Sliding Mode Control for Nonlinear Manipulator Systems

Beibei Zhang[a], Xiao Yang[b], Dongya Zhao[a*],
Sarah K. Spurgeon[c], Xinggang Yan[d]

a. College of Chemical Engineering, China University of Petroleum, Qingdao 266580, China

b. School of Automation, Beijing Institute of Technology, Beijing, 100081, China

c. Department of Electronic & Electrical Engineering, University College London, Torrington Place, London WC1E 7JE, United Kingdom (s.spurgeon@ucl.ac.uk)

d. School of Engineering & Digital Arts, University of Kent, Canterbury, United Kingdom (X.Yan@kent.ac.uk)
** Corresponding author' email: dyzhao@upc.edu.cn; dongyazhao@139.com*

Abstract: A sliding mode control is developed for nonlinear manipulator systems. An improved version of the exponential reaching law is presented to ensure the states converge to the sliding surface in finite time. In the presence of system uncertainty and external disturbances, the system states converge to a small region centered at the origin within a finite time and thereafter will asymptotically converge to the equilibrium point. The robustness and convergence properties of the proposed approach are demonstrated from both the theoretical point of view and also using simulation studies.

Keywords: nonlinear sliding mode, manipulator system, tracking error, exponential reaching law, robustness, residual set

1. INTRODUCTION

Increasing labour costs coupled with the need to improve product quality place the development of manipulator systems in the spotlight (Motoi, Shimono, Kubo, & Kawamura, 2014). The development of appropriate control systems to provide high accuracy is a key design challenge. The presence of system uncertainty and disturbances is a significant issue in achieving high control performance as there is likely to be some discrepancy between the actual manipulator system and the mathematical model used to develop the control system. In conventional controller design, the control algorithm can be based on a nonlinear compensation method but this approach is complex and costly for implementation (Chen, Zhang, Wang, & Zeng, 2007).

A great deal of research has been conducted to develop control strategies for manipulator systems. One such approach is sliding mode control (SMC) which is attractive because of its excellent robustness to both system uncertainty and disturbances (Moura, Elmali, & Olgac, 1997). The classical approach to sliding mode controller design defines a linear sliding mode dynamics. The sliding mode is attained in finite time following the reaching phase and asymptotic convergence to the equilibrium point takes place in the sliding phase. However, in practice, the asymptotic properties of the linear sliding mode control approach limit performance and high gain may be needed to achieve fast convergence, which may lead to control input saturation (Barambones & Etxebarria, 2001; Levant, 2007). To overcome these problems, nonlinear sliding mode approaches have been developed (Barambones & Etxebarria, 2002; Kao, Wang, Xie, Karimi, & Li, 2015; Zhang, Su, & Lu, 2015) where the so-called terminal sliding mode (TSM) control approach is an attractive one (Man, Paplinski, & Wu, 1995; Wu, Yu, & Man, 1998). This approach not only achieves finite-time convergence in the sliding phase, but also provides better robustness when compared with traditional linear SMC.

TSM control has been applied in numerous industrial applications including robotic manipulator tracking control, chemical process control and marine power engineering (Wang, Chai, & Zhai, 2009). However, the existence of negative fractional powers in TSM control may lead to singularity problems (Tang, 1998; Yu, Yu, & Zhihong, 2000; Zhao, Zhu, & Dubbeldam, 2014). To avoid this phenomenon, non-singular terminal sliding mode (NTSM) has been developed (Feng, Yu, & Man, 2002). This approach is successful in avoiding singularity but in order to reduce chattering, a boundary layer technique is employed and the properties of finite-time stability are lost. To avoid this problem, a continuous NTSM control with a rapid power reaching law has been proposed which achieves global finite-time stability (Yu, Yu, Shirinzadeh, & Man, 2005).

In this paper, a novel nonlinear sliding mode control is proposed and applied in manipulator systems which seeks to keep the advantages of rapid convergence whilst avoiding the singularity problem of TSM control. An improved version of the continuous exponential reaching law of (Fallaha, Saad, Kanaan, & Al-Haddad, 2011) is proposed to achieve faster tracking control of manipulator systems. The robustness analysis is performed by using Lyapunov theory. Simulation results demonstrate the superiority of the proposed approach including continuous control, eliminating problems with singularity, faster convergence and greater robustness in comparison with the TSM control with a discontinuous reaching law as proposed in (Fallaha et al., 2011).

Notations: For $x = col(x_1, x_2, \cdots, x_n) \in R^n$, $\chi = col(\chi_1, \chi_2, \cdots, \chi_n) \in R^n$:

$$\boldsymbol{sig}(\boldsymbol{x})^{\gamma} = \left[|x_1|^{\gamma} sign(x_1), \cdots, |x_n|^{\gamma} sign(x_n) \right]^T ,$$

$$\frac{2\boldsymbol{\chi}}{\pi} \boldsymbol{sin}\left(\frac{\pi}{2\boldsymbol{\chi}} \boldsymbol{x} \right) = \left[\frac{2\chi_1}{\pi} sin\left(\frac{\pi}{2\chi_1} x_1 \right), \cdots, \frac{2\chi_n}{\pi} sin\left(\frac{\pi}{2\chi_n} x_n \right) \right]^T ,$$

$$|\boldsymbol{x}|^{\gamma} = \left[|x_1|^{\gamma_1}, \cdots, |x_n|^{\gamma_n} \right]^T , \ \boldsymbol{x}^{\gamma} \boldsymbol{y} = \left[x_1^{\gamma} y_1, \cdots, x_n^{\gamma} y_n \right]^T ,$$

$$\boldsymbol{sign}(\boldsymbol{x}) = \left[sign(x_1), \cdots, sign(x_n) \right]^T ,$$

$\lambda_{\min(\max)}(.)$ represents the minimum (maximum) eigenvalue of a matrix.

2. PROBLEM FORMULATION

Consider an n-joint rigid manipulator system (Siciliano, Sciavicco, Villani, & Oriolo, 2011)

$$M(\boldsymbol{q})\ddot{\boldsymbol{q}} + C(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + G(\boldsymbol{q}) = \boldsymbol{u} + \boldsymbol{d} \tag{1}$$

where $\boldsymbol{q}(t) \in R^n$ denotes the angular displacement vector. $M(\boldsymbol{q}) \in R^{n\times n}$ represents the inertia matrix, $C(\boldsymbol{q},\dot{\boldsymbol{q}}) \in R^{n\times n}$ denotes the centrifugal and Coriolis term and $G(\boldsymbol{q}) \in R^n$ is the gravity vector. $\boldsymbol{u} \in R^n$ is applied joint torque and $\boldsymbol{d} \in R^n$ represents the external disturbance which is bounded $\|\boldsymbol{d}\| \le D$.

Assume that in manipulator dynamics (1) has known and unknown parts

$$\begin{aligned} M(\boldsymbol{q}) &= M_0(\boldsymbol{q}) + \Delta M(\boldsymbol{q}) \\ C(\boldsymbol{q},\dot{\boldsymbol{q}}) &= C_0(\boldsymbol{q},\dot{\boldsymbol{q}}) + \Delta C(\boldsymbol{q},\dot{\boldsymbol{q}}) \\ G(\boldsymbol{q}) &= G_0(\boldsymbol{q}) + \Delta G(\boldsymbol{q}) \end{aligned} \tag{2}$$

where $M_0(\boldsymbol{q}) \in R^{n\times n}$, $C_0(\boldsymbol{q},\dot{\boldsymbol{q}}) \in R^{n\times n}$ and $G_0(\boldsymbol{q}) \in R^n$ denote the known nominal parts and $\Delta M(\boldsymbol{q})$,and $\Delta C(\boldsymbol{q},\dot{\boldsymbol{q}})$ denote the unknown uncertain terms.

According to the expression (2), the manipulator dynamics (1) become

$$M_0(\boldsymbol{q})\ddot{\boldsymbol{q}} + C_0(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + G_0(\boldsymbol{q}) = \boldsymbol{u} + \boldsymbol{\rho}(t) \tag{3}$$

where $\boldsymbol{\rho}(t)$ represents all the uncertain dynamics and external disturbances in the system, and

$$\boldsymbol{\rho}(t) = -\Delta M(\boldsymbol{q})\ddot{\boldsymbol{q}} - \Delta C_0(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \Delta G_0(\boldsymbol{q}) + \boldsymbol{d} \tag{4}$$

Define the position error as

$$\boldsymbol{x}(t) = \boldsymbol{q}(t) - \boldsymbol{q}_d(t) \tag{5}$$

where $\boldsymbol{q}(t) = [q_1(t), \ldots, q_n(t)]^T$ is the position vector, $\boldsymbol{q}_d(t) = [q_{d1}(t), \ldots, q_{dn}(t)]^T$ is the desired trajectory of the manipulator and $\boldsymbol{x}(t) = [x_1(t), \ldots, x_n(t)]^T$ represents the position error.

TSM controllers and NTSM controllers have been designed for manipulator systems and achieve good trajectory tracking (Wu et al., 1998; Yu et al., 2005). The sliding surface with TSM and NTSM are given by

$$\boldsymbol{s} = \dot{\boldsymbol{x}} + \kappa \boldsymbol{x}^{\frac{q}{p}} \tag{6a}$$

$$\boldsymbol{s} = \boldsymbol{x} + \frac{1}{\kappa} \dot{\boldsymbol{x}}^{\frac{p}{q}} \tag{6b}$$

The corresponding first derivatives can be expressed, respectively, as

$$\dot{\boldsymbol{s}} = \ddot{\boldsymbol{x}} + \kappa \frac{q}{p} \boldsymbol{x}^{\frac{q}{p}-1} \dot{\boldsymbol{x}} \tag{7a}$$

$$\dot{\boldsymbol{s}} = \dot{\boldsymbol{x}} + \frac{1}{\kappa} \frac{p}{q} \dot{\boldsymbol{x}}^{\frac{p}{q}-1} \ddot{\boldsymbol{x}} \tag{7b}$$

where $\kappa \ge 1$, $q < p$ and q , p are positive, odd numbers. From (6a) and 6(b), if $\boldsymbol{s} = 0$, TSM surface and NTSM surface are equivalent and finite time stable (Feng et al., 2002). According to (7a) and (7b), TSM has a singularity problem due to $p/q - 1 < 0$ and NTSM avoids the singular problem because of $0 < q/p - 1 < 1$ (Yu et al., 2005). Note that convergence speed of the NTSM control is slower in a neighbourhood of the origin when compared with a TSM control (Chao-Xu & Xing-Huo, 2014).

The objective of this paper is to design an improved sliding mode control for manipulator trajectory tracking in order to achieve rapid convergence and remove singularity problems.

3. NONLINEAR SLIDING MODE DESIGN

Before introducing the sliding mode design, a nonlinear function is initially defined

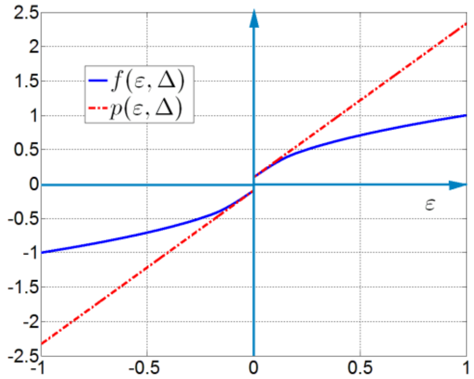
$$f(\varepsilon, \Delta) = \begin{cases} |\varepsilon|^{\alpha} sign(\varepsilon), & |\varepsilon| > \Delta \\ \dfrac{2\Delta}{\pi} sin\left(\dfrac{\pi}{2\Delta} \varepsilon \right) + \alpha \Delta^{\alpha-1} \varepsilon, & |\varepsilon| \le \Delta \end{cases} \tag{8a}$$

$$\Delta = \left(\frac{\pi}{2}(1-\alpha)\right)^{\frac{1}{1-\alpha}} \quad (8b)$$

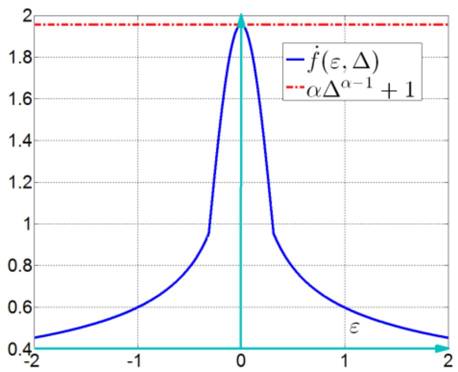
where $\varepsilon \in R_+$ is a scalar, $sign(\varepsilon)$ is the standard signum function and $\alpha \in R_+$ and $\Delta \in R_+$ are parameters in the region $(0,1)$. The derivative of $f(\varepsilon,\Delta)$ is given as

$$\dot{f}(\varepsilon,\Delta) = \begin{cases} \alpha|\varepsilon|^{\alpha-1} sign(\varepsilon), |\varepsilon| > \Delta \\ cos\left(\frac{\pi}{2\Delta}\varepsilon\right) + \alpha\Delta^{\alpha-1}, |\varepsilon| \le \Delta \end{cases} \quad (9)$$

The function $f(x)$ and $\dot{f}(x)$ are continuous. From (8) and (9), one can see that $f(x)$ and $\dot{f}(x)$ are bounded by $|p(\varepsilon,\Delta)| = (\Delta^{\alpha-1}+1)|\varepsilon|$, $p(\varepsilon,\Delta) = (\Delta^{\alpha-1}+1)\varepsilon$ and $\alpha\Delta^{\alpha-1}+1$, respectively (See Appendix A). This is illustrated in Fig. 1.



(a) $f(\varepsilon,\Delta)$ and its bound function $p(\varepsilon,\Delta)$



(b) $\dot{f}(\varepsilon,\Delta)$ and its bound $\alpha\Delta^{\alpha-1}+1$

Fig. 1. $f(\varepsilon,\Delta)$ and $\dot{f}(\varepsilon,\Delta)$ with their bound ($\Delta \approx 0.3129$, $\alpha = 0.6$)

In terms of the properties of $f(\varepsilon,\Delta)$ and $\dot{f}(\varepsilon,\Delta)$, a nonlinear sliding surface is defined by

$$\boldsymbol{s} = \dot{\boldsymbol{x}} + \Lambda_1\boldsymbol{x} + \Lambda_2\boldsymbol{\Psi}(\boldsymbol{x},\boldsymbol{\delta}) \quad (10)$$

where $\boldsymbol{\Psi}(\boldsymbol{x},\boldsymbol{\delta}) = [f(x_1,\delta_1),\cdots,f(x_n,\delta_n)]^T$, $\Lambda_1 = diag\{\Lambda_{11},\Lambda_{12},\cdots,\Lambda_{1n}\}$, $\Lambda_2 = diag\{\Lambda_{21},\cdots,\Lambda_{2n}\}$ and $\boldsymbol{s} = [s_1,s_2,\cdots,s_n]^T$, $\boldsymbol{\delta} = [\delta_1,\delta_2,\cdots,\delta_n]^T$, $0 < \delta_i < 1$.

Lemma 1: If the tracking error exhibits a sliding mode on the sliding surface (10) in finite time, the error state $\boldsymbol{x}$ will converge to a small region with $\|\boldsymbol{x}\| \le \|\boldsymbol{\delta}\|$ in finite time T_{s1} and then converge to zero asymptotically,

$$T_{s1} = \frac{1}{(1-\alpha)\lambda_{min}(\Lambda_1)} \times \left\{Ln\left(V(x(0))^{\Phi} + 2^{-\Phi}\omega\right) - Ln\left(V(\delta_i)^{\Phi} + 2^{-\Phi}\omega\right)\right\} \quad (11)$$

where $\omega = \lambda_{min}(\Lambda_2)/\lambda_{min}(\Lambda_2)$ and $\Phi = (1-\alpha)/2$.

Proof: In the sliding phase, $\boldsymbol{s} = 0$. Also, it should be noted that the nonlinear function $f(x_i,\delta_i)$ is a piecewise continuous function. Therefore, calculation of the settling time must be considered in the following cases.

1) Case A: in the region $|x_i| \ge \delta_i$, when $s_i = 0$, from (10)

$$\dot{\boldsymbol{x}} + \Lambda_1\boldsymbol{x} + \Lambda_2\boldsymbol{sig}(\boldsymbol{x})^{\alpha} = 0 \quad (12)$$

Consider a Lyapunov function candidate $V_1 = 0.5\boldsymbol{x}^T\boldsymbol{x}$ and substitute the expression (12) into the first derivative to yield

$$\begin{aligned} \dot{V}_1 = \boldsymbol{x}^T\dot{\boldsymbol{x}} &\le -\lambda_{min}(\Lambda_1)\sum_{i=1}^{n} x_i^2 - \lambda_{min}(\Lambda_2)\sum_{i=1}^{n}|x_i|^{\alpha+1} \\ &\le -2\lambda_{min}(\Lambda_1)V - 2^{\frac{\alpha+1}{2}}\lambda_{min}(\Lambda_2)V^{\frac{\alpha+1}{2}} \end{aligned} \quad (13)$$

By solving the differential equation, the settling time in the region $|x_i| \in [\delta_i, x_i(0)]$ can be obtained as

$$T_{s1} = \frac{1}{(1-\alpha)\lambda_{min}(\Lambda_1)} \times \left\{Ln\left(V(x(0))^{\Phi} + 2^{-\Phi}\omega\right) - Ln\left(V(\delta_i)^{\Phi} + 2^{-\Phi}\omega\right)\right\} \quad (14)$$

where $\omega = \lambda_{min}(\Lambda_2)/\lambda_{min}(\Lambda_1)$ and $\Phi = (1-\alpha)/2$. Therefore, the error state $\boldsymbol{x}$ will converges to a small region with $\|\boldsymbol{x}\| \le \|\boldsymbol{\delta}\|$ in finite time.

2) Case B: in the region $|x_i| < \delta_i$, $s_i = 0$ and the relationship between $\dot{\boldsymbol{x}}$ and $\boldsymbol{x}$ can be described as

$$\dot{\boldsymbol{x}} + \Lambda_1\boldsymbol{x} + \Lambda_2\left\{\frac{2\boldsymbol{\delta}}{\pi}\boldsymbol{sin}\left(\frac{\pi}{2\boldsymbol{\delta}}\boldsymbol{x}\right) + \alpha\boldsymbol{\delta}^{\alpha-1}\boldsymbol{x}\right\} = 0 \quad (15)$$

Similarly, consider the same Lyapunov function candidate $V_2 = 0.5\boldsymbol{x}^T\boldsymbol{x}$ and substitute from (15) to yield

$$\dot{V}_2 = -\sum_{i=1}^{n} \Lambda_{1i} x_i^2 - \sum_{i=1}^{n} \left\{ \Lambda_{2i} \left[\frac{2\delta_i}{\pi} sin\left(\frac{\pi}{2\delta_i} x_i \right) x_i + \alpha \delta_i^{\alpha-1} x_i^2 \right] \right\}$$
$$= -\sum_{i=1}^{n} \left\{ \left(\Lambda_{1i} + \Lambda_{2i} \alpha \delta_i^{\alpha-1} \right) x_i^2 \right\}$$
$$= -2\lambda_{min} \left(\Lambda_1 + \Lambda_2 \alpha diag \left\{ \delta_i^{\alpha-1} \right\} \right) V_2 \tag{16}$$

Let $\psi = \lambda_{min}(\Lambda_1 + \Lambda_2 \alpha diag\{\delta_i^{\alpha-1}\})$. The following expression results

$$V_2 = V_2\left(x(0)\right) e^{-2\psi t} \tag{17}$$

Q.E.D..

4. CONTROLLER DESIGN

In this section, the control law is designed by using the proposed novel nonlinear sliding function with an improved version of the exponential reaching law. Stability of the resulting closed-loop system is established by using Lyapunov theory.

As in the literature (Man & Yu, 1997), it is assumed that

$$\left\| \boldsymbol{\rho}(t) \right\| \le b_0 + b_1 \left\| \boldsymbol{q} \right\| + b_2 \left\| \dot{\boldsymbol{q}} \right\|^2 \tag{18}$$

where b_0, b_1 and b_2 are positive constants. This assumption will be used in the sliding mode control design.

Theorem 1: For the manipulator system, with the nonlinear sliding surface defined by (10), if the control law is designed as

$$\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}_1$$
$$\boldsymbol{u}_0 = M_0(\boldsymbol{q})\left(\ddot{\boldsymbol{q}}_d - \Lambda_1 \dot{\boldsymbol{x}} - \Lambda_2 \dot{\boldsymbol{\Psi}}(\boldsymbol{x})\dot{\boldsymbol{x}} \right) + C_0(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + G_0(\boldsymbol{q}) \tag{19}$$
$$\boldsymbol{u}_1 = -\left(b_0 + b_1 \left\| \boldsymbol{q} \right\| + b_2 \left\| \dot{\boldsymbol{q}} \right\|^2 \right) \frac{L(\boldsymbol{s})}{\sqrt{n}} M_0(\boldsymbol{q}) \boldsymbol{sig}(\boldsymbol{s})^{\varphi}$$

where $0 < \varphi < 1$, $L(\boldsymbol{s}) = diag\{1/l_{11}, \cdots, 1/l_{nn}\}$, $l_{ii} = \delta_i + (1-\delta_i)e^{-\gamma|s_i|^p}$, $\gamma > 0$ and $p > 0$, $i = 1, 2, \cdots, n$.

(i) For the nominal manipulator system where $\boldsymbol{\rho}(t) = 0$, the tracking error asymptotically converges to zero.

(ii) For the uncertain manipulator system where $\boldsymbol{\rho}(t) \ne 0$, the error trajectories will converge to a neighborhood of the nonlinear sliding surface $\boldsymbol{s} = 0$ as

$$\left\| \boldsymbol{s} \right\| \le 2^{\frac{1}{2\varphi}} \left(2\delta_i^2 - 2\delta_i + 1 \right)_{max}^{\frac{1}{2\varphi}} \tilde{\beta}_0^{\frac{1}{\varphi}} \tag{20}$$

Proof: Consider the Lyapunov function candidate $V_3 = 0.5\boldsymbol{s}^T\boldsymbol{s}$. By differentiating V_3 with respect to time and substituting from (19), it follows that

$$\dot{V}_3 = \boldsymbol{s}^T M_0^{-1}(\boldsymbol{q})\left(\boldsymbol{u}_1 + \boldsymbol{\rho}(t) \right) \tag{21}$$

From (19), the following inequality (see Appendix B) results

$$\sqrt{n} \le \frac{\sqrt{n}}{\sqrt{2\left(2\delta_i^2 - 2\delta_i + 1 \right)_{max}}} \le \left\| L(\boldsymbol{s}) \right\| \le \frac{\sqrt{n}}{\delta_{imin}} \tag{22}$$

where $\delta_{imin} = min\{\delta_i\} \in (0,1)$.

(i) For $\boldsymbol{\rho}(t) = 0$, substituting the inequality (19) and (22) into (21), it follows that

$$\dot{V}_3 \le -2^{\frac{\varphi+1}{2}} b_0 V_3^{\frac{\varphi+1}{2}} \tag{23}$$

According to finite time theory, the nonlinear sliding mode will be reached in finite time. The settling time can be obtained as $T_r \le |2V(0)|^{(1-\varphi)/2} / (b_0(1-\varphi))$. Thereafter, the manipulator tracking errors enter the sliding mode phase and asymptotically converge to the equilibrium point.

(ii) For $\rho(t) \ne 0$, substitute the designed control law (19) and inequality (22) into (21) to yield

$$\dot{V}_3 \le -\left(\frac{\left(b_0 + b_1 \left\| \boldsymbol{q} \right\| + b_2 \left\| \dot{\boldsymbol{q}} \right\|^2 \right) \left\| \boldsymbol{s} \right\|^{\varphi}}{\sqrt{2\left(2\delta_i^2 - 2\delta_i + 1 \right)_{max}}} - \left\| M_0^{-1}(\boldsymbol{q}) \right\| \left\| \boldsymbol{\rho}(t) \right\| \right) \left\| \boldsymbol{s} \right\| \tag{24}$$

Hence, finite-time stability can be guaranteed if $\|\boldsymbol{s}\|^{\varphi} > \left\| M_0^{-1}(\boldsymbol{q}) \right\| \sqrt{2\left(2\delta_i^2 - 2\delta_i + 1 \right)_{max}}$. Assume there exists condition $\left\| M_0^{-1}(\boldsymbol{q}) \right\| \le \tilde{\beta}_0$, then the final residual set given by

$$\left\| \boldsymbol{s} \right\| \le 2^{\frac{1}{2\varphi}} \left(2\delta_i^2 - 2\delta_i + 1 \right)_{max}^{\frac{1}{2\varphi}} \tilde{\beta}_0^{\frac{1}{\varphi}} \tag{25}$$

can be reached in finite time.

5. SIMULATION

The dynamic equations of a two-link manipulator can be described as (Feng et al., 2002; Yu et al., 2005).

$$\begin{bmatrix} \alpha_{11}(q_2) & \alpha_{12}(q_2) \\ \alpha_{21}(q_2) & \alpha_{22}(q_2) \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} -\beta(q_2)\dot{q}_1 & -2\beta(q_2)\dot{q}_1 \\ 0 & \beta(q_2)\dot{q}_2 \end{bmatrix}.$$
$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_1 \end{bmatrix} + \begin{bmatrix} \gamma_1(q_1, q_2) \\ \gamma_2(q_1, q_2) \end{bmatrix} g = u + \rho(t)$$

$$\begin{aligned} &\alpha_{11}(q_2) = (m_1 + m_2) r_2^2 + 2m_2 r_1 r_2 \cos(q_2) + j_1 \\ &\alpha_{12}(q_2) = m_2 r_2^2 + m_2 r_1 r_2 \cos(q_2) \\ &\alpha_{12} = \alpha_{21} \\ &a_{22} = m_2 r_2^2 + j_2 \\ &\beta(q_2) = m_2 r_1 r_2 \sin(q_2) \\ &\gamma_1(q_1, q_2) = (m_1 + m_2) r_1 \cos(q_2) + m_2 r_2 \cos(q_1 + q_2) \\ &\gamma_1(q_1, q_2) = m_2 r_2 \cos(q_1 + q_2) \end{aligned} \tag{26}$$

where q_1 and q_2 are the angular positions of the joints of the manipulator, $\boldsymbol{\rho}(t)$ represents the system uncertainty and disturbance effects.

The parameter values are selected as $r_1 = 1m$, $r_2 = 1m$, $m_1 = 1kg$, $m_2 = 1kg$, $j_1 = 5kg \cdot m$, $j_2 = 5kg \cdot m$, $g = 9.81m/s^2$. The reference signals are given by $q_{d1} = 1.25 \; -1.4e^{-t} + 0.35e^{-4t}$, $q_{2d} = 1.25 + e^{-t} - 0.25e^{-4t}$. For simulation purpose, the uncertainty and disturbance is given by $\boldsymbol{\rho}(t) = [sin(2\pi t), 0.5 sin(2\pi t)]^T$, The control parameters are chosen to be: $\Lambda_1 = diag\{5,5\}$, $\Lambda_2 = diag\{5,5\}$, $\alpha = 0.6$, $b_0 = 4$, $b_1 = 1$, $b_2 = 1$, $\gamma = 1$, $p = 1$, φ=0.6, The initial position and the initial velocity are assigned as $[q_1, \dot{q}_1, q_2, \dot{q}_2]$= $[1.0 \quad 0 \quad 1.5 \quad 0]$. The simulation results are shown in Fig. 2.

To verify the superiority of the proposed method, the results are compared with the fast terminal sliding mode control with exponential reaching law described in (Fallaha et al., 2011). The rapid terminal sliding mode surface and its derivative are given by

$$\boldsymbol{s} = \dot{\boldsymbol{x}} + \Lambda_1 \boldsymbol{x} + \Lambda_2 \boldsymbol{sig}(\boldsymbol{x})^\alpha, \dot{\boldsymbol{s}} = \ddot{\boldsymbol{x}} + \Lambda_1 \dot{\boldsymbol{x}} + \Lambda_2 \boldsymbol{x}_\alpha \qquad (27)$$

where $\boldsymbol{x}_\alpha = [\alpha \,|\, x_1 \,|^{\alpha-1} \, \dot{x}_1, \cdots, \alpha \,|\, x_n \,|^{\alpha-1} \, \dot{x}_n]^T \in R^n$. Because $\alpha - 1 < 0$, when $x_i = 0$, $i = 1, 2 \cdots, n$, singularity will occur. Therefore, to avoid this problem, the terminal sliding mode with the exponential reaching law can be designed as

$$\begin{aligned} &\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}_1 \\ &\boldsymbol{u}_0 = \begin{cases} M_0(\boldsymbol{q})(\ddot{\boldsymbol{q}}_d - \Lambda_1 \dot{\boldsymbol{x}} - \Lambda_2 \boldsymbol{x}_\alpha) + C_0(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} & \\ +G_0(\boldsymbol{q}) & , x_i \neq 0 \\ 0 & , x_i = 0 \end{cases} \qquad (28) \\ &\boldsymbol{u}_1 = -kL(\boldsymbol{s})\boldsymbol{sign}(\boldsymbol{s}) \end{aligned}$$

The simulation parameters are assigned as $\Lambda_1 = diag\{5,5\}$, $\Lambda_2 = diag\{5,5\}$, $\gamma = 1$, $p = 1$, $k = 4$, φ=0.6. The initial position and the initial velocity are assigned as $[q_1, \dot{q}_1, q_2, \dot{q}_2] = [1.0 \quad 0 \quad 1.5 \quad 0]$. The simulation results are shown in Fig. 3.

From Fig.2 (a), (b) and Fig.3 (a), (b), it can be observed that the error convergence with the proposed nonlinear sliding mode control is superior to that achieved by the conventional TSM control. Fig.2(c), shows that the proposed control is continuous and reduces the chattering in comparison with the discontinuous TSM control in Fig.3(c). Comparing Fig.3(c) and Fig.3(c), one can see the singularity phenomenon with the sharp increase of u when x is near the $x = 0$ axis is weaker especially in the case of the control input of joint 2. From Fig.2 (d), one can see that the tracking errors first reach a neighbourhood of the sliding mode in finite time $t_1 = 0.326s$ and $t_2 = 0.267s$ respectively, and the residual set is $\| s \| \leq 6.77 \times 10^{-3}$. In Fig.3 (d), the tracking errors first reach a neighbourhood of the corresponding TSM in finite time $t_1 = 0.906s$ and $t_2 = 0.726s$. Furthermore, the existence of chattering makes the tracking performance deteriorate so as that the residual set is $\| s \| \leq 3.69 \times 10^{-2}$, which is larger than with the proposed approach. Therefore, the proposed method provides a faster response with smooth control and improved tracking accuracy in comparison with the existing TSM control.

6. CONCLUSIONS

In this paper, a nonlinear sliding surface has been proposed. It is shown that the error states converge to an arbitrarily small region centred at the origin within a finite time and thereafter asymptotically converge to the equilibrium point. A nonlinear sliding mode control law has been developed using an improved version of the exponential reaching law. The method is applied for control of a robot manipulator system. The residual set defined by the sliding mode has been determined. The simulation results show that the proposed method has more precise tracking, faster convergence, and stronger robustness against system uncertainty and external disturbances than an existing approach. Due to the smooth control signal, the proposed approach reduces the chattering effectively and has a wider range of applications. Future work will focus on application of the results experimentally and in industry.

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Appendix A. The proof of the upper bounds (8) and (9)

When $|\varepsilon| > \Delta$ ($0<\Delta<1$, $0<\alpha<1$)

$$\begin{aligned} \left|f(\varepsilon,\Delta)\right| &= |\varepsilon|^{\alpha} \le |\varepsilon|^{\alpha-1}|\varepsilon| \le \Delta^{\alpha-1}|\varepsilon| < \left(1+\Delta^{\alpha-1}\right)|\varepsilon| \\ \left|\dot f(\varepsilon,\Delta)\right| &= \alpha|\varepsilon|^{\alpha-1} \le \alpha\Delta^{\alpha-1} \le \alpha\Delta^{\alpha-1}+1 \end{aligned} \quad \text{(A.1)}$$

When $|\varepsilon| < \Delta$ ($0<\Delta<1$, $0<\alpha<1$)

$$\begin{aligned} \left|f(\varepsilon,\Delta)\right| &\le \left|\frac{2\Delta}{\pi} sin\left(\frac{\pi}{2\Delta}\varepsilon\right)\right| + \Delta^{\alpha-1}|\varepsilon| \\ &\le \frac{2\Delta}{\pi}\left|sin\left(\frac{\pi}{2\Delta}\varepsilon\right)\right| + \Delta^{\alpha-1}|\varepsilon| \\ &\le \frac{2\Delta}{\pi}\cdot\frac{\pi}{2\Delta}|\varepsilon| + \Delta^{\alpha-1}|\varepsilon| \\ &\le \left(1+\Delta^{\alpha-1}\right)|\varepsilon| \\ \left|\dot f(\varepsilon,\Delta)\right| &= \left|\cos\left(\frac{\pi}{2\Delta}\varepsilon\right) + \alpha\Delta^{\alpha-1}\right| \le \alpha\Delta^{\alpha-1}+1 \end{aligned} \quad \text{(A.2)}$$

Therefore, $|f(\varepsilon,\Delta)| \le (\Delta^{\alpha-1}+1)|\varepsilon|$, $|\dot f(\varepsilon,\Delta)| \le \alpha\Delta^{\alpha-1}+1$.

Q.E.D..

Appendix B. The proof of inequality (28)

From the expression (25), it follows that

$$L(\boldsymbol{s}) = diag\left\{\frac{1}{l_{11}},\cdots,\frac{1}{l_{nn}}\right\} \quad \text{(B.1)}$$

where $l_{ii}(s_i) = \delta_i + (1-\delta_i)e^{-\gamma|s_i|^P}$, $i = 1,2,\cdots,n$, $\gamma > 0$ and $p > 0$.

As $0 < \delta_{imin} < \delta_i \le l_{ii}(s_i) < 1$, then

$$\sqrt{n} < \left\|L(s)\right\| \le \frac{\sqrt{n}}{\delta_{imin}} \quad \text{(B.2)}$$

Also, according to the equation $l_{ii}(s_i) = \delta_i + (1-\delta_i)e^{-\gamma|s_i|^P}$, it follows that

$$\begin{aligned} l_{ii}(s_i) = \delta_i + (1-\delta_i)e^{-\gamma|s_i|^P} &\le 2\sqrt{\frac{\delta_i^{\,2} + (1-\delta_i)^2 e^{-2\gamma|s_i|^P}}{2}} \\ &\le \sqrt{2\left(2\delta_i^2 - 2\delta_i + 1\right)} \end{aligned} \quad \text{(B.3)}$$

To obtain (28), consider

$$\begin{aligned} \left\|L(s)\right\| &\ge \sqrt{\sum_{i=1}^{n}\frac{1}{2\left(2\delta_i^2-2\delta_i+1\right)}} \ge \frac{\sqrt{n}}{\sqrt{2\left(2\delta_i^2-2\delta_i+1\right)_{max}}} \\ &\ge \sqrt{n} \end{aligned}$$

(B.4)

Q.E.D..

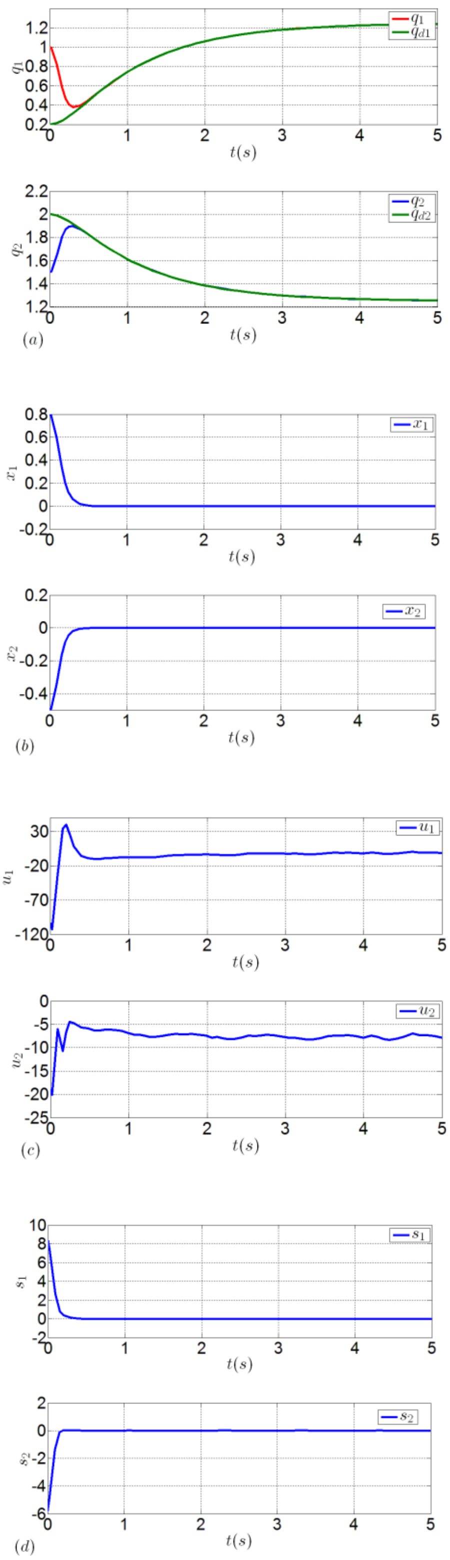

Fig.2 State tracking using the proposed nonlinear SMC for the manipulator (a) position q, (b) error x, (c) control inputs u, (d) switching function s.

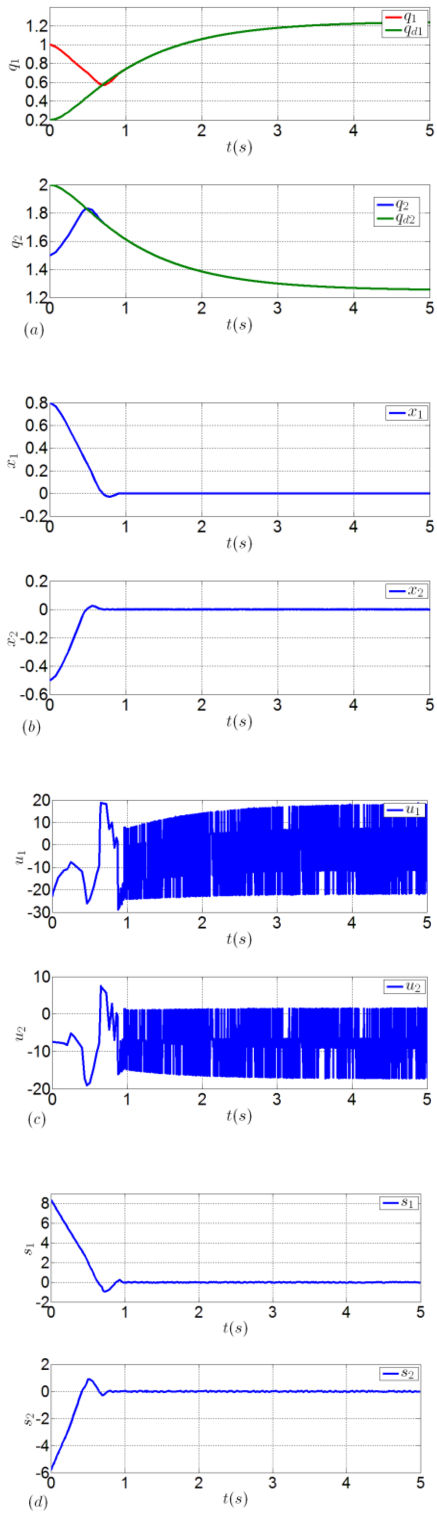

Fig.3 State tracking using the NTSM controller for the manipulator (a) position q, (b) error x, (c) control inputs u, (d) switching function s.