

**A Design Approach to Research in Technology Enhanced
Mathematics Education**

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Declaration

I hereby declare that the work presented in this thesis is my own.

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Abstract

This thesis explores the prospect of a design science of technology enhanced mathematics education (TEME), on three levels: epistemological, methodological and pedagogical. Its primary domain is the identification of scientific tools for design research in TEME. The outputs of this enquiry are evaluated by a demonstrator study in the domain of secondary school mathematics.

A review of existing literature establishes a need for a design perspective in TEME research, but at the same time suggests a need for a consensual *epistemic infrastructure* for the field: a shared set of rules, processes and representations which bound and support its scientific discourse. Three constructs are proposed towards such an infrastructure: design narratives, design patterns, and the cycles of design research in which they are embedded. The first two are representations of domain design knowledge; the latter is a description of a design-centred scientific process.

The three constructs identified at the epistemological level are operationalised as a methodological framework by projecting them into a specific research setting of the demonstrator study. Appropriate methods and procedures are identified for collecting data, organising and interpreting them as design narratives, and extracting design patterns from these narratives.

The methodological framework is applied in the demonstrator domain to the question of learning about number sequences. A review of the educational research on number sequences identifies challenges in this area related to the tension between learners' intuitive concept of sequences and the dominant curricular form. The former appears to be recursive in nature and narrative in form, whereas the latter is a function of index expressed in algebraic notation. The chosen design approach combines construction, collaboration and communication. It highlights the need for representations and activities which lead learners from intuitive concepts to formal mathematical structures.

Three interleaved themes connect the primary and the demonstrator domains: narrative, systematisation and representation. Narrative emerges as a key element in the process of deriving knowledge from experience. Systemisation concerns the structured organisation of knowledge. The tension between the two calls for representations which support a trajectory from the intuitive to the structural.

The main outcome of this study is a methodological framework for design science of TEME which combines design narratives and design patterns into structured cycles of enquiry. This framework is supported both theoretically and empirically. Inter alia, it is used to derive a contribution towards a pedagogical pattern language of construction, communication and collaboration in TEME.

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Chapter 1 Introduction

This chapter begins with some thoughts that inspired and motivated my study. It then presents the central questions and themes of the thesis, lists its settings, process and methods, identifies its aims and surveys its structure.

1.1 Overview

When my son first learnt to write numbers he would occupy himself for hours with a game of sequences. He would write down a number, then add 2, and repeat until the page was full. Then he would do the same with 3, or 4. Before long, this was not a challenge, and multiplication replaced addition. In due course, this game brought the inevitable question – how far can it go (if we ignore the size of the page)? This story is a familiar one. Yet, for many students, by the time they are confronted with number sequences in a school, this naïve fascination is lost. Occasionally we observe such magical moments of spontaneous mathematical learning. We see children working out of their intuitions, driven by a passion for knowledge, and either developing these intuitions into structured knowledge or challenging and refining them. Yet how can one design for such moments? Furthermore, how can such a question be approached as a subject of scientific enquiry?

The ability to learn and the passion to do so may be innate human characteristics. Education, on the other hand, concerns identifying specific structures of knowledge and directing learning towards these by assembling resources and activities under cultural, institutional, social, and psychological constraints. In other words, education is designed learning, and as such, an incredibly complex and inherently multidisciplinary endeavour. Any framework attempting to address this domain needs to identify methodological tools which allow it to confront these complexities. Such tools need to balance the need for a crisp directive for action with a rich representation of context, intentions and possible solutions.

This study originated in my interest in providing children opportunities for learning mathematics. My background in Computer Science led me to relate to this as a design problem, and try to systematise it as such. This attempt gave rise to an enquiry into the prospects and challenges of a design science of technology enhanced mathematical education (TEME), resulting in a multi-layered analysis, reflecting on design as an object, a method and an outcome of study. I see design as a problem-solving activity, and therefore my reflections on design need to be anchored in a particular problem domain. The arguments I make regarding the primary domain of this thesis are evaluated by applying them to a demonstrator domain. The demonstrator domain I chose concerns the use of technology to support learning about number sequences through construction, collaboration and communication.

Three interleaved themes connect the primary and the demonstrator domains: narrative, systematisation and representation. Jerome Bruner (1986; 1990; 1991; 1996) argues that narrative is a powerful cognitive and epistemological construct (section 4.2). I see narrative as a key element in the process of *Situated abstraction* (section 6.3.1), forming a path from experience to knowledge. Yet structured knowledge is often perceived as propositional: a formalism which defines terms, states axioms and rules, then derives theorems and proves them. Its structures are static and timeless, devoid of time and person. This would appear to be

antithetical to narrative form, which is always personal, contextual and time-bound. To reconcile these forces, we need representations which capture the essence of narrative and transpose it to structured forms.

In the primary domain, this proposition motivates a search for suitable representations for capturing tacit design knowledge subjecting it to scientific process. In the demonstrator domain, it motivates the design of representations and activities by which learners derive mathematical forms of thinking, acting and conversing from their experiences.

The remainder of this chapter expands on these themes. Sections 1.2, 1.3 and 1.4 introduce the design approach to educational research, and develop these themes in its context. Section 1.5 outlines the topic of the demonstrator study, while section 1.6 describes its settings. Section 1.7 notes the research process and methods of the study, section 1.8 lists its aims, and section 1.9 reviews the structure of the rest of the thesis.

1.2 Education as a design science

The choice of design as my primary domain of enquiry naturally draws me towards the emerging paradigm of design based research in education (Brown, 1992; Noss & Hoyles, 1996; diSessa & Cobb, 2004). Chapter Two traces this tradition back to the seminal work of Herbert Simon (1969). Simon defines design in broad terms: *“everyone designs who devises courses of action aimed at changing existing situations into desired ones”* (Simon, 1969, p. 129), and calls for a discipline of design science. Simon argues that the world we experience is mostly man-made (sic), but that it nevertheless needs to be investigated scientifically. However, such an investigation is radically different than that of the natural world, not least because we cannot detach ourselves from the object of our study. I argue that a design science approach implies a value-laden scientific agenda, a change of methods and awareness of subjective issues such as representation. I present this argument in section 1.4 and elaborate it in Chapter Two.

1.3 Systemisation of design

A quest for the systematisation of design serves two purposes: exposing it to scientific enquiry and opening it up as a public resource. The current discourse of educational design tends to oscillate between two extremes: high abstractions on the one hand and anecdotes on the other. From a scientific point of view, abstractions tend to offer “truisms” which are hard to refute and anecdotes are so specific that they are hard to critique on a theoretical level. From a practitioners’ perspective, the abstractions give little practical guidance and anecdotes offer little confidence when their conclusions are transferred to a new problem. A design science of education should be based on a linguistic framework which offers an intermediate level of systematisation, rising above anecdotes but remaining grounded in reality. Such a framework would allow us to capture the structure of educational situations, the challenges they engender, as well as the means of addressing them, in forms which should empower learners and teachers to control their practice as much as it allows researchers to inspect it scientifically.

The empowering nature of systematisation, and its relation to issues of language and communication, links the primary domain of this work with the demonstrator domain. This

analogy is demonstrated in the words of Hans Freudenthal, who sees the capacity to systematise as a primary educational goal:

Systematization is a great virtue of mathematics, and if possible, the student has to learn this virtue, too. But then I mean the activity of systematizing, not its result. Its result is a system, a beautiful closed system, closed, with no entrance and no exit. In its highest perfection it can even be handled by a machine. But for what can be performed by machines, we need no humans. What humans have to learn is not mathematics as a closed system, but rather as an activity, the process of mathematizing reality and if possible even that of mathematizing mathematics. (Freudenthal, 1968, p. 7)

I find this passage inspiring when seeking a systematisation of design. I am perusing an understanding of the *activity* of systematising design, not just in its outputs. My own path led from intuitive solutions to problems in designing for mathematics learning to a first sketch of a language of design patterns for TEME. But as much as this sample of design patterns should be useful for others pursuing proximal objectives, my success should be measured in the signposts I provide for others to follow a similar trail, expanding the language of patterns or creating their own.

The notion of narrative is another link between the primary and the demonstrator domains of my work. Narrative, in this thesis, is considered in its epistemic capacity. Bruner (1986; 1990; 1991; 1996; Bruner & Lucariello, 1989) identified narrative as the predominant vernacular form of constructing, representing and communicating meaning. Narratives are not merely descriptions of *what* happened – they provide an implicit or sometimes explicit explanation of *why* it happened. A narrative, in a nutshell, is an account of *something* happening to *someone* in some *circumstances*. A well-formed narrative must maintain coherence of *temporality* and *causality* (Gergen, 1998). Polkinghorne (1995) explains:

"Narrative descriptions exhibit human activity as purposeful engagement in the world. Narrative is the type of discourse composition that draws together diverse events, happenings, and actions of human lives into thematically unified goal-directed processes." (Polkinghorne 1995, p 5)

I see narrative as essential for describing design experiences and processes, as an initial step towards their systematisation. I also see narrative as playing a role in communicating design knowledge to broad audiences. With respect to the demonstrator problem domain, I wish to design activities which will guide participants along trajectories from intuitions expressed in narrative form to a structured understanding of number sequences. An issue I wish to explore is: how can the epistemic power of narrative be harnessed in the construction of systematic knowledge? What is required from such narrative, and what is required from the activity encompassing it? Specifically –

- How can design knowledge be expressed in narrative form, and what is the value of such a representation?
- How can systematised structural representations of design knowledge be derived from narrative forms?

1.4 *Design as a method, an object and an outcome of study*

The opening section of this chapter highlighted Herbert Simon's (1969) call for a science of design, as a seminal voice in the background of the emerging Design Research paradigm (Chapter Two). This section broadens the perspective on design science and design-based research, raising some methodological questions which lead into the outline of my methodological approach.

Simon distinguishes between the natural sciences and the sciences of the artificial, challenging the view of the latter as 'practical' science or 'vocational arts'. At the core of the study of the artificial, Simon places the science of design (section 1.2). Simon's concept of design science entails more than a shift in the subject of study. It calls for a change in scientific agenda. Natural science is concerned with what *is*, design science asks what *ought to be*. Whereas Neurobiology and psychology may also ask how humans learn, some argue (as discussed in Chapter 2) that the main concern of Educational research is how they *ought to learn* and how they *can be helped to learn*. The first may claim to be value neutral and objective, but the questions of education, by their imperative nature, are evidently derived from the observers' (often implicit) ethical, social and community agenda. Sections 1.1 and 1.3 point at the values underpinning my work – the aesthetic pleasure and the empowering nature of systematisation. Such values precede the work itself, and thus should be read as underlying premises rather than arguments to be discussed.

The second implication that Simon (1969) draws from the imperative nature of design science regards the method of problem decomposition. The process of decomposing complex problems into simpler ones is one of science's powerful ideas. Design science is interested in purpose, intent and the shaping of the world to these ends. Therefore, Simon proposes function as the appropriate axis of decomposition. The functional focus also leads Simon to what he calls the generator-test cycle (1969, p. 149) as a viable method of achieving decomposition while acknowledging the networks of interdependencies between components. The design process iteratively generates solutions and then tests them against an array of functional requirements. This cycle maps directly to the iterative process of design-based research.

A third key concern of Simon's is the place of representation in science. A focus on function introduces the human agent who interacts with the objects of study. The agent's perception of the object is no less crucial to its function than the nature of the object itself. Ontologically XIX and 19 may be the same, but epistemologically they are radically different: try to compute XIX^{II}. Whereas natural science strives for representational invariants, design science is deeply concerned with the way problems under investigation are represented in order to illuminate our capacity to solve problems.

The last couple of decades have witnessed the growing popularity of design research as a valuable methodology for educational research. Design based research is a methodology for the study of function. Often referred to as design research or design experiments, it is concerned with the design of learning processes, taking account of the involved complexities, multiple levels and contexts of educational settings. The primary aim is to develop *domain-specific theories* in order to understand the learning process. DiSessa & Cobb (2004) claim that design studies can – and should – make significant theoretical contributions by addressing the gap

between theory and practice. They suggest that design research may offer ontological innovations – new constructs for describing and discussing educational phenomena.

The goals of TEME are ambitious, and its challenges complex. Design research offers a sophisticated response to these challenges, but sophistication is often hard to communicate. This difficulty is amplified by the shortage in clear consensual frameworks, as argued in Chapter Two. At its core, a design science of TEME requires an epistemic infrastructure: a set of conventions defining the rules and boundaries of discourse of the scientific community, and the logical system by which claims are presented and justified. This study attempts to make a modest contribution towards such an infrastructure, by proposing several epistemic elements and deriving methodological tools from them.

1.5 The Demonstrator Domain: Learning About Number Sequences

As illustrated in the story at the top of this chapter, children often engage themselves with spontaneous games of number sequences. Many teachers, as well as several national curricula, use games of pattern-spotting in number sequences as an entry into algebra. The challenge arises when one tries to move on from informal games to the understanding of mathematical structure. A number of researchers, including Noss et al (1997) and Radford (2000), point to the difficulties students encounter in shifting from pattern spotting to structural understanding. Students often tend to base their conclusions on superficial or incidental patterns they observe in the sequence, rather than on arguments referring to its structure. I ask what are the known difficulties encountered by students in this domain and how they can be explained. I am interested in the possible contribution of the exploration into number sequences to the construction of advanced mathematical concepts such as function and convergence.

My choice of the demonstrator problem domain, and the specific educational goals of my design, is derived from several parallels with the primary domain of this thesis. The first among these are the values guiding my work. Several writers (Ernest, 2007; 2002; 2000; Sriraman and Steinhorsdottir, 2009) have proposed social justice and personal empowerment as a motivation for teaching mathematics. Papert (1971) challenges the positioning of mathematics as the privilege of a small elite, arguing that education should enable all learners to *be* mathematicians and adopt *mathematical ways of thinking*. Resnick (2007) calls for a focus on creative thinking, as a key for success in today's society. In line with these thoughts, I perceive mathematical creativity as a source of personal empowerment, and see skills of mathematical discourse as vital for a healthy modern society. Yet I hold that the only viable basis for engaging any individual with them is an appreciation of the intrinsic beauty of the system and the ideas of mathematics.

A second parallel between the two domains is the focus on representation. Many educational studies highlight the importance of representation, even if rarely in a direct reference to Simon (with the notable exception of Kafai, 1995). In mathematics, Noss & Hoyles (1996) observe that the issue of selecting and constructing representations is key to learning, and the potentials of alternative representations have been a prevailing concern of the constructionist tradition. These themes are present even in the story that opens this chapter. The child in question

constructed his understanding of sequences through an activity, specifically – a game. He would not have been able to do so in the absence of a representational system (Hindu-Arabic numerals) and the technology to support it (pen and paper).

Chapter Six reviews some of the difficulties associated with number sequences, as reflected in the literature. I see many of these difficulties as rooted in a failure to provide learners with representations that would allow them to use their intuitions as a basis for formal concepts, rather than requiring them to abandon them. I claim that the naïve view of number sequences is recursive, expressed as a relationship between consecutive terms rather than its general rule (e.g. describing the sequence represented by the function $f(n) = 2n + 1$ as "add 2"). If this claim is true it should be possible to design activities which enable children to formalize their recursive intuitions and derive a mathematical symbolism from them. Such symbolism, whether its representational form be algebraic or not, should allow children to express and respond to structural arguments about sequences. A central aim of the demonstrator study is to design a set of activities which build upon children's recursive intuitions of number sequences, and encourage them to develop these intuitions into structured formal arguments.

The use of technology in learning environments has become ubiquitous in recent years, taking diverse forms, derived from underlying, often implicit, models of learning. The examples I choose focus on functionalities which are more directly related to the processes of learning, namely construction, communication, collaboration and, eventually, reflection. The design challenge posed by these examples is to produce tools and activities which support social and cognitive processes of learning; supporting the individual construction in the sense of providing stimulating and effective learning activities, and supporting the social process through classroom practices and web-based collaboration systems. Inevitably, the epistemic processes themselves need to be understood to a degree which will enable valid and effective design.

1.6 The research setting

The empirical work of the demonstrator study was supported by the *WebLabs* project, directed by Professors Richard Noss and Celia Hoyles, as detailed in Chapter Five. Some of the analytical work was supported by the *Learning Patterns* project, directed by Dr. Niall Winters and Professor David Pratt. Table 2 in section 5.2 enumerates the inputs of members of project members to my design, data collection and analysis. The *WebLabs* project explored the collaborative construction and communication of mathematical and scientific knowledge in communities of young learners (age 10-14), by designing new learning activities and the tools to support them. To this end, it employed and enhanced two technologies: the ToonTalk programming environment as a constructionist medium and a bespoke collaborative knowledge-building medium called *WebReports*. The technological platform used by the *WebLabs* projects manifests a particular educational approach. The nature of the project was such that the underlying pedagogy and the supporting technology shaped and reshaped each other. Within this context, my individual responsibilities included leading the strand of activities on number sequences and the development of the *WebReports* system.

The main bulk of the analysis presented in this thesis was done after the completion of the WebLabs and Learning Patterns projects. This includes the construction of the main epistemological and methodological arguments.

1.7 Process and methods

The principal concern of this study is a scientific enquiry into the process of techno-pedagogic design in TEME. The main source of data for this enquiry was my personal experience in such a process, in the problem domain of number sequences. The craft of designing tools and activities for learning calls for domain-specific methodologies, which provide validation of designed artefacts with respect of their intended purpose.

Three constructs play a prominent role in this thesis, with respect to the scientific process of studying design in TEME: design narratives, design patterns, and the cycles of design research in which they are embedded. The first two are representations of domain design knowledge; the latter is a description of the process by which they are derived. The primary study reviews the rationale for these forms and their historical development. It then proposes an operational framework for their use. The demonstrator study applies these constructs within framework in a TEME problem domain.

1.7.1 Cycles of Design Research in TEME

Chapter Four identifies methodological characteristics common to many design studies. Among these are a commitment to practical as well as theoretical contributions, a highly interventionist and agile attitude, and a cycle of iterative research. This cycle includes phases of theory, design, implementation, execution (experiment / practice), articulation of experience, interpretation, evaluation and analysis, and feedback to both theory and design. This cycle is embedded in a meta-cycle, which includes a framing phase, an empirical phase and a retrospective analysis phase.

The demonstrator study was conducted through three cycles of design experiments. Tools and activities were evaluated during trials with small groups of six to ten children in London schools. Analysis of empirical observations provided insights into subjects' learning trajectories and informed the subsequent cycle of design. The three design iterations were followed by a phase of retrospective analysis.

1.7.2 Design Narratives

Design narratives systematise an innate form of extracting knowledge from experience. They provide accounts of critical events from a personal, phenomenological perspective, providing a first tier of interpretation by affording rich contextualised descriptions of design experiments. They focus on design in the sense of problem solving, describing a problem in the chosen domain, the actions taken to resolve it and their unfolding effects. Design narratives offer an account of the history and evolution of a design over time, including the research context, the tools and activities designed, and the results of users' interactions with these. They portray the complete path leading to an educational innovation, not just its final form – including failed attempts and the modifications they espoused. Chapter Four makes a case for their

appropriateness as an epistemic form in design research, and Chapter Five realises this form as a methodological instrument.

Chapter Seven presents the empirical part of my work, as a set of nine design narratives. These narratives are organised thematically according to the three main activities designed and tested in the demonstrator problem domain: basic number sequences, the guess my robot game, and convergence and divergence. Each activity was motivated by specific learning objectives. The design of the activity co-evolved with that of the technological tools that support it, both in terms of the construction environment and in terms of the collaboration platform. The narratives encompass all these elements and trace them chronologically within and across iterations.

1.7.3 Design patterns

Design patterns capture recurring features across narratives, encapsulating critical challenges and forces pertaining to a domain of learning design, the interactions between them and possible methods of solution. Design patterns are seen as abstractions of *design knowledge*: they attempt to capture generic principles, while acknowledging our epistemic need to maintain a link between systematised concepts and the context from which they were derived. A design pattern encapsulates a problem, the context in which it arises, and a possible solution based on examples of practice. This structure promotes a functional decomposition of the problem domain, and offers a representation of design knowledge which yields itself to theoretical scrutiny as well as pragmatic implementation. Thus it appears to be a good candidate form for a design science of education. However, to qualify as science, the process of deriving patterns from case narratives needs to be transparent and the patterns themselves need to be rigorously validated. Chapter Four reviews the origin of and motivation for design patterns, and traces their trajectory from architecture through software engineering to educational research. Chapters Four and Five propose criteria for adapting design narrative as a scientific form, and a structured process for extracting patterns from narratives.

Chapter Eight presents a set of seven detailed design patterns, and outlines seven more. Together with the design narratives in Chapter Six, these form an initial draft of a pattern language for TEME.

1.8 Aims of this study

Having presented the context of my research and the key issues it explores, I now have the means to define the scope and objectives of my work. The overarching intent of this work is to contribute to the understanding of education as a design science, highlight the implications of such a paradigm, and propose ways to theorize design in a manner which draws on and informs educational research.

This intent is manifested on three levels: epistemic, methodological and pedagogical. I aim to:

- Identify potential elements of an **epistemic infrastructure** for a design science of TEME.
- Combine and elaborate the elements identified into a coherent **methodological framework** in a given research TEME context.

- Apply the methodology in the problem domain of learning about number sequences and demonstrate its potential by producing a contribution towards a language of **pedagogical patterns** for TEME.

These aims are elaborated in Chapter Three, drawing on the foundations laid in Chapter Two.

1.9 Structure of the Thesis

The present chapter illuminated the main themes and approaches which guide this study and enumerated its aims. These aims build on each other, and together form a multi-level argument which runs through the thesis. Figure 1 provides a high-level map of this argument. The rest of the thesis works systematically through it.

My point of departure is the intuitive observation differentiating learning, as a spontaneous indigenous human capacity, and education, as a complex and challenging realm of design. This observation motivates the question that inspires my work: how to design for “magical moments of learning”? Reformulated as a subject for scientific enquiry, this question is translated into a cascaded study of design in TEME: first, as an epistemological question, leading to a methodological question, and evaluated by applying the outcomes of the first two investigations to a pedagogical challenge.

The initial review of the field of design-based research in TEME identifies a lack of consensual epistemic infrastructure. This motivates the first aim of my thesis: to identify elements of such an infrastructure. Three constructs emerge from a review of existing approaches, in the field of education and beyond: the cycles of design research, and the representational forms of design narratives and design patterns. Recognising the need to operationalise these abstract constructs leads to the second aim of my thesis: identifying a possible methodological framework which utilises these constructs. This aim is addressed by situating the epistemic elements in a concrete research setting. First, the setting is described. Then appropriate methods are enumerated for collecting data, organising it into design narratives, deriving design patterns from these and inducing theoretical contributions from the resulting sketch of a pattern language.

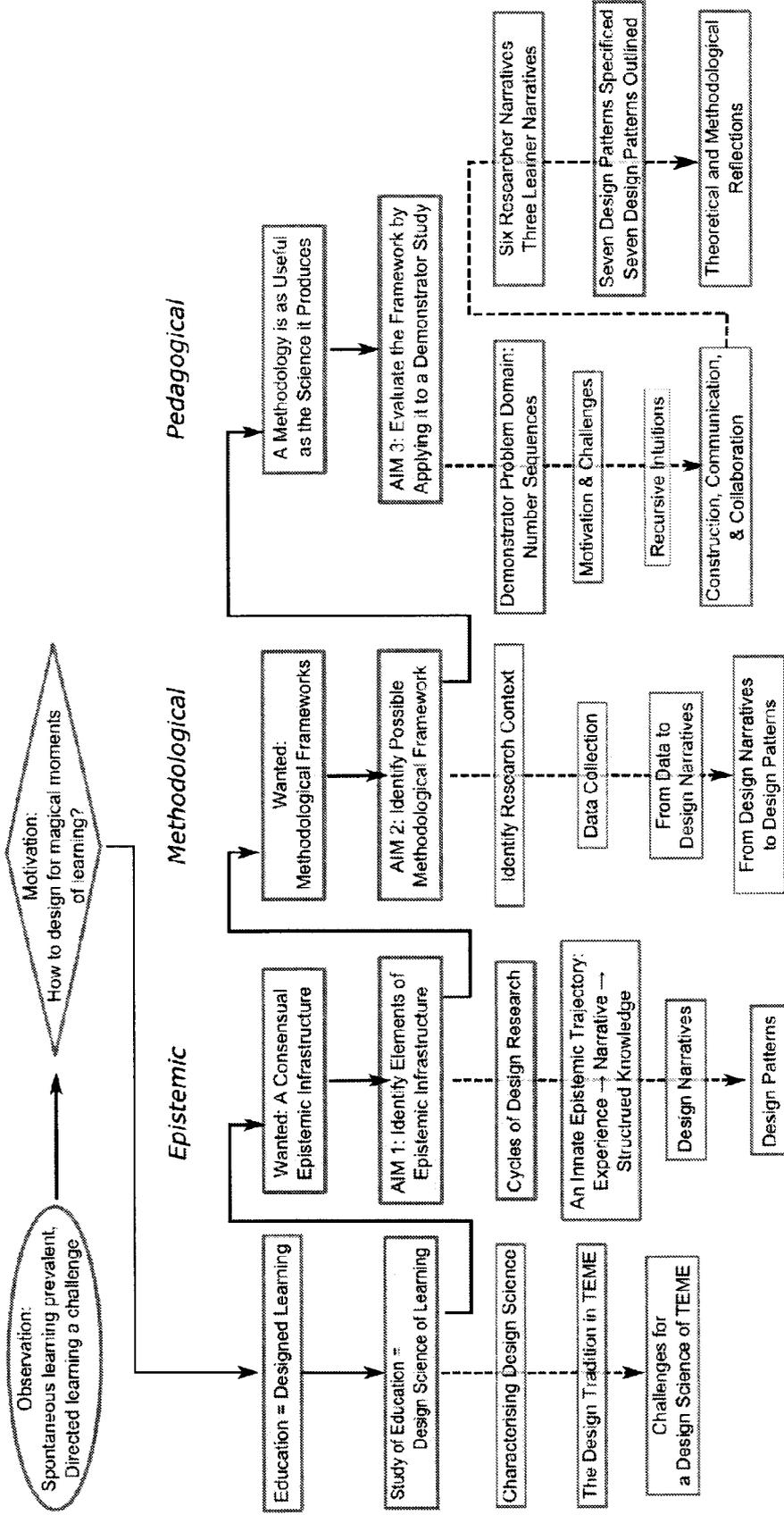


Figure 1: A birds-eye view map the argument of this thesis. Inspired by a desire to design for learning, the theory first contemplates the prospects of education as a design science of learning, and then traces the implications of this proposal through three levels: epistemic, methodological and pedagogical.

Chapter Two provides the theoretical backdrop, by reviewing the traditions of design research in education and in related fields.

Chapter Three builds on the foundations provided by Chapter Two to list the aims of the thesis in detail.

Chapter Four presents the three central constructs and delineates the conditions for their use as scientific instruments.

Chapter Five describes the methodology for the demonstrator study, by projecting these three constructs into the specific research settings.

Chapter Six reviews the challenges and prior research in the demonstrator domain, deriving several conjectures which will underlie the design of activities and technologies.

Chapter Seven presents a set of nine design narratives, recounting the design experiments conducted in the demonstrator domain.

Chapter Eight derives a set of design patterns from these narratives, seven of which are articulated in detail and seven outlined.

Chapter Nine concludes the thesis by discussing the major findings and mapping them to the aims.

Chapter 2 Technology Enhanced Mathematics

Education as A Design Science of Learning

This chapter argues for a design science perspective in technology enhanced mathematics educational research. It presents the principles and origins of such an approach, and identifies methodological weaknesses which need to be addressed. The emerging issues are formulated as concrete aims in the following chapter.

2.1 Introduction

The last decade has witnessed a growing trend towards design based research in education and, in particular, on the use of technology in education (Barab & Squire, 2004; Barab, Thomas, Dodge, Squire and Newell, 2004; Bell, Hoadley and Linn, 2004; Béguin, 2003; Brown, 1992; Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Collins, 1992; Collins, Joseph, & Bielaczyc, 2004; Edelson, 2002; Lesh and Sriraman, 2005; O'Donnell, 2004; Reeves, 2006; Sandoval and Bell, 2004; Wittmann, 1995). Design based approaches focus on the process of developing innovative tools and activities as means of understanding learning and advancing educational practice. While this trend has moved towards centre stage in recent years, its roots go back to the 1960s. My study is positioned within this tradition, and draws its methods from it. This chapter sets out to provide a theoretical foundation for these methods.

This chapter focuses on the concept of design and its relations with research in technology enhanced mathematics education (TEME). In order to frame the discussion, I begin by tracing the roots of the notion of a design science back to the seminal work of Herbert Simon (1969), noting some of his key insights which are still remarkably relevant and innovative. I then review the evolution of design-based methods in educational research. I see three research perspectives that relate to design: design as an *object* of study; design as an *outcome* of study and design as a *method* of study. All three are manifested in this study. I review the methodological implications of a design perspective, and highlight some of the challenges in this domain.

Christopher Alexander defines design as: *"The process of inventing physical things which display new physical order, organization, form, in response to function"* (Alexander, 1964, p.1). Middleton et al. characterise the activity of design as *"a subtle but complex interaction between the designer and contextual constraints and is accomplished by proposing the form of an artifact, system or process, which in turn drives its behaviour, which in turn can be compared with its desired function"* (2008, p. 22, original emphasis). Herbert Simon summarises: *"everyone designs who devises courses of action aimed at changing existing situations into desired ones"* (Simon, 1969, p 129).

TEME is concerned with bringing about change in learners' mathematical knowledge by using technology to enrich the social and individual environment of learning. Over the last few decades, many studies have shown a positive correlation between the use of technology and attainment in mathematics (Wenglinsky, 2005; 1998; Kulik, 2003; 1994). Yet most of these studies emphasise that this link is far from universal. It is contingent on the details of the design

of technology, as much as on the educational activities in which it is embedded. For example, Wenglinsky (1988) found that the use of simulation and higher order thinking skills software gave students an advantage of up to 15 weeks over the control group, but students who used drill and practice software performed worse than students who did not. In view of these findings, the role of research in TEME is to identify *how* technology could be designed to promote given educational goals. Much of the technology used in education is designed to streamline well-tested educational practices. Despite the immediate appeal of this approach, it is criticized for failing to capitalise the full potential of technology. Balacheff and Kaput (1996) claim that digital technologies hold the potential of direct manipulation of mathematical objects and relations. This offers learners a much deeper and more widely accessible experience of mathematics than possible with traditional technologies. A more radical position argues that digital technologies have espoused a cultural shift in the way mathematics is done, and what counts as mathematics, and this change should be reflected in mathematics education (Noss and Hoyles, 1996; Kaput, Noss and Hoyles, 2002).

Consequently, research in TEME should strive to realise the potential of technology to “create democratizing infrastructures which will redefine school knowledge” (Kaput, Noss and Hoyles, 2002, p33). This is a difficult task; it requires the assimilation and integration of deep knowledge from diverse domains of expertise including mathematics, computer science, epistemology and pedagogy. I see all these domains of knowledge as diverse facets of *design knowledge*. The mathematical dimension pertains to the selection and connection of mathematical content, e.g. the sequencing of curricula. The question of pedagogy is a question of designing instructional structures; and so on. Due to the complexity of each of the various bodies of knowledge it is often hard to communicate concepts and ideas across domains. Furthermore, each community relies on its own jargon and lore. For example, when software engineers speak of ‘encapsulation’ they mean something completely different from what educational researchers would when using the same term. The result of this fragmentation of knowledge is that most designs emerge from a particular viewpoint, often a restricted one, rather than an inherently interdisciplinary one.

2.2 What is a Science of Design, and what value does it bring to TEME?

Many scientific disciplines turn their attention to questions of learning: cognitive and developmental psychology, linguistics, neurology and computer science, to name a few. The science of education is distinguished by its focus on how learning is induced and directed to a specific agenda. Diana Laurillard identifies the key challenge for TEME research as “how to identify and provide what it takes to learn” (Laurillard, 2008, p 140). This distinction identifies TEME science as a study of *designed* learning.

Herbert Simon (1969) distinguishes between the natural sciences and the sciences of the artificial. While the former have been the flagships of intellectual activity since the days of Newton, the latter are habitually suppressed as ‘practical’ sciences or ‘vocational arts’. Yet most of our lives are situated amidst the artificial. At the core of the study of the artificial, Simon places the science of design. He asserts that design thinking is a defining feature of the human

mind. Whether driven by survival instincts or by intricate desires, we are continuously engaged with the problem of manifesting desired situations under the constraints imposed by our environment. Thus –

... the proper study of mankind is the science of design, not only as the professional component of a technical education but as a core discipline for every liberally educated person (ibid, p 159)

Simon's (1969) concept of design science entails more than a shift in the subject of study. It calls for a change in scientific agenda, or more broadly what Kelly et al (2008) call the *commissive space*: the substrate of explicit and implicit rules and assumptions which bind the discourse of a scientific community. Whereas natural science is concerned with what *is*, design science asks what *ought to be*. As Kelly et al (2008) argue, "a central question for educational research is how to design interventions that move beyond 'what is?' or confirming 'what works?' to designing 'what strategy or intervention might work better?'" (p. 3). When shifting our focus from engineering to social subjects – such as learning mathematics – the *value* aspect of design sciences becomes salient. Arguably, while other sciences ask how humans learn, the study of education is concerned with how their *learning may be improved and directed*. The questions of education, by their imperative nature, are evidently derived from the observers' (often implicit) ethical or political agenda. Reeves, Herrington and Oliver (2005) go further in claiming that a "realist" approach is fundamentally unsuitable for studying artificial phenomena such as education. Such an approach would assume that the objects under observation are governed by immutable laws of nature, whereas the *raison d'être* of artificial systems is human intentions, and by extension their values and beliefs. The same intentions, values and beliefs motivate researchers investigating these systems, and ignoring them would create a dissonance.

The second implication that Simon (1969) derives from the imperative nature of design science regards the method of problem decomposition. All the sciences proceed, to an extent, by decomposing complex problems into simpler ones. The common method of decomposition in natural science is structural. Complex entities have been described in terms of their parts and the relations between them. Design science is interested in purpose, intent and the shaping of the world to these ends. Therefore, Simon proposes *function* as the appropriate axis of decomposition. As an illustration of the difference, a structural description of a cricket bat would be "a processed plank of willow wood, weighing 1.1 to 1.4kg, approximately 96cm long, flat and wide (up to 10.8cm) at one end, rounded at the other". A functional description would note that it is an instrument used in the game of cricket in order to hit a ball. Such a functional perspective leads Simon to what he calls the *generate-test cycle* (Simon, 1969, p 149; Newell and Simon, 1976) as a viable method of achieving decomposition while acknowledging the networks of interdependencies between components. The design process generates solutions which it iteratively tests against an array of functional requirements. Taken as a method of scientific inquiry, this translates into the design based research approaches described in section 2.2.1.

The third theme I take from Simon concerns the place of representation in science. From the perspective of a natural science, representation is tangential to the phenomena being studied. The nature of gravity is invariant to the symbolic system by which we measure it, describe it or bring it into our calculations. Newton's development of Calculus had a radical impact on our ability to understand – and make use of – gravity, but the nature of gravity itself remained

unaffected. Design science, on the other hand, is concerned precisely with these processes of understanding and using. It strives to illuminate our capacity to solve problems, and this capacity is strongly dependant on the way we represent the problems under investigation. Whether we see mathematical structures as rules of nature or human constructions, our ability to learn and teach these structures is contingent on their representation. Indeed, representation is a key concern of many studies in TEMEs, both within the design research tradition (cf. Radford, 2000; Kaput, Noss and Hoyles 2000; diSessa, 1977; Noss, Healy and Hoyles, 1997) and from other perspectives (Jewitt, 2003; Ainsworth, Bibby and Wood 2002; Morgan 2001; Moreno, 1995). Yet, as I argue below, representation is also an open methodological concern; the advancement of design science relies on identifying appropriate forms for formalising and communicating design knowledge.

Lesh and Sriraman (2005) advocate the re-conceptualisation of the field of mathematics education research as that of a design science. They identify two aspects of the subjects investigated by TEME research which define it as a design science: they are predominantly man-made, and they are (or embody) complex systems. Lesh, Kelly and Yoon (2008) claim that the interlinked processes of learning and teaching mathematics and science education are more complex than many of the systems studied by the physical sciences.

2.2.1 The Tradition of Design Based Research in TEME

The common design-based approach to TEME research takes the concept of design as a practical imperative, seeing the aim of educational research as bringing about a positive change of learners' experiences and knowledge, and referring to design as a method of action. The design element in a design study may refer to the pedagogy, the activity, or the tools used. In some cases, the researchers will focus on iterative refinement of the educational design while keeping the tools fixed, whereas in others they may highlight the tools, applying a free-flowing approach to the activities. In yet others they will aspire to achieve a coherent and comprehensive design of the activity system as a whole.

In recent years, design-based research (DBR) of this kind has become a popular methodology of educational science. Middleton et al (2008) describe DBR as design processes subjected to standards of scholarship recognised by the scientific community. This definition hides a dual agenda: on one hand, producing better artefacts – material and other - by utilising theory, on the other, advancing theory through the design and use of new artefacts (Bell, 2004). DBR aims to “(a) help design innovations (b) explain their effectiveness or ineffectiveness, theoretically, and (c) re-engineer them where possible, while adding to the science of design itself” (Kelly et al, 2008, p. 5). At the same time “design-based research can advance theories of learning because educational designs embody conjectures about learning that can be empirically refined” (Sandoval, 2004, p. 213). Juuti and Lavonen (2006) identify pragmatism (in the sense of Peirce, 1935) as a philosophical foundation for design based research, leading to an action-oriented conception of knowledge. Cobb et al (2003) identify five characteristics of design experiments:

- The purpose of design experimentation is to develop a class of theories about both the process of learning and the means that are designed to support that learning.

- It is a highly interventionist method of study. The researcher is a participant-observer with flexible control of many of the research parameters.
- Design experiments always have two faces: prospective and reflective. On the prospective side, designs are implemented with a hypothesized learning process and the means of supporting it in mind in order to expose the details of that process to scrutiny. On the reflective side, design experiments are conjecture-driven tests, often at several levels of analysis.
- Together, the prospective and reflective aspects of design experiments result in iterative design. As conjectures are generated and tested, sometimes confirmed, at others refuted, new conjectures are developed and subjected to test.
- Theories developed during the process of experiment are modest not merely in the sense that they are concerned with domain-specific learning processes, but also because they are accountable to the specific activity of design.

As discussed below, DBR stems from multiple roots (Bell, 2004). Yet the characteristics identified by Cobb et al. (2003) describe, by and large, a reflective practice of design shared across the field. These are necessary but not sufficient conditions for a design science of TEME.

Ann Brown (1992) puts forth two substantive arguments in favour of design-based educational research. The first argument is methodological. The complexity of classroom situations does not lend itself to the procedures of laboratory research. Strict control of experiments and isolated variables are unattainable. Under these circumstances, Brown (1992) suggests that we adopt in education the procedures of design sciences such as aeronautics and artificial intelligence. Similarly, Hoadley (2004) characterises educational settings as dominated by multitudinous interdependent variables that would be hard to control in randomised experiments. Lesh, Kelly & Yoon (2008) suggest that mathematics education occurs in complex systems – involving multiple agents, partially conflicting goals, feedback loops, second-order effects and emergent properties. Hoadley (2004) argues that under these conditions, the premises of randomised control experiments are violated, and the results of standard experimental results are at least hard to interpret and at worst meaningless and misleading.

The second argument questions the fundamental goals of TEME research. To what extent are we driven by a pure quest for knowledge, and to what extent are we committed to influencing educational practice? If we see contribution to good practice as a primary goal, then the outputs of our research should have direct bearing on it. This argument is echoed in the call for a socially responsible study of education (Reeves, Herrington and Oliver, 2005). The authors argue that a study of education must be socially relevant, and in order to do that it should not focus on how education works, but on how to make it work better. It should be measured by its practical impact as well as its disciplinary rigour.

Critics of the design based approach take issue mainly with the first argument, questioning the scientific value and lack of “evidence” of inherently irreproducible experiments (Shavelson et al, 2003). The response to this critique is twofold: first, DBR modestly accepts its limitations, confining its offerings to *humble theories* (Cobb et al., 2003) or *domain specific instructional theories* (diSessa and Cobb, 2004). As a side note, the observations above beg the question whether ostensibly scientific methods are practicable and can offer any greater validity. At the

same time, one needs to be as stringent and self-critical when analysing data – precisely because we do not enjoy the protection of standardized statistical tests. DBR typically compensates for lack of statistical validity by calibrating diverse methods and data sources, and by focusing on process-oriented causal explanations (Maxwell, 2004; Gravemeijer and Cobb, 2006).

A more subtle criticism of the design-based approach scrutinizes it on its own terms: does this approach live up to its commitment to offer a contribution to TEME practice? On the one hand, the conditions of most design experiments do not resemble those of a normal classroom, if only due to the presence of a dedicated, highly informed researcher in the class. As argued by Alan Collins:

Typically the experiments are carried out by the people who designed some technological innovation, so that they have a vested interest in seeing that it works. They typically look only for significant effects (which can be very small effects) and test only one design, rather than trying to compare the size of effects for different designs or innovations. Further more such experiments are so variable in their design and implementation that it is difficult to draw conclusions about the design process by comparing different experiments. Finally they are carried out without any underlying theory and so these results are for the most part uninterpretable with respect to constructing a design theory of technological innovation (Collins, 1992, p. 24 in Issroff and Scanlon, 2002).

On the other hand, the reported data and analyses typically include case-studies and theoretical generalizations derived from them. Despite their growing popularity, cases are criticized for being unsystematic, with evidence on their effect on practice being fragmented and fragile (Markovits and Smith, 2008). Theoretical works on the other hand are often judged by practitioners as too abstract to be applicable (Hirschhorn and Geelan, 2008). Furthermore, there is a fundamental difference between the nature of knowledge produced by design experiments and that of traditional methods of social science. Whereas the latter strive to establish beyond doubt the *existence* of phenomena, design research aims to *explain* phenomena, while maintaining a cautious stance on the determinism of their appearance. In order to enhance our ability to solve new problems as they arise, we need to go beyond the investigations of “what works” and ask “*why* it works”. In the words of Ann Brown: “*a ‘Hawthorne effect’ is what I want: improved cognitive productivity under the control of the learners, eventually with minimal expense, and with a theoretical rationale for why things work*” (Brown, 1992, p 167). This would appear to be at odds with Collins above. The key is in the ‘theoretical rationale for why things work’. To allow the requirements for empirical evidence to be relaxed, design research needs to raise the bar of theoretical justification. It is not enough to show the correlation of an educational design and a learning effect, especially where the experimental conditions are unique and sample size small. Design research needs to show a detailed rationale for how and why the specific configuration of design elements gave rise to the perceived effects.

Perhaps the most critical remarks on design studies in education come from two of its foremost proponents and promoters. diSessa & Cobb (2004) warn against the drift of design research

away from theory. They argue that theory is essential, both from a perspective of research and of practice. Furthermore, they claim that design studies can – and should – make significant theoretical contributions by addressing the gap between theory and practice. First, they describe four categories of theory: *Grand Theory*, *Orienting Frameworks*, *Frameworks for action* and *Domain specific instructional theories*. All of these are important for educational design, but cannot be applied readily to concrete situations. In the words of the authors, it is fine to say that one should build on students' contributions, but totally unclear how to do this. They answer by suggesting that design research may offer *ontological innovations* – new linguistic constructs for describing and discussing educational phenomena. Schwartz, Chang & Martin (2008) suggest these should be termed “epistemic innovations”, since they pertain to our knowledge of reality rather than to the structure of reality itself.

It is interesting to note the parallels between DBR and *Participatory Design*. Participatory design (PD) is “a set of theories, practices, and studies related to end users as full participants in activities leading to software and hardware computer products and computer-based activities” (Muller, 2002). Similar to DBR, PD is a highly iterative process, which sees design as a process of mutual learning between users and designers (Nesset and Large, 2004; Béguin, 2003). From its highly political origins in the Scandinavian workplace democracy movement, PD had grown to be recognised as an effective method of generating artefacts with good fit for purpose. However, as noted by Druin (2002) and Carrol et al (2000), this effectiveness comes at a price: users need to be trained to participate in the design process, and need to commit to its success. The designer needs to devote considerable attention to the interaction with and guidance of the users. This investment is justified where the focus is on optimising the design process and the artefacts it produces. It may be overwhelming when design is a means of testing and developing educational theory. Consequently, most design studies place the learner in the role of an informant rather than a full design partner Druin (2002).

2.2.2 Reconnecting Theory and Practice

A primary role of TEME research should be to understand and inform educational practice (Hoadley, 2008; Laurillard, 2008b; Wittmann, 2001; 1995). The link between practice and theory has been found lacking in both directions. Practitioner involvement in educational research is often associated with the Action Research paradigm (Carr and Kemmis, 1986; Cohen, Manion and Morrison 2007). Action research is a powerful tool for driving change at the local level. It has also been promoted as a method for professional training. Action research shares many of the characteristics of Design research: it involves theory-guided problem-solving and theorisation of the consequences of problem-solving; it is participatory, collaborative and iterative. As Reeves, Herrington and Oliver (2005) argue, action research has clear merit but there is much more potential value in design research, which combines seeking practical solutions to classroom problems with the search for transferable and generalisable design knowledge that others may apply. At the other extreme stands the “evidence-based” attitude in policy, funding and research. The evidence-based qualifier has emerged as a euphemism for randomised control experiments. In a highly quoted paper, Slavin (2002:15) proclaims: “At the dawn of the 21st century, educational research is finally entering the 20th century. The use of randomized experiments that transformed medicine, agriculture, and technology in the 20th century is now beginning to affect educational policy”. In an equally well-noticed paper, Feuer, Towne and

Shavelson (2002:8) posit: "For example, when well-specified causal hypotheses can be formulated and randomization to treatment and control conditions is ethical and feasible, a randomized experiment is the best method for estimating effects". As argued above, such conditions are rarely available when studying educational innovations in real-life settings. As a compromise, Burke Johnson and Onwuegbuzie (2004) propose mixed methods research and argue for a pragmatist philosophy.

Wittmann (2001) claims that the majority of scientific research in education is irrelevant to practice. Such an argument resonates with the observations noted above regarding the applicability of randomised control experiments. Instead, he posits, research in mathematics education should put at its centre "the design of substantial learning environments around long-term curricular strands" (Wittmann, 2001, p. 4), arguing that research, development and teacher education should be consciously related to them in a systematic way. This is not a trivial requirement. Mellar, Oliver and Hadjithoma-Garstka, (2009) review the tensions between research and practice. They find that research is perceived by practitioners as providing too much detail, or conflicting evidence, does not address their immediate concerns or does not acknowledge the reality of their experiences. Ironically, they conclude, "the same characteristics that make it hard to draw general principles from the work can also make it credible to practitioners". Although this review focused on the context of e-Learning in higher education, it is reasonable to assume these tensions are just as relevant to TEME. Laurillard (2008a) observes that technology has the potential to address many of the hard challenges in education, yet at the same time it amplifies the difficulty of linking theory to practice: "the solutions technology brings, in their most immediate form, are solutions to problems education does not have" (Laurillard, 2008b:139). Technology evolves at a rapid pace, and brings with it rapid changes in social practices and epistemic structures. Consequently research is denied a breadth of historical perspective or the comfort of stable control conditions. Laurillard suggests that in order for innovation to be relevant to practice, it needs to be led by practitioners. Yet for it to be reliable and effective, innovation needs to be conducted in a quasi-scientific process. Taken together, these observations suggest that TEME research should engage practitioners as design researchers, in line with Schön's ideal of reflective practice (1987). A commitment to developing theory which is relevant for practice, and practice which is informed by theory, resonates with the dual agenda of DBR mentioned above. It supports diSessa and Cobb's emphasis on domain specific instructional theories (2004), although arguably it should not preclude theoretical contributions of broader remit.

Demanding a strong link between theory and practice suggests a pragmatist stance, as noted by Barab and Kirshner "The goal of these researchers, educators, and designers moves beyond offering explanations *of*, and onto designing interventions *for*. In fact, and consistent with pragmatists such as Dewey, Pierce, and James, to some degree it is the latter functional constraint that constitutes what is a useful explanation *of*." (Barab and Kirshner, 2001:11; original emphasis). Such a stance sees knowledge as instrumental, its value derived from the action it engenders. One corollary of this stance is a flexibility and pluralism in theoretical and methodological choices. Design research assumes that more than one theory may be required to describe, explain, or predict a single phenomenon (Kelly et al., 2004). Since the primary commitment is towards action, design science will prioritise a comprehensive and proactive understanding of the situation over theoretical aesthetics. Lesh et al. (2004:138) adamantly

reject *the* “the naïve wishes of those who hope to use design science as a methodology for translating (a single) theory into practice” and “the naïve claims ... that each research project should be based on (a single) theory”, arguing that design science is inherently multidisciplinary. This theoretical pluralism complements the methodological pluralism mentioned above.

2.2.3 Methodological Characteristics of Design Research in TEME

A design approach to TEME research is inherently amenable to synergy with other paradigms. It borrows methods and results from other fields insofar as these can inform the design process. The complexity of the experimental situations and the difficulties in extracting controlled data demand that methodological tools be selected, adjusted and calibrated per case. Design methods can also be utilised in studies where the dominant paradigm is derived from a different tradition, as a means of testing specific conjectures.

As noted above, the theoretical and methodological diversity of DBR follows from its fundamental premises. This diversity also has historical roots: DBR is a young paradigm, and its advocates have come from a range of traditions (Bell, 2004), including developmental psychology, cultural psychology, cognitive sciences, mathematics and science, artificial intelligence, and computer science. Nevertheless, over time a common life-cycle of research projects has emerged (Winters and Mor, 2009; Cobb and Gravemeijer, 2008; Gorard, Roberts and Taylor, 2004; Cobb et al, 2003; Bannan-Ritland, 2003). Researchers conduct a series of teaching experiments. These experiments are run on a small scale, to facilitate elaborate interpretation. The consequences of interpretation then feed into the next round of design. Thus, in subsequent iterations, the design is refined and at the same time the interpretation is validated. The immediate products of the design process – tools, practices and methods – are often seen as transient since the settings in a design experiment are idiosyncratic: the subjects are often a small selected group, the researcher is highly involved in the experiment in all its stages, and her knowledge advances as it proceeds. Thus, it is reasonable to expect the results not necessarily to be representative and the products of the process are therefore often discarded between iterations. Most of the analysis is done by qualitative means. The cycles of experimentation are followed by a phase of retrospective analysis. Middleton et al. (2008) put forward a more elaborate, seven phase model they term the “complete design cycle” [sic]. This cycle goes through identification of grounded models, development of artefacts (tools or practices), feasibility study, prototyping, field study, definitive test, dissemination and assessment of impact. Lesh, Kelly and Yoon (2008) propose a multi-tiered variant of this model, gradually scaling the experimental setting in order to balance cost and adaptability with validity and applicability.

These examples portray a vibrant community, reflective with respect to its practices, which is gradually converging towards its *commissive space* (Kelly et al, 2008) as discussed above. Yet at the same time it illustrates the need for a clear framing of methodological standards. Bell argues that “One might expect to find widespread theoretical or methodological coherence among efforts purporting to be design experimentation, but that is largely not the case” (Bell, 2004:243), suggesting that this is primarily due to a failure in communicating practices between groups. The craft of DBR is by and large transferred by daily communication in research teams, akin to an apprenticeship model. Practical knowledge sharing between groups is insufficient.

Many authors note the acute need for documentation and standardisation of procedures (e.g. Middleton et al. 2008; Bannan-Ritland and Baek 2008; Cobb and Gravemeijer, 2008; Kelly, 2004) yet often the same authors respond by anecdotal description of the practices employed by their group. These descriptions suffer from the same weakness often attributed to the results of DBR studies: it is hard to discern the generic from the context-specific, and there is no clear algorithm for choosing the most appropriate methods and procedures for any given research situation. Some frameworks provide good guidelines for a particular aspect of analysis. For example, the *conversational framework* affords systematic analysis of a system with respect to "what it takes to learn" (Laurillard, SHE 2008) while a tool such as the activity checklist (Kaptelinin, Nardi and Macaulay 1999) provides a thorough HCI view. As argued above, DBR demands the integration of multiple perspectives (Lesh, Kelly, Yoon, 2004), but relatively little is offered in form of guidance as to the means and criteria for such integration.

Arguably, the sparse articulation of the methodological frameworks of DBR has contributed to the resistance it has met both in research and in practice. The highly context-sensitive nature of DBR presents a challenge for replicability (Bell, Hoadley and Linn, 2004), which raises questions regarding scientific validity as well as applicability. Such concerns need to be met by a coherent *argumentative grammar*: a logical system by which claims are presented and justified, independently of their content (Kelly, 2004; Cobb and Gravemeijer, 2008). The argumentative grammar of randomised experiments, prevalent in many sciences, assumes regularity of phenomena and the inability to observe internal states. Such assumptions exclude the unique, innovative and irregular from the scope of study (Maxwell, 2004). By contrast, since DBR is interested in change, "keeping an eye open for unexpected knowledge is a central methodological heuristic" (Smith, diSessa and Roschelle, 1993: 66). Consequently, its argumentative grammar stresses process-oriented explanations (Maxwell, 2004; Cobb and Gravemeijer, 2004). This grammar needs to include forms for providing an *audit trail* (Creswell and Miller, 2000; Lincoln and Guba 1985): a mechanism of tracking back from conclusions to evidence. An artefact may embody a local theory, but without transparent scientific process it does not offer a test of the theory (Middleton et al 2008). Some researchers have suggested design narratives as a useful tool in this respect (Barab et al, 2008; Bell, Hoadley and Linn, 2004).

Finally, the argumentative grammar should provide forms for articulating claims about educational design, in a manner that affords scientific debate. Specifically, the functional tenor of design science would suggest a need for argument structures which associate characteristics of artefacts with their function. Various constructs have been proposed for capturing abstractions of design in education: design principles (Fuhrmann, Kali and Hoadley, 2008; Kali, 2006; 2009; Bell, Hoadley and Linn, 2004) and design patterns (Retalis et al, 2006; Haberman, 2006; Derntl and Motschnig-Pitrik, 2005; Goodyear et al. 2004) on the more theoretical end, and activity sequences (Dalziel 2006) EML and IMS-LD (Koper and Tattersall, 2005; Koper and Olivier, 2004) on the practical / technical side. Some initial work has been done on comparing different frameworks (Mor & Winters, 2007; McAndrew, Goodyear and Dalziel, 2006) but more is needed. Two questions arise: how do the different frameworks respond to the requirements for a scientific argumentative grammar of design, and how do they relate to each other, potentially contributing to a cohesive and comprehensive discourse.

2.3 Further Challenges for a Design Science of TEME

The previous sections presented a case for pursuing a design paradigm in TEME research. The motivation and merits of such an approach were reviewed, along with some open theoretical and methodological issues. These issues are compounded by practical and cultural obstacles. Such challenges are in no way surprising for a young field of enquiry. Nevertheless, they should be considered if we aspire for this field to establish itself and have an impact outside of a small community. These issues affect the impact and the cumulativeness of the paradigm. The first is extrinsic, related to the communication between this paradigm and others. The second is intrinsic, and refers to knowledge building within the community.

2.3.1 Impact

Relevance and resonance are two factors affecting the impact of research on practice, policy and other research. Relevance refers to the potential impact, the extent to which the outputs of a study could inform other works. Resonance indicates actual impact, the extent to which a study is noticed and considered in other works. While all sciences may be scrutinised along these parameters, design science by its nature has a greater commitment to impact. Pure sciences may perhaps appeal for the protection of their “aesthetic value”; a mathematical proof is not measured by its usefulness, but by its elegance. By contrast design science, as argued above, takes a pragmatist stance, identifying the value of knowledge with its effect on action.

It is hard to directly assess the impact of a particular discipline. Nevertheless, there is a subjective sentiment among some researchers involved in design research, suggesting that their findings are harder to publish and are not well-received by practitioners and policymakers. This impression finds some support by negation, when noting the dominance of the “evidence-based” attitude in policy, funding and research, discussed above. Such an uncompromising interpretation of evidence leaves little room for the agile, interventionist, process-oriented attitude of design research.

Despite the apparently difficult atmosphere, not all the fault lies externally. Several of the characteristics of design research observed above have contributed to its plight. The most notable factor is the lack of a clearly articulated consensual alternative methodology. Without a concise description of what constitutes evidence, and how it is used to predict effect, it is hard to argue for the utility of design science.

A second ‘self-inflicted wound’ is the tendency of writers in the field to attach cautious qualifiers to their results. Circumspection is a desirable quality in scientific discourse, and when there are limitations on the scope of results these should be communicated honestly. Yet a result is worth reporting only if it is generalisable to some extent, and that ‘some extent’ needs to be as clear as the result. diSessa and Cobb (2004) warn against a drift towards the anecdotal or the ethereal, and promote local, modest theories. Such theories are potentially of great value, but only inasmuch as their scope and preconditions are rigorously specified.

2.3.2 Cumulativity

The ideal of cumulative scientific progress is epitomised in the famous saying attributed to Isaac Newton: "If I have seen further than other men, it is because I have stood upon the shoulders of giants". Cumulativity is fundamental to the culture of research in mathematics and natural sciences; one is expected to prove theorems based on previously proven ones. An alternative proof is only merited if it displays a superior technique or is more robust. Not all sciences place the same emphasis on cumulativity; in the social sciences one is often encouraged to "find a voice" (e.g. Winter, 1998) and distinguish oneself by critique and rejection of prior art. Where should a design science of TEME position itself on this spectrum?

Mathematics itself is universal and consistent; it does not endure multiple truths. Yet the teaching and learning of mathematics are complex human activities, where individual trajectories play an important role. The nature of design science, as portrayed above, also displays a duality; the functional-pragmatist agenda dictates a stress on cumulativity, an aspiration not to solve the same problem twice. Yet the value dimension suggests a personal stance, and the act of design itself is often associated with individual creativity. Ideally, the design research paradigm should try to balance these forces, acknowledging the individual voice but seeking transferable elements of design knowledge, and clearly distinguishing between the unique and the modular. In order to do these, we need an argumentative grammar that allows researchers to articulate their personal experiences and extract from them generalisable claims. Such claims need to be expressed in a form which allows them to be scrutinised and aggregated into larger conceptual networks, connecting the experiences of multiple individuals and research groups.

Many of the issues discussed above make this a challenging goal. The shortage of consensual methodological and argumentative frameworks, the reluctance of researchers to commit to generic claims and the focus on innovative practice in unique circumstances stand out as contributing factors. One notable obstacle arises from the inherently multi-disciplinary nature of design research. As noted, researchers in the field have arrived from a wide range of traditions. Consequently, findings are reported using the language of these diverse traditions and their epistemic frameworks. The result is a "translation gap" between contributions which may be concerned with similar problems. In order to bridge this gap, we need forms of articulation which are structured around the problems being addressed, the forms of solution and the scope of relevance. The theoretical method used to analyse empirical observations is important, but subjecting it to the functional axis would allow us to establish link along the functional and across the analytical.

2.4 Summary and Conclusions

This chapter opened with an historical reflection on the notion of design science. It asked if and how this notion may be applied to educational research, and specifically to the use of technology in mathematics education. Three relations between design and research were identified: design as an object of study, design as a method of study and design as an outcome of study.

From this perspective, some of the prevailing design-centric approaches in educational research were reviewed. This review highlighted some of the gaps in these frameworks, most of which were noted by their respective practitioners, and some of which emerge from juxtaposition.

A notable theme emerging from this review is the need for extending and articulating the epistemic infrastructure of design research in mathematics education. Several weaknesses were identified in this respect, contributing to issues of relevance, resonance and cumulativity.

2.4.1 Towards a Model of Design Knowledge in TEME

The product of design science should be the systematisation of design knowledge. In light of the observations above, this chapter proposes a characterisation of design knowledge in technology enhanced mathematics education as:

Problem driven, solution oriented, value laden: design is always concerned with “changing existing situations into desired ones”. Thus, design knowledge always departs from an undesired situation, i.e. a problem, and aims to move to a desired one, i.e. a solution. The ascription of desirability measures to world states is a matter of values. This is evident in the case of TEME, which is always directed towards change: conceptual, behavioural and social.

Situated in context: the orientation towards states of the world also entails that the specifics of the circumstances to which an act or product of design apply are crucial. Problems and solutions are only valid with respect to a particular context, and that context needs to be articulated. Indeed, the field of TEME is often partitioned by context: primary vs. secondary or tertiary, formal or informal, etc.

Holistic (inherently inter-disciplinary): a focus on solving problems entails attention to all aspects of the issues under consideration. When the primary axis of decomposition is functional, it cuts across structural distinctions. Problems are dissected into sub-problems, but might retain the structural character of the whole. Thus, for example, the problem of designing an on-line course can be decomposed into the design of separate sessions, but each one will still need to consider social, cognitive and pedagogical factors.

2.4.2 Motivation for a Design Perspective on TEME Research

Simon’s (1969) observations above provide the initial motivation for considering a design science of TEME: learning is a natural phenomenon, whereas education is a conscious human endeavour directed at change. As such, it falls within the scope of Simon’s definition of design; the key challenge for the science of education is to understand and support “what it takes to learn” (Laurillard, 2008b, p 140).

Some researchers suggest that a design approach would be better adapted to the complex and dynamic nature of the circumstances and questions studied by educational science. Furthermore, this approach has the potential to offer a cohesive paradigm, bridging across practice and multiple theories. These advantages are made even more salient in the face of the rapid pace of change induced by technological developments, calling for agile, responsive and proactive approaches to educational research.

2.4.3 Towards an Epistemic Infrastructure for a Design Science of TEME

Design approaches to educational sciences project a young and vibrant research tradition, stemming from multiple roots and evolving simultaneously in multiple locations. Nevertheless, the review presented here identifies the emergence of shared practices and pockets of expertise. Among the common methodological characteristics are a dual focus on practical and theoretical contributions, a highly interventional and agile attitude, and a cycle of iterative research. This cycle includes phases of theory, design, implementation, execution (experiment / practice), articulation of experience, interpretation, evaluation and analysis, and feedback to both theory and design. The products of this cycle are validation or critique of existing theory, evidence regarding the effectiveness of artefacts and practices in well-defined settings, and innovations in practice and theory. A frequent by-product of research is the synthesis of multiple frameworks.

The goals of TEME are ambitious, and its challenges complex. Design research offers a sophisticated response to these challenges, but this sophistication makes it hard to communicate. This difficulty is amplified by the shortage of clear consensual methodological frameworks, as noted by many leaders in the field. Internal critics of the paradigm have called for a discussion of its commissive space and argumentative grammar. The former refers to the explicit and implicit rules and assumptions which bound the discourse of a scientific community, and the latter to the logical system by which claims are presented and justified, independently of their content. Together, these contribute to the epistemic infrastructure of the field. Some desirable features of such an infrastructure emerge from the discussion -

Accessibility: the arguments made by researchers should be readable by the scientific community, both the immediate and the broader scope of neighbouring fields, as well as practitioners, and policy makers. All these parties should be able to judge the validity of claims and interpret the results to their needs.

Transparency and traceability: the full cycle of a design study should be observable by an external reviewer, and most importantly the path that leads from theory to conjecture through experience and back to theory.

Expressiveness: the forms used for communicating design research should allow for the articulation of all that is needed to support the above requirements. They need to be able to capture process and product, connecting personal experience and generic abstractions.

Functional-pragmatist orientation: the mechanisms used to organise and communicate knowledge in the design science of TEME need to be aligned with the nature of this knowledge. Given the pragmatist foundations and functional axis of design knowledge, the research community needs means for organising this knowledge accordingly. For example, indexing findings by the problems they solve more than by the means they use to do so, by the conditions under which they are relevant more than by their academic heritage.

Cumulativity: finally, the forms of presenting claims and arguments need to afford easy aggregation of knowledge, building new results on the basis of prior art. At the same time, this

demand needs to be balanced with an acknowledgement of individual and local voice, creativity and the uniqueness of any given human situation.

2.4.4 Questions to be Addressed by this Thesis

The review of the possibility of TEME as a design science, its motivation and current state gave rise to several observations regarding the epistemic infrastructure of this paradigm. These observations are related to issues pertaining to the impact and prospects of such a paradigm. It follows that the emerging arguments should be put to an empirical test.

The main question arising is: does an epistemic infrastructure exist, such that it satisfies the requirements listed? Specifically –

- How do current methodological tools map to these requirements? Can they be combined to provide a more comprehensive coverage?
- Which new tools can be incorporated to address the issues noted and how?
- Once a comprehensive epistemic infrastructure is identified, properly articulated and justified, will it fulfil the promise of a design science of education? Will it answer the critique of current practices in design research of TEME?

The following chapters contribute towards an answer to these questions, by providing elements of such an infrastructure and testing them in a genuine research context.

Chapter 3 Aims and Method of Enquiry

This chapter revisits the aims outlined in Chapter one, in light of the review presented in Chapter 2. These aims are elaborated to propose a programme of research, and consequently characterise the overarching method of enquiry.

3.1 Introduction

Chapter Two presented an argument for a design science of technology-enhanced mathematics education (TEME), and reviewed existing traditions in this field. A key issue which emerged from this discussion was the need for a clearly articulated consensual epistemic infrastructure for this paradigm. This issue gave rise to several questions. The thesis intends to make a contribution towards such an infrastructure, by acknowledging existing frameworks, identifying gaps within them, and proposing some constructs to address them. The resulting framework is tested by applying it to a genuine problem domain, namely the use of technology to foster learning about number sequences through construction and collaboration.

Chapter 1 opened with the characterisation of education as designed learning, thus establishing a multi-faceted link between design and epistemology, or the creation of knowledge. This link is a motif that flows through this thesis. Epistemology can be read through three lenses: genetic, normative and generative; how we construct knowledge, how knowledge should be produced, and how we can design for the creation of knowledge. Piaget notes that "For many philosophers and epistemologists, epistemology is the study of knowledge as it exists at the present moment; it is the analysis of knowledge for its own sake and within its own framework without regard for its development" (Piaget, 1970, online). By contrast, "Genetic epistemology attempts to explain knowledge, and in particular scientific knowledge, on the basis of its history, its sociogenesis, and especially the psychological origins of the notions and operations upon which it is based" (ibid). The shift from the universal to the anthropocentric is completed by the school of design research in education. In his monograph *Toward an Epistemology of Physics*, diSessa posits: "A theory of knowledge and its development ought to be significant for education." (diSessa, 1993, p205). Epistemology has been repurposed from the study of knowledge as an abstract universal, to its evolution in the individual, social and historical human context, and finally to the question of how such dynamics can be facilitated and perfected.

The pragmatist nature of a design stance to educational research suggests a tight dependency between the three levels of epistemology: the method by which we study education (its normative epistemology) needs to link our understanding of how people learn (genetic epistemology) to how we design artefacts for learning (generative epistemology). The specific aims of this thesis are derived from this realisation. The aims identified in Chapter 1 can now be situated in this context and elaborated in view of the observations from Chapter 2. Thus, an overarching theme of this thesis can now be restated as:

To consider the study of technology-enhanced mathematics education as a design science; Highlight the implications of such a paradigm, and propose ways to theorize design in a manner which draws both on educational research and computer science.

Chapter 2 has taken the first steps in this direction, by proposing a characterisation of a design science of education, and arguing the case for its necessity. At the same time, that chapter has identified several issues which need to be addressed for the promise of a design science of education to be realised. These issues relate to the (normative) epistemic infrastructure of the paradigm, and its ability to link genetic to generative epistemologies. Consequently, the aims of this thesis can now be elaborated to address the question of a design science of TEME at three levels: epistemic, methodological and pedagogic.

Aim 1: *To identify potential elements of an **epistemic infrastructure** for a design science of technology enhanced mathematics education.*

Chapter Two identified the need for articulating an epistemic infrastructure for a design science of TEME. Such an infrastructure should delineate the paradigm's commissive space (Kelly et al, 2008) and argumentative grammar (Cobb and Gravemeijer, 2004). The former refers to the explicit and implicit rules and assumptions which bound the discourse of a scientific community, and the latter to the logical system by which claims are presented and justified, independently of their content. Chapter Two also identified some desired qualities of this epistemic infrastructure. It should be -

- **Accessible** to the scientific community, practitioners and policy makers.
- **Transparent and traceable** so that the full cycle of a design study should be observable by an external reviewer.
- Sufficiently **expressive** to allow articulation of all that is needed to support the above requirements.
- Organised with a **functional-pragmatist orientation**, indexing findings by the problems they solve rather than the resources (physical or theoretical) they utilise.
- Promote **Cumulativity** and afford easy aggregation of knowledge, building new results on the basis of prior art.

This thesis aims to explore two forms of reporting and analysing design experiences in education which claim to afford systematisation and effective communication of design knowledge in this domain: design narratives and design patterns. The theoretical issues arising from Chapter Two have practical implications in terms of the craft of research. The cycles of design research, and the embedding of Design narratives and design patterns within them, are at the apex of the forms of practice and associated representation discussed in Chapter four.

Aim 2: *To combine and elaborate the elements identified into a coherent **methodological framework** in a given research context*

In order to operationalise the proposed elements of epistemic infrastructure, the forms of design narratives and design patterns need to be embedded in the cycles of design research and appropriate tools of data collection and analysis recruited to support the mechanics of conducting experiments, collecting design narratives, eliciting patterns, and developing and substantiating theoretical innovations.

In specifying these tools and procedures, a methodological framework is deduced from the epistemic infrastructure. The details of such a framework are contingent on the specifics of the

research setting. This thesis takes the learning of number sequences in lower secondary level as its application domain. The experimental setting involved small groups of children in informal activities complementing their school curriculum, using the ToonTalk programming environment as a constructive medium and the WebReports collaboration system as a communicational medium. This thesis aims to present, justify and evaluate a methodological framework based on design narratives and design patterns which is appropriate for this setting, and consider the breadth of its applicability to similar situations.

Aim 3: *Apply the methodology in a problem domain and demonstrate its potential by producing a contribution towards a language of **pedagogical patterns** for technology enhanced mathematics education*

Aim 1 and Aim 2 relate to the nature and the process of a design science of education. The claims arising from these discussions need to be judged both logically and empirically. The empirical test of a scientific programme is in its execution. Thus, Chapters Six, Seven, and Eight constitute a “mini-thesis”, employing the outcomes of the previous chapters as a framework for a design study in a demonstrator problem domain. The fruits of this enquiry should, in a modest way, shed light both on processes of learning (genetic epistemology) and on effective ways of facilitating such processes (generative epistemology). Reflections on the process and outcomes of this enquiry should provide initial evidence towards evaluating the normative claims in Chapters Two and Four. The outputs of the design study will be captured as set of design narratives and design patterns, contributing towards a (pedagogical) pattern language for TEME. This contribution will be assessed in terms of its theoretical and practical potential.

Sections 3.2, 3.3 and 3.4 elaborate these three aims, and section 3.5 outlines the method of enquiry by which they will be pursued.

3.2 Notes on Aim 1: Towards a Design Science of Technology Enhanced Mathematics Education

Chapter Two noted the distinction between asking “*how do people learn*”, and, “*how do we improve the conditions for people to learn?*” While the former could be investigated from the perspective of natural or social sciences, the latter is amenable to enquiry from a design science stance. Several immediate questions emerge from this observation:

- What is the nature and character of a design science of mathematics education?
- What relationship between educational research and practice is implied by a design science perspective, and what advantages does it bring to both?
- What are the epistemological structures of a design science of mathematics education – how is knowledge sought and truth established?

These questions were considered in Chapter Two. The emerging conclusions provide the foundations for the rest of this thesis, and at the same time act as conjectures to be explored empirically. Following the arguments in Chapter Two are valid, it should be possible to conduct a study along the lines sketched there which would produce innovative results and offer a

significant contribution in terms of both practice and theory. Chapter Four addresses this objective.

The review of the possibility of education as a design science, its motivation and current state gave rise to several observations regarding the epistemic infrastructure of this paradigm. The main question arising is: does there exist an epistemic infrastructure that satisfies the requirements listed? Specifically –

- What are the existing conventions regarding appropriate forms for conducting design-based research in education?
- What are the requirements on methodological tools implied by the characterisation of the design paradigm? How do current tools map to these requirements? Can they be combined and enhanced to provide a more comprehensive coverage?
- Which new tools can be incorporated to address the issues noted and how?

Chapter Four considers these questions and presents some contributions towards an epistemic infrastructure. These are used by Chapter Five to propose a concrete methodological framework. These observations are related to issues pertaining to the impact and prospects of the underlying paradigm. Indeed, the demonstrator study reported in Chapters Six, Seven and Eight provides a test of a design science of education as reflected in this framework. The primary innovation in this framework is the combination of design narratives and design patterns as a means for analysing and reporting design knowledge in mathematics education. The specific contribution in terms of these structures is the focus of Aim 2.

3.3 Notes on Aim 2: Towards a Pattern-based Methodology for a Design Science of Technology Enhanced Mathematical Education

A consideration of the epistemic infrastructure for design research of TEME, in response to Aim 1, suggests the combination of design narratives and design patterns as a contribution towards an epistemic infrastructure for a design science of mathematics education. This proposal rests on several claims regarding the qualities of such representations, when embedded in appropriate research practices, namely that they -

- Allow the process and outcomes of techno-pedagogical design to be captured in a manner that affords analysis, critique, and refinement.
- Provide an intermediate level of abstraction that links theory to practice and facilitates a design-centric discussion of the problem domain.
- Combine narrative and structured representation, linking intuitive and rich descriptions with rigorous semantics.

Chapters Two, Four and Five provide theoretical warrants for these claims. Empirical evidence is required to further substantiate them. Such preliminary evidence, as a proof of concept and feasibility, is presented in Chapters Seven and Eight in the form of a contribution towards a pattern language in the domain of TEME. This language is to include a system of design patterns and supporting design narratives, with theoretical and practical implications. It intends to –

- Capture and link theoretical and practical innovations in a manner that is transferable to other circumstances, while clearly delineating the boundaries of transferability.
- Articulate these innovations in a form that is accessible to both academic and practitioner communities.
- Uphold a scientific standard of transparency and traceability.
- Afford cumulativity, by providing forms for integrating other relevant bodies of knowledge and encapsulating outputs in a form that can be integrated and reused elsewhere.

3.4 Notes on Aim 3: Design Patterns for Learning about Number Sequences by Construction, Communication and Collaboration

In order to gauge the value of the epistemic constructs and the methodological framework emerging as a response to Aims 1 and 2, they need to be applied to a genuine problem domain. The problem domain chosen for this thesis is the design of tools and activities for learning about number sequences, in an extra-curricular lower-secondary school setting. Although the main effort in this thesis is oriented towards general theoretical, disciplinary and methodological questions, the demonstrator study should yield meaningful, if modest, results in its domain.

The demonstrator study reported in Chapters Six, Seven and Eight follows groups of young teens exploring questions about number sequences through collaborative construction activities. These activities utilise two integrated computational media: WebReports, a web-based collaboration system, and ToonTalk, a game-like programming environment. Consequently, the theoretical and practical contributions derived from these experiments, and expressed through design narratives and design patterns, should bear immediate relevance to the problem domain. Some of the outputs will be specific to the domain of learning about number sequences through construction and collaboration, and some will have wider relevance.

The demonstrator study will:

- Identify obstacles to learning in the domain.
- Raise pedagogical conjectures addressing these obstacles.
- Design tools and activities manifesting these conjectures.
- Evaluate the success of these tools and activities.
- Capture the insights emerging from this evaluation in the form of design patterns.
- Note how the outcomes reflect back on the initial pedagogical conjectures.

3.5 Method of Enquiry

The three aims declared in Chapter One, further detailed in section 3.1, and elaborated in subsequent sections, approach the question of research in mathematical education on three levels of abstraction. The first considers the epistemic level, the next explores methodological issues, and the third level engages in concrete pedagogical issues. These three levels are linked in two directions: the arguments in each level form the basis for the next, and the evidence

observed at each level feeds back to support the previous one. Thus, each aim applies a different method of enquiry, while relating to the others.

Aim 1 is investigated through theoretical review and discussion, with an emphasis on constructing a coherent argument by critical consideration of the existing literature.

Aim 3 is addressed through a cycle of experimental field work, analysis and reflection.

Aim 2 mediates between these two by deriving and projecting the consequences of Aim 1 into the specific settings of Aim 3, monitoring the effectiveness of the derived constructs and adjusting them in response to the empirical observations.

3.6 Conclusions

This chapter defined the aims of the thesis and elaborated them in view of the theoretical review presented in Chapter Two. Each aim was expanded into a set of specific research questions. Each aim addresses the notion of research in technology enhanced mathematics education on a different level, while referring to and feeding back into the other levels. Consequently the methods of enquiry differ between levels while being tightly interlinked.

Chapter 4 A Pattern-Based Approach to Design Research in Technology Enhanced Mathematics Education

This chapter responds to Aim 1, identified in Chapter three: To identify potential elements of an epistemic infrastructure for a design science of technology enhanced mathematics education. The proposed elements include a cycle of enquiry, design narratives, and design patterns. These constructs are reviewed and critiqued. The next chapter uses them as a basis for a methodological framework.

4.1 Introduction: In Search of an Epistemic Infrastructure for a Design Science of Mathematical Education

Chapter two put forth a case for a design science of mathematics education, reviewed the existing tradition in this paradigm, and highlighted some theoretical and methodological challenges. An observation of the state of the field suggested a need for a clearly articulated consensual epistemic infrastructure. Some requirements for this infrastructure were identified, in terms of accessibility, transparency and traceability, expressiveness, orientation, and cumulativity. These requirements gave rise to several open questions (section 2.4.4):

- How do current methodological tools map to these requirements? Can they be combined to provide a more comprehensive coverage?
- Which new tools can be incorporated to address the issues noted and how?
- Once a comprehensive epistemic infrastructure is identified, properly articulated and justified, will it fulfil the promise of a design science of education?

This chapter offers a contribution towards the overarching goal of establishing an epistemic infrastructure, by addressing some of these questions. I begin by describing a cycle of design research, common to several accounts (Middleton et al, 2008; Gravemeijer and Cobb, 2006; Lesh and Sriraman, 2005; Cobb et al, 2003) reviewed in Chapter 2. In line with these accounts and others, this cycle is situated within a broader framework.

Each stage in these cycles calls for appropriate methodological tools. In the sections that follow, I present such tools and discuss their relative merits and challenges. I then focus on a specific set identified as suitable for this study, focusing on design narratives and design patterns. I conclude with some challenges associated with these.

4.1.1 Common Cycles of Scientific Process

As discussed in Chapter 2, design research in mathematics education is commonly described as a cyclic process. At the core of this process is the design experiment, which oscillates between theoretical and practical innovations. Most authors situate the cycle of design experiment within the context of their research setting. However, when the various descriptions (McKenney et al, 2006; Gravemeijer and Cobb, 2006; Cobb and Gravemeijer, 2008; Gorard, Roberts and Taylor,

2004; Cobb et al, 2003; Bannan-Ritland, 2003) are compared and synthesised and the contextual details blurred, a stable image emerges (Figure 2).

As in most scientific endeavours, a design experiment would typically start from a theoretical stance, which the researcher would project into a particular problem domain to derive a conjecture. This conjecture is examined by designing artefacts (tools and practices) that embody it. The artefacts are implemented and used in action, ideally in a realistic educational setting. The researcher collects evidence of the successes and tensions arising from the use of the artefacts, with respect to the learning aims. This evidence is interpreted, analysed and evaluated, and the results fed back into a revised theory.

Although scientific enquiry often stems from a theoretical stance, the cyclic nature of design experiments suggests that other options are just as valid, e.g. a study led by a teacher and originating from her classroom experience.

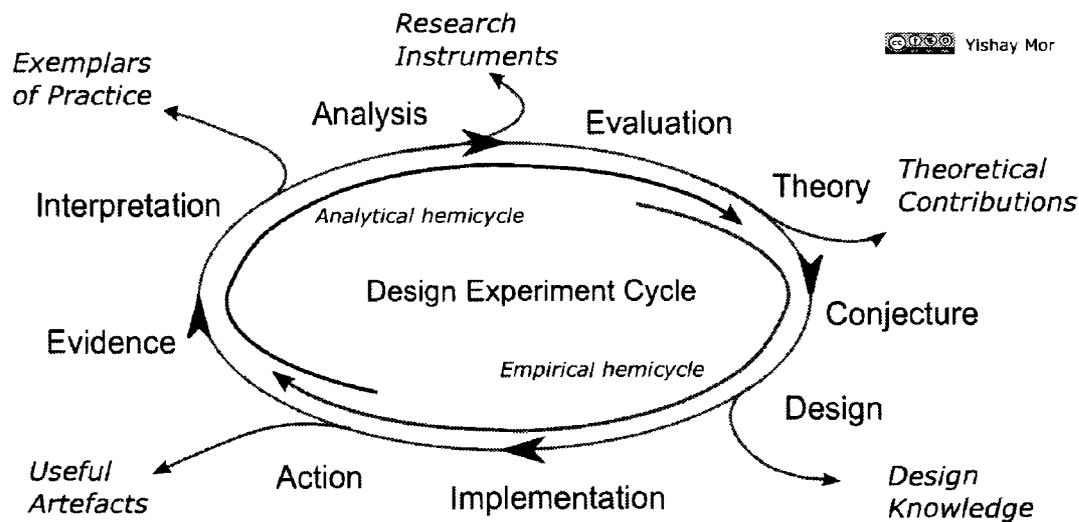


Figure 2: The design experiment cycle, synthesized from the literature reviewed in Chapter 2. Research conjectures are derived from theory, and explored by design, implementation and field testing of new artefacts (tools and activities). The evidence from the field trials is collected, interpreted, analysed and evaluated – feeding back into theory and on to the next iteration. This cycle is embedded in the broader framework described in Figure 3.

As argued in Chapter 2, a design science of education pursues a double-edged agenda to produce theoretical as well as practical innovations. Figure 2 illustrates how these aims are reflected in the outputs derived from the different phases in the cycle. The empirical hemicycle proceeds from theory to action through design and implementation, ultimately producing artefacts (technological tools, curricular materials, teaching methods, etc.) which should be useful for practitioners operating in similar situations. The evidence collected from the action phase and its interpretation produce exemplars of practice, which provide practitioners with valuable insights as to how to make effective use of the artefacts. As noted by Schwartz et al. (2008), the study of innovative artefacts demands innovative research instruments. The development of such instruments is a likely by-product of the interpretation and evaluation phases. Some of these instruments are specific to the situation being studied, but others are useful for peers studying similar situations.

The pinnacle of the analytical hemicycle, starting from the end of the action phase, is the contribution to an updated theoretical stance. This contribution has two facets: a reflection on the underlying theory, validating or challenging the premises of the experiment, and local theories and ontological (diSessa and Cobb, 2004) or epistemic innovations (Schwartz et al., 2008) referring to the specific problem domain. These theoretical innovations feed back into the design process, along with direct input from the analytical outcomes. The outputs of the design phase are representations of design knowledge derived by projecting the theory into the problem domain, and adjusting to meet the pragmatic constraints imposed by the learning context. Given appropriate representations, such design knowledge should be valuable beyond the unique situation being studied.

Similar iterative models are common in human-centred design (Maguire, 2001). Gasson (2003) proposes a dual-cycle model, iterating between inquiry, design, development and evaluation. Sharples et al (2002) describe the socio-cognitive engineering methodology for design of human-centred technology. At its core is a cycle which includes specifying a design concept; generating a space of possible system designs; specifying the functional and non-functional aspects of the system; implementing and deploying the system. Such models provide valuable reference points for the empirical hemi-cycle of the design experiment. Drawing on learner-centred design, Kolodner et al (2004; 2003) combine case-based reasoning and problem-based learning, to derive a proposal for learning by design. This pedagogy employs a similar cycle as a basis for learners' activity. As such, it is not concerned with producing any lasting outputs other than the change in the learners' knowledge.

Design experiments are embedded in a broader cycle of design research (Middleton et al., 2008; Gravemeijer and Cobb, 2006; Pratt, 1988). This cycle include a preliminary phase where the research problem is framed, an empirical phase consisting of an iterative design experiment, like that described above, and a longitudinal reflective phase of retrospective analysis (Figure 3). Hevner (2007) describes a similar three-cycle model in the context of a design science approach to information systems research.

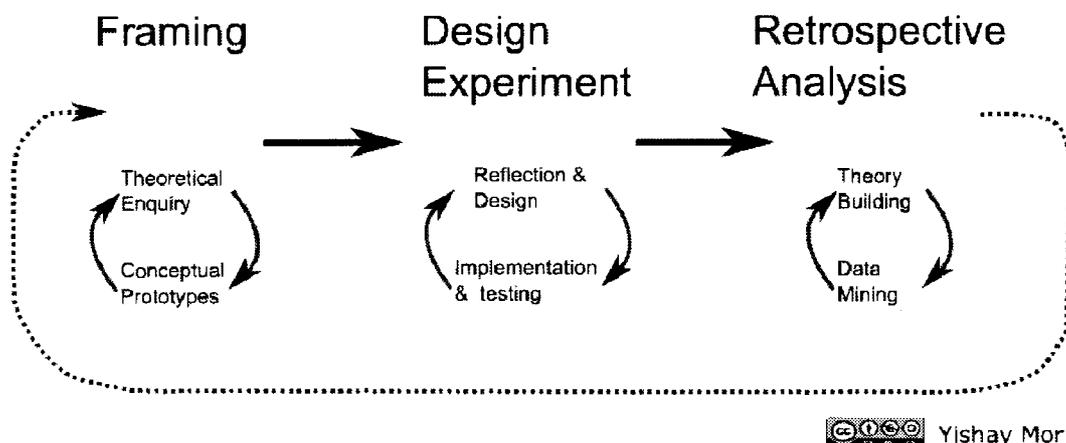


Figure 3: Design research meta-cycle, synthesized from the accounts of several experts in the field. The life cycle of a design study begins with a framing phase, iterating between theoretical enquiry and prototyping. This is followed by multiple iterations of the design experiment cycle described in Figure 2. The study is concluded by a retrospective analysis phase, considering data from across the multiple empirical iterations.

The framing and retrospective analysis phases are by and large context independent; they are conducted by the researcher at the comfort of her desk and thus are not constrained by the experimental setting. In fact, the retrospective analysis does not differ significantly in structure from similar phases in other research paradigms, although the actual sources of data and methods of analysis do. The framing phase does reflect the unique nature of design science, in its underlying premise of the link between knowledge representation and artefacts. This principle, which underlies many of the research questions, also motivates the researchers' own process of understanding and interpreting theory. The framing phase oscillates between reviewing existing literature for theoretical concepts and reifying these concepts in quick prototypes. Such prototypes are primarily used as a means of understanding the theory, as a form of "armchair experiments", and would most often be discarded before the next phase. Reeves (2006) describes a similar meta-cycle, and notes a crucial distinction between run-of-the-mill predictive research and design research. The first is strictly committed to producing theoretical outputs, and thus its meta-cycle ends when a robust academic result is achieved, leaving its interpretation to policy makers and practitioners. The dual agenda of design research entails that the theoretical outputs of retrospective analysis need to be complemented with practical outputs and both are fed back into the next iteration of design experiments.

The design experiment phase iterates along the path described above. The number and nature of iterations varies, but they are expected to be expansive; each iteration extends the scope or validity of the previous ones (Middleton et al, 2008). At some point the cycles of the design experiment are concluded, and the study shifts to a retrospective analysis, taking in a long perspective covering multiple iterations and calibrating with other studies. This phase is also iterative; building theories and mining the history of the project for supporting data. In reality, the boundaries between the three phases are often blurred. Pratt (1998) describes a four-iteration structure used in his study:

- **Iteration 0: Bootstrapping.** A 'blue sky' phase, free-form exploration of the platform and the tools it offers. In the case of the demonstrator study, this involved prototyping of ToonTalk tools and a web-based collaboration system.
- **Iteration 1: Exploratory.** Initial development and testing of tools within the platform / environment. The emphasis is still on basic usability and intuitive indicators of learning potentials.
- **Iteration 2: Developmental.** The designs have achieved a level of maturity which allows the researcher to shift attention to questions of learning and specific aims within a well-defined educational context and content domain.
- **Iteration 3: Analytical.** Relatively minor changes to design, and careful attention to questions of data collection, analysis and theorisation.

Each iteration in Pratt's framework consisted of two steps: design-development and use-evaluation. The lessons learnt in the second step of each iteration defined the agenda for the first step of the next. Such a structure is typical of many design based studies in education. It

reflects the emergence in tandem of the researcher's own understanding of the tools he or she develops and the evidence collected regarding the influence these tools have on students' understanding of the subject matter. Pratt's iteration 0 may overlap with the framing phase, and iteration 3 leads naturally into the retrospective analysis.

The two cycles presented above (Figure 2 and Figure 3) provide a "wireframe" for an epistemic infrastructure for design research in mathematics education. In order to complete the model, we need to identify suitable forms of representing knowledge in each node of these cycles, and the means for moving from one node to the next. These would include research methods and "translation schemes" for converting knowledge from one representation to another. Together, these would provide the fabric of argumentative grammar.

The next section considers a possible path from experience to formal knowledge. This model derives elements of a scientific methodology from a naturalistic process of knowledge construction.

4.2 A Possible Path from Experience to Knowledge

The empirical hemicycle of a design experiment has a variety of analogues to refer to. The act of deriving conjectures by projecting theory into context is common to most scientific practices, and the act of expressing design knowledge in artefacts is rooted in the crafts of teaching and educational development. Although there is a lot to be explored in terms of formalising these arts, it is the analytical hemicycle that warrants primary attention. This hemicycle is unique to design research, and consequently less well documented.

This section seeks to articulate a framework for the analytical hemicycle, by reference to an apparent innate process by which we extract knowledge from experience. The main motivation for grounding a scientific framework in innate epistemic dynamics is lucidity, the main challenge is rigour; tracing a natural process should make it easier to understand but since such a process does not emerge from a scientific tradition, it runs the risk of compromising standards of research.

The model, which I call the *narrative transposition epistemic model* (Figure 4), traces the construction of knowledge through a series of transpositions from one representation to another. This process starts from our physical presence in the world. As we experience events, they are encoded in our episodic memory (Wheeler, Stuss and Tulving, 1997). The ability to store and replay sequences of experiences is a defining feature of human cognition. It is in the root of our ability to reflect on the past and plan for the future (Atance and O'Neill, 2005) "mental time travel involves awareness not only of what has been but also of what may come." (Tulving, 2000, p20). The selection of experiences we store in memory, indeed – our perception of these experiences – is influenced by our social and cognitive functions (Simons, Hannula, Warren and Day, 2007; Balctetis and Dunning, 2006; Simons and Rensink, 2005) yet these choices are not conscious or deliberate. As sequences of events are committed to episodic memory, we compare them to our previous memories. This comparison invokes perceptual cognition (Biederman and Vessel, 2006): recurring perceptions arouse episodic memories of past events, while novel experiences stimulate our attention. In our attempt to make sense of our experiences, we order them into narratives: sequences of events bound by temporal and implied

causal links (Bruner and Lucariello, 1989; Bruner, 1991). This is the *mode transposition* phase: in the process of *narratisation* the mental imagery of episodic memory is transposed into narrative mode. As these narratives accumulate in our memory, those that are closest to each other are *fused*, creating new narratives with less contextual detail and a more general scope. This process is shaped by the discourses in which we engage, by providing both substance and structure. The surrounding culture provides additional experiences – encoded as narratives – with which we fuse our own. At the same time, it provides us with accepted genres for articulating knowledge in different levels of abstraction. Each genre is a system of conversational meta-rules (Bruner, 1991; Sfard, 2007). This is the *genre transposition* phase: as common patterns emerge and singular details fade away in the process of *fusing*, the narrative shifts from an imaginative genre to a paradigmatic one (Bruner, 1986); “my cat Felix died” becomes “all cats are mortal”. Thus, knowledge is abstracted from experience. Yet the fine threads from these abstractions to the experiences from which they were derived are never fully severed; these abstractions remain situated in the context of our activities (Noss & Hoyles, 1996).

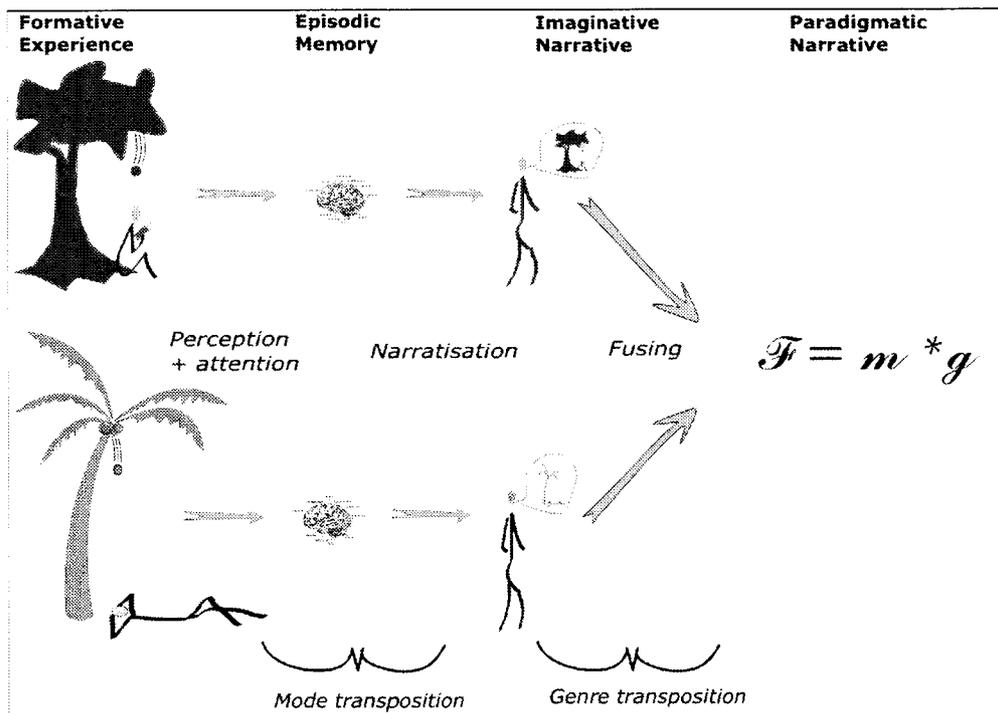


Figure 4: Schematic of narrative transposition epistemic model. Experiences are encoded in episodic memory by mode transposition of perceptions. Episodic memory is organised by narratisation, and encoded as imaginative narratives. These are fused in a process of genre transposition into paradigmatic narratives.

Nota bene: the narrative transposition model is offered as an epistemological claim, not a grand theory of learning. I am not claiming that this is *the* way we construct knowledge, only that it is *a* way, and one worthy of attention. The importance of this model is in the trajectory it offers from experience, through narrative, to structured propositional knowledge: experience is encoded in episodic memory, interpreted by the selection and sequencing of events in narrative, and generalised and abstracted by fusing of narratives and genre transposition. This epistemic model can be read using three lenses: genetic, normative and pragmatic; i.e., how we construct knowledge, how knowledge should be constructed, and how we can design for the construction of knowledge. Recognising that the extraction of propositional knowledge from narrative is non-

trivial, this issue is addressed in detail below. Chapter 6 applies the pragmatic lens as a guide for designing tools and activities, and Chapter 7 applies the genetic lens to interpret learners' interaction with them. Sections 4.4 and 4.5 focus on the normative perspective, deriving methodological tools and argumentative grammar from the model. Before doing so, the remainder of this section elaborates the base model, and section 4.3 notes some parallels and differences with other approaches.

4.2.1 Narrative and Knowledge

Narrative, in this thesis, is considered in its epistemic capacity. Bruner (1986; 1990; 1991; 1996; Bruner & Lucariello, 1989) identified narrative as the predominant vernacular form of representing and communicating meaning. Narratives are not merely descriptions of *what* happened – they provide an implicit or sometimes explicit explanation of *why* it happened. A narrative, in a nutshell, is an account of *something* happening to *someone* in some *circumstances*. A well-formed narrative must maintain coherence of *temporality* and *causality* (Gergen, 1998). Temporality refers to the chronological ordering of events. Narrative intelligence theory (Mateas & Sengers, 1999), suggests that the identification of temporal affinity of events also plays a strong role in learners' inferences of causality, an important component in the construction of meanings. The sequencing of events is referred to as the *plot*. Gergen (1998) adds that events are carefully selected to support an endpoint. Yet perhaps the most important part of a narrative is typically left unstated: its moral; the narrative's implicit endpoint. A story is told for a purpose – establishing norms, conveying knowledge, or raising a question. It is the implicit layer that holds the narrative together – the causal relationships along the way and the climactic moral at the end. Without them, all we have is an arbitrary list of events. As Mar asserts, "If a well-crafted story contains mention of an event or character, it is assumed that this element is in some way relevant to the goals of the protagonist." (Mar, 2004, p. 1416). To summarise, Narrative is a form of language which includes a context (setting), a protagonist, a plot, and an implicit moral. Narrative also has an affective facet, which includes elements such as voice (the storyteller's presence), and genre (her choice of style and cultural frame of reference). This facet is outside the scope of this thesis.

Schank and Abelson (1995) argue that stories about one's experiences, and the experiences of others, are the fundamental constituents of human memory, knowledge, and social communication. They call for a shift towards a functional view of knowledge, as Schank (1995) explains: "intelligence is really about understanding what has happened well enough to be able to predict when it may happen again" (p. 1). Such knowledge is constructed by indexing narratives of self and others' experiences, and mapping them to structures already in memory. While Schank and Abelson come from an AI perspective, their theory is supported by recent psychological studies. Atance and O'Neill (2005) define episodic future thinking as the ability to project oneself into the future to pre-experience an event. This, they claim, is a uniquely human phenomenon which precedes semantic future thinking (Atance and Meltzoff, 2005), and provides the developmental basis for skills such as planning and causal reasoning. They found that episodic future thinking emerges around the age of four, and is related to children's abilities to construct and comprehend verbal accounts of experiences.

The model above seems to be supported by recent advances in neural psychology (Spreng et al, 2008; Mar et al, 2006; Mar, 2004; Holyoak & Kroger, 1995; Young & Saver, 2001; Addis et al, 2004; Mason and Just, 2004). Xu et al (2005) link context to brain regions responsible for global semantic processes such as inference, coherence, conceptual association and text integration. Other findings point to a strong link between narrative comprehension and theory-of-mind processing (Mar, 2004), suggesting that the cognitive modelling of the storyteller and the protagonists is a critical constituent in understanding a story. The neural evidence shows that similar mechanisms are invoked in narrative comprehension and construction. It also suggests an embodied element: reading a story which involves actions or physical experiences activates the same regions of the brain that are involved in control or perception of similar experiences (Mar et al, 2006). While some parts of this model have already found a large body of supporting data, others still call for further validation. Yet the picture we see is strongly consistent with existing theories of learning. The invocation of physical experience in narrative comprehension and construction supports an embodied view of learning (Lindblom and Ziemke, 2003; Lakoff and Núñez, 2000; Núñez et al, 1999). In particular, it suggests a mapping of the aspect schema (Lakoff & Núñez, 2000) to the exposition, plot and closure of a narrative. Promising as it may seem, a critical discussion of this literature is beyond the scope of this thesis.

These links need to be explored carefully, both theoretically and empirically. They are presented here as corroborating perspectives, but are not fundamental to the central arguments of this thesis.

4.3 The Narrative Turn in Mathematics Education Research

Narrative Enquiry is a well-established paradigm in educational research and teacher training (Conle, 2000; Clandinin, 2007). It is a methodology which "... rests on the epistemological assumption that we as human beings make sense of random experience by the imposition of story structures" (Bell, 2002). Narrative methods are seen as pathways into mathematics education (e.g. Smith, 2006). Much of the work in this vein focuses on narrative as a means of expressing issues of identity and culture, as Bailey (2007:103) sees narrative inquiry as "a journey during which researchers come to know more deeply about their lives and who they are as people" and Kaasila (2007) promotes the value of a narrative view of teacher education in highlighting the personal process of becoming a teacher and construing professional identity. Others also see the notion of *identity* as pivotal: learning is directed by the need to transfer oneself from an actual to a designated identity (Healy and Sinclair, 2007; Sfard and Prusak, 2005). Identities are defined by the stories, narratives, we tell ourselves about ourselves. To understand how learners come to *be* mathematical, how mathematics becomes part of their identity, we need to look at their stories.

This focus on *being* appears at odds with the pragmatist attitude of design science, which forefronts *doing*. Indeed, some authors, such as Markovits and Smith (2008) prefer to use the term cases in order to distance themselves from what they see as "narratives for entertainment". Yet the epistemic model considered above does not preclude knowledge in the pragmatic sense. This model highlights a particular trajectory from experience to knowledge, which involves two phases: the encoding of experience as narrative, and the fusing of narratives through genre transposition. All it requires in order to fit a pragmatist paradigm is that the

narratives we choose to inspect are those dealing with change in the world rather than with change in the protagonist's self-perception. I will refer to such narratives as *design narratives* in order to distinguish them from *identity narratives*.

Design narratives are reviewed in the next section. Notwithstanding their fundamental differences, the Design Narrative approach shares a common trait with Narrative Enquiry and case based research. All three attempt to redress the tension between research and practice, discussed in section 2.2.2. The principle underlying these approaches is the provision of rich and accurate descriptions of reality as experienced by the researcher-practitioner. This principle appears to be very much in line with the pragmatist ideal. However, all approaches risk reducing their discourse to anecdotes. Regardless of whether one considers the plural of anecdote to be data or not,¹ the accumulation of anecdotes does not constitute a theory. As noted above the moral, or conclusion, of a narrative is implicit. Scientific discourse demands that this be made explicit so that it be exposed to scrutiny. Furthermore, a clear method needs to be identified by which knowledge is generalised across individual cases or narratives and captured in a form which can be applied to new situations. The model above suggested an innate process which operates by fusing narratives and genre transposition. Section 4.5 proposes a similar model as a scientific method.

4.4 Design Narratives

As discussed in the previous chapter, design research operates "at the edge of chaos"; research settings and problems are complex, messy and often unique. This creates a challenge in terms of the replicability expected of a scientific experiment. Several authors have noted this difficulty and proposed the construct of *design narratives* as a means of addressing it (Bell, Hoadley and Linn, 2004; Hoadley 2002; Barab et al, 2008). The main argument in favour of design narratives is that they provide a "thick description" of the design experiment, allowing critics to assess the validity of the researchers' claims, and trace them back to evidence. At the same time, design narratives provide sufficient contextual information for those who wish to conduct a similar experiment in proximal settings, be they fellow researchers or practitioners wishing to apply the research findings.

Design narratives are accounts of critical events from a personal, phenomenological perspective. They focus on design in the sense of problem solving, describing a problem in the chosen domain, the actions taken to resolve it and their unfolding effects. They provide an account of the history and evolution of a design over time, including the research context, the tools and activities designed, and the results of users' interactions with these. They portray the complete path leading to an educational innovation, not just its final form – including failed attempts and the modifications they espoused. Narrative, notes Hoadley (2002:454), "is only one way of

¹ "The plural of anecdote is not data" is an idiom popular among sceptics in on-line discussions. An attempt to trace it leads to many references to Frank Kotsonis, although other opinions persist and it is hard to assert a definitive attribution. Several sources note that the original is actually "The plural of anecdote is data", attributed to the Berkeley political scientist Raymond Wolfinger.

making sense of design-based research” but “to really convey what happened, though, requires a story.”

Despite the prevalence of the narrative form in reports of design research (Bannan-Ritland, 2003) it raises several methodological and practical issues. In the words of Shavelson et al. (2003:25), “there is nothing in narrative form that guarantees veracity”. Practically, narrative accounts do not fit well into academic publication format (Reeves et al, 2005). One apparent source of methodological vagueness is the lack of upfront discussion of the narrative tools used by researchers. With a few notable exceptions (e.g. Barab et al, 2008) most studies intuitively use a narrative style of report without explicitly formulating it as a methodology. The term design narrative itself is rarely used, although many papers are in essence design narratives. Even when the form is discussed, it lacks a rigorous definition: what is the core structure of a design narrative? How are its boundaries set? How are events selected and details filtered out? How should we judge if the narrative warrants the researchers’ claims? Section 4.4.1 takes a closer look at these questions.

Another source of difficulty lies within the inherent nature of narrative. In a well-crafted narrative, the message of the story is left implicit (Mor and Noss, 2007). This feature may be epistemically powerful, as it provokes the reader to infer the message and construct her own logical structure to support it. However it is incompatible with scientific discourse, which demands that the path from evidence to arguments to conclusions be exposed to peer scrutiny. The implication is that design narratives are incomplete as a scientific form, and need to be accompanied by a representation of the derived knowledge. Bell, Hoadley and Linn (2004) propose design principles (Kali, Levin-Peled and Dori, 2009), while section 4.5 suggests design patterns. Both are structured abstractions of design knowledge. Whereas design principles are arguably self-contained, and thus more readily accessible, I find the structure of design patterns more amenable to scientific cumulativeness (as discussed in section 2.3.2), while at the same time retaining a sense of narrative.

Finally, it is important to remember the interpretive quality of narrative. A narrative is not a neutral recount of events; it is the outcome of the narrator’s immediate attempt at making sense of events, a conjecture regarding the semantics of occurrences. Arguably, this is common to all manner of organising evidence: the statistical analysis of a randomised experiment reflects the researchers’ choice of parameters and variables. Yet in the case of statistical analysis, another researcher using the same choice of material could have produced the same result. A narrative is unique to its narrator. This subjectivity may be appropriate in design research, where the researcher is part of the phenomena, but nevertheless needs to be accounted for.

4.4.1 Towards a formalisation of design narratives

In order for design narratives to provide an effective form of discourse for design research in education, they need to be shaped in a way that would adhere to scientific standards, acknowledge the agenda of design science, and retain the essential qualities of narrative. This may seem a tall order, but in fact carefully designed forms and procedures for design narratives could allow us to align these forces.

A scientific standard demands a transparent audit trail from reliable data to conclusions, and a clear articulation of refutable claims. Where subjectivity is inevitable, it should be reported

honestly. A design science stance dictates a functional (pragmatic) focus linked to a value dimension, attention to context and representation, and an awareness of the complexity of human situations. Narrative form entails a clear context description, a protagonist, a plot – a temporally and semantically linked sequence of events – and an implied moral. Combining these three delineates the requirements for design narratives as a scientific instrument. A design narrative is defined by a single problem to be solved or task to be accomplished. In this thesis, I distinguish between two types of design narratives: researcher narratives (RNs) and participant narratives (PNs).

Researcher narratives recount a pedagogical problem and its resolution from the researcher's point of view. They are first person accounts of the researcher's experience and observations, in the course of a design experiment. In most cases, the focus is on the design and development of activities, social practices and supporting technology. These elements are seen as an integral unit, under the socio-technical stance that these are inseparable and any partial description would be meaningless for our purpose.

Participant narratives follow the participants in a design experiment – teachers and learners – as designers, contending with problems they encountered in the context of an activity, their use of the resources provided in confronting this problem, and the indications of their learning gains in the process. These are third person accounts based on the learners' written and verbal articulations and my observations.

In a researcher narrative, the protagonist is the researcher, and the narrative would typically be her first-voice account of events. In a participant narrative, the protagonists are teachers or learners participating in the experiment. Although it is often not feasible to expect a full first-person account, an effort should be made to capture the participants' voice.

The two types of narratives are interdependent; the problems encountered by learners and their resolution are the drivers of their learning trajectory. The researcher's problem, from a bird's eye view, is to provide learners with an effective set of problems and the means for resolving them, so as to direct their learning trajectory. Thus, the PNs illuminate and substantiate the RNs.

A design narrative should:

- Provide an account of an aspect of a design experiment, from the perspective of the designer / researcher or that of a participant, and, as much as possible, capturing their voice.
- Clearly delineate the context of the design experiment and its educational goals.
- Present a documented record of the researchers' / participants' actions and their effect.
- Incorporate data collected and processed in appropriate scientific methods.
- Decouple reporting events from their evaluation and reflection.
- Be followed by a statement of the derived conclusions, linking them clearly and explicitly back to the narrative.

The conclusion derived from a design narrative is a design claim, i.e. a statement about how to achieve a particular educational effect in a particular context. This claim is external to the design narrative, but it guides the narrator's choice of which events to include in the narrative. Consequently, there can be multiple narratives of the same experiment. All are just as valid, as long as they meet the criteria.

Bruner identifies *Canonicity and Breach* as a defining quality of narrative, arguing that “for to be worth telling, a tale must be about how an implicit canonical script has been breached...” (Bruner, 1991, p 11). In the case of design narratives this implies they should either capture a new solution to a known problem, or a new problem. The uniqueness of the single narrative is complimented by its *Accrual* (Bruner, 1991): the manner in which it connects with other narratives to form a coherent body of knowledge.

Bruner (1991) enumerates ten qualities of narrative: Narrative diachronicity, Particularity, Intentional state entailment, Hermeneutic composability, Canonicity and breach, Referentiality, Genericness, Normativeness, Context sensitivity and negotiability and Narrative accrual (Nardi, 2007; Sinclair, Healy and Sales, 2009). *Canonicity and breach* and *Accrual* have been mentioned above as criteria for delineating the whole set of narratives. The others serve as guidelines in the construction of the narratives themselves. These principles require adaptation in order to comply with the norms of scientific discourse, as illustrated in Table 1.

	Bruner (1991)	Adaptation
Narrative diachronicity	“narrative comprises an ensemble of ways of constructing and representing the sequential, diachronic order of human events ... its unique pattern of events over time” (p. 6)	Events are clearly dated, with reference to documentation, and reported in chronological order.
Particularity	A narrative reflects on the generic via the specific. It is an account of an incident, not any incident.	Incidents are selected to provide clear and traceable examples of general phenomena. The selection criteria need to be verifiable.
Intentional state entailment	The actions and events portrayed in a narrative must be relevant to the characters beliefs, desires, theories, values, etc. These cannot be observed directly, yet the story derives its meaning from their induction.	First person design narratives include the researchers’ account of their intentional state. Any conjectures regarding the intentional state of other agents are supported by interviews and observations.
Hermeneutic composability	The interpretation of a story and the extraction of meaning from it is inseparable from its text, and is part of the implied contract between author and perceiver.	The researcher’s interpretation needs to be opened for scrutiny. It is appended to the narrative, while the body of the narrative is kept free of interpretation.
Referentiality	To be accepted a narrative does not need to be a verified recount of reality, but it must convince the reader that it could have been a recount of reality.	Reference had to be convincingly true to actual events.
Genericness	A narrative is associated with a Genre, which provides a framework for its interpretation	The genre of design narratives emerges from the growing tradition of design research in mathematics education.
Normativeness	The problem in the centre of a narrative illuminates a norm by its resolution or in the absence of resolution by contrast.	Design narratives highlight innovations, thus proposing a variation of norms to account for changing circumstances or tensions between existing norms.
Context sensitivity and negotiability	Assumed background knowledge which modulates the narrative’s interpretation and the meaning it implies.	The key features of the context are listed, to allow the reader to evaluate it.

Table 1: Bruner’s (1991) qualities of narrative, with adaptations to serve as guidelines for constructing design narratives

Referring back to section 4.1.1 and Figure 2, the construction of design narratives is a suitable instrument for the interpretation of the raw evidence arising from the empirical actions. The resulting narratives should be useful in themselves, as exemplars for practitioners and peers. However, in terms of the design research process, they need to be processed further in the course of analysis and evaluation. Following the epistemic model presented in section 4.2, this processing can take the form of genre transposition, shifting from the imaginative form of a design narrative to a paradigmatic one. The next section argues for the use of design patterns as a paradigmatic form suitable for the analysis and evaluation of the outcomes captured in the design narratives.

4.5 Design Patterns

The *Design patterns* paradigm (Alexander et al, 1977) was developed as a form of design language within architecture. This was done with the explicit aim of externalizing knowledge to allow the accumulation and generalization of solutions and to allow all members of a community or design group to participate in discussions relating to design. These patterns were organized into coherent systems called *pattern languages* where patterns are related to each other. The use of design patterns never achieved a large following among professional architects, but the idea has been embraced in several other disciplines, starting with software engineering. In their seminal book Gamma et al. (1995) argue:

One thing expert designers know *not* to do is solve every problem from first principles. Rather, they reuse solutions that have worked for them in the past. ... Consequently, you'll find recurring patterns of classes and communicating objects in many object-oriented systems. These patterns solve specific design problems and make ... designs more flexible, elegant, and ultimately reusable. They help designers reuse successful designs by basing new designs on prior experience. A designer who is familiar with such patterns can apply them immediately to design problems without having to rediscover them. (Gamma et al, 1995, p1)

Appropriating the ideas of Christopher Alexander, they provided a standard template for software design patterns and a taxonomy of 26 patterns. Since then, pattern books, conferences and web sites have proliferated and spread into every aspect of software related design and production. These patterns and pattern languages enable designers to share, discuss and aggregate their knowledge across wide, scattered and diverse communities. More recent examples of areas where collections of design patterns have been created include hypermedia (c.f. German & Cowan 2000), interaction design (c.f. Erickson, 2000 and Borchers, 2000; 2001), and computer-mediated interaction (Schümmer and Lukosch, 2007). Recently, the concept of design patterns has made its first strides in educational domains. One such domain is that of educationally oriented software systems, such as e-learning systems (Derntl & Motschnig-Pitrik, 2004); another is the design of computer science courses (Bergin, 2000).

The original definition of a design pattern positions it as a high-level specification of a method of design which specifies the context of discussion, the particulars of the problem, and how these can be addressed by the designated design instruments. In *Pattern Languages* Alexander writes:

Each pattern describes a problem which occurs over and over again in our environment, and then describes the core of the solution to that problem, in such a way that you can use this solution a million times over, without ever doing it the same way twice. (Alexander, 1977, p. x)

And in *the Timeless Way of Building* (1979) he elaborates:

Each pattern is a three-part rule, which expresses a relation between a certain context, a problem, and a solution.

As an element in the world, each pattern is a relationship between a certain context, a certain system of forces which occurs repeatedly in that context, and a certain spatial configuration which allows these forces to resolve themselves.

As an element of language, a pattern is an instruction, which shows how this spatial configuration can be used, over and over again, to resolve the given system of forces, wherever the context makes it relevant.

The pattern is, in short, at the same time a thing which happens in the world, and the rule which tells us how to create that thing, and when we must create it. It is both a process and a thing; both a description of a thing which is alive, and a description of the process which will generate that thing. (Alexander, 1979, p 247)

In other words, a pattern has three facets: descriptive, normative, and communicative. It is an analytic form, used to describe design situations and solutions; a meta-design tool, used to highlight key issues and dictate a method of resolving them; and a communicative tool enabling different communities to discuss design issues and solutions. The tension between these three aspects is visible in Alexander's work, and in much of the literature that followed. I will touch upon this issue shortly.

The original collection by Alexander et al (1977; 1979) can arguably be positioned on the *normative* end of the scale, in the sense that an ethical stance can be interpreted from the collection. As Ericksson puts it: "Alexander's Pattern Language is not value neutral" (Ericksson, 2000). On the other hand, Alexander's Mexicali project is taken as an emblem of participatory design, where patterns are used to facilitate design and empower users – who make their own choices (Dearden et al, 2002). In this case, patterns are predominantly a social tool allowing the expert to communicate knowledge to the families designing their own home. One could claim that there is an ethical agenda here as well. The difference is that in this case it is stated frankly, and given explicitly as the premise – not the conclusion.

Such pattern languages seem to be quite alien to the *descriptive* pattern languages, prevalent in software design. This contrast may stem from Alexander's strong convictions – which need not be shared by many software designers. On the other hand, they may be inherent in the nature of some fields. While in urban planning and architecture it is clear that almost any decision has a political and socio-economic context, it is hard to see such a context in the design of, for

example, network routing protocols. However, this distinction should be made with great caution. Design is rarely as value-neutral as we perceive it. The designers' personal, subliminal values are always in the background. Even the software example we used has its ethical dimensions: are the protocols open or closed? Do they allow for encryption? Do they have 'government backdoors'? Such decisions which are often made off-hand have significant consequences in terms of civil liberties. The value dimension of patterns becomes more salient as we move from the core of a technological system (e.g. network protocols, data storage algorithms and database structures) towards the user. Interface and interaction design is laden with such implicit value decisions: does the interface empower the user, or harness her to organizational needs? Is it gender or culturally biased? Does it marginalize users with disabilities? Such questions are generally pushed aside. Perhaps the most notable exception was the Scandinavian participatory design movement in the 1970s, mentioned above, which set forth out of an explicit design agenda of democratizing technology and empowering workers (Kensing & Blomberg, 1998; Asaro, 2000).

I have explored two interrelated dimensions of design patterns: the functional axis (what are they used for) and the value axis. It still remains to mention the *subject* axis – or, what are the patterns *of*? Alexander's patterns are structural – they describe spatial configurations (Alexander, 1977). So are the 'Gang of Four' software design patterns, which describe ensembles of classes in object-oriented programming (Gamma et al, 1995). Other languages aim to design *Actions* (Ericksson, 2000) or *Activity Systems* (Guy, 2004). Digiano et al (2002), for example, interweave three levels in a language of *collaboration design patterns*: whole activity patterns, which describe the dynamics of human interaction, data patterns, which describe the structure and relationships of the artefacts exchanged in the process, and support patterns, detailed patterns which enable higher-order patterns to flow smoothly. The next section discusses the use of pattern languages in educational contexts.

4.5.1 Use of Pattern Languages in Educational Contexts

As mentioned above, the idea of design patterns originated in architectural theory, but the computer science community was the first to embrace it. It is not surprising that this is also where it has made the greatest impact with respect to education. The design patterns approach has manifested itself through three main trends. The first is the growing trend of *Pedagogical Design Patterns* (Anthony, 1996; Bergin 2000; Eckstein, Bergin & Sharp, 2002). The second is the development of software design patterns for educational technology (Dearden, Finlay, Allgar & Mcmanus, 2002; Avgeriou, Vogiatzis, Tzanavari and Retalis, 2004). The third is the search for patterns in related practices, such as evaluation and assessment (Barre, Chaquet & El-Kechai, 2005). Nevertheless, it is important to note that the first reference to learning is made by Alexander himself. In his seminal book (Alexander et al, 1977) he describes a pattern called "Network of Learning". The premise of this pattern is that in a society that emphasises teaching, learners become passive and unable to think or act for themselves. He argues that creative and active individuals can only grow up in a society that focuses on learning instead of teaching. The solution he proposes is to replace the structures of compulsory schooling in a fixed place, with decentralised processes of learning which engage learners through contact with many situations and people all over the city: workshops, teachers at home, professionals willing to take on the young as helpers, older children, museums, youth groups, scholarly seminars, industrial

workshops, old people, and so on. This argument resonates with Illich's call for "deschooling society" (1971) and conviviality (1973).

Pedagogical design patterns apply the concept of design patterns to pedagogical design. The fundamental claim behind this effort is that many experienced practitioners in education have tried and tested methods of solving recurring problems or addressing common needs. Among the pioneers in this field were Anthony (1995) and later the pedagogical patterns project (<http://www.pedagogicalpatterns.org/>), initiated by a group of experienced software engineering and computer science educators (Bergin, 2000; Eckstein, Bergin & Sharp, 2002). They proposed a set of patterns dealing with issues ranging from the design of a college course to specific principles of computer science instruction and to concrete problems and their solutions.

A second arena that has seen a proliferation of design patterns over recent years is web-based educational technologies. Notable examples in this field include the E-LEN project (<http://www2.tisip.no/E-LEN/>) and several initiatives within the IMS-LD framework (<http://www.imsglobal.org>). Most of the work in this area is focused on the engineering aspects of designing, developing, deploying and evaluating good technology for web-based instruction (Frizell & Hubscher, 2002; Hernández-Leo et al, 2006; Bailey et al, 2006).

This strain of work is done mainly in the context of developing large scale technological systems to support organizational and vocational learning or web-delivered higher and further education. Due to this context, much of the work is highly technical. Many of the valuable innovations have a strong engineering flavour to them (e.g. Bailey et al, 2006), which might deter teachers and educational researchers. The interaction between student and instructor is assumed to be mediated exclusively by web-based communication channels. Under such circumstances, most of the effort goes into designing the representation and organization of educational content and the mechanisms by which learners interact with it (Frizell & Hubscher, 2002). Design patterns are also situated in this context, with the engineer of educational technologies as the user in mind (Avgeriou et al, 2003; Garzotto et al, 2004; Kolås & Staupe, 2004). From this perspective, pedagogical issues are often assumed rather than discussed. A noteworthy exception is Goodyear (2004). In an attempt to distance himself from the dominant approaches in e-learning, Goodyear focuses on what he calls networked learning, where technology is used to promote connections between learners and foster communities which make efficient use of their resources. In this context, Goodyear emphasises patterns as a means of empowering practitioners to utilize accumulated design knowledge. His patterns are succinct and written in plain language. Another study oriented towards educators is Dearden et al. (2002; 2002b). They point to the strong ideological and methodological parallels between Alexander's original vision of pattern language and the paradigm of participatory design. They propose the 'facilitation' model developed by Alexander et al (1985) in the Mexicali project as an alternative to the dominant approach of using patterns to deliver expert knowledge to novices.

Finally, design patterns have recently been used in the context of assessment, evaluation and analysis of learning and learning systems. Pachler et al (2009) apply a pattern-based methodology in the domain of formative e-assessment. Wei, Mislevy, and Kanal (2008) provide an extensive collection of patterns for language assessment. Gibert-Darras et al, (2005) offer a

pattern language for assessing students' problem solving abilities in the context of a basic Java course. The standard Alexandrian argument holds here as well: assessing students' performance is a hard job, into which a lot of research has been done and many practitioners have accumulated insights through experience. Patterns allow us to offer this knowledge in a useful form to novice teachers.

To conclude, with the exception of Dearden et al (2002a; 2002b), Goodyear (2004) and Bergin et al (2000, 2002), most studies which utilize design patterns in education are concerned with the hard issues of creating good educational technology and authoring content within technological systems. Without downplaying the importance of such endeavours, I see a potential for pattern languages which refer both to the technological and the pedagogical aspects of designing environments and opportunities for learning. They should elucidate grounded principles for the design of content, activities and tools, and the relationships between them. To achieve this, they would need to be integrated with a range of tools and representations for capturing design knowledge.

Another uncharted territory is the pedagogical dimension of known software design patterns. For example, the STREAMS pattern, presented in section 8.7, is widely used in application and systems programming. When approached from an epistemic-pedagogical perspective, it has other qualities of note such as its correspondence with intuitive notions of sequences and infinity (section 7.2.1). Another example is the *model-view-controller*, or *MVC*, pattern (Krasner & Pope, 1988; Gamma et al, 1995). This pattern separates the representation and manipulation of information from its structure and content. MVC is perhaps one of the most powerful and widely used patterns in interface design. From a pedagogical perspective, it resonates well with the discussion of representations (Balacheff & Kaput, 1966; Radford, 2000). Indeed, this pattern is utilized in the design of ToonTalk (Kahn, 1996). However, most educators involved in constructionist activity design are not aware of it. Furthermore, the pattern's common descriptions are focused on the engineering aspects and do not expose its epistemic qualities. For educational designers to leverage the benefits of this pattern, it needs to be expanded in a manner that will bring together both worlds, that of software engineering and that of educational design.

4.5.2 Design patterns between timelessness and innovation

Salingaros (2000) argues for the timelessness of Alexander's pattern language. Yet if we see the design, production and usage of tools as a dynamic process of social learning, timelessness is lost. In the iterations of a design experiment, patterns are constantly shifting and reforming. Even the act of eliciting and formalizing a pattern may lead to its own evolution, as it is adopted and modified by the community. Paradoxically, the more successful a pattern language is, the less stable it would be. In part, this discrepancy with Alexander's theory can be attributed to the nature of digital technology. The design of urban environments is constrained by characteristics of the physics and geography of our world, the structure of the human body and the workings of the brain, all of which are slow to change. Hence, most of the patterns that Alexander identified in the 1970s are just as relevant today as they probably were 200 years earlier. In the case of digital artefacts, the extent of human imagination is often the only constraining factor. A second difference is the breadth and depth of design activity. When building a house I can design its

layout, but I must choose among available tools and materials. With digital artefacts, this distinction is blurred. Furthermore, digital technologies give widespread access to domains of design which were previously laborious and professionalized.

The observations of the previous section suggest a change of emphasis on the role of design patterns. Alexander's patterns may have aimed at capturing age-old design knowledge and make it available to a wide audience. In the domain where age-old knowledge does not exist, patterns should instead aim at identifying elements of effective practice as they emerge, capturing them as objects for discussion, scrutiny and manipulation. Alexander's architectural design patterns are informed by theories of construction, engineering and human psychology. In much the same way, pedagogical patterns should be based on a theoretical layer concerning pedagogy and epistemology. Whereas, for example, a software design pattern may need to include justification in terms of computational efficiency and robustness, a pedagogical design pattern should include its epistemological, psychological or social dynamic rationale. Unfortunately, this is rarely the case. Since most pedagogical patterns are developed by skilled practitioners (or software engineers) who have their formal grounding in computer science rather than educational sciences, they are informed by solid intuitions much more than educational theory.

Furthermore, design patterns could offer a potent analytical tool for design-based study of technology-enhanced mathematical learning. Design patterns offer a method for gradual abstraction out of personal experiences. This method is particularly relevant in the design-based research context, where personal reflection on design experiments is considered valid data. Design patterns retain critical elements of narrative form, and add structure which allows us to see links, hierarchies and systemic forces.

4.5.3 The Promise of Design Patterns

Section 4.4 suggested a need for semi-formal notation to be used in conjunction with design narratives to help capture the design knowledge derived from them. Several forms have been suggested for capturing abstractions of design knowledge in education, among them design principles (Kali, Spitulnik and Linn, 2004; Kali, 2008; 2005; Kali, Levin-Peled and Dori, 2009), scripts (Miao et al., 2005; Kobbe et al., 2007) and sequences (Dalziel, 2006). McAndrew, Goodyear and Dalziel (2006) compare a few of these. This section considers the qualities of the design pattern form, which make it a suitable candidate for complementing design narratives as a component in an epistemic infrastructure for a design science of education.

The core of a design pattern can be seen as a local functional statement: "for problem P, under circumstances C, solution S has been known to work". Such a structure reads like a direct generalisation of the narrative form of "something happened to someone under some circumstances", when that narrative is a record of a problem solving effort – in other words, a design narrative.

By forefronting the problem, the structure of a design pattern acknowledges the functional axis of decomposition and the value dimension, identified in section 2.2 as tenets of design science emerging from Herbert Simon's work. These features are further expressed in the links between patterns, inherent to the pattern format. Christopher Alexander (1999) explicitly highlights what

he calls the "moral" and "generic" qualities of pattern languages, and asks whether these are present in the way the idea has been appropriated by computer science.

Complexity and context-dependence are characteristics of design based research in education which emerge from the discussion in Chapter 2. The design patterns approach is sensitive to these issues, and reflects them by restricting solution statements to compact classes of problems in clearly delineated contexts. In this sense, a design pattern can be seen as a representation of a local theory or a modular ontological innovation, to rephrase diSessa and Cobb (2004).

The modest nature of design patterns can also be seen as an expression of a pragmatist philosophy, suggested by several authors as the foundation of design-based research. This philosophy supports the notion of ontological innovations, which diSessa and Cobb (2004) derive from the need to address the gap between practice and theory. Design patterns were described as abstractions of expert knowledge; they generalise from successful practice without detaching from its context. As such, they offer a two-way bridge between practice and theory: opening practical wisdom to theoretical scrutiny and allowing theory to be projected into practice. A pragmatist perspective leads many design researchers to seek holistic frameworks, calling on diverse mixes of theories and methodologies in the service of comprehensive solutions. The core structure of design patterns is conducive to such an approach, as it demands precision in description of problem, context and solution, and subjects to them theory and evidence.

The functional, holistic, compact form of design patterns also makes them promising candidates to serve as boundary objects in design-level interdisciplinary discussions. Following Bowker and Star (1999), there is a growing acknowledgement that practitioners from different communities interfacing in a joint enterprise may inhibit distinct activity systems (Tuomi-Gröhn and Engeström, 2003). Consequently, the conceptual spaces that these communities form around the joint enterprise would diverge, impeding communication and coordinated resolution of emerging issues. Boundary objects are artefacts that might help to calibrate the diverse perspectives towards a shared canon of knowledge situated in common problems (Noss et al., 2007; Bakker et al. 2006). As argued in Chapter 2, educational design is an inherently multi-disciplinary activity. An effective study of design – whether scientific or practical – demands linguistic and symbolic tools which will enable boundary crossing and facilitate discussion between the various interested communities, to ensure that solutions and analysis take into account all factors they deem significant. Design patterns – if carefully crafted as products of interdisciplinary discussion – may emerge as such boundary objects.

Finally, design patterns have been used extensively in object-oriented programming for over a decade. Apart from their popularity amongst software designers, recent studies indicate measurable benefits in terms of cognitive load (Kolfschoten et al., 2006), software quality (Guéhéneuc et al., 2006) and system maintenance (Prechelt et al. 2001). Evaluating the effect of design patterns is neither trivial nor conclusive; as noted by Khomh and Guéhéneuc (2008) they can also have negative consequences. Furthermore, it would be irresponsible to suggest a simple analogy from software development to education. Nevertheless, such results do raise the possibility of added value for both educational design as a practice and its scientific study.

4.5.4 Challenges for the Design Pattern Approach

Section 4.5.3 argued for the potential of design patterns as components in an epistemic infrastructure for design research in mathematics education. Design patterns, introduced by Alexander in 1977, have been appropriated for education more than a decade ago (Eckstein, Manns and Voelter, 2001). Yet so far they have not witnessed wide-spread adoption in educational practice or research. To an extent, this could be attributed to sociological factors: for many years, the design pattern approach was predominantly visible only within the community of architects. The impact of the approach in the world of software design was only noticeable after the publication of Gamma et al.'s book in 1995. Design patterns entered the world of education through the "back door" of educational technology and computer science education. However, in education the design patterns approach does face some fundamental challenges that impede its acceptance. Notably, it raises issues of validity, resonance, cumulativity and innovation.

Validity refers to the scientific confidence which can be attributed to a pattern. Much of the pattern literature positions itself as expert insights rather than a component of scientific research. Thus, common patterns are supported by intuitive arguments and personal conviction derived from professional experience. Christopher Alexander himself declares: "I am a scientist" (2008), but does not present the scientific method by which his patterns were derived. In order to be accepted as part of a scientific discourse, patterns need to include a clear audit trail linking them to data, and incorporate reference to relevant theoretical warrants. They also need to be situated within and in relation to other forms of scientific practice.

The issue of resonance relates to the impact design patterns have in the relevant communities. This is something of a "chicken and egg" issue: as a linguistic form, a tool for conversation within and across communities, the more widespread patterns are, the more useful they become. Yet without a critical mass acknowledging their usefulness, they remain esoteric and marginal. While a substantial literature of pedagogical patterns appears to be building up, there is a shortage of introductory texts, initiating new audiences into the art of reading, writing and using patterns. More crucially, one of the core virtues of design patterns is an obstacle to their adoption: as noted above, design patterns inhabit a space between theory and practice. This allows them potentially to bridge between the two, but it also raises the risk of being perceived by practitioners as too abstract and by theoreticians as trivial. In order to overcome these obstacles, design patterns need to strengthen their link both to practical examples (e.g. through associated design narratives) and to theoretical warrants and consequences. Design patterns need to be embedded within a wider scientific and practical discourse, and their relationship with other formats and methods established.

The issue of cumulativity concerns the extent to which existing knowledge is used as a foundation for new developments. By definition, design patterns describe the common elements of recurring solutions to recurring problems. It would seem that knowledge aggregation is inherent in patterns, as they attempt to provide a shareable, reusable representation of expert knowledge. Yet the literature is dominated by collections of novel patterns, while reference to prior collections or evidence of their use is sparse. In computer science, the commercial proliferation of pattern language books (e.g. Wiley publishers software design patterns series)

suggests that at least the knowledge captured by these languages is reused, if not accumulated. Unfortunately, when it comes to educational design patterns the evidence of use is inconclusive at best. Several factors might possibly contribute to the lack of cumulativeness. The lack of cumulativeness may be related to a common positioning of pattern writing as a “discovery” process, rather than one of “construction”. A prevalent view assumes that patterns are “out there” in the world, and the task of the pattern author is to uncover them. This view ignores the fact that for many authors of patterns, the process of identifying and articulating a pattern is in itself a trajectory of learning, interpretation, and construction of personal local theories. In such cases, the author might be more interested in expressing her own patterns than in using others’. Scientific cumulativeness is a property of a community, and needs to be embedded in its culture. In order to achieve this property, design patterns need to become part of an on-going discourse in a design science community attuned to its cycles of enquiry.

The issue of cumulativeness, and in particular the observation of pattern-writing as a trajectory of learning, leads to the tension between effectiveness and validity on one hand and innovation on the other. This tension is inherent in design research (Schwartz, Chang and Martin, 2008) and is amplified by the original positioning of design patterns as a record of established practice. As the title of “The Timeless Way of Building” (Alexander, 1979) suggests, Christopher Alexander saw design patterns as capturing timeless qualities of architectural solutions which have been observed consistently across time and location. By contrast, design research in education deals with innovation – identifying novel solutions, and using these as tests for educational theories. This is even more the case when dealing with technological elements, where the problems themselves are fluid and the context constantly shifting. The tension between established “good practice” and innovation echoes the tension between the dual facets of design science, advancing theory on the one hand and supporting practice on the other. Alexander himself may hint at a way to reconcile these forces. In *A pattern language* (Alexander, 1977) he assigns three degrees of confidence to different patterns. Borchers (2000) notes that such ranking helps the community distinguish and position each pattern on the scale between the universal and the experimental. Some patterns aim to highlight truly timeless solutions, others to provoke questions and new perspectives by proposing radical changes. A good pattern language should strive to balance these, clearly identifying each pattern on this scale. Furthermore, articulating reputable solutions as patterns has its own value for innovation, as it allows us to scrutinize and refine them, and to transfer them to new settings. Educational and technological innovations often focus on one aspect of a problem situation, at the risk of naïve assumptions along other dimensions. Design patterns allow us to base innovations on established knowledge, confining uncertainty to the conjecture being tested. When combined with suitable complementing representations (such as design narratives) and supported by meticulous research practices, design patterns hold the potential for aligning cumulativeness, effectiveness and innovation, while balancing practical value and scientific validity.

4.6 Bringing it all Together: Attaching the Representations to the Research Cycle

Section 4.1.1 presented a cycle of design experiments in education, and identified a need for representations of knowledge appropriate for each phase, and the means of transitioning

between them. Section 4.2 reviewed an innate process of extracting knowledge from experience, as a possible analogy for such representations and transitional mechanisms. Using such an analogous model has two advantages. First, it lends credibility to the scientific process by grounding it in a natural process that is known to work, while reinforcing it with formal structures which add scientific rigor. Furthermore, it enhances its readability by drawing on familiar dynamics, thus making it more transparent to academic scrutiny and more accessible to practitioners.

Section 4.4 introduced design narratives and section 4.5 complemented them with design patterns. The latter were presented as representations of design knowledge with a growing degree of abstraction. Using the innate learning model as a guideline, it is possible to set these representations into the design experiment cycle and see the transitions between them as genre transpositions. Figure 5 illustrates one possible embedding; the events of the implementation and action phases are captured by design narratives, augmented with any data that can support them. Design narratives are used as a basis for pattern extraction, producing initial design patterns. These patterns are substantiated by reference to theory and additional supporting evidence. The resulting mature patterns are offered as outputs to the community, and at the same time fed back into the design and implementation phases in preparation for the next iteration of empirical work.

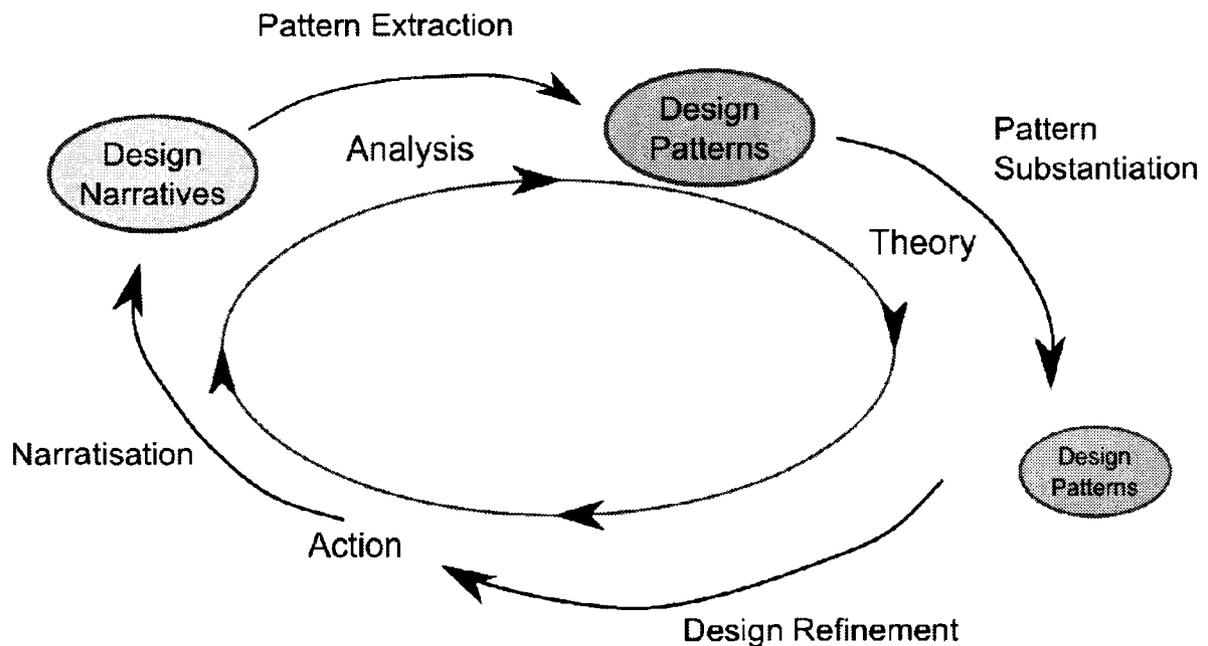


Figure 5: Design narratives and design patterns set in the design experiment cycle

4.7 “Yes, but is it methodological?”²

The coming of age of the design based research (DBR) paradigm in education has been marked by emerging methodological standards (Cobb et al, 2003; Shavelson et al, 2003) which are characterised as iterative, process-focused, interventionist, collaborative, multileveled, utility oriented, and theory driven. The applicable methods of data collection and analysis are derived from the nature of research. Controlled variables and pre- and post- tests give way to intensive process-oriented observations. A mixture of mainly qualitative methods is used, including video and audio recording of student activities, analysis of texts and artefacts produced in the course of these activities, interviews and ethnographic field notes. The collaborative nature of DBR poses an apparent challenge to doctoral research, given the requirement to demonstrate individual contribution. This tension can be negotiated by defining clear responsibilities and keeping an account of the process, e.g. as a research journal.

Drawing parallels between DBR and other methodological paradigms provides a guideline for evaluating specific research tools. DBR shares the situatedness of *action research*. Similar to grounded theory, it strives to make theory relevant by maintaining a strong link to practical experience. It resonates with the subjective stance and meticulousness of phenomenology. Juuti and Lavonen (2006) compare DBR to other paradigms, such as didactical engineering (Douady, 1985; Artigue, 1994), noting that such approaches aim to produce theory-informed educational design, whereas DBR is equally concerned with theoretical innovation. What sets DBR apart from these other paradigms, particularly in the case of educational research, is, perhaps, its fundamental interest in questions of epistemology. The situation which is studied is not the focus of attention: it is a window into participants’ experiences and through them to common human traits. Putting human experience in the centre, while acknowledging the subjectivity of the participant-observer, links DBR with the phenomenological approach. Yet phenomenology sees the detailed description of the personal experience as its main concern (Denscombe, 2003). For DBR it is a step on the way to modest and careful theoretical claims. These derive further support from triangulation with other empirical results, often from diverse fields of research.

Mapping the links between design-based educational research and other paradigms suggests that some of the specific tools used by these traditions may be relevant to the present study. Yet there are two unique characteristics of DBR which would require careful adaptation of any tool. First, the focus of inquiry in DBR is human learning and how it is affected by educational design. Obviously, any research tool used would have to be calibrated towards this perspective. Second, DBR is distinguished by a tight, dynamic and continuous interaction of design, experimentation and analysis. In contrast with other approaches, it is not unusual for data collected in one week to be quickly analysed and used to redesign the session for the following week. This requires agile methods of collection and analysis which can fit into such an intensive cycle.

Data is analysed at several levels of granularity. The microgenetic level (Brown, 1992) offers immediate and specific indications regarding the relations between details of design and incidents of learning. These serve as input for the dynamic adjustment of designs, and at the

² The title of this section paraphrases Kelly (2004): *Design Research in Education: Yes, but is it Methodological?*

same time for the accumulation of data to support higher-level theoretical statements. Intermediate analysis takes place between iterations, revisiting the design of tools and activities in light of the evidence collected during and after the experiment. This level of analysis supports adjustments to design within the existing structure. Such adjustments allow researchers to reduce 'noise' by removing superfluous usability flaws, and at the same time test conjectures that emerge from the data by designing elements which relate specifically to predictions derived from them.

Reflective analysis is a critical phase of a design study, which takes in the totality of microgenetic and longitudinal observations made over several iterations, and reflects on them in light of existing theory to produce new fundamental claims. The complexity arising from the real-life nature of the research settings, along with the highly interventionist method of study, call for detailed personal longitudinal accounts of design and experimentation processes. Such accounts, argue Shavelson et al (2003), tend to take a narrative form.

4.7.1 Observation-in-action

Augmenting the principles of DBR with the ideas of situated abstraction discussed below (section 6.3.1) suggests an emphasis on thinking-in-change (Noss & Hoyles, 1996) within an activity system. The notion of thinking-in-change emerges from the view that knowledge is not transferred instantaneously from teacher to learner, but is dynamically constructed through constructive activity and reflective communication. This implies that in order to study learning, we need to observe the process by which it proceeds: pre and post comparison does not suffice. If knowledge germinates or coagulates in the course of activity, we need instruments capable of capturing critical incidents as they occur. Pratt (1998) argues that this leads to a focus on description and interpretation, in the tradition of ethnography or anthropology. Yet as he notes elsewhere (Ainley, Pratt and Hansen, 2006) the interventionist nature of DBR leads to a methodology based on perturbing thinking (Noss and Hoyles, 2006): confronting the learner with new situations, provoking her to act, and observing how her thinking changes in response. The ideal of the participant-observer is extended to a principle of observation-in-action: the researcher disrupts the learner's experience, provokes her to act or express herself verbally, and tries to capture the minute indications of thinking-in-change as they occur. This principle applies to all levels of granularity, from the microgenetic to the reflective.

It is important to acknowledge the absence of 'pure' observations in a DBR setting. There are no 'one way mirrors'. When the observer is a key participant, any action he takes in the face of learners is an intervention. Taking note of a learner's casual articulation lends it significance in that learner's mind. Rather than trying to ignore this factor, it is better to incorporate it into the design and practice of research, while accepting the implied limitations. Instead of pretending to make unobtrusive observations, it is better to say: 'that's interesting. Let me take note of that. Can you explain?' Inevitably, such an approach blurs the distinction between observation, interview and intervention. Special care is required in order to collect robust observations while minimizing the disruption to the learning process. This balance is achieved by combining a multitude of observation methods, some passive (e.g. video), some active (e.g. in-activity probes) and some post-event (e.g. stimulated recall interviews). The exact choice of tools is

dependent on the research context. Chapter 5 describes the methods which were found appropriate for the demonstrator study.

4.8 Conclusions

This chapter started by highlighting some methodological challenges facing a design science of technology enhanced mathematics education. These challenges all reflect the overarching need for a clearly articulated consensual epistemic infrastructure.

As a first contribution towards the formulation of such an infrastructure, two concentric cycles of scientific process were presented. These cycles were abstracted from the reports of various research groups regarding their practices.

While these cycles plot a general framework for conducting research, they need to be realised by providing appropriate representations for expressing design knowledge in their various phases, along with procedures for managing the transitions between phases. A particular innate trajectory from experience to knowledge was reviewed as a basis for these representations and transitions. This mechanism proceeds by narratisation of experience and genre transposition. Genre transposition fuses similar narratives, thus abstracting similarities and eliminating detail. In this process temporal relations are replaced by semantic ones.

By analogy with this model, two representations were proposed: design narratives and design patterns. The former serves the interpretive phase of the design experiment cycle, in which the researcher organises the data and records the unfolding of events in the empirical phase. The latter serves the analytical and conjectural phases, allowing researchers to articulate abstractions of design knowledge derived from the experiment.

Several issues were identified with the intuitive notion of narrative, leading to a proposal for a formalisation of design narrative as a form of scientific discourse. Design patterns emerged as a promising form for encoding design knowledge in educational research, but likewise, several challenges were identified. These challenges can be met by providing a rigorous methodological apparatus, which would include the detailed format of each representation and the procedures for transitioning between them, along with measurable criteria for validity. Such a framework will be the subject of Chapter Five.

Chapter 5 Methodological Framework, Methods and Settings

This chapter addresses Aim 2 in Chapter Three, by describing the methodological framework and the set of methods used in the demonstrator study defined by Aim 3. The framework and methods are derived by projecting the approach detailed in Chapters Two and Four onto the specific research settings. These settings are specified in this chapter as well.

5.1 Introduction: a Methodological Framework for a Design Study of Learning about Number Sequences

The primary research question of this thesis, as expressed in Chapter Three, Aims 1 and 2, concerns the formulation of methodological tools for design research in technology enhanced mathematics education. This question was addressed in Chapters Two and Four. Chapter Two argued for a design-based approach to research in technology enhanced mathematics education (TEME). Chapter Four extended this argument, to propose elements of an epistemic infrastructure for design research in this field. In order to assess the validity and utility of these elements, they need to be applied to a concrete, genuine and non-trivial research problem. Such a demonstrator problem is defined by Aim 3 in Chapter Three: “the design of tools and activities for learning about number sequences, in an extra-curricular lower-secondary school setting”. Chapters Six, Seven and Eight communicate a study of this problem in the context of secondary school extra-curricular activities with number sequences.

The current chapter operationalises the elements and provides the methodological framework along with a set of suitable research methods, with respect of the given questions and settings of the demonstrator study. The primary constructs presented in Chapter Four were the two cycles of design research, design narratives and design patterns. This chapter describes the research setting chosen to illustrate these constructs. It then explains how they interacted in the context of this setting, and identifies the processes by which design narratives and design patterns were produced and validated.

The demonstrator study was conducted between September 2002 and December 2006, with some refinement of analysis continuing beyond that date. The work as a whole followed the design research meta-cycle (Figure 6) described in Chapter Four.

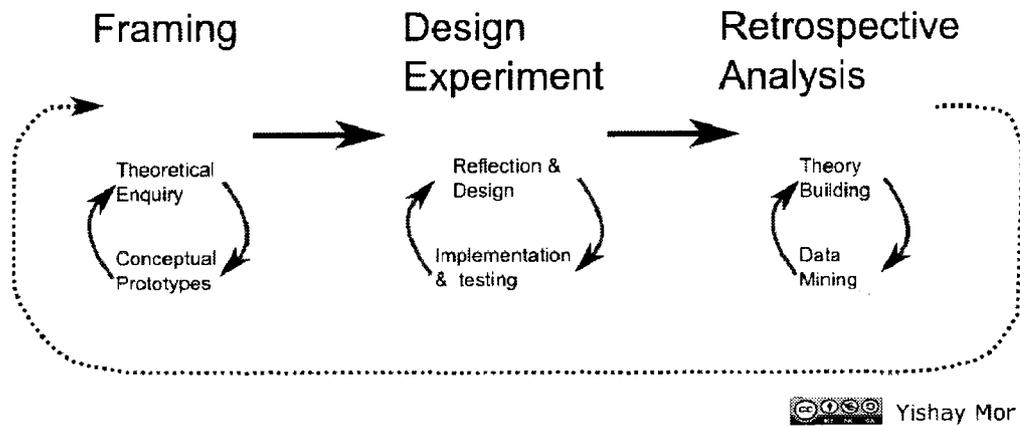


Figure 6: Design research meta-cycle, presented in Chapter Four, reproduced for convenience

The framing phase of the research occurred between September and December 2002. This phase included a scoping study, construction of conceptual prototypes in ToonTalk and an initial outline of activity and tool designs. The scoping study provided the basis for the review presented in Chapter Six. However, this review evolved in tandem with the design throughout the following phases.

The empirical design experiments proceeded from December 2002 to June 2005. They involved three major iterations and several intermediate adjustments. The first iteration was a loosely designed pilot study and was dominated by a trial and error strategy. The final iteration enjoyed a stable design, shifting the focus to data collection. Consequently, most of my comments on the process of design refer to the first two iterations, while the epistemic observations rely mainly on evidence from the third year. These experiments followed the design experiment cycle (Figure 7) discussed in Chapter Four. Each new iteration was motivated by reflections on the previous one, where the first iteration was driven by reflections on the prototyping from the framing phase. These reflections were supported by data collected and interpreted in a variety of methods, as detailed in section 5.3. The empirical observations were supplemented by revisiting the literature to provide a sound basis for the next revision of design.

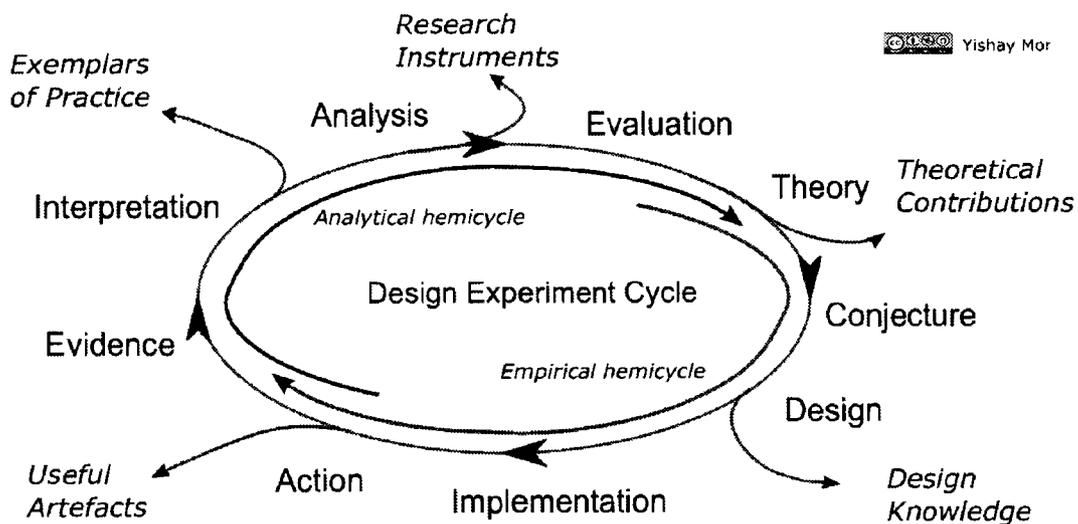


Figure 7: Design experiment cycle, presented in Chapter Four, reproduced for convenience

The three design iterations were followed by a phase of reflective analysis, the main bulk of which was performed between July 2005 and December 2006. In this phase, I took a step back to scrutinize my study as a whole, distancing myself from the specific details of classroom incidents to take note of general themes and meta-questions. It is at this stage that design narratives and patterns came into play. Design narratives allowed me to consolidate observations within and across the three iterations, and promoted a systematic interpretation of the data collected in the experimental cycles. The design patterns derived from these narratives highlight the effective methods and tools which emerge from my reflections across the three years of empirical work. These patterns point at key issues and obstacles in the chosen domain of knowledge, and suggest possible ways to address them. Patterns are grounded in theoretical arguments and validated by a range of qualitative methods. Section 5.4 presents the processes by which design narratives were chosen and developed. This process is based on Bruner's (1991) ten principles. Section 5.5 notes how design patterns were derived from the design narratives and substantiated, using a six step process which captures the key design elements, systemises and substantiates them. This process is followed by a phase of structural manipulations ("refactoring") which strengthen the coherence of the pattern language as a whole.

5.2 Experimental Setting

The context of the demonstrator study was defined by the *WebLabs* project (www.weblabs.eu.com, European Union, Grant # IST-2001-32200) directed by Professors Richard Noss and Celia Hoyles. WebLabs provided both the research setting and the technological infrastructure for my study. WebLabs aimed to explore new ways of constructing and expressing mathematical and scientific knowledge in communities of young learners. The project's approach brought together two traditions: *constructionist learning* as described by Papert & Harel (1991) and collaborative *knowledge-building* in the spirit of Scardamalia & Bereiter (1994). The nature of the project was such that the underlying pedagogy and the supporting technology shaped and reshaped one another. The initial configuration of the infrastructure reflected the initial pedagogical conception. After the first round of experiments, the concept was adjusted to accommodate the lessons learnt regarding the potentials and limitations of the platform, and in turn the platform was reconfigured to adapt to the pedagogical change.

Part of the analysis presented in Chapter 6 and Chapter 7 was initiated during the course of the Learning Patterns project³, directed by Dr. Niall Winters and Professor David Pratt.

The work reported in this thesis includes my individual contributions to these projects, along with independent analysis conducted after their completion. Nevertheless, this work benefited from my interaction and collaboration with colleagues and team members. Table 2 enumerated the inputs of various team members in different stages of my research process.

Content	Project	Significance	Input from team members
Literature review on number sequences.	WebLabs	Initial input for Chapter 6.	Reviewed by Richard Noss, Celia Hoyles, Ken Kahn and Gordon Simpson.

³ <http://lp.noe-kaleidoscope.org>, Kaleidoscope JEIRP

Design of number sequence tools and activities.	WebLabs	Recounted in Chapter 7.	Reviewed by Richard Noss, Celia Hoyles, Ken Kahn and Gordon Simpson.
Implementation, testing and evaluation of number sequence tools and activities.	WebLabs	Recounted in Chapter 7.	Field notes, audio and video recordings assisted by Gordon Simpson and Constantia Xenofondos. Procedures and collected data vetted by Richard Noss and Celia Hoyles. Independent evaluation conducted in several European sites provided informal input, not included in this thesis.
Design of WebReports system	WebLabs	Recounted in Chapter 7.	Conducted in continuous consultation with Richard Noss, Celia Hoyles and Gordon Simpson. Periodic review and feedback from WebLabs partners. Initial versions implemented with the assistance of Gordon Simpson. Final version implemented by Jesper Holmberg.
Case studies	WebLabs / Learning Patterns	Initial input for Chapter 7.	Presented at project meetings and published on-line, received feedback from team members in both projects.
Design Patterns	Learning Patterns	Initial input for Chapter 8.	Presented at project meetings and published on-line, received feedback from team members and participants. Presented and reviewed at writer's workshop at EuroPLoP 2008.

Table 2: Input from team members of the WebLabs and Learning Patterns projects

5.2.1 Classroom setting

The empirical part of this study was conducted in two locations in London over three consecutive years, from 2002 to 2005. The experiments involved several groups, as detailed in Table 3. All groups were instructed and observed by Gordon Simpson and myself. The collaborative activities involved interaction with other sites in Oxford, Sofia, Nicosia and Lisbon. Most of the activities were also tested and evaluated independently in each of these sites. The analysis in this study focuses on published materials from the London groups, with reference to other sites when relevant. Interviews and observations were collected from the London sites alone.

It. ⁴	Gr.	Period	Location	F	M	Age	Sessions
1	I	Autumn 2002	After school club, central London.	3	3	11	10 weekly x 90 minutes, 1 full day workshop.
2	II	Autumn 2003	Lunchtime club, central London.		6	10-11	10 weekly x 50 minutes, 1 full day workshop.
2	III	Autumn 2003	In lieu of ICT class, central London.	4	4	13	10 weekly x 50 minutes, 1 full day workshop.

⁴ See section 5.2.3 for a discussion of the nature of the different iterations.

3	IV	Autumn 2004 & Spring 2005	After school club, north London.	10		13- 14	16 weekly x 90 minutes, 2 full day workshops.
3	V	9 June 2005		2	3	12- 13	One full-day workshop (previous sessions with Ken Kahn).

Table 3: Experiment Groups

5.2.2 Technological setting

The technological approach of the Weblabs project consisted of two tightly related components: a programming environment for students to construct models of their ideas and a web-based collaborative environment for them to share them. ToonTalk (www.toontalk.com) was chosen as the programming platform, while the WebReports collaborative system was designed by me and developed by our team in the course of the project. The technological platform used by the Weblabs projects manifests a particular educational approach.

Section 5.2.2.1 provides a brief overview of ToonTalk, and 5.2.2.2 notes the fundamental features of WebReports, as they pertain to the data collection and analysis.

5.2.2.1 The ToonTalk programming environment

ToonTalk is a language and a programming environment designed to be accessible by children from a wide range of ages, without compromising computational and expressive power (Kahn, 1996; 1999). It does this by embedding complex programming constructs in a video-game setting as shown in Figure 8. In ToonTalk, every programming structure is concretised as an animated cartoon object: *robots* (labelled 2 in Figure 8) stand for programs, *boxes* (labelled 3) for data structures, *birds* (5) for message sending, *nests* (6) for message receiving, *scales* for comparisons, *trucks* for process spawning, and *bombs* for process termination. The *toolbox* (11) contains the data types and operators, while the *notebook* (12) provides a standard library of stored procedures.

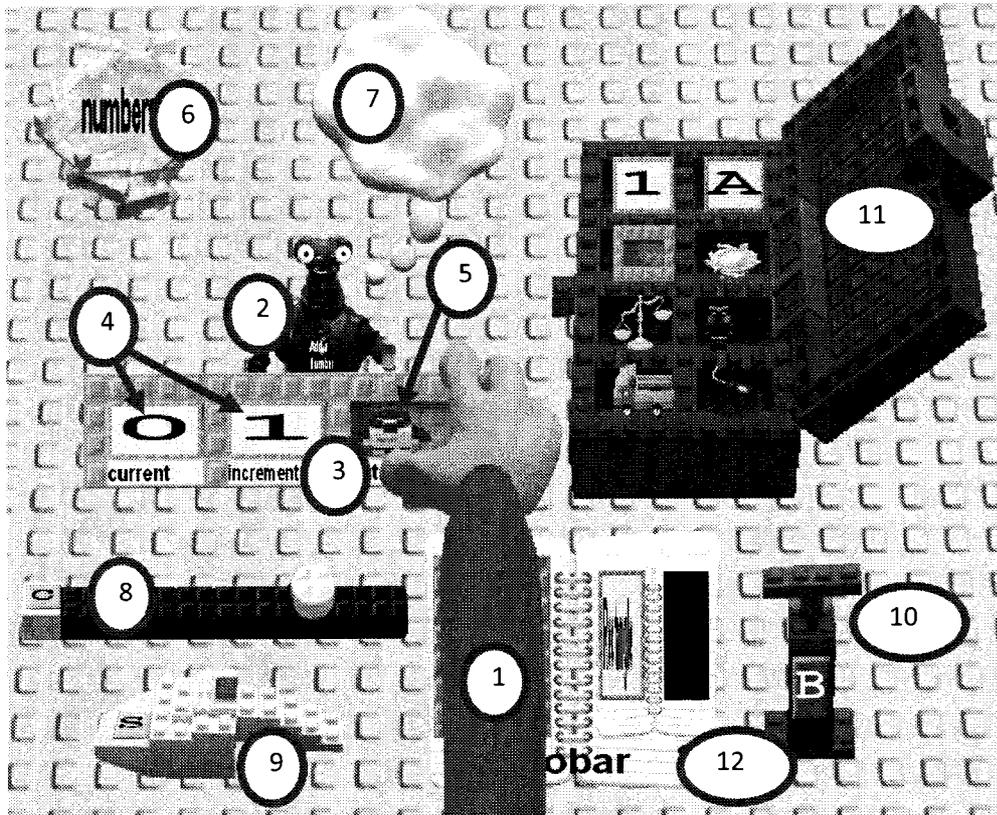


Figure 8: The ToonTalk Environment, showing the programmer's "hand" (1), a robot to be trained (2), and various tools used in training.

The user directly manipulates objects using a virtual *hand* (labelled 1 in Figure 8), or with tools such as the *magic wand* for copying (labelled 8), *vacuum cleaner* (9) for cutting, pasting and erasing or *bicycle pump* (10) for changing object size. Programs are created by *training* a robot – directly leading it through the steps of a task it is required to perform. The robot remembers what it is trained to do, but only for the specific set of values with which it was trained. These are stored in the robot's *thought bubble* (7). The robot's memory can then be generalised by vacuuming: erasing the values and leaving an empty slot for 'any value'. Thus the concept of variables is introduced implicitly through the programming metaphors. Distinct program units (or modules) communicate using the bird-nest mechanism: a robot which completes a computation can pass the result to a bird, which would carry it to another robot for further processing. The user or a robot can load a robot and its parameters on a *truck*, sending it to build a new *house* and perform its computation there. Sub-routines are manifested by training a robot to compute a function, send the response with a bird and invoke a bomb to explode its house. Needless to say, this mode of programming is very different from that used in traditional text-based languages, and induces different patterns and styles of problem solving.

5.2.2.2 WebReports

WebReports (Figure 9) was a web-based collaborative authoring system designed by myself and developed with the help of members of the WebLabs project. This system supported publication of user generated content and social dynamics around that content, embodying many of the

principles of *web2.0* long before the term was coined. The details of the system's functionality and its evolution are the focus of section 7.5.1.

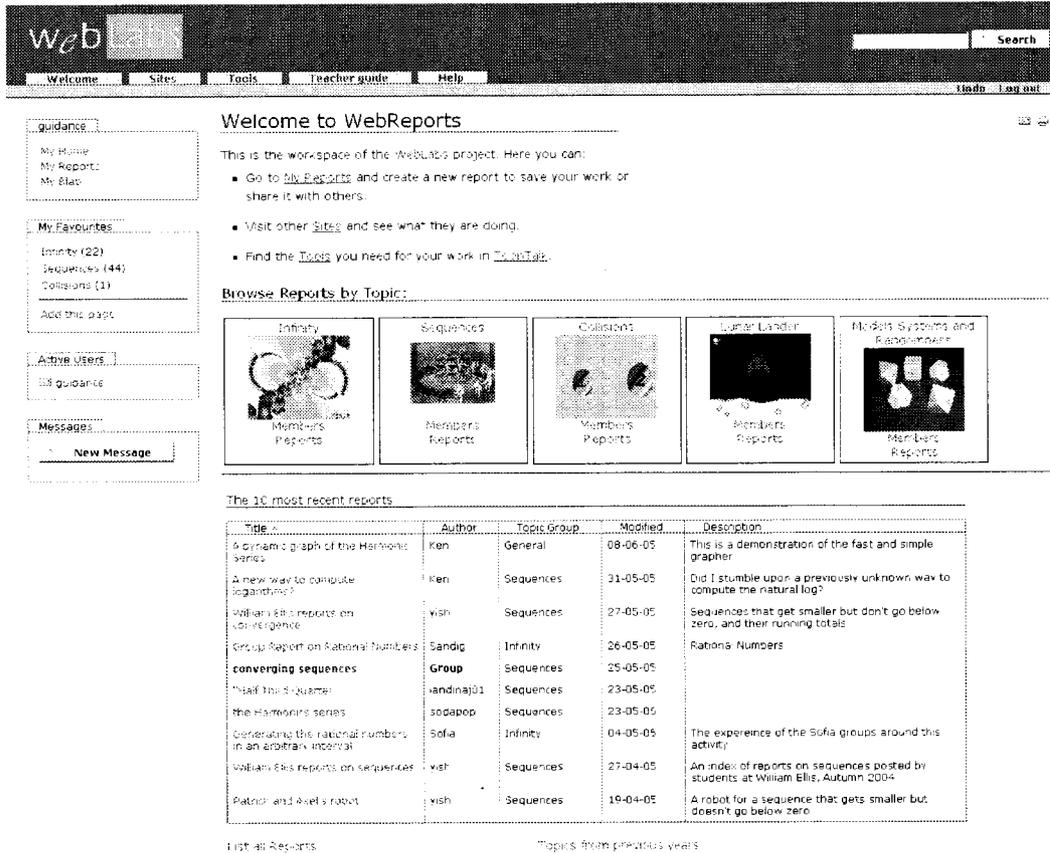


Figure 9: WebReports front page. This is the view a user sees after logging in. It includes a personal navigation bar on the left, links to main content areas and a listing of recently published content.

The atomic element of the WebReports system was the *webreport*:⁵ an online multi-media document created by a member of the community – researcher, teacher or student. The basic requirements for webreports were set in the initial bid of the WebLabs project (WebLabs, 2001). This states that webreports should be:

- Collaboratively constructed, dynamic, web-based multimedia reports of evolving understandings of a knowledge domain.
- Include working models, along with multi-media descriptions, interpretations and reflections, textual and graphical illustrative explanations, and reflective notes and guidance hints.
- Provide a link that, when followed, will reconstruct the original object in ToonTalk for further editing and inspection.

While the details of design and implementation were only realised over the course of the project's first year, this set of requirements already identifies a rich source of data, which can potentially provide insights into the process and outcomes of participant's learning trajectories.

⁵ I use WebReports to refer to the system, and webreports to refer to the actual documents.

5.2.3 Iterative design

The experimental phase of my study followed the four iteration structure (Pratt, 1998) described in Chapter Four, with the necessary modifications implied by the characteristics of my research setting.

5.2.3.1 Iteration 0, Autumn 2002: Bootstrapping

The aims of the bootstrapping phase were to gain some basic insights regarding the affordances and suitable uses of the chosen technology, and use them as a basis for designing an initial set of tools and activities.

In terms of ToonTalk programming, this implied an intensive period of experimentation with ToonTalk development. On the surface, this effort was dedicated to the development of tools for presenting and manipulating number sequences, which children would use in the course of their activities. Tools emerged from an initial sketchy scenario of educational activity, to which more details could be added as the tool design matured. In turn, the elaboration of the scenario into a concrete plan placed new demands on the tools, and often called for altogether new tools. Many of the tools developed in this process primarily served my own learning as well as group discussions among WebLabs researchers, and were consequently radically altered before they reached the classroom.

Alongside the ToonTalk tools and activity design, I explored the design of the collaborative medium to support these activities and the WebLabs project in general. Using an Extreme Programming approach (Beck, 1999), I began by constructing a mock-up and using it to discuss various usage stories with fellow researchers. Following this, a first prototype was developed by myself and Gordon Simpson. This prototype was the collaborative environment for the activities of the exploratory phase. It also served researchers as a platform for collaborative design of tools and activities.

5.2.3.2 Iteration 1, Spring 2003: Exploratory

The aim of the exploratory phase was to validate and elaborate the initial designs of tools and activities. The details of the experiment group are listed in Table 3 above. Evaluation in this phase was focused on the tools' usability and aptness, and less on learning trajectories. The methods used were predominantly heuristic (Nielsen & Molich, 1990; Nielsen, 1994): several researchers observed children working through the tasks and took notes. These notes were used by me to identify design flaws. My conjectures regarding these flaws and possible remedies were discussed with colleagues in the project team and eventually a new design of tools and activities emerged. I was assisted in observations by Gordon Simpson and Constantia Xenofondos in London and Eugenia Sendova, Liliana Moneva and George Gachev in Bulgaria. Gordon Simpson, Ken Kahn, Richard Noss, Celia Hoyles and Eugenia Sendova contributed to the design discussions.

Evaluation takes *activity* as a unit of analysis (Kaptelinin and Nardi, 2006). Thus the resulting conjectures could lead to redesign of the tasks, the mediating tools, or the learning environment. For example, observing that task A is harder than expected and B easier could

suggest that the order of these two needs to be changed. On the other hand, seeing that constructing a particular tool is too taxing for learners could lead to the provision of that tool ready-made. Design and evaluation were tightly interwoven in this phase. Often contradictions or tensions were identified in one session leading to the redesign of a task or a tool, which were tested the following week with another group or even with the same.

In order to sustain this responsiveness, the collaborative system used in this phase was based on a Wiki. This meant that the system could be rapidly reconfigured, albeit at the cost of cognitive overhead to the users. The fluidity of the system design meant that the data collected was extremely messy and hard to use as conclusive evidence for learning. Nevertheless, the insights gained from this iteration identified the main themes to be monitored in the next two.

5.2.3.3 Iteration 2, 2003 / 2004: Developmental

The developmental iteration took place in autumn 2003 and spring 2004 (see Table 3 above). This iteration marks the shift of focus from technology and activity design to the trajectories of learning. At this stage, the design was stable enough to afford the collection of reliable data. Modest predictions were derived from the heuristic observations of the previous stages and compared with empirical results.

In the transition from the previous iteration to this, the fragmented exploratory task designs were replaced by a coherent plan of activity spanning two school terms. This activity plan was driven by clearly defined learning aims, and milestone tasks directed at evaluating these aims. Design benefited from a clear plan on the one hand and grounded intuitions regarding tool adequacy on the other, resulting in tools with good fit-for-purpose. Consequently, the time spent on technical issues was minimized and more attention could be devoted to tracing the learning trajectories.

The collaborative platform was also replaced. The Wiki that had been used as a malleable prototype served as a model, and along with the specific requirements derived from the activity plan, fed into a detailed specification of a highly structured platform (implemented by Jakob Tholander and Jesper Holmberg from The Royal Institute of Technology, Sweden - KTH). The key principle in its design was functional minimalism: include only the features and options that are needed to support the activities and practices of the community.

The tight link between pedagogic and technological design was maintained, but at a higher level. Frequent adjustments of design – pedagogic, technological and methodological – were generally avoided. Instead, systematic refinement was driven by the evaluation of the plan of activities as a whole.

5.2.3.4 Iteration 3, 2004 / 2005: Analytical

The analytical phase relied on stable and mature activities and tools. While occasional local refinements were not ruled out, the main design challenge was to identify and adapt methods of data collection and analysis, suitable for the context of study and the technology developed to support the activities. The specific instruments in this phase are the focus of section 5.3.3.

5.2.3.5 2005 / 2006: Retrospective analysis

With the completion of the empirical cycles of design and evaluation, the time had come for retrospective analysis from a broad perspective. This phase was thematically driven, tracing several questions through the data collected across all iterations. The aim of this analysis was both to substantiate claims regarding the learning process and to elucidate elements of transferable design knowledge.

The epistemic themes were mapped to existing theories of mathematical learning and the evidence was interpreted in that context. Consequently, the instruments used were geared towards the evaluation of mathematical performance and mathematical discourse.

Some of the prominent design elements were captured using a framework of design patterns (Alexander, 1979). This work was partially supported by the Learning Patterns project (Kaleidoscope JEIRP), directed by David Pratt and Niall Winters.

5.3 Collection and Management of Data

Chapter Four characterised design research methodologies as iterative, process-focused, interventionist, collaborative, multileveled, utility oriented, and theory driven. As a corollary the appropriate methods of data collection highlighted process-oriented observations, using a mixture of mainly qualitative methods, including video and audio recording of student activities, analysis of texts and artefacts produced in the course of these activities, interviews and ethnographic field notes. This versatile process-oriented approach to data collection dominated my research.

5.3.1 Sources of Data

My study draws on three classes of data: design data, student productions, and classroom observations. Design data refers to the details of the tools and activities which I have designed. It encapsulates not only the final form or design, but also the path which led to it. Student productions are the actual artefacts and (multi-modal) texts produced by students in the course of activities. These include written text, ToonTalk models, spreadsheets, charts and graphics. Classroom observations include audio, video and field notes recorded by me and my colleagues during activities.

5.3.1.1 Design Data

Design data were collated from WebLabs on-line guidance and project reports which I authored. Data reflecting the design process was retrieved from on-line discussions with my colleagues, various design drafts and sketches, and from my research journal.

5.3.1.2 Student Productions

Student productions were predominantly collected from their webreports. The WebReports system was designed to support students' investigation, with the important side effect of capturing their reflections, in their own words, at key points of the learning process. I make extensive use of this source. Apart from the text, students' reports included graphics and

programs. All of these were treated as means of expression that provide a window on students' evolving concepts. The nature of ToonTalk allowed me to see through the code and interpret the explorative process by which it was conceived. In several cases, paper-based tasks or questionnaires were also collected.

5.3.1.3 Observations

By and large, observations were done during active facilitation of activities, and are thus highly participatory. Video and audio collection was often restricted by the experimental settings. Consequently, *in-activity probes* play a central role in observational data: short (up to 5 minutes) unstructured interviews taken while students were engaged in an activity. These probes aimed at capturing snapshots of the process of knowledge construction. In order to enhance the reliability of these data, they were calibrated with field notes and with the observations of my colleagues.

5.3.2 Data Cataloguing

Design data were filed chronologically and thematically. Student productions in the form of webreports were automatically catalogued in a searchable database. Paper based productions were filed chronologically and thematically. Observational data were indexed, annotated and coded were relevant. Audio and video recordings were scanned for critical incidents, which were transcribed. Transcriptions emphasised the content of expressions, with less attention given to gesture and emotive facets of discourse.

Data were analysed in two modes: situated and reflexive. Situated analysis refers to the attempt to interpret data as it unfolds in order to respond to emerging issues. At the micro scale, this could mean on-the-spot design adjustments. More often, this would lead to adjustments and resampling, from one session to the next or between design iterations. The reflexive mode concerns in depth analysis of data after the completion of an experiment, with the added perspective of time and the opportunity to compare data across longer spans of activity. Situated analysis focused on identifying indicators of learning and conflict, proposing preliminary explanations, and verifying these by subsequent interventions. The themes which emerged from this mode were theorised and used as a frame of reference for the reflexive mode. Reflexive analysis tracked these themes across incidents and data-forms.

5.3.3 Data collection

Section 5.3 opened with some principles of data collection and analysis in design based research, and some challenges they implied. These challenges are amplified by field conditions: a nearby building site might render audio recordings worthless, bad lighting conditions eliminate the possibility of video recording, a planned pre-test is cancelled by unexpected change of school schedule. In the midst of these is the complex position of the participant-observer, trying to interact with learners and record these interactions at one and the same time.

Several principles emerged in my attempt to meet these challenges: redundancy, triangulation, and nearest substitute.

Redundancy means that anything available is collected from the scene. Recordings, notes, produced artefacts, scraps of paper scribbled on by learners or researchers: everything is saved, even though little may be used. Many of these items may have no use as data, but they can still contribute as memory aids when constructing post-hoc descriptions.

The inevitable sparsity and discontinuity of data is a challenge to validity. Triangulation (Bell, 1998; Denscombe, 2003) tries to meet this challenge by juxtapositioning evidence obtained by different methods. For example, the interpretation of an interview transcript can be supplemented by field notes describing the context in which it was taken and by analysis of the artefacts constructed by the learner prior to the interview.

The term 'nearest substitute' acknowledges the pragmatics of the research setting, and accepts the use of instruments which are as close as possible to the ideal. When video recording is ineffective, it is replaced by audio. When even that is infeasible, verbatim notes of key oral expressions are taken as soon as possible after the event.

The actual instruments of data collection used in this study included:

- Pre- and post-trial written evaluations and interviews.
- Notes and recordings (video and audio) of learners' face-to-face discussions and classroom presentations.
- Stimulated recall interviews (Lyle, 2003), in which students were provoked to share their reflections on the activities and their products, and express the conceptualizations they have developed through them.
- Task-based interviews (Koichu and Harel, 2007) in which a learner is presented with a task and prompted to discuss it as she performs it. This method is designed to test specific conjectures of thinking-in-change in near-laboratory settings.
- In-activity probes (Mor et al, 2005), short interviews – typically up to five minutes – conducted while a student is engaged in an activity and referring to it. The use of this tool aims at capturing the process of knowledge construction and allows students to express their situated abstractions (section 6.3.1) in the context that they are formed.
- The multimodal (Jewitt, 2003) text of learners' webreports and task worksheets, including their comments on peer reports.
- The ToonTalk code produced by learners, as published in webreports or collected from their workspace.
- Field notes recording students' work process as they perform tasks or participate in discussions.

Appendix II offers examples of the various data types. Sections 5.3.3.1, 5.3.3.2 and 5.3.3.3 highlight some specific issues which emerged in the process of collecting these data.

5.3.3.1 Observational instruments: video and audio recordings and field notes

Due to the sound and light conditions at schools, video recording on site was typically inefficient. Video was used predominantly during five full-day workshops, collecting four to six hours of

footage in each. Recordings were assisted by Richard Noss, Gordon Simpson and Constantia Xenofondos. The footage was scanned and indexed to help identify key incidents which were later transcribed and analysed in greater detail. The choice and processing of segments for deeper analysis was intertwined with the development of the design narratives, as described in section 5.4.1.

Audio recordings were used rarely in group I, regularly in groups II, III and V, and extensively in group IV. The three common formats were group discussions, work sessions and interviews, discussed below. Table 4 provides a summary of recordings by group, data and format. Group discussions ranged from five to thirty minutes. Longer discussions were recorded at the inauguration and conclusion of every segment of activity. Shorter discussions were often opportunistic: exploring an issue which emerged from the students' work. Discussions held at the lab (during workshops) were recorded on video. Altogether, between four and ten discussions were recorded per group.

Work sessions were incidents where one of my colleagues or myself worked with an individual or a small group of students on a particular programming task. Such sessions would typically run for five to ten minutes, during which a recording device was placed unobtrusively before students. Students were briefed on data collection and use before obtaining their consent, and were always notified when a recording device was active.

group	topic ⁶	session	recordings	format
II	basic sequences	22 Jan 2004	1 (23 min)	group work session
II	Plotting sequences	12 Feb 2004	1 (20 min)	group work session
III	convergence	13 Feb 2004	1 (30 min)	group discussion
III	convergence	04 March 2004	1 (10 min)	group discussion
III	convergence	23 April 2004	1 (33 min)	group discussion
III	convergence	30 April 2004	1 (32 min)	group discussion
IV	sequences	9 Nov 2004	4	pretask interviews
IV	sequences	16 Nov 2004	9	pretask interviews and IAPs ⁷
IV	sequences	23 Nov 2004	5	IAPs

⁶ The topic column refers to activity titles, as described in Appendix I

⁷ IAP = in activity probe. See 5.3.3.2.

IV	sequences	30 Nov 2004	7	work sessions, discussions
IV	sequences	6 Dec 2004	13	(workshop) discussions, work sessions,
IV	sequences	14 Dec 2004	10	IAPs
IV	convergence	22 march 2005	8	intro discussion, IAPs
IV	convergence	12 April 2005	7	discussions
IV	convergence	19 April 2005	10	IAPs
IV	convergence	17 May 2005	8	IAPs
IV	convergence	23 May 2005	7	(workshop) discussions, IAPs
IV	convergence	24 May 2005	10	conclusive interviews, IAPs, discussions
V	convergence	5 June 2005	30	IAPs, discussions

Table 4: Summary of audio recordings

Shorthand notes were taken during sessions, with verbatim quotes from learners' oral expressions, along observations of their activity and affective state. These notes were elaborated into detailed reports shortly after the session, and supplemented by relevant excerpts from their produced texts and code and transcripts of audio recordings when available. These reports were then reviewed by fellow researchers who had been on site, and any discrepancies in observations were discussed.

5.3.3.2 Interviews

Several forms of interview were used, predominantly in conjunction with other instruments.

Stimulated recall interviews (Lyle, 2003) provided a form of post-hoc observation, enhancing the understanding derived from products or passive observations, by eliciting learners' perspective on their actions. They were used mainly as a follow-up to written assessments (Phillips et al, 2000). At key points in the activity sequence, learners were presented with worksheets or questionnaires. After these were reviewed, some or all of the students would be interviewed and asked to elaborate their responses and explain the thought process behind them.

Task based interviews were used mainly in the early iterations, to test specific conjectures regarding cognitive and epistemic facets of tool design. A single student would be asked to perform a task and explain her actions. This technique is similar to the 'think-aloud' method used in usability studies (Phillips et al, 2000; Barendregt et al, 2003).

In-activity probes (Mor et al, 2005) denote a technique which emerged from the WebLabs project. I would approach a learner in the course of a scheduled activity and conduct a short interview on her actions at the moment. This technique proved to be a highly powerful and cost-effective instrument, allowing me to substantiate observations and induce learners to articulate their thinking-in-change. To an extent, it used a standard instructor practice in constructionist classrooms as a form of a task-based interview. This allowed data collection to be streamlined into the flow of activity. Yet it calls for special care: observation needs to be non-suggestive. If students mistake an act of observation for intervention, they might misinterpret it as affirming their conjectures. The distinction between observation and intervention is easy to maintain, by stating explicitly (“I’m not saying if this is right or wrong, I just want to hear what you think”), or by gesturing at a recording device. Often the probe was followed by an intervention, for example, after students were interviewed, they would occasionally ask for feedback on their responses, which would lead to a didactic discussion of their work.

5.3.3.3 Texts and artefacts

Texts and artefacts produced by learners, in various media and modalities, were used both for formative and summative evaluation.

Paper and pencil questionnaires were used in early iterations for summative evaluation. A worksheet with questions reflecting the subject domain was presented to learners before a section of activities, and a similar one used at its end. With time, this instrument was modulated in two significant ways.

The first change was driven by the realization, discussed in section 4.7.1, that any act of evaluation is inevitably an intervention. Consequently, these questionnaires were integrated into the design of activities, as a means of introducing a subject or as a prelude to a concluding group discussion.

The second change was a typical example of the dynamics of iterative design. As the WebReports platform matured, new possibilities of using it emerged. One of these was the Active Worksheet, a webreport template which serves the two-fold process of directing students’ attention towards the ideas and explorations in which we are interested as well as providing valuable data through the students’ responses to the questions posed. Using a webreport for this purpose was a practical convenience, but it also allowed learners to embed various digital artefacts in their text, such as graphs, sketches and ToonTalk objects.

Webreports were originally conceived as a deliberate form of intergroup communication, strongly facilitated by teachers and researchers. Eventually they became a medium for both personal and collaborative expression, reflection and action (section 7.5.1). As such, they were an indispensable window (in the spirit of Noss and Hoyles, 1996) on the process and outcomes of learning. Thus, with time webreports became a prime source of data. Table 5 shows a summary of webreports produced by students per group and activity. Group I was not counted because the reports they produced were too rough, and were used mainly to infer user interface requirements for the WebReports system. Group III was only introduced to WebReports in the course of the convergence activities. Group reports were collaboratively authored by students, assisted by me or my colleague Gordon Simpson, at the end of an activity segment.

Group	Number of Students	Number of webreports		
		basic sequences	Convergence	group reports
II	6	15	4	2
III	8		12	3
IV	10	26	15	5
V	5		4	1

Table 5: Webreports by group and activity

In addition to these, over 50 reports were collected from the *Guess my Robot* activity segment (section 7.3). These were collected from eight schools, over two years.

5.3.4 Analysis

The first level of analysis is oriented towards the fundamental question of epistemic outcomes. Before asking how students learned, and how the design of activities and tools contributed to learning, we need to verify that they did actually learn something. The primary sources of evidence in this respect are the mathematical objects they produce and the mathematical arguments they articulate. Products were analyzed in terms of their aptness (Jewitt and Kress, 2003) for the task in hand, and their complexity and sophistication.

The term *Mathematical argument* is taken to signify any deliberate expression aimed at conveying mathematical ideas or claims to an audience. From a situated abstraction perspective (section 6.3.1), mathematical arguments are not restricted to the conventional formal language of mathematical science. More often they would be stated in a form or medium derived from the context of activity. Consequently the method of analysis assumes that learners engaged in an activity of mathematical nature will attempt to make mathematical arguments, and researchers should aim to identify and understand them. In my analysis I look for the mathematical meanings that are constructed and expressed using the tools provided within the context of activities. My guiding assumption is that text is articulated for a purpose, and should be interpreted in the context of that purpose. In other words, in order to understand what the author of a report *meant*, we need to observe *what* she published in the context of *why* she published it.

The next level of analysis strives to unpack the process of learning and its relation to the activities learners were engaged in and tools which they used. In order to do this, special attention is given to learners' articulation as they confront the tasks. These are tracked in their reflective notes in webreports, in discussions between them, and by in-activity probes.

Both levels of analysis are present in two grain sizes: as an instrument for immediate, week-to-week refinement, and as part of the transition from one iteration to the next. The third level is that of reflective analysis, subsequent to all iterations. This level is focused on identifying and elaborating cross-cutting themes, and takes the outcomes from the first two as its raw data.

Chapter Four highlights the epistemic role of narrative, as a mediator between experience and paradigmatic knowledge. This observation prompts the use of design narratives as a scientific tool. It also serves as an analytic guideline: learners' expressions are interpreted as narratives, whether they are presented in words, image or code.

5.4 From Data to Design Narratives

The data harvested throughout the cycles of design experiments are eclectic, opportunistic, and sporadic: the challenging and unpredictable research environment requires that any possible form of data is used, resulting in inconsistent quality and quantity of usable data. Often the most interesting events are unplanned, thus calling for "ad-hoc" responsive data collection. The transition to the retrospective phase calls for systematic organisation of these data. Design narratives provide a means to this end.

Bruner (1990; 1991) identifies narrative as the predominant tool by which humans organise events to derive meaning. The instrument of design narratives, as described in Chapter Four, aims to formalise this innate process into a more methodical one. In order to provide the transparency expected of a scientific method, several questions need to be considered:

- How is the set of design narratives pertaining to a study selected?
- How are events to be included in these narratives chosen?
- How are the factual claims contained in the narratives verified?

The remainder of this section addresses these questions.

5.4.1 Selecting and Constructing Design Narratives

The design narratives listed in Chapter Seven were selected by me in a two-phase process: first, a large set of candidate narratives was derived from the data. A subset was then selected for inclusion in this chapter and elaborated, by the procedures described below.

The initial set of candidate narratives was compiled so as to ensure chronological and thematic coverage. Data was catalogued by year and activity, and then scanned to identify incidents which illuminate the central themes defined in Chapter Three: designing for learning about number sequences by *Construction, Communication and Collaboration*.

This set was culled so that each narrative would capture either a problem or a solution which was unique. Where duplicates were identified, priority was given to the candidate that was more representative and better supported by data. A second selection criterion was the need to balance researcher narratives and learner narratives. As elaborated in section 4.4.1, researcher narratives refer to the process of designing tools and activities for learning, whereas learner narratives relate to learners' experiences with the tools and the activities.

A candidate for the initial set typically emerged from a single document: a design specification, project report, video or audio recording, web report, etc. During the selection process, additional sources were listed for each narrative. Once a narrative was chosen to be included in the final set, these sources were used to calibrate the data from the initial source and fill in any gaps. A template was used, to ensure that all the necessary elements are present in each

narrative. This template is described in detail in section 5.4.2. The next section explains the rationale behind it.

Consequently, this process addressed two of Bruner's principles, as discussed in 4.4.1: the individual narratives' *canonicity and breach*, and the *accrual* of the collection as a whole.

5.4.2 Structure and Form of Design Narratives

Section 4.4 considered Bruner's (1991) ten qualities of narrative, and their mapping to design narratives as a form of scientific discourse. The principles of *canonicity and breach* and *accrual* guided the collation of the whole set of narratives. *Genericness* was manifested in a stylistic alignment with the discourse norms of the community. The remaining seven principles were translated into concrete guidelines. The design narratives were selected and developed to ensure they provide:

- *Diachronicity*: an account of a single, compact, thread of events.
- *Particularity*: a detailed description of representative incident, rather than cumulative generalisations.
- *Intentional state entailment*: the protagonists' known or assumed intentions are declared explicitly. These are either the educational aims behind a particular activity, or the learner's task derived from such an activity. As noted in section 4.4.1, intentional states are inferred, not observed. My own inferences are presented as an epilogue to each narrative, so as to open them to criticism.
- *Hermeneutic composability*: similar to intentional states, the body of the narratives was left free of commentaries, but my reflections were included in the epilogues. These reflections weave the narratives into a larger story.
- *Referentiality*: in contrast to fictional narratives, the design narratives in this study need to refer convincingly to real events. Furthermore, they provide an "audit trail" (Creswell and Miller, 2000; Lincoln and Guba 1985) by listing the sources used in their construction.
- *Normativeness*: The normative claims derived from the narratives are expressed in Chapter Eight in the form of design patterns.
- *Context*: the common context of all narratives in this study is provided section 5.2, and elaborated for each narrative in its preface.

These principles were manifested in the design narrative template (Table 6). The core of this template is a *STAR* structure: *Situation, Task, Actions, Results*. The *Situation* element conveys the context, the *Task* element puts forth the intentional state, *Diachronicity* guides the *Actions* section, and the normativity of the actions is implied by the *Results*. The *STAR* structure is augmented by a *Sources* section which addresses referentiality, and by a *Reflections* section, which makes explicit hermeneutic inferences, intentional state entitlements and normative claims. The primary purpose of formulating these principles as a template is to provide a checklist for those requirements. This structure should also, inter alia, improve the readability of the narratives by signposting their constituents.

Component	Description
Overview	Brief summary of the situation, task and main outcomes.
Sources	List of data sources consulted in the composition of this narrative.
Situation	The circumstances in which this narrative is situated, in terms of time, place, environment and participants.
Task	The objectives that the protagonists set to achieve or the problem to solve.
Actions	The sequence of actions the protagonists took in order to address the task.
Results	The actual effects of the protagonists' action, as demonstrated by evidence.
Reflections	Lessons and questions derived from the narrative.

Table 6: Design narrative template

5.5 From Design Narratives to Design Patterns

In order to include design patterns as elements of a scientific discourse, a clear path needed to be marked from narratives to patterns, and mechanisms established for validating them. This process included the following steps:

1. A prominent design feature was identified in a design narrative, by linking it to a pedagogically effective outcome, or to the resolution of a critical problem.
2. The design feature was captured using a core template of *Problem, Context, and Solution*. The source design narrative was noted.
3. Other narratives were searched for additional support.
4. The problem was expressed as a configuration of forces. The articulation of the problem as a "system of forces" stems from Alexander's original work (section 4.5). Borchers (2000:4) defines forces as: "Aspects of the design that need to be optimised. They usually come in pairs contradicting each other".
5. The initial context of the pattern was defined by the situational characteristics common to all supporting narratives.
6. The solution was articulated in the most specific detail that was still consistent with all supporting cases.

Sections 5.5.1 and 5.5.2 provide further detail operationalising these steps.

The same collection of design narratives could, theoretically, have given rise to a different set of design patterns. Again, the primary yardstick was the question derived from Aim 3 in Chapter Three: how to design for learning about number sequences by *Construction, Communication and Collaboration*. Thus, the initial set of patterns expressed insights addressing this question directly as they emerged from the design narratives.

The identification and articulation of the initial set of patterns was followed by a phase of organising and refactoring the pattern language as a whole. The links between patterns were identified and noted, and new patterns were derived by structural manipulations, such as:

- *Specification*: when the substantiation process (section 5.5.1) indicated that a pattern's empirical support was weak, the pattern's scope was narrowed down to fit the evidence.

- *Decomposition*: where peer review (section 5.5.1) indicated that patterns were too complex or too sensitive to contextual factors they were broken into several more robust components, each expressed as a separate pattern.
- *Extraction*: design features which recurred in several patterns were expressed as a new pattern and noted as a component in the others.
- *Generalisation*: where the distinction between two patterns was unclear, they were merged and expressed as a pattern of a higher level of abstraction, and the source patterns noted as its extensions.

This process was iterated until it produced a stable collection of linked patterns. Patterns which lacked sufficient empirical support, or were poorly connected to the collection, were removed from the collection but saved for future consideration. The guiding objective was to collate a coherent set of patterns, offering a solid base for a potential language of patterns for technology-enhanced environments for learning mathematics through Construction, Communication and Collaboration. The patterns which were produced by this process were then substantiated further by eliciting empirical and theoretical support from the literature. Finally, visual aids such as metaphoric illustrations and structural diagrams were added to enhance the patterns' text.

5.5.1 Pattern Substantiation

The process of developing the design patterns was accompanied, and to an extent guided, by constant monitoring of their quality along two dimensions: validity and communicativeness. Patterns aim to provide a language for a design science of learning. Consequently, they need to be judged both in terms of the scientific validity of the claims they encapsulate, and in terms of their ability to communicate these claims. Three mechanisms were used to substantiate the patterns: expert review, theoretical alignment, and empirical calibration. These terms are defined here, while the details of their implementation are noted in section 8.1. A further degree of credibility should be attained by documented independent application of the patterns by others. This will only be possible after they are published.

Expert review subjects design patterns to the scrutiny of experts in relevant fields. Expert review scrutinised the validity of the claims embodied in the pattern, and tested the extent to which these claims are clearly communicated.

Theoretical alignment identified relevant literature and articulated its relation to the patterns. In terms of validation, it provided theoretical support for the pattern. In terms of communicativeness, it linked the pattern to the language of the relevant communities.

Empirical calibration enumerated independent occurrences of the patterns. Appleton (2000) expresses the common perception in pattern communities that a record of good practice can only be considered a pattern once it has been shown to be a recurrent phenomenon. He notes the rule of three, which is often considered a necessary criterion for patterns (Fincher, 1999;

Kohls and Panke, 2009): “A pattern can be called a pattern only if it has been applied to a real world solution at least three times”⁸.

In order to ensure the progress of the design patterns along this path, I defined a sequence of phases which patterns were expected to traverse (Table 7).

Phase	Definition	Criteria
Seed	The initial core of the pattern was observed and noted.	Pattern description includes short summary, and an outline of problem, context and solution. Pattern supported by single design narrative.
Alpha	Pattern documented in a form accessible to immediate peers who have direct access to the source observations.	Pattern reviewed by at least three team members / colleagues who are familiar with the context from which it emerged. Comments from these peers were addressed to their satisfaction. Pattern description includes detailed specification of context, problem and solution.
Beta	Pattern is suitable for review by relevant communities.	Pattern presented at a conference or writers’ workshop to peers who are not familiar with the originating context. Comments from these peers addressed and pattern written as a coherent and self-contained document. Pattern supported by two or more design narratives. The pattern description is extended to include theoretical and empirical support and related patterns.
Release	Pattern is suitable for publication.	Pattern is included in a reviewed publication, such as academic or professional journal, conference proceedings or official report. Alternatively, pattern is cited by others. Fully specified, and includes credible empirical and theoretical support.

Table 7: Phases in pattern development

Seed patterns were identified in the course of the WebLabs project or shortly after. They were initially reviewed by my team members in the WebLabs and Learning Patterns projects. The feedback from this review informed the individual pattern development, as well as the refactoring of the collection as a whole. Patterns which emerged as valid and suitable for further development were classified as “alpha”. These were refined and presented at various conferences and at a EuroPLOP⁹ writers’ workshop. These presentations elicited expert feedback both in the subject domain of the demonstrator study and in the general field of techno-pedagogical design patterns. Refactoring and editing of patterns in response to this feedback has brought them to a potential release state. This thesis will be their first reviewed publication. With respect of independent applications, while anecdotal evidence exists, a systematic evaluation of the patterns’ use is beyond the scope of this study.

5.5.2 Pattern Template

A review of pattern language and collections revealed that each one defined a common template for all the patterns it includes. Such a template is useful for users of the collection, as it enables them to search and apply patterns as needed. At the same time, a carefully designed

⁸ The rule of three originates in the Portland Pattern Repository, <http://c2.com/cgi/wiki?RuleOfThree>

⁹ <http://hillside.net/europlop>

template safeguards rigour by prompting the pattern author to address all the important aspects of the patterns.

In order to derive a template for this thesis, I reviewed the templates of a number of pattern languages and collections (Schümmer and Lukosch, 2007; Bjork and Holopainen, 2004; Avgeriou, Papasalouros, Retalis and Skordalakis, 2003; Eckstein, Bergin and Sharp, 2002; Duyne, Landay and Hong, 2002; Gamma, Helm, Johnson and Vlissides, 1995; Alexander, Ishikawa and Silverstein, 1977; and others). Appleton (2000) identifies ten elements which he claims should be clearly recognisable in any pattern: *name, problem, context, forces, solution, examples, resulting context, rationale, related patterns, and known uses*. Appleton’s main concern is pattern languages for object oriented programming, and indeed some of his arguments are specific to that context. For example, the resulting context element assumes a deterministic domain. Other elements, such as forces, are contested by other reviews (Fincher, 2002). Name, problem, solution, examples and related patterns are common to practically all languages. The template chosen for this thesis was based on these elements, with adaptations motivated by the discussion in section 4.5. The details of this template are enumerated in Table 8.

Alexander describes a design pattern as “a three-part rule, which expresses a relation between a certain context, a problem, and a solution” (Alexander, 1979, p247). Consequently, most, if not all, pattern authors include *problem, context and solution* as core components of their templates. These sections can be seen as generalised forms of the respective *task, situation, and actions* sections of the design narrative template, thus assisting the transition (or *genre transposition*) between them. The *support* section in the template responds to the issue of validity, raised in section 4.5.4. Its structure reflects the discussion in section 5.5.1. The last three components – *liabilities, notes and related patterns* – are found in many collections and address the practical needs of designers wishing to apply the patterns. While all patterns in this thesis are specified in terms of the template components, the elements underneath each component may occasionally be omitted.

Component	Element	Description
Name		Design patterns are intended to be used as part of a pattern language, as an infrastructure for design-level conversation. As such, the name of a pattern should be amenable for use as a noun in such a conversation. It should be short, memorable and indicative of the big idea of the pattern.
The problem		The problem description provides a clear and succinct presentation of the motivation for the pattern, its <i>raison d'être</i> . The educational research literature tends to shy away from the term “problem”, as it may suggest a negative attitude. This thesis takes the position of the design literature, and specifically the computer science tradition, where problems are seen as opportunities for action. From a design perspective, where there is no problem, nothing needs to be changed, no act of design is required and consequently there are no questions for research.

	Synopsis	A single sentence highlighting the essence of the problem.
	Illustration	A graphic or photo intended to provide the reader with a vivid mental image for the pattern.
	Forces	The objectives and constraints constituting the problem. Often the complexity of noteworthy problems emerges from a tension between forces: objectives which seem to contradict, or are incompatible with the constraints.
Context		A fundamental premise of design research in TEME is that problems and solutions are rarely universal. The scope of any statement needs to be qualified if it is to be meaningful.
	Origins	The origins of a pattern define its context in the most reliable manner: the characterisation of the common features of the situation descriptions of the design narratives from which it was derived.
	Expansion	Based on the theoretical analysis of the pattern, it may be justifiable to claim a broader scope of applicability than can be directly deduced from the originating narratives. Such a claim is speculative until verified experimentally or by independent application, and should be noted with caution.
	Boundaries	Specifying the boundaries of a pattern – where it does not apply – is arguably as important as where it does. These boundaries need to be clearly marked where there is a risk that a pattern would be applied erroneously.
Solution		The solution is the centrepiece of the pattern. In scientific terms, it is the claim that under certain conditions the described actions will have a particular effect which addresses the problem. The solution would ideally be articulated at a level of detail which allows immediate implementation, and yet is applicable beyond the specific experiences from which it is described. However, in a hierarchy of patterns, several levels of abstraction will be represented and the respective solutions will be positioned appropriately.
	Pedagogical aspects	Features of the pattern pertaining to its pedagogical structure and driven by its learning objectives, irrespective of the technical implementation.
	Technical aspects	Technological requirements derived from the pedagogical aspects and necessary for their success.
	Diagram	When appropriate, a diagram will be added to elucidate the solution. In contrast to the illustration, which communicates at an intuitive or metaphoric level, the diagram is like a blue-print for implementation.
Support		If the solution is the claim, then the support section is its theoretical and empirical substantiation. This section is not common in non-scientific uses of design patterns, e.g. in software engineering pattern languages. Where a partial equivalent would be the "known uses" section. The latter is however more in way of demonstration than

		evidence.
	Theoretical	The theoretical alignment, as defined in section 5.5.1, justifies the pattern by reference to domain theories.
	Empirical	The empirical calibration, as defined in section 5.5.1, justifies the pattern by reference to data from cases where it appears to have had a positive effect. These would primarily be derived from the design experiment at hand and supplemented by external documented cases.
Liabilities		Liabilities are the potential negative effects and points of failure associated with a pattern.
	Risks	Risks refer to possible direct negative side effects. These need to be averted, often by applying a related pattern.
	Limitations	Limitations note the patterns shortcomings, setting the readers expectations in perspective, and highlighting the additional conditions needed for this pattern to be effective.
Notes		Additional comments, reflecting on the pattern at a theoretical level or linking it to related work.
	Potential extensions	Conjectures as to how the pattern could be extended and further developed to provide better results or cover a broader context.
Related Patterns		Note hierarchical, structural and lateral links to other design patterns.
	Uses (makes use of)	Patterns used as components by the present pattern.
	Used by	Pattern which use the present pattern as a component.
	Extends	Patterns at a higher level of abstraction, elaborated by the present pattern.
	Extended by	Patterns at a greater level of specificity, elaborating the present pattern.
	Conflicts	Patterns excluding, excluded by or otherwise clashing with the present pattern.
	Associated	Patterns used in conjunction with the present pattern.

Table 8: The structure of the pattern template: all components are present in all patterns, but elements appear as relevant. Components are marked by headers.

The template presented here is the outcome of a long iterative process, which accompanied and succeeded the Learning Patterns and Planet projects. An initial draft template was synthesized from the templates of other languages and presented to project members as a basis for discussion. The agreed template resulting from this discussion was implemented as a bespoke software tool and used by myself and others to record and examine collections of patterns in various domains. I revised this template several times in response to feedback from the project members as well as workshop participants who used it: sections were added, removed and renamed. Finally, the mature format available at the end of these projects was inspected and refined in light of the epistemic and methodological considerations raised in Chapter 4.

Nevertheless, it is offered not as an ultimate, definitive format but as a contribution to the design research community which should serve as a starting point for other pattern efforts. Each group or individual embarking on such an effort will need to review, debate adopt and adapt a

suitable template format. For example, Sharp, Manns and Eckstein (2003) offer a perceptive account of the evolution of the template used by the pedagogical patterns project.

5.6 Summary and Conclusions

This chapter addressed Aim 2 by tracing the methodological framework and instruments of the demonstrator study, from data collection and management, through interpretation and systemisation of observations as design narratives and on to the formalisation of research outcomes as design patterns. This methodological framework was derived by projecting the principles and constructs proposed in Chapters Two and Four onto the research question and in the context of the research settings of the demonstrator study. Thus, this chapter bridges between the primary study of the thesis and the demonstrator study which validates it.

The chapter began with a description of the experimental setting: the classroom environment, the technological setup, and the process of iterative design. It proceeded to list the methods of collecting, cataloguing and analysing data used in this context. Finally, it articulated the process by which design narratives were constructed from the data, and design patterns extracted from the narratives.

Taken together, the result is a full specification for implementation of the analytical hemicycle of the design experiment cycle proposed in Chapter Four (Figure 7), and of the retrospective analysis phase of the design research meta-cycle (Figure 6). While the principles and constructs presented in previous chapters claim to be generic (to a degree), the instruments described in this chapter are a single instance of their application to a given problem domain and experimental circumstances.

Three classes of data were identified: design data, student productions, and classroom observations. Design data include any record of the design process and its product. Student productions refer to multi-modal texts and artefacts produced by students in the course of activities. Classroom observations denote any account or recording of students activities. The main focus was on process data, with occasional pre / post assessments where relevant. The challenges of a messy environment were addressed by:

- **Redundancy:** collect any bit of evidence offered by the scene of activity.
- **Triangulation:** juxtapositioning evidence obtained by different methods.
- **Nearest substitute:** accept the limitations of the research setting, use pragmatically available data which is closest in form to the ideal.

The primary sources for design are project reports, design documents, teacher manuals and research journals. The primary sources for student productions are student webreports, ToonTalk code and paper-based written tasks. All texts and artefacts were read as mathematical arguments expressed in narrative. Acknowledging the impossibility of separating observation from intervention, data collection was integrated with activity design – e.g. pre tests became motivators for new topics. Products were assessed in terms of aptness, complexity and sophistication of argument. The primary sources for classroom observations were field notes, video and audio recordings. Interview data included (individual and group) stimulated recall

interviews, task-based interviews and in-activity probes. The latter played a central role in observational data.

A structured process of selection and construction of design narratives was identified, using Bruner's ten principles as guidelines. These principles, adapted to the needs of scientific form, were expressed in the design narrative template.

Design patterns were extracted from design narratives through a six step process devised to capture the key design elements, systemise and substantiate them. This was followed by a phase of refactoring: structural manipulations which give the pattern language as a whole greater coherence.

Chapter 6 The Demonstrator Problem Domain: Number Sequences, Representation and Structure

The purpose of this chapter is to inform theoretically the demonstrator study defined by Aim 3 in Chapter Three. It provides the motivation for number sequences as a curricular topic, considers some of the known challenges in this domain, and proposes some possible causes and remedies for them. The focus of this review is pragmatic: although it does aim to offer a theoretical contribution in this field, the main objective is to guide educational design. The resulting tools and activities are evaluated in Chapter Seven and systematised in Chapter Eight.

6.1 Overview

This chapter draws on the educational research literature to guide the design of a set of activities for learning mathematics. The point of departure for this enquiry is the belief, expressed by Bob Burn, that "Learning or growth in mathematics consists of a transition from experiences of the particular, through pattern recognition or problem solving, to perceptions of a generic" (Burn, 2005, p 269). With this in mind, this chapter proposes a path of learning which starts from the observation of patterns in number sequences, progresses through investigations of the structure and behaviour of sequences, and finishes with explorations of limits and convergence.

The argument itself proceeds as follows. I first review the fundamental motivations for using number patterns as an entry point to mathematics. I review some difficulties associated with the transition from patterns to algebraic and functional structures. I identify several barriers in this domain, and consider these in a broader theoretical perspective. These barriers include the tension between closed-form and recursive representation of sequences, the tension between relating to a sequence as an object and as a process, and the tension between normative and naïve interpretation of mathematical symbolism. The notion of narrative emerges as a theme which cuts across these issues. Reflections on these observations led me to suggest a framework of activities in which to construct and discuss models of number sequences in a medium which allows them to create mathematical narratives. I argue that programming can, under certain circumstances, provide the medium of construction. I also emphasize the social context of programming, and the need to support it in the design of collaborative media.

6.2 Number patterns, sequences and mathematical structure

6.2.1 Why patterns?

This section presents some motivations for focusing on number sequences as a domain of exploration. In many countries pattern recognition and generalisation are considered fundamental to mathematical thinking, and serve as a fruitful pathway into algebraic expression. The English National Curriculum¹⁰ for key stage 2 (age 7-11) states that students should:

10 From the curriculum online website, <http://www.nc.uk.net/>

recognise and describe number patterns, including two- and three-digit multiples of 2, 5 or 10, recognising their patterns and using these to make predictions; make general statements, using words to describe a functional relationship, and test these; recognise prime numbers to 20 and square numbers up to $10 * 10$; find factor pairs and all the prime factors of any two-digit integer.

Whereas at key stage 3 (age 12-14) they should be taught to:

generate common integer sequences (including sequences of odd or even integers, squared integers, powers of 2, powers of 10, triangular numbers)

find the first terms of a sequence given a rule arising naturally from a context ...; find the rule (and express it in words) for the n th term of a sequence

generate terms of a sequence using term-to-term and position-to-term definitions of the sequence; use linear expressions to describe the n th term of an arithmetic sequence, justifying its form by referring to the activity or context from which it was generated

Similar statements can be found in other curricula, such as that of Victoria, Australia (Stacey, 1994) and South Africa (Dept. of Education, 2002). The common theme that emerges from these examples, as well as numerous research papers (Sasman et al, 1999; Zazkis & Liljedahl, 2002; Mason, 1996), is that seeing and interpreting patterns is a foundational mathematical skill. This skill does not constitute mathematical thinking in itself, unless it is complemented by generalization, which in time manifests itself in algebraic formalism. As asserted by Lee (1996, p. 102):

... [it is] much of a challenge to demonstrate that functions, modelling, and problem solving are all types of generalizing activities, that algebra and indeed all of mathematics is about generalizing patterns.

Lee also quotes Whitehead (1947):

The history of the science of algebra is the story of the growth of a technique for representing of finite patterns.

The notion of the importance of pattern is as old as civilization. Every art is founded on the study of patterns.

Mathematics is the most powerful technique for the understanding of pattern, and for the analysis of the relationships of patterns (Lee, 1996, p. 103).

And in the words of John Dossey (1998):

From whence does algebra grow? It grows from the study of growth itself. One of the first places students see growth is when they look at patterns and patterns of numbers. (Dossey, 1998, p 20).

Following this line, it seems reasonable to progress from patterns to the study of the structure of number sequences. However, Zazkis & Liljedahl (2002) note that most research focuses on either fundamental counting sequences or on advanced mathematical thinking (AMT). In the first category we find studies such as Steffe (1988; 1994) and Olive (1991) which illuminate the construction of the basic number sequence at an early age. These highlight developmental issues which are outside the scope of my study. The AMT studies illuminate the fundamental difficulties facing learners and teachers. I review some of the questions arising from this stream of research in section 6.2.4 and 6.2.5. In between these extremes, a common theme is the potentials and pitfalls arising from observing number patterns. These are the focus of the next section.

6.2.2 Sequences as a Foundation for Advanced Mathematical Thinking

Sequences are often seen as a path into functions and algebra. Much of the research is concerned with questions of limits and convergence (Davis and Vinner, 1986, Tall and Schwarzenberger, 1978, Cornu, 1991, Oehrtman, 2001, Zehavi et al, 2001, Srisakanda, 2003, Floris, 2004; Alcock and Simpson, 2005a; 2005b; Przenioslo, 2005; Roh, 2007), typically in the context of advanced high-school and college curricula. Some of the studies reported below, though proposing broader claims, stem from questions regarding limits and convergence (Confrey & Smith, 1994, Cottrill et al, 1996). The notion of convergence of sequences is a well-documented stumbling block for students of all ages. For instance, many first year undergraduate students continue to believe that a sequence cannot reach its limit or that the limit is the last term in a sequence (Eade, 2003). Many of the university-level studies (e.g. Alcock and Simpson, 2005a; 2005b; Przenioslo, 2005; Roh, 2007) focus on students' mastering of the Bolzano - Cauchy ($\epsilon - N$) definition of limit. Przenioslo (2005) notes that the majority of university students formed their concepts at secondary school, suggesting that the problem should be tackled there. Barbé et al (2005) apply the anthropological theory of didactics as a lens on the teaching of limits in Spanish secondary schools. They identify several curricular and institutional tensions which result in school experiences that rarely transcend the technical and fail to motivate the content taught. As much as such observations may be convincing, they appear endemic to school mathematics and do little to illuminate the specific topic. Furthermore, the intricate theoretical framework used to express these observations may be powerful in describing the state of play, but it is unclear how to use it to derive imperative insights. Przenioslo (2005) and Roh (2007) present specific educational designs as normative claims, although they are in effect design narratives. The result is over-generalised, and lacks clear distinction between generic and context dependent design elements. However, Burn (2005) points to a more fundamental flaw with the underlying attempts to lead students to "discover" the definition. In his words, "Never, in my experience, has a student proposed the standard definition of limit" (Burn, 2005, p. 270).

The Bolzano – Cauchy definition is treated as the source of the concept of limit, where, Burn (2005) argues, it is in fact the outcome of a historical process, a tool devised for a purpose. He calls on historical examples to propose activities which motivate the definition by necessitating it for the solution of computational problems:

Fundamental definitions do not arise at the start but at the end of the exploration, because in order to define a thing you must know what it is and what it is good for. (Freudenthal, 1973, p. 107, in Burn, 2005, p. 294)

Mamona-Downs (2001) offers what is, in effect, a three-step design pattern for teaching the formal concept of limit:

1. Initiate and develop intuitions by raising issues in a classroom discussion.
2. Introduce the formal definition and analyse it by reference to the issues raised in (1), using a particular representation.
3. Use the formal definition and the representation as a measure for evaluating opinions raised in (1).

Mamona-Downs (2001) proposes some examples of realistic problems to motivate the classroom discussion. However, the realism of these problems can become an obstacle as students' real world experiences conflict with the mathematical ideal. The result may be a reinforcement of the uneasiness with the notion of infinity, which Tall & Schwarzenberger identify as the source of many students' difficulties with the topic of limits "as if it were all a piece of mathematical double-talk, having no real-life meaning" (Tall & Schwarzenberger, 1978, p. 45). The first problem is the disbelief in the existence of infinity (Boero et al, 2003). A second common issue is seeing infinity as "the last number" (Falk, 1994). Although students intuitively accept the existence of a process that "goes on forever" (potential infinity), many researchers see this as an obstacle in the way to the harder conception of "Actual infinity" (Fischbein, 2001; Sierpiska, 1987). The distinction between potential and actual infinity resonates with the process-object issue which is the focus of section 6.2.5. Potential infinity is a property of the sequence as a process, while actual infinity is a property of the object.¹¹ Cornu (1991) notes that even the definition of limit is read as a process (give me an ε , I'll give an N) which implies a weaker qualifier than the formal mathematical statement (for every ε , there exists an N). Another hurdle is children's extensively documented difficulty in expressing statements and reasoning about rate of change (Bezuidenhout, 1998; Carlson et al, 2003; Confrey & Smith, 1994; Wilhelm & Confrey, 2003). In particular, students' initial conception is predominantly linear. Any ascending function is plotted as a linear growth, and any descending one as a linear decline (Davis & Vinner, 1986). As a response, Sacristán (1997) stresses the importance of acquiring awareness of the behaviour of sequences as well as the vocabulary for argumentation in this domain. Similarly, Salvit (1997) advocates a property-oriented perspective: reflecting on the co-variation approach mentioned above (Confrey & Smith, 1994) and the alternative correspondence view (Thompson, 1994), Salvit claims it is not enough to identify patterns of change in the processes of sequences – these need to be associated with the properties of the sequence as an object. Such voices highlight the issue of representation, elaborated in section 6.2.4.

¹¹ To be precise, the set: see Falk (1994), which notes that infinity is not a property of a sequence at all.

The emphasis on limits reflects the difficulty this subject poses for many students, but to some extent it may also express personal and institutional agendas. If the basic epistemology of number sequences were better understood and supported, students would perhaps not find the advanced topics so challenging. Helping students develop a more complex view of sequences could hopefully place them in a better position to grasp advanced mathematical concepts, such as convergence and infinity.

Kieran (1997) reviews several examples of how activities originating in observation of patterns in numeric or graphical sequences can create opportunities for introducing algebraic thinking. Mason adds an interesting observation: *"The reason for emphasizing expression of generality in number patterns is only to provide experiences which highlight the process"* (Mason, 1996, p. 65). In other words, we could perhaps just as well base a curriculum on graphical or auditory patterns. The focus on numbers is a matter of convenience. There is an implicit argument along these lines: *"Starting from numeric expressions, we are already in the representational system of algebra. We hope that the transition to letters would not be too hard."* Yet the actual effect might be quite the contrary. As observed, cautiously, by Sfard and Linchevski:

"The episodes discussed in this article give rise to the suspicion that most of the time algebraic formulae are for some pupils not more than mere strings of symbols to which certain well-defined procedures are routinely applied" (Sfard & Linchevski, 1994, p 223)

Identifying a pattern is a primal capacity we share with most cogitative creatures. Explaining it is a high-level meta-cognitive skill which needs to be painstakingly cultivated. As many researchers have noted (Arzarello, 1991; Arzarello et al., 1993; Mason, 1996; Lee, 1996; Rico et al., 1996), students erroneously base their conclusions on superficial or incidental patterns they observe in the sequence, rather than on arguments referring to its structure. When students do progress beyond base intuitions, their arguments are predominantly empirical: they spot a pattern, then test it on several cases, and if it fits the test cases they are satisfied with it. Although the use of structural reasoning increases modestly with age, empirical reasoning remains widespread (Küchemann & Hoyles, 2005). Noss, Healy & Hoyles (1997) consolidate the findings of multiple researchers in this area, and assert that while most students are able to identify a great variety of patterns, many of these patterns do not readily lend themselves either to the expression of a functional relationship or to an algebraic representation in any straightforward way. Students who are able to apply a correct method to any number of specific cases often cannot articulate a general pattern or relationship in natural language, and expression in algebraic symbolism is even more problematic. Even when algebraic notation is used, it is often the outcome of the activity rather than a tool. Students who do produce algebraic representation rarely verify them or refer to them in their verbal arguments. They conclude:

Taken together, the evidence suggests that algebraic formulation is often disconnected from the activity which precedes it, a meaningless extra that neither illuminates the problem nor provides a means for validating its solution. Algebra is viewed as an endpoint, a problem solution in itself rather than a tool for problem solving. (Noss, Healy & Hoyles, 1997, p. 204).

6.2.3 From pattern spotting to formal thinking

The previous section presented a commonly accepted path into mathematics: begin with patterns, generalize these into rules, elucidate these rules by rephrasing them in formal language, and use this as a basis for functional and algebraic thinking. Pattern recognition is a fundamental neuro-cognitive skill (Eysenck, 2001; Juola, 1979) and the use of patterns as an entry point to mathematics is intuitively appealing. It makes sense to utilise it as a starting point. Yet, as we saw, the transition to formal algebra is not straightforward. Perhaps the primary element of a language is its lexicon: the set of symbols it uses and their interpretation. In the case of mathematics, the accepted lexicon is algebraic symbolism. Several researchers have commented on learners' difficulties in acknowledging the conventional interpretation of algebraic symbols.

Radford (2000) notes the "dual life" of algebraic symbols: on the one hand they are signifiers, pointing to abstract mathematical objects and concepts, while at the same time they are tools which allow us to perform actions. When we write:

$$y = 3x + 1$$

We use x and y to express the idea of unknown quantities with a known relationship, as we use '=' to signify equivalence. Yet the same expression is also an instrument to infer the cost of 3 theatre tickets, which includes a £1 fixed booking fee. Radford demonstrates how some difficulties arise from failure of assigning the correct meanings to the signs, e.g. by associating letters with fixed numeric values according to lexical order. Nunes, Schliemann and Carraher (1993) show how other difficulties arise from extreme dissociation of symbols from concrete meanings, when students learn to perform symbolic manipulations at a purely syntactic level ignoring the context from which the problems are derived. MacGregor and Stacey (1993) focus on students' understanding of letters as variables in algebraic notation. In an effort to go beyond others' broad statements, they note several common difficulties, such as interpreting any letter as 1 or replacing it by its alphabetical counterpart ($a = 1$, $b = 2$, etc). When attempting to explain their observations, they attribute them primarily to inadequate teaching materials. Thus, textbooks which try to build on intuitions by using letters as abbreviations (w for weight and h for height) reinforce this type of misinterpretation, and textbooks rich in puzzles reinforce the alphabetical value misinterpretation. This conclusion is obviously an important one, both for the designers of educational materials and for teachers. Yet a second look at their examples suggests two additional factors at a deeper epistemological level: the tendency to interpret symbols as specific signifiers and the emergence of alternative representations using standard symbols.

The first factor relates to a fundamental characteristic of human discourse. As Radford (2000) demonstrates, students talk metaphorically about the general through the particular. Even when asked to generalize, they will use statements such as "take any number, like 127". This phenomenon is part of what I refer to below as narrative form: telling a "story", where the general is implied by the specific. Likewise, even when contemplating a general rule, students will perceive symbols as referring to specific objects rather than to unknown members of a class. Even three-year-old children can comprehend the relationship between a symbol and a specific concrete object it signifies (DeLoache, 2002; 2004), yet much older children find it hard to

associate a symbol with an unknown quantity, even when they are capable of performing operations on these unknowns (Schliemann et al, 1998). The phenomena reported by MacGregor and Stacey (1993), such as identifying letters with specific measures (h for Johnny's height), alphabetic value ($h = 8$) or "any letter is 1" can be seen as special cases of the inbred tendency to refer to the specific. This conforms to the *letter as object* interpretation noted by Küchemann (1981), who reports on the findings of the longitudinal CSMS study (1974 – 1979) which analyzed written tests and interviews administered to English school children aged 13-15. He notes six common forms of interpreting letters in algebraic expressions and maps them to four levels of mathematical understanding. At the lower levels the dominant interpretations are: *letter evaluated* – assigning a value to the letter from the onset; *letter not used* – the letter is ignored or acknowledged but not used; *letter as object* – the letter is read as a referent to a specific object. At the third level children use letters as referring to *specific* or *generalised unknowns*. In the case of specific unknown, children acknowledge the idea of a symbol denoting an unknown quantity, but still see this as a fixed, unique value. In the case of *generalised unknowns*, they have expanded this view to include sets of values under some constraints. Only at the highest level do children recognise letters as *variables*: unknowns which can be manipulated and put in relations without specifying their value. Although Küchemann suggests some correspondence between these levels of understanding and Piagetian stages, he emphasizes that there is no simple correlation between levels and age or school experience.

It is interesting to compare this view with the historical perspective. As argued by Burn "*where current psychological research does not map out a path consonant with student intuitions, historical enquiry can reveal actual steps of success in learning*" (Burn, 2005, p. 271). Sfard & Linchevski (1994) identify five stages in the historical evolution of Algebra. The first three they group as *Generalized Arithmetic* and the last two as *Abstract Algebra*. The Generalized Arithmetic phase is characterized by forms of algebra which are extensions of recipes for doing arithmetic. At first, in the *operational* phase, these are "rhetorical", verbal recipes for performing useful calculations. Diophantus (circa 250 AD) was the first to introduce a scheme using letters to signify specific unknowns, signalling the shift from the *operational* to the *structural* phase. Yet Diophantus' *Arithmetica* deals with specific problems. The general methods are implicit or explained with reference to examples. Diophantus' symbolism is in fact a form of shorthand: abbreviations he adopts to simplify his verbal (or narrative) mathematics. Such use of symbols is equivalent to what Küchemann identifies as *letter as object*. Radford (2004) claims that the Greeks were concerned with the nature of the mathematical object, whereas the Renaissance scholars were interested in the process of its coming to be. Yet, without a language for referring to unknowns, the objects referred to were concrete and specific. The form of mathematical discourse before the emergence of algebraic symbolism was narrative: one told the story of performing a particular computation – and the reader was expected to infer the general method implicitly. According to Sfard and Linchevski (1994), the *structural* stage is followed by the *functional* phase, represented by the figure of François Viète (1540-1603), who is considered to have introduced the algebraic symbolism we use to this day. Only then did the view of an algebraic expression as a function emerge, in the sense of Vergnaud (1983), to be explained below. This historical phase is equivalent to Küchemann's distinction of *letter as variable*, in the highest levels of algebraic understanding. Yet algebra is still treated as a method of performing computation on specific unknowns. It is only in the hands of the British formalists

(e.g. George Peacock, 1791 – 1858) that it aspired to become a “*science which treats the combinations of arbitrary signs and symbols by means defined through arbitrary laws*” (O'Connor & Robertson, 1996). Even then, the approach was operational. An abstract and structural Algebra is a product of the last two centuries. Given this perspective, Küchemann's observations are far from surprising. Why should we expect a child to instantaneously master a method of thought which took the best minds of humanity centuries to construct? It is impressive enough that the child can mimic the mindset of Diophantus! Furthermore, there is a striking resemblance between the formalist view of Algebra as a science and the detached, alienating and inaccessible experience described by Noss, Healy and Hoyles (1997) and Sfard and Linchevski (1994). An Algebra which deals with the arbitrary manipulation of arbitrary signs through arbitrary rules may be an idealized endpoint for science, but it is definitely not a readily approachable art.

Returning to the analysis of the observations made by MacGregor and Stacey (1993), the second factor I suggested – the emergence of alternative representations – is somewhat more subtle. MacGregor and Stacey offer the student's answer $h = h + 10$ as an example of a misinterpretation that they call *general referent*; a letter that takes upon itself a specific attribute, applied to different objects (in this case, height). However, drawing on Kieran's observations regarding the use of the equality sign (Kieran, 1981) one can elaborate on this distinction. Kieran notes that young students use the equality sign ($=$) as a *do something symbol*, an instruction to perform an operation rather than a declaration of a relationship. This leads to confusion when confronted with expressions such as $3 + 4 = 5 + 2$. Likewise, it is possible that in the $h = h + 10$ case students were using the algebraic symbols as a *procedural* rather than *declarative* system. The statement reads “to obtain Jack's height, add 10 to that of John's” rather than “Jack's height is the same as John's plus 10”. MacGregor and Stacey hint to this possibility, when they refer to influences of other symbol systems, such as computer programming.

Some researchers have advocated that curricular reform should acknowledge these issues. For example, Smith and Thompson (2007) advocate an early emphasis on developing children's ability to conceive of, reason about, and manipulate complex ideas and relationships, as an equal component to numerical reasoning and computation. Such a capacity will serve as a foundation for a broad range of “Algebras”: Algebra as modelling, as pattern finding, as the study of structure. Their basic claim is: “*if students are eventually to use algebraic notation and techniques to express their ideas and reasoning productively, then these ideas and reasoning must become sufficiently sophisticated to warrant such tools.*” Smith and Thompson (2007) propose achieving this by promoting pre-algebraic quantitative reasoning. Children are confronted with sophisticated problems, in verbal form, and encouraged to solve them by constructing arguments about the relationships between the quantities represented in the situation. One class of such problems, which is most relevant to the current discussion, focuses on *Patterns of difference*: situations which involve corresponding repeated, or even continuous, additive change of two quantities. Conceptualizing a pattern of differences entails grasping a *collection* of changes as an object of consideration, which is a first step towards a functional view of change. Situations which involve a single set of differences – as in arithmetic progression – facilitate the conceptualization of single-variable functions, whereas situations which require a co-ordinated view of two sequences promote the conceptualization of dual-variable functions. Although Smith and Thompson do not mention this, one could also see a path leading along an

axis of functional complexity by manipulating the situation from patterns of differences to patterns of products and beyond. Such an approach is demonstrated by Zazkis and Liljedahl (2002), who suggest that it may be possible to design activities which use sequences as a bridge from additive to multiplicative conceptual fields.

Smith and Thompson propose a radical curricular reform to highlight quantitative reasoning. They conclude that teaching quantitative reasoning requires (a) selection of a sequence of problem situations and (b) providing support for students' reasoning in the selected domain. Such an agenda draws on two broadly supported assumptions: first, that the current curriculum leads many students to a superficial and procedural view of mathematics, and second, that concept formation is embedded in meaningful activities. The former theme has been illustrated above, and the latter will be discussed below. Yet within these premises there is still a wide degree of freedom regarding the methods of instruction. Smith and Thompson focus on verbal (or textual) problems and verbal argumentation. They implicitly posit that this context is in tune with students' pre-algebraic intuitive thinking. They mention, but do not elaborate on, the potential of visualization. They note that when students' verbal descriptions lose clarity, they can be asked to depict their ideas using a diagram. While they refer to other researchers (e.g. Confrey, 1991; Schwarz, Yerushalmy & Wilson, 1993) who have demonstrated a more systematic use of graphical tools, they do not explain how such tools can be employed systematically to foster quantitative reasoning. More important, they do not describe the pathway from quantitative reasoning to algebraic competency.

6.2.4 Closed form vs. recursive process

The previous section inspected the lexical level of mathematical language, a core set of symbols. The next level is the grammar of the language, the system by which the symbols are combined to express statements. In the case of mathematics, the standard is the algebraic formula. In the case of sequences, this entails the closed form representation: $a_n = f(n)$. Some of the aforementioned researchers have suggested that one of the obstacles to developing an understanding of this structure is students' tendency towards a recursive view. That is, the identification of the relationship between consecutive terms rather than its general rule of the sequence (Zazkis & Liljedahl, 2002; Orton & Orton, 1999), as expressed by Mamona-Downs: *"Sequences are rarely considered by students as functions on N , and hence tend to be regarded as processes rather than mathematical objects"* (Mamona-Downs, 2001, p. 262). This is sometimes referred to as a scalar view, as opposed to a functional view. This distinction is explained by Vergnaud (1983), using as an example the typical question *"Johnny bought 4 apples at 15p an apple, how much did he pay?"* Vergnaud argues that young children will tackle such a question by applying a unary operation (rather than a binary rule of composition), i.e. applying "x4" as an operation on 15 or "x15" as an operation on 4. The crucial observation is that the two possibilities are not symmetrical: The first involves scaling of a given measure. It can be obtained by repeated addition, as in *"one apple – 15, two apples – 30..."*. This is referred to as a scalar operator. On the other hand, the operator "x15" is derived from the functional relationship mapping the domain of apples to the co-domain of pennies. As an illustration of the fundamental difference between the two, consider the absurdity of *"4 apples at 1p/apple, 4 apples at 2p/apple, ..."*. An immediate implication is that the primary, scalar, multiplying scheme is derived from counting. This conjecture is supported by Steffe (1988) using a developmental

framework that builds on the work of Piaget (1952). Steffe delineates four stages in the conceptualization of counting sequences, starting from the *initial number sequence* (1, 2, 3, 4...), through the *tacitly nested counting sequence* – where some elements are hidden or otherwise skipped but remain implicitly present, the *explicitly nested number sequence* – e.g. counting in twos and fours, and the *generalized number sequence* which expands the iterative operation from addition of constants to other functions. Multiplicative structures emerge from the explicitly nested sequence:

Just as the iterable one was the abstraction of the repeated application of the “one more item” operation when double counting by ones, the *iterable composite unit* is the result of the abstraction of the repeated application of the “one more four” (say) when double counting by fours. (Olive, 2001).

The functional view of multiplication is more general and computationally powerful than the scalar view. Consequently, the functional form is seen as the correct, rigorous mathematical concept of a sequence: a function $f:N \rightarrow R$. Children’s failure to transcend the scalar view of a sequence – as derived from counting sequences – to the functional view is seen as a challenge.

Two observations are common to most of the studies above: first, that number-pattern spotting is a predominant solution strategy, and second, that the recursive form is a predominant description strategy. Indeed, pattern spotting lacks the definitiveness of a formal argument, and the recursive form does not generalise easily to functions of the real numbers ($f:R \rightarrow R$). Yet the association between *recursive* and *lack of structure* suggests that in some cases, researchers might be confusing structure with representation. Consider the sequence:

1, 4, 7, 10...

It can be represented in closed form, as:

$$a_n = 1 + 3*n$$

Or recursively as:

$$a_n = a_{n-1} + 3, a_0 = 1$$

Both are functions. One is a function of the natural numbers, the other a function of the previous term. Yet whereas the former is at odds with naive intuitions, the latter stems from them. Note that the recursive expression above is a formalisation of the observed intuitive view, but not an intuitive formalisation in itself. In fact, formal concepts of recursion are well-known stumbling blocks for most students (Goldwasser and Letscher, 2007; Ginat and Shifroni, 1999; Wu, Dale and Bethel, 1998; Harvey, 1992). Furthermore, this form is known as *tail recursion*: a single application of the recursive call, as the last operation in deriving a result. Tail recursion is algorithmically equivalent to iteration, in the sense that it could be expressed as a loop. The distinction made here is between an iterative process and a recursive structure. While the sequence may be computed by repeated addition, naming it “the add three sequence” borrows the process to denote the object, by describing the general term as a recursive structure.

From a design point of view, the challenge is to construct learning environments that are contiguous with existing knowledge, rather than seeking to replace it. Weigand's (1991) iteration sequences are defined by a formula which derives the next term from the current, similar to my notion of recursive form. Weigand notes that the class of iteration sequences is much broader than the arithmetic and geometric sequences that dominate the school curriculum.

To recap, we have considered two alternatives for the representation of number sequences in formal algebra: the closed form notation, which describes a term of the sequence as a function of its index, and the recursive form, which describes a term as a function of its predecessor. This observations leads to the question: which approach is more supportive of students' learning – working *from* these intuitions or *against* them?

Some of the difficulties and achievements reported in the literature may be more telling of the activity and tool design on the researchers' part than of students' epistemology. For example, Confrey & Smith (1994) argue convincingly for a co-variation approach to functions. Whereas the traditional approach sees a function as a relationship between x and y values, described as $y = f(x)$, a co-variation approach focuses on the advancement from one term to the next in parallel sequences of x 's and y 's. Typically, the x 's increase by 1. The object of study is the pattern that links one y value and the next, what Confrey and Smith call a unit. They define unit as "the invariant relationship between a successor and its predecessor". This approach emphasizes rate of change, and leads the authors to extend the epistemological definition of rate. In their framework, a rate is a "unit per unit comparison". In other words, the relationship between the units of x and those of y . The concept of rate evolves through elaboration of this relationship. The most primitive notion of rate is that of equality. This notion extends to additive rates (linear functions), and in turn to multiplicative rates (exponentials). More complex relationships can be built upon these primal notions by combining them, and finally by applying fractions to the units on both columns, extended to rational numbers. Confrey and Smith claim that the unit / rate view is "natural" and provides a good pathway into calculus. They note that multiplicative rates are less common in children's experiences, but not foreign to them.

Nevertheless, the evidence they present relies on the particular representation afforded by the technology they use (Function Probe). This tool includes a tabular view of a function, which lists the unit of advancement in one column (typically, the natural numbers) and the corresponding function values in the next. They fail to note the possibility that students see the unit column as redundant in this setup, and focus on the function value column and the change from one cell to the next. In other words, rather than co-variation of values in domain and co-domain, they see a recursive variation in a single domain. Furthermore, Thompson and Saldanha (1998) note while the tabulation is effective in investigating sequences, it is inhibitive to the concept of continuous functions, since it promotes discretisation of the function. They propose to extend the framework of co-variation to include simultaneous change in graphs – thus visualising continuous change.

The observations above regarding scalar and recursive conceptions of sequences give rise to several design imperatives, which will be investigated in the following chapters. The prevalence of recursive intuitions suggests that it would be beneficial to use them as a stepping stone to recursive formalisations, and from there to algebraic language. Furthermore, as noted by Noss,

Healy and Hoyles (1997), many of the patterns identified by students are beyond the expressive power of school algebra. Students are able to apply the right rule in action, but find it insurmountable to express it in natural language – let alone formal algebra. Yet some of the sequences which are hard to describe in closed form lend themselves to a simple recursive definition. Could this be a case where we should be providing students with a *learnable mathematics* (Noss, 2001), rather than struggling with their difficulties with standard Maths?

The studies mentioned above show compelling evidence that children’s approach to sequences is not structural. While I reject the fungibility of unstructured and recursive, I propose that the issue lies with the type of recursive thinking employed. Children identify and understand the recursive *process*, but not the recursive *structure*. The next section explains this distinction.

6.2.5 The process-object duality of sequences

The previous section concluded with the distinction between understanding the process of generating a sequence and seeing its structure. This observation stems from the common distinction between viewing a mathematical entity as a process or as an object. This duality features prominently in Cottrill et al.’s (1996) theory of understanding the limit concept. They base their analysis on APOS (Action-Process-Object-Schema) theory (Dubinsky, 1992), which was also applied by McDonald, Mathews & Strobel (2000) to the conceptualization of number sequences. They identified two concepts of sequence; one, which they called SEQLIST, focuses on the sequence as a syntactic object – numbers with commas between them. The other – SEQFUNC – embodies the function-oriented view of a sequence, as discussed above. While their empirical observations appear to be valid and interesting, it may have been easier to describe them outside of the scope of the APOS theory. Perhaps it would be simpler to note the distinction – which others have made in the general context of understanding algebraic symbolism – between a meaningless manipulation of signs and a structural comprehension of concepts. The need to fit a strict theory over the data leads McDonald et al to weak interpretations – for example, they take evidence for students reaching the “object phase” in the conceptualization of SEQFUNC from the fact that when asked to explain the relationship between sequences and functions, students responded “*a sequence is a function*”, a phrase they may have recited, out of a sense of the interviewers’ expectations, without necessarily understanding it. McDonald et al note that possessing a schema of function might in some cases obstruct the construction of a schema of SEQFUNC. It is possible that possessing a schema of APOS – or any other theory – might, in some cases, obstruct the understanding of the epistemological dynamics of a particular situation.

Sfard (1991) starts from Piaget’s observation regarding the trajectory from an operational to a structural view of number. However, she notes that the transition between these views (both epistemologically and phylogenetically) is spiral rather than linear; “again and again, processes performed on already accepted abstract objects have been converted into compact wholes, or *reified*” (Sfard, 1991, original emphasis). The process of reification captures an intuitive operational notion of a mathematical process, transforms it into a formal entity – which can now be processed by formal tools to allow for structural arguments. Once this is achieved, the newly acquired objects can shift into the intuitive realm, where new operations can be performed on them, and in turn reified into higher-level objects. Sfard’s observations illuminate the question

raised in the previous section – should learning design work *against* or *from* intuitions? In a classical Piagetian framework, formal concepts replace primal intuitions. In a dynamic framework, as proposed by Sfard, they evolve from them, gradually and iteratively. Given this view, it makes sense to refine and adjust intuitions rather than dismiss them. Consequently, students need to hold, and be aware of, both the process and the object view in tandem. Some observations about a sequence are better made and discussed in the context of a process and others when viewing it as an object. For example, I presented above the claim by Cottrill et al (1996) that the conceptualization of convergence requires a coordinated schema of two processes: the procession of the sequence terms and the change in their values. By contrast, David Tall, in his extensive investigation of the process-object problem in relation to a wide range of mathematical topics, coined the term *procept*. Tall and Gray (1993) locate the root of many difficulties in learning and performing mathematics in a failure of reification. In their words, a failure to transcend a procedural view and achieve a *proceptual* one; they define a *procept* as “a combined mental object consisting of a process, a concept produced by that process, and a symbol which may be used to denote either or both”, and put forth a strong claim:

The successful child becomes more successful because the mathematics of flexible procepts is easier than the mathematics of inflexible procedures. The gap between success and failure is widened because the less successful are actually doing a qualitatively harder form of mathematics.

Tall and Gray (1994) are in agreement with Sfard (1991) regarding the spiral dynamics of learning. They too describe a progression from process to concept to higher-level process using the concepts of the previous step as building blocks, yet they claim that there is a meta-step to take beyond the hierarchy of concepts:

A proceptual view which amalgamates process and concept through the use of the same notation therefore *collapses the hierarchy* into a single level in which arithmetic operations (processes) act on numbers (procepts). (Tall & Gray, 1994, p. 22, original emphasis)

Vergnaud (1996) also acknowledges the importance of the relationship between concept, process and symbol – or, in his words, concept, scheme and signifier. However, his emphasis is different. On the one hand, the triadic relationship is not as explicit as in the procept view: Vergnaud's *concept-in-action* can include processes as well as the objects, properties and relationships. On the other, Vergnaud puts forth a theory of *conceptual fields*. While Tall and Gray (1993; 1994) presented isolated cases, Vergnaud posits that the useful unit of analysis is a conceptual field:

a set of situations, the mastering of which requires several interconnected concepts. It is at the same time a set of concepts, with different properties, the meaning of which is drawn from this variety of situations. (Vergnaud, 1996, p 225)

Vergnaud names several such conceptual fields: Additive structures, Multiplicative structures, Elementary algebra and Number and space. Each such field is characterized by the common

nature of the problems, procedures, and schemes within it, and by the fact that these draw upon a common pool of representations and concepts. Vergnaud's theory of *Conceptual Fields* adds a socio-cultural dimension. It stems from the premise that learning, in the sense of developing representations and theorems within those representations, is aimed at providing humans – as well as animals – with better and better strategies to deal with the situations they encounter. From this it follows that the knowledge which emerges is tightly bound to the situation in which it was acquired and the actions available to the learner. This is what Vergnaud calls *Theorem-in-action*: “a proposition that is held to be true by the individual subject for a certain range of situation variables”. This type of knowledge does not have to be logically consistent, only locally consistent in the context of the situation.

A similar proposal is put forth by Smith, diSessa and Roschelle (1994; diSessa, 1988), who pointedly argue that both experts and novices maintain simultaneous complimentary, overlapping and conflicting partial models and task-specific problem solving techniques in any given domain. They move fluently and flexibly between them, making mistakes and recovering from them, and adaptation and reconfiguration of the pieces replace the need for dramatic paradigm shifts. This *knowledge in pieces* is acquired through the iterative refinement of effective solutions to concrete problems, and is situated in their context. The *knowledge in pieces* view challenges the perception of knowledge as extensive theories which can be proved or refuted, yet it is consistent with the concept of scripts (Abelson and Schank, 1977; Bruner, 1990). A script is a recipe for solving problems. It includes the context in which it is applicable, the sequence of operations to carry out, and the expected implications. Clearly such scripts would be piecemeal and situated, akin to *knowledge in pieces*. Likewise, they do not need to be logically consistent with each other, but only need to maintain internal consistency. Perhaps the only difference between the approaches of Smith et al (1994) and that of Bruner is that the former focus on the meta-structures of knowledge, highlighting scientific domains, whereas the latter is interested in the representation – internal and external – of knowledge, mainly in daily, developmental, contexts.

The seemingly different perspectives of Sfard (1991), Tall and Gray (1993; 1994), and Vergnaud (1996) can be synthesised using the notion of webbing (Noss & Hoyles, 1996, ch. 3). Tall and Gray (1993) use the example of shifting from the process of counting to the concept of adding. Once a child has surmounted this transition, she can learn new procedures of addition and connect them to her concept of addition. The tripolar concept evolves into a node in a conceptual web, which links symbols, concepts, and procedures. Indeed, the meaning of addition is perhaps not in any particular item but in a cluster of densely connected nodes, encapsulating a host of different facts and experiences. While the conceptual field framework is a convenient means of organizing knowledge and learning experiences, the web metaphor affords more richness in terms of describing cross links between fields, while at the same time maintaining the distinction between concept, process and symbol.

As an aside, theories of embodied cognition (Núñez & Lakoff, 1998; Núñez, Edwards & Matos, 1999; Lakoff & Núñez, 2000) relate process to concept at a more fundamental level. They ground mathematical concepts in bodily experiences. The primal organization of experiences is facilitated by perceptual-conceptual primitives called *image schemata*, such as the container schema, which defines the concepts of “in” and “out”. These basic concepts are then elaborated,

extended and interpreted by means of *conceptual metaphor*, thus giving rise to higher and higher abstractions. For example, the container schema evolves into concepts of groups and logical relationships. Lakoff and Núñez (2000) devote significant attention to concepts of sequences and infinity, which are derived – according to their theory – from the experience of sequentiality in human action. While an elaborate discussion of the theory of embodied cognition is beyond the scope of this study, it is worth paying attention to the mechanism of conceptual metaphor: mapping of known structures to new domains, in order to create conceptual structures that allow us to operate on new problems. If indeed this mechanism is fundamental to our learning apparatus, then it is reasonable to expect it to play a role even beyond the bodily experiences. Thus, the spiral process described by Sfard (1991) and Tall and Gray (1994) can be explained as iterative application of conceptual metaphor. At the same time, the embodied approach to mathematical learning may provide the missing link in our previous discussion of narrative: if indeed learning originates from bodily experience, the first act of abstraction needs a representation that is as close as possible to that experience. Narrative is an obvious candidate (section 4.2). Yet the conceptual metaphors – which are essential for the formation of new concepts – can only be made explicit in the course of a critical discussion.

Douady (1985) adds yet another dimension to the discussion. In the context of what she calls *tool-object dialectic*, Douady provides a detailed account of the classroom dynamics by which new concepts emerge from existing ones. When confronted with a new problem situation, students first attempt to solve it with the tools – concepts and procedures – they already possess (the *Explicit Old* phase). When this fails, they devise ad-hoc adaptations of tools to better fit the problem (the *Implicit New* phase). In the following phases the new tools can be institutionalized and obtain a status of an object through discussion and teacher guidance. They are named, formalized and established as shared knowledge. Now the teacher can set new tasks which assume the availability of the new tools. Here a second pattern, or principle, is manifested: *interplay between settings*: introducing new situations, or problems, which are similar enough for the students to acknowledge the applicability of the newly acquired tools but divergent enough to promote generalization. As with the spiral progression of processes and objects mentioned above, so can the dynamics of *tool-object-dialectic* and *interplay between settings* be explained in terms of conceptual metaphor.

6.2.6 Interim conclusions

Section 6.2.1 presented the rationale for a path into mathematics via number sequences which proceeds from spotting patterns to observing and manipulating structure. The next two sections reviewed the obstacles which learners are known to encounter in this domain; Section 6.2.4 highlighted the tension between indexed and recursive representations of sequences, and section 6.2.5 considered their process-object duality. Most of these difficulties can be seen as pertaining to the process of initiation into a mathematical discourse, adopting its set of symbols and their interpretation, the rules by which they are combined, and the context and focus on conversation. Section 6.3 explores these themes in a broader context.

6.3 Representational systems and mathematical conceptualization

Section 6.2 concluded with questions regarding the relation between the ways in which we represent mathematical objects and the ways we understand them. The empirical work reported in the following chapters focuses on computational representations, following a constructionist tradition. I review prior research in this direction in section 6.3.3. To set that discussion in a broader perspective, I present several illustrative snapshots from the history of mathematical notations. These lead me to the notion of *situated abstraction*, as a framework for learners' construction of meaning, and to *narrative* as a process by which it proceeds.

Luis Radford (2002a) asks the following question: "How ... are we going to deal with objects that cannot be directly perceived such as numbers and mathematical objects?" and offers Frege's answer: with symbols. Since the mathematical concept of addition does not have any physical presence, we use the symbol '+' to refer to it. This distinction made by Radford (or Frege) needs clarification. In what sense is the symbol '7' different from the symbol 'giraffe'? The former does not signify any particular group of objects, but all groups which have the property of seven-ness, but in much the same way the symbol giraffe signifies all objects with a property of giraffe-ness. The difference is that no one has seen a 7, whereas many of us have seen a giraffe. Hence, as Radford and others warn us, there is a danger of mistaking the symbol 7 for the mathematical object it refers to – a danger that is not likely in the case of giraffes. Radford focuses on the process of *objectification* as the loci of meaning-making (from Husserl's *objectivity*): a "process aimed at bringing something in front of someone's attention or view". This is, in a way, a social equivalent of Sfard's *reification*. In Radford's framework, an unstructured mathematical entity takes its form as an object through the need to signify it in the course of discussion. This process involves the creation and use of signs – not just written symbols, but gestures, body motion and rhythm (Radford 2005a; 2005b). Is this social dimension truly necessary? Isn't it sufficient to understand the individual epistemic processes of connecting symbols, processes and concepts? Radford (2004) answers convincingly that symbols are always a product of a socio-cultural system, and need to be understood in this context. To illustrate this point, he presents an historical account of how algebraic symbolism emerged in the Italian Renaissance as a consequence of a specific combination of social, economic and technological conditions. The implied claim is that the emergence of this symbolism not only supported but in fact made possible certain forms of structural thinking.

Radford's claim can be supported by a wealth of historical, archaeological and anthropological examples. From ancient Mesopotamia (Denise Schmandt-Besserat, 1992) through the New Guinean forest (Saxe and Esmonde, 2004; 2005) and to the modern primary school (Nunes and Moreno, 1998), the common theme that emerges is the interdependency of cultural and cognitive development, which is modulated by the specifics of the environment – natural, economic or technological (Kaput, Noss & Hoyles, 2001). Representational systems emerge via a long process of abstraction from the concrete objects. Thus this abstraction is by necessity *situated* in the context in which it originated, to an extent that it is sometimes *material* (as tokens or coins are material abstractions of value). As the representational system evolves so does the communities' ability to engage in complex activities, and the individuals' ability to

participate in and reason about these activities. The complexity of these activities is often manifested in the use of higher-level mathematics.

The dynamics described above are consistent with the ideas of *situated abstraction* and *webbing* (Noss & Hoyles, 1996; Noss Healy and Hoyles, 1997), discussed in the next section.

6.3.1 Situated Abstraction

The previous section highlighted the historical and cultural links between representation and understanding of mathematical concepts. The *situated abstraction* approach (Noss and Hoyles, 1996; Noss et al, 1997) considers similar questions from a design perspective. The idea highlights the dynamics of constructing knowledge from activity, by inserting or populating an abstraction with meaning – in the shape of special cases, particular values, or familiar contexts (or, in the special case of the *mathematical* situation, with mathematical objects and relationships). Abstraction is achieved within, not above, this context. Mathematical knowledge is constructed and expressed with available tools (physical, linguistic, digital or social) that may not map trivially to standard mathematical notation. Abstraction does not proceed by replacing one ‘theory’ with another, but by building layers of concepts one on top of the other. Even those layers that may appear to be highly generalized are still rooted in the learner’s personal experience. These concepts are connected in a web of relationships – some logical, some associative, some anecdotal.

Neuropsychological evidence gives direct support to the ideas of layering and webbing, using a different terminology. For example, Addis et al (2004) talk about ‘specific and general autobiographical memories’ (p. 1740), and show that these activate the same regions in the brain. Or, in the words of Mason & Just (2006): “Text attributes at the discourse level enter into combinations with other information to allow a reader to weave individual sentences into an integrated narrative structure. The resulting conceptual structure incorporates pragmatic information and connects the text with the reader’s world knowledge”. Such observations are very much in line with the findings of Noss et al (2007), who argue that in diverse situations people make sense of abstract mathematical representations by populating them with imaginary narratives derived from their own daily experience. Such narratives capture the essential knowledge emerging from familiar activity, and create a bridge between the personal experience and its abstract mathematical manifestation.

The notion of narrative has emerged over and over again in previous sections. The idea of situated abstraction gives it a concrete role in the dynamics of mathematical learning. The next section explores this role.

6.3.2 Mathematics and Narrative in Education

Section 6.2.3 discussed the narrative nature of children’s vernacular epistemology of mathematical ideas, and drew historical parallels. Section 6.2.5 projected these observations onto the context of this study, by associating the narrative mode with the process view of sequences. Chapter Four reviewed, in greater depth, the notion of narrative and its epistemological role. Given the strong cultural and neurological grounding of narrative, it seems that we should strive to embed narrative structure in the design of systems or activities which

are aimed at meaning-making. However, narrative approaches to computer-enhanced learning are often focused on designing systems that support narrative-based learning (Mott et al, 1999; Decortis & Rizzo, 2002; Decortis, 2004), i.e. systems that support the production of imaginary narrative as the site of learning. Nehaniv (1999) argues for a wider view, claiming that any design that does not acknowledge the "narrative grounding" of humans will appear to its users as bizarre, unintelligent and unintelligible.

... it is desirable to take into account that humans are temporally grounded, narratively intelligent beings. Their evolutionary heritage leads them to expect that the actions of others are embedded in a context of past history and future events." (Nehaniv, 1999, p. 102)

Likewise, Laurillard et al (2000) highlight the importance of embedding narrative structure in the design of multi-media resources, where non-linearity risks impeding learners from maintaining a personal narrative line and thus increasing cognitive costs. It is the responsibility of teachers and designers to maintain narrative flow in order to allow learners to maintain a focus on the development of sound arguments: "With such design features, the non-linear medium is able to afford something more than mere browsing: it will afford structured, meaningful learning" (p. 18).

Applying these ideas to mathematics education highlights the tension between the acceptable forms of mathematical language and those of daily language. This tension manifests itself on two levels: first, the fundamental structures of statements, and next in the norms of establishing truth. At its fundamental level, mathematical language is propositional. It defines terms, sets axioms and rules, then states theorems and proves them. Its structures are static and timeless. Spoken language, on the other hand, is predominantly narrative, which by its nature is chronological and dynamic. The tension between spoken language and mathematics was demonstrated lucidly by Wittgenstein:

In mathematics we have propositions which contain the same symbols as, for example, "write down the integral of...", etc., with the difference that when we have a mathematical proposition time doesn't enter into it and in the other it does. Now this is not a metaphysical statement. (Wittgenstein, 1989, pp 34)

When we say 'two plus two equals four', the truth value of this statement is independent of when we say it or who 'we' are. Yet how do we present such a statement to a young child? One might say:

You had two marbles, and I gave you two more, so now you have four.

When we attempt to humanise the mathematical statement, we unconsciously transform it from the propositional form to the narrative. Something (transfer of marbles) happened to someone (the child and me) under some circumstances (say, sitting around the kitchen table). In that event, two groups of two were magically exchanged for one group of four.

Such conversions from propositional to narrative do not disappear as the subject matter becomes more elaborate. Let us review one more example:

$$\{S_i\} \rightarrow C \equiv \text{for each } \varepsilon \text{ there exists an } N \text{ such that for every } n > N, |S_n - C| < \varepsilon$$

How do we explain such a statement to a student? Perhaps:

Let's look at the sequence we had yesterday $\{1, 1/2, 1/3, \dots\}$. Go far enough along the sequence and you will reach a point such that all subsequent terms lie within a very small range of 0. Now, we can make that small range as small as we want.

Again, in our attempt to make the mathematical idea accessible, the propositional is rendered as a narrative. A static structure, 'devoid of time and person', is placed in a specific context, and becomes a string of events happening to 'you'. However, this symbiosis is short lived. Very quickly, the student is asked to abandon the narrative discourse and pick up the propositional form, to use algebraic symbolism in its static interpretation, a demand expressed by Solomon and O'Neill (1998): "Mathematics can be embedded in a variety of texts in a variety of styles from dialogue [...]. This, however, is quite distinct from linguistic features constitutive of mathematical discourse itself: mathematics cannot be narrative for it is structured around logical and not temporal relations" (p. 217). Solomon and O'Neill reject the idea that "Children could re-invent mathematics by abstracting it from the world around them" (p. 217): for them mathematics is a strict social practice, with distinct rules of genre. This requirement, they readily admit, gives rise to a dissonance between the students' interpretation of the symbols and the one expected by their teachers. But is the static, disembodied form a necessary feature of mathematical language? The historical examples above suggest that there are other possibilities for a notational infrastructure for mathematics, and that the static formalism may have been optimised for static media (Kaput et al, 2002).

Healy & Sinclair (2007), in a studied response to Solomon & O'Neill (1998), argue that the latter's position overlooks the possible role of narrative in more personal acts of understanding. Many testimonies show an alienated experience of mathematics. This barrier can be breached by allowing space for learners' personal narratives, relating mathematical meanings to their own experiences and reflecting on their individual learning trajectories. Solomon & O'Neill (1998) see this as a debate between "an emphasis on authorship and creativity versus an emphasis on understanding genre" (p. 210). Yet one could argue that the chasm runs deeper: it is a question of what is the mathematics we wish to teach: a practice, or a phenomenon, a noun or a verb? Should children learn to *see* mathematics or to *do* mathematics? Perhaps both, but then – which comes first? Solomon & O'Neill (1998) present an example of two texts by William Rowan Hamilton, one from his published letters and the other from his more formal publications. But while they see the former as literature and the latter as mathematics, Healy & Sinclair (2007) see one as a window on the process of doing mathematics, and the other as the output of that process. They inspect various reports of mathematicians' personal experiences, and find that all have temporal structure and carry a strong sense of voice.

Bruner (1991) distinguishes between scientific knowledge, which is organised by logical principles, and cultural assets, what he calls "folk psychology", which he argues are "organized narratively" (p. 21). He calls for a shift of attention which would honour both forms of

knowledge. Nevertheless, this distinction does not preclude representing and learning of scientific and logical knowledge in narrative forms. Indeed, Bruner (1986) notes two modes of thinking, mapped to two genres of narrative – paradigmatic and imaginative. Paradigmatic narrative is top down, seeks generality and demands consistency. Imaginative narrative is bottom up, seeks specificity and demands coherence. Several researchers have suggested that in order to provide learners with tools for coping with unfamiliar problems, they need to share the experiences of those who possess such tools. Burton (1996) argues that this points to a need to facilitate learners' authoring of their accounts of how they came to know mathematics. These narratives are personal, i.e. imaginative, as they are general and paradigmatic. Nardi (2008) makes a dual use of narrative in her study of university level mathematics: first, as a lens for interpreting students' and teachers' work, and then as a format for reporting her findings. Livingston (2006) calls for an educational approach to mathematical proof that acknowledges the context in which proofs are constructed and the personal path taken by those who prove. Although he does not refer explicitly to the notion of narrative, we find many parallels in his situated view of proof. Morgan (2001) also distinguishes between mathematical 'facts' and 'activity'. Inspecting several mathematical texts, she identifies elements of temporality and personalisation, similar to the constituents of narrative we noted. Morgan argues that rather than rejecting such style as 'inappropriate', we should ask: what are the criteria for a personal narrative to qualify as an account of mathematical activity?

The model of narrative comprehension presented in Chapter Four provides further support for these arguments. We saw how developing a theory-of-mind is fundamental in narrative comprehension. Likewise, if we want children to learn to think and act like mathematicians they need to develop a theory of mathematical mind: the ability to imagine "how a mathematician approaches this problem", and what better way than through mathematical narratives? Furthermore, our minds are geared towards extracting causal structures from the temporal sequencing of a narrative. "The queen died, then the king died" is transformed to "if queen dies, then king dies" (with apologies to E.M. Forster, 1927). So, counter to Solomon & O'Neill's claim, it may be possible that children will invent mathematical structure by abstracting it from the *narratives* around them – be it those they receive, or those they construct. Indeed, O'Neill et al (2004) find a surprising correlation between children's performance in generating narratives at the age of three to four, and their mathematical abilities two years later. This correlation is unique: general language skills were neither predictive of mathematical achievement nor were narrative skills predictive of spelling skills or general knowledge. They suggest that the same skills which underlie narrative comprehension form the basis of mathematical thinking: inference of relationships and logical chains.

This approach is in agreement with many of the assertions of the communicational approach (Sfard, 2000; 2001; Sfard & Kieran, 2001; Kieran, Forman & Sfard, 2002; Sfard & Lavi, 2005; Ben-Yehuda, Lavy, Linchevski and Sfard, 2005; Ryve, 2006), which sees learning as the "*process of changing one's discursive ways in a certain well-defined manner*" (Sfard, 2001, p 26), and consequently thinking as "*a special case of the activity of communicating*" (ibid, p. 27). The tension between vernacular narrative forms and mathematical forms of discourse has been discussed, under a slightly different emphasis by Paul Cobb and his colleagues (Yackel & Cobb, 1995; Whitenack, Cobb & McClain 1995). They describe the process of establishing *socio-mathematical norms* – forms of discourse that are regarded as valid in a mathematics classroom:

what counts as proof, what constitutes a valid argument, which questions are worth asking. One particular aspect of disparity is the relationship between the general and the particular. In narrative discourse the specific is used to reflect on the general, whereas mathematical statements use the general to make claims on the specific.

It is also worth noting the distinction Scardamalia and Bereiter (1987) make between knowledge telling and knowledge transforming. Observing the difference between novice and mature writers, Scardamalia and Bereiter identify two modes of text generation. Novice writers exhibit the more simplistic Knowledge Telling model, which provides a step-by-step mechanism for externalising known facts retrieved from memory in response to prompts. This process may produce syntactically well-formed narratives, but they are likely to be shallow in the meaning they convey. By contrast, expert writers engage in a process of knowledge transformation. In this process text is generated with the explicit aim of establishing a shared understanding of domain knowledge between the writer and the reader. Thus, this process necessitates a coordinated manipulation of both a domain knowledge model and a corresponding rhetorical model, resulting in their co-evolution. In other words, in this mode constructing narratives not only transfers knowledge between the narrator and her audience – it transforms the knowledge structures of both. The question that arises is how do we guide learners away from knowledge telling and towards knowledge transforming?

Sfard & Prusak (2005) see the notion of *identity* as pivotal: learning is directed by the need to transfer oneself from an actual to a designated identity. Identities are defined by the stories, narratives, we tell ourselves about ourselves. It is interesting to note the parallel to the neurological evidence presented above linking narrative comprehension to theory-of-mind (Mar, 2004). Sfard & Prusak see a process by which actions (“A lifted a table”) are reduced to assertions (“A is strong”). Such a process would be crucial to our formation of a theory-of-mind, allowing us to draw inferences about future actions of the subject. Referring to Wittgenstein, Sfard and Prusak warn against understanding narrative as a recount of activity. Narrative originates in activity, but it is an independent activity in its own right, aimed at structuring our understanding of the world, and in particular constructing our identity.

With this in mind, it seems natural to return to Wittgenstein’s notes on the nature of mathematics: Mathematics is invented to suit experience and then made independent of experience (Wittgenstein, 1939 / 1989, p. 43). The theory of situated abstractions suggests that this is an idealization: our concepts may be abstracted further and further away from experience, but they are never detached. In this trajectory from action to abstract, narrative plays a key role – both in its external and in its internal manifestations.

Turning to the design of technology-enhanced environments for mathematics education, the above observations suggest that we should seek representations which embody narrative qualities, and provide environments and activities which afford narrative expressions as a step towards formalisation and abstraction. Sinclair, Healy and Sales (2009) demonstrate how such considerations are manifested in the case of dynamic geometry environments (DGEs). The nature of DGEs allows learners to observe the unfolding behaviour of mathematical objects over time, thus provoking learners’ construction of narratives in which these objects are the

protagonists. Such narratives should be embraced as part of the pedagogic path, if learners are to make mathematics a “land of their own” (Healy and Sinclair, 2007).

Dynamic geometry environments, and similarly computer algebra systems, place the learner in the position of a director in a theatre of mathematics: orchestrating props and actors and sequencing events to tell mathematical stories. Programming is unique in this context, because it is a form of production and of expression at the same instance. By describing events, the programmer engenders them. This description is narrative in the sense that it is contextual, sequential and carries implicit meanings. Yet it is mathematical in the strictness of its grammar and in its lack of ambiguity. Thus programming carries a potential to serve as a powerful mediating form between intuition, action and abstraction. In particular, I argue that some forms of programming maintain a stronger sense of narrative structure, and should be exploited in this capacity. The *Streams* pattern is one such example. Section 6.3.3 sets the scene of programming as a medium for learning mathematics, and then introduces the Streams pattern.

6.3.3 Computational representations

Section 6.3 opened with a few examples which demonstrate how representational systems emerge via a long process of abstraction from the concrete objects, and how the evolution of these systems is linked to the social and individual construction of mathematical concepts. Such a process is not limited to primitive or prehistoric cultures. The evolution of computer culture is rich with examples of such dynamics. Symbols – words or icons – are invented to represent a specific object or action. Over time, these symbols are used as metaphors for other entities. This “borrowing” of a symbol expands its meaning, and inevitably abstracts it. As a ubiquitous example, consider the folder icon used in most graphical interfaces.

Computer scientists have always recognized the potential of computer programs as a representation of human thought. In fact, a common lore is that programming languages are designed to communicate ideas among humans, not computers:

Just as everyday thoughts are expressed in natural language, and formal deductions are expressed in mathematical language, methodological thoughts are expressed in programming languages. A programming language is a medium for communicating methods, not just a means for getting a computer to perform operations – programs are written for people to read as much as they are written for machines to execute. (Abelson and Sussman, 1987)

Computer programming offers a unique expressive power that has never before been available in human history. In previous eras, representational systems evolved through a long historical process. As Abelson and Sussman (1987) explain, the computational medium allows us to *design* representational systems as best fit our needs. The choice of representation has no effect on the computer. It may, however, have profound effects on the human reader (as well as the writer) of the program. As Abelson and Sussman note, *some* very challenging problems become trivial once the right language is chosen – or created. (Note the emphasis on *some*; many problems are inherently hard – regardless of representation).

The linkage between computer programs and human thought was taken a few steps further by the AI community in the late 1960s. Computer programs were accepted as a useful metaphor for modelling the human mind. This is what Searle (1980) criticizes as: "... *the claim that the appropriately programmed computer literally has cognitive states and that the programs thereby explain human cognition*", which is the *Strong AI* assumption. This assumption infiltrated the cognitive sciences, where computational models of mind have become common.

Another side effect of the AI community's contribution was the emergence of functional and logic programming languages in education. Most notably, LOGO was developed in BBN and MIT by the same people who were promoting LISP and SCHEME (Papert, 1973; Abelson et al, 1975). LOGO marked the dawn of what later came to be known as the constructionist approach (Papert, 1981; Noss & Hoyles, 1996), which highlights the epistemic potential of new representational systems, derived from the use of new technologies (Papert & Harel, 1991; Kaput, Noss & Hoyles, 2001; diSessa, 2004), but shift the focus to programming as a medium of *activity*, following the agenda set by Papert: "I believe with Dewey, Montessori and Piaget that children learn by doing and thinking about what they do" (Papert, 2005). And elsewhere:

Constructionism [...] shares constructivism's connotation of learning as "building knowledge structures" irrespective of the circumstances of the learning. It then adds the idea that this happens especially felicitously in a context where the learner is consciously engaged in constructing a public entity, whether it's a sand castle on the beach or a theory of the universe. (Papert & Harel, 1991, online)

Constructionism turns from the analytical focus of constructivist and socio-cultural theories to a prescriptive approach. If learning occurs in activities, by interacting with objects through mediating instruments, then why not create such activities, objects and instruments that would best facilitate it? Given a topic, one may think of creating an environment with artefacts or objects that have some relationship to the topic and where learning may occur by exploring the environment. Such an idea is at the core of the concept of the Microworld, introduced by Minsky and Papert (1971). This is an environment that is built around a given domain, which has to be explored by interacting with the program. A detailed history of the concept of microworlds can be found in Noss and Hoyles (1996).

The constructionist approach has bred various educational programming languages and environments such as Boxer (diSessa & Abelson, 1986), StarLogo (Resnick, 1996; 1997), NetLogo (Wilensky & Reisman, 2004; Blikstein, Abrahamson & Wilensky, 2005), ToonTalk (Kahn, 1996; 2000; 2004), Squeak (Ingalls, Kaehler, Maloney, Wallace & Kay, 1997; Masuch & Rüger, 2005), Stagecast (Cypher & Smith, 1995), Agentsheets (Repenning & Fahlen, 1993), MOOSE crossing (Bruckman, 1997), Alice (Cooper et al, 2000; Conway et al, 2000) and Scratch (Resnick et al, 2003; Maloney et al, 2004; Monroy-Hernández and Resnick, 2008). Guzdial (2004) offers a thoughtful comparative discussion of several of these languages, mainly descendents of Logo. Kelleher & Pausch (2005) offer an extensive taxonomy of many other languages.

The demonstrator study reported here uses ToonTalk as a platform for exploring mathematical ideas in the domain of number sequences. ToonTalk stands out in its computational model, its programming interface and its execution model. Computationally, ToonTalk is a concurrent

constraint-based programming language. Concurrent means that multiple code elements can be executed simultaneously. Constraint-based means that these elements are predominantly expressed as rules, identifying a particular configuration of objects in the world and a sequence of actions to execute, once such a configuration has been observed. In terms of its programming interface, ToonTalk is a language and a programming environment designed to be accessible by children of a wide range of ages, without compromising computational and expressive power. Following a video game metaphor, the programmer is represented by an avatar that acts in a virtual world. Through this avatar the programmer can operate on objects in the ToonTalk world, or train a robot to do so. Training a robot is the ToonTalk equivalent of programming. The programmer leads the robot through a sequence of actions, and the robot will then repeat these actions whenever presented with the right conditions. Finally, ToonTalk's standard mode of execution is animated: the robot displays its actions as it executes them. ToonTalk's unique qualities make it an interesting candidate to support a new genre of mathematical narrative. The structure of routines (or 'robot's training' in the ToonTalk terminology) displays narrative features, with a predefined context, a sequence of events, and a desired outcome. The method of programming has, arguably, elements of narrative construction: the programmer defines the context and then walks the 'robot' through the moves. Observing code in execution provides for possibilities of narrative comprehension, unparalleled in other languages where only the effects of the execution are visible.

The affordances of ToonTalk, or any other technology, do not guarantee any manifestation of desired events. A window's handle affords opening, yet people may choose to turn on the air conditioning. In order to prompt a particular use of technology, and hopefully a desirable learning trajectory, one needs to supplement the medium with the design of tools and activities within it. The next section introduces the *Streams software design pattern* which suggests a narrative representation of number sequences in a programming medium.

6.3.3.1 Streams: Computer-aided explorations of sequences

Traditionally, the predominant representation of a mathematical sequence in computer programming was as a *list*: an ordered set of items. This abstract definition needs to be implemented with respect to the particulars of the chosen language. Common educational implementations attempt to capture the essence of the formal definition of a sequence, as a function $f:N \rightarrow R$. These representations are static – at any given point in time, their content is fixed. Furthermore, while lists are not limited in length and can be extended on the fly, any actual list at any given time is, of course, finite. This could be a source of epistemic conflict. While we talk about infinite lists, the objects we manipulate are inherently finite, and the algorithms used are geared towards finiteness. Furthermore, emphasising the $f:N \rightarrow R$ formalisation of sequences risks a conflict with students' recursive intuition. Most of the attempts to use interactive computer representations of sequences were aimed at notions of infinity and convergence (Kidron, 2002; 2003; Li & Tall, 1993; Guin & Trouche, 1999; Giraldo, Carvalho, & Tall, 2003). It is not surprising that these studies report mainly on experiments in using CAS (Computer Algebra Systems) and graphing. Although visualization by graphs is usually helpful, Giraldo et al (2003) note that resolution limitations of displaying graphs on screen might hinder conceptions of continuity. The focus on advanced topics also leads to assumptions about prior knowledge. For example, Kidron (2002) uses Mathematica as a modelling environment.

Although the results are impressive, the tool seems to be suitable only for more advanced students.

Sacristán (1997) proposes an alternative approach that uses recursive programs as a representation of infinite sequences. She focused on establishing intuitions by visualising the sequences, an approach which proved successful. However, she stresses the need to supplement this approach with alternative representations in terms of numeric values. This is achieved by allowing students to manipulate the code that instantiated the visual representation directly. Seeing the visualisation and the sequence unfold together gradually, allowed the students to consider the sequence as a process and object and helped them to identify local structure. Such a dual view, as noted in section 6.2.5, is fundamental to mathematical thinking. Sacristán's design called for task specific programs, which balanced functional richness with code simplicity, so that students could observe the visualisation and then tweak the code that produces it. Unfortunately, simplicity is often achieved at the price of generality. A function created to display one sequence cannot be used to plot another. Arguably, students can overcome this by a minor code change. Yet this would require them to deal with the code's inner workings, which may be tangential to the learning goals. It is possible to design components that can be used "off the shelf" and modified when the need arises, but this entails significant engineering effort.

One possible alternative is offered by the *Stream* design pattern, commonly used in software engineering. A stream is a dynamic representation of a sequence. It generates the terms one by one, as these are needed. In object oriented languages (such as JAVA or C++) it is implemented by an object with a `read()` method (function) which retrieves the next term every time it is invoked. Shapiro (1988) explains why 'streams' are most useful in concurrent systems, those in which many processes are executed in parallel. Streams provide a structured mechanism for dividing work between processes using an assembly line metaphor: every process sends out a stream of outputs, which are passed as a stream of inputs to the next. While process II is busy, say, with the 5th term, process I is already generating the 6th, and process III can work on the 4th. Streams are used as a fundamental mechanism in UNIX for communicating between applications and operating system processes (SUN, 2005) and are the primary input-output framework in Java (Eckel, 2002). Abelson & Sussman (1996, section 3.5) highlight the temporal dimension of streams:

If time is measured in discrete steps, then we can model a time function as a (possibly infinite) sequence. ... we will ... model change in terms of sequences that represent the time histories of the systems being modelled. To accomplish this, we introduce new data structures called streams. From an abstract point of view, a stream is simply a sequence.

Abelson & Sussman propose streams as a representation of a specific category of sequences, namely discrete time functions. Yet from an embodied perspective (Lakoff and Núñez, 2000) such sequences form the basis of our understanding, and are projected by conceptual metaphor to the broader class. Here lies the possible epistemic value of streams: in providing a time-aware representation for sequences, it may be possible to reintroduce narrative. Nevertheless, the considerations in designing educational tools are radically different from those in systems programming. For example, whereas the designers of the operating systems would value the

opacity of their Stream implementation, we would emphasise learners' need to see the mechanism's inner workings. Such issues are addressed by adapting the pattern to an educational context, as discussed in Chapter Eight.

6.4 Conclusions

Section 6.2 opened this chapter with a discussion of pedagogical and epistemic issues concerning number sequences. I focused on the difficulties of formulating a structural view of sequences, and related them to the issues of process-product duality, and recursive vs. closed form. The questions that emerged from this discussion were: how much of the difficulty is inherent in human cognition, how much is a consequence of a particular representation and how much is a result of the dominant pedagogic tradition? These questions led into an inquiry into the relationship between communication, representation and meaning in mathematical learning. The communicational paradigm offered a solid framework for this discussion, but has some limitations when approached from a design perspective. These limitations are addressed by the concept of situated abstraction and its implications. The amalgamation of these two approaches led me to consider narrative as a fundamental social and cognitive epistemic force. In mathematical narratives (6.3.2) the context and plot are mathematical processes. Protagonists may be the humans engaged in these processes or the mathematical objects participating in them. The media by which narrative is communicated can transcend the verbal to include graphical or coded elements.

Bringing narrative into the domain of constructionist learning and educational programming raises questions regarding its manifestation in computer-based representations, specifically how to represent number sequences. I propose the Streams design pattern as a promising candidate.

The different approaches reviewed above appear to be divided between those which highlight language (e.g. the communicational approach) and those that focus on interaction with objects (e.g. constructionism). Yet this division is misleading: if we want to guide learners along the path from the perception of their experiences to the articulation of structured knowledge, we need to consider the source of experience as well as the drivers of extracting knowledge from it. The union of the various approaches suggests a need for a synergy of construction, communication and collaboration. Construction provides learners something to talk about, collaboration gives them a reason to talk about it, and sustained communication, guided by a teacher, establishes norms of mathematical language.

Chapter 7 Design Narratives

This chapter responds to Aim 3 identified in Chapter three: to apply the methodology in the problem domain and demonstrate it's potential by producing a contribution towards a pattern language for technology enhanced mathematics education. This chapter presents a series of design narratives, as a first tier of interpretation of the empirical work. These narratives are the basis for the design patterns in Chapter 8, but also provide direct insights into the practice of techno-pedagogic design.

7.1 Introduction

The design narratives (DNs) presented in this chapter form the main empirical content in this thesis. As discussed in section 4.4, these design narratives are the first tier of interpretation, not the data themselves. Following the distinction in section 4.4.1, these include researcher narratives (RNs) and participant narratives – in this case learner narratives (LNs). Each narrative recounts a particular incident, where this incident could span a single session, a few weeks or a few months. Some of the narratives overlap and some zoom in on a particular aspect of others.

The demonstrator study consisted of three major activities, each one examined through three major iterations. As discussed in section 5.4, the data collected in the course of these experiments was processed into an extensive set of narratives, from which a smaller set was selected and elaborated. Table 13 (in Appendix III) lists the data sources drawn upon in the construction of narratives, and the sources section in each narrative notes the specific sources it is based on. The RNs and LNs were selected to cover the three activities and three iterations, from a researcher and a learner perspective. Consequently, they are organised in four groups. The first three groups of DNs cover one activity over three years, the last focuses on the design of the technological infrastructure to support the three activities. The process by which the narratives were derived from the data follows the procedure enumerated in Section 5.4.1. The structure of the narratives complies with the common template defined in Section 5.4.2. Table 13 lists the data sources used in the construction of these narratives.

7.2 Basic Sequence Activities

7.2.1 RN 1: Streams – tools and activities

Overview

This RN recounts the first two iterations of designing activities in the domain of number sequence, from the initial prototyping during the first year of the study, through preliminary classroom trials and preparation for the second year. The initial experiment with computing π prompted the use of the STREAMS¹² design pattern as a useful metaphor for exploring number sequences. Although the *computing π* activity was never realised, STREAMS proved to be useful for a range of other activities.

Sources

WR1, D3.2.1, WR2, Monkeys

Situation

This narrative covers the period of autumn to spring, 2002. The first part of the work described here was mainly desk research, done at the WebLabs office. The initial experiments were conducted with two to six children (group I), age 10 and 11, either at the WebLabs office or at a north London city learning centre.

Task

Identify learning challenges in the domain of numbers sequences, and devise tools and activities which will lead learners to engage with these challenges in a meaningful and educationally effective manner.

Actions

In autumn 2002 I began preliminary prototyping (“iteration 0”) of activities in the domain of number sequences. The motivation for these activities was to use number sequences as an entry path into issues of cardinality and randomness. For example, cardinality could be introduced by comparing the length of the sequence of natural numbers to that of the odd numbers, and randomness by searching for patterns in a random sequence.

As a first activity to experiment with, I chose constructions of the number π . The idea was to start the activity with guided programming of the classic formula (the Madhava–Leibniz series):

$$\pi = 4*(1 - 1/3 + 1/5 - 1/7 + \dots)$$

¹² I use STREAMS to refer to the software design pattern, and stream or streams to refer to concrete objects implementing it.

Next, students would be challenged to search for “better” formulas and program them. The motivation for this theme was to approach the topic of converging series from an unconventional angle: through a particular series that was designed to converge to a desired value. From here they could branch off to many other activities, e.g., searching for patterns in the digits of π ¹³. My intention was to provide learners with tools and instructions for implementing the Madhava–Leibniz series. As a first step, I set out to implement it in ToonTalk myself. This task proved more onerous than I expected, by and large due to my inexperience in ToonTalk.

I decided to share the robot I constructed, and the difficulties I encountered, with my colleagues using an initial prototype of the WebReports system. The primary motivation in publishing this report was to elicit feedback from more experienced peers, as part of my own learning process. An expected side benefit was to gain first-hand experience with the learning dynamics envisioned for WebLabs participants. Figure 10 shows the robot as it was embedded in the report.

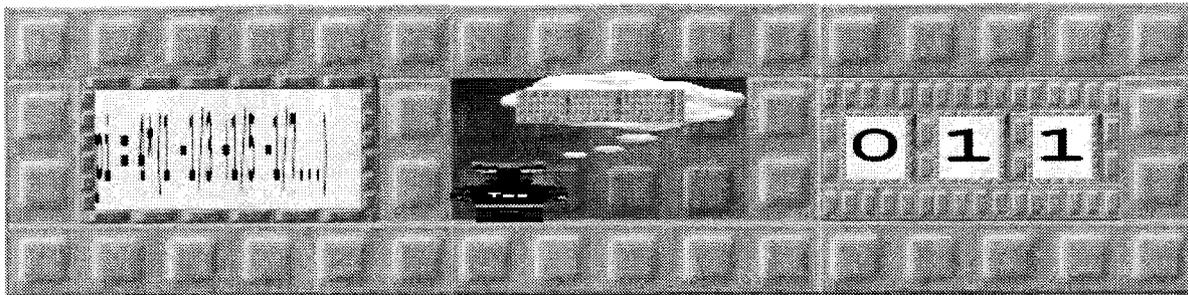


Figure 10: π robot, as embedded in the WebReport (WR1)

Dr. Ken Kahn commented on the report, suggesting a different approach, which he attributed to systolic programming in concurrent Prolog (Shapiro, 1988). The essence of this solution was to break down the problem into simple, modular tasks performed in parallel by independent robots (or teams of robots), in effect implementing the STREAMS design pattern (6.3.3.1). The discussion that followed among the researchers, via comments on the report, highlighted the potential of this approach in enabling learners to see and manipulate a sequence as a whole rather than focusing on term-to-term transitions. Consequently, I adopted STREAMS as a central metaphor, and began experimenting with ideas for various activities. Two properties of ToonTalk made it highly amenable to STREAMS processing:

Concurrency: ToonTalk robots and teams of robots can operate concurrently and independently. This means that while one team is producing a stream of numbers, another can modulate it and a third can consume it. This is in contrast to most languages which are sequential, and would need a mechanism of passing control from one process to another.

Constraint-driven: ToonTalk robots repeat the actions they were trained to do as long as their constraints are met. This means that a robot trained to produce or consume a

¹³ In effect, these plans never materialized, but they paved the way to the successful designs of later iterations.

single item will by default do the same for an infinite stream, until forcefully stopped. This is in contrast with other languages, where loops or recursions need to be coded explicitly.

My plan was to provide learners with a toolkit of components for streams of numbers and characters, and a set of activities using these to explore the selected mathematical themes. Example tools included:

- **Constant number stream generator:** generates the same number over and over again.
- **Function filter:** applies the same function to a stream of incoming numbers and sends out the stream of results.
- **Add-up:** receive a stream of numbers and output the stream of their partial sums.
- **Random character generator:** initialised with a box of characters, repeatedly select a character from the box at random and copy it to the output stream.
- **Graph display:** receive a stream of numbers and display its terms on a bar graph.

Example activities included:

- Generate the natural numbers using the Constant and Add-up tools.
- Generate the odd / even numbers
- Find your name in a random character stream

I aligned my design with the standard set by the Playground project (Kahn, Noss, Hoyles and Jones, 2006; Goldstein, Kalas, Noss and Pratt, 2001) to provide these tools as *anima-gadgets*: modular ToonTalk components, which hide their functionality behind a graphical or animated representation, and can be combined by physical gestures, such as touching one to the other.

The immediate consequence of this design was that ToonTalk's concurrency, an enabling factor for the Streams approach, now became a challenge: each anima gadget would process the terms at its own pace, causing bottlenecks at some points and overflows at others. In order to address this issue I needed to add control elements which regulate the flow of data through the chain of stream components. I also needed to add code to manage the connecting of gadgets. Both these issues were tangential to the core functionality of the tools, but required significant effort and led to significantly more complex code.

Students using the prototype tools in exploratory experiments found them confusing: they needed to follow too many simultaneous events, the control mechanisms were unintuitive, and the chain of data flow from one component to another was hard to trace. Some of these problems could be addressed by *flipping* the anima gadgets: exposing their inner working and following their step-by-step execution. However, the complexity of the code required to address the technical issues made it inaccessible to novice programmers.

After considering various forms of implementation, I realised that the most straightforward way to ensure that students understand the working of the tools was to let them construct them from scratch. Instead of providing coded components, I provided the design pattern for coding them. This was done by setting a task, allowing students to discover the naïve solution, analysing that solution for its deficiencies, and then introducing STREAMS as a superior method of solution.

Results

The initial idea of computing π served as a motivation for my explorations, but as these led me in different directions this idea never materialised. Although I did implement the algorithm, the result was unusable due to problems with ToonTalk's representation of large numbers. A discussion of these problems with the project team eventually led to innovations in ToonTalk's design (Kahn, 2004), which provided a satisfactory solution. Yet by that time I had moved on to a different design of activities.

The main outcome of the computing π design was the identification of STREAMS as a central metaphor for number-sequence activities. However, my first attempts at implementing STREAMS in ToonTalk, and designing a set of activities using it, resulted in failure. Creating prototypes of the sequence/stream tools was simple enough. The difficulties arose when I tried to convert these into ToonTalk "anima-gadgets". While the technical challenges were surmountable, their resolution prompted a new set of obstacles in terms of activity design. Initial trials indicated that although learners found each individual tool simple enough, they struggled with constructing ensembles or interpreting the output from such constructions. The difficulties were both technical and conceptual. The conceptual challenge of understanding the underlying mathematical structure was compounded by the lack of programming skills. As a result, the activities gravitated towards simulation rather than construction: learners were instructed to assemble specific parts in a specific manner, observe the outcome and interpret it. Such activities failed to excite the learners, and did not seem to offer a valuable learning experience.

Acknowledgment of these issues led to a change of design strategy: instead of providing learners with ready-made tools, I should provide them with the design knowledge to create their own tools. However, at first this approach failed as well and was almost abandoned. Careful examination revealed that the fault was an issue of usability and interface design rather than techno-pedagogical design (see RN 6: Eager Robots).

Reflections

The apparent outcome of my initial attempts was a series of failures, concluding with a very partial success. Arguably, these failures paved the way for later successes, as described in the following design narratives. Yet in retrospect, the question of their inevitability presents itself. By and large these failures were part of my learning trajectory: I had extensive experience in programming and modest experience teaching, but no previous knowledge of ToonTalk and no record of teaching middle school mathematics. My inexperience with ToonTalk led me to "program with an accent": forcing my habits from other languages on an environment where they were ineffective. On the other hand, it also allowed me to bring a fresh view and stretch the boundaries of the language. My explorations in ToonTalk were driven by a naïve design of activities. Only after I learnt to "think in ToonTalk" could I identify pedagogically effective designs.

Section 6.2.2 noted the tension between potential and actual infinity, while section 6.2.5 related it, and other difficulties students have with number sequences, to the process-object duality of sequences. The idea of representing number sequences as streams emerged as a response to

these issues, drawing on the observations in 6.2.4 which highlighted the recursive nature of the intuitive conception of sequences.

Any manifestation of infinity in a computational medium is inevitably potential (as opposed to actual), since the computer's memory and processing power is finite. The STREAMS pattern is as close as possible to this intuitive concept of infinity; it will continue providing terms indefinitely until it is interrupted. It can also possibly provide a bridge towards the conception of actual infinity: since it is not possible to count the length of a stream (as is possible with lists), the stream object itself represents all terms of the sequence – *ad infinitum*. Again, the power of STREAMS is not in the representation of any specific infinite process – but in the possibility of combining and manipulating infinite processes.

Although the implementation of STREAMS in the activities above utilized specific features of ToonTalk, there is nothing about STREAMS inherently specific to ToonTalk. All these activity designs could be implemented in a variety of languages with minor variations. In fact, in order to make the most effective use of Streams for exploring sequences, it would perhaps be best to develop a new language, which would allow learners to construct streams by manipulation of visual representations. The message is in the design pattern of streams as a method of representing number sequences, and in its notable educational value, not in the details of implementation in a specific environment. This value is derived from its affinity to intuitive conceptions, and from its affordance for decomposing complex processes.

This design narrative offers an example of the fluidity of design which characterised much of the early iterations of the demonstrator study – a shift in the nature of activities and tools from inception to implementation. It begins with an attempt to design an activity using the calculation of the number π as an arena for exploring issues of randomness and cardinality, and ends with an initial framework for using the Streams pattern to support exploratory activities in number sequence. None of the original plans were realised, yet the overall aim remained constant and in the end was satisfied. Even the final set of tools and activities was fluid: it was a subset of a broader range of options, dictated partially by coincidental circumstances. Such fluidity is common to many design studies. Consequently, it was the patterns and principles that emerge as a sustainable outcome from the experiment, not the concrete designs.

The parallels between my research process and the learning trajectory of the participants in the study appear in both directions. I had used the tools and social practices designed for learners to support my enquiry, and then guided them in conducting their own miniature design study. In itself, the link between normative and genetic epistemology is not new; many educational initiatives aim to direct learners to act and think as “young scientists”. Yet this common approach assumes that we, as scientists, know the proper way of constructing knowledge and the best children can do is to imitate us. What emerges here is the possibility that the new possibilities afforded by technology may change both the genetic and the normative epistemology, and both should evolve in tandem.

In line with the tradition of constructionism I tried to provide learners with a *microworld* for learning: a compact environment, populated with selected tools, carefully designed so that through using them in exploratory activities learners will engage with complex concepts. In the end, the tools I provided were not the gadgets I had planned to construct, but the design

patterns by which learners could construct their own. This suggests yet another parallel between my mode of research and the learners' practices, which needs to be explored further: can patterns reduce task complexity for researchers and learners alike?

A caveat is in order. This narrative provides an account of the preliminary iteration of the demonstrator study. The data collected at that stage was sparse and inconsistent. It serves mainly to illustrate the research process and flag some conjectures for further exploration, and should be read mainly as a backdrop to the following narratives.

7.2.2 RN 2: Final Form of Basic Sequences Activities

Overview

In their final form, the basic sequences activities combined construction, guided exploration and discussion, utilising STREAMS and the webreports system to facilitate learning of complex concepts related to number sequences.

Sources

D8.2.1_8.3.1_Sequences, AppG_Sequences(Yishay).v2.02-03, sequences_yishay02-03, WR3, WR4, WR5, WR6

Situation

This narrative recounts iterations 2 and 3 of the design experiment, from autumn 2003 to spring 2005. It involved approximately 25 children in two sites in London (groups II, III, and IV). Groups II and IV participated as an extra-curricular activity, and group III in lieu of their ICT class.

Task

Design a set of activities using STREAMS by which learners would:

1. Develop a non-algebraic language for describing, discussing and reasoning about polynomial (and maybe some non-polynomial) sequences.
2. Develop an understanding of:
 - a. The generation of number sequences,
 - b. The rules that sequences rely on and
 - c. How sequence generation relates to the ToonTalk environment (robots, birds, etc).
3. Gain some insight into the relationship between the structure of the programming constructs (e.g. the number of chained robots) and the type and complexity of the corresponding sequence.
4. Engage in a semi-structured conversation about the structure and qualities of numeric sequences in order to develop students' ability to make conjectures, suggest more than one solution to a problem, evaluate arguments and reason.

Design and develop the necessary resources to support these activities.

Actions

In 2003 I formulated a stable design of a sequence of activities aimed at establishing basic competencies required for exploring number sequences using the WebLabs infrastructure. The first tier of competencies included the programming skills for modelling sequences as number streams in ToonTalk and the use of the WebReports system as an individual and collaborative work area. These provided a basis for learners to develop their skills in manipulation and analysis of number sequences, and establish socio-mathematical norms.

Drawing on my analysis of the results from iterations 0 and 1 I minimised the set of ready-made components, and instead focused on a path by which students would construct their own. First, they constructed an Add 1 robot – this robot generates the natural numbers in situ by repeatedly adding 1 to a number in a box (Figure 11).

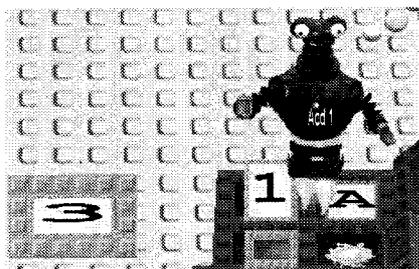


Figure 11: Add 1 robot: repeatedly add 1 to the value in the box

This solution is trivial to implement, but raises the problem of transience – the result of one round of computation is overwritten by the next. This problem serves as a motivation for introducing STREAMS (albeit not by name): learners train a new robot, called Add-a-Num with a two-hole box, one for the computation variable, and one for a bird. The bird is used to carry a copy of the results, so that these can be observed, recorded and manipulated.

Once learners completed this task, they published their robot in a webreport. They were provided with a template which prompted them to describe the robot and their experience in constructing it, and identify the range of sequences it can produce. The webreport was used both as a personal store of work in progress and as a means of sharing findings. Next, learners were asked to train an Add-up robot which received the stream of numbers from the first robot and outputs the stream of partial sums (Figure 12).

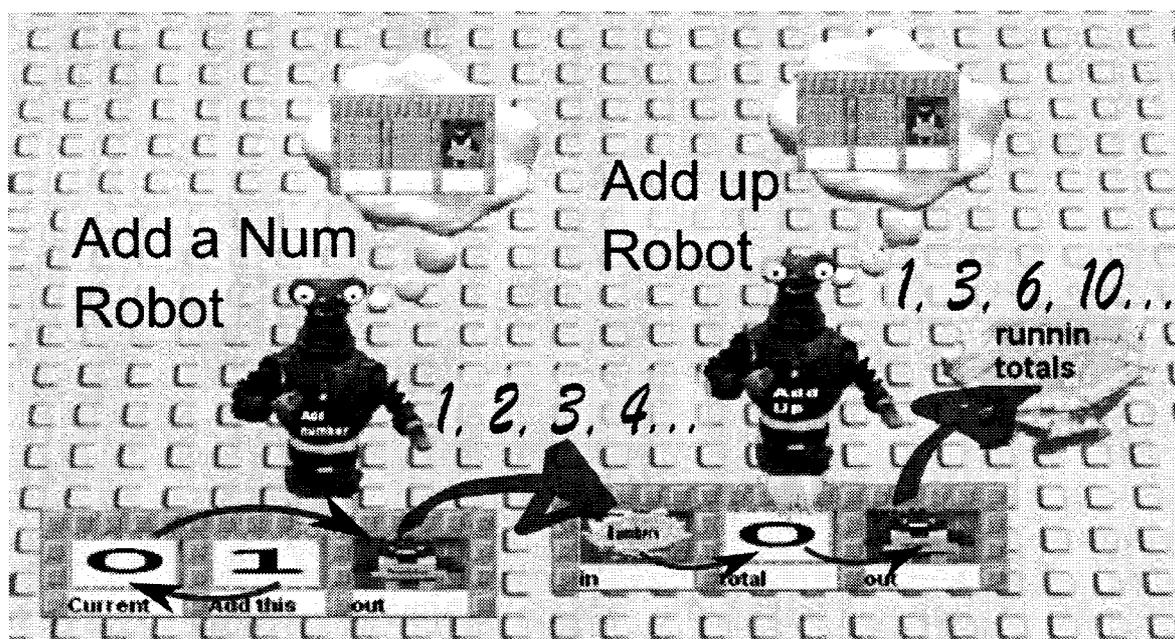


Figure 12: Add a num robot produces stream of natural numbers, Add up robot consumes that stream and produces a stream of their partial sums

This task was also supported by a template, but this time the template included a ToonTalk box with a partially assembled robot and task instructions (Figure 13). This box helped learners in the transition between environments and reduced the work they needed to do which was not directly related to the mathematical learning aims.

Training *Add up*

Train a robot to add up all the terms in a sequence. Call your robot *Add up*.

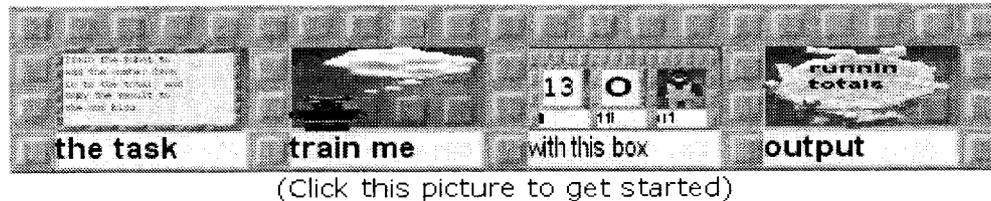


Figure 13: Add-a-num task box, containing partially trained robot and instructions.

After I tested these activities in 2003, they were also adopted by teachers in Portugal, Cyprus and Bulgaria.

In 2004 I augmented the *Add-a-Num* and *Add-up* activities with two assessment tasks and published teacher guidance notes. Most of my efforts that year were directed to understanding the impact these tools and activities have on participants' learning.

Results

The potential of STREAMS as a tool for learning about number sequences is demonstrated by students' ability to deconstruct complex processes and distinguish process, parameters and product, in the complexity of sequences they created and analysed and in the language they developed for discussing sequences (as demonstrated in section 7.3).

The extensive adoption of the specific design of activities, tools and resources (templates, teacher guides, programming aides) testifies to their effectiveness.

These activities served as a basis for the *Guess my Robot* and *Convergence* activities, reported in sections 7.3 and 7.4. I will therefore allow the results of those activities to reflect back on this one, rather than providing more direct evidence at this point.

Packaging tasks in ToonTalk boxes proved to be highly effective in streamlining activity flow. It also evolved into a powerful communicational convention. It set a standard for programming, packaging and publishing ToonTalk models, such that students could learn by example, avoiding the need for us to articulate conventions explicitly. Figure 14 shows Luminardi's *Add-a-number* robot, as he had published it in his report. Appropriating the scheme introduced in the active worksheet for his needs, he replaced box labels to describe how his tool should be used. Establishing such conventions as norms was crucial to the facilitation of communication in later phases. It ensured that models shared by one student would be readable by another. Furthermore, they were important, even from an individual perspective, as they allowed students to easily revisit work they had done, reuse tools they had constructed or reflect on the evolution of their ideas.

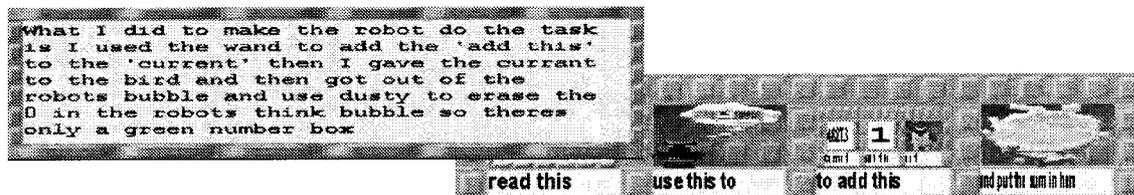


Figure 14: Luminardi's add a number robot

While these norms emerged at the level of programming style, they evolved into a standard of mathematical discourse. Students quickly got into the habit of attaching written descriptions to their models, labelled 'description' or 'read this'. Superpat313 published his robot in a similar way (see Figure 15). He followed the same convention (from the description in the leftmost hole to the nest in the rightmost). In contrast to Luminardi's product, the labels on Superpat313's report suggest that he was more focused on presentations ('what I did', 'my trained robot') than on usage.

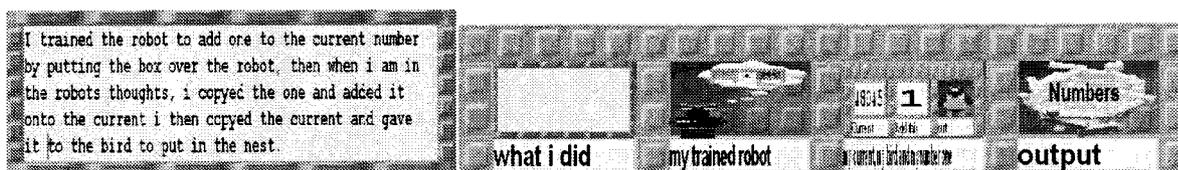


Figure 15: Superpat313's add a number robot and its description

As expected, most of the descriptions by students were procedural. Nevertheless, they constitute a first step towards students' reflective articulation of their work. For example, students used the box we provided as a package for the task to package their completed models. By doing so, they adopted the programming conventions we set without us having to impose them explicitly. Students appropriated the packaging scheme to their needs – changing the box labels and adding box cells (holes in ToonTalk terminology) as needed. This packaging convention goes beyond aesthetics. For example, it standardized the use of STREAMS, and prompted students to attach a reflective text to their model. Putting their ToonTalk constructions in a box prompted students to reflect on their constructions, their structure and functionality, thus transforming them from objects to play with to objects to think with (Papert, 1980; Turkle & Papert, 1992) and, along the social dimension of learning, to *objects to talk with*.

Reflections

Several factors suggest that the design of tools and activities was pedagogically and technically effective; the engagement and enthusiasm of students, their ability to confront complex mathematical concepts, the adoption of this design by my colleagues as a basis for other activities. This success seems to stem from a careful combination of the ToonTalk and WebReports environments. Yet almost no fundamental element of design relies on any specific feature of ToonTalk or WebLabs. Had I replaced either one of these tools, the technical details of implementation would change – but not its pedagogical core. This does not mean to suggest that any arbitrary tool would work, but that any tool which satisfies certain requirements could work. A rigorous examination of the transferable factors of success is the focus of Chapter eight.

7.3 *Guess My Robot*

7.3.1 RN 3: Guess My Robot

Overview

Guess my Robot (GmR) is a game in which players exchange number sequence challenges encoded as ToonTalk robots. It was conceived as a short recreational interlude between “serious” activities, but emerged as a pivotal element of the number sequence activities in six sites and four countries.

Sources

D8.2.1_8.3.1_Sequences, AppG_Sequences(Yishay).v2.02-03, sequences_yishay02-03, WR3, GmR1, GmR2, GmR3, GmR4, GmR-PS, EB2.3

Situation

This activity has been tested over 3 consecutive years, in 6 sites and 4 countries. The first experiment in 2002/3 included 8 students in London and Sofia. In 2003/4 the experiment expanded to 33 students from 6 sites (in different European countries), 15 girls and 18 boys, ages 10 (2), 11 (10), 12 (16), 13 (2) and 14 (3). Most of the data were collected from this iteration. The last iteration in 2004/5 involved 10 boys in London, age 13, and 5 children in Cyprus (age and gender unknown).

Task

To engage students in an activity that would provoke them to:

- Develop a shared mathematical (not necessarily algebraic) language for describing, discussing and reasoning about sequences.
- Gain a proficiency in manipulating mathematical tools to generate and analyze sequences.
- Acknowledge the duality of a sequence as a structure and a process.
- Confront fundamental issues of mathematical argumentation – conjecture, hypothesis testing, proof and equivalence.

To provide the tools and resources required for such an activity to be conducted simultaneously across several remote sites.

Actions

GmR is a game of number sequences. Players exchange challenges of number sequences coded as ToonTalk robots. Each player can participate as a *proposer* or as a *responder* (or both). A proposer trains a robot to generate a numerical sequence, and then publishes its first few terms as a ToonTalk “box” in a webreport. A responder downloads the proposer’s box from the webreport, and tries to reconstruct the robot that produced it. If successful, the responder posts

a comment on the proposer's webreport, with the solution robot. The proposer then confirms or refutes the solution. At this point the players can engage in a discussion regarding the challenge and its solution, or move on to the next challenge.

Guess my Robot was first played between children in four London and four in Sofia in autumn 2002. First, each group played a paper version of the game. Next, they played it within the group using ToonTalk robots. Finally, they used the WebReports system to exchange challenges between sites.

At that time, the WebReports system was at its second prototype, which was essentially a customised Wiki. Proposers created their personal game pages, and attached their challenge using the wiki's standard mechanism. Responders attached their response robot, and edited the page to add their comments behind the challenge text. Figure 16 shows an example game page, with a challenge from a challenge from Tedi in Sofia, a response from Ozzy in London, and a brief follow-up discussion.

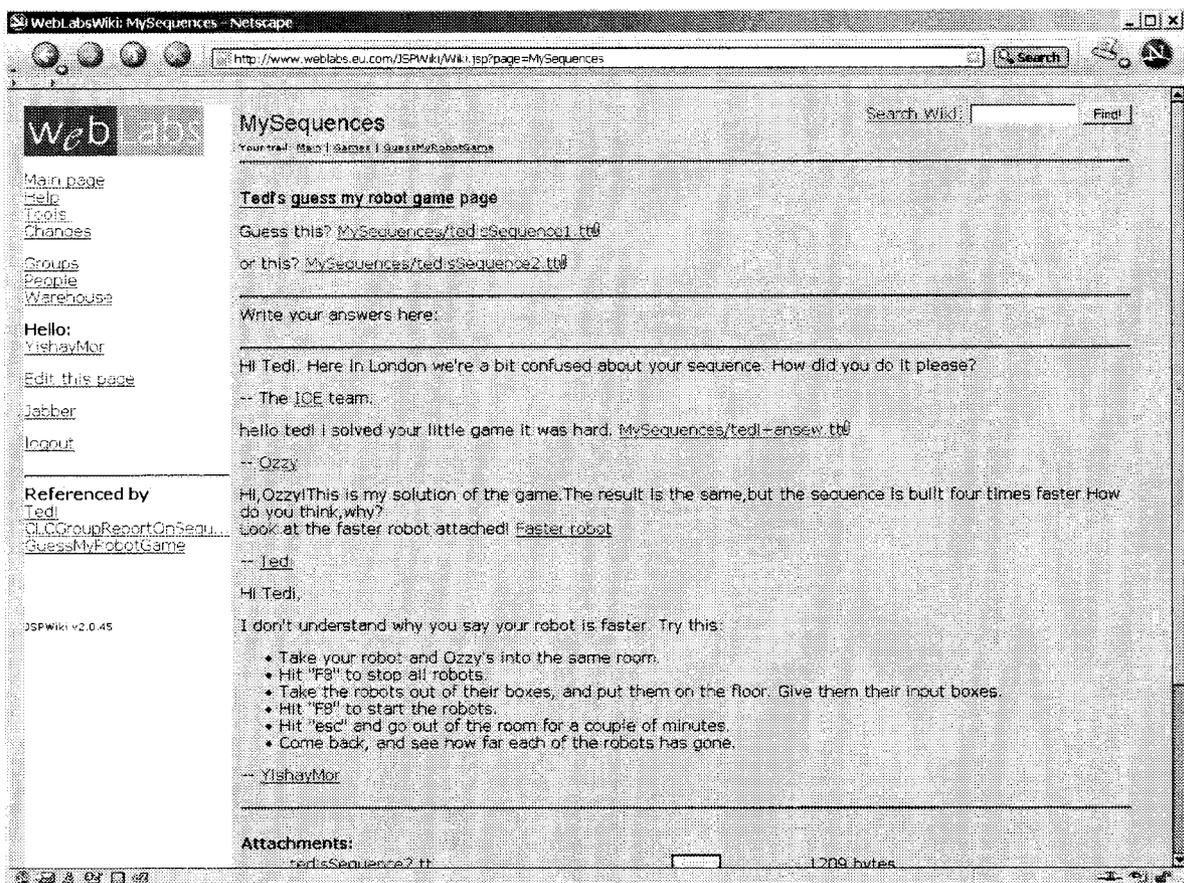


Figure 16: Tedi's GmR page, using the 2nd prototype of the WebReports system. The page links to his challenges, and includes responses from Ozzy in London and the ensuing discussion

In the original specification, GmR was listed as one out of eight activities. The initial trial indicated that it had a greater potential than expected, yet at the same time that it required sustained interaction to realise this potential. Consequently, I decided to expand this activity in subsequent years – both in terms of the time dedicated to it and in terms of the number of participants. This decision had two implications: first, that other activities would be deferred,

and second that the WebReports interface would need to adapt to the needs of Guess my Robot.

Giving more weight to GmR led to an overall path of activities as described in Figure 17. This path begins with the Add-a-Num and Add-up activities described in Section 7.2, follows through the GmR game, a collaborative reflective report, and an assessment task (Small Change challenge). It then branches to various advanced activities, such as the Fibonacci sequence, designed by my colleagues (Mousoulides and Philippou, 2005) and Convergence and Divergence, described in Section 7.4.

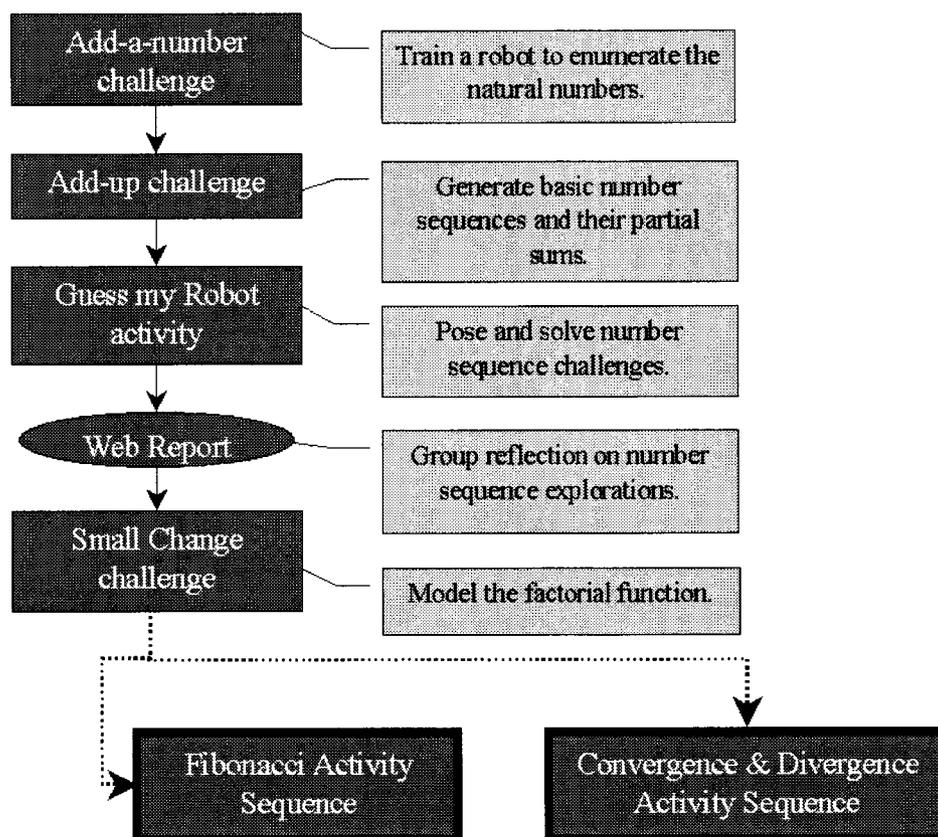


Figure 17: Number Sequences activities path.

Aiming for broader participation called for bespoke interface features. The flexibility of the Wiki had allowed me to respond immediately to observations by adjusting the game design. Yet this flexibility also implied that the game relied on players' compliance with conventions and technical abilities, and on my close monitoring of these. All these assumptions seemed unjustified with a larger audience. As part of the shift to the stable version of the WebLabs platform, two features were designed to support GmR: a template mechanism, and streamlined embedding of ToonTalk objects. Both are described in Section 7.5.1. Using these features, I created the GmR template shown in Figure 18, which proposers would use to post their challenge.

Guess my Robot Template

Created by [gardon](#) - Topic Group: [Sequences](#) - Created: 06-01-04 - Modified: 06-12-04

[New Report from this template](#)

<your name here>'s Guess My Robot Page

Replace <your name here> with your name, or code name.

My Sequence:

Replace the box below with your own:

1	2	3	4
---	---	---	---

Hints:

If you think they are needed.

Solution:

After someone posts a Robot that generates your sequence, mention him / her and post your original robot here.

Figure 18: The Guess my Robot template directed students to post their challenges in a common form.

The game was managed by a central game page, created each year, which would link to a page listing the rules of the game, and link to active proposers' challenge pages. Figure 19 shows a typical challenge page, and Figure 20 shows a response posted to that challenge.

Guess my robot

Created by [B. Santos](#) - Created: 03-03-04 - Modified: 17-03-04

Bárbara's Guess My Robot Page

My Sequence:

22	44	59	118	133
----	----	----	-----	-----

Hints:

Solution:

After someone posts a Robot that generates your sequence, mention him / her and post your original robot here.

Figure 19: Example of Guess my Robot challenge. Barbara posted her challenge using the template

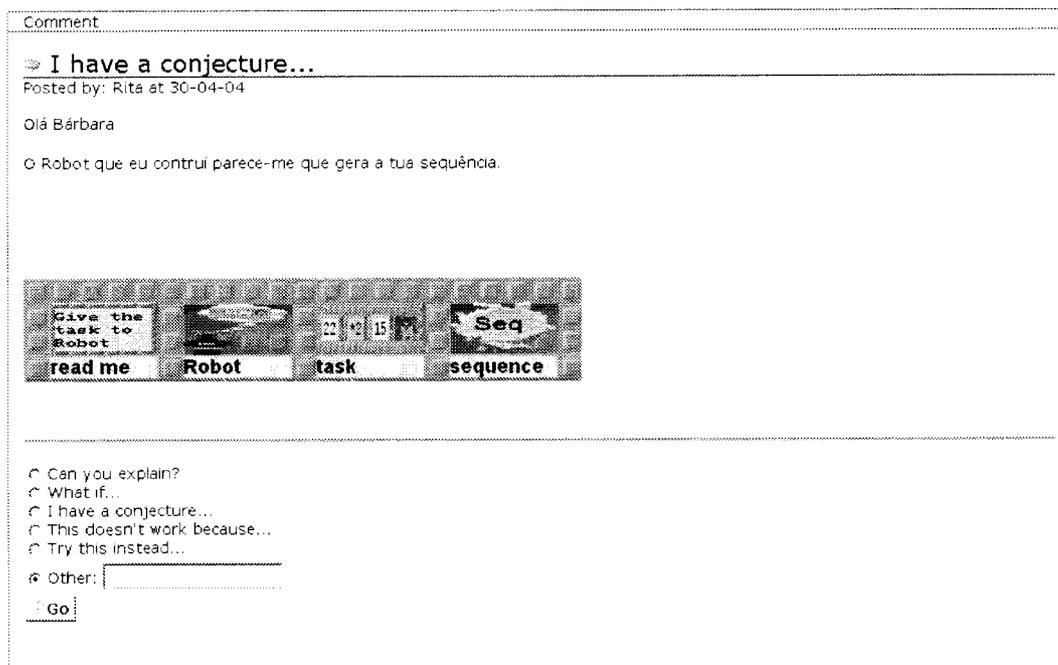


Figure 20: Example of Guess my Robot response. Rita responded to Barbara's challenge above with a robot she programmed to generate the same sequence.

Results

The *Guess my Robot* activity served as a model for other activities designed and conducted by colleagues Michelle Cerulli and Gordon Simpson in the domain of function graphs (Simpson, Hoyles and Noss, 2006) and randomness (Cerulli, Chiocciariello and Lemut, 2007).

An analysis of the challenges and responses posted by students in 2003/2004 reveals a trend towards more difficult or complex sequences, with a concurrent trend away from sequences that they found too hard to solve. It also suggests convergence to a common game culture, including a close compliance with the rules of the game and emergent programming conventions. This analysis focused on the mathematical structure of GmR challenges with respect to length and character of engagement. I collected 45 challenge pages from participants in seven groups across four countries. Challenges were coded by the type of sequence, adherence to the rules of the game, and number of responses.

I identified seven dominant classes of sequences proposed by students. Three of these classes are familiar from the standard school setting: the trivial sequence, i.e. the natural numbers; arithmetic and geometric progression. The remaining four were considerably more complex than the structures most students encounter at school. Table 9 lists the seven classes, with examples, and respective numbers of challenges and responses. Column B notes how many challenges were posted in each category, column C counts the number of challenges with at least one response, and column D the overall number of responses to challenges in the category. The response counts are naturally biased towards the categories with the higher number of challenges. To compensate for this, columns E and F provide normalised figures.

	description (A)	challenges (B)	responded (C)	responses (D)	C/B (E)	D/B (F)
Trivial	e.g. 1, 2, 3, 4	2	0	0	0.0	0.0
arithmetic	$a_n = a_{n-1} + p$	3	2	4	0.7	1.3
geometric	$a_n = a_{n-1} * q$	6	4	9	0.7	1.5
combined	$a_n = a_{n-1} * q + p$	12	6	12	0.5	1.0
interleave	$a_n = b_n, a_{n-1} = C_n$	6	3	4	0.5	0.7
compound	a sequence built as a function of another sequence	4	2	4	0.5	1.0
complex	multiple operations, cannot be reduced to any other category	1	0	0	0	0
unknown	could not uncover sequence rule.	11	0	0	0	0
total		45	17	33		

Table 9: Summary of challenges and responses. Column C refers to the number of challenges in a class which received at least one response, and column D counts the overall number of responses. The last two columns note the number of responses normalized to the overall number of challenges in that class. The values were highlighted to expose the highest values, where gold denotes the highest value and blue and green the runners-up.

Notably, when modelling sequences of their choice, the more complex classes were the more popular among students. Furthermore, as demonstrated by Figure 21 this tendency towards complexity appeared to increase over time.

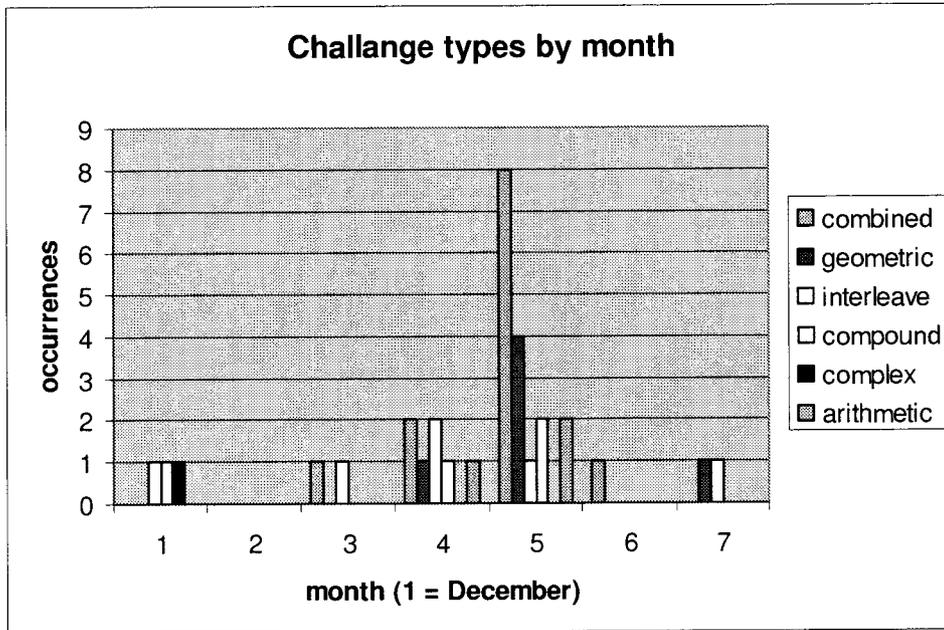


Figure 21: Challenge types over time

In an attempt to quantify this impression, I devised two rough estimates of sequences' computational difficulty and perceived complexity. This analysis showed that most challenges were in the mid-range of complexity, and (in a slightly less profound manner) difficulty. This observation was amplified when tracked over time (Figure 22). The number of challenges in the low-difficulty range (ranked under 3) drops to zero, whereas the mid- and high- ranges show significant growth, with the strongest and most persistent trend in the former.

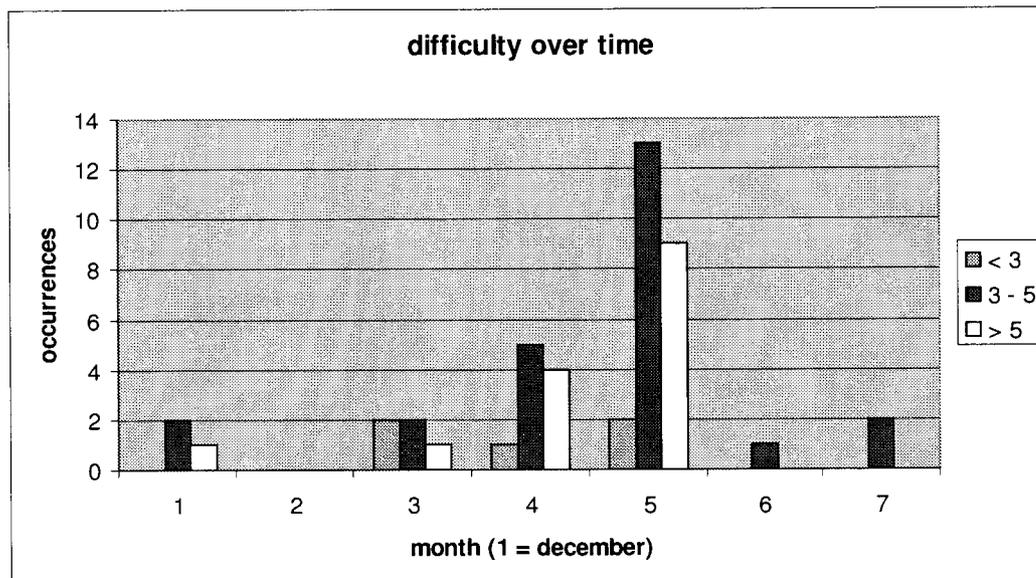


Figure 22: Difficulty of challenges over time

To summarise, students developed a culture of posting challenges which were “hard but not too hard”, i.e. would demand an effort to solve, yet that effort was likely to be rewarding: pushing the limits of their peers' capabilities but not exceeding them. This result appears to be consistent

with Vygotsky's notion of zone of proximal development (Vygotsky, 1930 / 1980). Within these boundaries, the mathematical level of challenges posed and solved exceeded the school standards for this age.

There were 12 cases in which the same student posted a second or third challenge. In eight of these the subsequent challenge was more complex than the first, and in one case the student shifted from an unknown type (i.e. challenge no one could solve) to one of the advanced types. Given the small sample size, this evidence should be taken with some reserve. However, it does seem to support the conjecture that students converge towards challenges that are "hard but not too hard".

Reflections

To a large extent, the success of GmR game can be attributed to the structure of the activity, which encouraged students to formalize their intuitive view, rather than try to ignore their intuitions and replace them by unrelated knowledge. The evidence suggests that the intuitive view is primarily *recursive* in form (section 6.2.4). Students define sequences as a function from one term to the next ($a_n = f(a_{n-1})$) rather than the "school maths" view of a sequence as a function of the natural numbers ($a_n = f(n)$). By acknowledging this preference and building on it, students were given freedom to develop their own formalizations and understandings, allowing them to engage with mathematical structures far more complex than they would in their regular curriculum.

GmR was also successful in engendering mathematical discourse, and engaging participants with sophisticated notions of equivalence and proof, as demonstrated in the following LNs.

Even when mathematical ideas are meticulously embedded in an activity, there is a risk that students might remember the fun of the activity and "miss the maths". In order to structure and reify knowledge, students need to reflect on their actions and experiences. The mathematical ideas need to be made explicit and transformed from the intuitive realm to the consciously articulated. This realization motivated the design pattern of POST LUDUM discussion (noted in section 0): upon completing this activity, students convened to report on their experiences and what they had learnt from them. The proclaimed aim of this discussion was to produce a consensus webreport, an account of the activity and its consequences undersigned by all team members. This discussion was facilitated to start off from activity narratives, but eventually shift focus to the mathematical issues. The discussion also provided motivating questions for future activities.

7.3.2 LN 1: Rita's GmR

Overview

The main protagonist in this narrative is Rita, a 14-year-old girl from Lisbon, and her interactions with peers in Sofia and Nicosia and with researchers in London, through her participation in the GmR game.

Sources

GmR1, GmR3, GmR5

Situation

This narrative follows the interactions between two groups of students playing the "Guess my Robot" game, one in Sofia and the other near Lisbon. The Sofia group consists of 6 boys and girls, aged 11-12, working with *WebLabs* researchers. They have been working with *ToonTalk* for several months, approximately once a week for a couple of hours. The second group is from a village south of Lisbon. Paula, a teacher and researcher in the *WebLabs* team, worked with a school group (aged 12-13) there during the first project year. Researchers in both groups act as teachers, guiding the students through the mathematical ideas as well as through the programming skills. At the same time, the researchers facilitate interactions, by pointing children to interesting peer reports and helping them to add a few words in English to their own reports.

Task

Rita found the 'guess my robot' activity, and decided to pose her own challenge. As the responses to her challenge appeared, she engaged in dialogue with her peers, and her aims shifted from the overt definition of the game to an attempt to provide formal proof for the mathematical equivalence of two robots.

Actions

Rita entered the GmR game by posting the sequence (see Figure 23): 2, 16, 72, 296, 1192 ...

Rita's guess my robot

Created by [Rita](#) - Created: 27-01-04 - Modified: 25-04-04

Rita's Guess My Robot Page

My Sequence:

I created this sequences of numbers. My five firsts terms of the sequence are:

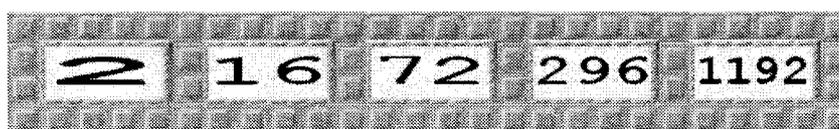


Figure 23: Rita's "Guess My Robot" challenge

A few days later, the Sofia *WebLabs* group held a session, and some of the students tried solving Rita's challenge. Nasko, a 12 year-old boy, posted his response. He had built a robot that produced Rita's five terms, but also realised that the same robot could be used to generate other sequences by changing its initial inputs. He appended one such sequence to his response, along with a two-part challenge for Rita:

1. What is the input of my robot?
2. Can your robot generate it?

When Rita came to her next session she was very excited to find comments on her page – and from children on the other side of Europe. Having examined Nasko's solution, she responded to his comment:

Congratulations, you found a solution for my sequence! But you used a different procedure of mine.

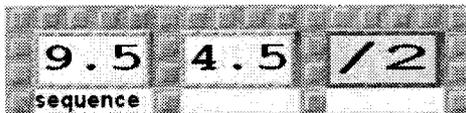
...and included her original robot. Next, she considered Nasko's dual challenge, and posted her response to it:

I have a conjecture...

Posted by: Rita at 10-02-04

About your questions:

- For your new sequence I think that is the input for your robot:



- For my robot I make this:

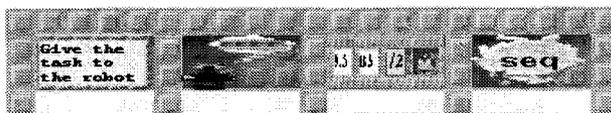


Figure 24: Rita's response to Nasko's dual challenge

Rita received a second response to her challenge from Ivan. She congratulated his solution, but added:

If you use one bird with a nest you can get all the terms of the sequence and not only the last. Moreover, if you make like that it was more easy, for me, to understand which of the numbers produced for robot are the terms of sequence. You can try make that?

At this point, the researchers from London posted a comment, asking Rita, Nasko and Ivan to explain how they constructed their challenges and how they worked out the solutions. To this, Rita responded:

I try explain

To create my sequence I thought thus: My first term is 2 and each one of the other terms is gotten of the previous one adding 2 and multiplying 4 to the result.

I created a box with 4 holes. In the first hole I put the first term (2), in the second hole I put the number that I wanted to add (2), in the third hole I put the number that I wanted to multiply (x4) and in the fourth hole I put a bird. I gave the box to the robot and I went in to the robot thought.

In the robot thought [ym: box], I copied with magic wand the first 2 and gave to the bird, I copied with magic wand the second 2 and put in the first hole, I copied with magic wand x4 and put in the first hole. I Clicked in Esc and I left the robot thought.

I cleaned the first number of the robot thought box and I tested my robot... And my sequence born...

As she continues to explain how she solved Nasko's challenge, the text shifts from a procedural listing of coding actions to a description of mathematical reasoning, while still retaining a strong sense of narrative:

For the Nasko challenge I thought thus:

In the Nasko's task to my sequence he used 2 like first term, 14 is the number that he used for add a 2, and to get the second term (16), and for to multiply 4. Then, I think to the Nasko's sequence the first number of the task it has that to be 9.5 because is the first term of him sequence, the second number of the task it has a number that he add to 9.5 for to get 14 (second term), this number is 4.5.

In my sequence he use the x4 for to get the third term ($16 + 14 \times 4$), then in him sequence I think this: $14 + 4,5$ "I don't know what" it has that to be 16.25 or 4,5 "I don't know what" it has that to be 2.25. But 2.25 is half of 4.5, then in third hole of the task I need to put /2.

After that I tested this task in Nasko's robot and it works.

The London researchers followed with a new question: "*We think your robots will generate the same sequence forever, but how can we be sure?*" Rita responded to this in the spirit of the GmR challenges: she constructed a robot that accepts two streams of numbers, and outputs the stream of term-by-term differences. She posted this robot, and noted:

... the difference's robot make the difference between the same terms of the two sequences.

That new sequence is a zero's sequence, that show to us that the Rita's robot and the Nasko's robot make the same sequence.

If we made the same of Rita's sequence and Ivan's sequence we get the same result.

This led us (Rita and myself) to a discussion regarding the validity of this construction as a proof of the robots equivalence. At some point, Rita argued:

I think not! [*ym: this robot is not a proof*] And I hope that some mathematician has demonstrate that. It isn't any reason for the 100 001 term will be different if the 100 000 previous terms were equals.

In Sofia, this thread of conversation was the motivation for a classroom discussion in which the teachers introduced the formal algebraic analysis of the sequences, and a proof of the robot's equivalence. This discussion was captured by the Sofia team, and shared through the comments on Rita's challenge. A couple of weeks later, Rita received another response, this time from Cyprus. After congratulating them on their solution, Rita notes:

I can prove that my sequence and your sequence are equal with the process of algebraic representation used by Sofia group.

Rita's sequence:

$$A_1 = 2$$

$$A_{n+1} = (A_n + 2) \times 4,$$

but if I using the distributive property of the multiplication relatively to the addition I can write that:

$$A_1 = 2$$

$$A_{n+1} = A_n \times 4 + 8$$

that is the algebraic representation of the Cyprus's sequence. then I can prove that two sequences are equal.

Rita had discovered the algebraic formalism through her interaction with peers in a distant land and made it, in the words of Healy and Sinclair (2007), "a land of her own". This process was meditated by the WebReports technology but driven by an activity which engendered a genuine need; algebra emerged as a tool for a purpose, not an esoteric ritual.

Results

The webreports system allowed learners to comment on any page using a free-form WYSIWYG editor. This allowed them to express their mathematical ideas in a personal narrative, as well as the path by which they arrived at these ideas. Using this feature, learners expressed and developed arguments which they could not yet formalize, and shared their learning process.

Through their participation in the game, and in the discussions provoked by researchers' questions, Rita and her peers gradually shift from an intuitive grasp of mathematical concepts, expressed in narrative form, to a structured mathematical understanding expressed in algebraic formalism. The link between the two is provided by the coded representation in ToonTalk, which retains intuition and at the same time enforces formalism.

ToonTalk objects are seamlessly woven into the learners' discourse, using the embedding features of the WebReports system.

The mathematical structures and arguments exchanged between participants are well beyond the level common in their school classes.

Reflections

This narrative demonstrates the potential of the GmR game. It illustrates how the design of the game, and the environment that supports it, contributed to participants' learning trajectories. The narrative is representative and singular at once: each element highlighted by this story can find parallels in many of the other game instances, yet few – if any – other examples manage to capture so vividly all these elements together.

Rita and her responders demonstrated construction and analysis of complex mathematical structures. Very quickly they learn to distinguish between process, parameters and process in the formation of sequences. Their conversation gravitates to an advanced socio-mathematical norm. Rita and her peers shift from a focus on mathematical skill to sophisticated notions of proof and equivalence. This shift is prompted by the researchers' questions, but facilitated by their writing. Pugalee (2004; 2001) shows how students' writing about their mathematical problem solving promotes metacognitive behaviour, and that students who wrote descriptions of their thinking were significantly more successful in the problem solving tasks than students who verbalized their thinking. The examples above suggest that such writing takes a narrative form. This is no surprise: as discussed in Chapter Four and Six, we use narrative to organise our experiences and extract meaning from them. When these are experiences of solving mathematical problems, the meaning extracted is mathematical meta-cognitive knowledge. However, it is important to remember the distinction (noted in section 6.3.2) between knowledge telling and knowledge transforming. The initial narratives produced by Rita and her peers certainly fall into the first category. It is through the sustained reflective interaction, prompted by the post-game discussions, that they shift to the latter.

The effects of the game on participants' learning are linked to its structure and to the qualities of ToonTalk and WebReports as the media by which it was conducted. However, there is nothing unique about these media which was essential to the game and its success. The specific tools are interchangeable; it is the orchestration of the design patterns which they embodied, as discussed in Chapter eight, which led to the educational effects.

7.3.3 LN 2: Joe999's GmR

Overview

Joe999, an 11-year-old boy, programmed a robot to enact his process of discovery as support for a mathematical argument he posted in his webreport.

Sources

WR7, GmR6, GmR7, GmR8

Situation

Joe999 was an 11-year-old boy from London. His group worked with Ken Kahn on a different activity, and was not involved in the Guess my Robot game.

Task

Joe999 joined the GmR game independently, and decided to solve a challenge I had posted. After posting his solution verbally, he programmed a robot to present it.

Actions

At some point, Joe999 started using the system's messaging facility to chat with me. His messages were social in nature. I tried to divert the conversation to activity-related content. Eventually, I invited him to join the game. Joe999 found a challenge posted by me, and Ken showed him how to load the box into ToonTalk and how to use the wand to copy and subtract numbers. After some hard work, he managed to solve the challenge. Joe999 was very proud of his achievement and was confident he could train a robot to build it but had only time to write a short comment.

Yish. After 10 minutes I figured out how to do the sequence. You take away 3.5. Then you find half of 3.5 and take that away from 11 and continue this sequence.

To which I responded:

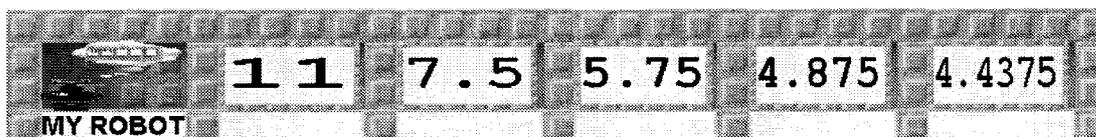
Can you explain?

Don't just talk. ToonTalk. Instead of telling my you figured it out, build a robot (or chain of robots) that produces this sequence.

See you soon!!

- Yishay

Joe999 picked up the gauntlet, and trained a robot.



This robot did not produce the sequence – it acted out the story of how Joe999 had solved the puzzle. The robot calculates the differences between the terms of the sequence and arranges them in a box:

.4375	.875	1.75	3.5
Is Half Of	Is Half of	Is Half of	This Number

Then it prints:

```
Conclusion: You are halving the
number You halved before.
I have shown this in this box.
Good sequence though Yish (^_^)
```

A few weeks later, Joe999 published his own challenge. When mariaf, a participant from Nicosia posted a verbal solution, Joe999 responded:

You need to make this sequence with a robot!!!

Results

Joe999 had used ToonTalk as a narrative medium. He had turned the execution of a program into a domain-specific genre. Without any guidance, he had used programming as a way of making a mathematical argument. This argument was narrative in structure, yet precise and succinct in nature. It is contextualized – by the ToonTalk environment and then by the packaging of the robot; it has a plot – the robot goes through a carefully chosen sequence of actions and events; it acts as an avatar for Joe999, expressing his voice when typing “I have shown this in this box. Good sequence though Yish”.

Joe999’s initial response does not comply with the norms of the game, but his response to mariaf’s solution shows that he is keen to align himself with the rules, as he has inferred them from my prompts.

Reflections

Joe999 self-driven engagement with GmR shows genuine interest in the game and the mathematical ideas embodied in it. Joe999 follows a trajectory from exploratory game-play to formal argumentation. The fact that his argument is expressed as ToonTalk code is tangential to its mathematical validity, but critical to Joe999’s learning. ToonTalk, as a medium, and its integration with the WebReports system, allowed Joe999 to express himself in ways which other notations could not. These media afforded a smooth transition from action through intuition to structured mathematical argument.

A key factor in this transition was Joe999’s ability to articulate his ideas in narrative form, first in words and then in code. Joe999 had used ToonTalk as a narrative medium. Yet at the same time, this form of expression leaves no room for ambiguity. After all, as anyone who has ever

programmed a computer knows, if you are not completely accurate in your coding, the result will be anything but what you intended it to be.

Finally, Joe999's code has a moral. The purpose of the protagonist's (robot's) actions in the story is not their immediate outcome (a box of numbers, a block of text), but the implicit transfer of an idea. Here perhaps lies the difference between a knowledge telling and a knowledge transforming narrative: when the narrator's intention is merely to recount events, the resulting text has the syntactic structure of narrative, but lacks any underlying meaning. When a narrative is constructed to convey an implicit idea, the narrator needs to engage simultaneously with the domain knowledge model and the rhetoric model, and maintain coherence in both.

7.4 Convergence and Divergence

7.4.1 RN 4: $1/n$ vs. $1/2^n$

Overview

My high-level design of number sequences activities concluded with a section dedicated to convergence and divergence of sequences and series. Iteration 0 and 1 were hindered by the difficulties listed in Sections 7.2.1 and 7.3.1. The second iteration of the convergence and divergence activities benefited from a mature techno-pedagogical framework, but encountered unexpected difficulties which revealed insights regarding learners' perception of the underlying mathematical concepts.

Sources

1overN.24jan05, SessionReport.27.feb.reciprocals, SessionReport.05.03.04.reciprocals-task-in-a-box, SessionReport.04.02.13.reciprocals-pre-activity, WR8, WR9, WR10, WR11, WR12, WR13

Situation

This narrative follows the refinement of activity design during 2003 (iteration 2), focusing on a group of eight children in London (group III).

Task

The convergence and divergence segment of activities was perceived as the pinnacle of the number sequence activities: a highly advanced topic which would express the programming and mathematical capacities learners had developed through the basic numbers and GmR activities. During the first year, this intention was left unsatisfied, as the preliminary segments demanded much more effort than expected. Having completed these activities successfully in the second year, I proceeded to experiment with convergence and divergence. The aim was to model converging sequences as ToonTalk streams, plot their graphs in Excel, and then use an Add-up robot to generate and explore their sum series.

Actions

The topic was introduced by a question:

Can a sequence get smaller and smaller but not go below 0?

This question was first presented in a pre-activity questionnaire, then opened for group discussion, and finally used as the topic for a group webreport. Once the group had identified two such sequences, I asked:

What happens to the sums of these sequences?

As expected this raised a controversy: some thought that the sum series would converge, others thought it would diverge (without using these exact words). The disagreement on this issue provided the pretext for a set of tasks, formulated as an empirical study:

1. Choose a converging sequence.
2. Predict the shape of its graph.
3. Model the sequence as ToonTalk Stream.
4. Collect the terms produced by these robots and plot their graph in Excel.
5. Compare the outcome to the prediction.
6. Predict the behaviour of the sum series.
7. Attach the Add-up robot to the modelled sequence, to produce a stream of partial sums.
8. Plot the graph of the partial sums sequence, and compare to prediction.

I directed learners to model the reciprocals sequence: $\{1/n\}$, under the assumption that it is the most simple and intuitive converging sequence. My hidden intention for students to discover that the sums of this sequence diverge, allowing them to generalise this outcome to all converging sequences, and then confront this claim with the inverse powers of 2: $\{1/2^n\}$.

Programming the reciprocals in ToonTalk seemed like a simple enough task:

Provide the robot with a box containing a counter and a bird. In each iteration, the robot:

1. Take a number "1" from the toolbox
2. Copy the value of the counter, and divide the "1" by this value.
3. Give the result to the bird (as the next term of the stream).
4. Increment the counter.

To my surprise, learners found this task challenging. I tried various measures for reducing the difficulty of the task, focusing on the main mathematical concept and eliminating extrinsic technical obstacles:

- Supply the input box needed for training the robot, to provide learners with a starting point for the task.
- Package the task and the instructions in a ToonTalk box, to reduce the confusion when shifting between environments, as in Figure 25.
- Provide a robot that divides 1 by 2, ask learners to generalise it so that it will increment the denominator and repeat, thus producing the reciprocals, as in Figure 26.

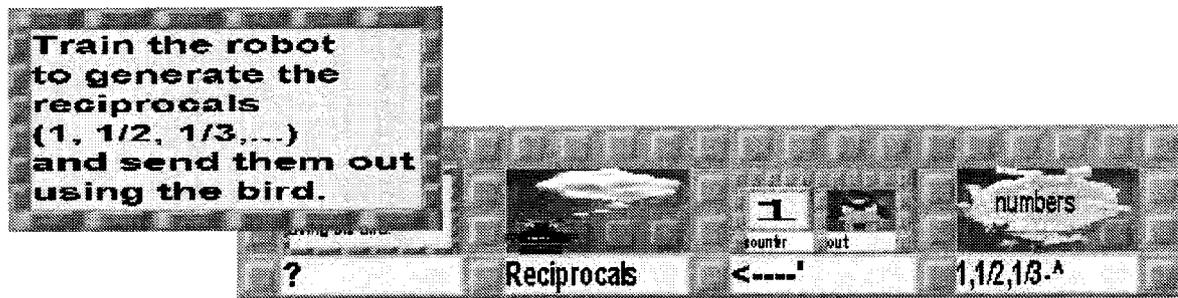


Figure 25: Reciprocals task, packaged in a ToonTalk box.

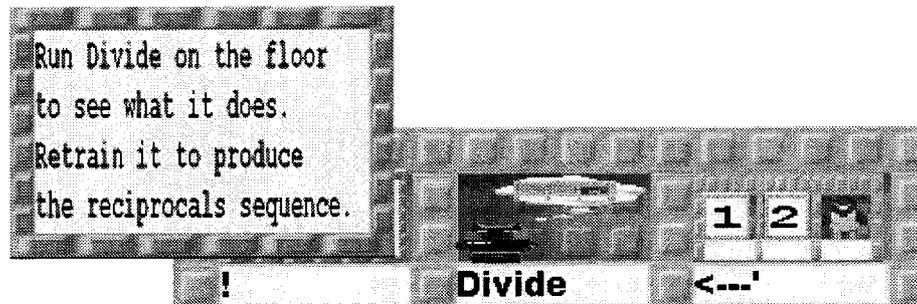


Figure 26: Divide robot – needs to be generalised to produce the reciprocals.

While these modifications helped, none of them made the task as easy as I had anticipated. Students still found this task much harder than training the Add 1 and Add up robots, even though it would appear to be of similar complexity: same order of number of actions, parameters, etc.

The solution was provided by a chance incident: while working on this task, a pair of girls accidentally trained a robot to produce the sequence $1/2^n$. This robot was later referred to as the Halfer robot. The surprising thing about this incident was that although it would appear to be of similar structural complexity to the Reciprocals robot – more or less the same number and type of actions, same number of variables, etc. – students found it very easy to construct and understand. Observing their implementation, I noted that they produced the sequence as a recursive function: $a_n = a_{n-1}/2$.

The implications for the activity design were obvious: I had intended to open with $\{1/n\}$, which would lead to the conjecture that all sum series diverge, and then present $\{1/2^n\}$ as a counter example. In fact, the basic design was almost indifferent to the order in which sequences were presented. I switched to using $\{1/2^n\}$ as the initial sequence and $\{1/n\}$ as the counter-example.

In the final version of the activity design, tested in 2004, learners chose their own sequence to explore and were later shown a counter example as appropriate. Practically all learners expressed their sequences as recursive functions, and consequently modelled power series which converged.

The various attempts I had made to reduce the complexity of the task also made their way into the final design: tasks were packaged in ToonTalk boxes, embedded in report templates, and

included partially programmed components to minimise learners' work on mathematically insignificant activities.

I was somewhat anxious about the next step: chaining `Reciprocals` to `Add up` to obtain the harmonic series. Learners had no problem following the process and immediately proceeded to chain the robots to the excel tool, in order to collect the sequence data and plot it. In the classroom discussion that followed learners described their findings in an accurate and sophisticated manner.

Results

Despite the difficulties, learners were deeply engaged in these activities, and showed significant learning gains.

Learners' initial concept of sequences was strictly linear. In the course of these activities they learned to identify and describe non-linear structures.

Learners initially found the notion of a sum series very difficult to work with, often confusing the sequence terms with its partial sums. This difficulty was predicted by the literature. Having constructed the sequences, combined them with an `Add-up` robot to produce the partial sum series, and plotted both – this confusion seemed to disappear.

Learners' initial conjectures and arguments were simplistic in content and in form. As the activities progressed, their reports showed a growing sophistication of mathematical discourse.

Apart from the reciprocals task, most students found the level of challenge of most individual tasks reasonable and enjoyable. However, the minor "wrinkles" in activity and tool design meant that the sequence of activities as a whole took much longer than expected.

The lessons learnt from this iteration informed the final design, which was tested in 2004 with notable success. Following this iteration, I developed detailed on-line worksheets for each task in this segment of activities. These worksheets were presented as `WebReport` templates, with embedded `ToonTalk` boxes containing task instructions and partially trained robots.

The most important change of design was that I decided to side-step the question of which sequence to explore first ($\{1/n\}$ vs. $\{1/2^n\}$): in the next version, learners would propose their own sequence to explore. Based on their choice, I would then select the counter example.

Reflections

Some of the improvements to design which emerged from this trial of the convergence and divergence activities could be classified as straightforward issues of usability. These include the packaging of tasks as `ToonTalk` boxes, and the use of partially trained robots. Although my insights emerged from this experience, they are not specific to it, and could have been derived from others.

The most important outcome of this trial was of a different nature. The order in which the sequences were presented may be a very subtle change in design, but it had a profound impact on learners' achievements. This change resolved the difficulty learners had modelling their first

converging sequence, a critical step in the learning trajectory. Unlike the other issues, this is not an issue of general usability: it is related to learners' innate understanding of sequences. As evidenced by the analysis of GmR challenges, the naïve concept of sequences is iterative, or recursive: a function from one term to the next. This is in contrast with the "schoolbook" concept of a sequence as a function of the natural numbers. Analysis of learners' pencil and paper texts indicates that the preference for the recursive form is not related to the ToonTalk medium. Yet this medium, and the STREAMS pattern, gave the recursive form an added advantage. Thus, the design of activities for learning – my generative epistemology – exposed insights regarding innate processes of learning – the genetic epistemology of number sequences. These insights fed back into my design, thus ratifying my conjectures, and at the same time producing a more effective learning design. My observations regarding the epistemology of number sequences are not derived from the pedagogical structure alone, or from the technical aspects of the media – but from their interaction.

Finally, the power of the design of these activities is in that it explicitly positioned learners as design researchers: exploring a mathematical question by designing and manipulating representations for it. This idea was expressed vividly by the group who participated in the final iteration. As part of a summative interview, they were asked what they found interesting and what they had learned. One boy explained:

um, like the debates were great, about um, is there a limit and can we prove it and um, also I've learnt a lot more on how to prove things in Algebra, with the a_k for the terms and everything about which I didn't know before. I've learnt a lot more about that.

And concluded:

You gotta have a proper method instead of just, like, um, try and fail

While his friend remarked:

If you program a robot to do it rather than having to write down each thing you can get a lot more results done and also you can see exactly sort of what's going on its not just like "just DO it" and like "that's just what you have to do". You have to build the robot yourself and so you know exactly what it's doing.

And later in the same interview:

It's about logically thinking things through, rather than just um, like you being told that this is the equation you have to do and say "oh, yeah". So, what we're being told is right, instead we sort of have to discover for ourselves, and you then have to think through things more logically. Think more in depth into things.

7.4.2 LN 3: Sodapop's convergence

Overview

Sodapop, a participant in the 2004/05 trial of the convergence and divergence activities, explores the sequence $\{100 * \frac{1}{2}^n\}$. He posts his conjectures, and reflects on his mistakes.

Sources

WR14, WR15, WR16

Situation

Sodapop was a 14-year-old boy from London (group IV) who participated in the convergence and divergence activity in 2004/05 (iteration 3). This narrative focuses on a couple of sessions in which he explored a sequence and published his findings in a webreport.

Task

Following the instructions in the activity worksheet, Sodapop's task was to:

1. Choose a sequence that "gets smaller and smaller but never goes below zero"
2. Predict the shape of the sequence's graph
3. Model the sequence and test this prediction
4. Predict the behaviour of the sequence's partial sums series.
5. Connect the sequence to the Add-up robot and test the prediction
6. Publish a report with the robots, graphs, and reflections on the activity.

Actions

Sodapop chose to explore the sequence:

$$a_n = 100 * \frac{1}{2}^n \quad (n=0..∞)$$

He modelled it in ToonTalk by training a robot to receive a number, repeatedly half it and send out a stream of the results. Having tested his robot, Sodapop packaged it in a box (Figure 27) similar to the task boxes provided in our worksheets and posted it to his report. His box included a description of the robot:

This robot halves numbers. The sequence never goes below 0.

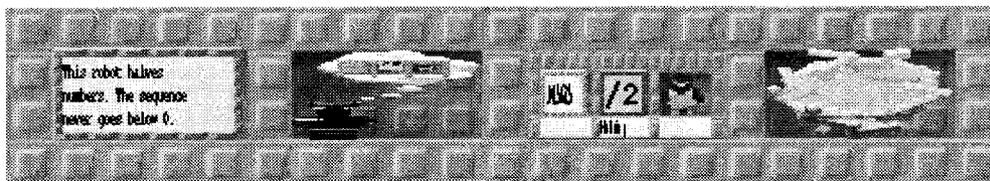


Figure 27: Sodapop's robot, packaged for publication.

Sodapop collected the outputs from his robots and plotted their graph in Excel. He saved the graph as an image, and used the Paint program to overlay his prediction of the sum series on it. He embedded this image in his report and proceeded to test his conjectures.

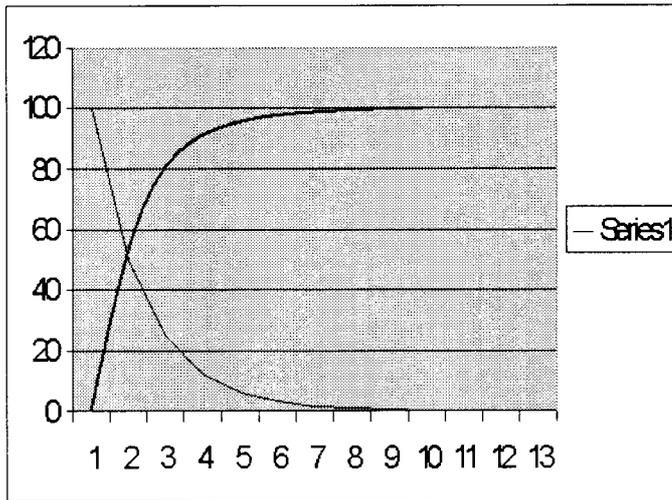


Figure 28: Graph of Sodapop's sequence, overlaid with his prediction for the sum series

Sodapop generated the partial sum series by connecting his robot to an Add-up robot he had constructed in the past. He then plotted its terms, and observed that his conjecture was partially correct. He posted the graph (Figure 29), along with his observations:

This is the real graph that was produced by the cumulate total of the halving-a-number robot. it looks like the top of my graph but i made the fatal mistake of thinking it started at zero. I also said it wouldn't go over 100, which was very wrong.

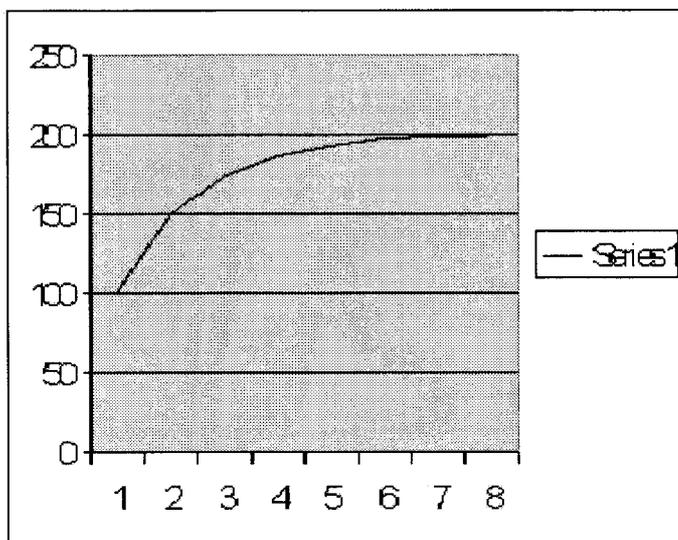


Figure 29: Sodapop's graph of the sum series of his sequence

Sodapop concludes his report with a statement of his finding and a reflection on his errors:

After lengthy research and a detailed experiment, I have concluded that if the primary source was an integer between 99 and 101 (not including those numbers) that the cumulative total can never go above 200. This is because if you have 0.1 and you double it and add it together you will get 0.15 so every time you do this you will get another number after the decimal place. So you will constantly get more numbers after the decimal place, but the numbers closest to the decimal place will not be getting any larger.

Results

Sodapop had no difficulty with any part of the task, and found it highly engaging. He even extended the task on his own initiative, using the paint program to draw his predicted graph.

Sodapop's initial prediction was based on visual intuition: his sum-series graph is a mirror image of the sequence graph. After observing the actual data from his construction, he seeks a more rigorous explanation. This explanation is presented in narrative form ("you have 0.1 and you double it.."), wrapped in a narrative of his discovery ("after lengthy research and detailed experimentation"). The first is a step towards a mathematical argument, the second aims to establish his identity as a mathematician.

The argument-as-narrative shows a genre transposition in process: it makes general claims ("every time you do this") and the moral is stated explicitly. Nevertheless, it still lacks the crispness of a formal propositional argument. It is unclear if Sodapop claims that each iteration will only add more digits without changing all the existing ones or that only the first digit after the decimal point remains unchanged. As demonstrated in the story of Rita's GMR (7.3.2), guiding questions by teachers or peers challenging this narrative might have assisted Sodapop in completing the transposition to a propositional form.

Sodapop perceives the sequence recursively, and distinguishes between its structure and parameters. The robot is a direct implementation of: $a_n = a_{n-1}/2$, where a_0 is assigned the value 100 by providing that number in the input box. In his report Sodapop describes it as "A Halving sequence", and does not refer to its actual values. In his argument he changes the value of a_0 to his convenience, implying that it is not fundamental to the nature of the sequence.

Sodapop acknowledges and elaborates the mistakes he made. At a meta-cognitive level, such reflexive behaviour is conducive to learning. Learners are often reluctant to publically acknowledge their mistakes for fear of losing face.

Reflections

Sodapop interweaves words, images (graphs) and ToonTalk objects in his text, drawing on all available resources to present mathematical arguments and share the path by which he reached it. The possibility of mixing media and referring directly from the verbal text to the objects he used plays a critical part both in his learning and in his ability to articulate his ideas.

Sodapop expresses his ideas in narrative. The text begins with an exposition, providing the context in an initial situation of failure: the protagonist had made "a fatal mistake". The tone is

personal and dramatic. It suggests a voyage of discovery. All this is superfluous in terms of mathematical argumentation. Yet to allow students to make mathematics “a land of their own” (in the words of Healy & Sinclair, 2007), these elements must have their space. As for the central mathematical argument, it can only be understood if read as a narrative. It is framed as a protagonist (you) engaged in a sequence of events: “*you* have 0.1 *and you* double it...”. As in any good narrative, the final conclusion is left unsaid: if starting from 0.1 the sum series is bounded by 0.2, then starting from 100 it is bounded by 200. It is our role as teachers to accustom the student to make the whole line of reasoning visible, but in order to do so we need to better understand his naïve modes of reasoning and expression. The narrative mode should be seen as a starting point on the path to mathematical rigour, not its enemy.

Although Sodapop’s narrative does contain elements of identity, it is predominantly a tale of a problem to be solved and the lessons learnt in the attempts to do so. Just as his mathematical arguments do not qualify as scientific discourse, neither does his narrative. Still, both are notable steps in the right direction. In effect, Sodapop had conducted a design study of a mathematical question, and reported on his findings in a design narrative. Through the construction of this narrative, Sodapop had reflected on his process of inquiry and identified its strengths and limitations. Again, the shift in his intentions is accompanied by a shift from a knowledge telling narrative to a knowledge transforming one: his first narrative reports on his actions, and thus does not engage with the domain knowledge model. His later one aims to convince the reader of a mathematical claim, and hence adopts a knowledge transforming mode.

7.5 Technological Infrastructure

7.5.1 RN 5: Supporting Number Sequences Activities with the WebReports platform

Overview

The WebReports platform which I designed (presented in section 5.2.2) provided a wide range of services in support of the variety of WebLabs activities. This design narrative focuses on the evolution of the system with respect of the needs of the number sequences activities, as described in sections 7.2, 7.3 and 7.4.

Sources

D3.2.1, webreports_gordonyishay.02-03, WebReports

Situation

This narrative follows the evolution of the WebReports system from its inception in September 2002 to its final form in 2005.

Task

My initial intention was to provide a web-based platform which would allow groups of students, aided and guided by their teachers, to publish carefully crafted accounts of their work, and to discuss their work on-line with peer groups in remote sites.

This aim quickly proved unrealistic. Instead, an intricate list of required features emerged from the needs of the activities as their designs matured. Taken together, these comprise a demand for a system which affords individual record of and reflection on mathematical construction, leading to collaborative construction and activity-centred communication.

Actions

I began developing the WebReports system in August 2002. In preparation for the WebLabs inaugural meeting, I constructed an html/javascript mock-up, demonstrating the basic proposed UI features and workflows (Figure 30). The main objective of this was to provide an object for experimentation and discussion with project team members.

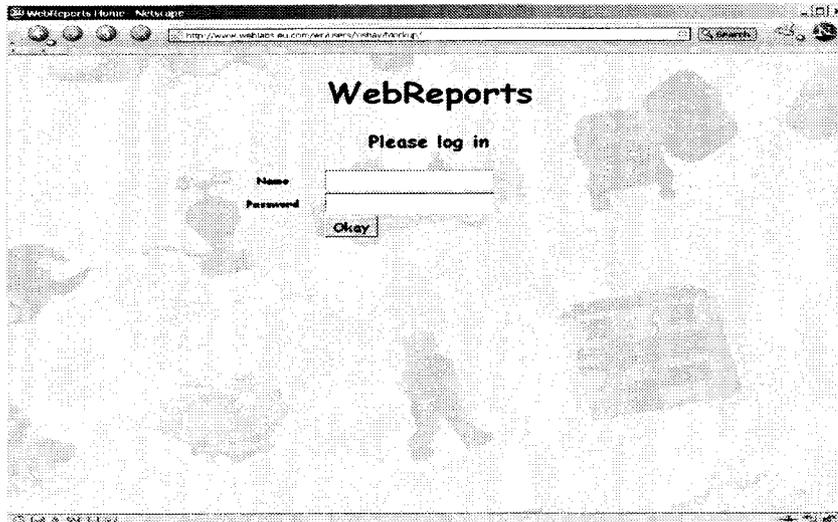


Figure 30: Login page of "mockup" used to demonstrate WebReports design concepts to project members

Based on this discussion, I developed the first prototype of the system over the next few weeks. It consisted of a web server and a small set of custom html tags (Figure 31). The supported features included those accepted as most important, but also those which generated the most debate. Thus, the prototype provided an opportunity to continue the debate with reference to tangible examples. Among these features were several templates for authoring reports and a software mechanism called "notes", to be used for commenting. Notes were to serve both as an object of discussion and a mechanism for our collaborative interchange. They were implemented as special tags which could be inserted into web pages. Where they are present, the web server generated a special link which allowed the reader of the page to annotate it.

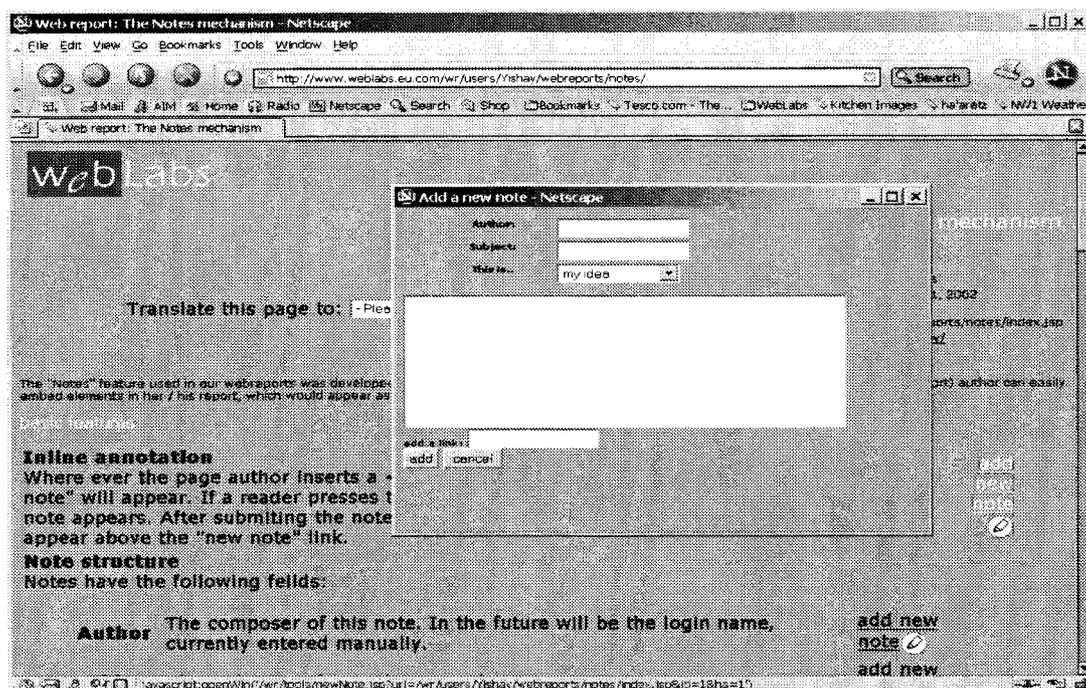


Figure 31: Notes mechanism developed as first prototype of the WebReports system

Experiments with the notes mechanism revealed that its rich features were not accessible to less technically versed learners. While it was possible to edit reports in a WYSIWYG editor, the note tags were added at code level and publishing was done by FTP. Such technicalities meant that posting reports required a level of technical competence beyond that of most teachers and students. Furthermore, many features we identified as necessary for the support of prolonged activities were judged to require a substantial development effort. With a view on the scheduled classroom experiments, it seemed reasonable to search for an existing technology which would support most of our needs and would be easy to install, use, adapt and maintain. The system was also required to support rapid redesign of content and structure as the activity plans evolve. This suggested Wiki technology as a suitable approach. The JSPWiki (<http://jspwiki.org>) platform provided an easy mechanism for registering users, creating and editing collaborative reports, embedding user media in pages, and maintaining versions of reports. Most students found the wiki markup easy to use, even entertaining (although some of the adults found it challenging). The Wiki was used successfully by students in several countries to collaborate on number sequence activities, as illustrated in section 7.3.1. It was also used by researchers to share their designs and tools as an aid for collaborative planning of the following year's activities. The main weakness of the Wiki was that it was too permissive: it did not impose structure, neither on the site as a whole, nor on individual pages. Instead, social conventions needed to be constantly negotiated and enforced. On the other hand, the Wiki's malleability was invaluable in research terms. It afforded experimentation with different feature sets and educational practices, and eventually led to a clear definition of needs for the final phase of design and development.

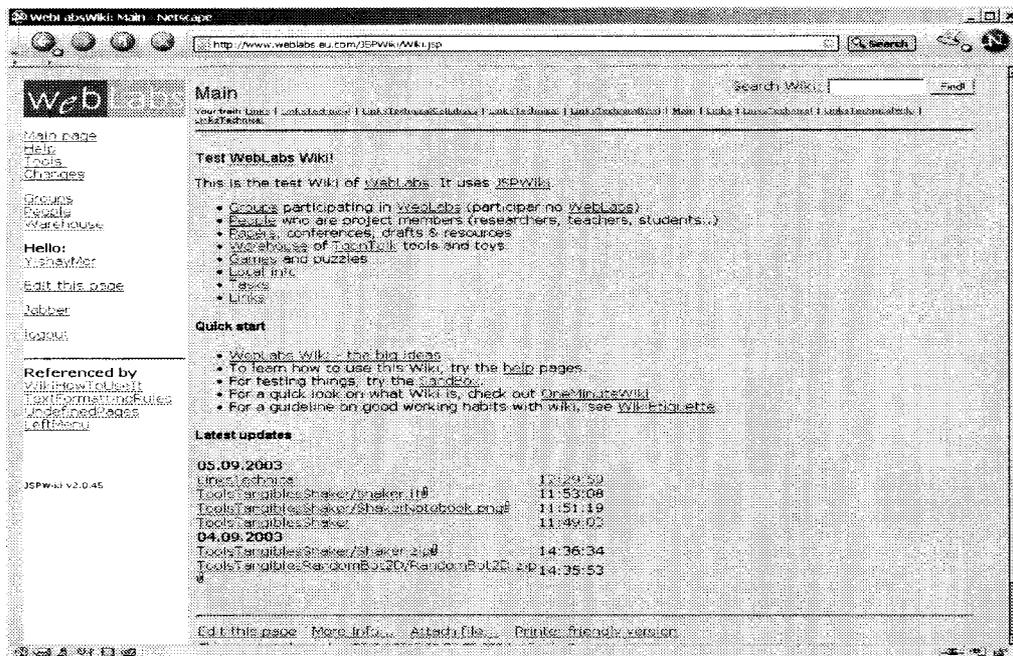


Figure 32: Second prototype, based on the JSPWiki platform

The final version of the WebReports system was in use for the last 18 months of experiments. During this period, approximately 400 registered users published nearly 600 reports. Many more reports were posted and shared but not made public. At its core, WebReports was a content management system, based on the open source Plone platform, reduced and enhanced to include the exact functionalities required for the chosen educational activities (Figure 33).

Among its critical elements were a multi-dimensional navigation scheme, a multi-media document structure and commenting tool, and mechanisms to support active learning through templates, tutorials and tools.

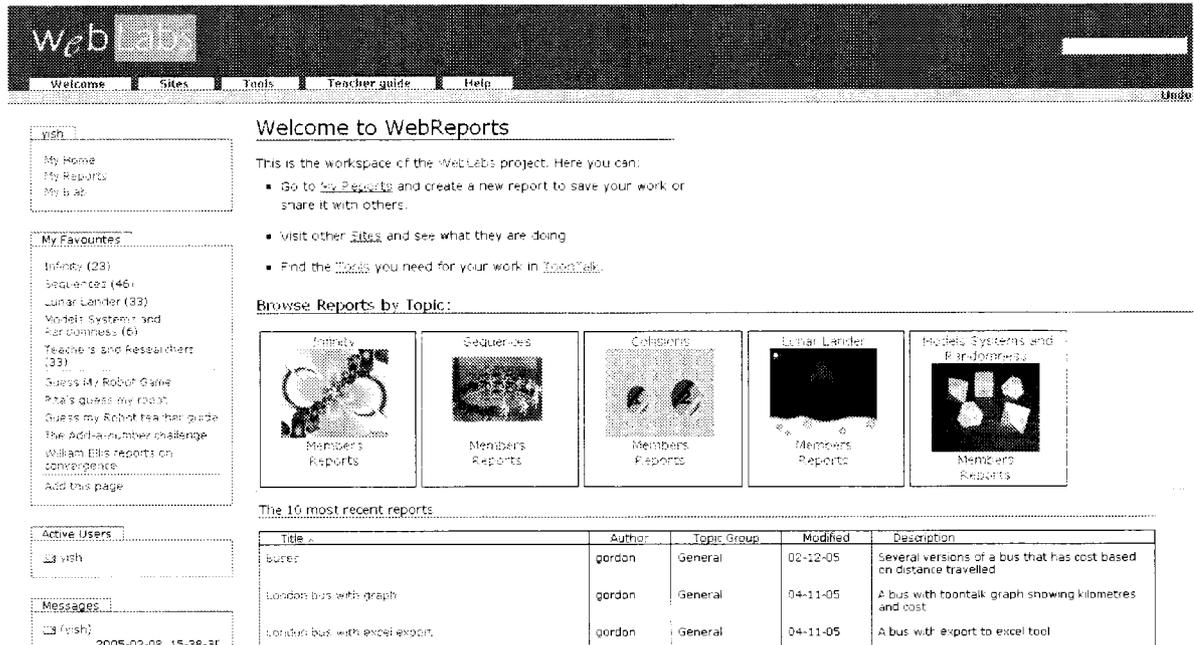


Figure 33: Final form of the WebReports system

Results

One of WebLabs' declared aims was to combine individual constructionist learning with collaborative knowledge building. The initial interpretation of this aim was naïve: it was hoped that children's excitement would drive them to spontaneously share their experiences and comment on each other's observations. Unsurprisingly, this did not happen. Success came with the shift to activity design which intentionally promoted discussion and encouraged learners to make the most of personal explorations. The result was the WebLabs common activity framework (Figure 34), which I developed in collaboration with Celia Hoyles, Richard Noss and Gordon Simpson (Mor et al, 2006). Each activity is initiated with a mathematical question, which is discussed by the group. Individual programming tasks emerge from this discussion. The products of these tasks are published by students in their individual reports, along with any observations and conjectures derived from them. Students then comment on each others' reports and confer in a group discussion, sharing their new knowledge. The discussion results in a group report which reflects the consensual understanding of the group. Where disagreements remain unresolved, they are noted as minority opinions. The group report is then exchanged with a remote group, working through the same topic. Both groups compare their findings and try to resolve any differences. This discussion raises new questions for investigation, which lead into a new cycle of activity.

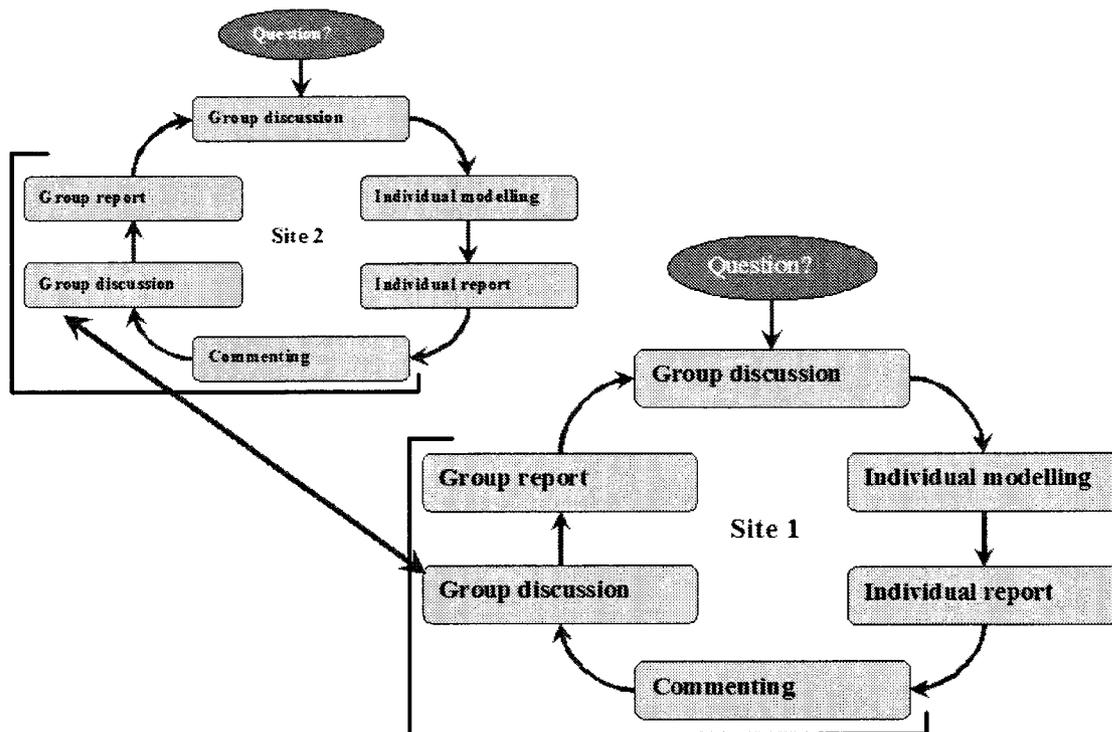


Figure 34: The Weblabs common activity framework (adapted from Mor et al., 2006). A motivating question triggers a classroom discussion, out of which conjectures are derived and explored by individual modelling. Students publish a report, which goes through a round of peer-comments, leading to a group discussion and a consensual report. The process may be mirrored in a remote site, with cross-commenting between groups. The final report leads to new questions for exploration.

The Weblabs common activity framework continuously flows between individual and social spaces of learning. Group discussions lead into individual explorations, which in turn inform more educated discussions. This flow is mirrored in the design of media. Each webreport began its life-cycle from a shared template, often tailored to questions derived from a specific group discussion. By using this template to create a report, the student transports it – and herself – from the public to the private space. The report is located in the student’s personal folder and remains there until she chooses to share it. This report will serve as the starting point for individual modelling in ToonTalk, as well as an intermediate storage for models and as a research diary of sorts. Students move back and forth between the ToonTalk environment and the WebReports system throughout the individual modelling phase, drawing tools from the system to their personal environment, consulting tutorials, or uploading drafts and components of their work. As they proceed with their tasks, they note down observations. Eventually these mature to a report which hopefully contains a coherent argument. That argument is expressed in text, graphics and code. Once they are comfortable with it, they click the ‘publish’ button and the report is listed in the group and topic indices. With this, they are relocated from the individual realm of learning to the social. The interweaving of individual and social was not limited to the textual components of the reports. Students used graphs, diagrams and code as means of communication. The Objects to think with became Objects to talk with. Indeed, the GmR game described in section 7.3 was based on the principle of expressing and exchanging

ideas embodied in code. The commenting mechanism of the WebReports system was designed specifically to meet this challenge.

To recap, the synergy between individual and collaborative learning is expressed in the flow of reports from personal to shared space and in the typification of group reports. It is also expressed in the construction and exchange of digital artefacts which capture students' emerging knowledge in diverse representations. The features which were designed to facilitate on-line collaboration eventually restructured our classroom practices. Group discussions were often initiated by reviewing personal reports. The availability of live models within these reports enabled us to bring the models into a live classroom discussion as well.

The constant flow between individual and collaborative learning spaces is illustrated in the convergence and divergence activity described in section 7.4.

Reflections

The common activity framework and its technological embodiment may seem intuitive and straightforward. The structure presented here is the form it took by the end of the third design iteration. This was preceded by many messy drafts of both the pedagogical structure and of the supporting technological infrastructure. The refinement of practice was echoed by the evolution of technology, as demonstrated in the elaboration of the types and uses of webreports, described in the previous sections. Another example is the mechanism of embedding code in reports. The first prototypes of the WebReports system required a somewhat complex sequence of actions to embed a ToonTalk model in a report page. Having realized that this is an action which all users need to perform on a regular basis, I directed significant effort at streamlining it. New ideas needed to be tested at minimal development costs in order to allow these changes to emerge. This was achieved by using open source systems which allowed enough flexibility to experiment and explore various ideas before fixing them as system features. Social conventions often bridged the gap between development iterations.

The principle of narrative spaces is yet another example of the co-evolution of the system and our understanding of its potentials for learning. ToonTalk was not intended to be used as a narrative tool, yet an awareness of the epistemic importance of narrative enabled us to identify when it was instrumentalized in this manner. Likewise, the flexibility of the WebReports system was initially a consequence of implementation constraints: having used a Wiki for prototyping, the ability to impose structure was limited. This limitation gave learners the ability to express themselves in narrative form. The analysis of learning highlighted the virtues of this ability, and its contribution to individual and social learning. Consequently, it was consciously retained and enhanced in the design of the final version of the system.

7.5.2 RN 6: Eager Robots

Overview

The disappointing results from the first trial of the stream-based activities led me to question their fundamental design, and by extension – its epistemic basis. At the same time, I considered usability factors which may have presented obstacles.

Sources

1overN.24jan05, Lessons_Learnt_Convergence_IOE

Situation

This design narrative concerns the transition between iteration 1 and 2.

Task

My first attempt at using STREAMS was far from encouraging. Learners succeeded in constructing the individual robots, and with significant effort and guidance managed to chain them to produce the sum series. However, they were not able to explain the outcomes or derive any mathematical understanding. The results were so poor that I did not even consider conducting interviews or collecting observations. As a temporary measure, I reverted to a non-modular design, where learners constructed a single-function robot for each task. This approach appeared to be somewhat more successful, but the actual causes of success and failure were unclear.

During the summer break I set out to redesign the tools and activities in light of the outcomes of iteration 1.

Actions

My main conclusion was that the practice of chaining robots was too complicated. I decided to relax my requirements for generality and modularity, in favour of the mathematical learning aims. I redesigned the task based on the successful second session trial. I went as far as speculating on the deep epistemological reasons for this failure, and generally the inadequacy of requiring modular programming.

At the same time, I considered a possible interface design factor affecting the “readability” of the chain of robots. In the original design of ToonTalk, it was expected that only one robot (or team) would operate in a house. Different teams, possibly communicating by birds, would be sent by trucks to other houses. To make the actions of the robots easier to perceive by the programmer, robots were designed to be “eager” to show what they were doing, shifting location to perform their work in the programmer’s field of view. This “eagerness” of robots to show their actions is very useful in the original design. However, when several robots are working concurrently in the same house, and competing to display their actions, the resulting image is extremely confusing (Figure 35).

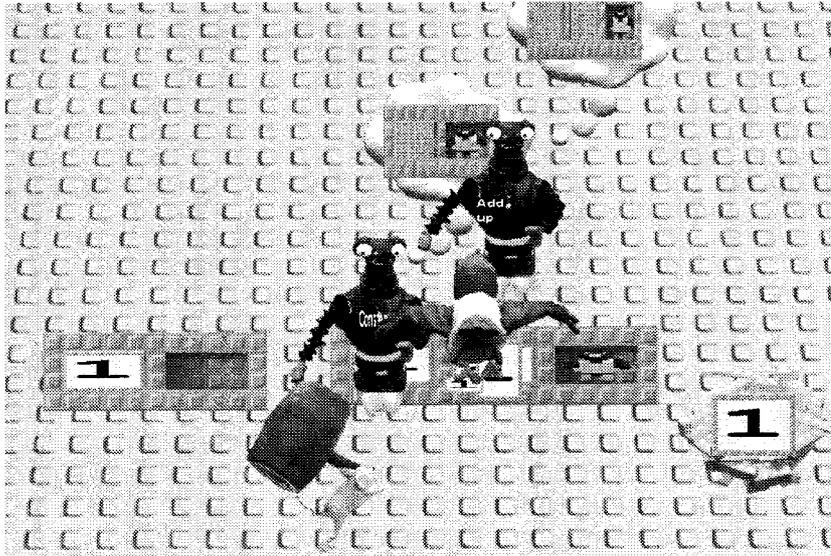


Figure 35: "Eager" robots. In the original design of ToonTalk, robots would always move into the viewport to show the user their actions. As a result, two or more robots working concurrently were impossible to follow.

The solution was to make robots less "eager" to display their work; restricting each robot to run only in the view frame in which it was invoked. Thus, by placing several chained robots in a reasonable distance on the floor, the programmer could move from one to another, and observe each robot's actions without being distracted by its peers (Figure 36).

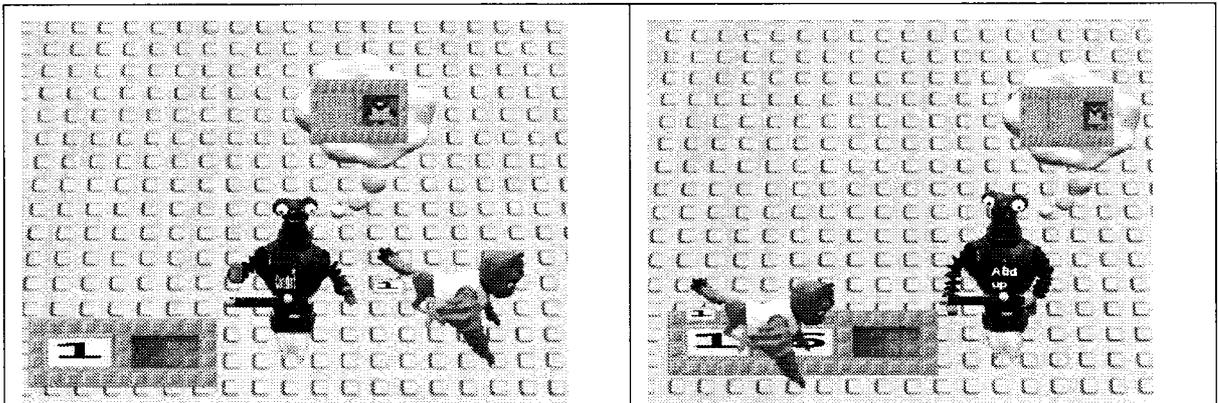


Figure 36: Not-so-Eager robots. In response to my suggestion, Dr. Ken Kahn modified ToonTalk's behaviour, so that robots would remain within their original space. The user could scroll across the floor to observe each robot's actions.

Results

Turning off robots' eagerness seemed a subtle change of interface design. Yet its effect was clearly seen in the results of iteration 2. Learners in several sites used the stream based activity design to successfully explore a range of mathematical subjects, as described in sections 7.2, 7.3 and 7.4. The non-modular design was abandoned, along with its epistemological justification.

Reflections

The primary lesson implied by this narrative is one of modesty and caution. In my disappointment at the failure of my design, I resorted to far-reaching conjectures regarding

learners' perception and epistemology. These conjectures were refuted by a minor change to the interface of the tool I used. Had I used a commercial product, such a change would not be possible. It was only my rapport with ToonTalk's designer which allowed me to test the "eager robots" hypothesis, and differentiate interface design from mathematical epistemology.

By extension, this incident highlights the difficulty of designing technologically enhanced environments for mathematics education; this is an endeavour in which the technology, mathematics and education are inseparable. The designer needs to have the capacity to manipulate all variables in order to optimise the products of design – both in terms of the theoretical and the practical outputs. The appropriation of the STREAMS design pattern in my activity design embodies mathematical and pedagogical ideas, but these could only manifest themselves through a suitable user interface.

The accessibility of chains of non-eager robots, in contrast to eager ones, can be interpreted as an issue of narrative structure. In the non-eager case, the programmer can follow the narrative of the program's execution. She can observe one robot after another playing out their role. In the eager version, it is never clear "what" is happening to "whom".

7.6 Conclusions

This chapter presented nine design narratives: six RNs and three LNs, as a demonstration of the design narrative construct and its role in the cycle of design research in technology enhanced mathematics education (TEME).

The narratives presented here cover the three main subject themes and the supporting infrastructure, over the three years of design experiments. This coverage is far from comprehensive; the narratives were chosen to illustrate a variety of scale, data sources, and analytical foci.

These narratives expose insights at different levels. Some are unique to the situation at hand, some generalise to a broader scope within TEME, and some reflect on the nature of design science, as practiced in this study. The situation-specific outcomes should be read in the scope of the narrative they emerge from, and thus will not be repeated here. Several generalisable observations stand out: the potential of the STREAMS design pattern; the importance of combining construction, communication and collaboration; the role of narrative; and the recursive intuition of sequences. However, generalising from a single example is a dangerous practice, as is the proclamation of “universal truths”. Instead, local theories need to be identified by recognising patterns across narratives: this is the focus of the next chapter.

STREAMS proved apposite to teaching and learning about number sequences. It allowed learners to distinguish between process, parameters and product, construct complex mathematical entities from simple blocks, appreciate the characteristics and behaviour of classes of sequences, and articulate sophisticated arguments. At a meta-level, this observation raises the question of software design patterns as educational tools: design patterns might make it possible to provide participants in constructionist learning environments with the knowledge of how to create their tools, rather than use pre-fabricated tools. This should be an advantage where the tool embodies a mathematical idea in its very structure, or when understanding the workings of the tool is required for understanding its effects and uses. This conjecture calls for further research, which is beyond the scope of this study.

A theme which emerges through the LNs is the role of learners’ construction and publication of narratives as a mediator between personal experience and shared structured knowledge. This theme calls for some distinctions and qualifiers.

The first distinction concerns the intentional mode of the narrator. Some narratives aim to flatly recount events, others interpret them, and others use events as elements of an (perhaps implicit) argument. Examples of the first mode are the texts in the descriptions Luminardi and Superpat313 provide with their robots in RN2 (7.2.2): *“I trained the robot to add one to the current number by putting the box over the robot, then when I am in the robots thoughts, I copied the one and added it onto the current I then copied the current and gave it to the bird to put in the nest”*. Rita’s response to the researcher’s first question (7.3.2) is an example of the second mode: *“I try explain. To create my sequence I thought thus: My first term is 2 and each one of the other terms is gotten of the previous one adding 2 and multiplying 4 to the result.”* Joe999’s robot (7.3.3) is a (non verbal) example of the last mode. This robot acts out Joe999’s actions, as a proof for the claim “you are halving the number you halved before”. These modes

are correlated to Scardamalia and Bereiter (1987) models of text composition. The first mode maps to knowledge telling, while the last two correspond to knowledge transforming. In other words, narratisation in itself does not imply meaning making. A superficial recount of actions does not extract any knowledge from them. It is only when the narrative is produced as part of an inquisitive conversation that it engenders meaningful learning. Sodapop's webreport (7.4.2) shows a transition from the second mode to the third. He first offers a reflective account of his actions, then presents a mathematical claim and provides a supporting argument in narrative form. Scardamalia and Bereiter (1987) identify the knowledge transforming model with experienced writers. This does not seem to hold here: learners were not given any specific guidance in writing, and did not have enough time to develop their skills independently. What appears to differentiate the different modes of narrative is the narrator's intention in the act, derived from the conversational context. When producing the narrative is seen as an end in itself, or a means to demonstrate that a technical task was completed as required, it does not embody any knowledge. When it is a means to communicate ideas it becomes a tool for manipulating and developing these ideas.

A second distinction is between narrative as an epistemic structure and a conversational form. Rita's response to the researcher's questions is a reflective account of her experience. It is an example of the genre transposition model proposed in section 4.2. By contrast, when Joe999 says *"After 10 minutes I figured out how to do the sequence. You take away 3.5. Then you find half of 3.5 and take that away from 11 and continue this sequence"* and when Sodapop says *"if you have 0.1 and you double it and add it together you will get 0.15 so every time you do this you will get another number after the decimal place"* they are using the form of narrative to express a mathematical claim or argument. Although their expressions are formulated in the mould of "something happening to someone" these are not references to real protagonists or specific events. Narrative form is used simply as a convenient and familiar formula for communicating meaning. The transition from narrative structure to narrative form can perhaps be seen as a step in the genre transposition to propositional expression of generalised knowledge. When using narrative as a syntactic formula, the narrators have already detached themselves from the concrete experience, abstracting away from it and making generic claims. The next step is to acknowledge that the narrative form does not offer the rigour and precision required and seek alternative formalisms. An example of this can be seen in Rita's response to the Cypriot team.

Finally, examples such as Joe999's robot or Sodapop's integration of graphs in his report exemplify the diverse modalities in which narrative can be expressed. Alshwaikh (2008) suggests a narrative dimension to the interpretation of mathematical diagrams. Sinclair, Healy and Sales (2009) propose a narrative interpretation of dynamic geometry environments. Acknowledging the importance of narrative in the process of learning mathematics entails that we need to develop frameworks which allow us to apply a narrative lens to a wide range of student (and teacher) expressions – but also clearly discriminate where this lens is relevant and where it is not.

All the activities, in their final form, encourage learners to assimilate mathematical concepts through a combination of construction, communication, and collaboration. The technological infrastructure evolved to support the streamlined integration of these modes of action. Construction challenges preconceptions and seeds new ideas, but these need reflection to be

transformed to structured knowledge. Communication can drive reflection, but only if it can draw on substantial relevant experiences. Collaboration is a powerful motivator of sustained communication, and consequently reflection.

The synergy of construction, communication and reflection echoes the fundamental epistemic model proposed in Chapter Four. It highlights the importance of narrative in constructing mathematical knowledge, through a transition from action to narrative to propositional discourse – guided by the activity design and the teacher’s interventions. Nevertheless, careful reading across the DNs in this chapter raises several distinctions which call for further investigation.

Section 6.2.4 distinguished between the closed form representation of number sequences and the recursive form. Several of the LNs demonstrate how the naïve concept of number sequences tends towards the recursive form. As noted in 6.2.4, this does not entail a comprehensive understanding of recursion: the structures articulated by learners are tail recursions, equivalent to iterative processes, and do not refer to issues of embedding or termination. Allowing learners to encode their intuitions in ToonTalk evinced how this concept is no less mathematical than the indexed view. Activities only gained momentum once their design acknowledged learners’ intuitions and was adjusted to harness them, rather than try to fight them. On the other hand, the initial iterations of design were necessary in order to expose those intuitions.

Among the more general themes that surface are the co-evolution of technology and pedagogy, the interdependence of interface and substance, and consequently the fluidity of design and the need for flexibility and malleability.

The initial choice and design of technology was driven by embryonic pedagogical ideas, yet only the interaction with these ideas through the technology allowed them to mature. The technological choices influenced the direction in which the pedagogy evolved, and in turn the pedagogy dictated the trajectory by which technology developed.

This symbiosis between technology and pedagogy, or epistemology, was expressed in the fundamental, structural layers – but also at the level of the interface. The best ideas, expressed through the best technology, will fail if the interface by which the user interacts with the technology is not tuned to the underlying concepts and methods. This observation reiterates Herbert Simon’s emphasis on representation as a characteristic of design science, noted in Chapter two.

One theme that presents itself as a question for further research is the notion of learners as design scientists. Several parallels were drawn between my process of research, and students’ learning. These parallels should be further elaborated, and perhaps used explicitly as a basis for educational design.

Finally, a lesson that shines through these design narratives is that no element of design, technological as pedagogical, has guaranteed effects on its own – only their careful assembly. Solutions are context dependent. This calls for means of analysing the narratives to identify and articulate individual elements of effective design, and then synthesising those to devise solutions for novel problems. ToonTalk, WebReports, and their integration were conducive to effective

learning. But the same effects could have been obtained with alternative media, as long as these would have exhibited certain qualities. These qualities transpire as the narrative's corollaries. Yet in their current form they are scattered and isolated. Chapter eight addresses this need by offering a collection of design patterns, as a step towards a pattern language of construction, communication and collaboration in technology enhanced mathematics education.

Chapter 8 Design Patterns

This chapter responds to Aim 3 identified in Chapter three: to apply the methodology in the problem domain and demonstrate its potential by producing a contribution towards a pattern language for technology enhanced mathematics education. This chapter presents a collection of design patterns, and the links between them, as derived from the design narratives in Chapter 7.

8.1 Introduction

Chapter four presented several elements for an epistemic infrastructure for a design science of technology enhanced mathematics education (TEME): a design experiment cycle, embedded in a wider cycle of design research (section 4.1.1) and the constructs of design narratives (section 4.4) and design patterns (section 4.5). Chapter five elaborated these elements into a methodological framework oriented towards a particular research context. This framework proceeds from a theoretical review to an empirical study, reported as a set of design narratives, and ultimately processed into a collection of design patterns. Chapters six and seven demonstrated the first two steps with respect of the chosen problem domain, and this chapter completes the cycle with a set of derived patterns.

The nature of the methodology is such that the elucidation of one design pattern often gave birth to several new ones, making the circumscription of the pattern collection a challenge. A common solution among pattern writers, which I adopted here, is to focus on a subset of the patterns and provide “thumbnails” of the others: short descriptions which outline their essence, so that they can be referred to without a full specification.

The patterns in this chapter are all structured in accordance with the template described by Table 8 in section 5.5.2. They were derived by the methods described in section 5.5 and validated by the process described in section 5.5.1. As a convention, all references to patterns are in SMALL CAPS.

8.1.1 Choice of Patterns and Relations between Them

The design patterns presented in this chapter were all derived from the design narratives of Chapter 7, by the process defined in section 5.5. This process yielded close to twenty patterns. A full specification of the whole set would have been outside the scope of this study. I therefore selected a representative sub-set as a demonstration of a prospective pattern language for TEME. The patterns I chose to elaborate are those which appeared to be more innovative. Other patterns are included as thumbnails to provide a coherent picture. The space covered by these patterns spans two dimensions: scope and focus, as illustrated in Figure 37. The shaded boxes represent patterns which are fully specified, whereas the hollow ones are presented as thumbnails. The scope axis varies between the concrete and the general. The more general patterns apply in a broader range of situations, but call for further elaboration before they can be implemented. The more concrete patterns are extensions of the general ones which can be readily implemented, but in a relatively narrow range of circumstances. The focus axis shifts the balance between the pedagogical and the technical. All patterns have both pedagogical and technical aspects, but their weight changes. For example, TASK IN A BOX is aimed at solving a

particular usability issue, and therefore is in the concrete-technical quadrant. By contrast, POST LUDUM describes a pedagogical activity structure which should be useful in most collaborative learning situations and is almost independent of technological choices.

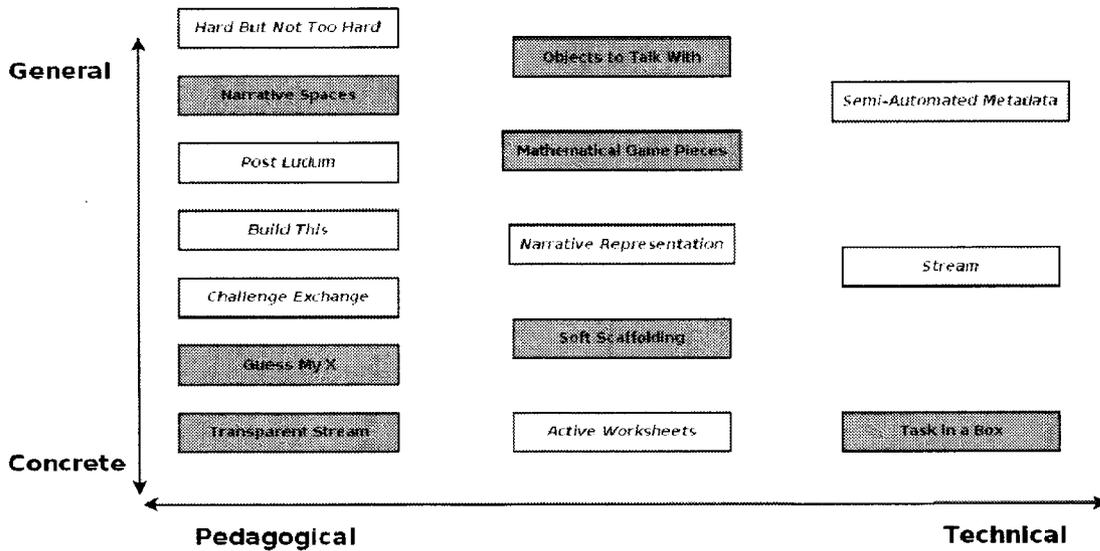


Figure 37: Distribution of patterns across axes of scope and focus. The shaded patterns are those described in detail, whereas the unshaded ones are referred to by other patterns and listed as “thumbnails” in section 0

Design patterns are linked to form pattern languages. Patterns are used by others as components, or elaborate others by adding detail and projecting them into a more focused scope. Figure 38 offers a map of the eight patterns presented in this chapter, along with eight patterns they referred to, but which are not specified in full detail. The latter are listed by name and summary, or “thumbnails” as these are often called in the pattern literature, in section 8.9. The map represents the hierarchical, structural and lateral links between patterns, as specified in Table 8. For example, GUESS MY X extends MATHEMATICAL GAME PIECES and uses OBJECTS TO TALK WITH as a component.

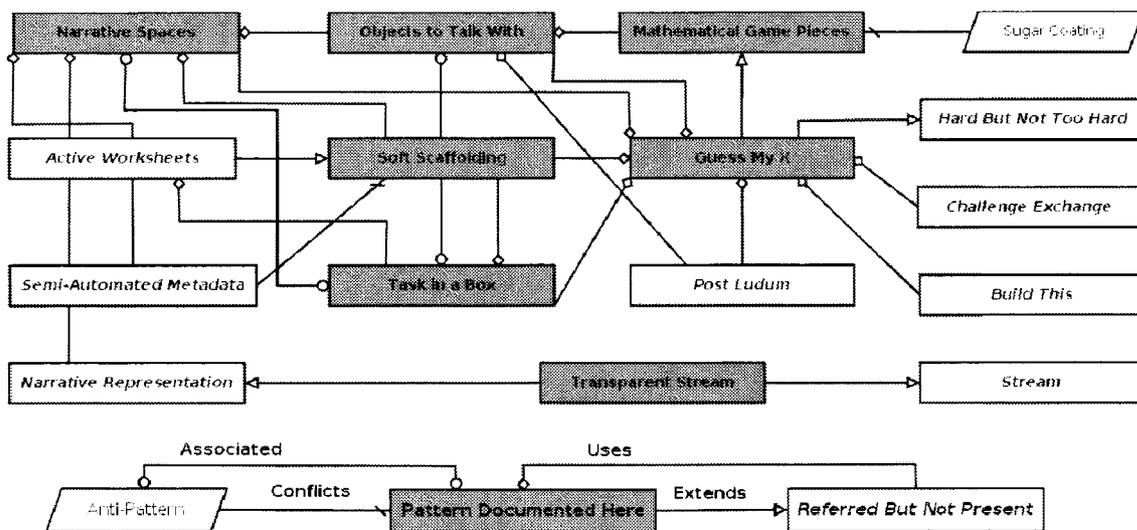


Figure 38: The design patterns in this chapter and the relations between them.

8.1.2 From Design Narratives to Design Patterns

The general method of deriving design patterns from design narratives was enumerated in section 5.5. Some patterns were derived directly from the design narratives, or even in the course of the preliminary analysis. Others emerged from the refactoring process of the language as a whole. The TRANSPARENT STREAMS pattern originated as a pedagogically-oriented extension of the STREAMS pattern prevalent in software engineering. Figure 39 illustrates the derivation paths of patterns from initial sources, and Table 10 lists the design narratives supporting each pattern.

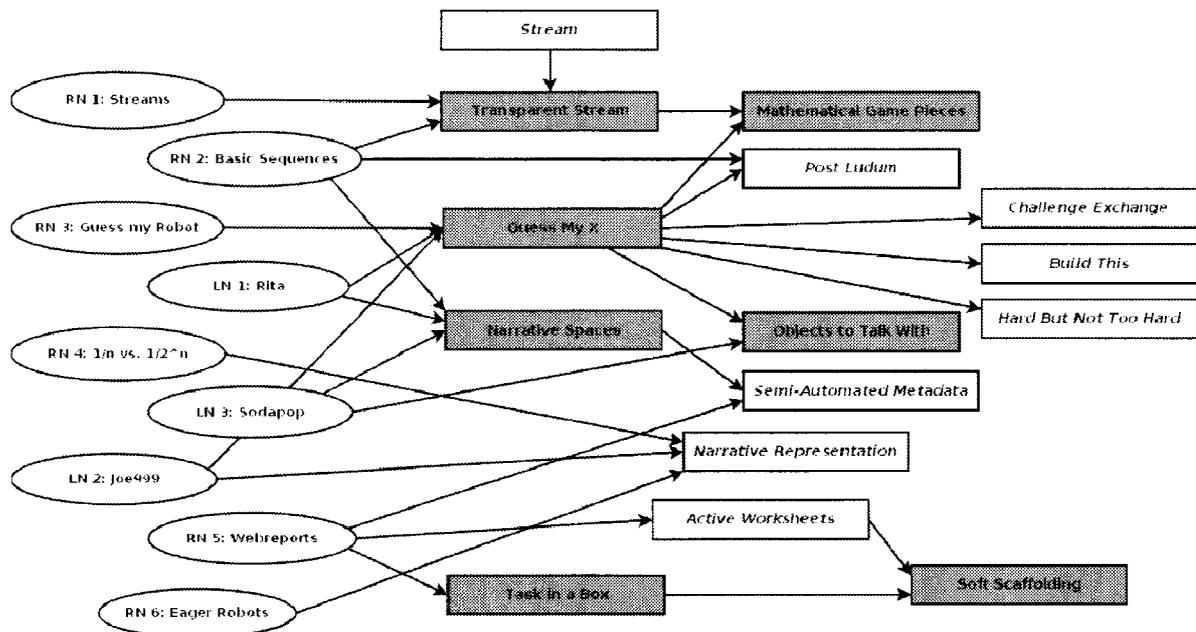


Figure 39: Design patterns initial sources. The ovals on the left note the design narratives in Chapter 7. Arrows link to the patterns derived from them. Other patterns were derived from these in the course or refactoring.

Design Pattern	Source Narratives / Patterns
MATHEMATICAL GAME PIECES (8.2)	RN 3 (7.3.1), LN 1 (7.3.2), LN 2 (7.3.3)
SOFT SCAFFOLDING (8.3)	RN 2 (7.2.2), RN 3 (7.3.1), LN 1 (7.3.2), LN 2 (7.3.3)
NARRATIVE SPACES (8.4)	RN 2 (7.2.2), RN 3 (7.3.1), LN 1 (7.3.2), LN 2 (7.3.3),
OBJECTS TO TALK WITH (8.5)	RN 2 (7.2.2), RN 3 (7.3.1), LN 1 (7.3.2), LN 2 (7.3.3), LN 3 (7.4.2)
GUESS MY X (8.6)	RN 3 (7.3.1), LN 1 (7.3.2), LN 2 (7.3.3)
TRANSPARENT STREAM (8.7)	RN 1 (7.2.1), RN 2 (7.2.2), RN 3 (7.3.1), RN 4 (7.4.1), LN 3 (7.4.2)
TASK IN A BOX (8.8)	RN 5 (7.5.1), RN 3 (7.3.1),
NARRATIVE REPRESENTATIONS	RN 4 (7.4.1), LN 2 (7.3.3), RN 6 (7.5.2)
POST LUDUM	RN 2 (7.2.2), RN 3 (7.3.1), LN 1 (7.3.2), LN 2 (7.3.3)
HARD BUT NOT TOO HARD	RN 3 (7.3.1), LN 1 (7.3.2), LN 2 (7.3.3)
CHALLENGE EXCHANGE	RN 3 (7.3.1), LN 1 (7.3.2), LN 2 (7.3.3)
BUILD THIS	RN 3 (7.3.1), LN 1 (7.3.2), LN 2 (7.3.3)
ACTIVE WORKSHEETS	RN 2 (7.2.2), RN 5 (7.5.1), LN 1 (7.3.2), LN 3 (7.4.2)

Table 10: Design Patterns supporting cases

The evolution of the GUESS MY X pattern demonstrates the process by which patterns were derived from narratives. The initial trials of the Guess my Robot game indicated that this activity is highly successful in promoting an understanding of the relationship between process and product, and in establishing a discourse of complex mathematical notions of equivalence and proof. These indications were, however, limited to the domain of number sequences and associated with the specific implementation of the game. As a first step of analysis, a design narrative of the game and its outcomes was drafted and shared with the WebLabs team. Based on this narrative, two members developed similar games in different domains – Guess my Graph (in the domain of functions and graphs) and Guess my Garden (in the domain of randomness and probability). Comparing these three games highlighted which features of the design are specific to a particular instance and which are part of a common pattern.

Consequently, a first version of the pattern was drafted. However, this draft was unreasonably complex in its details and needs to be broken down into more compact elements. First, commonalities with other games (e.g. the *Programming Building Blocks*) suggest more general patterns such as CHALLENGE EXCHANGE and BUILD THIS, which GUESS MY X was described as an extension of. On the other hand, some elements of the pattern were described as components which can be used independently, such as OBJECTS TO TALK WITH, LEAGUE CHART and ACTIVE WORKSHEET.

As a second iteration, the data was systematically reviewed to derive comprehensive versions of the guess my robot researcher and learner narratives, as part of the set of design narratives collated for this thesis. These were then analysed to validate, elaborate and refine the design patterns presented here. Thus, while the initial draft of design narratives shared with team members may have followed a knowledge telling model, the process in this iteration was a knowledge transforming one. The construction of design narratives involved the manipulation of two domain knowledge models: a model of learning in a mathematical subject area, and a model of designing technology for learning. The patterns were then revisited to ensure they provide an explicit articulation of the later.

Section 5.5.1 lists three mechanisms of pattern substantiation: expert review, theoretical alignment and empirical calibration. The patterns listed here were submitted to *expert review* by team members of the WebLabs and Learning Patterns projects, followed by consultation with senior researchers in mathematics education and learning technology, and latter presented to relevant professional and academic communities at workshops and conferences. Theoretical alignment was linked to the review in Chapter 6. Empirical calibration drew on experiments with the various groups listed in section 5.2.1, as noted in Table 3. Occasionally, additional empirical support was derived from design narratives (case studies / case stories) from the Learning Patterns and Planet projects. The *support* section of each individual pattern includes a short note on its origins, notes the calibrating examples and the theoretical warrants.

8.1.3 Structure of this Chapter

Each section from 8.2 to 8.8 describes a single pattern. For convenience, each section starts on a new page. Section 0 provides the “thumbnails” of patterns used or mentioned in the course of the fully specified patterns. Section 8.10 concludes with some observations and general comments.

8.2 Mathematical Game Pieces

Mathematical content is often injected artificially into games or other activities, as SUGAR-COATING. This has a dual effect of ruining the game and alienating the mathematics. By contrast, for many mathematicians, mathematics is the game.



8.2.1 The problem

How do you design (or choose) a game to convey mathematical ideas in an effective and motivating manner? How do you judge if a proposed game is an adequate tool for teaching particular mathematical concepts?

8.2.1.1 Forces

- A game used in education has to provide a good game experience, or else it is “just another boring task”.
- Learners need to engage with the mathematical content that the game aims to promote.
- The chosen representations need to be consistent both with the game metaphors and with the epistemic nature of the content domain.

8.2.2 Context

This is a high-level pattern, with a broad scope. It is relevant for -

Educators wishing to use games as part of their teaching, either evaluating existing “educational” games, appropriating “entertainment” games, or designing and developing their own games.

Developers wishing to develop games for the educational market.

Origins

This pattern is based on the experience of the WebLabs project, and a review of many other games designed for learning mathematics.

Extension

This pattern takes a liberal interpretation of the term game, and would probably apply to many design scenarios which are not commonly considered as games. The key factor defining the situations where this pattern applies is that the designer is striving for an entertaining, self-motivating activity which affords learners a high degree of autonomy.

Boundaries

This pattern would probably not be applicable where the mathematical content is secondary to other learning aims, such as passing an exam.

8.2.3 Solution

- Identify an element of the mathematical content to be addressed in this game.
- Identify the game's design space: its genre / format, graphical style, typical props and artefacts, central metaphors, etc.
- Find a visual, animated or tangible representation of this element which is consistent with this design space.
- Design the game so that these objects have clear PURPOSE AND UTILITY as game elements in the gameplay structure.

The objects representing the mathematical content should have a meaningful intrinsic role in the game. Manipulating these objects can be part of the game rules or goals, or understanding them could be a necessary condition for success.

If the game includes or is followed by communication between participants, then the mathematical game pieces should become OBJECTS TO TALK WITH.

8.2.4 Support

Theoretical

This reasoning behind this pattern can be traced back to the constructionist approach. One of the principles of this approach is that effective, deep learning of mathematical concepts is driven by self-driven engagement with representations of these concepts in activities where there is a genuine need for these concepts or their representation (diSessa and Abelson, 1981; Papert and

Harel, 1991; Papert, 1996; Noss & Hoyles, 1996). In most constructionist settings, these activities involve the construction of artefacts. The rationale is that there is a parallel between the construction of external and mental structures; the manipulation of mathematical representations to create complex artefacts leads to the internalisation of complex mathematical concepts.

This pattern is based on the observation that the basic principle holds even when the learner is not required to construct new artefacts in the conventional sense. Habgood (2007) considers the effectiveness of digital games as learning media, and highlights the importance of *intrinsic integration* of the domain content into the game design. However, he rejects the focus of previous writers on *intrinsic fantasy*: a dependency between the skill being learned and the fantasy, or narrative, of the game. Habgood claims that the fantasy is seen by players as a superficial quality of game, whereas the core of gaming experience is in the game mechanics: the underlying system of rules governing gameplay. Consequently, he proposes a definition of intrinsic integration which demands a tight dependency between the learning content and the mechanics – embodying the learning content within the structure of the game world and the players’ interaction with it, thus providing a reified representation of the content which is inherently explored through the core mechanics of the game.

Empirical

This pattern originates in the Guess my Robot game (section 7.3). It was initially calibrated by reference to several games developed by my colleagues at the WebLabs projects. These included the motion graphing activities designed by Gordon Simpson (Figure 40) and the Guess my Garden game (Figure 41), designed by Michele Cerulli.

The motion graphing activities included two games. In the first, students controlled the speed of an animated car, predicted and draw the corresponding position–time and velocity–time graphs. In the second, they controlled the initial location, speed and acceleration of an animated rocket to produce graphs which they posted online and challenged their peers to replicate. In both cases, the numerical measures of speed and acceleration were linked to their time graph representation via an animated vehicle. All three elements played a meaningful role in the game.

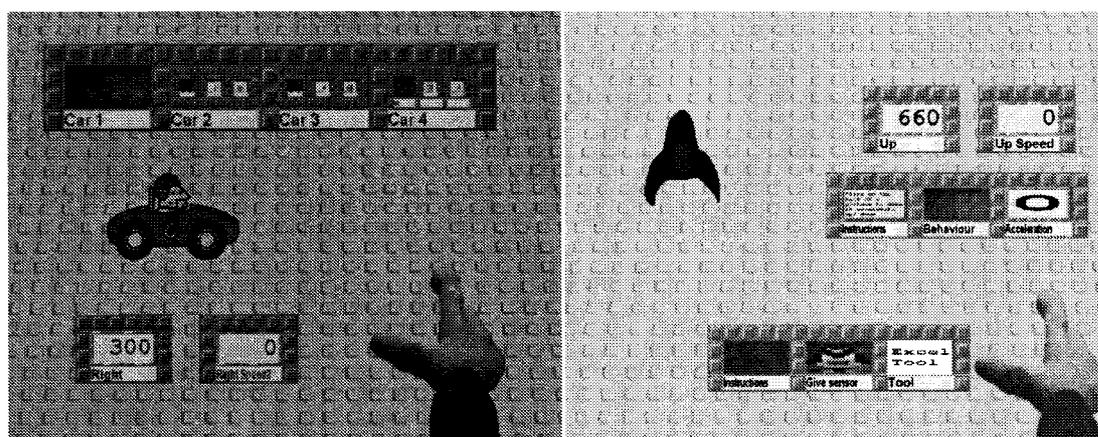


Figure 40: Motion graphing games, reproduced with author’s permission from Simpson, Noss and Hoyles (2006)

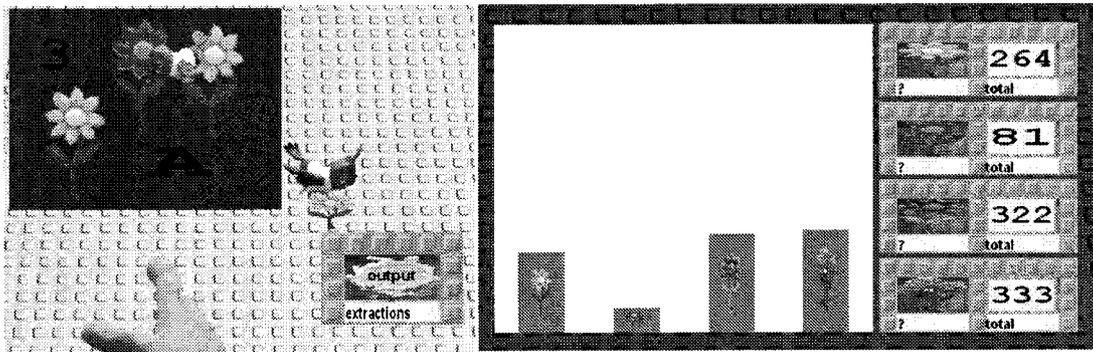


Figure 41: Guess my Garden game, reproduced with author's permission from Cerulli, Chiocciariello and Lemut (2007)

Learning Patterns project provided further support for this pattern by calibrating it with games developed independently and unawares of the WebLabs project. In Chancemaker (Pratt, 1988) users manipulate the odds of various chance devices, such as dice and roulette wheels. The game pieces are representations of probability (Figure 42).

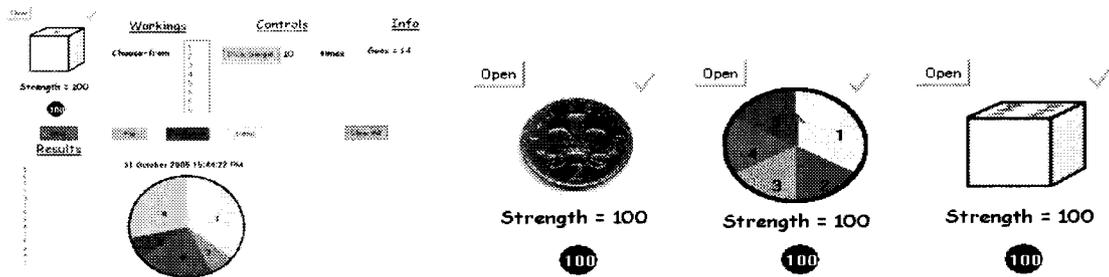


Figure 42: Chancemaker gadgets are probability game pieces

In Programming Building Blocks (<http://www.thinklets.nl>) the object of the game is to reconstruct a 3D shape from its 2D projections. The mathematical content is the focus of the game, and the objects used in the game are straightforward representations of that content (Figure 43).

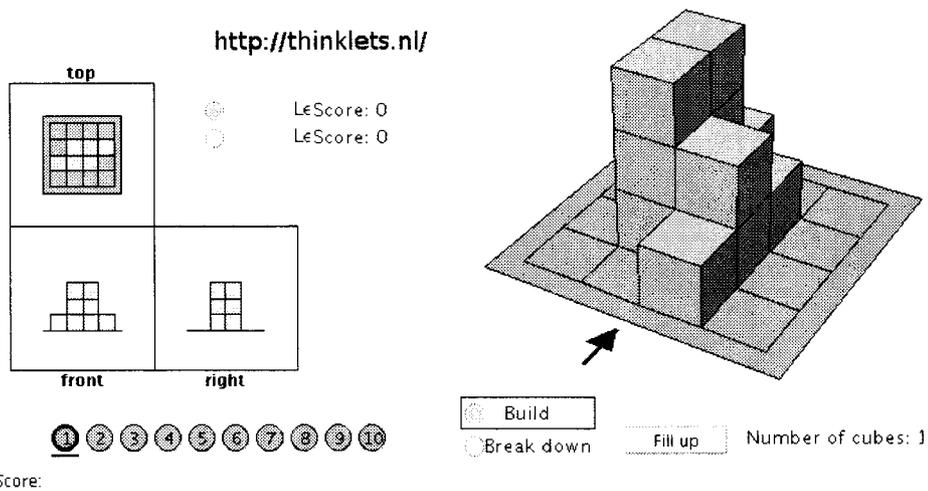


Figure 43: Building blocks constructions as 3D geometry game pieces

8.2.5 Liabilities

Risks

In some cases, the mathematical game pieces of a game present themselves in a straightforward manner. In others, significant effort is required to uncover them. This is a consequence of the relative high level of abstraction of this pattern, and can be addressed by choosing one of the sub-patterns which extends it in a more concrete manner.

The effectiveness of this pattern is contingent on learners using the mathematical game pieces in the way envisioned by the designer. However, players of a game may reinterpret the objects they are given, and miss the designer's intentions. To reduce this risk, it is important to conduct early and frequent experiments with game prototypes, and to embed the game within a coherent sequence of educational activities, e.g. follow it with a POST LUDUM discussion.

Limitations

In itself, this pattern does not guarantee that learners will find the game – and even more so the underlying mathematical content – engaging and entertaining. Once the genre and general structure of the game is identified, designers should consult the relevant collections of game and interaction design patterns to ensure a high-quality gaming experience.

This is a very high level pattern which needs to be elaborated per specific game and content classes. For example, in a quest type game it might spawn different sub-patterns than in puzzle type games. Likewise, factual and procedural knowledge might lead to different strategies than meta-cognitive skills. Nevertheless, it is useful as a guideline for evaluating design proposals. Its absence marks a game as a weak tool for learning.

8.2.6 Related Patterns

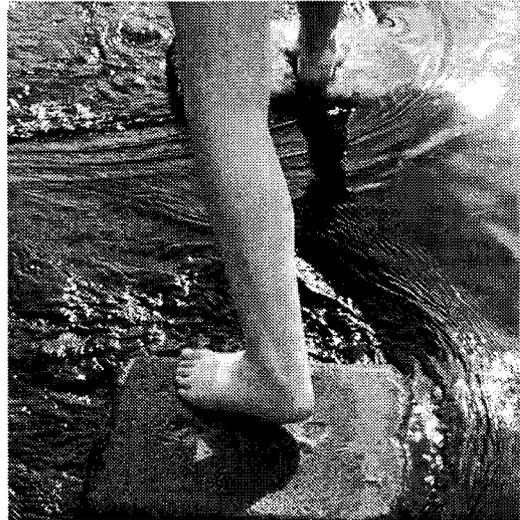
Used by: OBJECTS TO TALK WITH.

Conflicts: *SUGAR-COATING* (anti-pattern)

Elaborated by: GUESS MY X

8.3 *Soft scaffolding*

Technology should be designed to scaffold learners' progress, but an interface that is too rigid impedes individual expression, exploration and innovation.



8.3.1 The problem

Scaffolding is a powerful tool for accelerating learning. It is a fundamental principle in many interactive learning environments, such as OISE's Knowledge Forum, and is a guiding principle in Learner-centred approaches (c.f. Quintana et al, 2004). However, scaffolds can become straitjackets when they are too imperative.

How do you provide direction and support while maintaining the learners' freedom, autonomy and sense of self, as well as the teachers' flexibility to adapt?

8.3.1.1 Forces

- The role of the educator, and by extension educational tools, is to direct the learner towards a productive path of enquiry.
- If the educational tool adamantly leads the learner through a set sequence, it risks failure on several accounts:
 - There is less leeway for innovations, explorations or personal trajectories of learning, and "fortunate mistakes".
 - Learners feel deprived of personal voice, and their motivation may falter.
 - It is hard to bypass design flaws discovered in the field or adjust to changing circumstances.

8.3.2 Context

Scaffolding is a term commonly used in educational design to describe structure that directs the learner's experience along an effective path of learning. Many interactive learning support systems provide some form of scaffolding, either explicitly designed or as a side effect of other features.

Origins

This pattern emerged from the design of the WebReports system, and specifically its template mechanism. As such, it is directly applicable to similar scenarios, namely the design of tools for learners to report, reflect on and discuss their learning experiences.

Extension

The concept of scaffolding applies to any system which attempts to guide users along a trajectory of learning. This pattern is potentially relevant to any such system.

Boundaries

This pattern would not be productive in situations where learners need to follow very strict procedures, or when the activity is tangential to the learning aims, for example when asked to provide personal details for administrative purposes. In such cases, there is no value in personal expression, exploration and innovation.

Likewise, it is not likely to be useful in learning scenarios which call for unrestricted creativity and complete freedom of action.

8.3.3 Solution

Provide scaffolding which can easily be overridden by the learner or by the instructor. Let the scaffolding be a guideline, a recommendation which is easier to follow than not, but leave the choice in the hands of the learner.

- When providing a multiple-selection interface, always include an open choice, which the user can specify (select 'other' and fill in text box).
- When the user is about to stray off the desired path of activity, warn her, ask for confirmation, but do not block her.
- When providing templates for user contributions, include headings and tips but allow the user to override them with her own structure.

8.3.4 Support

Empirical

The ACTIVE WORKSHEETS used in the WebReports system (section 7.5.1) provided participants a structure to work with, but allowed them to take control and change this structure as their confidence grew (Figure 44).

Explore

Can you think of a way to use your robot to produce these sequences?

Sequence	Explain
	If yes, explain how you would do it. If you think it is impossible, explain Why. Add any ToonTalk object that helps support your argument.
2, 3, 4, 5...	
-1, -2, -3, -4...	
-7, -6, -5, -4..	
2, 4, 6, 8...	
5, -1, -7...	
Write down a sequence of your own, which can be generated by your robot.	
Write down a sequence of your own, which cannot be generated by your robot.	

Explain

How would you explain to a friend what kind of sequences your robot can generate, and how it can be used to generate those sequences?

Describe one sequence that cannot be generated by it, and explain why.

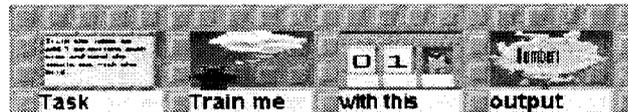
How would you convince a friend of your claims?

Figure 44: Example of Active worksheet. Learners were given a template in which to report on their exploration, but could edit it freely and replace the structure with their own.

The ToonTalk tool packaging convention, which was the basis for TASK IN A BOX, prompted learners to package their own productions in a particular way by providing them with useful examples. It did not block them from developing their own packaging style, but the ToonTalk-WebReports interface did give precedence to conventionally packaged constructions (Figure 45).

Training *Add number*

Train a robot to repeatedly add 1 to produce the numbers 0, 1, 2, 3.... Call the robot *Add number*.



(Click the box to get started, post your robot here when you're done)

Figure 45: Task-in-a-box demonstrating the ToonTalk packaging convention. Learners received their tasks in the recommended style for submitting their answers, but could edit and modify it to their preference.

8.3.5 Liabilities

Risks

One of the advantages of “rigid” or “hard” scaffolding is that it ensures that the content would be structured in a predictable manner, thus making it amenable for automated processing and allowing software systems to provide more intelligent support for learners.

Limitations

Any medium of expression has its inherent characteristics, determining what modes of use would be more immediate. Thus the choice of medium is an implicit choice of hard constraints which determine the scope of potential soft scaffolds. For example, if the chosen medium only supports plain text, then the designer cannot give the user a choice of embedding images.

8.3.6 Notes

The forces of this pattern are present in face-to-face learning situations. Experienced educators resolve them by providing ADAPTIVE SUPPORT; varying the learners' freedom in response to their confidence. This could be implemented by intelligent tutoring systems, but simple learning environments lack this flexibility, and tend to compensate by being over-directive.

Since its inception, this pattern has been recognised in a wide variety of TEME situations. In fact, when using off-the shelf technologies the scaffolding provided is often soft by necessity, since the technology was not tailored for a predefined use.

8.3.7 Related Patterns

Used by: GUESS MY X; NARRATIVE SPACES

Extended by: ACTIVE WORKSHEET

Conflicts: SEMI-AUTOMATED META-DATA

8.4 Narrative spaces

Constructing and interpreting narrative are fundamental epistemic mechanisms, by which humans make sense of events and observations. To leverage it, we must give learners opportunities to express themselves in narrative form.



8.4.1 The problem

How can the epistemic power of narrative be harnessed by educators and learners in the construction of mathematical meaning?

8.4.1.1 Forces

- Narrative is a powerful cognitive and epistemological construct (Bruner 1986; 1990; 1991).
- Mathematics appears to be antithetical to narrative form, which is always personal, contextual and time-bound.

8.4.2 Context

Digital environments for collaborative learning of mathematics.

Origins

This pattern emerged from the design of the WebReports system, and the observations of the ways in which it was used to support the learning on mathematics. It is therefore immediately applicable to similar environments, i.e. systems that support construction, collaboration and communication as drivers of learning mathematics.

Expansion

It may be easy to extend this pattern to other subject domains. However, it is important to stress its essential significance in the domain of mathematics, where narrative is often neglected.

Boundaries

Narrative assumes an identifiable narrator and audience. In environments where the learner does not have an opportunity to interact with other learners or instructors, this pattern is not likely to be effective.

8.4.3 Solution

Provide learners with a narrative space: a medium, integrated with the activity design, which allows learners to express and explore ideas in a narrative form. The effectiveness of a narrative space is contingent on its coordinated manifestation in two dimensions: a social and pedagogical space, and a communication medium. This medium should provide the capacity, or seamlessly integrate with other media, which support other forms of expression as appropriate for the learning activity: propositional statements, algebra, graphs, etc.

Pedagogical aspects

From a pedagogical perspective, learners need to be encouraged to express themselves in narrative structures and form, while making a clear distinction between narrative and mathematical formalism. Narrative expression includes both the articulation of learner experiences as reflective personal narratives and the phrasing of claims and arguments in a narrative style. The former is important as a means for learners to derive knowledge from their experiences; the later is a step in the formalisation of this knowledge.

Technical aspects

The technical implications of narrative space appear along two axis: the provision of capacity for learners to express themselves in narrative forms, both verbal and non-verbal, and the inclusion of narrative markers in interfaces to improve their readability.

- Allow for free-form text, e.g. by supporting SOFT SCAFFOLDING.
- Choose NARRATIVE REPRESENTATIONS when possible.

Mark narrative elements in the medium:

- Clearly mark the speaker / author, to support a sense of voice.
- Date contributions to support temporal sequentiality ('plot').
- Use semi-automated meta-DATA to provide context.

8.4.4 Support

Theoretical

Narrative learning environments (NLEs) have been gaining attention in recent years (Dettori et al, 2006). Narrative approaches to computer-enhanced learning are often focused on designing systems that support narrative-based learning (Mott et al., 1999; Decortis and Rizzo, 2002;

Decortis, 2004), i.e. systems that support the production of imaginary narrative as the site of learning. Nehaniv (1999) argues for a broader view, claiming that any design that does not acknowledge the ‘narrative grounding’ of humans will appear to its users as bizarre, unintelligent and unintelligible. Likewise, Laurillard et al. (2000) highlight the importance of embedding narrative structure in the design of multi-media resources, where non-linearity risks impeding learners from maintaining a personal narrative line and thus increasing cognitive costs.

Healy & Sinclair (2007) highlight the role of narrative in more personal acts of understanding. Many testimonies show an alienated experience of mathematics. This barrier can be breached by allowing space for learners’ personal narratives, relating mathematical meanings to their own experiences and reflecting on their individual learning trajectories. Sfard and Prusak (2005) go further, arguing that learning mathematics is a matter of establishing identity, and this is done through narratives we tell about ourselves. Several researchers have suggested that in order to provide learners with tools for coping with unfamiliar problems, they need to share the experiences of those who possess such tools. Burton (1996) argues that this points to a need to facilitate learners’ authoring of their accounts of how they came to know mathematics. Chapman (2008) suggests a similar approach to teacher education, highlighting stories of teaching mathematics.

Considering these observations with respect to TEME suggests that systems not designed explicitly for narrative production need to be narrative-aware. Users should be provided with adequate flexibility that enables them to express their ideas in narrative form. The constituents of narrative should be apparent in the interface design to make the content easier to grasp.

Empirical

The design of the WebReports system (section 7.5.1) embodied this pattern at several levels. One of the formative design decisions in the development of the system was to prefer open formats of reports and comments over rigidly structured ones. This format was utilised in the activity design, allowing authors to express themselves in narrative form. The value of this mode of expression is demonstrated in Rita’s guess my robot (section 7.3.2) and Sodapop’s report on convergence (section 7.4.2).

The interface of WebReports demonstrated a narrative meta-structure, by highlighting the author identity and contextual elements upfront.

ToonTalk’s narrative qualities may have contributed to its success in elucidating abstract ideas. ToonTalk’s programmes are represented by robot figures which work through the procedures step by step. Observing such a programme in action puts the mathematical concept into a narrative form that retains the rigour of algebraic symbolism, albeit in a radically different format. In the basic sequences activity (section 7.2.2), students built two ToonTalk robots. The first generates the natural numbers, and can be generalized to any arithmetic progression. The second takes a sequence as its input and produces a series of its partial sums.

Students found unexpected ways to use ToonTalk to narrate, or ‘narratize’ their mathematical arguments, as demonstrated in the case of Joe999’s guess my robot (section 7.3.3).

Similar phenomena can be observed in the case of Scratch and its community site (Monroy-Hernández and Resnick, 2008; Resnick, 2007; Maloney et al, 2004). Like ToonTalk, Scratch is a programming language designed to be used creatively by children of all ages. The Scratch language is coupled with a community web site which allows users to present their creations, comment on others' work, collaborate on projects and "remix" each others' code. Many users use Scratch to present a narrative of some form or other, and the project notes and discussions on the site both demonstrate narrative spaces.

8.4.5 Liabilities

Risks

One of the key characteristics of narrative is that it conveys knowledge implicitly; in the words of Walter Benjamin –

"Actually, it is half the art of storytelling to keep a story free from explanation as one reproduces it (...) the psychological connection of the events is not forced on the reader. It is left up to him to interpret things the way he understands them." (Benjamin, 1968)

This quality grants narratives their epistemic power and aesthetic beauty, but it is also an obstacle in the way of scrutiny and rigor. In order to have a critical discussion of claims, they need to be articulated explicitly and precisely. Narrative form is also at odds with a basic principle of mathematics: that the construction of new knowledge by de-contextualised symbolic manipulation. Consequently, it is important to remember that narrative, especially in its imaginative form, is an important step in knowledge construction – but not an end point.

8.4.6 Notes

Section 4.3 made a distinction between identity narratives and design narratives. This distinction can be extended to the realm of learning mathematics. An identity narrative tells a story of coming to be mathematical; a design narrative tells a story of doing mathematics: solving a problem mathematically. The design of educational activities should allow learners to share narratives of both types. As discussed in section 6.3.2, educators sometimes tend to discount learners' narrative articulations as "non-mathematical" or blatantly wrong. Instead, these articulations need to be acknowledged, and then transposed – in a process of dialogue – to a paradigmatic form.

8.4.7 Related Patterns

Uses: NARRATIVE REPRESENTATIONS; OBJECTS TO TALK WITH; SEMI-AUTOMATED META-DATA; SOFT SCAFFOLDING

Used by: GUESS MY X

8.5 Objects to talk with

Natural discourse makes extensive use of artefacts: we gesture towards objects that mediate the activity to which the discussion refers. This dimension of human interaction is often lost in computerized interfaces. When providing tools for learners to discuss their experience, allow them to easily include the objects of discussion in the discussion.



8.5.1 The problem

Several approaches to mathematics education highlight the importance of conversation and collaboration. The communicational approach (Sfard, 2008) equates thinking with communication, and sees learning mathematics as acquiring certain rules of discourse. Yackel and Cobb (1995) talk of the establishment of socio-mathematical norms through classroom discourse. Hurme and Järvelä (2005) argue that networked discussions can mediate students' learning, allowing students to co-regulate their thinking, use subject and metacognitive knowledge, make metacognitive judgments, perform monitoring during networked discussions and stimulates them into making their thinking visible.

Most computer-mediated discussion tools are strongly text-oriented, prompting users to express their thoughts lucidly in words or symbols. Yet two important elements of natural conversation are lost: the embodied dimension, i.e. gestures, and the ability to directly reference the objects of discussion.

8.5.1.1 Forces

- Conversation is a powerful driver of learning, it:
 - Prompts learners to articulate their intuitions and in the process formulate and substantiate them.

- Establishes mathematical norms of discourse.
- Enables learners to share knowledge and questions.
- Conversation is even more powerful when building on personal experience or constructing or exploring mathematical objects.
- However, text based conversation media may obstruct learners, by forcing them to describe verbally the objects of enquiry which they would naturally gesture at.

8.5.2 Context

Web-based collaboration and communication systems.

Origins

The pattern emerged from the design of the WebReports system, where learners shared and discussed ToonTalk code, charts, spreadsheets and other artefacts created in the course of investigating mathematical questions.

Expansion

This pattern is likely to apply to most interfaces which allow learners to converse about a common activity.

Boundaries

In some situations conversation originates from the production a shared, on-line, object, e.g. the co-authoring of a document. In such cases this pattern is somewhat trivial. In other cases the object of discussion is considered common knowledge (e.g. the Eifel tower) and its representation would be superfluous. Thus, the pattern is primarily relevant to contexts which involve two media: one used for construction, or production of artefacts, and the other used for conversations about these artefacts.

8.5.3 Solution

TOME activities often involve the use or construction of digital artefacts. When providing tools for learners to discuss their experience, ensure that they can include these artefacts in the discussion. Where possible, the discussion medium should allow embedding of these artefacts. Otherwise, the medium should support a visual (graphical, symbolic, animated or simulated) 1:1 representation of these objects.

Pedagogical aspects

In POST LUDUM discussions, the game's MATHEMATICAL GAME PIECES should become the OBJECTS TO TALK WITH. If the game is supported by a NARRATIVE SPACE, this emerges from the game flow.

Technical aspects

When providing a NARRATIVE SPACE, allow the user to seamlessly embed the objects of discussion in the flow of narrative, so that learners can refer to these objects in a naturalistic manner.

8.5.4 Support

Theoretical

As discussed in section 6.3.3, the creation and manipulation of artefacts as part of the learning process is a core principle of the constructionist paradigm. Conversation about personal experiences and derived observations, as a driver of learning, is fundamental to social constructivist approaches. This pattern brings the two together.

The idea of semiotic mediation – the role of artefacts as embodiments of historical-cultural knowledge – is a central theme in Vygotsky's theory (Vygotsky, 1987; Wertsch, 1998). This idea has been considered in the context of designing learning technology by the activity-theoretic tradition in HCI (Kaptelinin, 2003). At the same time, it is useful in understanding the personal and historical development of mathematical knowledge (Radford, 1998; 2005). Interfaces which do not afford the inclusion of artefacts or their visual representation lose the power of semiotic mediation.

Empirical

This pattern identifies one of the WebReports system's primary design objectives. When developing the final version of the system, significant effort went into providing streamlined integration, which would allow students to select objects in ToonTalk and with a few clicks embed them in a webreport. The embedded objects were represented by their graphical image. When clicked, this image would invoke the original ToonTalk object in the viewers' ToonTalk environment. Likewise, when the activity involved graphs, learners could embed these in their report.

The implementation of this pattern was critical to the success of activities such as Guess My Robot (section 7.3), where players embedded challenges and responses in their reports in the form of ToonTalk objects, and the convergence and divergence activity (section 7.4) where they embedded graphs of their sequences. Rita and Sodapop's LNs (sections 7.3.2 and 7.4.2) show how learners refer directly to the objects they had constructed in articulating their mathematical and meta-cognitive arguments.

8.5.5 Related Patterns

Used by: GUESS MY X; TASK IN A BOX; ACTIVE WORKSHEET; SOFT SCAFFOLDING; NARRATIVE SPACES; POST LUDUM

Uses: MATHEMATICAL GAME-PIECES

8.5.6 Notes

The wide range of patterns which use this one indicate that it is indeed a fundamental component, applicable to most systems aiming to support discussion and collaborative learning.

8.6 Guess My X

Use a CHALLENGE EXCHANGE game of BUILD THIS puzzles to combine construction and conversation, promoting an understanding of process-object relationships and leading to meta-cognitive skills such as equivalence classes, proof and argumentation.



8.6.1 The problem

A teacher wants to design a game for learning concepts, methods and meta-cognitive skills in a particular mathematical domain. This game should use a combination of available technologies.

Many complex concepts require an understanding of the relationship between the structure of an object and the process which created it. Novices may master one or the other but find it challenging to associate the two. Constructing objects helps build intuitions, and discussing them espouses abstracting from intuitions and establishing socio-mathematical norms (Yackel & Cobb, 1995).

Learning mathematics is fundamentally learning to be a mathematician. It requires the learner to internalize a range of mathematical skills as regular habits: computation, analysis, conjecturing and hypothesis testing, argumentation and proof. For this to happen, the learner needs to be deeply engaged in meaningful mathematical inquiry, problem solving and discussion. Games provide a natural setting for the kind of “flow” needed, but how do we ensure that the focus of this flow is mathematical activity and discourse?

8.6.1.1 Forces

- Many mathematical domains require learners to understand the relationship between mathematical objects and the process which generated them. This is a challenge for many learners.
- The teacher needs a non-invasive monitoring mechanism to assess students' performance.

- The communicational approach (Sfard, 2006) sees learning mathematics as acquiring a set of language rules and meta-rules. In order to achieve this, learners need to engage in meaningful and sustained discussion of mathematical topics.
- The classroom hierarchies and the perception of a teacher as more knowledgeable causes learners to be cautious and restrained in their mathematical discourse.

8.6.2 Context

This pattern is the basis for activities in which students repeatedly share and discuss digital artefacts. It therefore assumes a classroom supported by a technological environment which provides a shared and protected web space (e.g. wiki or forum), common tools (programming environment, spreadsheets, etc.) and sufficient access time for all students.

The game relies on sustained interaction over a period of several sessions. It can be used as a short introduction to a topic, but the greater meta-cognitive benefits may be lost if not enough time is allowed for conversations to evolve.

Also works for several groups collaborating over a web-based medium.

8.6.3 Solution

GUESS MY X (GMX) is a pattern of game structure, which can be adapted to a wide range of mathematical topics. It extends CHALLENGE EXCHANGE to encourage discussion and collaborative learning, and to break down classroom hierarchies. It uses BUILD THIS to engender reflection and discussion about the relationships between mathematical objects and the processes that produce them. The core of the pattern is described in Figure 46.

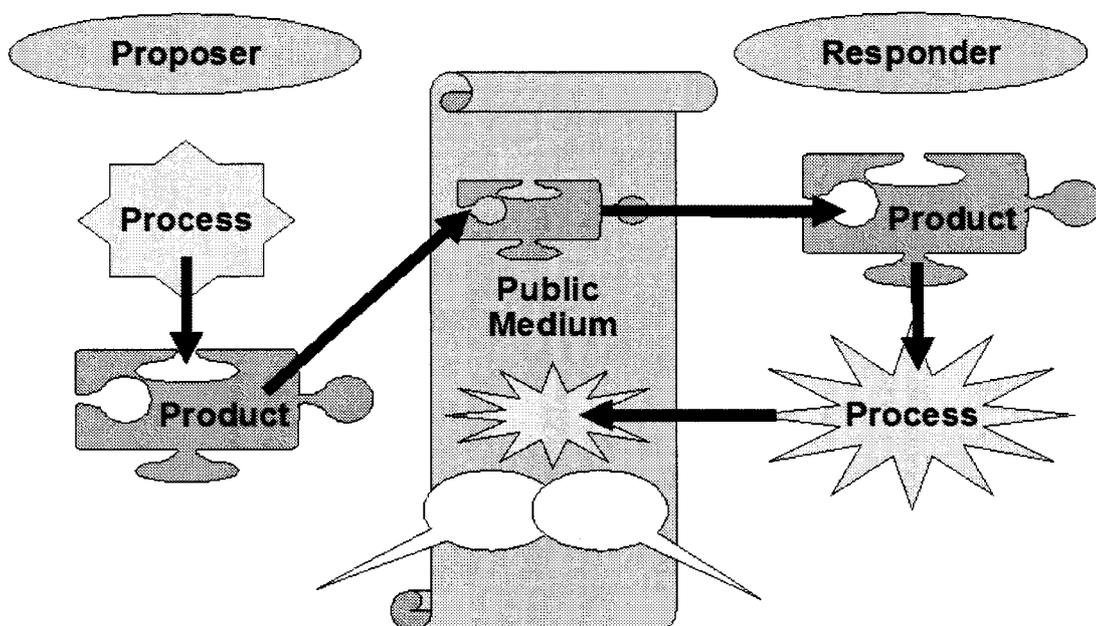


Figure 46: Schematic diagram of GmX

GMX involves players in two roles, *proposers* and *responders*, and a facilitator. An implementation of the game would specify a domain of mathematics and rules for constructing processes in that domain. A proposer sets a challenge, in the form of a mathematical object which she constructed. The explicit rules of the game define the nature of the process by which this object can be created, but not its details. A proposer would construct such a process, and capture its product. The proposer then saves the process model in a private space and publishes the product as a challenge. Responders then need to “reverse engineer” the process from the product. If they succeed, they publish their version as a response to the challenge. The proposer then needs to confirm the responder’s solution or provide evidence for the contrary.

8.6.3.1 Set-up phase

Before the game begins, the teacher needs to verify that the players have a minimal competence in analysing and constructing the mathematical objects to be used.

1. Teacher introduces the rules of the game and the game environment.
2. Teacher simulates one or two game rounds during a whole class discussion.
3. Students may need to initialize their game space on the chosen collaborative medium.

If the game uses separate media for construction and communication, consider using a TASK IN A BOX to streamline the transition between them.

8.6.3.2 Game session

The game sessions for the proposer and the responder are different, although the same player can play both parts in parallel.

1. Proposer initiates the game, by constructing an object according to the game rules and publishing it. She then waits for responses.
2. Responder chooses an attractive challenge, and attempts to resolve it. If she believes she has succeeded, she responds to the challenge by posting the object she constructed and the method she used.
3. Proposer reviews the response, and confirms or rejects it. If the response is rejected, an argument needs to be provided.

8.6.3.3 Play session

Each play session involves a single iteration of the game. Students tend to prolong their interaction in the game, by providing secondary challenges, etc. The iterations can be asynchronous, with a time gap of several days between turns.

The communication medium chosen for the game should afford NARRATIVE SPACES for the proposer and the responder. Although the rules of the game are limited to the exchange of

mathematical objects, the ability to augment these with personal narratives is crucial for personal reflection as well as for collaborative knowledge building.

8.6.3.4 Set-down phase

The POST LUDUM discussion should highlight the issue of the evaluation function and its resolution.

Pedagogical aspects

It is important to keep the mathematical content explicit from the start. The game is not a SUGAR-COATING to disguise the mathematics: it is a game with MATHEMATICAL GAME-PIECES. The rules of the game are intentionally left vague, in the sense that the evaluation function used to determine the responders' success is not fully specified. This requires students to negotiate what constitutes a correct answer, and in doing so collaboratively refine the underlying mathematical concepts. These negotiations can lead to discussions of issues such as proof, equivalence and formal descriptions.

Technical aspects

The quality and extent of these discussions depends on the scaffolding and provocations provided by the teacher, but a necessary condition for them to emerge is that the medium of the game provide a NARRATIVE SPACE, where the MATHEMATICAL GAME-PIECES of the game can become OBJECTS TO TALK WITH.

8.6.4 Support

Theoretical

The basic structure of GMX draws on the popular "Guess my rule" game, which has traditionally been used by teachers as an introduction to functions and to formal algebraic notation. Carraher and Ernest (2003) note that children of all ages enjoy this game, and can be drawn by it into discussions of algebraic nature. Guess my rule has been adapted to TEME environments, using Logo and spreadsheets (c.f. Healy & Sutherland, 1990).

Matos, Mor, Noss and Santos (2007) argue that the particular design of GMX promotes the emergence of a community of practice (Lave and Wenger, 1991) among its players. Wenger (1998) proposes three dimensions of practice as the property of a community:

- Mutual engagement: a sense of "working together".
- Joint enterprise: having some object as an agreed common goal.
- Shared repertoire: agreed resources for negotiating meanings.

These elements are provided by the rules of the game and the supporting collaborative environment. The joint enterprise is mathematical, and thus prompts learning in this direction. The loosely specified rules of the game lead participants to establish their own rules, which contribute to the community's sociomathematical norms Yackel & Cobb (1995).

Empirical

GmX emerged from the Guess my Robot game (section 7.3). This game was first tested in 2002, and then refined and re-tested in the two subsequent years, with promising results. The pattern then served as a model for other activities designed and conducted by colleagues Michelle Cerulli and Gordon Simpson in the domain of function graphs (Simpson, Hoyles and Noss, 2006) and randomness (Cerulli, Chiocciariello and Lemut, 2007). Pratt et al (2009) note that GmX was used to develop the guess-my-die game (Pratt, Johnston-Wilder, Ainley and Mason, 2008) in the domain of statistics.

8.6.5 Liabilities

Risks

The game requires flexibility in time to allow learning dynamics to emerge. The game can be played by individuals, pairs or teams. The number and spread of participants can also vary. However, it is crucial to allow enough time for a culture to emerge. This can be achieved by interleaving the game with other activities, e.g. playing it for the last 10 minutes of each session over several weeks.

The fact that the game dynamics are driven by participants makes the educators' role subtle and critical. The educator needs to facilitate fruitful interactions, and monitor these to divert them to high standards of mathematical discourse.

Limitations

GmX assumes a degree of social and technical sophistication which suggests it would be suitable for young teens and above. It can, however, be adapted for younger children.

It is suitable for concrete, well-bounded content domains, such as computation, modelling or analysis. It uses these as a stratum for developing meta-cognitive skills of problem solving, analysis, argumentation and general mathematical discourse.

8.6.6 Notes

Both proposers and responders tend to converge to challenges which are HARD BUT NOT TOO HARD. When the environment encourages social cohesion, players seem to reciprocate 'good' challenges.

8.6.7 Related Patterns

Extends: MATHEMATICAL GAME-PIECES;

Uses: CHALLENGE EXCHANGE; BUILD THIS; TASK IN A BOX; OBJECTS TO TALK WITH; NARRATIVE SPACES;

Leads to: POST LUDUM;

8.7 *Transparent Stream*

8.7.1 The Problem

Number patterns and number sequences offer a path into complex mathematical topics such as functions, algebra and cardinality. Pattern spotting in number sequences is intuitive and appealing to most learners. However, many learners find it challenging to transcend intuition and formulate rigorous analytical concepts. Computational representation and manipulation of sequences can help learners in this transition, if it is compatible both with intuitive and formal views of sequences.

Forces

- Number patterns and sequences are accessible and appealing to many learners. A possible source for the intuitive attractiveness of sequences is their narrative nature.
- Formal analytical concepts of number sequences are a viable basis for many other complex mathematical concepts.

But –

- The formal representation of number sequences, which is needed for their mathematical manipulation, is un-intuitive and confusing for most learners. This may be due to the fact that the formal representation does not retain the narrative character of sequences.
- Manipulations on sequences are computationally intense, and thus take considerable time when performed on paper.

Programmable representations of sequences alleviate these problems, because –

- They enforce a formalism which may be more accessible to learners
- They allow manipulation of large finite sequences in reasonable time

However, the common representations of sequences as lists or tables of numbers –

- Decouples the terms of the sequence from the process of their production, thus breaking away from intuition.
- Do not capture the narrative character of sequences.
- Have no way of conveying the infinite nature of mathematical sequences.

8.7.2 Context

Origins

This pattern emerged from the search for effective methods of designing exploratory activities in the domain of number sequences, using the ToonTalk language. The range of activities envisioned was extensive, including analysis, cardinality, convergence, complexity and number theory. The decision to extend the STREAM pattern was based on its well established robustness

in software engineering. Some of these activities were implemented and others only designed at a conceptual level. Nevertheless, the evidence from these experiments suggests the pattern should be valuable for teaching many subjects related to number sequences.

Boundaries

The pattern utilises the features of object based concurrent languages. Although it can be implemented in many languages which do not support these qualities, the result may be cumbersome.

8.7.3 Solution

The STREAM design pattern (also known as “pipeline processing”) is widely acknowledged in software engineering as an efficient and robust aid for managing large quantities of sequential data (Abelson and Sussman, 1996; Shapiro 1988). For example, in the Java programming language, it is the standard approach for handling program input and output and inter-process communication (SUN, 2005; Eckel, 2002).

In this approach, data is processed by combining modular computational components. Each component can receive a single element, perform a limited computation on it, and pass it on. The sequence as a whole is generated or manipulated by repeatedly invoking a chain of such components. A component that produces elements without any input is often called a “source”, one that consumes elements but does not generate any output (e.g. stores the element to disk or displays them on screen) is a “sink”, and one that modulates or manipulates a flow of elements is a “filter”. A chain of components may be invoked by a source “pushing” elements or by a sink “pulling” them.

Figure 47 shows an example Stream-based design of a hypothetical traffic monitoring application. The application needs to collate raw data from a real-time traffic sensor, and continuously update a histogram displaying traffic density per time slot.

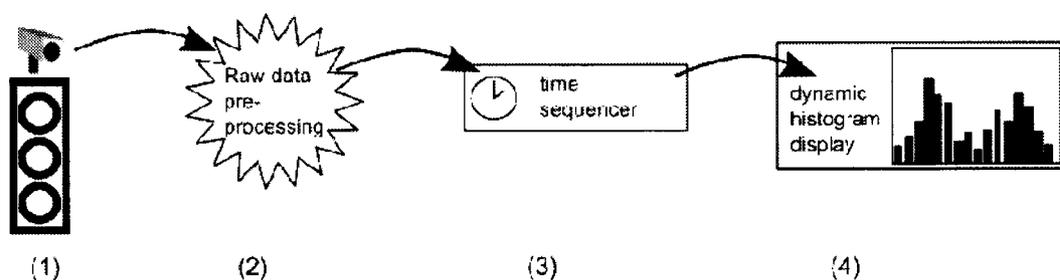


Figure 47: Schematic view of a traffic monitoring application using the Streams design pattern. A sensor (1) generates a stream of real-time raw data. This is passed through a pre-processing filter (2), which converts it to a stream of structured data, on to a time sequencer (3) which aggregates the data by time-slots. Finally, a dynamic histogram (4) acts as a sink for the stream of time-aggregated data and displays traffic density per time slot.

The STREAMS pattern is potentially a powerful representation for exploring and learning about number sequences. However, in order to realise this potential several modifications and extensions are needed. These are the essence of the TRANSPARENT STREAMS pattern.

8.7.3.1 Basic structure

The TRANSPARENT STREAMS pattern provides a framework for representing and operating on number sequences in programmable media. The fundamental elements of this framework are:

- A unit source stream, e.g. a constant stream of 1s, or a stream of the natural numbers.
- Function filters, which perform a mathematical operation on a stream of numbers (or several streams).
- Post-processing sinks, which display or store streams of numbers, or delegate their processing to external programs.

Examples of function filters include:

- Memory-less unary operator filters, which apply a fixed function to each term of the sequence they consume and outputs its outcome. E.g. a “times 2” filter would multiply every term of the incoming sequence by 2.
- Aggregators, e.g. a filter that adds the incoming terms to a running total and outputs a series of partial sums.
- Conditional filters, e.g. a filter that eliminates odd terms and passes on the even ones.
- Multi-stream operator filters, e.g. a stream difference filter which subtracts each term of one sequence from the respective term of the other and outputs the stream of results.

Examples of post-processing sinks include:

- Graph display of stream terms as they come in
- A spreadsheet conduit, which collates the terms from a stream to a file readable by a spreadsheet application.

Pedagogical aspects

TRANSPARENT STREAMS can be used as a simulation tool for teacher demonstrating mathematical ideas, or by learners exploring these ideas independently. A simulation scenario would call for refined tools which can be easily activated on the fly – even at the expense of limited control.

A constructionist scenario would require very little in terms of pre-configured tools. In this case, it is the learners themselves who should internalise the TRANSPARENT STREAMS pattern, and use it in the course of their activities. These activities would typically include:

- **Constructing number sequences:** Learners can use a unit source and basic operation filters to construct complex sequences, and thus appreciate their structure.
- **Analysing number sequences:** Learners are given a complex source sequence, and use various filters to uncover its underlying structure, or raise and test conjectures about its properties.
- **Modelling sequential phenomena:** Learners use a simulation of, or data logged from observed phenomena represented as a source stream and use various filters and sinks to analyse and represent it.

Technical aspects

STREAMS can be implemented in most languages, although they are perhaps better suited for object-oriented concurrent languages. However, in order to be pedagogically effective they need to be enhanced with a user interface which allows learners to observe both their inner workings and their external effects in a controlled manner; hence the qualifier “transparent” in this pattern.

The user interface of TRANSPARENT STREAMS should allow the learner to:

- Inspect each component in isolation, both behaviourally and structurally. A behavioural inspection allows the learner to provide the components with input data and observe the respective outputs. A structural inspection allows the user to observe how the component maps inputs to outputs internally.
- Easily connect components to form chains of processing.
- Regulate the speed of flow, i.e. control the pace by which sequence terms are “pushed” by the source or “pulled” by the sink through a chain of components.
- Zoom in and out seamlessly, from observing a single component to the combined effect of a chain in action.

In a simulation scenario, additional requirements may arise, for example that the components have a visual signature which makes them easy to identify.

In a constructionist / exploratory setting, it is important that each individual component is simple enough for the learners to be able to construct it themselves, even if it is provided as a tool. Complexity emerges from the combination of simple components.

8.7.4 Support

Theoretical

As noted section 6.2, children’s intuitions of sequences tend to be predominantly recursive. The stream design pattern allows us to capture this intuition and formulate it as code.

Section 6.2 also noted the importance of maintaining a cohesive process-product view of sequences. In traditional list programming, the process (generating the sequence) is decoupled from the product (an enumeration of the first n terms). Similarly, the relationship between successive terms of the sum series is lost, as each one is computed independently from the original list. A stream is an object which ties together the process and the product.

A final rationale for the stream approach concerns difficulty of conceptualising infinity. Several researchers (Tirosh, 1991; Li & Tall, 1993; Falk and Lavy, 1989; Dubinsky et al, 2005) have commented on the tension between potential and actual infinity. Any manifestation of infinity in a computational medium is inevitably potential, since the computer’s memory and processing power is finite. The stream pattern is as close as possible to this intuitive concept of infinity; it will continue providing terms indefinitely until it is interrupted. It can also possibly provide a bridge towards the conception of actual infinity: since it is not possible to count the length of a

stream (as is possible with lists), the stream object itself represents all terms of the sequence – ad infinitum. Again, the power of streams is not in the representation of any specific infinite process – but in the possibility of combining and manipulating infinite processes.

Empirical

The TRANSPARENT STREAMS pattern emerged from, and was refined through, the basic sequences activity (section 7.2). It was then used as the fundamental programming approach to sequences throughout the Guess my Robot (section 7.3) and convergence (section 7.4) activities.

It was then used by my colleague Gordon Simpson in the design of Guess my Graph (Simpson, Hoyles and Noss, 2006) and by my colleague Ken Kahn in the design of activities for the domain of infinity and cardinality (Kahn, Sendova, Sacristan and Noss, 2005).

8.7.5 Liabilities

Risks

While the STREAMS pattern is very robust as a general purpose programming technique, the pedagogical effectiveness of TRANSPARENT STREAMS is highly sensitive to the specific features of the user interface, as demonstrated by the story of Eager Robots (section 7.5.2).

TRANSPARENT STREAMS reinforce the naïve recursive perception of sequences, i.e. as functions from one term to the next rather than a function of the index. This perception gives precedence to sequences which are easier to formulate in recursive form, as demonstrated by the story of $1/n$ vs. $1/2^n$ (section 7.4.1).

Limitations

Representing a sequence as a stream would be ineffective for scenarios which require processing of large sections of the sequence at once, or rely on random access to sequence terms by their index.

8.7.6 Notes

Potential extensions

In application programming, streams are used to process non-numeric data, such as text or sensor readings. Such uses have potential educational value, either in learning domains where they emerge naturally or as motivational problems for mathematical modelling.

8.7.7 Related Patterns

Extends STREAMS; NARRATIVE REPRESENTATION

8.8 Task-in-a-box

When using environment A to provide tasks in environment B, package these tasks in a compact form that can be embedded in A and unpacked in B. Each task package should include the task description, any tools required to perform it, and the mechanism for reporting back the results.

8.8.1 The problem

Switching from one environment to another loses the activity context. This is a source of confusion. Often learners are bemused by the need to transform the task from one representational framework to another. As a result, learners' time is lost on a cause that rarely contributes to the learning aims.



Forces

- Complex tasks may require multiple tools or work environments.
- Often the optimal tool for a particular learning task cannot be embedded in the medium used to orchestrate learning activities.
- Setting up the tools and work environment needed to perform a task may demand considerable time, which is tangential to the task itself and the associated learning objectives.
- When the task is defined in one environment but executed in another, learners need to rely on their memory for the task details – thus increasing cognitive load, or constantly switch between environments – increasing work load.

8.8.2 Context

Complex learning tasks mediated by multiple media.

Origins

This pattern emerged from the integration of WebReports and ToonTalk in the WebReports project.

Expansion

This pattern should potentially apply to any scenario where the medium in which learning tasks are delivered or discussed is different from the medium in which they are performed. E.g. a task conducted using spreadsheets but delivered to learners via a VLE.

Boundaries

This pattern is not applicable when the task execution media is too complex or specialised to allow it to be packaged, e.g. when it includes large mechanical tools.

8.8.3 Solution

When the learning environment incorporates a conversation medium (CM) which is detached from the task medium (TM), learners need to “switch context” when embarking on a task – both physically and mentally. Switching context loses the thread of activity. Learners need to reconstruct a representation of the task in a new environment. This adds on to the general disorientation of shifting from one language to another. To avoid this problem:

- Create a task object (**TO**) in **TM**, which includes:
 - A description of the task.
 - The data and tools required to perform the task.
 - A container for the task outcomes.
- Create a metaphoric (iconic) representation which allows you to refer to an object in **TM** from within **CM**.
- Make the iconic representation a control in **CM** which invokes its referred object in **TM**.
- Create a convention for representing tasks which uses the iconic representation above.
- When describing the task in **CM**, accompany the description with an iconic representation of the same task in **TM**.
- Instruct users to invoke this representation to transfer themselves, with the task, from **CM** to **TM**.
- Provide the users with a mechanism for transferring the **TO** back from **TM** to **CM** when the task has been completed and the results stored in the outcome container.

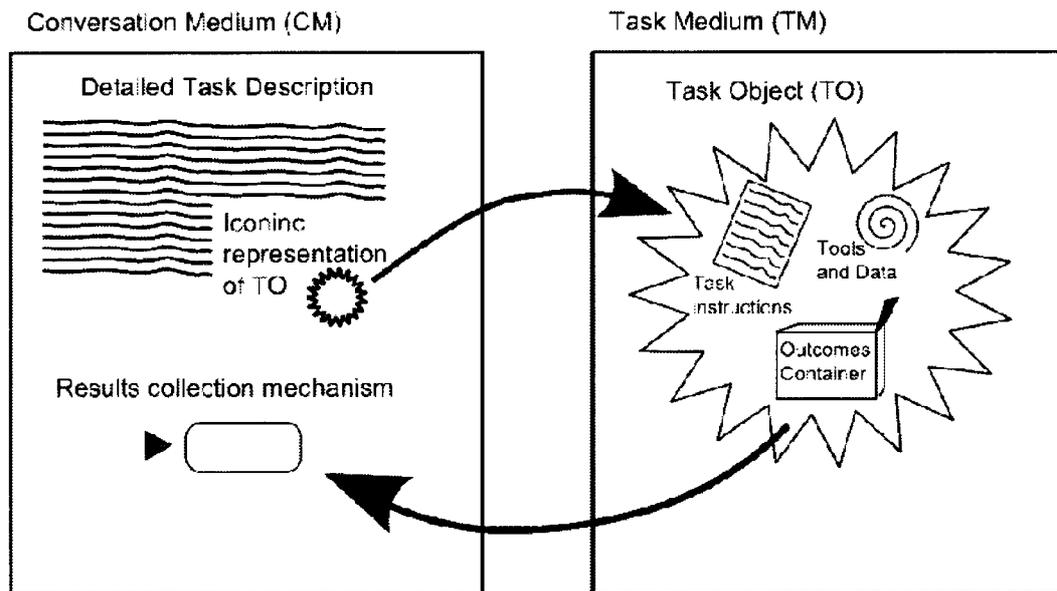


Figure 48: Schematic diagram of the Task-in-a-Box pattern.

8.8.4 Support

Empirical

The WebReports system employed ACTIVE WORKSHEETS which included ToonTalk boxes. These encapsulated programming tasks described in the worksheet text (Figure 49).



Figure 49: Example ToonTalk 'task in a box' from the WebLabs project

8.8.5 Liabilities

Risks

An over-regimented implementation of this pattern can lead to mechanistic task execution, and consequently hindering creativity and reflexivity. Preferably, the task object (TO) should serve as a SOFT SCAFFOLDING. The task, and by extension the instructions and tools included in the TO, should be designed to challenge the learner with respect to the learning aim and minimise tangential obstacles.

Limitations

Sometimes, mapping an idea to a new representation is a worthy learning experience in its own right. If this is the case, then it needs to be made explicit.

8.8.6 Notes

A fully streamlined implementation of this pattern will typically require bespoke modifications of both the user interface and the underlying application interfaces of both the LOM and the TEM. However, it can usually be readily applied with minimal compromises in a variety of scenarios, even without any coding. The context section above included an example of a task involving spreadsheets and delivered via a VLE. In this example, a teacher can use the spreadsheet file as a TO, by creating a sheet with the task summary, various utility formulae, and a table for collating results. A second sheet would include the data to be processed, and a third dedicated as the learners' work area. This spreadsheet would be uploaded to the VLE and linked from the task description page.

If the LOM is a NARRATIVE SPACE, then the TOs can be used as OBJECTS TO TALK WITH.

8.8.7 Related Patterns

Uses SOFT SCAFFOLDING

Used by ACTIVE WORKSHEETS

Associated NARRATIVE SPACE, OBJECTS TO TALK WITH.

8.9 Thumbnails

The following patterns have been mentioned in the description of other patterns above, but their full articulation is beyond the scope of this thesis. Initial drafts of these patterns can be found at <http://lp.noe-kaleidoscope.org/outcomes/patterns/>

NARRATIVE REPRESENTATIONS	Prefer forms of representation which have narrative qualities, or afford narrative expression. These are not only textual representation, but also visual or animated forms which include elements of context, plot, and voice.
POST LUDUM	Follow a game (or any other exploratory activity) with a discussion in which participants are prompted to articulate their learning experience and acquired knowledge. Such a discussion brings intuitions to the surface, strengthening structured knowledge and exposing discrepancies.
HARD BUT NOT TOO HARD	A challenge has to be set at a level which is slightly above the participants' current level. A challenge too easy will be perceived as boring, while a challenge too hard will result in frustration – both leading to disengagement.
CHALLENGE EXCHANGE	A self-regulating mechanism for implementing HARD BUT NOT TOO HARD: allow learners to set challenges for one another.
BUILD THIS	A type of challenge / game, where learners are shown an object and asked to reconstruct it.
ACTIVE WORKSHEET	Scaffold learners' work by an on-line worksheet which they edit as they progress through a task. Tools needed for the task are embedded in the worksheet. Learners populate the worksheet with products of their work, along with explanatory narrative, responses to questions, etc.
SEMI-AUTOMATED META-DATA	Having meta-data attached to objects is very useful when you want to find something, but attaching that meta-date is tedious. This problem is particularly acute in game-based learning environments. Meta-data may be essential to your ability to analyse learning, but expecting learners to bother with it in the

	<p>course of playing a game is highly unrealistic.</p> <p>Design the interface for object management, and activities in which it is used, in such a way that meta-data is either:</p> <ul style="list-style-type: none"> • Inferred: Can automatically be derived from the workflow in which the object is created / invoked. • Motivated: Offers clear and immediate benefits to the user.
<p><i>SUGAR-COATING</i> (anti-pattern)</p>	<p>Mathematical (or any other subject) matter is injected into a game in a disconnected way, so that the game is the sugar-coating used to help the learner swallow the bitter content.</p> <p>As a result the game loses its appeal, and the learner receives the message that the educational content is inherently un-enjoyable.</p> <p>Bruckman (1999) and Resnick (2006) argue vigorously against the trend towards sugar-coating prevalent in “edutainment”.</p>

8.10 Conclusions

This chapter presented seven patterns, contributing towards a language of patterns for techno-pedagogical design for mathematics learning and teaching, and outlined another seven. These design patterns reflect an approach which sees the technological and pedagogical dimensions inseparable in the design of educational activities and tools. Thus, while some patterns emphasise certain features of technology and others highlight structures of activity, they all relate to some extent to both.

Chapter Seven concluded with several themes pertaining to the design of technology for technology enhanced mathematics education. These themes capture the insights emerging from the design narratives, yet are expressed in a fairly abstract manner, in which they are neither refutable nor readily implementable. The patterns presented in this chapter systematically unpack these themes. Each pattern can be read as a practical imperative or as a local theory. As a practical imperative, it serves as a recommendation for the educational designer to be sensitised to challenges which may arise in a given context, and some possible ways of dealing with them. As a local theory, it is a claim that problem P under conditions C can be addressed by solution S. Such a form opens the door for systematic scientific discourse of design.

The first theme derived from the theoretical discussion in Chapter Six, and examined through the narratives in Chapter Seven, is the power of combining construction, communication and collaboration, which was reified in patterns such as OBJECTS TO TALK WITH, BUILD THIS, CHALLENGE EXCHANGE, and POST LUDUM. This theme is tightly related to the issue of

narrative, which manifests itself in patterns such as NARRATIVE SPACES and NARRATIVE REPRESENTATIONS.

Whereas these themes have a fairly general remit for TEME, the demonstrator study focused on learning number sequences. The theoretical review engendered a proposal to design educational experiences that allow formal concepts to flow from intuitions, rather than against them. This principle, combined with the two above, is the primary source of GUESS MY X. In the case of number sequences, the theoretical review claimed that naïve intuitions are recursive, consequently motivating the use of the STREAM pattern. However, this pattern needs to be elaborated to make it suitable for educational purposes. The result is the TRANSPARENT STREAM, which provides functional modularity and prompts learners to acknowledge the relationship between process, parameters and product and to classify sequences by their behaviour.

When dealing with technology it is inevitable to consider the user interface. An interface, by definition, is the layer through which users access a technology's functionality. In the case of TEME, technology is designed to engage the learner with mathematical ideas. Yet it cannot do so without interfaces which will provide access to these ideas. Such interfaces need to combine usability and pedagogy, as demonstrated by patterns such as SOFT SCAFFOLDING, TASK IN A BOX, SEMI-AUTOMATED META-DATA and ACTIVE WORKSHEETS.

On the other hand, technology designed for learning should embody fundamental educational principles and values. This ideal is exemplified by patterns such as MATHEMATICAL GAME PIECES and HARD BUT NOT TOO HARD. Both these patterns stem from two beliefs: first, that the origins of mathematical thinking are in a fundamental human desire to identify patterns in the world and explain them, and second, that the role of education is to nourish and guide this desire.

Above all, these patterns demonstrate the immense complexity of designing for learning. This complexity calls for further efforts towards identifying methodical frameworks for describing, aggregating and mapping design knowledge. The prime example of this complexity is the GUESS MY X pattern. At first, guess my robot may seem a simple game with surprising effects. The detailed analysis embodied in the GUESS MY X pattern, along with its 'ancestor' patterns – such as CHALLENGE EXCHANGE and BUILD THIS, suggests that the game's success is not a happenstance, but rather a result of an intricate assemblage of multiple design elements relating to the tools, the activity and the ways in which they interact.

Some of these patterns embody ideas already present in the literature: the importance of supporting mathematical discourse, fading of scaffolding, shared learning objects, as well as principles of interface and game design. The value of the patterns, and of the empirical work preceding them, is not in the rediscovery of such principles but in the insights into their practical application and in the description of the webs that link them – addressing the critical gap between theory and practice.

Chapter 9 Summary and Conclusions

This chapter concludes the thesis. It sets off by reviewing the main findings and associating them with the three aims defined in Chapter Three. These are followed by notes on the limitations of this study, directions for future research and some final comments.

9.1 Overview

Chapter One began with a story of spontaneous learning, and asked how we could design for such events to occur. This simple question was the seed for an enquiry into the nature of education as a science of designed learning, with specific attention to technology enhanced mathematics education (TEME). The characterisation of education as designed learning established a multi-faceted link between design and epistemology, or the creation of knowledge. This link is a motif that weaves through this thesis. Design emerged as a method of study, an object of study and an outcome of study, leading to an overarching theme of this thesis (section 3.1):

To consider the study of technology-enhanced mathematics education as a design science; highlight the implications of such a paradigm, and propose ways to theorize design in a manner which draws both on educational research and computer science.

The pragmatist nature of a design stance to educational research entailed a tight dependency between the three levels of epistemology (section 3.1): the method by which we study education (its normative epistemology) needs to link our understanding of how people learn (genetic epistemology) to how we design artefacts for learning (generative epistemology). The specific aims of this thesis were derived from this realisation, to:

- Identify potential elements of an **epistemic infrastructure** for a design science of TEME.
- Combine and elaborate the elements identified into a coherent **methodological framework** in a given research TEME context.
- Apply the methodology in a problem domain and demonstrate its potential by producing a contribution towards a language of **pedagogical patterns** for TEME.

Chapter Three elaborates these Aims. Chapters Two and Four address Aim 1 by synthesising the design research literature with insights from other design disciplines. Chapters Four and Five address Aim 2 by defining a demonstrator study in the use of technology to support learning about number sequences through construction, collaboration and communication in secondary school. Chapters Six to Eight address Aim 3 by presenting the results of applying the methodology to the demonstrator study.

Sections 9.2, 9.3 and 9.4 relate the findings of this study to each one of these aims. Section 9.5 notes some of its limitations, and section 9.6 considers questions for future research.

9.2 Findings related to Aim 1: to identify potential elements of an epistemic infrastructure for a design science of TEME

The design science perspective on TEME was motivated by the observation that the study of education is distinguished from other fields concerned with human learning in its attention to *designed learning* (section 2.2). Where other sciences ask how and what humans learn, the science of education asks what they should learn and how best to support them to this end. This argument was expanded and supported by reference to the literature, in particular to Herbert Simon's seminal work. Simon (1969) distinguishes between the natural sciences and the sciences of the artificial, the sciences of design: natural science asks what *is*, the science of design asks how to bring about what *ought to be*. Three characteristics of design science, derived from Simon's work, were found to be highly relevant to the study of education: a value-driven agenda, a functional axis of decomposition and the role of representation. The value dimension is reflected in education's remit to improve the life of individuals and communities. A functional axis of decomposition means that the objects of educational research should be categorized and analysed according to their effect rather than their structure. The role of representation is generally accepted in the learning sciences. Iterative methodologies are gaining ground, in particular where technology is involved.

A design approach was found to be better adapted to the complex and dynamic nature of the circumstances and questions studied by educational science (section 2.2). This approach has the potential to offer a cohesive paradigm, bridging across practice and multiple theories. These advantages are even more salient in the face of the rapid pace of change induced by technological developments, calling for agile, responsive and proactive approaches to educational research.

Simon's argument for an inherently iterative method of inquiry resonates well with the growing tradition of design-based research (DBR) in TEME (2.2.1). DBR was characterised as pragmatic, committed to theoretical as well as practical innovation, highly interventionist, iterative. DBR produces modest theories – constrained to well-defined domains of learning and contexts of design. A design science of TEME produces design knowledge (section 2.4) characterised as: problem driven, solution oriented, value laden; Situated in context; and holistic, i.e. inherently inter-disciplinary.

A review of the field identified the need for a clearly articulated consensual epistemic infrastructure (section 2.3); the explicit and implicit rules and assumptions which bound the discourse of the scientific community, and a logical system by which claims are presented and justified, independently of their content. Some desirable features of such an infrastructure emerged from the discussion (section 2.4.3): arguments should be *accessible* to the scientific community, as well as by practitioners and by policy makers; the processes of design study should be *transparent and traceable*; the forms used for communicating results should be sufficiently *expressive* to articulate all that is needed to support the above requirements. These forms should promote *cumulativity* and aggregation of knowledge, and its organisation should follow a *functional-pragmatist orientation* aligned with the functional axis of design knowledge.

A common cycle of design experiments, and an encapsulating meta-cycle of design research, were synthesised from the literature (section 4.1). Narrative was identified as a powerful epistemic form (section 4.2), leading to a proposal for a formalisation of design narrative as a form of scientific discourse (section 4.4). Design patterns, organised into pattern languages, emerged as a promising form for encoding design knowledge in educational research (section 4.5), but nevertheless, several challenges were identified (section 4.5.4). These challenges were the motivation for Aim 2, addressed in the next section.

The acknowledgment of education as designed learning calls for a design science perspective in educational research. Such an approach appears to be particularly well suited for TEME. A design science stance inspires a new agenda of pragmatic research which promotes innovation in theory and practice as one. Yet, for such an approach to succeed it needs to develop a coherent consensual epistemic infrastructure.

9.3 Findings related to Aim 2: To combine and elaborate the elements identified into a coherent methodological framework in a given research context

Aim 2 was addressed by projecting the theoretical arguments and epistemological elements of Aim 1 onto the setting of Aim 3, as reflected in the demonstrator study (section 3.4): “the design of tools and activities for learning about number sequences, in an extra-curricular lower-secondary school setting”. The result is a concrete framework which offers a contribution towards a pattern-based methodology for a design science of TEME.

A review of existing design approaches to educational research exposed the emergence of shared practices and pockets of expertise (section 4.1). Among the common methodological characteristics identified were a dual focus on practical and theoretical contributions, a highly interventionist and agile attitude, and a cycle of iterative research. This cycle includes phases of theory, design, implementation, execution (experiment / practice), articulation of experience, interpretation, evaluation and analysis, and feedback to both theory and design. The products of this cycle are validation or critique of existing theory, evidence regarding the effectiveness of artefacts and practices in well-defined settings, and innovations in practice and theory. A frequent by-product of research is the synthesis of multiple frameworks. This cycle is embedded in a meta-cycle, which includes a framing phase, an empirical phase and a retrospective analysis phase.

A need was identified for appropriate representations for expressing design knowledge in the various phases of these cycles, along with procedures for managing the transitions between phases. A particular trajectory from experience to knowledge, denoted *the narrative transposition model*, was proposed as a basis for these representations and transitions (section 4.2). This mechanism proceeds by narratisation of experience and genre transposition. Similar narratives are fused, thus abstracting similarities and eliminating detail. In this process temporal links are replaced by semantic relations.

By analogy to the narrative transposition model, two representations were proposed: design narratives (section 4.4) and design patterns (section 4.5). The former serves the interpretive

phase of the design experiment cycle, in which the researcher organises the data and records the unfolding of events in the empirical phase. The latter serves the analytical and conjectural phases, allowing researchers to articulate situated abstractions of design knowledge derived from the experiment.

Once the research setting was described, it was used to illustrate how these constructs interacted in practice. Special attention was given to the analytical half of the design experiment cycle, which describes the trajectory from experience to structured theory, focusing on the processes by which data was collected (section 5.3) and from it design narratives (section 5.4) and design patterns (section 5.5) were produced and validated. The result is a description of a methodological framework and a set of instruments for the demonstrator study, from data collection and management, through interpretation and systemisation of observations as design narratives and on to the formalisation of research outcomes as design patterns. These provided a full specification for implementation of the analytical hemicycle of the design experiment cycle and of the retrospective analysis phase of the design research meta-cycle.

The analytical hemi-cycle began with the collection of data during the design, implementation and trial of educational innovations. Three classes of data were identified: design data, student productions, and classroom observations (section 5.3.1). Design data included any record of the design process and its product. The challenges of a messy environment were addressed by redundancy, triangulation, and nearest substitute; collecting every available fragment of data, supporting claims by combining data from different sources, and identifying the pragmatically accessible forms of data closest to the laboratory ideal.

A structured process of selection and construction of design narratives was identified, using Bruner's ten principles as guidelines (section 5.4). These principles, adapted to the needs of scientific form, were expressed in the design narrative template.

Design patterns were extracted from design narratives through a six step process devised to capture the key design elements, systemise and substantiate them (section 5.5). This was followed by a phase of refactoring: structural manipulations which give the pattern language as a whole greater coherence.

diSessa & Cobb (2004) highlight the importance of theory in design studies, for research as well as for practice, and claim that design studies can – and should – make significant theoretical contributions by addressing the gap between theory and practice. Design patterns, and the design narratives from which they are derived, inhibit this space. A design experiment should be founded on solid theoretical principles, derived from existing literature. The narratives portray the projection of these principles into novel settings and provide insights into their implementation. The patterns generalize the lessons learnt, link them back to the theoretical foundations, validating, questioning, expanding and elaborating them. In the process well-known theoretical principles are translated to a network of pragmatic constructs which both illuminate the theoretical sources and provide the means for translating them into concrete action.

A methodology is only as good as the research it engenders. This recognition was the motivator for Aim 3 and the demonstrator study, discussed in the next section.

A design science of TEME needs to identify languages suitable for its discourse. This study has demonstrated that pattern languages, integrating design narratives and design patterns, offer powerful tools in this respect. These representations of design knowledge fit well within the cycles of design research. Design narratives systematise an innate form of extracting knowledge from experience. They support the first tier of interpretation by affording rich contextualised descriptions of problem solving. Design patterns capture essential features across narratives, encapsulating recurring challenges and forces pertaining to a domain of learning design, the interactions between them and possible methods of solution. Together they fill a gap between theory and empirical evidence, and maintain their links with both.

9.4 Findings related to Aim 3: Apply the methodology in the problem domain and demonstrate its potential by producing a contribution towards a pattern language for technology enhanced mathematics education

The demonstrator study (Chapters Six to Eight) focused on designing activities and technology to support learning about number sequences through construction, communication and collaboration. It addressed Aim 3 by applying the methodological framework devised in response to Aim 2 on the basis of the theoretical and epistemological foundations of Aim 1.

The demonstrator study followed the design research cycle and meta-cycle noted above. The framing phase produced a pragmatic review of the literature (Chapter Six). The aim of this review was to identify key challenges in the domain and raise conjectures as to ways in which they may be addressed. These conjectures were translated into the design of activities and the technology to support them. The iterative process of design, implementation and evaluation was captured as a set of design narratives (Chapter Seven). These design narratives were analysed to produce the collection of design patterns (Chapter Eight).

Number patterns and sequences are broadly accepted as intuitively appealing to learners, and a viable gateway to advanced mathematical subjects (section 6.2), yet the review exposed several pedagogical and epistemic issues concerning this topic. Foremost among these are the difficulties of formulating a structural view of sequences (section 6.2.3), specifically with respect of the process-product duality (section 6.2.5) and recursive vs. closed form (section 6.2.4). These difficulties were linked to a dissonance between intuitive perceptions and the prevalent school representations of mathematical objects. This observation was inspected in the broader context of the relationship between communication, representation and meaning in mathematical learning. Combining the communicational approach with the concept of situated abstraction led to considering narrative as a fundamental social and cognitive epistemic force (section 6.3.1). Narrative form would appear to be at odds with the nature of mathematics, raising the question of their reconciliation – which has been a central question of the demonstrator study (section 6.3.2). Bringing narrative into the domain of constructionist learning and educational programming raised questions regarding its manifestation in computer-based representations, specifically, how to represent number sequences (section 6.3.3). The STREAMS design pattern was

proposed as a promising candidate. The union of the various approaches calls for educational designs which weave construction, communication and collaboration.

The observations and conjectures emerging from the review of the literature were translated into design of activities and tools. These were tested and refined in two sites over three years. The process of design and the outcomes of testing were captured in Chapter Seven as series of nine design narratives. These design narratives form the empirical content of this thesis. They are the first tier of interpretation, not the data itself. Each narrative recounts a particular incident, defined by a single problem to be solved or task to be accomplished.

The primary sources for design data were project reports, design documents, teacher manuals and research journals. The primary sources for student productions data were student webreports, ToonTalk code and paper-based written tasks (section 5.3.3). All texts and artefacts were read as mathematical arguments expressed in narrative. Acknowledging the impossibility of separating observation from intervention, data collection was integrated with activity design. Products were assessed in terms of aptness, complexity and sophistication of argument. The primary sources for classroom observation data were field notes, video and audio recordings. Interview data included (individual and group) stimulated recall interviews, task-based interviews and in-activity probes. The latter played a central role in observational data.

Two types of design narratives emerged (section 4.4.1): researcher narratives (RNs) and participant narratives (PNs) – which were represented in Chapter Seven by learner narratives (LNs). RNs recount a pedagogical problem and its resolution from the researcher's point of view. The focus is on the integrated design and development of activities, social practices and supporting technology. PNs follow the teacher or learner as a designer, contending with a problem they encountered in the context of an activity, their use of the resources provided in confronting this problem, and the indications of their learning gains in the process. These are third person accounts based on the learners' written and verbal articulations and my observations. The two types of narratives are interdependent; the problems encountered by learners and their resolution are the drivers of their learning trajectory. The researcher's concern is to direct learning by providing learners with an effective set of problems and the means for resolving them. Thus, LNs illuminated and substantiated the RNs. Chapter Seven included six RNs and three LNs. These covered the three main subject themes and the supporting infrastructure, over the three years of design experiments. They were chosen to illustrate a variety of scale, data sources, and analytical foci, and to expose insights at different levels.

Several generalisable observations stood out (section 7.6): the potential of the STREAMS design pattern, the roles and characterisations of narrative, the importance of combining construction, communication and collaboration, and the recursive intuition of sequences.

STREAMS proved apposite to teaching and learning about number sequences (e.g. section 7.2.2). They allowed learners to distinguish between process, parameters and product, construct complex mathematical entities from simple blocks, appreciate the characteristics and behaviour of classes of sequences, and articulate sophisticated arguments.

The roles and characteristics of narrative in learning mathematics was explored through the texts and artefacts published by learners. Three distinctions emerged. The first distinction concerns the intentional mode of the narrator. Some narratives aim to flatly recount events, others interpret them, and others use events as elements of an (perhaps implicit) argument. These modes are correlated to Scardamalia and Bereiter (1987) models of text composition. The first mode maps to knowledge telling, while the last two correspond to knowledge transforming. When producing the narrative is seen as an end in itself, or a means to demonstrate that a technical task was completed as required, it does not embody any knowledge. When it is a means to communicate ideas it becomes a tool for manipulating and developing these ideas. A second distinction is between narrative as an epistemic structure and a conversational form. The transition from narrative structure to narrative form can perhaps be seen as a step in the genre transposition to propositional expression of generalised knowledge. When using narrative as a syntactic formula, the narrators have already detached themselves from the concrete experience, abstracting away from it and making generic claims. The third distinction noted the diverse modalities in which narrative can be expressed. Acknowledging the importance of narrative in the process of learning mathematics entails that we need to develop frameworks which allow us to apply a narrative lens to a wide range of student (and teacher) expressions – but also clearly discriminate where this lens is relevant and where it is not. These distinctions are offered as signposts for further investigations: they need to be explored in greater depth and across a wider range of data, bringing in appropriate interpretive frameworks. Such an endeavour is beyond the scope of this thesis.

Activities were designed to encourage learners to assimilate mathematical concepts through a combination of construction, communication, and collaboration. The technological infrastructure evolved to support the streamlined integration of these modes of action (section 7.5.1), captured in the common activity framework: Construction challenged preconceptions, seeding new ideas, but these needed reflection to be transformed to structured knowledge. Communication drove reflection by drawing on substantial relevant experiences. Collaboration was powerful motivator of sustained communication, and consequently reflection. The trajectory from construction through communication to reflection highlights the importance of narrative in constructing mathematical knowledge, through guided transition from action to narrative to propositional discourse.

Several of the LNs demonstrate the recursive nature of the naïve concept of number sequences (e.g. section 7.4.1). Allowing learners to encode their intuitions in ToonTalk afforded the transformation of this intuition into an alternative mathematical representation.

Among the more general themes that surfaced were the co-evolution of technology and pedagogy, the interdependence of interface and substance, and consequently the fluidity of design and the need for flexibility and malleability. This symbiosis between technology and epistemology was expressed in the fundamental, structural layers – but none the less at the level of the interface, as demonstrated by the initial failure of the STREAMS approach, reported in the Eager Robots RN (section 7.5.2).

An overarching conclusion from these design narratives is that no single element of design, technological or pedagogical, can be directly linked to a desired effect – only their careful

assemblage, adapted to the given context. This called for means of analysing the narratives to identify and articulate individual elements of effective design, and then synthesising them to devise solutions for novel problems. This need was addressed by a collection of design patterns (Chapter Eight), contributing towards a pattern language of construction, communication and collaboration in technology enhanced mathematics education. This collection included seven fully specified patterns, and outlined another seven. They reflected an approach which sees the technological and pedagogical dimensions inseparable in the design of educational activities and tools. Thus, while some patterns emphasised certain features of technology and others highlighted structures of activity, they all relate to some extent to both. The sample of patterns in Chapter Eight serves a dual purpose. For the educational designer it illuminates challenges which may arise in a given context, and some possible ways of dealing with them. For the research community, it opens the door for systematic scientific discourse of design for TEME.

The theme of combining construction, communication and collaboration was reified in patterns such as OBJECTS TO TALK WITH (section 8.5), BUILD THIS, CHALLENGE EXCHANGE, and POST LUDUM. This theme was linked to the issue of narrative, which manifested itself in patterns such as NARRATIVE SPACES (section 8.4) and NARRATIVE REPRESENTATIONS.

GUESS MY X (section 8.6) operationalised the abstract principle of deriving formal concepts from intuitions. The STREAM pattern captured naïve recursive intuitions of number sequences. Adapting it for educational settings produced TRANSPARENT STREAM (section 8.7), which provides functional modularity and prompts learners to acknowledge the relationship between process, parameters and product and to classify sequences by their behaviour.

Patterns such as SOFT SCAFFOLDING (section 8.3), TASK IN A BOX (section 8.8), SEMI-AUTOMATED META-DATA and ACTIVE WORKSHEETS addressed the need to combine usability and pedagogy at the user interface level. The patterns MATHEMATICAL GAME PIECES (section 8.2) and HARD BUT NOT TOO HARD demonstrated the value dimension of TEME design, embodying the belief that the role of education is to nourish and guide a fundamental human desire to identify patterns in the world and explain them.

Applying the methodological framework to a demonstrator study yielded a contribution towards a pedagogical pattern language of construction, communication and collaboration in TEME. This contribution includes thirteen design patterns, supported by nine design narratives. Seven of these patterns have been fully specified, while six others provided as thumbnails. The design patterns unpack the general themes emerging from the design narratives, translating them into theoretically and empirically substantiated context-aware guidelines for educational design. This collection of design narratives and patterns demonstrates the validity of the proposed approach in dealing with the immense complexity of designing for learning, and calls for further work in this vein.

9.5 Limitations of this study

As is expected in an endeavour of this scale, I can identify several junctions where in retrospect I would have taken a different path.

9.5.1 Evolution of the Methodological Framework

Had the methodological framework been available at the onset, the empirical work would have enjoyed a much smoother journey, and would have been likely to produce more elaborate results. However, it was precisely the deficiency of a clear methodology that motivated this study, implying a secondary role for the demonstrator study. Undoubtedly, there are rich opportunities for exploring the demonstrator domain as a primary interest, now that the methodological framework has been articulated.

One of the major challenges throughout the work on this thesis was maintaining the balance between the primary and the demonstrator domain. The functional-pragmatist orientation, identified as a tenet of design science applies here as well: the epistemic and methodological constructs I propose need to be judged not only by their internal coherence and aesthetics, but also by the action they enable. This functional value manifests itself in the quality of the outcomes in the demonstrator domain, but these are bounded by the availability of the methodological tools and the efforts required to develop them.

9.5.2 Further Validation of the Outcomes

The elements of an epistemic infrastructure described in this thesis, and the derived methodological framework, have been justified by theory and demonstrated in practice. However, these warrants still fall short of a robust validation. In order to stamp the epistemic and methodological constructs as scientifically credible they will need to be assessed by multiple researchers in varied circumstances, a requirement which lies beyond the nature of a PhD study.

Indeed, here lies a fundamental paradox of PhD research: while science is ideally a collaborative endeavour, conducted by research teams interacting within a community, a PhD is required to demonstrate individual capacity – and thus demands a clearly defined individual contribution to science. Versions of the epistemic and methodological tools described here have been used in practitioner workshops I facilitated as part of several research projects. The nature of those variants and the outcomes of their use were not found suitable for inclusion here, because it was impossible to delineate my individual contribution in such intensely collaborative contexts.

9.5.3 Data collection

My experiments were conducted under 'messy' conditions: school classrooms where video and audio recording was often inefficient. The study of innovative learning technologies called for innovative data collection methods. These inherent constraints were amplified by my personal "migration" from computer science to educational research. Suitable methods were eventually identified and adapted to the specific settings of this study. However, valuable observations from the early iterations were lost due to missing or low-quality recordings.

Again, this is partially due to the shortage of a clearly articulated and comprehensive methodological framework – which this thesis set to address. It is also likely a price I paid for my transition from a different domain of enquiry (computer science) to educational research.

9.5.4 Sensitivity to specific conditions

The availability of the WebReports and ToonTalk media, and the classroom practices which they inspired, created a social setting radically different from the typical classroom. In order to reduce the risks associated with testing innovative and potentially unstable technologies, experiment groups were small. Voluntary participation suggested prior interest and motivation. All these factors raise valid concerns regarding the possibility of replicating these experiments on a large scale with a randomly chosen population. However, the tools have been used by diverse groups across Europe, some larger in size, and the activities have been tested in other sites, although not in the same sequence. Nevertheless, all of these sites were under close attention of experienced researchers, which in turn had my dedicated support. While the epistemological and methodological arguments should extend beyond the specific context of this work, as should the derived design patterns, I wish to warn against any naïve attempt to extend the specific findings of the demonstrator study.

Perhaps this is a caveat common to many design studies in education. Design research deals with innovation, and thus inherently examines atypical conditions, yet aims to derive knowledge of how these innovations can be extended to broader situations. The organisation of research findings through design narratives and design patterns should offer a means of systematically capturing the unique and separating the generalisable. Another answer to this tension could be provided by “scaling up” iterations – as proposed by Lesh, Kelly and Yoon (2008). Such a model is not possible within the confines of PhD research.

9.6 *Open Questions and Future Research*

This section concludes the chapter, and with it the thesis, by marking some directions for extending the work reported here. Mirroring the three aims of this thesis, the notes that follow refer to epistemology, methodology and pedagogy.

9.6.1 Extending the Epistemological Argument

The images of design knowledge and design science portrayed in this thesis are a view “from the inside out”; they originate in a study of TEME, and the recognition that this study is built on a fragile foundation. Consequently, the epistemological claims expressed here are confined to the domain of TEME. It would appear that they could be extendible to a broader remit, yet the radius of this remit and the effects of such an expansion are unclear. Should education, in general, be studied as a design science? How would such a perspective interact with others, such as the sociology, history or philosophy of education? How much of the edifice constructed here will survive such an expansion, and which parts are domain specific and will need to be remodelled?

The positions expressed here regarding the nature of a design science of TEME grazes the border of philosophy of science and epistemology. Should they be expanded to a broader scope, this border needs to be explored and possibly crossed. The pragmatist foundation of design science and its implications need to be fully understood. Some current debates in these fields would appear to be relevant, for example the question of the relationship between knowledge-that and knowledge-how (Ryle, 1949; Stanley and Williamson, 2001), the relation between

knowledge and actions (Hawthorne and Stanley, 2008), the dependence of knowledge on the practical consequences for the knower (Stanley, 2007) and the value dimension of scientific enquiry (Kitcher, 2001).

9.6.2 Extending the Methodological Framework

The design experiment cycle presented in Chapter Four was divided into two parts: the empirical hemicycle, flowing from theory to practice, and the analytical cycle feeding back from practice to theory. The methodological tools presented thereafter focused on the latter, while the former was treated more as a craft than as a scientific process. Cross (2001) notes that much of the effort in the design science tradition has focused on “scientising design”, devising scientifically valid manners of producing useful artefacts. By contrast, he proposes a view of design as a discipline, a reflective practice with its own culture and forms of knowledge. The questions that follow are: should this distinction be maintained? Has scientised design fallen from favour due to a fundamental flaw or a failure of implementation? If the science of design needs to be segregated from its craft, how should the boundaries and interchange channels be defined? If they can be blended, what are the scientific tools suitable for the empirical arc of design research? It may be the case that previous attempts failed because they employed a positivist paradigm, incapable of responding to the dynamic complexity of design problems.

The power of narrative in constructing and communicating design knowledge is a theme that underlies much of this thesis. This power rests on an inherent paradox: narrative reflects on the general through the particular. The timeless is captured in the transient. If narrative is to be used as a scientific form, this paradox needs to be resolved. A possible explanation would be that we use narrative to understand domains too complex for symbolic logic. Another option would be to argue for a systematic network of knowledge representations, linking the abstract and formalistic to the vernacular and indigenous.

The process of extracting design narratives from data and design patterns from narratives, such that both would qualify as productive scientific representations, needs to be further interrogated. Section 4.4.1 offered a set of guidelines for design narratives based on Bruner’s qualities. These should be operationalised and examined in varied research contexts, and compared to other frameworks. Section 6.3.2 considered the literature on children’s’ use of narrative in constructing mathematical knowledge. The parallels to researcher’s systemisation of design knowledge need to be considered. In particular, the distinction Scardamalia and Bereiter (1987) make between knowledge telling and knowledge transforming. Section 8.1.2 reported that the actual process of developing the design narratives and design patterns iterated between the two, suggesting a link between these iterations and a shift from knowledge telling to knowledge transforming. This link needs to be further explored.

The specific tools of data collection and analysis used in this study were limited to the scope of the demonstrator study, and should be expanded and woven into the emerging framework for design based TEME research. In particular, the media used by learners to express their ideas offer an extraordinary richness of representation. While some elements of multimodal analysis (Jewitt and Kress, 2003) were incorporated into the analysis, a more comprehensive examination is called for.

The methodological framework presented in this thesis is subject to a weakness, inherent to PhD work; a methodology is ultimately part of the norms of a scientific community. A PhD, on the other hand, is characterised as an individual contribution to science. This tension will only be resolved once the thesis will be published. It is my hope and belief that I have made a convincing argument in favour of the epistemic and methodological tools presented here. My next goal will be to share these tools with the TEME research community, and make them accessible to other researchers. I expect that, as they are adopted and appropriated by others, their lacunae will be exposed.

9.6.3 Extending the Pedagogical Findings

The role of narrative, as a mediating form between experience and systemic knowledge, emerged as one of the main outcomes of the demonstrator study. This calls for further investigation. Section 7.6 proposed three distinctions concerning the roles and characteristics of narrative. These need to be elaborated and examined through a broader range of examples, both from an epistemic perspective – understanding learning, and from a design perspective – creating the conditions for learning. Section 7.6 noted that the transition from action through narrative to knowledge should be guided by activity design and the teacher’s interventions. This transition and the guidance it requires need to be described in greater detail, so that clear instructions can be derived for designers and teachers. In particular, the relationship and flow between narrative structure, narrative representation and other forms of representation needs to be elaborated.

The communicational framework described in Chapter Six bears a striking affinity to models of computational discourse analysis and comprehension developed in the realm of artificial intelligence (Grosz & Sidner, 1986). The field of artificial intelligence in education (AIED) has developed a variety of tools for modelling learner discourse. The advantage of these tools is in their mathematical precision, required by any model that needs to be processed by a machine. Such models could be valuable as practical methodological tools. At the same time, as noted by Rickel et al (2001), they could be informed by the sophistication of socio-cultural theories.

Another avenue worth considering is a dynamical model for the relationship between symbols, meaning and learning. A recent trend in cognitive psychology views development from a dynamical systems perspective (Vallacher & Nowak, 1997; Beer, 2000; Lewis 2000).

The notion of programming as narrative form raises further questions. There is a notable difference between the narrative present in most of the literature and that expressed in programmed code. The former is predominantly a recount of past events, whereas the latter is a recipe for affecting future events. This distinction needs to be elaborated. In a way, programming is a form of fantasy: but perhaps so is mathematics? The narrative qualities of different programming environments should be inspected, and correlated to the nature of mathematical discourse.

Chapter 8 presented a collection of techno-pedagogical design patterns. Seven of these were fully specified. Eight others are listed as “thumbnails” in section 8.9, initial drafts of which patterns can be found at <http://lp.noe-kaleidoscope.org/outcomes/patterns/>. This collection needs to be developed further in order to provide a comprehensive pedagogical pattern

language of construction, communication and collaboration in TEME. The draft patterns listed as thumbnails need to be specified to the same level of detail as those included here, and all patterns need to be validated by additional independent empirical evidence.

The use of the STREAMS pattern raises the question of software design patterns as educational tools: design patterns might make it possible to provide participants in constructionist learning environments with the knowledge of how to create their tools, rather than use pre-fabricated tools. This should be an advantage where the tool embodies a mathematical idea in its very structure, or when understanding the workings of the tool is required for understanding its effects and uses. This conjecture calls for further research.

One theme that presents itself as a question for further research is the notion of learners as design scientists. Several parallels were drawn between my process of research, and students' learning trajectories, for example in comparing researcher narratives and learner narratives. These parallels should be further elaborated, and perhaps used explicitly as a basis for educational design.

9.7 *Final comments*

This chapter marks the endpoint of a long journey, but as with every journey – it is just the prologue to the next. I end this study with more questions opened than answered, and some answers to questions I have not considered at the onset. What began as a naïve attempt to use technology to help children learn about number sequences, ended as an elaborate methodological proposition. The true value of this proposal will only transpire when it stops being mine, and becomes appropriated by and integrated into the conversation of the educational research community.

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Appendix I Final design of activities and the tools to support them

Overview

This appendix presents the final design of activities and the computational tools that support them. The activities were structured in three consecutive segments (Figure 50). Although I had designed them to be conducted sequentially, in some cases only one or two segments were used, and in others they were combined with other activities, in experiments led by my colleagues Ken Kahn, Gordon Simpson (Simpson et al, 2006) and Nicholas Mousoulides. The feedback from those variants informed subsequent iterations of design, yet a detailed discussion is beyond the scope of this study.

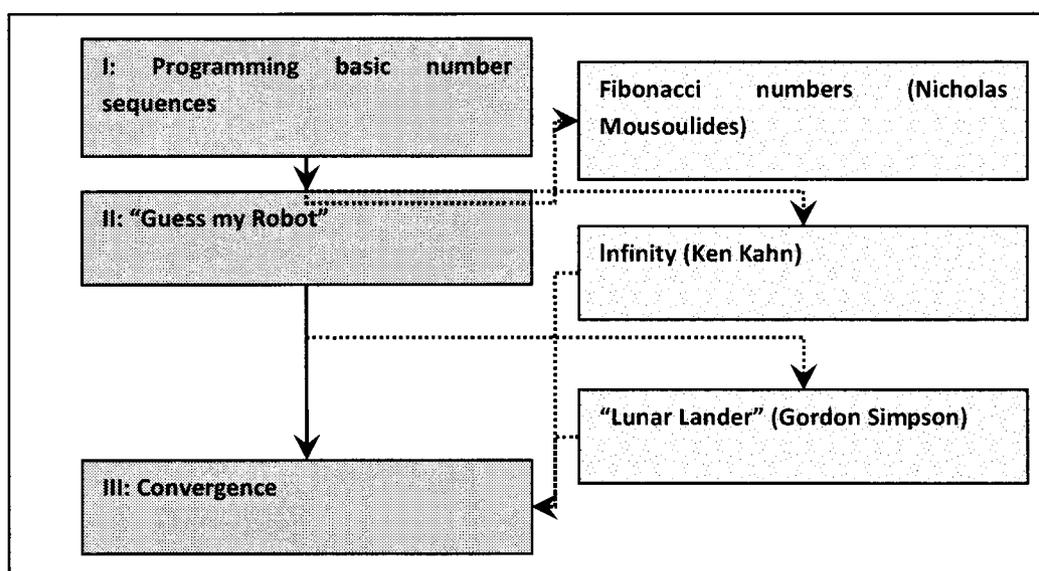


Figure 50: Sequencing of activity segments

Segments I and III consisted of several activities each, all of which followed a common pattern (Figure 51). Each activity was initially motivated by a discussion of an intriguing mathematical theme. Students were encouraged to propose conjectures or derive concrete questions to explore, which were then reformulated as concrete programming tasks. Students completed these tasks individually or in pairs and published their ToonTalk programs, along with their observations about them. These programmes were used as input to an instructor-led group discussion, driven by the joint production of a *consensus report*. By engaging students in a discussion they are provoked to reflect on their work. Mistakes are exposed and explained by peers, rather than a figure of authority. Students acknowledge the need to construct rigorous arguments for their claims, and negotiate socio-mathematical (Yakel & Cobb, 1995) and socio-technical norms.

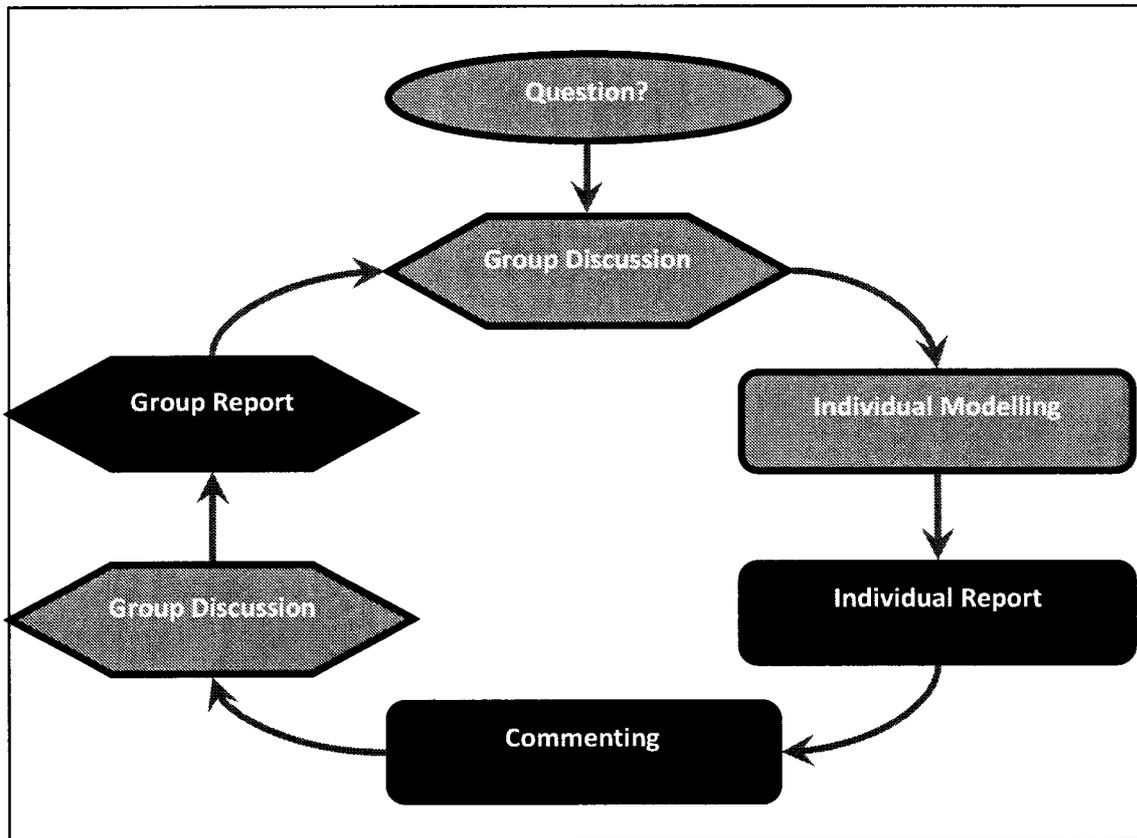


Figure 51: Pedagogic cycle (adapted from Mor et al, 2006). This cycle, common to most activities, combines individual construction and group collaboration and communication.

Ideally, at this point the webreport would be reviewed by another group, perhaps in another country, and an inter-group discussion would ensue, using the WebReports “comment” mechanism. In reality, we rarely succeeded in orchestrating such a discussion due to pragmatic limitations. Alternatively, where possible, the class was split into two or more groups for the concluding discussion and webreport authoring. Each group then elected a representative to present their webreport to the whole class, using the electronic whiteboard.

The above pattern attempts to combine the power of individual constructionist learning (Papert & Harel, 1991) and collaborative knowledge-building (Scardamalia & Bereiter, 1994). This design also aims at making the most of the new media at our disposal, without forgoing the established power of traditional educational methods. In a way, the students are mimicking the practice of a research and development team, the only difference being that their broad research path is mapped out beforehand.

The following sections present the three activity segments I led in detail, focusing on the selection of main tasks and the educational rationale behind them. While the discussion emphasises the final form of activities and tools, as tested in group V (5.2), the evolution of design which led to it is noted when relevant.

Segment I: Programming basic number sequences

This segment of activities was conceived as an initiation into ToonTalk programming and the WebLabs educational culture. It was designed to induct students into a culture of exploring structure in number sequences, assuming a minimal programming competence. The main aim of this initial set of activities is to initiate students into reasoning and arguing about the structure of number sequences. This was achieved by allowing students to develop an alternative, non-algebraic language for mathematical discussion; a language which emerged from their collaborative discourse and stemmed from their experience in constructing programs which generate and modulate sequences. The segment consisted of two main tasks, leading to a conclusive group webreport (Figure 52). This was followed by two optional puzzles which served as tools for evaluating students' learning.

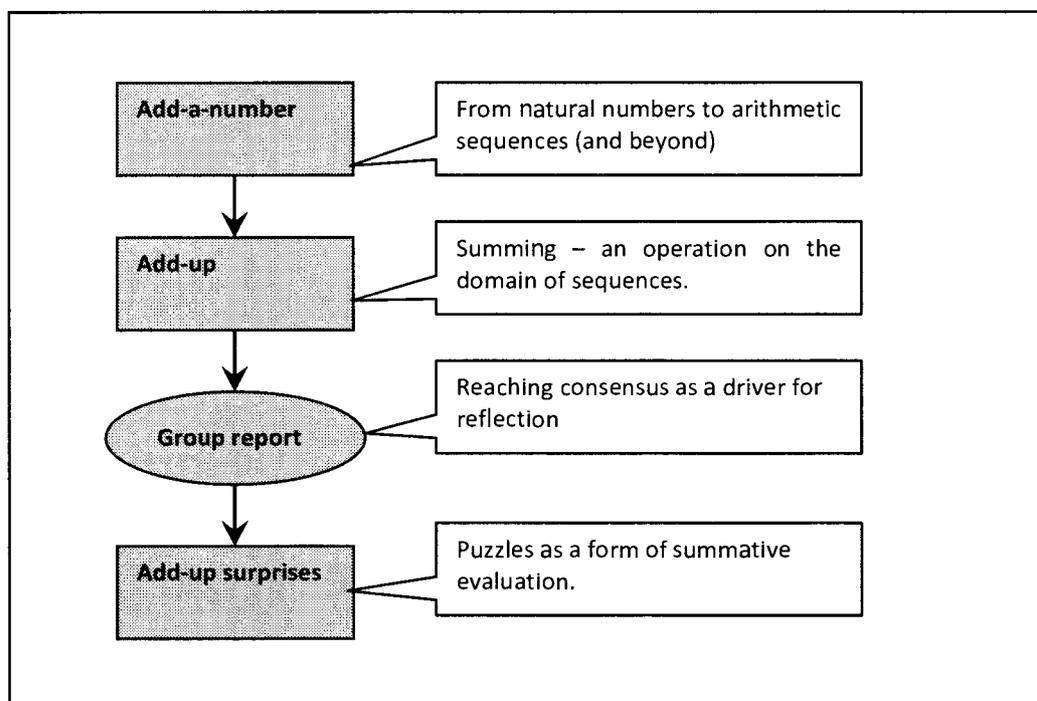


Figure 52: Segment I activities (programming basic sequences)

As noted in 6.2, pattern recognition and generalisation are considered fundamental to mathematical thinking, and a fruitful pathway into algebraic thinking. Yet students encounter difficulties in shifting from pattern spotting to structural understanding, erroneously basing their conclusions on superficial or incidental patterns they observe in the sequence, rather than on arguments originating in its structure. The naïve view of sequences is often recursive (6.2.4), phrased in terms of a progression from one term to the next (rather than as a function of the index). While some researchers see this as an obstacle to formulating a structured mathematical view, I took it as an opportunity. Thus, the representation of sequences I chose was designed to capture the recursive view and reformulate it in an unambiguous symbolism. Specifically, the aims of this segment were:

- To develop a non-algebraic language for describing, discussing and reasoning about polynomial (and maybe some non-polynomial) sequences.
- To develop an understanding of:

- The generation of number sequences,
 - The rules that sequences rely on and
 - How sequence generation relates to the ToonTalk environment (robots, birds, etc).
- To gain some insight into the relationship between the structure of the programming constructs (e.g. the number of chained robots) and the type and complexity of the corresponding sequence.

Add a number

The “Add-a-number challenge” was the first programming task presented to students. Its motivating question was posed more as a ToonTalk puzzle than a mathematical one. Students were asked “how would you train a robot to count (1, 2, 3, 4, and so on)”? As expected, they proposed a construction similar to the one in Figure 53.

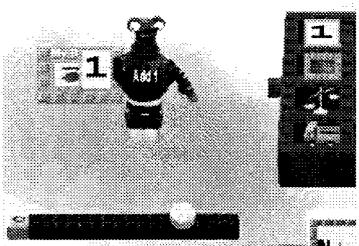
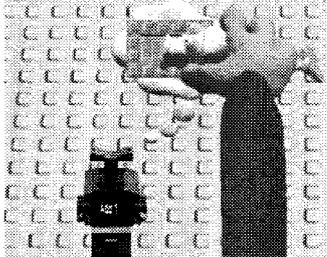
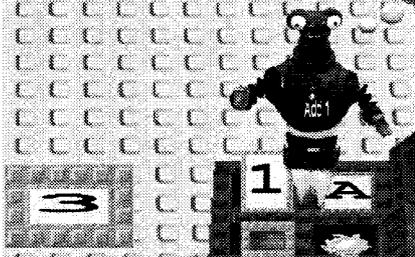
		
<p>Train the robot to take a number 1 from the toolbox and drop it on the input, to increment it.</p>	<p>Generalise the program by erasing the value of the input from the robot’s memory.</p>	<p>Give the robot its input box. The robot will continually repeat the actions it has been taught.</p>

Figure 53: Training a robot to count. This is the naïve method, typically proposed by students as an initial solution.

However, this solution has a serious flaw: the robot does not maintain a record of the numbers it generates. All computations are done “in place”: the same memory register – or ‘hole’ in ToonTalk term – is used to store the value of the current sequence term before and after the progression operation. Thus, the only available term of the sequence is the last. This problem provided a motivation for introducing birds: ToonTalk’s message-passing mechanism (section 5.2.2.1).

Discussing this solution led naturally into the first the programming task: train a robot to generate the natural numbers, and send them to a nest. To scaffold students’ work, they were presented with an active worksheet: a webreport template that includes the instructions for the task and questions related to it. Students created a webreport of their own by clicking a button on this page. As they worked through the tasks, they filled in the answers and added their models and observations. The unique feature which makes the worksheet “active” is that the ToonTalk tools required for the task are embedded in it, and at the click of the mouse students can load them into their programming environment. In this particular case, the worksheet contained a *task-in-a-box* (Figure 54): a ToonTalk box containing task instructions, an untrained robot, an input box, and output nest. The TASK-IN-A-BOX design pattern is discussed in section 8.8.

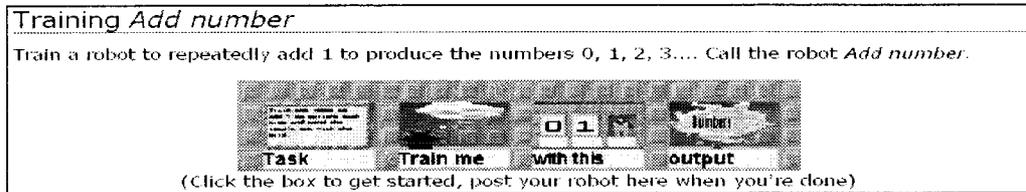


Figure 54: Add-a-number task-in-a-box. The box, embedded in a report template, includes a description of the task and the tools needed for its resolution.

The input box contains two holes with numbers: an *increment* and the *current*¹⁴ value. A third hole contains the *output* bird. The robot needs to be trained to perform two actions: drop a copy of the *increment* over *current* (thus adding them), and then hand a copy of *current* to the *output* bird. This is the first occurrence of the TRANSPARENT STREAM design pattern (section 8.7): the numbers are sent out to the nest, one after the other, *ad infinitum*.

From a mathematical point of view, a stream generates the natural numbers by repeated application of the *successor* function. By constructing this procedure, manipulating it and using it as a building block in larger constructions, students reified the natural numbers as the product of this process; obviously not as strong a formalisation as Peano's axioms, but perhaps a step in that direction. A second important mathematical concept was prompted by a unique affordance of ToonTalk. Most programming languages distinguish clearly between constants and variables. Code is written for the general case ("*any n*") and tested for specific cases (or written for a singular setting). ToonTalk employs *programming by example*. This means that robots are trained for specific values, which can then be generalized by "erasing" the specific value from the robot's memory. In the case of the Add-a-number robot, the first generalization is required immediately: after the robot runs once, the value of *current* is no longer 0, and needs to be generalized if we want the robot to continue counting past 1. However, by generalising the *increment* as well, students can use the program to generate any arithmetic progression! The next part of the worksheet asked students to predict which sequences can be generated by their robot and which cannot (Figure 55).

¹⁴ I use the `courier` font to denote code elements.

Explore
Can you think of a way to use your robot to produce these sequences?

Sequence	Explain
	If yes, explain how you would do it. If you think it is impossible, explain Why. Add any ToonTalk object that helps support your argument.
2, 3, 4, 5...	
-1, -2, -3, -4..	
-7, -6, -5, -4..	
2, 4, 6, 8..	
5, -1, -7...	
Write down a sequence of your own, which can be generated by your robot.	
Write down a sequence of your own, which cannot be generated by your robot.	

Explain
How would you explain to a friend what kind of sequences your robot can generate, and how it can be used to generate those sequences?

Describe one sequence that cannot be generated by it, and explain why.

How would you convince a friend of your claims?

Figure 55: Add-a-number questions. These were included in the active worksheet of the task, to prompt reflection on the mathematical nature of the robot.

These questions aimed to promote students' mathematical conjecturing and argumentation, and specifically to raise their awareness of the relationship between the *procedural* and the *structural* facets of sequences. Table 11 shows answers from group V to the fifth question. Such responses were typical for this task. Note that although all answers were correct, some were incomplete and a majority suggested reprogramming the robot (rather than changing its parameters).

	response	note
1.	yes you use -6's	incomplete
2.	Do same as above, but start with 5 and take away 6 in the process	reprogram
3.	Take 5 as the current, then take -6 and copy it to the current each time.	reprogram
4.	yes just make the current 5 and add this -6	complete
5.	Start with 5 in current and replace the add this with -6	complete
6.	Start on 5 and + -6 each time	reprogram
7.	Get a 5 and a -6. Copy the -6 and add it to the 5.	reprogram

Table 11: Responses to the "5, -1, -7.." question

After reflecting on these examples, students were asked to provide one more sequence that their robot can generate and one that it cannot. The latter question was probably the hardest, in order to

say that the robot *cannot* produce a sequence one has to argue about the nature of the *class* of sequences it *can* produce. While many students provided valid examples, none offered any justification as to why their sequence could not be generated by the add-a-num robot. This balance changed after a group discussion (see example in Table 12).

<p>you cannot do:</p> <ul style="list-style-type: none"> - square numbers - anything where you times or divide - in can't go up in prime numbers - any sequence with two stages - triangular numbers 	<p>It can only go up (or down) in the same number each time.</p> <p>That's why it can't do all those sequences on the left.</p>
---	---

Table 12: Response to "sequences you can not generate" from a group report¹⁵

Add up

Once students have posted their Add-a-number robot and answered the questions, they were introduced to the next task: train a robot to add up the terms of a sequence (the Add-up robot). Mathematically, this robot embodies the concept of a partial sum series, and implements it as a function on the domain of sequences: for any given sequence, it produces the sequence of its partial sums. In concrete terms, when the Add-up robot is given the nest of the first sequence to, it sums the numbers coming in to that nest, and sends the results out to its output nest (Figure 56).

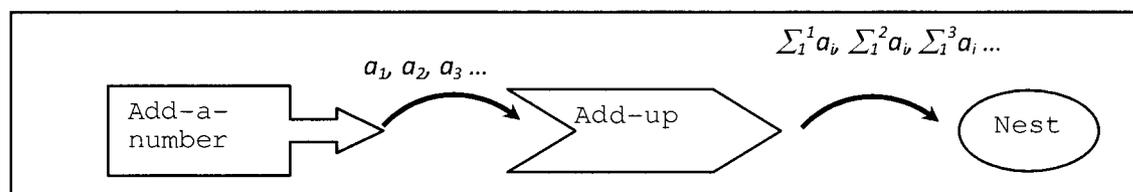


Figure 56: "Chaining" Add-a-number to Add-up

Again, I relied on ToonTalk's features and utilised the streams pattern. ToonTalk is a concurrent language, which means that several programs (robots) can run concurrently. This allows students to generate a sequence and add up its terms at the same time, while keeping the two processes clearly distinguishable. Directing students to this pattern addressed two aims. First, initial experiments had shown – as suggested by the literature – that students tend to become confused between source sequences and the corresponding sequences of partial sums. This confusion causes difficulties in reasoning about limits, and sequence behaviour in general. Secondly, by using one sequence as an input to a process which generates another one, both process and object perspectives were involved, and students were encouraged to construct links between them.

¹⁵ http://www.weblabs.org.uk/wlplone/Members/yish/my_reports/Report.2004-12-06.1107

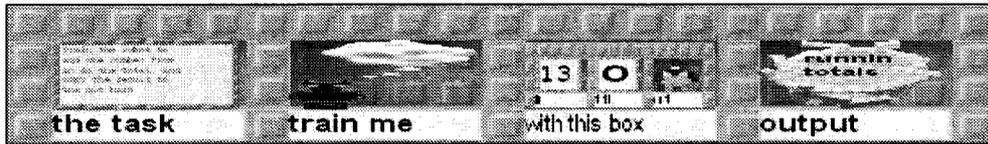


Figure 57: Add-up task-in-a-box

Using a new worksheet, which also includes a task-in-a-box (Figure 57), students were asked to construct the Add-up robot and post it on their webreport. They then chained it with the Add-a-number robot (Figure 58), and experimented with summing different sequences. Next, they were asked to answer some questions regarding the chain of robots. The Add-a-number and Add-up phase of activities was concluded by a group discussion. The discussion was motivated by the goal of composing a consensus webreport based on the individual webreports for Add-a-number and Add-up. Using an active worksheet displayed on an interactive whiteboard, the Add-a-number robot was constructed by the group. One of the students demonstrated how to train the robot, and where others disagreed, they discussed their solutions until a consensus was reached. After the robot was trained and posted, the students continued to discuss the answers to the questions in the worksheet. Occasionally students' individual webreports were reviewed to refresh their memory. This process was iterated for the Add-up robot.

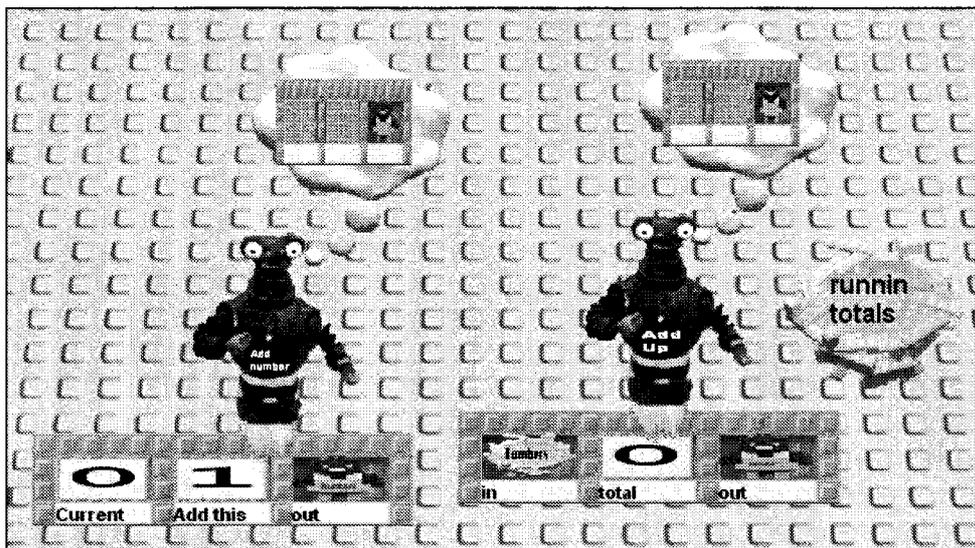


Figure 58: Chaining Add-a-number to Add-up.

Add up surprises

This supplemental task tests the depth of students' command of the techniques developed in the previous two. While the previous two activities were foundational and indispensable in terms of subsequent activities, this one was not. In some groups it was used as an assessment task, in some as a bonus activity for faster students, in others it was skipped altogether.

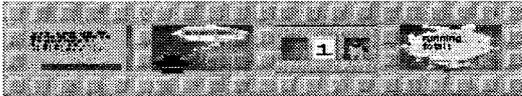
This task was set up as a puzzle game, or riddle. Students were challenged to use the Add-up tool they had constructed to generate unexpected sequences: powers of 2 and the Fibonacci sequence (see Figure 8). When used primarily as a means of keeping advanced students occupied, it was positioned as a recreational activity, and any assessment or evaluation was done by non-intrusive

observations. To be used as a formal assessment tool, adherence to the elaborate response template is expected. To allow this flexibility, the response template was separated from the game page, which only contains the task-in-a-box, links to the response template and a concealed 'hint'. To retain the game-like style, this hint is itself enigmatic: 'You **ONLY** need Add up! No other robots! Let the serpent eat its tail...'

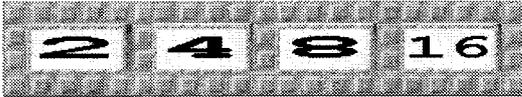
Add Up surprises 🏠 🍷 📄

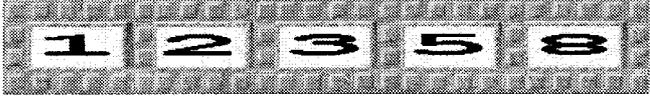
Created by [yish](#) - Topic Group: [Sequences](#) - Created: 04-11-04 - Modified: 28-02-05

Challenge: Can you generate these sequence with Add-Up?



I was playing with my *Add Up* robot, and discovered I can generate these sequences with it. Can you?





Use [this template](#) to publish your response.

[Click here for a hint.](#)

Figure 59: Add-up surprises puzzle, used as bonus task and an assesment tool conculding segment I.

Whatever the scenario, this activity marks a transition from linear and quadratic sequences to a broader, perhaps more complex, set. It is worth noting that some students 'discovered' such sequences during the first two activities. However, these would typically be generated by replacing the additive in Add-a-number with a function. In the case of this challenge, students are drawn to engage not only with richer mathematical structures, but also with richer processes for generating these structures. This point requires elaboration.

Let us use the notation $[a|b|c\dots]$ to denote a ToonTalk robot's box with the inputs a in the first hole, b in the second, c in the third and so on. An Add-a-number robot receives a box of the form:

$$[a|b|bird],$$

and computes the sequence:

$$S_0 = a; S_n = S_{n-1} + b,$$

which in closed form would be written as:

$$S_n = a*n + b.$$

In ToonTalk this behaviour is achieved by 'dropping' a copy of b over S_n . This is in fact shorthand for applying the unary function $+b$ to S_n . By replacing b with a function $f(x)$, we generalize to a broader set of recursive (or iterative) sequences:

$$S_0 = a; S_n = f(S_{n-1})$$

This class includes the powers of 2, for example. Set $f(x) = 2x$ (in ToonTalk terms, place '*2' in the second hole). However, the second sequence in the challenge – the Fibonacci numbers – can not be produced by a unary function, and thus cannot be obtained as an outcome of `Add-a-number`. Furthermore, the challenge requires the student to use `Add-up` to generate the sequences. `Add-up` operates on a sequence as an object. Using `Add-a-number` engaged students with questions of functional relations between numbers, whereas using `Add-up` elevated the discussion to functional relations between sequences – or, in other words, second order functions. In the case of this challenge, the question they are asked to solve is:

Given the operation $S_n = S_{n-1} + T_n$, What is the sequence T I need in order to produce a sequences $S = \{2, 4, 8, 16...\}$ and $S = \{1, 3, 5, 8...\}$?

The solution requires them to extend their recursive intuitions from the domain of numbers to the domain of sequences, identifying $T = S$ and then $T = \{T_i \mid T_n = S_{n-1}\}$.

Concluding notes on Segment I

The main mathematical theme of this segment might seem simplistic: generating the natural numbers and then summing them. In fact, this was very much an open-ended realm for explorations, which has the potential to challenge and extend students' basic intuitions of numbers, sequences and functions. By starting from very simple premises, students had the opportunity to engage with confidence and initiative.

It is important to acknowledge that students were required to learn much more than the mathematical content. At the same time, they needed to assimilate a new educational culture and gain control of a range of technological assets. While the mathematical knowledge was the primary goal of these activities, ignoring the necessity of the other aspects of learning is a recipe for failure.

Several of these tools are fundamental to the activity set as a whole and warrant special attention. On the social interaction side, these include the active worksheets and task-in-a-box. On the individual construction side, the key elements are the streams pattern and the method of chaining robots.

The streamlined design of the `Add-a-number` and `Add-up` tasks did not emerge overnight. It was preceded by many versions during the bootstrapping and exploratory iterations (5.2.3). The first attempt produced tools which were specific to constrained classes of sequences, and were modulated by changing parameter values. These tools were clumsy, and afforded only a limited variety of activities. One of the problems which emerged from the preliminary trials was the representation of the sequences produced by robots. Discussing this issue with my colleagues led to identifying the Streams approach as most suitable for ToonTalk, due to the language's concurrent nature and the "birds" mechanism. The first implementation of this approach was oriented towards providing students with black-box tools for generating, manipulating and displaying number

sequences. The set highlighted mathematical elegance: it was based on a `Constant` tool, which generated a stream of numbers of a single value, and `function` applicators, which would consume a stream (or two streams) of inputs, apply a designated function and produce a stream of outputs. The activities designed at this phase involved chaining these components to explore various questions. A set of stream visualisation tools was also provided for observing the resulting sequences. An example activity used two different constructions of the even numbers to compare their cardinality to that of the integers.

This implementation suffered from several shortcomings. First, the ToonTalk interface is very powerful in displaying a computational process, or in hiding it altogether. The implementation called for a partial disclosure and fine-grained control of parameters such as speed of calculation, which was hard to achieve. More crucial, the pilot study suggested that this approach requires a conceptual leap which students find very demanding. The conclusion which emerged was that students would understand the workings of tools better if they constructed them from scratch, with ample guidance and support, and at the same time would find tasks more interesting if they were posed as programming puzzles. These assertions are at the basis of the final design of tools and activities.

Segment II: “Guess my robot”

Following the basic sequences activities, students moved on to a game called *Guess my Robot* (GmR). This game was a pivotal activity in our explorations of number sequences. Most students entered it with very little formal knowledge of sequences, and minimal ToonTalk experience. After GmR they move on to more advanced topics, such as the Fibonacci sequence, convergence and divergence, and cryptography. GmR was one of the most successful and widely reported activities of the WebLabs project (Mor & Sendova, 2003; Mor, 2004). It was used by teachers and researchers in four different countries, sometimes as part of the flow of activities described here and sometimes as an independent activity. It has also inspired two other games, one in the domain of probability, the other in the domain of graphs.

In GmR, *proposers* (students) invent a rule for a number sequence and model it as a ToonTalk *robot* (procedure) that generates that sequence. They then collect the first few terms of its output in a ToonTalk *box* and embed it in a web report. *Responders* can click on the image of the box, and explore its contents in their own ToonTalk environment. They use a variety of tools to uncover the rule of the sequence. Once they succeed, they respond to the challenge by posting a comment on the report, which includes a robot they have created for generating the same sequence.

The design of this game was motivated by the desire to engage students intensely, both individually and as a community, with structural features of number sequences. It aspired to promote a *mathematical community of practice*, in which issues such as algebraic equivalence are legitimate topics of conversation, and ToonTalk code is a shared form of *mathematical narrative*. Specifically, it aimed to:

- Develop a shared mathematical (albeit not necessarily algebraic) language for describing, discussing and reasoning about sequences.
- Gain a proficiency in manipulating mathematical tools to generate and analyze sequences.

- Acknowledge the duality of a sequence as a structure and a process at once.
- Confront fundamental issues of mathematical argumentation – conjecture, hypothesis testing, proof and equivalence.

The Guess my Robot game

The game would start with a *proposer* training a robot to generate a numerical sequence, and publishing its first few terms as a ToonTalk “box” in a WebReport, using the template shown in Figure 60.

Guess my Robot Template 🔍 🗨️ 🏠

Created by **gordon** - Topic Group: **Sequences** - Created: 06-01-04 - Modified: 06-12-04

[New Report from this template](#)

<your name here>'s Guess My Robot Page

Replace <your name here> with your name, or code name.

My Sequence:

Replace the box below with your own:

1
2
3
4

Hints:

If you think they are needed

Solution:

After someone posts a Robot that generates your sequence, mention him / her and post your original robot here.

Figure 60: Guess my Robot template This directed students to post their challenges in a common form, which also allowed the game to transcend language barriers.

Responders then attempted to build a robot that produces this sequence, and thus show that they have worked out the underlying rule.

Guess my Robot Game



Created by [yish](#) - Topic Group: [Sequences](#) - Created: 02-12-04 - Modified: 07-12-04

This is the main page for the guess my robot game. Before you start, please read the Rules of the Game.

Challenges

This table should list all game pages published this year. If yours is not on it, send a message to [yish](#).

game page	author	published
Sobbe-Guess my robot	sebid	06-12-04
guess the robot	BOB	06-12-04
my puzzle	mark	06-12-04
Guess my robot	Achilles	06-12-04
guess my robot	sodepop	06-12-04
Guess the sequence	Titan	06-12-04
guess my robot	superpa313	06-12-04
Guess my Robot	yish	02-12-04
Gordon's guess my robot challenge	gordon	02-12-04

Posting your own challenge

To post your own challenge, use the [Guess my Robot Template](#).

Figure 61: Guess my Robot main page, which included links to game resources and a "league chart" to amplify the social dynamics of the game.

The game activity was orchestrated through a main game page, shown in Figure 61. This page lists the existing challenges and points to the rules of the game (Figure 62) and the template.

Guess my robot rules

Created by [yish](#) - Topic Group: [Sequences](#) - Created: 08-03-04 - Modified: 03-12-04

Portuguese | Greek

The rules of *Guess My Robot*:

Each player can "challenge" all others to guess the robot he / she has built, or "respond" by trying to guess how someone else's robot works.

To challenge:

1. Create a robot that generates a sequence of numbers like "1,3,5,7,9..." - only smarter :-)
2. Create a game page: go to [Guess my Robot Template](#) and click *New Report from this template*.
3. Let your robot run and collect the sequence in a box.
4. Add the box to your game page (if you don't know how to do that, try this [tutorial](#))
5. Add any hints or tips you think might be necessary to guess your robot.
6. Publish your page.
7. Add a comment to the [Guess My Robot](#) page, with a link to your page.

To respond

1. Create a robot that generates the same sequence as the "challenge" you are responding to.
2. Add it to a comment on the challenging game page .

When your challenge is responded to

If the robot that someone posted does the same thing as your original robot say something nice, upload your original, and think of a tougher challenge.

Figure 62: Guess my Robot rules

Figure 63 shows a typical challenge page, and Figure 64 shows a response posted to that challenge.

Guess my robot

Created by [B. Santos](#) - Created: 03-03-04 - Modified: 17-03-04

Bárbara's Guess My Robot Page

My Sequence:



Hints:

Solution:

After someone posts a Robot that generates your sequence, mention him / her and post your original robot here.

Figure 63: Example of Guess my Robot challenge. Bárbara posted her challenge using the template

Comment

⇒ **I have a conjecture...**

Posted by: Rita at 30-04-04

Olá Bárbara

O Robot que eu contruí parece-me que gera a tua sequência.



Can you explain?
 What if...
 I have a conjecture...
 This doesn't work because...
 Try this instead...
 Other: _____

Figure 64: Example of Guess my Robot response. Rita responded to Bárbara's challenge above with a robot she programmed to generate the same sequence.

The small change challenge

Similar to the *Add-up surprises* challenge above, this task was positioned as a puzzle-game or riddle. In this case, the challenge is to re-program one of the two robots in a chain (Add-a-Num and Add-up) to produce the factorial sequence (Figure 65).

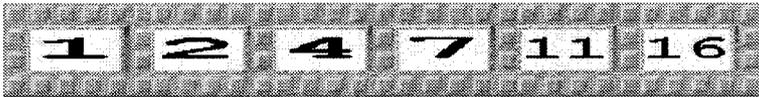
The Small Change challenge

Created by [yish](#) - Topic Group: [Sequences](#) - Created: 04-10-04 - Modified: 05-01-05

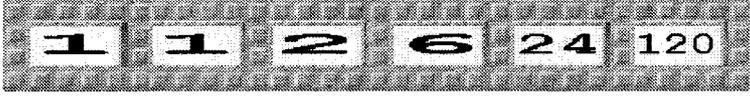
I've created this chain of robots:



When I run it, it generates this sequence:



I trained ONE robot in the chain slightly differently and now it generates this sequence:



Can you do the same?

Figure 65: Small change challenge: modify the chain of robots from segment 1 to produce the factorials.

This task was used as an intermediary evaluation tool for the *guess my robot* game. From this perspective, one of the primary motivations for this task was the findings of Küchemann and Hoyles (2005) regarding the factorial task:

4! means $4 \times 3 \times 2 \times 1$.

5! means $5 \times 4 \times 3 \times 2 \times 1$.

a. Is 5! exactly divisible by 3? Explain your answer.

b. What does 100! mean?

a. Is 100! exactly divisible by 31? Explain your answer.

Küchemann and Hoyles' findings were used as a baseline for evaluating students' performance in this task, by presenting them similar questions and keeping an open ear for situated abstractions which emerge from their experience in solving the puzzle.

Segment III: Convergence and divergence

Background

The notion of convergence of sequences is a well-documented stumbling block for students of all ages. For instance, many first year undergraduate students continue to believe that a sequence cannot reach its limit or that the limit is the last term in a sequence (Eade, 2003). Although most of the literature focuses on difficulties with the formal ($\epsilon - \delta$) definition of limit, Tall & Schwarzenberger (1978) suggest that the root of these difficulties may lie in an uneasiness with the notion of infinity "*as if it were all a piece of mathematical double-talk, having no real-life meaning*". This segment of the activity plan aimed to address this issue, by allowing students to explore their theories through modelling with familiar tools, and thus bypassing the difficulties inherent to the $\epsilon - \delta$ formalism.

The pilot study in 2002-2003 identified a difficulty in expressing clear statements about the behaviour of sequences, e.g. their rate of change, which echoes the findings of Sacristán (1998, 1999). It is important to observe the behaviour of a sequence's to obtain insights into the convergence or divergence of the sequence itself and that of its corresponding series.

In this segment of activities, students investigated questions such as: Can a sequence get smaller and smaller and never go below 0? What happens when you sum the terms of such a sequence? How can you describe the differences between sequences, and how do these differences affect the convergence of the corresponding sum series?

Having constructed their tools for exploring sequences and used them extensively in the previous segments, students would by now have achieved a level of mastery which allows them to express ideas effortlessly, as required by the ideal of tool conviviality.

This segment aimed to translate tool conviviality into mathematical conviviality: allowing students to manipulate and construct mathematical arguments about sequences just as effortlessly as they manipulate the code which generates these sequences. In the process, they should transcend their initial pattern-spotting tendencies and find intrinsic motivation for formal methods. The intent behind the design of this segment was to challenge students' intuitions without alienating them, by creating the conditions for them to acknowledge contradictions in their intuitions as these emerged from their actions and individual explorations. Such contradictions prompted them to seek a rigorous method of formulating their ideas and assessing them. Exposing this trajectory in a public medium promoted robust socio-mathematical norms. Specifically, this segment aimed to prompted students to -

- Experience surprises arising out of the tension between intuitions of infinity and evidence revealed through activity.
- Develop a non-algebraic language for describing, discussing and reasoning about convergence, divergence and limits. This language will be derived from discussions of the programming activities, but will be extrapolated to address the underlying mathematical concepts.

- Develop students' ability to evaluate their own and their peers' arguments and reasoning.
- Begin to appreciate the nature of divergent and convergent sequences.
- Discriminate between empirical evidence and formal argumentation, while using both in exploratory activities.

The structure of this segment (Figure 66) was inspired by the classical notion of computer science as empirical enquiry (Newell and Simon, 1976). It explored mathematical concepts by means of generate-test cycle: students raised conjectures regarding the behaviour of sequences, and then generated examples to test these conjectures. Based on their experience, they were naturally inclined to generalise their findings. At this point, I would present them with a counter example. This counter example led them to acknowledge the limitations of empirical inquiry in mathematics, and look for a formal solution.

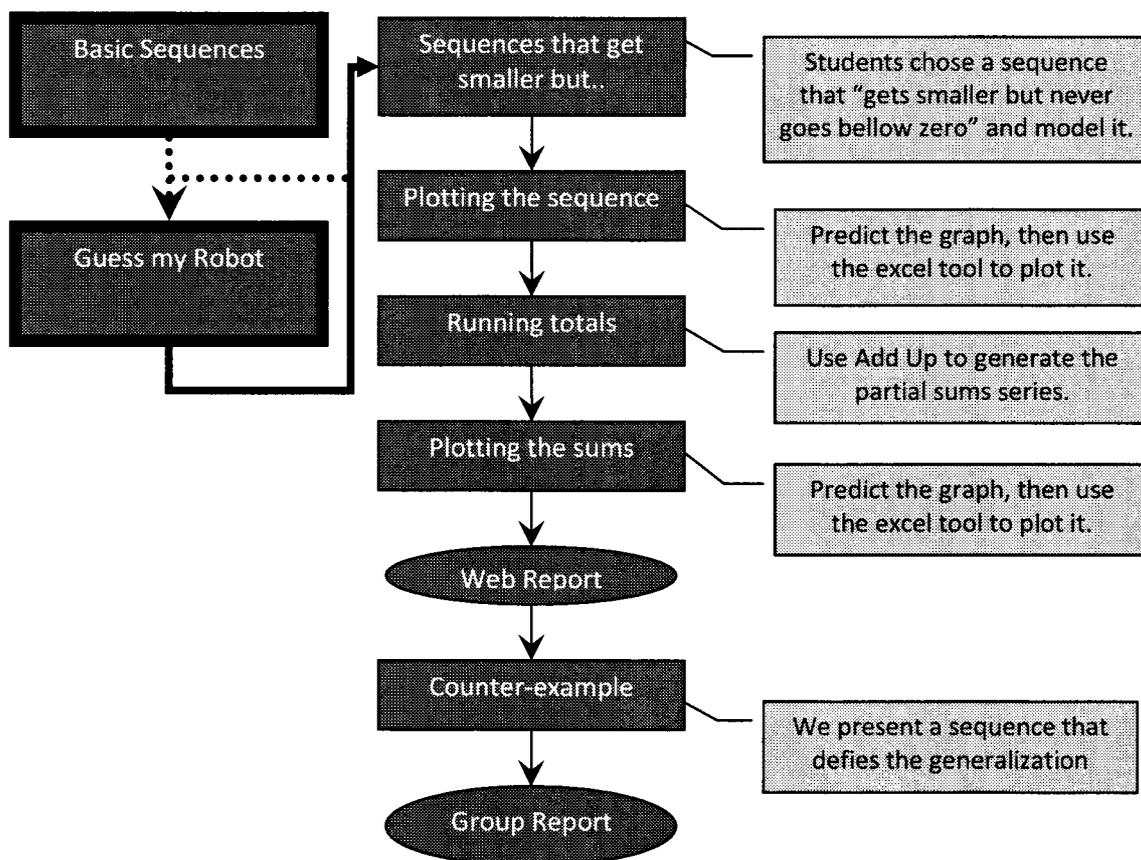


Figure 66: Convergence activity segment

Pre-activity questionnaire

The activity was initiated with a questionnaire on convergent sequences. This questionnaire prompted students to reflect on the idea of convergence and articulate their initial intuitions. It provides a record of these intuitions, which can be referred to later.

This questionnaire was used as a pre-test assessment of students' knowledge, and compared with a similar post-test questionnaire. It was supplemented with stimulated recall interviews. The questionnaire was designed to assess:

- Are students capable of providing an example sequence, and does their example meet the criterion of "gets smaller but does not go below 0"?
- Do they predict the general shape of the sequences and series graphs correctly?
- Do they predict the convergence / divergence of the sequences and series correctly?
- Are their arguments structural?

Group discussion

After completing the questionnaire, students discussed their responses to the first question:

Give an example of a sequence that goes on and on, getting smaller and smaller but never going below 0.

Predict the shape of its graph.

Students were asked to compare their answers and evaluate each other's arguments. The pilot study suggested that most students choose the halving sequence ($a_n = 1/2^n$, or $a_n = a_{n-1} / 2$) with different starting points. Students were alerted to the question of similarity of these examples, and were asked to expand and generalize this class.

Next, students were presented with the question of the sum series:

What happens to the sequence of running totals of your sequence? Does it converge? What would its graphs look like?

They were encouraged to raise different conjectures and argue for them. These led into the task of investigating their conjectures empirically using ToonTalk and graphs. Students' articulations in the course of the discussion were compared to their written answers.

The empirical cycle: predict, model, reflect, publish

Students were asked to model their sequence and test their conjectures. They started by publishing their conjecture about the behaviour of the sequence in a webreport, using a template (Figure 67).

My sequence ⌵ ⌵ ⌵

Created by yish - Topic Group: Sequences - Created: 22-03-05 - Modified: 22-03-05

[New Report from this template](#)

The sequence I modelled is: _____

Here is the robot that generates it: _____

Here is its graph: _____

Here is the graph of its running totals (using the Add up robot): _____

My observations: _____

Figure 67: Report template for segment III, prompting students to post their sequence, their conjectures, and the results of their exploration.

Next, they modelled the sequence in ToonTalk, and uploaded it to their report. Students were reminded to follow the programming conventions set down in the previous activities (basic sequences, guess my robot):

- Use the Stream design pattern; the sequence is modelled by a robot which generates its terms one by one and sends them out with a bird.
- Package the robot in a box, along with its input box and output nest.
- Add a text to this box, describing the robot – its use and mode of operation.

After they uploaded their robot, students returned to the ToonTalk environment and experimented with it. They started by running it for a few minutes and observing its output.

The next task was to connect the sequence robot to a graphing tool and plot the sequence. In groups II, III and IV this was done using the “data to Excel” tool (Figure 68) and in group V using the ToonTalk graphing component (Figure 69).



Figure 68: Export data (to Excel) tool

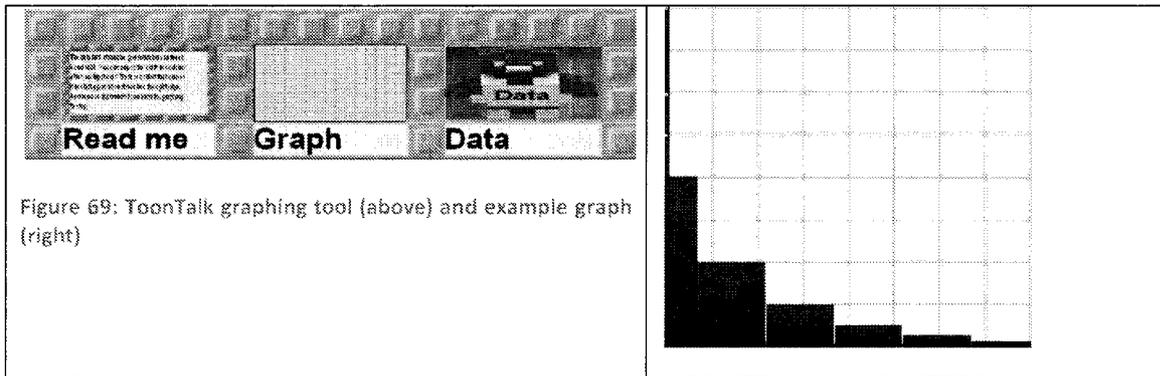


Figure 69: ToonTalk graphing tool (above) and example graph (right)

Using excel, students tend to use line graphs in Excel, where bar graphs would be more appropriate (Figure 70). Line graphs are somewhat misleading when plotting a sequence. Strictly speaking, the lines between the data points are mathematically meaningless. Epistemologically, it seemed that students found it easier to link the sequence to its partial sums when seeing corresponding bars, but the data was not conclusive.

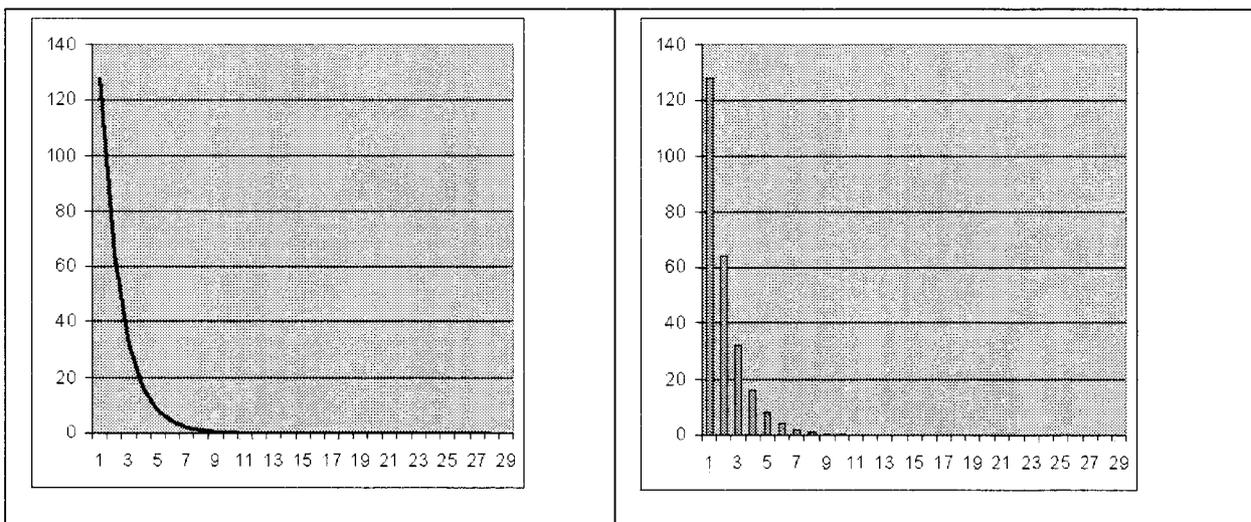


Figure 70: Line vs. bar graphs of a sequence in Excel

Once they have published the graph for their sequence, students were asked to predict the shape of the graph of its running totals. Most students intuitively used basic graphics software for this task (such as Microsoft Paint), and embedded their prediction sketch in their report accompanied by a narrative description.

Having published their prediction, students naturally move on to testing it empirically. In order to do so, they used the Add-up tool (Figure 71) they had constructed in segment I, which sums the terms of the sequence and generates a stream of the running totals. This was then chained to either of the graphing tools, to plot the sequence.

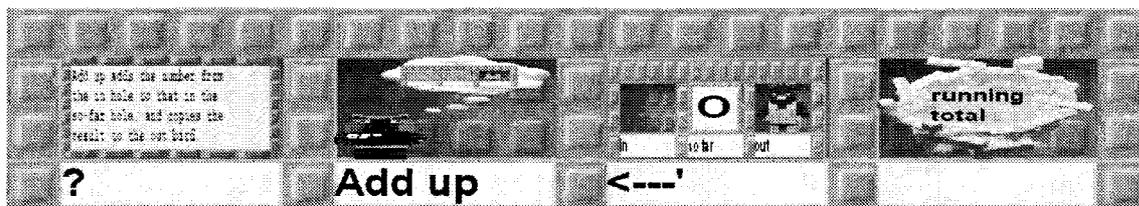


Figure 71: Add-up tool

At this point the students have completed their exploration of the sequence they have chosen. They proceeded to complete their webreport, commenting on the correlation between their prediction and their results, and any other observations they had made. They were encouraged to go beyond the empirical, and provide some rationale for their findings.

The counterexample

By far the most common example students chose is “the halving sequence”¹⁶, $a_n = a_{n-1} / 2$, or in its closed form $a_n = 1/2^n$. The partial sums of this sequence converge, as do the partial sums of most other examples provided by students¹⁷. This leads students to generalize incorrectly and assume that for any sequence that converges to zero, its partial sums series will converge. This calls for an introduction of a counter-example. The simplest instance of such a counter example is the reciprocals sequence, $a_n = 1/n$, and its partial sums – the harmonic series. Introducing this counter example was not intended to discredit students’ intuitions, but to provoke their awareness for the need to monitor these intuitions by formal methods.

A retrospective note is due here. Early versions of this sequence (groups I, II and III) directed students to first experiment with the reciprocals, assuming that it is easier to model. In reality, students found modelling the reciprocals surprisingly challenging. So challenging, that it led to the development of the first *task-in-a-box* (Figure 72). This task-in-a-box includes an untrained robot and an input box for its training. The input box includes two numbers: a denominator and a numerator. Students need to train the robot to copy them out, then divide the copy of the numerator by the copy of the denominator, give the result to the bird, and finally increment the denominator.

¹⁶ This name was consistently proposed by students in different groups.

¹⁷ One rare but interesting exception is sequences that converge to a constant greater than zero, e.g. $a_n = \text{sqrt}_+(a_{n-1})$. Students are easily led to notice that in such cases, the sum series converges to a linear function, and explain this phenomenon by referring to the sequence’s structure. When the limit constant is subtracted from the sequence terms, we obtain a new sequence which converges to zero. In typical student examples, the partial sums of this sequence will converge.

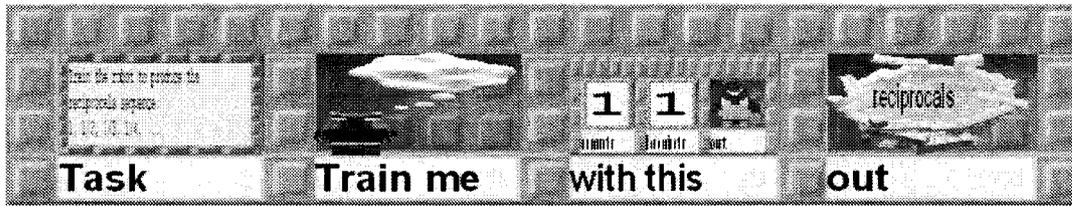


Figure 72: Task-in-a-box for reciprocals robot. The input box was provided to point the students in the direction of an easy to implement solution.

Some students still found the task too challenging, and were eventually provided with a second task-in-a-box, which includes a half-trained robot (Figure 73). This robot has been trained to do everything but increment the numerator. Students needed to run the robot on the floor, observe it, and then complete its training. The one operation they need to add is the one that defines the sequence. Thus, the programming task was reduced to its mathematical core.

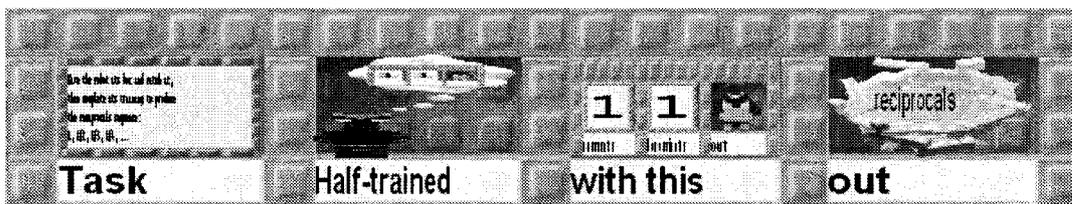


Figure 73: Task-in-a box for reciprocals robot, with half trained robot. This limited the programming task to the fundamental mathematical idea.

The extreme difficulty of this task was a frustrating mystery to me, until in group III, two girls who were not paying attention to my instructions effortlessly modelled the “halving sequence” instead ($a_n = 1/2^n$). This incident exposed the fault of my design, a fault originating in an erroneous epistemic assumption. My assumption was based on a closed form view of the sequence, i.e. as a function of its index. From this perspective, modelling the reciprocals is trivial while modelling the inverse powers of two demanding. If, as argued in 6.2.4, students’ intuitive view of sequences is predominantly recursive, then the converse holds. The “halving sequence” is generated by repeatedly halving the current term (hence its name), whereas the reciprocals sequence does not lend itself to a reasonable recursive representation.

After training the reciprocals robot, students repeated the cycle they had conducted with their sequence: predict its graph, plot it and compare to their prediction, predict the behaviour of the sum series and explore it using Add-up and graphing.

In the case of the sum series – the harmonic series – it was useful to allow time to explore the generation of the sequence before plotting it. When students predicted that it does not exceed a limit, they were asked to “go out of the room” and let the robot run unwatched for a few seconds. Repeating this process several times engendered a strong intuition that the series is unbounded. Students were then asked to report this finding in their webreport, and compare it to the sequence they had investigated earlier. Many found the graphs instrumental in explaining why one series converges and the other does not.

Group discussion & report

The convergence segment of activities concluded with a group discussion, in which students were asked to draw general observations from the activity. To start off, they were asked to report on the sequences they explored, their experiences and findings. Following this, they were asked to derive consensual statements they wish to share with their peers in other sites. The resulting group report was supplemented, before publication, with links to the students' personal reports. Ideally, other groups would comment on the group report, and an inter-site dialogue will ensue. Such a dialogue is highly inductive in terms of knowledge-building. It will tend to drive students more and more towards formal argumentation; first, out of the attempt to promote their views. And secondly, out of a playful desire to "outsmart" their peers. This, however, did not happen: it is very hard to orchestrate events across sites so that two groups would reach the end of the segment at the same time. As a compromise, group IV was split into two sub-groups and each commented on the other's report.

Conclusions

This appendix reviewed the design of a complex and lengthy sequence of activities, starting from basic programming skills and leading to advanced mathematical topics. These activities make extensive use of ToonTalk and WebReports as media for individual and collaborative exploration of number sequences. Through this exploration, learners were introduced to functional and algebraic thinking, and to mathematical argumentation. The design of tools and activities was tightly bound to the learning aims, so that it was impossible to distinguish technical from mathematical proficiency: the students' programming skills developed in tandem with their mathematical sophistication.

Appendix II Examples of data sources and methods of analysis

The examples shown here are drawn from the first phase of group IV's basic number sequences activities (7.2). Students were administered a pre-trial questionnaire, and then proceeded to work on the add-a-number task. While students were working on the task, they were invited one by one for a stimulated recall interview, in which their answers on the questionnaire were reviewed. As they were working on the task itself, Gordon Simpson and I conducted in-activity probes. After each session we produced a session report based on our field notes. After completing the task, the students published a webreport which included their models, observations and answers to a few questions.

Name: _____

1. How would you explain to a younger student what a *number sequence* is?

2. Look at the following number sequence: 2, 4, 6, 8, ...

a. What is the 1st term? _____

b. What is the term after 8? _____

Figure 74: Fragment of Group IV's basic sequences pre-trial questionnaire (3 & 9 Nov. 2004)

Figure 74 shows the top of the first page of the pre-trial questionnaire as it was printed and presented to students. Students completed this questionnaire in class, and the completed forms were collected and processed by the next session. Figure 75 shows a section from the coded responses to the questionnaire in Figure 74. The relevant code significations, with examples from the whole group, are provided in Figure 76. These examples are presented in their original form, apart from the student names which have been modified.

	Michael		Sam		Harold		Luthar	
1	A number sequence is about 3 or more numbers, in a line that have something in common E.g. 1, 2, 3, 4 you keep on adding one to the	C	some numbers that follow a certain pattern	P	A number sequence is like a row of numbers which follow each other(sic), like 1, 2, 3, 4 or 2, 4, 6, 8. Because they all go up by the same amount	A	It's a series of numbers that have rule that will change them	R

	numbers					
2		C		C		C
a	2		2	2		2
b	10		unclear - either 10, or 16	10		10

Figure 75: Coded responses to the pretest (9 Nov. 2004)

1: How would you explain to a younger student what a <i>number sequence</i> is?		
1	P	<p>A sequence is a pattern</p> <p>A pattern of numbers progressing by the same amount each time</p> <p>A number sequence has a pattern</p>
1	A	<p>A sequence is an arithmetic progression.</p> <p>a number sequence is when you keep adding a number to another one</p>
1	R	<p>A sequence is defined by a rule</p> <p>It's a series of numbers that have rule that will change them</p>
1	N	<p>No significant definition</p> <p>A number line</p>
1	O	<p>Other notable statement</p> <p>A number sequence is about 3 or more numbers, in a line that have something in common E.g. 1, 2, 3, 4 you keep on adding one to the numbers</p>
2. Look at the following number sequence: 2, 4, 6, 8, ... a. What is the 1st term? b. What is the term after 8? c. What is the 10th term? d. What is the 100th term?		
2	C	Correct, know what a "term" is
	a-	

	d		
	a-		
2	d	W	Wrong, do not know what "term" means

Figure 76: Coding table with examples

Figure 77 lists a raw transcription of the first few minutes from a stimulated recall interview conducted with Luthar during the following session (note that both Luthar and I had mistaken the date).

	Y	I've started recording you, now, ok, so, just Luthar, right, and what date is it today, 13 th ?
	L	Think it's the 12 th , I'm not sure.
	Y	12 th , OK, 12 th of November. What we're going to do, is this. We're going to go through this questionnaire which you filled last week, OK, and I'm going to ask you to clarify a few of your answers here, and then we're going to, I'm going to give you another task to work on, and we're going to talk a bit about that, OK?
	L	OK, fine
	Y	Um, right, so, number sequence is a series of numbers that have a rule that will change them. OK, can you explain what you mean by that?
	L	Um, it's like, uh, well, if I use an example like if you have, if it's the first one in the sequence, and then it's the number 3, it could be n plus 3, could make the sequence, so it's, um, uh, it's a, like a chain of numbers that, with each one changing to a rule, so not just randomly.
	Y	OK. I just, I [unclear] maybe. Sorry, OK, sorry, back to this, OK. So here, how did you work out the 100 th term, so you write 20 times 10. Why did, why does 20 times 10 help you work out the 100 th term of this sequence?
	L	Um, [whispers] true [whispers] not sure, cos, for instance, if you take 1, then if you times that by 2, you get 2 so if you want to work out the 100 th term, you times, so 1 by 100 makes it 100, so you times, um,
	Y	Can you? How would you name this sequence, how would you call, what would you call it?
	L	2n
	Y	2n. Right, so does that, when you call it 2n does that help you explain what the 100 th term is?
	L	Yeah, because if n is 100, then you times it by 2. That would have been easier, but.

Figure 77: Stimulated recall interview with Luthar (16 Nov. 2004)

Figure 78 presents the text of the session report produced by me after the session during which the interview in Figure 77 was conducted.

Group IV.

Visit 16th November, 2004

1 hour (afterschool) session 3:30-4:30 pm

Six 13-14 yr-old boys.

One computer per student.

Teacher: David Croston

Students: Sam, Paul, Jon, Luthar, Alan (new), Aaron (missed last week)

Students created accounts on WebReports by following through a demonstration on the projector (minimum of hassle, these kids are more net savvy than our previous groups). We gave a very brief introduction of the system. Then we showed them the add-a-number webreport template and explained they had to download the ToonTalk object and complete the task. Note that this was their first session in 'freeplay'. We gave a demonstration in ToonTalk of what the task consisted of, introduced bird-nests (they hadn't seen them before) etc then set them going on the task. Yishay conducted the pretest interviews with Sebby, Luke and Adam (now transcribed by Sophie). Gordon helped students with the task and conducted some brief in-activity probes (audio files filed and labelled).

There were four major issues that came up during the programming:

Using the wand. In order to complete the task, students need to **copy** the *add this* onto *current*, and then **copy** *current* and give it to the bird. In most first attempts, students would **move** one or both of these number pads instead of copying them, resulting in the numbers not being available to the robot the next time he tries to run. With a bit of help, students came up with the idea of using the wand.

Training robots once or more than once. Some students still aren't sure about whether they need to train a robot's actions once or more than once (this is a common problem when students are first learning). One student trained their robot to go through two iterations of copying *add this* and *current* etc, "just to make sure he got the idea". This is related to the next point – robots often stop after one iteration because they need to be generalised, and students sometimes infer from this that robots will only run once if they were trained once.

Generalising. Students have to generalise their robot by erasing (or sucking) the *current* after training. Most got this idea, although some were initially confused as to why the robot stopped after one iteration. Had a good discussion in the group at the end about this topic (i.e. exactly what an erased number pad means in a robot's thought bubble).

Leaving things in the input box. Perhaps a less important issue, but some students trained their robot to take the bird out of the input box and put it on the floor, before giving it the *current* number. After

the robot runs the first time, he will get dusty and vacuum up the bird (because robots are tidy). One student asked if you can train a robot to not be tidy, and I had to tell him this wasn't possible!

Towards the end of the session we asked students as a group how to program the robot. Patrick did most of the instructing, as he was the only one to have successfully programmed his robot (with Gordon's help). Note that David answered quite a few of the questions we were directing at students!

Note: got a few permission slips back for the 6th, David is holding on to them.

Technical issues:

Dinesh opened up the ability for students to download (from webreports, but actually it was from any site I think) which was good. Unfortunately he has to turn this on before every session, and turn it off afterwards. Similarly he hides the ToonTalk icon between sessions and we have to rely on him being there and setting things up for us before each session which is not ideal.

The upload tool is not working. Could be either because FTP is blocked, or because the ftp exe hasn't been given the privileges to run. Should be able to work around by uploading the locally saved tt file using the old browse mechanism on the webreports site (which uses HTTP).

Figure 78: Session report (16 Nov. 2004)

Figure 79 shows Luthar's first webreport, published after he completed the add-a-number activity.

Add a number

Created by **luminardi** - Topic Group: **Sequences** - Created: 16-11-04 - Modified: 06-12-04

Training Add number

I trained a robot to repeatedly add 1 to produce the numbers 0, 1, 2, 3.... Call the robot *Add number*.



Explain how you trained the robot:

What I did to make the number's go up by one each time is I made the robot add the 'add this' to the current then I copy it and give it to the bird

Explore

Can you think of a way to use your robot to produce these sequences?

Sequence	Explain - in words	Explain - in ToonTalk
----------	--------------------	-----------------------

	<p>If yes, explain how you would do it.</p> <p>If you think it is impossible, explain why.</p>	<p>Add any ToonTalk object that helps show how you did it.</p> <p>Describe what a robot must do to produce a sequence.</p>
2, 3, 4, 5...	yes just make the current 1	
-1, -2, -3, -4...	yes just make the add this -1	
-7, -6, -5, -4..	yes just make the current -7	
2, 4, 6, 8...	yes just make the add this 2	
5, -1, -7...	yes just make the current 5 and add this -6	
Write down a sequence of your own, which can be generated by your robot.	22,1,-20,-41	
Write down a sequence of your own, which cannot be generated by your robot.	2,5,8,11	

Explain

How would you explain to a friend what kind of sequences your robot can generate, and how it can be used to generate those sequences?

Describe one sequence that cannot be generated by it, and explain why.

How would you convince a friend of your claims?

Figure 79: Luthar's first webreport

Finally, Figure 80 offers the raw transcription of an in-activity probe conducted with Luthar as he was working on the next task (Add-up, section 7.2). The total length of this probe was a minute and a half.

Y	So, you already explained why in the in box, it actually tells you how many numbers you've added,
---	---

	ok.
L	Yeah
Y	And then the total, what does the total show you?
L	And that shows, what the, in, all the numbers, up to this one have made the [unclear] so the in box.
Y	OK, what, can you say anything about the relationship between the number you have in total, and the number you have in in?
L	Probably something to do with 3, because we've got a sequence of triangle numbers, so, probably this is something the, relate, is this times that 3? Uh, no.
Y	What is this and what is that? That...
L	Total is, this is total, and that was in, so that...
Y	Yes
L	So total, the in times 3 doesn't equal the total, I thought it might.
Y	It's not in times 3, you say,
L	No
Y	So what would it be?
L	Not sure. I could do total divided by in to find out. Not sure how you do that in ToonTalk.
Y	Do you think there'll be a constant?
L	It'll probably be changing every time, because the number that it's adding by changes every time, so.
Y	Alright

Figure 80: In-activity probe with Luther (6 Dec, 2004)

Luther has just finished constructing a robot which takes the natural numbers sequence as an input, and produces a sequence of their partial sums. The purpose of this probe is to establish what Luther had noticed, and understood, regarding the mathematical features of the mathematical process embodied in the robot and the resulting structure. The questions provoke him to suggest various conjectures (e.g., the output is the “times 3 table”) and verify or refute them. Inter alia, he makes several mathematical arguments. The most significant is the last: the resulting sequence cannot be linear because “the number that it’s adding by changes every time”.

Appendix III Data sources for the design narratives in Chapter 7

This appendix provides a table (Table 12) of the data sources used for constructing the design narratives in Chapter 7. The first column notes the identifier used to refer to this source in the narratives' *sources* section. The second column provides the full name of this source. The third column notes the date it was last updated. The fourth lists its location and the fifth adds any relevant comments.

ID	Name	Date	Location	Notes
D3.2.1	WebLabs deliverable 3.2.1: Technical Infrastructure	Sept 2004	Archive	
Monkeys	Numbers, BIG Numbers, and Infinity	16-19 February 2003	Archive	Presentation at project meeting
D8.2.1_8.3.1_Sequences	Learning evaluation in the Number Sequence Activities	Sep 2005	Appendix I	Written by me as part of the WebLabs project final report
AppG_Sequences(Yishay).v2.0-03	TM spec 7: Functions, Sequences, Convergence and Divergence	Sep 2003	Archive	Specification of tools and activities at the end of iteration 1.
sequences_yishay02-03	Sequences Activity Pilot	Sep 2003	Archive	Project presentation
loverN.24jan05	Collated notes on convergence / divergence activities	24 January, 2004	Archive	
SessionReport.27.feb.reciprocal	Reciprocals Session report 1	27 February, 2004	Archive	
SessionReport.05.03.04.reciprocals-task-in-a-box	Reciprocals Session report 2	05 March, 2004	Archive	

SessionReport.0 4.02.13.reciprocal als-pre-activity	Reciprocals pre-activity report	13 February, 2004	Archive	
GmR-PS	Guess my Robot Pedagogical Scenario	06 Sept 2004	Archive	Analysis of type and complexity of sequences produced and analysed by participants
EB2.3	Vale de Milhaços report	17 May 2004	Archive	
webreports_gor donyishay.02- 03	Web Reports Mechanism	Sept 2003	Archive	Project meeting presentation
WebReports	The WebReports system		http://www.weblabs.org.uk/w/plone	
loverN.24jan05	1/n, 1/2", the object in focus, eager robots	24 January 2005	Archive	
Lessons_Learnt _Convergence_ IOE	Lessons Learnt: Activity sequence 2	07 May, 2004	Archive	
WR1	Web report: The Story of Pi	Sep. 23, 2002	Archive	
WR2	Web report: Streams, infinity and robots	Oct. 16, 2002	Archive	
WR3	Sequences Teacher guide: overview	24 June 2005	http://www.weblabs.org.uk/w/plone/Groups/sequences/group_guidance_index.html	
WR4	The Add-a-number challenge	30 Dec 2004	http://www.weblabs.org.uk/w/plone/Members/yish/my_reports/Report.2003-12-18.2548	
WR5	The Add-up challenge	30 Dec 2004	http://www.weblabs.org.uk/w/plone/Members/yish/my_reports/Report.2003-12-18.4817	
WR6	Add up surprises	28 Feb	http://www.weblabs.org.uk/w/plone/Members/yish/my_reports/Report.2003-12-18.4817	

WR7	Report on Joe999 GmR incident	2005 04 May 2004	rs/yish/my_reports/Report.2004-11-04.4023/	
WR8	Reciprocals task template	03 March, 2004	http://www.weblabs.org.uk/wlplone/Members/yish/my_reports/Report.2004-05-04.5906/	
WR9	Initial group report on convergence	13 February, 2004	http://www.weblabs.org.uk/wlplone/Members/yish/my_reports/Report.2004-02-13.4220/	
WR10	Reciprocals and Harmonic	29 April, 2004	http://www.weblabs.org.uk/wlplone/Members/yish/my_reports/Report.2004-04-29.5327/	
WR11	Personal reports on convergence	28 May, 2004	http://www.weblabs.org.uk/wlplone/Members/yish/my_reports/Report.2004-05-28.3232/	
WR12	Harmonic Sequence Task	03 March, 2004	http://www.weblabs.org.uk/wlplone/Members/yish/my_reports/Report.2004-03-03.3328/	Task worksheet for harmonic sequence
WR13	Reciprocals Task	03 March, 2004	http://www.weblabs.org.uk/wlplone/Members/yish/my_reports/Report.2004-03-03.3222/	Task worksheet for reciprocals sequence
WR14	My Sequence	22 March, 2005	http://www.weblabs.org.uk/wlplone/Members/yish/my_reports/Report.2005-03-22.1925/	Webreport template for convergence & divergence
WR15	Reciprocals and Harmonic	23 June, 2005	http://www.weblabs.org.uk/wlplone/Members/yish/my_reports/Report.2005-06-23.3745/	Reciprocals worksheet, 2005
WR16	Halving Sequence	22 March, 2005	http://www.weblabs.org.uk/wlplone/Members/yish/my_reports/Report.2005-03-22.2632/	Sodapop's report on his "halving" sequence
WR17	Gets-smaller sequence	22 March, 2005	http://www.weblabs.org.uk/wlplone/Members/Sarah/my_reports/Report.2005-06-09.2248/	Sarah's report on her sequence
GmR1	GmR game page, 2003-2004	18 March 2004	http://www.weblabs.org.uk/wlplone/Members/yish/my_reports/Report.2004-01-06.5353/	
GmR2	GmR game page, 2004-2005	2 February, 2005	http://www.weblabs.org.uk/wlplone/Members/yish/my_reports/Report.2005-02-03.4559/	

GmR3	GmR rules	8 March, 2004	http://www.weblabs.org.uk/wlplone/Members/yish/my_reports/Report.2004-03-08.1130	
GmR4	GmR teachers' guide	25 June, 2004	http://www.weblabs.org.uk/wlplone/Members/yish/my_reports/Report.2004-06-25.4000/	
GmR5	Rita's GmR challenge	27-01-04 to 25-04-04	http://www.weblabs.org.uk/wlplone/Members/Rita/my_reports/Report.2004-01-27.1557/	
GmR6	Yishay's GmR challenge	14 January, 2004	http://www.weblabs.org.uk/wlplone/Members/yish/my_reports/Report.2004-01-14.1024	
GmR7	Joe999's response	12 February, 2004	http://www.weblabs.org.uk/wlplone/Members/Joe999/my_reports/Report.2004-02-10.0501	
GmR8	Joe999's GmR challenge		http://www.weblabs.org.uk/wlplone/Members/Joe999/my_reports/Report.2004-03-04.2738/	

Table 13: Data sources for the design narratives. The first column notes the identifier used to refer to this source in the narratives' sources section. The second column provides the full name of this source. The third notes the date it was last updated. The fourth lists its location and the fifth adds any relevant comments.

Appendix IV Published Papers

This appendix includes three of my publications which have been mentioned in the text of the thesis:

Matos, J. F., Mor, Y., Noss, R., & Santos, M. (2005). Sustaining Interaction in a Mathematical Community of Practice. In *Fourth Congress of the European Society for Research in Mathematics Education (CERME-4)*. Sant Feliu de Guíxols, Spain.

Mor, Y., Noss, R., Hoyles, C., Kahn, K. & Simpson, G. (2006) Designing to see and share structure in number sequences. *International Journal for Technology in Mathematics Education*, 13, 65-78.

Mor, Y., & Winters, N. (2007). Design approaches in technology enhanced learning. *Interactive Learning Environments*, 15, 61-75.

These publications are strictly supplemental to the thesis. They are provided here for the readers' convenience. All my publications can be accessed at:

<http://www.yishaymor.org/publications/>