorem.

Theorem 5.2. *Let δ, ε, λ and η be positive numbers and N be a positive integer. Let $\Omega \in C^2$. Then there exists a solution to system (5.1)–(5.4) such that $\varrho \in W^{2,q}(\Omega)$ for all $q < \infty$, $\varrho \geq 0$ in Ω, $\int_\Omega \varrho \, dx = M$, $\mathbf{u} \in X_N$, $\vec{Y} \in W^{1,2}(\Omega)$ with $\log Y_k \in W^{2,q}(\Omega)$ for all $q < \infty$, $Y_k \geq 0$ a.e. in Ω and $\vartheta \in W^{2,q}(\Omega)$ for all $q < \infty$, $\vartheta \geq C(N) > 0$.*

The strategy of the proof is the following:
(i)   We rewrite the system above for $\tau = \log \vartheta$ and $Z_k = \log Y_k$, $k = 1, 2, \ldots, n$.
(ii)   We fix $\mathbf{u}$ in the space X_N and use it to find a unique smooth solution $\varrho = \varrho(\mathbf{u})$ to (5.1); here we may follow [23] verbatim.
(iii) We find a unique solution to a system of equations which is nothing but a linearization of the above system (5.2)–(5.8) written in the new variables.
(iv)   We apply a Schauder type of fixed-point theorem for the momentum, the internal energy and the species equations and we deduce the existence of $\mathbf{u} \in X_N$ and $\log \vartheta \in W^{2,q}(\Omega)$, $\log Y_k \in W^{2,q}(\Omega)$; this part follows similarly to [20] provided some a-priori estimates are valid.

6 Existence of solutions for the full approximation

Step 1. We define the operator

$$\mathcal{S} : X_N \to W^{2,p}(\Omega),$$

$1 \leq p < \infty$, $\mathcal{S}(\mathbf{u}) = \varrho$, where ϱ solves the approximate continuity equation (4.1) with the Neumann boundary condition. We then claim the following lemma.

Lemma 6.1. *Let the assumptions of Theorem 5.2 be satisfied. Then the operator $\mathcal{S}$ is well defined for all $p < \infty$. Moreover, if $\mathcal{S}(\mathbf{u}) = \varrho$, then $\varrho \geq 0$ in Ω and $\int_\Omega \varrho \, dx = \int_\Omega \overline{\varrho} \, dx = M$. Additionally, if $\|\mathbf{u}\|_{X_N} \leq L$, $L > 0$, then*

$$\|\varrho\|_{2,p} \leq C(\varepsilon, p, \Omega, M)(1 + L), \quad 1 < p < \infty.$$

The above lemma is an analogue of [23, Proposition 4.29], so we omit the proof.

Step 2. We consider the mapping

$$\mathcal{T} : X_N \times W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega) \to X_N \times W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega)$$

with

$$\mathcal{T}(\mathbf{v}, \vec{S}, \sigma) = (\mathbf{u}, \vec{Z}, \tau), \quad \vec{S} = (S_1, \dots, S_n)^t, \ \vec{Z} = (Z_1, \dots, Z_n)^t,$$

where (we take always $\varrho = \varrho(\mathbf{v})$ as the unique solution to (5.1) with $\mathbf{u}$ replaced by $\mathbf{v}$)

$$\int_\Omega -\mathbf{S}_\eta(e^\tau, \nabla \mathbf{u}) : \nabla \mathbf{w} \, dx = \int_\Omega \left(\frac{1}{2} \varrho(\mathbf{v} \otimes \mathbf{v}) : \nabla \mathbf{w} - \frac{1}{2} \varrho \mathbf{v} \cdot \nabla \mathbf{v} \cdot \mathbf{w} \right) dx$$
$$+ \int_\Omega (\pi(\varrho, e^\sigma) + \delta \varrho^\beta + \delta \varrho^2) \operatorname{div} \mathbf{w} \, dx + \int_\Omega \varrho \mathbf{f} \cdot \mathbf{w} \, dx, \tag{6.1}$$

which is fulfilled for any $\mathbf{w} \in X_N$,

$$- \operatorname{div}\left(\sum_{l=1}^n \widehat{D}_{kl}(e^\sigma, e^{\vec{S}}) e^{S_k} e^{S_l} \nabla Z_l + (\varepsilon(\varrho + 1)e^{S_k} + \lambda)\nabla Z_k \right) + \lambda Z_k$$
$$= \omega_k + \varepsilon \overline{\varrho} - \varepsilon \varrho e^{S_k} - \operatorname{div}(\varrho \mathbf{v} e^{S_k}) + \varepsilon \operatorname{div}(e^{S_k} \nabla \varrho), \quad k = 1, \dots, n \tag{6.2}$$

and

$$- \operatorname{div}(\kappa_{\delta,\eta}(e^\sigma)(\varepsilon + e^\sigma)\nabla \tau) = - \operatorname{div}(\varrho e(\varrho, e^\sigma, e^{\vec{S}})\mathbf{v}) - \pi(\varrho, e^\sigma) \operatorname{div} \mathbf{v}$$
$$- \mathbf{S}_\eta(e^\sigma, \nabla \mathbf{v}) : \nabla \mathbf{v} + \delta e^{-\sigma} + \delta \varepsilon(\beta \varrho^{\beta-2} + 2)|\nabla \varrho|^2$$
$$+ \sum_{k=1}^n c_{vk} \operatorname{div}\left(e^\sigma \sum_{l=1}^n e^{S_k} e^{S_l} \widehat{D}_{kl}(e^\sigma, e^{\vec{S}})\nabla Z_l + e^\sigma(\varepsilon(\varrho + 1)e^{S_k} + \lambda)\nabla Z_k \right),$$

with the boundary conditions

$$\left(\sum_{l=1}^n \widehat{D}_{kl}(e^\sigma, e^{\vec{S}}) e^{S_k} e^{S_l} \nabla Z_l + (\varepsilon(\varrho + 1)e^{S_k} + \lambda)\nabla Z_k \right) \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{6.3}$$

$$\kappa_{\eta,\delta}(e^\sigma)(\varepsilon + e^\sigma)\nabla \tau \cdot \mathbf{n}|_{\partial\Omega} + (L + \delta e^{\sigma(B-1)})(e^\sigma - \vartheta_0^\eta) + \varepsilon \tau + \lambda e^{q\sigma} \tau = 0. \tag{6.4}$$

The existence of a unique solution to (6.1)–(6.4) is a consequence of the Lax–Milgram theorem. Since the right-hand sides and the boundary terms are of lower order and sufficiently smooth, the operator $\mathcal{T}$ is compact. The continuity of $\mathcal{T}$ is straightforward.

In what follows we will show that there exists a bound (independent of t) for all fixed points to

$$tT(\mathbf{u}, \vec{Z}, \tau) = (\mathbf{u}, \vec{Z}, \tau) \tag{6.5}$$

for $t \in [0, 1]$. Thus, due to a well-known version of the Schauder fixed-point theorem (see, e.g., [7, Theorem 9.2.4]) we finish the proof of Theorem 5.2. In fact, we shall show that these estimates are independent of N, η and λ which will be used in the subsequent limit passages.

6.1 Uniform estimates

We now denote $\vartheta = e^\tau$, thus $\vartheta > 0$, and we can in particular divide the energy equation by ϑ. Similarly we denote $Y_k = e^{Z_k}$ and have also $Y_k > 0$, but we do not know yet if $\sum_{k=1}^n Y_k = 1$. However, we denote

$$\sigma_Y = \sum_{k=1}^n Y_k$$

and in what follows, we show certain estimates of $\sigma_Y - 1$ (we get the norm small, provided λ is small). Take $t \in [0, 1]$ and consider the fixed points as in (6.5), i.e. consider

- the continuity equation

$$\varepsilon\varrho + \operatorname{div}(\varrho\mathbf{u}) = \varepsilon\Delta\varrho + \varepsilon\overline{\varrho}, \quad \nabla\varrho \cdot \mathbf{n}|_{\partial\Omega} = 0;$$

- the momentum equation

$$\int_{\Omega} -\mathbf{S}_\eta : \nabla\mathbf{w}\,\mathrm{d}x = t\int_{\Omega}\left(\frac{1}{2}\varrho(\mathbf{u}\otimes\mathbf{u}):\nabla\mathbf{w} - \frac{1}{2}\varrho\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{w}\right)\mathrm{d}x$$
$$+ t\int_{\Omega}(\pi + \delta\varrho^\beta + \delta\varrho^2)\operatorname{div}\mathbf{w}\,\mathrm{d}x + t\int_{\Omega}\varrho\mathbf{f}\cdot\mathbf{w}\,\mathrm{d}x; \tag{6.6}$$

- the species mass balance equations

$$-\operatorname{div}\left(\sum_{l=1}^{n} Y_k\widehat{D}_{kl}\nabla Y_l + (\varepsilon(\varrho+1)Y_k + \lambda)\frac{\nabla Y_k}{Y_k}\right) + \lambda\log Y_k$$
$$= t(\omega_k + \varepsilon\overline{\varrho}_k - \varepsilon\varrho Y_k - \operatorname{div}(\varrho\mathbf{u}Y_k) + \varepsilon\operatorname{div}(Y_k\nabla\varrho)), \quad k = 1,\dots,n; \tag{6.7}$$

- the internal energy equation

$$-\operatorname{div}\left(\kappa_{\delta,\eta}\frac{\varepsilon+\vartheta}{\vartheta}\nabla\vartheta\right) = -t\operatorname{div}(\varrho e\mathbf{u}) - t\pi\operatorname{div}\mathbf{u} + t\delta\varepsilon(\beta\varrho^{\beta-2}+2)|\nabla\varrho|^2 - t\mathbf{S}_\eta : \nabla\mathbf{u} + t\frac{\delta}{\vartheta}$$
$$+ \sum_{k=1}^{n} c_{vk}\operatorname{div}\left(\vartheta\sum_{l=1}^{n} Y_k\widehat{D}_{kl}\nabla Y_l + \vartheta(\varepsilon(\varrho+1)Y_k + \lambda)\frac{\nabla Y_k}{Y_k}\right), \tag{6.8}$$

with the boundary conditions

$$\left(\sum_{l=1}^{n} Y_k\widehat{D}_{kl}\nabla Y_l + (\varepsilon(\varrho+1)Y_k + \lambda)\frac{\nabla Y_k}{Y_k}\right)\cdot\mathbf{n}|_{\partial\Omega} = 0,$$
$$\kappa_{\eta,\delta}\frac{\varepsilon+\vartheta}{\vartheta}\nabla\vartheta\cdot\mathbf{n}|_{\partial\Omega} + t(L + \delta\vartheta^{B-1})(\vartheta - \vartheta_0^\eta) + \varepsilon\log\vartheta + \lambda\vartheta^{\frac{B}{2}}\log\vartheta = 0. \tag{6.9}$$

We first add up the equations for the species (6.7) for $k = 1,\dots,n$ and test the obtained sum by $\xi \in C^\infty(\overline{\Omega})$. We get

$$\varepsilon\int_{\Omega}(\varrho+1)\nabla\sigma_Y\cdot\nabla\xi\,\mathrm{d}x + \lambda\sum_{k=1}^{n}\int_{\Omega}\nabla\log Y_k\cdot\nabla\xi\,\mathrm{d}x + \lambda\sum_{k=1}^{n}\int_{\Omega}\log Y_k\xi\,\mathrm{d}x$$
$$= t\int_{\Omega}(\varepsilon\overline{\varrho} - \varepsilon\varrho\sigma_Y - \operatorname{div}(\varrho\mathbf{u}\sigma_Y) + \varepsilon\operatorname{div}(\sigma_Y\nabla\varrho))\xi\,\mathrm{d}x.$$

On the other hand, the continuity equation can be written as

$$\varepsilon\int_{\Omega}\varrho\xi\,\mathrm{d}x - \int_{\Omega}\varrho\mathbf{u}\cdot\nabla\xi\,\mathrm{d}x = -\varepsilon\int_{\Omega}\nabla\varrho\cdot\nabla\xi\,\mathrm{d}x + \varepsilon\int_{\Omega}\overline{\varrho}\xi\,\mathrm{d}x.$$

Therefore, multiplying the latter by t and subtracting the two equations above, we may write

$$t\int_{\Omega}\left[\varepsilon(\sigma_Y - 1)\varrho\xi - \varrho(\sigma_Y - 1)\mathbf{u}\cdot\nabla\xi + \varepsilon(\sigma_Y - 1)\nabla\varrho\nabla\xi\right]\mathrm{d}x + \varepsilon\int_{\Omega}(\varrho+1)\nabla\sigma_Y\cdot\nabla\xi\,\mathrm{d}x$$
$$= -\lambda\sum_{k=1}^{n}\int_{\Omega}\nabla\log Y_k\cdot\nabla\xi\,\mathrm{d}x - \lambda\sum_{k=1}^{n}\int_{\Omega}\log Y_k\xi\,\mathrm{d}x.$$

Now, taking $\xi = (\sigma_Y - 1)$, we obtain

$$\varepsilon\int_{\Omega}(\varrho+1)|\nabla\sigma_Y|^2\,\mathrm{d}x - \int_{\Omega}\varrho\mathbf{u}\cdot\frac{\nabla(\sigma_Y - 1)^2}{2}\,\mathrm{d}x + t\int_{\Omega}(\varepsilon\varrho(\sigma_Y - 1)^2 + \varepsilon(\sigma_Y - 1)\nabla\varrho\cdot\nabla(\sigma_Y - 1))\,\mathrm{d}x$$
$$= -\lambda\sum_{k=1}^{n}\int_{\Omega}\nabla\log Y_k\cdot\nabla\sigma_Y\,\mathrm{d}x - \lambda\sum_{k=1}^{n}\int_{\Omega}\log Y_k(\sigma_Y - 1)\,\mathrm{d}x. \tag{6.10}$$

Multiplying the continuity equation by $\frac{(\sigma_Y - 1)^2}{2}$, we obtain

$$\varepsilon\varrho\frac{(\sigma_Y - 1)^2}{2} + \text{div}(\varrho\mathbf{u})\frac{(\sigma_Y - 1)^2}{2} = \varepsilon\Delta\varrho\frac{(\sigma_Y - 1)^2}{2} + \varepsilon\bar{\varrho}\frac{(\sigma_Y - 1)^2}{2}, \tag{6.11}$$

thus, multiplying (6.11) by t, integrating it over Ω and subtracting from (6.10), we see that

$$t\varepsilon\int_\Omega \varrho\frac{(\sigma_Y - 1)^2}{2}\,dx + t\varepsilon\int_\Omega \bar{\varrho}\frac{(\sigma_Y - 1)^2}{2}\,dx + \varepsilon\int_\Omega (\varrho + 1)|\nabla(\sigma_Y - 1)|^2\,dx$$

$$= -\lambda\sum_{k=1}^n \int_\Omega \nabla\log Y_k \cdot \nabla\sigma_Y\,dx - \lambda\sum_{k=1}^n \int_\Omega \log Y_k(\sigma_Y - 1)\,dx.$$

Hence we get an estimate for $\sum_{k=1}^n Y_k$:

$$\varepsilon\left(t\|\sigma_Y - 1\|_2^2 + \|\nabla\sigma_Y\|_2^2\right) \le C(\varepsilon)\lambda^2\sum_{k=1}^n\left(\|\nabla\log Y_k\|_2^2 + \frac{1}{t}\|\log Y_k\|_2^2\right), \tag{6.12}$$

which will become very important in the derivation of estimates following from the entropy inequality.

Next, using $\mathbf{w} = \mathbf{u}$ in (6.6), we obtain

$$-\int_\Omega \mathbf{S}_\eta : \nabla\mathbf{u}\,dx = t\int_\Omega \left([\pi + \delta(\varrho^\beta + \varrho^2)]\,\text{div}\,\mathbf{u} + \varrho\mathbf{f}\cdot\mathbf{u}\right)dx. \tag{6.13}$$

Integrating (6.8) over Ω and using (6.9), we get that

$$\int_{\partial\Omega} \left(t(L + \delta\vartheta^{B-1})(\vartheta - \vartheta_0^\eta) + \varepsilon\log\vartheta + \lambda\vartheta^{\frac{B}{2}}\log\vartheta\right)dS$$

$$= -t\int_\Omega \mathbf{S}_\eta : \nabla\mathbf{u}\,dx + t\int_\Omega \delta\vartheta^{-1}\,dx - t\int_\Omega \pi\,\text{div}\,\mathbf{u}\,dx + t\varepsilon\delta\int_\Omega |\nabla\varrho|^2(\beta\varrho^{\beta-2} + 2)\,dx.$$

We add up these two expressions and use the renormalized continuity equation, to get

$$\int_{\partial\Omega} \left(t(L + \delta\vartheta^{B-1})(\vartheta - \vartheta_0) + \varepsilon\log\vartheta + \lambda\vartheta^{\frac{B}{2}}\log\vartheta\right)dS$$

$$+ t\varepsilon\delta\int_\Omega \left(\frac{\beta}{\beta - 1}\varrho^\beta + 2\varrho^2\right)dx - (1 - t)\int_\Omega \mathbf{S}_\eta : \nabla\mathbf{u}\,dx + \varepsilon\int_{\partial\Omega} \log\vartheta\,dS$$

$$= t\int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\,dx + t\int_\Omega \frac{\delta}{\vartheta}\,dx + t\varepsilon\delta\int_\Omega \left(\frac{\beta}{\beta - 1}\bar{\varrho}\varrho^{\beta-1} + 2\bar{\varrho}\varrho\right)dx. \tag{6.14}$$

Moreover, dividing (6.8) by ϑ, we derive

$$-\text{div}\left(\kappa_{\delta,\eta}\frac{\varepsilon + \vartheta}{\vartheta^2}\nabla\vartheta\right) - \kappa_{\delta,\eta}\frac{(\varepsilon + \vartheta)}{\vartheta}\frac{|\nabla\vartheta|^2}{\vartheta^2}$$

$$= -t\varrho\mathbf{u}\left(\frac{\nabla e}{\vartheta} - \frac{\pi}{\vartheta}\frac{\nabla\varrho}{\varrho^2}\right) - t\,\text{div}(\varrho\mathbf{u})\left(\frac{e}{\vartheta} + \frac{\pi}{\varrho\vartheta}\right) + \sum_{k=1}^n c_{vk}\,\text{div}\left(\sum_{l=1}^n Y_k\widehat{D}_{kl}\nabla Y_l + ((\varepsilon\varrho + 1)Y_k + \lambda)\frac{\nabla Y_k}{Y_k}\right)$$

$$- \sum_{k,l=1}^n c_{vk}Y_k\widehat{D}_{kl}\nabla Y_l \cdot \nabla\log\vartheta - \sum_{k=1}^n c_{vk}(\varepsilon(\varrho + 1)Y_k + \lambda)\frac{\nabla Y_k}{Y_k} \cdot \nabla\log\vartheta$$

$$- t\frac{\mathbf{S}_\eta : \nabla\mathbf{u}}{\vartheta} + t\frac{\delta}{\vartheta^2} + t\delta\varepsilon(\beta\varrho^{\beta-2} + 2)\frac{|\nabla\varrho|^2}{\vartheta}. \tag{6.15}$$

Now we multiply (6.7) by $\xi = \log Y_k$, sum up with respect to k and integrate over Ω:

$$\sum_{k,l=1}^n \int_\Omega \widehat{D}_{kl}\nabla Y_k \cdot \nabla Y_l\,dx + \sum_{k=1}^n \int_\Omega \left(\varepsilon(\varrho + 1)\frac{|\nabla Y_k|^2}{Y_k} + \lambda|\nabla\log Y_k|^2 + \lambda(\log Y_k)^2\right)dx$$

$$= t\sum_{k=1}^n \int_\Omega [\omega_k + \varepsilon\bar{\varrho}_k - \varepsilon\varrho Y_k - \text{div}(\varrho\mathbf{u}Y_k) + \varepsilon\,\text{div}(Y_k\nabla\varrho)]\log Y_k\,dx. \tag{6.16}$$

Next we multiply (6.7) by $\xi = c_{vk} \log \vartheta - c_{pk}$, sum up with respect to k and integrate over Ω:

$$\sum_{k,l=1}^{n} c_{vk} \int_{\Omega} Y_k \widehat{D}_{kl} \nabla Y_l \cdot \nabla \log \vartheta \, dx + \sum_{k=1}^{n} c_{vk} \int_{\Omega} (\varepsilon(\varrho+1)Y_k + \lambda) \frac{\nabla Y_k}{Y_k} \cdot \nabla \log \vartheta \, dx$$

$$= \sum_{k=1}^{n} \lambda \int_{\Omega} \log Y_k [c_{pk} - c_{vk} \log \vartheta] \, dx$$

$$+ t \sum_{k=1}^{n} c_{vk} \int_{\Omega} [\omega_k + \varepsilon\overline{\varrho}_k - \varepsilon\varrho Y_k - \mathrm{div}(\varrho\mathbf{u}Y_k) + \varepsilon\,\mathrm{div}(Y_k\nabla\varrho)] \log \vartheta \, dx$$

$$- t \sum_{k=1}^{n} c_{pk} \int_{\Omega} [\omega_k + \varepsilon\overline{\varrho}_k - \varepsilon\varrho Y_k - \mathrm{div}(\varrho\mathbf{u}Y_k) + \varepsilon\,\mathrm{div}(Y_k\nabla\varrho)] \, dx. \tag{6.17}$$

Now, take (6.14) $- \int_{\Omega}$ (6.15) $dx -$ (6.16) $-$ (6.17):

$$\sum_{k,l=1}^{n} \int_{\Omega} \widehat{D}_{kl} \nabla Y_k \cdot \nabla Y_l \, dx + \sum_{k=1}^{n} \int_{\Omega} \left(\varepsilon(\varrho+1) \frac{|\nabla Y_k|^2}{Y_k} + \lambda|\nabla \log Y_k|^2 \right) dx$$

$$+ \lambda \sum_{k=1}^{n} \int_{\Omega} (\log Y_k)^2 \, dx + \int_{\Omega} \kappa_{\delta,\eta} \frac{(\varepsilon+\vartheta)}{\vartheta} \frac{|\nabla\vartheta|^2}{\vartheta^2} \, dx + t \int_{\Omega} \left(\frac{-\mathbf{S}_\eta : \nabla\mathbf{u}}{\vartheta} + \frac{\delta}{\vartheta^2} + \delta\varepsilon(\beta\varrho^{\beta-2}+2) \frac{|\nabla\varrho|^2}{\vartheta} \right) dx$$

$$- (1-t) \int_{\Omega} \mathbf{S}_\eta : \nabla\mathbf{u} \, dx + t\varepsilon\delta \int_{\Omega} \left(\frac{\beta}{\beta-1} \varrho^{\beta} + 2\varrho^2 \right) dx + t \int_{\partial\Omega} (L + \delta\vartheta^{B-1})\vartheta \, dS + t \int_{\partial\Omega} (L + \delta\vartheta^{B-1}) \frac{\vartheta_0^{\eta}}{\vartheta} \, dS$$

$$+ \int_{\partial\Omega\cap\{\vartheta>1\}} \left(\varepsilon \log \vartheta + \lambda\vartheta^{\frac{B}{2}} \log \vartheta \right) dS - \int_{\partial\Omega\cap\{\vartheta\le1\}} \left(\varepsilon \frac{\log\vartheta}{\vartheta} + \lambda\vartheta^{\frac{B}{2}-1} \log \vartheta \right) dS = \mathrm{RHS},$$

where

$$\mathrm{RHS} = t \int_{\partial\Omega} (L + \delta\vartheta^{B-1})\vartheta_0^{\eta} \, dS + t \int_{\partial\Omega} (L + \delta\vartheta^{B-1}) \, dS$$

$$- \int_{\partial\Omega\cap\{\vartheta\le1\}} (\varepsilon + \lambda\vartheta^{\frac{B}{2}}) \log \vartheta \, dS + \int_{\partial\Omega\cap\{\vartheta>1\}} (\varepsilon\vartheta^{-1} + \lambda\vartheta^{\frac{B}{2}-1}) \log \vartheta \, dS$$

$$+ t \int_{\Omega} \varrho\mathbf{f} \cdot \mathbf{u} \, dx + t \int_{\Omega} \frac{\delta}{\vartheta} \, dx + t\varepsilon\delta \int_{\Omega} \frac{\beta}{\beta-1} \overline{\varrho}\varrho^{\beta-1} \, dx + t\varepsilon\delta \int_{\Omega} 2\overline{\varrho}\varrho \, dx$$

$$+ t \int_{\Omega} \varrho\mathbf{u} \left(\frac{\nabla e}{\vartheta} - \frac{\pi}{\vartheta} \frac{\nabla\varrho}{\varrho^2} \right) dx + t \int_{\Omega} \mathrm{div}(\varrho\mathbf{u}) \left(\frac{e}{\vartheta} + \frac{\pi}{\varrho\vartheta} \right) dx$$

$$+ t \sum_{k=1}^{n} \int_{\Omega} [\omega_k \log Y_k + \varepsilon\overline{\varrho}_k \log Y_k - \varepsilon\varrho Y_k \log Y_k - \mathrm{div}(\varrho\mathbf{u}Y_k) \log Y_k] \, dx$$

$$+ t \sum_{k=1}^{n} \int_{\Omega} \varepsilon\,\mathrm{div}(Y_k\nabla\varrho) \log Y_k \, dx - t \sum_{k=1}^{n} c_{vk} \int_{\Omega} [\omega_k \log \vartheta + \varepsilon\overline{\varrho}_k \log \vartheta] \, dx$$

$$+ t \sum_{k=1}^{n} c_{vk} \int_{\Omega} [\varepsilon\varrho Y_k \log \vartheta + \mathrm{div}(\varrho\mathbf{u}Y_k) \log \vartheta - \varepsilon\,\mathrm{div}(Y_k\nabla\varrho) \log \vartheta] \, dx$$

$$+ \lambda \sum_{k=1}^{n} c_{vk} \int_{\Omega} \log Y_k \log \vartheta \, dx - \lambda \sum_{k=1}^{n} c_{pk} \int_{\Omega} \log Y_k \, dx$$

$$+ t \sum_{k=1}^{n} c_{pk} \int_{\Omega} [\omega_k + \varepsilon\overline{\varrho}_k - \varepsilon\varrho Y_k] \, dx = \sum_{i=1}^{25} I_i. \tag{6.18}$$

Finally, from (6.13) and the renormalized continuity equation we see

$$\|\mathbf{u}\|_{1,2}^2 + t\varepsilon\delta(\|\varrho\|_\beta^\beta + \|\nabla\varrho^{\beta/2}\|_2^2) + t\varepsilon\delta(\|\varrho\|_2^2 + \|\nabla\varrho\|_2^2) \le t\left|\int_\Omega (\pi\operatorname{div}\mathbf{u} + \varrho\mathbf{f}\cdot\mathbf{u})\,\mathrm{d}x\right|,$$

and from the standard elliptic regularity for the continuity equation we also have

$$\|\nabla\varrho\|_q \le C(\varepsilon)(1 + \|\varrho|\mathbf{u}|\|_q), \quad 1 < q < \infty.$$

Hence,

$$\|\mathbf{u}\|_{1,2}^2 + t\varepsilon\delta(\|\varrho\|_\beta^\beta + \|\nabla\varrho^{\beta/2}\|_2^2) + t\varepsilon\delta(\|\varrho\|_2^2 + \|\nabla\varrho\|_2^2) \le Ct^2\left(1 + \|\vartheta\|_{\frac{2\beta}{\beta-2}}^{\frac{2\beta}{\beta-2}}\right), \tag{6.19}$$

provided $\beta > 2$, and

$$\|\nabla\varrho\|_q \le C(\varepsilon)\left(1 + t\|\vartheta\|_{\frac{2\beta}{\beta-2}}^{\frac{\beta}{\beta-2}}\right), \quad 1 < q \le \frac{6\beta}{\beta+2}. \tag{6.20}$$

We can now estimate the terms on the right-hand side of (6.18). First, note that we may easily estimate the boundary integrals and the terms I_5 to I_8. In order to estimate terms I_9 and I_{10}, recall that we may write

$$e = \frac{1}{\gamma-1}\varrho^{\gamma-1} + \vartheta\sum_{k=1}^n c_{vk}Y_k = e_c + e_m.$$

We first estimate the part corresponding to e_c. We have

$$t\int_\Omega \varrho\mathbf{u}\left(\frac{\nabla e_c}{\vartheta} - \frac{\pi}{\vartheta}\frac{\nabla\varrho}{\varrho^2}\right)\mathrm{d}x + t\int_\Omega \operatorname{div}(\varrho\mathbf{u})\left(\frac{e_c}{\vartheta} + \frac{\pi}{\varrho\vartheta}\right)\mathrm{d}x$$

$$= -t\int_\Omega \varrho\mathbf{u}\cdot\frac{\nabla\varrho}{\varrho}\,\mathrm{d}x + t\int_\Omega \operatorname{div}(\varrho\mathbf{u})\left(\frac{1}{\gamma-1}\frac{\varrho^{\gamma-1}}{\vartheta} + 1 + \frac{\varrho^{\gamma-1}}{\vartheta}\right)\mathrm{d}x$$

$$= t\int_\Omega \operatorname{div}(\varrho\mathbf{u})\left(\log\varrho + \frac{\gamma}{\gamma-1}\frac{\varrho^{\gamma-1}}{\vartheta}\right)\mathrm{d}x$$

$$= t\int_\Omega (\varepsilon\Delta\varrho + \varepsilon\overline{\varrho} - \varepsilon\varrho)\left(\log\varrho + \frac{\gamma}{\gamma-1}\frac{\varrho^{\gamma-1}}{\vartheta}\right)\mathrm{d}x$$

$$= -\varepsilon t\int_\Omega \frac{|\nabla\varrho|^2}{\varrho}\,\mathrm{d}x - \varepsilon t\int_{\{\varrho\ge 1\}} \varrho\log\varrho\,\mathrm{d}x + \varepsilon t\int_{\{\varrho<1\}} \overline{\varrho}\log\varrho\,\mathrm{d}x$$

$$\quad - \varepsilon t\int_{\{\varrho<1\}} \varrho\log\varrho\,\mathrm{d}x + \varepsilon t\int_{\{\varrho\ge 1\}} \overline{\varrho}\log\varrho\,\mathrm{d}x - \varepsilon t\int_\Omega \frac{\varrho^\gamma}{\vartheta}\frac{\gamma}{\gamma-1}\,\mathrm{d}x$$

$$\quad - t\varepsilon\gamma\int_\Omega \frac{\varrho^{\gamma-2}|\nabla\varrho|^2}{\vartheta}\,\mathrm{d}x + \varepsilon t\frac{\gamma}{\gamma-1}\int_\Omega \varrho^{\gamma-1}\frac{\nabla\varrho\cdot\nabla\vartheta}{\varrho}\,\mathrm{d}x + \varepsilon t\frac{\gamma}{\gamma-1}\int_\Omega \frac{\overline{\varrho}\varrho^{\gamma-1}}{\vartheta}\,\mathrm{d}x. \tag{6.21}$$

Now, observe that the terms no. 1, 2, 3, 6, 7 of the right-hand side of (6.21) are non-positive, therefore they can be transferred to the left-hand side of (6.18). The terms no. 4, 5, 8, 9 can be then easily estimated by means of those terms and the rest of the terms from the left-hand side of (6.18). This part of the proof is a repetition of arguments from [20] so we skip the details. Next we take $I_9(e_m) + I_{10}(e_m) + I_{14} + I_{19}$ and get

$$t\sum_{k=1}^n \int_\Omega \left[\frac{\varrho\mathbf{u}}{\vartheta}\cdot\nabla(\vartheta c_{vk}Y_k) + \operatorname{div}(\varrho\mathbf{u})(c_{vk}Y_k) - \operatorname{div}(\varrho\mathbf{u}Y_k)\log Y_k\right]\mathrm{d}x + \sum_{k=1}^n c_{vk}\int_\Omega \log\vartheta\operatorname{div}(\varrho\mathbf{u}Y_k)\,\mathrm{d}x$$

$$= t\int_\Omega \varrho\mathbf{u}\cdot\nabla\sigma_Y\,\mathrm{d}x = -t\int_\Omega (\varepsilon\Delta\varrho - \varepsilon\varrho + \varepsilon\overline{\varrho})\sigma_Y\,\mathrm{d}x.$$

Then, to handle the first term, we use (6.12) and (6.19):

$$\varepsilon t\left|\int_\Omega \nabla\varrho\cdot\nabla\sigma_Y\,\mathrm{d}x\right| \le \varepsilon t\|\nabla\varrho\|_2\|\nabla\sigma_Y\|_2 \le C(\varepsilon)t\left(1 + \|\vartheta\|_{\frac{2\beta}{\beta-2}}^{\frac{\beta}{\beta-2}}\right)\sum_{k=1}^n \lambda(\|\nabla\log Y_k\|_2 + 1/\sqrt{t}\|\log Y_k\|_2),$$

which can be estimated by the left-hand side of (6.18). The second term can be estimated similarly, and the last term is non-positive.

Now, for $I_{11} + I_{16} + I_{23}$, recalling that $\sum_{k=1}^{n} \omega_k = 0$, we obtain

$$t \sum_{k=1}^{n} \int_{\Omega} \omega_k (\log Y_k - c_{vk} \log \vartheta + c_{pk})\, dx = t \sum_{k=1}^{n} \int_{\Omega} \omega_k (\log Y_k - c_{vk} \log \vartheta + c_{pk} + \log \varrho)\, dx$$

$$= t \int_{\Omega} \sum_{k=1}^{n} \frac{g_k \omega_k}{\vartheta}\, dx \le 0.$$

Further, for I_{12} we use again (6.12) to write

$$I_{12} \le t \sum_{k=1}^{n} \left(\int_{\Omega \cap \{Y_k \le 1\}} \varepsilon \overline{\varrho}_k \log Y_k\, dx + \int_{\Omega \cap \{Y_k > 1\}} \varepsilon \overline{\varrho}_k \log Y_k\, dx \right)$$

$$\le t \sum_{k=1}^{n} \int_{\Omega \cap \{Y_k > 1\}} \varepsilon \overline{\varrho}_k Y_k\, dx \le C\left(1 + \lambda \sum_{k=1}^{n} \left(\sqrt{t} \|\nabla \log Y_k\|_2 + \|\log Y_k\|_2 \right) \right) \qquad (6.22)$$

and the terms on the right-hand side of (6.22) can be now easily estimated by the left-hand side of (6.18).

Next, I_{13} may be split into two parts, namely

$$I_{13} = -t\varepsilon \sum_{k=1}^{n} \left(\int_{\Omega \cap \{Y_k > 1\}} \varrho Y_k \log Y_k\, dx + \int_{\Omega \cap \{Y_k \le 1\}} \varrho Y_k \log Y_k\, dx \right) \le Ct\varepsilon \int_{\Omega} \varrho\, dx \le C(M).$$

The term I_{15} can be estimated similarly as $I_9 + I_{10}$ for e_m. The term I_{17} can be easily estimated by means of $t \int_{\Omega} \frac{\delta}{\vartheta^2}\, dx$ from the left-hand side of (6.18).

For $\alpha > 0$ small, we may write

$$I_{18} = \varepsilon t \int_{\Omega} \varrho \log \vartheta \sum_{k=1}^{n} c_{vk} Y_k\, dx \le C\varepsilon t \int_{\Omega} \varrho |\log \vartheta|(\sigma_Y - 1)\, dx + Ct\varepsilon \int_{\Omega} \varrho |\log \vartheta|\, dx$$

$$\le C\varepsilon t \|\varrho\|_{\beta} (\|\sigma_Y - 1\|_2 + 1)\left(\int_{\partial \Omega} \vartheta^{\frac{B}{2}} \log \vartheta\, dS + \int_{\Omega} \kappa_{\delta,\eta}(\vartheta) \frac{|\nabla \vartheta|^2}{\vartheta^2}\, dx + \varepsilon \int_{\partial \Omega \cap \{\vartheta < 1\}} \frac{\log \vartheta}{\vartheta}\, dS \right)^{\alpha}$$

$$\le \frac{\lambda}{4} \sum_{k=1}^{n} \left(\|\nabla \log Y_k\|_2^2 + \|\log Y_k\|_2^2 \right) + \frac{\lambda}{4} \int_{\partial \Omega} \vartheta^{\frac{B}{2}} \log \vartheta\, dS + \frac{\varepsilon}{4} \int_{\Omega} \kappa_{\delta,\eta}(\vartheta) |\nabla \vartheta|^2\, dx + \frac{\varepsilon}{4} \int_{\partial \Omega \cap \{\vartheta < 1\}} \frac{\log \vartheta}{\vartheta}\, dS.$$

To treat I_{20}, we will use the continuity equation to replace the highest order terms of the density:

$$I_{20} = -t\varepsilon \sum_{k=1}^{n} c_{vk} \int_{\Omega} \operatorname{div}(Y_k \nabla \varrho) \log \vartheta\, dx$$

$$= -t\varepsilon \sum_{k=1}^{n} c_{vk} \int_{\Omega} \nabla Y_k \cdot \nabla \varrho \log \vartheta\, dx - t\varepsilon \sum_{k=1}^{n} c_{vk} \int_{\Omega} Y_k \Delta \varrho \log \vartheta\, dx$$

$$\le Ct\varepsilon \left(\int_{\Omega} \sum_{k=1}^{n} \frac{|\nabla Y_k|^2}{Y_k}\, dx \right)^{\frac{1}{2}} \|\sigma_Y - 1\|_2^{\frac{1}{2}} \|\nabla \varrho\|_4 \|\log \vartheta\|_6 - t \sum_{k=1}^{n} c_{vk} \int_{\Omega} Y_k (\varepsilon \varrho - \varepsilon \overline{\varrho} + \operatorname{div}(\varrho \mathbf{u})) \log \vartheta\, dx$$

$$\le \frac{\lambda}{4} \sum_{k=1}^{n} \left(\|\nabla \log Y_k\|_2^2 + \|\log Y_k\|_2^2 \right) + \varepsilon \sum_{k=1}^{n} \|\nabla \sqrt{Y_k}\|_2^2 + \frac{\lambda}{4} \int_{\partial \Omega} \vartheta^{\frac{B}{2}} \log \vartheta\, dS$$

$$+ \frac{\varepsilon}{4} \int_{\partial \Omega \cap \{\vartheta < 1\}} \frac{\log \vartheta}{\vartheta}\, dS + \frac{t\varepsilon}{4} \int_{\partial \Omega} \vartheta^{B}\, dS + \frac{\varepsilon}{4} \int_{\Omega} \kappa_{\delta,\eta} |\nabla \vartheta|^2\, dx.$$

Next

$$I_{21} = \lambda \sum_{k=1}^{n} \int_{\Omega} c_{vk} \log Y_k \log \vartheta \, \mathrm{d}x$$

$$\leq \frac{\lambda}{4} \sum_{k=1}^{n} \|\log Y_k\|_2^2 + \frac{\lambda}{4} \int_{\partial\Omega} \vartheta^{\frac{B}{2}} \log \vartheta \, \mathrm{d}S + \frac{\varepsilon}{4} \int_{\Omega} \kappa_{\delta,\eta}(\vartheta) |\nabla \vartheta|^2 \, \mathrm{d}x + \frac{\varepsilon}{4} \int_{\partial\Omega \cap \{\vartheta < 1\}} \frac{\log \vartheta}{\vartheta} \, \mathrm{d}S,$$

so, I_{18}, I_{20}, I_{21} may be estimated by the left-hand side of (6.18).

The terms I_{22}, I_{24} and I_{25} are easy or can be estimated as above. Now we may employ the standard elliptic theory (cf. [20]) to show that we can estimate $\log \vartheta$ and $\log Y_k$ in $W^{2,q}(\Omega)$ for any $q < \infty$, i.e. in particular, in $W^{1,\infty}(\Omega)$ independently of t. The proof of Theorem 5.2 is finished.

7 Proof of Theorem 4.2

7.1 Limit passage $N \to \infty$

Recall that in the previous section, we proved the following estimates (see (6.18) with $t = 1$). Note that the constant on the right-hand side is independent of N, η and λ, however, may depend on ε and δ:

$$\sqrt{\lambda} \|\vec{Y}\|_{1,2} + \sum_{k=1}^{n} \left\| \frac{|\nabla Y_k|^2}{Y_k} \right\|_1 + \|\nabla \vartheta^{\frac{B}{2}}\|_2 + \left\| \frac{\nabla \vartheta}{\vartheta^2} \right\|_2 + \|\vartheta^{-2}\|_1 + \|\vartheta\|_{B,\partial\Omega} + \left\| \frac{\log \vartheta}{\vartheta} \right\|_{1,\partial\Omega} + \|\varrho\|_\beta \leq C. \qquad (7.1)$$

Moreover, (6.19) and (6.20) together with (5.1) imply

$$\|\nabla^2 \varrho\|_2 + \|\mathbf{u}\|_{1,2} + \|\nabla \varrho\|_6 \leq C. \qquad (7.2)$$

Hence, we easily let $N \to \infty$ in the continuity equation (5.1), in the weak formulation of the momentum equation (5.2) and the weak formulation of the species mass balance equations (5.4).

However, we cannot pass to the limit so easily in the internal balance equation due to the presence of the term $\mathbf{S}_\eta(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u}$ which is bounded only in $L^1(\Omega)$. Nonetheless, we may proceed as in [20]: we may use as test function in the limit momentum balance the function $\mathbf{u}$ to show that

$$\lim_{N \to \infty} \int_{\Omega} \mathbf{S}_\eta(\vartheta_N, \nabla \mathbf{u}_N) : \nabla \mathbf{u}_N \, \mathrm{d}x = \int_{\Omega} \mathbf{S}_\eta(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u} \, \mathrm{d}x$$

which yields the strong convergence of the velocity gradient in $L^2(\Omega)$, and the passage to the limit in the weak formulation of the internal energy balance (5.7) can be performed. Let us stress that this is the only limit passage for which the strong convergence of the velocity gradient can be shown. It is caused by the fact that the momentum equation may be tested by the solution $\mathbf{u}$ only when $\eta > 0$.

7.2 Limit passage $\eta \to 0^+$

Before the next limit passage $\eta \to 0^+$ we must replace the internal energy balance by the total energy balance. Note that at this level, they are still equivalent (we may test the momentum equation by $\mathbf{u}$) but this might not be true after the limit passage $\eta \to 0^+$. Therefore, we use in the momentum equation as a test function $\mathbf{u}\psi$ with $\psi \in C^\infty(\overline{\Omega})$ and sum it with the weak formulation of the internal energy balance with the test function ψ. We get the weak formulation of the approximate total energy balance:

$$- \int_\Omega \left[\varrho e + \frac{1}{2}\varrho|\mathbf{u}|^2 + (\pi + \delta\varrho^\beta + \delta\varrho^2) \right] \mathbf{u} \cdot \nabla\psi \, dx$$

$$- \int_\Omega (\mathbf{S}_\eta \mathbf{u} \cdot \nabla\psi + \delta\vartheta^{-1}\psi) \, dx + \int_\Omega \kappa_{\delta,\eta} \frac{\varepsilon + \vartheta}{\vartheta} \nabla\vartheta \cdot \nabla\psi \, dx$$

$$+ \int_{\partial\Omega} [(L + \delta\vartheta^{B-1})(\vartheta - \vartheta_0^\eta) + \varepsilon \log\vartheta + \lambda\vartheta^{\frac{B}{2}} \log\vartheta]\psi \, dS$$

$$+ \sum_{k=1}^n c_{vk} \int_\Omega \left[\vartheta \sum_{l=1}^n Y_k \widehat{D}_{kl}\nabla Y_l \cdot \nabla\psi + \vartheta(\varepsilon(\varrho + 1)Y_k + \lambda)\frac{\nabla Y_k}{Y_k}\psi \right] dx$$

$$= \int_\Omega \varrho\mathbf{f} \cdot \mathbf{u}\psi \, dx + \delta \int_\Omega [\varepsilon|\nabla\varrho|^2(\beta\varrho^{\beta-2} + 2) + (\varrho^\beta + \varrho^2) \operatorname{div} \mathbf{u}]\psi \, dx.$$

Therefore, using the renormalized continuity equation we deduce

$$- \int_\Omega \left[\varrho e + \frac{1}{2}\varrho|\mathbf{u}|^2 + (\pi + \delta\varrho^\beta + \delta\varrho^2) \right] \mathbf{u} \cdot \nabla\psi \, dx$$

$$- \int_\Omega (\mathbf{S}_\eta \mathbf{u} \cdot \nabla\psi + \delta\vartheta^{-1}\psi) \, dx + \int_\Omega \kappa_{\delta,\eta} \frac{\varepsilon + \vartheta}{\vartheta} \nabla\vartheta \cdot \nabla\psi \, dx$$

$$+ \int_{\partial\Omega} [(L + \delta\vartheta^{B-1})(\vartheta - \vartheta_0^\eta) + \varepsilon \log\vartheta + \lambda\vartheta^{\frac{B}{2}} \log\vartheta]\psi \, dS$$

$$+ \sum_k^n c_{vk} \int_\Omega \left[\vartheta \sum_{l=1}^n Y_k \widehat{D}_{kl}\nabla Y_l \cdot \nabla\psi + \vartheta(\varepsilon(\varrho + 1)Y_k + \lambda)\frac{\nabla Y_k}{Y_k} \cdot \nabla\psi \right] dx$$

$$= \int_\Omega \varrho\mathbf{f} \cdot \mathbf{u}\psi \, dx + \frac{\delta}{\beta - 1} \int_\Omega (\varepsilon\beta\overline{\varrho}\varrho^{\beta-1}\psi + \varrho^\beta\mathbf{u} \cdot \nabla\psi - \varepsilon\beta\varrho^\beta\psi) \, dx + \delta \int_\Omega (2\varepsilon\overline{\varrho}\varrho\psi + \varrho^2\mathbf{u} \cdot \nabla\psi - 2\varepsilon\varrho^2\psi) \, dx$$

for all $\psi \in C^\infty(\overline{\Omega})$. Hence we may now use (7.1) and (7.2) to justify the limit passage $\eta \to 0^+$ and hence to verify the validity of (4.1)–(4.5). Theorem 4.2 is proved.

8 Proof of Theorem 3.5

To finish the proof of the main theorem we subsequently let $\lambda \to 0^+$, $\varepsilon \to 0^+$ and $\delta \to 0^+$.

8.1 Limit passage $\lambda \to 0^+$

The first limit passage in this section is still based on estimates (7.1) and (7.2). Note, however, that we cannot dispose with the estimate of $\vec{Y}$ in $W^{1,2}(\Omega)$ anymore. On the other hand, from (6.12) we still have

$$\|\sigma_Y - 1\|_6 \le C(\lambda), \tag{8.1}$$

with $C(\lambda) \to 0$ when $\lambda \to 0^+$. Therefore, combining estimate (8.1) with the second term on the left-hand side of (7.1), we get

$$\|\nabla\vec{Y}\|_{\frac{12}{7}} \le C, \tag{8.2}$$

with C independent of λ. Therefore, using (7.1), (7.2), (8.1) and (8.2) we may let $\lambda \to 0^+$ and get the following:
• the approximate continuity equation

$$\varepsilon\varrho + \operatorname{div}(\varrho\mathbf{u}) = \varepsilon\Delta\varrho + \varepsilon\overline{\varrho}, \quad \nabla\varrho \cdot \mathbf{n}|_{\partial\Omega} = 0; \tag{8.3}$$

- the weak formulation of the approximate momentum equation

$$\int_{\Omega}\left(\frac{1}{2}\varrho\mathbf{u}\cdot\nabla\mathbf{u}\cdot\boldsymbol{\phi}-\frac{1}{2}\varrho(\mathbf{u}\otimes\mathbf{u}):\nabla\boldsymbol{\phi}-\mathbf{S}:\nabla\boldsymbol{\phi}\right)dx-\int_{\Omega}(\pi+\delta\varrho^{\beta}+\delta\varrho^2)\operatorname{div}\boldsymbol{\phi}\,dx=\int_{\Omega}\varrho\mathbf{f}\cdot\boldsymbol{\phi}\,dx \quad (8.4)$$

for each $\boldsymbol{\phi}\in C_0^{\infty}(\Omega)$;

- the weak formulation of the approximate species balance equations

$$\varepsilon\int_{\Omega}Y_k\varrho\psi\,dx-\int_{\Omega}Y_k\varrho\mathbf{u}\cdot\nabla\psi\,dx+\int_{\Omega}\sum_{l=1}^{n}Y_k\widehat{D}_{kl}\nabla Y_l\cdot\nabla\psi\,dx$$

$$=\int_{\Omega}\left[\omega_k\psi-\varepsilon\varrho\nabla Y_k\cdot\nabla\psi+\varepsilon\operatorname{div}(Y_k\nabla\varrho)\psi-\varepsilon\nabla Y_k\cdot\nabla\psi+\varepsilon\overline{\varrho}_k\psi\right]dx \quad (8.5)$$

for any $\psi\in C^{\infty}(\overline{\Omega})$, $k=1,2,\ldots,n$;

- the weak formulation of the approximate total energy equation

$$-\int_{\Omega}\left[\varrho e+\frac{1}{2}\varrho|\mathbf{u}|^2+(\pi+\delta\varrho^{\beta}+\delta\varrho^2)\right]\mathbf{u}\cdot\nabla\psi\,dx$$

$$-\int_{\Omega}(\mathbf{S}\mathbf{u}\cdot\nabla\psi+\delta\vartheta^{-1}\psi)\,dx+\int_{\Omega}\kappa_{\delta}\frac{\varepsilon+\vartheta}{\vartheta}\nabla\vartheta\cdot\nabla\psi\,dx+\int_{\partial\Omega}[(L+\delta\vartheta^{B-1})(\vartheta-\vartheta_0)+\varepsilon\log\vartheta]\psi\,dS$$

$$+\int_{\Omega}\left[\vartheta\sum_{k,l=1}^{n}c_{vk}Y_k\widehat{D}_{kl}\nabla Y_l\cdot\nabla\psi+\vartheta\sum_{k=1}^{n}\varepsilon(\varrho+1)c_{vk}\nabla Y_k\cdot\nabla\psi\right]dx$$

$$=\int_{\Omega}\varrho\mathbf{f}\cdot\mathbf{u}\psi\,dx+\frac{\delta}{\beta-1}\int_{\Omega}(\varepsilon\beta\overline{\varrho}\varrho^{\beta-1}\psi+\varrho^{\beta}\mathbf{u}\cdot\nabla\psi-\varepsilon\beta\varrho^{\beta}\psi)\,dx+\delta\int_{\Omega}(2\varepsilon\overline{\varrho}\varrho\psi+\varrho^2\mathbf{u}\cdot\nabla\psi-2\varepsilon\varrho^2\psi)\,dx$$

$$(8.6)$$

for any $\psi\in C^{\infty}(\overline{\Omega})$.

8.2 Limit passage $\varepsilon\to 0^+$

At this step we loose all the control of the density, except for the L^1-bound due to the given mass. On the other hand, due to (8.1), we know now that $\sigma_Y=1$ which leads to the bounds (recall $Y_k\geq 0$)

$$0\leq Y_k\leq 1 \quad\text{for all }k\in\{1,2,\ldots,n\}.$$

Unlike the situation before, we may also deduce from the first term on the left-hand side of (6.18) an additional bound on $\nabla\vec{Y}$. Employing again $\sigma_Y=1$, we have

$$\int_{\Omega}\sum_{k,l=1}^{n}\widehat{D}_{kl}\nabla Y_k\nabla Y_l\,dx\geq C\|\nabla\vec{Y}\|_2^2.$$

Further, due to the Korn inequality and the form of the viscous tensor, we control the $W^{1,2}$-norm of the velocity. We can therefore estimate most of the terms on the right-hand side of (6.18) and get (cf. [20])

$$\|\vec{Y}\|_{1,2}+\|\vec{Y}\|_{\infty}+\|\nabla\vartheta^{\frac{B}{2}}\|_2+\|\vartheta\|_{B,\partial\Omega}+\|\vartheta\|_{3m}+\|\vartheta^{-2}\|_1+\|\vartheta^{-1}\|_{1,\partial\Omega}+\|\mathbf{u}\|_{1,2}\leq C\left(1+\int_{\Omega}\varrho\mathbf{f}\cdot\mathbf{u}\,dx\right). \quad (8.7)$$

However, due to lack of sufficient density estimate, the boundedness of the right-hand side of (8.7) has to be verified. Note that the momentum equation is in fact the same as in the case of the compressible Navier–Stokes–Fourier system studied in [20], so, we may apply the same technique to obtain the so-called Bogovskii-type of estimates. Following [20], we use as test function in (8.4) the function $\boldsymbol{\phi}$, solution to

$$\operatorname{div}\boldsymbol{\phi}=\varrho^{\frac{2}{3}\beta}-\frac{1}{|\Omega|}\int_{\Omega}\varrho^{\frac{2}{3}\beta}\,dx,\quad \boldsymbol{\phi}|_{\partial\Omega}=\mathbf{0}.$$

For more information on the Bogovskii operator, we refer the reader to [23, Lemma 3.17] and to [6]. In consequence of this testing we obtain the additional bound on ϱ, namely

$$\|\varrho\|_{\frac{5}{3}\beta} \le C.$$

Thus we may pass to the limit in (8.3)–(8.6) and get
- the weak formulation of the continuity equation

$$\int_\Omega \varrho \mathbf{u} \cdot \nabla \psi \, dx = 0$$

  for all $\psi \in C^\infty(\overline{\Omega})$;
- the weak formulation of the approximate momentum equation

$$-\int_\Omega (\varrho(\mathbf{u} \otimes \mathbf{u}) : \nabla \boldsymbol{\phi} + \mathbf{S} : \nabla \boldsymbol{\phi}) \, dx - \int_\Omega (\varrho \vartheta + \overline{\varrho^\gamma + \delta(\varrho^\beta + \varrho^2)}) \operatorname{div} \boldsymbol{\phi} \, dx = \int_\Omega \varrho \mathbf{f} \cdot \boldsymbol{\phi} \, dx \qquad (8.8)$$

  for all $\boldsymbol{\phi} \in C_0^\infty(\Omega)$;
- the weak formulation of the approximate species balance equations

$$-\int_\Omega Y_k \varrho \mathbf{u} \cdot \nabla \psi \, dx + \int_\Omega Y_k \sum_{l=1}^n D_{kl}(\vartheta, \vec{Y}) \nabla Y_l \cdot \nabla \psi \, dx = \int_\Omega \omega_k \psi \, dx$$

  for all $\psi \in C^\infty(\overline{\Omega})$, $k = 1, 2, \ldots, n$;
- the weak formulation of the approximate total energy equation

$$-\int_\Omega \left[\varrho \vartheta \sum_{k=1}^n c_{vk} Y_k + \frac{1}{2} \varrho |\mathbf{u}|^2 + \left(\varrho \vartheta + \overline{\frac{\gamma}{\gamma - 1} \varrho^\gamma + \delta(\varrho^\beta + \varrho^2)} \right) \right] \mathbf{u} \cdot \nabla \psi \, dx$$

$$- \int_\Omega (\mathbf{S}\mathbf{u} \cdot \nabla \psi + \delta \vartheta^{-1} \psi) \, dx + \int_\Omega \kappa_\delta \nabla \vartheta \cdot \nabla \psi \, dx$$

$$+ \int_{\partial\Omega} (L + \delta \vartheta^{B-1})(\vartheta - \vartheta_0) \psi \, dS + \int_\Omega \vartheta \sum_{k,l=1}^n c_{vk} Y_k D_{kl} \nabla Y_l \cdot \nabla \psi \, dx$$

$$= \int_\Omega \varrho \mathbf{f} \cdot \mathbf{u} \psi \, dx + \delta \int_\Omega \left(\frac{1}{\beta - 1} \varrho^\beta + \varrho^2 \right) \mathbf{u} \cdot \nabla \psi) \, dx \qquad (8.9)$$

  for any $\psi \in C^\infty(\overline{\Omega})$.

Here and in the sequel, the bar denotes the weak limit of the corresponding term, for instance

$$\overline{\varrho^\gamma} = \lim_{\varepsilon \to 0^+} \varrho_\varepsilon^\gamma$$

in the sense of the weak convergence in $L^{\frac{5\beta}{3\gamma}}(\Omega)$. Recall that since $\varrho \in L^2(\Omega)$, we also have (see, e.g., [9]) the continuity equation satisfied in the renormalized sense (cf. Definition 3.4).

Next, we should also show the strong convergence of the density. Since this can be shown exactly as in [20] and we meet similar problems in the next subsection when letting $\delta \to 0^+$, we skip here all details. Hence we may remove the bars in (8.8) and (8.9).

8.3 Limit passage $\delta \to 0^+$

This last subsection is devoted to the limit passage $\delta \to 0^+$, i.e. to the final step of the proof of Theorem 3.5. Similarly as in the previous section, we may use the entropy inequality to deduce a-priori estimates independent of δ. However, in this case, we must proceed more carefully.

The main idea is that unlike the previous limit passage, we subtract the total energy balance tested by a constant function from our entropy estimate and deduce estimate of the type

$$\|\nabla\vec{Y}\|_2^2 + \|\nabla\vartheta^{\frac{m}{2}}\|_2^2 + \|\mathbf{u}\|_{1,2}^2 + \|\vartheta^{-1}\|_{1,\partial\Omega} + \delta(\|\nabla\vartheta^{\frac{B}{2}}\|_2^2 + \|\nabla\vartheta^{-\frac{1}{2}}\|_2^2 + \|\vartheta^{-2}\|_1 + \|\vartheta^{B-2}\|_{1,\partial\Omega})$$
$$\leq C(1 + \delta\|\vartheta^{B-1}\|_{1,\partial\Omega}). \tag{8.10}$$

Recall that as $\sigma_Y = 1$, we also have $\|\vec{Y}\|_\infty \leq C$. From the total energy balance tested by a constant function we deduce

$$\|\vartheta\|_{1,\partial\Omega} + \delta\|\vartheta^B\|_{1,\partial\Omega} \leq C\Big(1 + \Big|\int_\Omega \varrho\mathbf{u}\cdot\mathbf{f}\,dx\Big| + \delta\|\vartheta^{-1}\|_1\Big). \tag{8.11}$$

To get rid of the δ-dependent terms we use once more the Bogovskii-type of estimates, i.e. we test the momentum equation by a solution $\boldsymbol{\phi}$ to

$$\operatorname{div}\boldsymbol{\phi} = \varrho - \frac{M}{|\Omega|}, \quad \boldsymbol{\phi}|_{\partial\Omega} = \mathbf{0}.$$

This allows us to get a bound

$$\delta\|\varrho\|_{\beta+1}^{\beta+1} \leq C$$

which can be employed to get rid of the δ-dependent terms in (8.10) and (8.11). It yields

$$\|\nabla\vec{Y}\|_2^2 + \|\vec{Y}\|_\infty + \|\nabla\vartheta^{\frac{m}{2}}\|_2^2 + \|\mathbf{u}\|_{1,2}^2 + \|\vartheta^{-1}\|_{1,\partial\Omega} + \delta(\|\nabla\vartheta^{\frac{B}{2}}\|_2^2 + \|\nabla\vartheta^{-\frac{1}{2}}\|)_2^2 + \|\vartheta^{-2}\|_1 + \|\vartheta^{B-2}\|_{1,\partial\Omega} \leq C \tag{8.12}$$

and

$$\|\vartheta\|_{3m} \leq C(1 + \|\varrho\|_{\frac{6}{5}}). \tag{8.13}$$

The details can be again found in [20].

Thus we need now additional δ-independent estimates of the density. To this aim, we employ again the Bogovskii-type estimates i.e. we test the momentum equation by a solution $\boldsymbol{\phi}$ to

$$\operatorname{div}\boldsymbol{\phi} = \varrho^\alpha - \frac{1}{|\Omega|}\int_\Omega \varrho^\alpha\,dx, \quad \boldsymbol{\phi}|_{\partial\Omega} = \mathbf{0}.$$

We obtain

$$\int_\Omega (\pi(\varrho,\vartheta) + \delta(\varrho^\beta + \varrho^2))\varrho^\alpha\,dx = -\int_\Omega \varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla\boldsymbol{\phi}\,dx$$
$$-\int_\Omega (\mathbf{S}(\vartheta,\nabla\mathbf{u}) : \nabla\boldsymbol{\phi} - \varrho\mathbf{f}\cdot\boldsymbol{\phi})\,dx + \frac{1}{|\Omega|}\int_\Omega (p(\varrho,\vartheta) + \delta(\varrho^\beta + \varrho^2))\,dx\int_\Omega \varrho^\alpha\,dx$$

and we show the estimates of the most restrictive terms. The convective term yields

$$\Big|\int_\Omega \varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla\boldsymbol{\phi}\,dx\Big| \leq C\|\varrho\|_{\gamma+\alpha}\|\mathbf{u}\|_6^2\|\varrho\|_{\frac{3(\gamma+\alpha)\alpha}{2(\gamma+\alpha)-3}}^\alpha,$$

which leads to the restriction $0 < \alpha \leq 2\gamma - 3$ for $\gamma > \frac{3}{2}$. The stress tensor can be estimated as

$$\Big|\int_\Omega \mathbf{S}(\vartheta,\nabla\mathbf{u}) : \nabla\boldsymbol{\phi}\,dx\Big| \leq C(1 + \|\vartheta\|_{3m})\|\mathbf{u}\|_{1,2}\|\boldsymbol{\phi}\|_{\frac{6m}{3m-2}} \leq C(1 + \|\varrho\|_{\frac{6}{5}})\|\varrho\|_{\frac{6m\alpha}{3m-2}},$$

which leads to the restriction $0 < \alpha \leq \frac{3m-2}{3m+2}\gamma$, $m > \frac{2}{3}$.

Thanks to this we get, in addition to (8.12) and (8.13),

$$\|\varrho\|_{\gamma+\alpha} \leq C, \quad \text{where } 0 < \alpha \leq \min\Big\{2\gamma - 3, \frac{3m-2}{3m+2}\gamma\Big\}$$

with $\gamma > \frac{3}{2}$ and $m > \frac{2}{3}$. Using these bounds, we may pass to the limit in our system to get the weak formulation of the continuity equation

$$\int_\Omega \varrho \mathbf{u} \cdot \nabla \psi \, dx = 0$$

for all $\psi \in C^\infty(\overline{\Omega})$, and in the weak formulation of the approximate momentum equation

$$-\int_\Omega (\varrho(\mathbf{u} \otimes \mathbf{u}) : \nabla \boldsymbol{\phi} + \mathbf{S} : \nabla \boldsymbol{\phi}) \, dx - \int_\Omega (\varrho\vartheta + \overline{\varrho^\gamma}) \operatorname{div} \boldsymbol{\phi} \, dx = \int_\Omega \varrho \mathbf{f} \cdot \boldsymbol{\phi} \, dx$$

for all $\boldsymbol{\phi} \in C_0^\infty(\Omega)$. Note that here we used that ϱ_δ converges weakly to ϱ in $L^{\gamma+\alpha}(\Omega)$, ϑ_δ converges strongly to ϑ in $L^q(\Omega)$ for any $q < 3m$. Again, $\overline{\varrho^\gamma}$ stands for the weak limit of ϱ_δ^γ in $L^{(\gamma+\alpha)/\gamma}(\Omega)$. Note also that all the δ-dependent terms tend strongly in $L^1(\Omega)$ to zero. It is also not so difficult to pass to the limit in the species balance equations to get

$$-\int_\Omega Y_k \varrho \mathbf{u} \cdot \nabla \psi \, dx + \int_\Omega Y_k \sum_{l=1}^n D_{kl} \nabla Y_l \cdot \nabla \psi \, dx = \int_\Omega \omega_k \psi \, dx$$

for all $\psi \in C^\infty(\overline{\Omega})$, $k = 1, 2, \ldots, n$, under the assumptions that the growth in the D_{kl} term with respect to temperature is below $\frac{3m}{2}$. Anyway, we get a stronger restriction below.

We get the weak formulation of the total energy balance in the form

$$-\int_\Omega \left[\varrho\vartheta \sum_{k=1}^n c_{vk} Y_k + \frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho\vartheta + \frac{\gamma}{\gamma-1}\overline{\varrho^\gamma} \right] \mathbf{u} \cdot \nabla \psi \, dx - \int_\Omega \mathbf{S}\mathbf{u} \cdot \nabla \psi \, dx$$

$$+ \int_\Omega \kappa \nabla\vartheta \cdot \nabla\psi \, dx + \int_{\partial\Omega} L(\vartheta - \vartheta_0)\psi \, dS + \int_\Omega \vartheta \sum_{k,l=1}^n c_{vk} Y_k D_{kl} \nabla Y_l \cdot \nabla\psi \, dx = \int_\Omega \varrho \mathbf{f} \cdot \mathbf{u}\psi \, dx$$

satisfied for any $\psi \in C^\infty(\overline{\Omega})$, under several additional restrictions. First of all, we must assume that

$$D_{kl}(\vartheta, \cdot) \leq C(1 + \vartheta^a)$$

for $a < \frac{3m-2}{2}$. In order to pass to the limit in the term $\varrho_\delta|\mathbf{u}_\delta|^2\mathbf{u}_\delta$, we must require that the density is bounded in $L^p(\Omega)$ for some $p > 2$. To pass to the limit in the term with the stress tensor, we must require that $\vartheta_\delta \to \vartheta$ strongly in $L^q(\Omega)$ for $q > 3$. This leads to the restrictions $m > 1$ and $\gamma + \alpha > 2$, i.e. $\gamma > \frac{5}{3}$ and $m > 1$. In order to finish the proof, we have to remove the bars over certain nonlinear quantities which requires that $\varrho_\delta \to \varrho$ strongly in $L^1(\Omega)$.

The last step of the proof of our main theorem is hence to show the strong convergence of the density. Here we follow the ideas of P. L. Lions [14] developed for the isentropic flows, used in the context of the heat conducting fluid in [16] for constant viscosities and in [22] for temperature-dependent viscosities. Due to the uniform L^2 bounds of the density, we can directly employ the renormalized solution to the continuity equation and no truncations in the test functions are needed.

We first verify the validity of the effective viscous flux identity in the form

$$\overline{(\varrho\vartheta + \varrho^\gamma)\varrho^\alpha} - \overline{(\mu(\vartheta) + \nu(\vartheta))\varrho^\alpha \operatorname{div} \mathbf{u}} = \overline{\varrho\vartheta + \varrho^\gamma}\,\overline{\varrho^\alpha} - \overline{(\mu(\vartheta) + \nu(\vartheta))\varrho^\alpha} \operatorname{div} \mathbf{u},$$

with $\alpha > 0$ from the estimates above. It can be shown exactly as in [21, Lemma 8] (with α instead of Θ). It is based on testing the momentum equation before and after limit passage by $\xi(x)\nabla\Delta^{-1}(1_\Omega\varrho_\delta^\alpha)$ and by $\xi(x)\nabla\Delta^{-1}(1_\Omega\overline{\varrho^\alpha})$, respectively, and on proving certain limit passages via a compensated compactness technique. Next we may verify (note that both ϱ and ϱ_δ belong to $L^2(\Omega)$) that the following versions of the renormalized continuity equation hold true:

$$\operatorname{div}(\varrho_\delta^\alpha \mathbf{u}_\delta) + (\alpha - 1)\varrho_\delta^\alpha \operatorname{div} \mathbf{u}_\delta = 0, \quad \operatorname{div}(\overline{\varrho^\alpha}\mathbf{u}) + (\alpha - 1)\overline{\varrho^\alpha \operatorname{div} \mathbf{u}} = 0,$$

both in the sense of distributions in $\mathbb{R}^3$. Hence (see [23, Lemma 4.39])

$$\operatorname{div}((\overline{\varrho^\alpha})^{\frac{1}{\alpha}}\mathbf{u}) = \frac{1-\alpha}{\alpha}(\overline{\varrho^\alpha})^{\frac{1}{\alpha}-1}(\overline{\varrho^\alpha \operatorname{div} \mathbf{u}} - \overline{\varrho^\alpha} \operatorname{div} \mathbf{u}).$$

Therefore

$$\operatorname{div}(((\overline{\varrho^\alpha})^{\frac{1}{\alpha}} - \varrho)\mathbf{u}) = \frac{1-\alpha}{\alpha}(\overline{\varrho^\alpha})^{\frac{1}{\alpha}-1}\frac{\overline{(\varrho\vartheta + \varrho^\gamma)\varrho^\alpha} - \overline{\varrho\vartheta + \varrho^\gamma}\,\overline{\varrho^\alpha}}{\mu(\vartheta) + \nu(\vartheta)}, \tag{8.14}$$

again in the sense of distributions in $\mathbb{R}^3$. Testing (8.14) by $\psi = 1$ reads

$$\int_\Omega (\overline{\varrho^\alpha})^{\frac{1}{\alpha}-1}\frac{\overline{(\varrho\vartheta + \varrho^\gamma)\varrho^\alpha} - \overline{\varrho\vartheta + \varrho^\gamma}\,\overline{\varrho^\alpha}}{\mu(\vartheta) + \nu(\vartheta)}\,dx = 0.$$

It is easy to verify that $\varrho_\delta \to 0$ in $L^1(\{\varrho^\alpha = 0\})$. Thus the monotonicity of $t \mapsto t^\gamma$ and the strong convergence of the temperature yield

$$\overline{\varrho^{\gamma+\alpha}} + \vartheta\overline{\varrho^{1+\alpha}} = \overline{\varrho^\gamma}\,\overline{\varrho^\alpha} + \vartheta\varrho\overline{\varrho^\alpha}.$$

As $\vartheta > 0$ a.e. in Ω, we obtain $\overline{\varrho^\alpha} = \varrho^\alpha$ which implies

$$\overline{\varrho^{1+\alpha}} = \varrho^{1+\alpha} \quad \text{a.e. in } \Omega,$$

whence the strong convergence of the density. The proof of Theorem 3.5 is now complete.

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