The Calculus of Signal Flow Diagrams I: Linear Relations on Streams

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Abstract

We introduce a graphical syntax for signal flow diagrams based on the language of symmetric monoidal categories. Using universal categorical constructions, we provide a stream semantics and a sound and complete axiomatisation.

A certain class of diagrams captures the orthodox notion of signal flow graph used in control theory; we show that any diagram of our syntax can be realised, via rewriting in the equational theory, as a signal flow graph.

Keywords: signal flow graph, string diagram, linear algebra, PROP, distributive law, Hopf algebra, Frobenius algebra

1. Introduction

Feedback and related notions such as self-reference and recursion are at the core of several disciplines, including Computer Science, Engineering and Control Theory. In Control, *linear dynamical systems* are amongst the most extensively studied and well-understood classes of systems with feedback. They are signal transducers with two standard interpretations: *discrete*, where—roughly speaking—signals come one after the other in the form of a stream, and *continuous*, where signals are typically well-behaved real-valued functions.

From the earliest days, diagrams played a central role in motivating the subject matter. Graphical representations were not merely intuitive, but also closely resembled physical manifestations (implementations) of linear dynamic systems, such as electrical circuits. While differing in levels of formality and minor technical details, the various notions share the same set of fundamental features—and for this reason we will group them all under the umbrella of *signal flow diagrams*. These features are: (i) the ability to copy signals, (ii) the ability to add signals, (iii) the ability to amplify signals, (iv) the ability to *delay* a signal (in the discrete, stream-based interpretation) or to *differentiate*/*integrate* a signal (in the continuous interpretation), (v) the possibility of feedback loops

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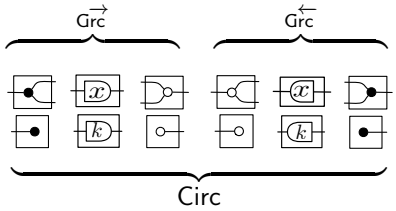
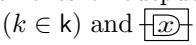
and (vi) the concept of directed signal flow. Notably, while features (i)-(v) are usually present in physical manifestations, (vi) seems to have been included to facilitate human understanding as well as to avoid "nonsensical" diagrams where the intended signal flow seems to be incompatible or paradoxical. Of course, physical electrical wires do not insist on a particular orientation of electron flow; both are possible and the actual flow direction depends on the context.

Signal flow diagrams were typically not considered as an interesting object of study per se, perhaps because of the perception that they lacked rigour: in the literature they are typically translated to sets of recurrence relations (under the discrete interpretation) or higher-order ordinary differential equations with constant coefficients (under the continuous interpretation). These are then solved using standard techniques. Diagrams are, instead, the main actor in our development, and we treat them rigorously as particular kinds of *string diagrams* [1] – i.e. graphical representations of arrows in a (symmetric) monoidal category. We introduce a graphical calculus of string diagrams, which we call *circuits*, consisting of the following operations, sequential ; and parallel $\oplus$ composition.



In this paper we concentrate on the discrete interpretation; thus circuits are given a *stream semantics*. The intuition is that wires carry elements of a field k that enter and exit through boundary ports. In particular, for circuits built from components in the leftmost three columns, which we refer to as being in $\mathsf{Circ}^{\rightarrow}$, the signal enters from the left and exits from the right boundary. Computation is synchronous, and at each iteration fresh elements are processed from input streams on the left and emitted as elements of output streams on the right. The basic components  ,  ,  ($k \in \mathsf{k}$) and  realise features (i)-(iv). The remaining components,  and  , are the units of  and  :  accepts any signal and discards it, while  constantly outputs the signal 0.

For circuits arising from the remaining columns, $\mathsf{Circ}^{\leftarrow}$, the signal flows in the opposite direction: from right to left. The behaviour is symmetric. Formally, the stream semantics of circuits in $\mathsf{Circ}^{\rightarrow}$ and $\mathsf{Circ}^{\leftarrow}$ consists of linear transformations, thus their behaviour is functional. Circuits in Circ—built out of *all* the components—do not, in general, yield functional behaviour. Signals no longer flow in a fixed direction: indeed feature (vi)—the notion of directed signal flow—plays no part in our definitions. The semantics of circuits in Circ is given by subspaces with relational composition, i.e., linear relations[1]. We

[1] We shall use the terms *subspaces* and *linear relations* interchangeably throughout the paper.

$\mathsf{k}[x]$	the ring of polynomials	$\sum_0^n k_i x^i$ for some natural n
$\mathsf{k}(x)$	the field of fractions of polynomials	$\frac{p}{q}$ for $p, q \in \mathsf{k}[x]$ with $q \neq 0$
$\mathsf{k}\langle x \rangle$	the ring of rationals	$\frac{\sum_0^n k_i x^i}{\sum_0^m l_j x^j}$ with $l_0 \neq 0$
$\mathsf{k}[[x]]$	the ring of formal power series	$\sum_0^\infty k_i x^i$
$\mathsf{k}((x))$	the field of Laurent series	$\sum_d^\infty k_i x^i$ for some interger d

Table 1: Rings and fields over a field k (k_i and l_j range over k).

must also use an extended notion of streams, *Laurent series*, typical in algebraic approaches [2] to signal processing—roughly speaking, these streams are allowed to start in the past. Passing from functions to relations gives meaning to circuits that contain feedbacks—taking care of feature (v)—which increases the expressivity w.r.t. $\overrightarrow{\mathsf{Circ}}$ in that certain infinite streams can be denoted: an example is the Fibonacci circuit (Example 7.2).

We obtain the stream semantics via both a universal property and an intuitive inductive definition. Furthermore, we provide a sound and complete axiomatization for proving semantic equivalence of circuits. To this end, we reuse the results of [3] that generalises our earlier work [4]. For $\overrightarrow{\mathsf{Circ}}$, we exploit the equational theory $\mathbb{HA}$ of Hopf algebras which is the theory of $\mathsf{k}[x]$-matrices, where $\mathsf{k}[x]$ is the ring of polynomials with coefficients from k. For $\overleftarrow{\mathsf{Circ}}$, we use the dual theory $\mathbb{HA}^{op}$. For the whole Circ, we work with the equational theory $\mathbb{IH}$ of Interacting Hopf algebras, which is the theory of linear relations over $\mathsf{k}(x)$, the field of fractions of $\mathsf{k}[x]$. Then, the passage to the stream semantics simply consists in interpreting polynomials and their fractions as streams, as outlined in Table 1. Using again a result in [3], also this interpretation is given by a universal property.

The theory of $\mathbb{IH}$—featuring two special Frobenius algebras [5]—plays a central role in the paper because it is rich enough to encapsulate linear algebraic arguments *within* the graphical theory, making further translations (e.g. to recurrence relations) redundant.

Orthodox Signal Flow Diagrams. The earliest reference for signal flow diagrams that we are aware of is Shannon's 1942 technical report [6]. They appear to have been independently rediscovered by Mason in the 1950s [7] and subsequently gained foundational status in Electrical Engineering, Signal Processing and Control Theory. Traditionally only diagrams that yield functional behaviours on ordinary streams are considered: to ensure this, circuits are restricted so that every feedback passes through at least one delay gate. A well-known theorem (see e.g. [8]) states that circuits in this form represent exactly those behaviours expressible by matrices with entries from $\mathsf{k}\langle x \rangle$, the ring of *rational* polynomials: those fractions where the constant term in the denominator is non-zero. A novel proof of this result was recently given by Rutten in [9], using coinductive

$$
\begin{tikzcd}
\overleftarrow{\mathsf{Circ}} \arrow[r, two heads] \arrow[d, hook] & \mathbb{HA}^{op} \arrow[r, "\cong"] \arrow[d, hook] & \mathsf{Mat}\,\mathsf{k}[x]^{op} \arrow[r, hook] \arrow[d, hook] & \mathsf{Mat}\,\mathsf{k}[[x]]^{op} \arrow[d, hook] \\
\mathsf{Circ} \arrow[r, two heads] & \mathbb{IH} \arrow[r, "\cong"] & \mathbb{SV}_{\mathsf{k}(x)} \arrow[r, hook] & \mathbb{SV}_{\mathsf{k}((x))} \\
\overrightarrow{\mathsf{Circ}} \arrow[r, two heads] \arrow[u, hook] & \mathbb{HA} \arrow[r, "\cong"] \arrow[u, hook] & \mathsf{Mat}\,\mathsf{k}[x] \arrow[r, hook] \arrow[u, hook] & \mathsf{Mat}\,\mathsf{k}[[x]] \arrow[u, hook]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{Circ} \arrow[r, two heads] & \mathbb{IH} \arrow[r, "\cong"] & \mathbb{SV}_{\mathsf{k}(x)} \\
\mathsf{SF} \arrow[r, two heads] \arrow[u, hook] & \mathbb{SF} \arrow[r, "\cong"] \arrow[u, hook] & \mathsf{Mat}\,\mathsf{k}\langle x\rangle \arrow[u, hook] \\
\overrightarrow{\mathsf{Circ}} \arrow[r, two heads] \arrow[u, hook] & \mathbb{HA} \arrow[r, "\cong"] \arrow[u, hook] & \mathsf{Mat}\,\mathsf{k}[x] \arrow[u, hook]
\end{tikzcd}
$$

Figure 1: A technical roadmap of results. The rings $\mathsf{k}[x]$, $\mathsf{k}(x)$, $\mathsf{k}\langle x\rangle$, $\mathsf{k}[[x]]$ and $\mathsf{k}((x))$ are described in Table 1. For a ring R, $\mathsf{Mat}\,R$ and $\mathbb{SV}_R$ denote the category of matrices and linear relations over R, respectively. In the diagrams, the double-headed arrows are the interpretation of syntax within an algebraic theory (i.e. quotienting w.r.t. a set of equations); the tailed arrows are embeddings and the arrows labeled with $\cong$ are isomorphisms. The middle row in the left diagram is the factorization of the stream semantics $\langle\!\langle\cdot\rangle\!\rangle\colon \mathsf{Circ} \to \mathbb{SV}_{\mathsf{k}((x))}$. The diagram on the right shows the status of the class of orthodox signal flow diagrams SF.

and coalgebraic techniques. We identify "orthodox" signal flow diagrams with a subclass SF of Circ and provide yet another proof, using the equational theory of $\mathbb{IH}$ (Theorem 7.4). In Figure 1 we summarise the results mentioned thus far.

Normal Forms. Another well-known fact in signal flow diagrams theory is a normal form: every circuit is equivalent to one where all delays occur in the feedbacks. The proof of this result (Proposition 7.7) becomes trivial after observing that feedbacks "guarded" by delays are a trace in the categorical sense [1].

This holds for circuits in SF. Circuits in Circ can be put either in span or cospan normal form (Proposition 5.4). The former consists of a circuit in $\overleftarrow{\mathsf{Circ}}$ followed by one in $\overrightarrow{\mathsf{Circ}}$ and the latter of a circuit in $\overrightarrow{\mathsf{Circ}}$ followed by one in $\overleftarrow{\mathsf{Circ}}$. In The Calculus of Signal Flow Diagrams II, following the development in [10], we shall exhibit deep connections between the two normal forms for Circ and facets of a canonical operational semantics. Normal forms also play a technical role in several results, notably in the proof of the realisability theorem.

Realisability. In the final part of the paper we compare the expressive power of our diagrammatic universe Circ and the class of orthodox signal flow diagrams SF. We prove *realisability* (Theorem 8.4): every circuit in Circ is equivalent to at least one suitably *rewired* circuit in SF. In general, circuits in Circ can be the rewiring of several different SF circuits, depending on the chosen orientation of signal flow (see Example 8.7). Thus Circ is not more expressive than orthodox signal flow diagrams; viewed as transducers, they define the same class.

What are, then, the advantages of keeping the direction of signal flow out of definitions? In his 1913 paper *On the Notion of Cause* [11], Russell criticised the prominence given to causal notions in the philosophical zeitgeist:

> [T]he reason why physics has ceased to look for causes is that in fact there are no such things. The law of causality, I believe, like much

> that passes muster among philosophers, is a relic of a bygone age, surviving, like the monarchy, only because it is erroneously supposed to do no harm.

In the last century, causality survived not just as a convenient fiction, suitable for throw-away explanations given to undergraduates: it is deeply embedded in conventional thinking about interacting systems across many fields. Our work shows that, for linear dynamical systems, discarding it is beneficial in several ways: the resulting formalism is simpler to define (compare the definitions of Circ and SF), it is compositional and—most importantly—reveals the beautiful underlying mathematical structure of $\mathbb{IH}$. Similar conclusions about the utility of pruning causality from mathematical models have recently been drawn by control theorists, in particular, Willems' *behavioural approach* [12] is an attempt to re-examine the central concepts of Control without giving definitional status to derivable causal information such as direction of flow.

Related Work. This journal version shares content with two conference publications. At CONCUR'14 [13] we presented the stream semantics, its relationship with $\mathbb{IH}$ and the connection between orthodox signal flow diagrams and matrices of rationals. In the PoPL'15 paper [10] we introduced an operational semantics, a full abstraction result (linking operational and stream semantics) and the realisability theorem. The realisability theorem appears in this paper; the operational semantics of Circ, and its relationship with the denotational story presented in [10], will appear in The Calculus of Signal Flow Diagrams II.

Some of the results shown in this paper rely on the isomorphism between $\mathbb{IH}$ and $\mathbb{SV}_{\mathsf{k}(x)}$ which follows from a more general theorem proved in [3]. The starting observation is the correspondence between Hopf algebras and matrices which was already shown in [14]. In this paper, we sketch the proofs of these two results—which will be formally published elsewhere—since they shed light on the modular structure of the involved equational theories.

Our methodology—using string diagrams, which originated in the study of free monoidal categories [1] as compositional syntax of interacting systems—forms part of the emerging field of *categorical network theory*. Amongst several recent works we mention the algebra of Petri nets with boundaries [15, 16], the algebra of stateless connectors [17], the algebra of $\mathsf{Span}(\mathsf{Graph})$ [18], Ghica's work [19] on asynchronous circuits, Baez and Fong's account of electrical circuits [20] and the ZX-calculus [21] for quantum circuits. Interestingly, the ZX-calculus shares the same basic algebraic features of $\mathbb{IH}$: two bialgebra and two Frobenius algebra structures.

Baez and Erbele's manuscript [22] is the most closely related: motivated by the continuous interpretation of signal flow diagrams, the authors independently give an equational presentation of the category of linear relations, which is equivalent to our equational theory $\mathbb{IH}$.

Finally, [9, 23] is an alternative categorical account of signal flow diagrams that focuses on coalgebras and coinduction, rather than string diagrams. The main difference with these works is that we give a formal syntax for circuits and

a sound and complete axiomatisation for semantic equivalence. These features are also present in the work of Milius [24], but its syntax is one-dimensional and diagrams are just used for notational convenience. Also, the circuit language is of a rather different flavour; most notably, it features primitives for recursion, which are not needed in our approach.

Structure of the paper. In §2 we present our string diagrammatic syntax and in §3 we recall the required categorical notions. In §4 we present the equational theory $\mathbb{HA}$ of $\mathsf{k}[x]$-matrices and in §5 the equational theory $\mathbb{IH}$ of linear relations over $\mathsf{k}(x)$. In §6 we introduce the stream semantics and prove soundness and completeness of $\mathbb{IH}$. In §7 we study orthodox signal flow graphs as a subclass of our string diagrammatic syntax. In §8 we prove the realisability theorem.

Notational conventions. $\mathbb{C}[a, b]$ is the set of arrows from a to b in a small category $\mathbb{C}$, composition of $f\colon a \to b$, $g\colon b \to c$ is denoted by $f\,;g\colon a \to c$. For $\mathbb{C}$ symmetric monoidal, $\oplus$ is the monoidal product and $\sigma_{X,Y}\colon X \oplus Y \to Y \oplus X$ the symmetry for $X, Y \in \mathbb{C}$. Given $\mathcal{F}\colon \mathbb{C}_1 \to \mathbb{C}_2$, $\mathcal{F}^{op}\colon \mathbb{C}_1^{op} \to \mathbb{C}_2^{op}$ is the induced functor on the opposite categories of $\mathbb{C}_1, \mathbb{C}_2$. If $\mathbb{C}$ has pullbacks, its span bicategory has the objects of $\mathbb{C}$ as 0-cells, spans of arrows of $\mathbb{C}$ as 1-cells and span morphisms as 2-cells. We denote with $\mathsf{Span}(\mathbb{C})$ the (ordinary) category obtained by identifying the isomorphic 1-cells and forgetting the 2-cells. Dually, if $\mathbb{C}$ has pushouts, $\mathsf{Cospan}(\mathbb{C})$ is the category obtained from the bicategory of cospans.

2. The Calculus of Signal Flow Diagrams: Syntax

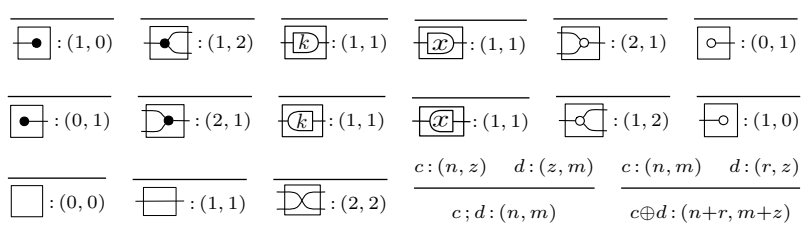
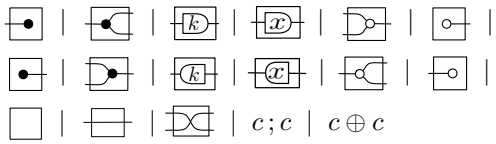


Figure 2: Sort inference rules.

In this section we define the string diagrammatic language that will be the focus of this paper and its sequels. Our presentation is syntactic: we consider diagrams to be certain (equivalence classes of) terms, rather than combinatorial structures. In part, this is for convenience: keeping the term structure of our diagrams allows the use of structural induction in proofs. Moreover, by keeping the link to syntax explicit, we are able to use standard programming language machinery: in The Calculus of Signal Flow Diagrams II we shall consider an *operational* semantics, complementing the *denotational* account in this paper.

Fix an arbitrary field k. The syntax, given below, does not feature binding nor primitives for recursion, while k ranges over k. As we shall see, the

indeterminate x plays a formal role akin to that in the algebra of polynomials.



$$c ::= \quad | \quad | \quad | \quad | \quad | \quad | \qquad (1)$$

$$| \quad | \quad | \quad | \quad | \quad | \quad | \qquad (2)$$

$$| \quad | \quad | \quad c\,;c \;\mid\; c \oplus c \qquad (3)$$

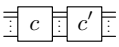
A *sort* is a pair (n, m), with $n, m \in \mathbb{N}$. We shall consider only terms that are sortable, according to the rules of Fig. 2. A simple inductive argument confirms uniqueness of sorting: if $c : (n, m)$ and $c : (n', m')$ then $n = n'$ and $m = m'$. We shall refer to sortable terms as *circuits* since, intuitively, a term $c : (n, m)$ represents a circuit with n ports on the left and m ports on the right.

Remark 2.1. *Recalling the intuition established in §1, we can consider circuits built up of the components in row* (1) *as taking signals—values in* k*—from the left boundary to the right: thus*  *is a* copier*, duplicating the signal arriving on the left;*  *accepts any signal on the left and discards it, producing nothing on the right;*  *is an* adder *that takes two signals on the left and emits their sum on the right, and*  *constantly emits the signal* 0 *on the right;*  *is an* amplifier*, multiplying the signal on the left by the scalar* $k \in \mathsf{k}$*. Finally,*  *is a* delay*, a synchronous one place buffer initialised with* 0*.*

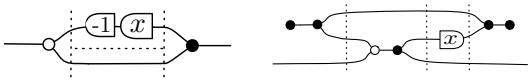
The terms of row (2) *are those of row* (1) *"reflected about the* y*-axis". Their behaviour is symmetric—indeed, here it can be helpful to think of signals as flowing from right to left. In row* (3)*,*  *is a* twist*, swapping two signals,*  *is the empty circuit and*  *is the* identity *wire: the signals on the left and on the right ports are equal. Terms are combined with two binary operators: sequential* ; *and parallel* $\oplus$ *composition.*

2.1. Circuit Diagrams and Symmetric Monoidal Structure

In the syntax specification we purposefully used a graphical rendering of the components. Indeed, we shall seldom write terms in the traditional way and instead represent them as diagrams. We adopt the common conventions:

$c\,;c'$ is drawn  $c \oplus c'$ is drawn  .

Example 2.2. *Consider the two circuits below.*



The first is a graphical representation of the term

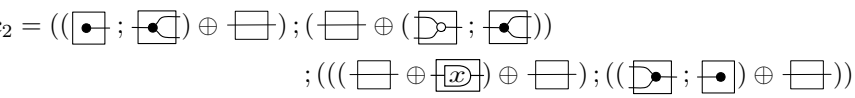
$c_1 =$ ( ; (( ; ) $\oplus$ )) ; 

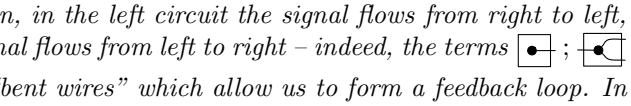
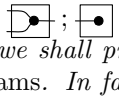
$$(c_1\,;c_3)\oplus(c_2\,;c_4)=(c_1\oplus c_2)\,;(c_3\oplus c_4)$$

$$\begin{array}{rl}
(c_1\,;c_2)\,;c_3=c_1\,;(c_2\,;c_3) & id_n\,;c=c=c\,;id_m\\
(c_1\oplus c_2)\oplus c_3=c_1\oplus(c_2\oplus c_3) & id_0\oplus c=c=c\oplus id_0\\
\sigma_{1,1}\,;\sigma_{1,1}=id_2 & (c\oplus id_z)\,;\sigma_{m,z}=\sigma_{n,z}\,;(id_z\oplus c)
\end{array}$$

Figure 3: Axioms of strict symmetric monoidal categories for Circ.

the second of the term

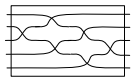
$c_2 =$ 

According to our intuition, in the left circuit the signal flows from right to left, while in the right, the signal flows from left to right – indeed, the terms  *and*  *serve as "bent wires" which allow us to form a feedback loop. In §6, we shall provide circuits with a formal semantics in terms of relations on* streams*. In fact, the two circuits above will have the same semantics, despite the apparent incompatibility in direction of signal flow – see Example 8.7.*

In Example 2.2 we used dotted lines to ease the passage from each diagram to the corresponding syntactic term. Indeed, the syntax carries more information than the diagrams (e.g. associativity). For our purposes, this information is redundant and is conveniently discarded by the graphical notation: we shall never again blemish our diagrams with dotted lines. More formally, our circuits are arrows of a symmetric monoidal category (SMC, see *e.g.* [1]).

Definition 2.3. The SMC Circ of *circuit diagrams* is defined as follows.

- objects are the natural numbers and the monoidal product $\oplus$ on objects is by addition. The unit object for $\oplus$ is 0.
- Arrows $n \to m$ are circuit terms of sort (n, m) quotiented by the axioms in Figure 3. Composition ; and monoidal product $\oplus$ of circuits are given by the corresponding syntactic operations in (3).
- The identities are $id_0 :=$  and $id_{n+1} := id_n \oplus$  . The symmetries $\sigma_{n,m} \colon n+m \to m+n$ are defined in the obvious way starting from $\sigma_{1,1} :=$  . For instance, $\sigma_{2,3}$ is (up-to the axioms of SMCs) the circuit below.



(4)

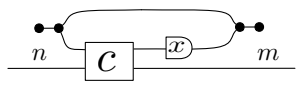
We identify two sub-categories of Circ: $\overrightarrow{\mathsf{Circ}}$ has as arrows only those circuits in Circ that are built from the components in rows (1) and (3) and $\overleftarrow{\mathsf{Circ}}$ only those circuits built from the components in rows (2) and (3). The notation

recalls the intuition that for circuits in $\overrightarrow{\mathsf{Circ}}$, signal flow is from left to right, and in $\overleftarrow{\mathsf{Circ}}$ from right to left. Formally, observe that $\overleftarrow{\mathsf{Circ}}$ is the opposite category of $\overrightarrow{\mathsf{Circ}}$: any circuit of $\overleftarrow{\mathsf{Circ}}$ can be seen as one of $\overrightarrow{\mathsf{Circ}}$ reflected about the y-axis.

We say that $c \in \mathsf{Circ}[n, m]$ is in *cospan form* if it is of shape $c_1 \,;\, c_2$, with $c_1 \in \overrightarrow{\mathsf{Circ}}[n, z]$ and $c_2 \in \overleftarrow{\mathsf{Circ}}[z, m]$ for some z. Dually, $d \in \mathsf{Circ}[n, m]$ is in *span form* if it is of shape $d_1 \,;\, d_2$, with $d_1 \in \overleftarrow{\mathsf{Circ}}[n, r]$ and $d_2 \in \overrightarrow{\mathsf{Circ}}[r, m]$ for some r.

2.2. Feedback and Signal Flow Diagrams

Beyond $\overrightarrow{\mathsf{Circ}}$ and $\overleftarrow{\mathsf{Circ}}$, we identify another class of circuits of Circ that adhere closely to the orthodox notion of *signal flow diagram* (see *e.g.* [7]), albeit without directed wires. Here, the signal can flow from left to right, as in $\overrightarrow{\mathsf{Circ}}$, but with the possibility of *feedbacks*, provided that these pass through at least one delay. This amounts to defining, for all n, m, a map $\mathsf{Tr}(\cdot)\colon \mathsf{Circ}[n{+}1, m{+}1] \to \mathsf{Circ}[n, m]$ taking $c\colon n+1 \to m+1$ to the n-to-m circuit below:



Above and henceforward, we use the shorthand notation $\boxed{z}$ for a circuit of the form id_z, for $z \in \mathbb{N}$. Intuitively, $\mathsf{Tr}(\cdot)$ equips the circuit c with a feedback loop carrying the signal from its topmost right to its topmost left port.

Signal flow graphs form a symmetric monoidal category SF, defined as the sub-category of Circ inductively given as follows:

(i) if $c \in \overrightarrow{\mathsf{Circ}}[n, m]$, then $c \in \mathsf{SF}[n, m]$

(ii) if $c \in \mathsf{SF}[n+1, m+1]$, then $\mathsf{Tr}(c) \in \mathsf{SF}[n, m]$

(iii) if $c_1 \in \mathsf{SF}[n, z]$ and $c_2 \in \mathsf{SF}[z, m]$, then $c_1 \,;\, c_2 \in \mathsf{SF}[n, m]$

(iv) if $c_1 \in \mathsf{SF}[n, m]$ and $c_2 \in \mathsf{SF}[r, z]$, then $c_1 \oplus c_2 \in \mathsf{SF}[n+r, m+z]$.

Equivalently, SF is the smallest sub-category of Circ that contains $\overrightarrow{\mathsf{Circ}}$ and is closed under the Tr operation. For instance, the right-hand circuit of Example 2.2 is in SF, whereas the left-hand is in $\overleftarrow{\mathsf{Circ}}$.

All three of $\overrightarrow{\mathsf{Circ}}$, $\overleftarrow{\mathsf{Circ}}$ and SF share a common sub-category – P with arrows only those circuits built from the components of (3). This can be seen as the category of *permutations* where $\mathsf{P}[n, m]$ is empty if $n \neq m$ and otherwise consists of the permutations on an n-element set. As we shall see, P plays a special role in our theory: all categories that we consider contain P as a sub-category.

3. Towards the Algebra of Signal Flow Diagrams

The categories Circ, P, $\overrightarrow{\mathsf{Circ}}$, $\overleftarrow{\mathsf{Circ}}$ and SF are all PROPs [25, 26]: a PROP (product and permutation category) is a strict symmetric monoidal category with objects the natural numbers, where $\oplus$ on objects is addition. Morphisms

between PROPs are identity-on-objects strict symmetric monoidal functors: PROPs and their morphisms form the category **PROP**.

In this section we introduce the tool kit that we will exploit to give a denotational semantics and an equational theory to Circ (and its sub-PROPs). First, §3.1 presents symmetric monoidal theories as a way of freely constructing PROPs from generators and equations. Then, in §3.2 we show how, following [26], PROPs can be composed together to express richer equational theories.

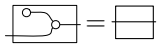
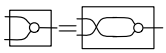
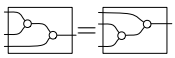
3.1. Symmetric Monoidal Theories

A one-sorted *symmetric monoidal theory* (SMT) is a pair (Σ, E) where Σ is the *signature*: a set of *operations* $o\colon n \to m$ with *arity* n and *coarity* m. The set of Σ-*terms* is obtained by composing operations, the identity $id_1\colon 1 \to 1$ and symmetry $\sigma_{1,1}\colon 2 \to 2$ with ; and $\oplus$: given Σ-terms $t\colon n \to z$, $u\colon z \to m$, $v\colon r \to s$, we construct Σ-terms $t\,;u\colon n \to m$ and $t \oplus v\colon n + r \to m + s$. The set E of *equations* consists of pairs of Σ-terms $(t, t'\colon n \to m)$. Given an SMT (Σ, E), one (freely) obtains a PROP where arrows $n \to m$ are Σ-terms $n \to m$ modulo the laws of SMC and equations $t = t'$ where $(t, t') \in E$.

We have already encountered four PROPs freely generated from SMTs with no equations: P is freely generated by the empty theory, $\overrightarrow{\mathsf{Circ}}$ by (1), $\overleftarrow{\mathsf{Circ}}$ by (2), and Circ by both (1) and (2) together—note that components in (3) are built-in by definition of SMT. Instead, SF is not generated by any SMT, as $\mathsf{Tr}(\cdot)$ cannot be expressed as an operation with an arity and coarity.

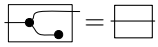
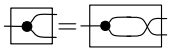
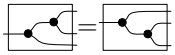
Below we introduce three more simple examples of SMT, this time with equations. They constitute the "building blocks" for richer theories that will be constructed, in a modular fashion, throughout the paper.

Example 3.1 (The SMT (Σ_M, E_M) of commutative monoids)**.** *The signature* Σ_M *consists of multiplication*  $: 2 \to 1$ *and unit*  $: 0 \to 1$. *Equations* E_M *assert identity* (A1)*, commutativity* (A2) *and associativity* (A3)*.*

 (A1)  (A2)  (A3)

We call $\mathbb{M}^w$ *the PROP freely generated from* (Σ_M, E_M).[2]

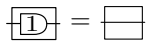
Example 3.2 (The SMT (Σ_C, E_C) of commutative comonoids)**.** Σ_C *consists of operations*  $: 1 \to 2$ *and*  $: 1 \to 0$ *and* E_C *consists of:*

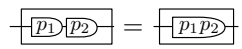
 (A4)  (A5)  (A6)

We call $\mathbb{C}^b$ *the PROP freely generated from* (Σ_C, E_C)*. Modulo the white vs. black colouring, the circuits of* $\mathbb{C}^b$ *can be seen as those of* $\mathbb{M}^w$ *reflected about the* y*-axis. This observation yields that* $\mathbb{C}^b \cong \mathbb{M}^{w\,op}$*.*

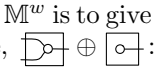
[2]The notation w emphasizes the white colouring of circuits in Σ_M.

Example 3.3 (The theory (Σ_R, E_R) of multiplication in $\mathsf{k}[x]$)**.** *Recall that* $\mathsf{k}[x]$ *denotes the ring of polynomials over* k*. The* Σ_R *contains an operation* $\boxed{p}\colon 1 \to 1$ *for each* $p \in \mathsf{k}[x]$ *and* E_R *consists of the following, where* p_1, p_2 *range over* $\mathsf{k}[x]$*.*


(A7)


(A8)

We call $\mathbb{K}[\mathbb{X}]$ *the PROP freely generated from* (Σ_R, E_R)*.*

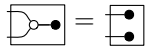
Rather than using SMTs, one can also define PROPs "directly": an example is the PROP of functions $\mathbb{F}$ where arrows $k \to l$ are functions $\{0, \dots, k-1\} \to \{0, \dots, l-1\}$. There is an isomorphism between $\mathbb{M}^w$ and $\mathbb{F}$: to give an arrow $c\colon n \to m$ in $\mathbb{M}^w$ is to give the graph of a function $\{0, \dots, n-1\} \to \{0, \dots, m-1\}$. For instance,  $\oplus$ $\colon 2 \to 2$ encodes $f\colon \{0,1\} \to \{0,1\}$ constant at 0.

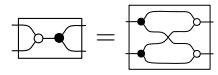
3.2. Constructing Richer Theories: Sum and Composition of PROPs

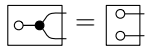
The SMTs introduced so far in this section were quite simple. Throughout our development, we will deal with more involved cases, which will be convenient to treat as the combination of basic theories. In this section we describe two PROP operations allowing for this modular reasoning: sum and composition.

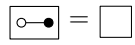
The sum of PROPs $\mathbb{T}$ and $\mathbb{S}$ is given by the coproduct $\mathbb{T} + \mathbb{S}$ in **PROP**. In order to compute $\mathbb{T} + \mathbb{S}$, it is useful to note that PROPs are also objects of the coslice category $\mathsf{P}/\mathbf{PRO}$. Here **PRO** is the category of strict monoidal categories (called PROs) with objects the naturals and tensor product on objects addition; morphisms of PROs are strict identity-on-objects monoidal functors. Morphisms of PROPs are thus simply morphisms of PROs that preserve the permutation structure. Working in the coslice is quite intuitive: e.g. P is the initial PROP and to compute the coproduct $\mathbb{T} + \mathbb{S}$ in **PROP** one must identify the permutation structures. When $\mathbb{T}$ and $\mathbb{S}$ are PROPs freely generated from (Σ_T, E_T) and (Σ_S, E_S) respectively, it then follows that $\mathbb{T} + \mathbb{S}$ is the PROP generated by $(\Sigma_T + \Sigma_S, E_T + E_S)$. For instance, $\overrightarrow{\mathsf{Circ}} + \overleftarrow{\mathsf{Circ}}$ is simply Circ.

The sum $\mathbb{T} + \mathbb{S}$ is the least interesting way of combining PROPs, because there are no equations that express compatibility conditions between $\mathbb{T}$ and $\mathbb{S}$ when "interacting" in $\mathbb{T} + \mathbb{S}$. Such interactions are common in algebra: for instance, a ring is given by a monoid and an abelian group, subject to equations telling how the former structure distributes over the latter. Another example, which will play a fundamental role in our work, is the PROP of co/commutative *bialgebras*: it consists of $\mathbb{M}^w + \mathbb{C}^b$ quotiented by the following set of equations, expressing the interaction between the monoid and comonoid structures.


(A9)


(A10)


(A11)

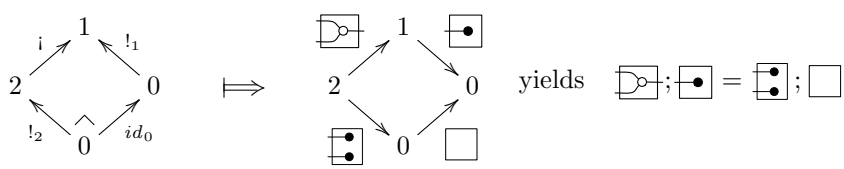

(A12)

In [26] Lack shows that this interaction arises through a notion of PROP composition, expressed in terms of distributive laws of monads. As shown by

Street [27], the theory of monads can be developed in an arbitrary bicategory. In this perspective, small categories are monads in the bicategory $\mathsf{Span}(\mathbf{Set})$. Similarly, PROPs can be described as monads on P in the bicategory $\mathsf{Prof}(\mathbf{Mon})$ of strict monoidal categories, profunctors and natural transformations [26].

Now, any two PROPs $\mathbb{T}$ and $\mathbb{S}$ can be composed via a distributive law $\lambda\colon \mathbb{S}\,;\mathbb{T} \Rightarrow \mathbb{T}\,;\mathbb{S}$ between the associated monads. λ makes $\mathbb{T}\,;\mathbb{S}$ into a monad, yielding a PROP whose arrows can be seen as *pairs* $(f,g)\colon n \to m$, where $f\colon n \to z$ is an arrow of $\mathbb{T}$ and $g\colon z \to m$ one of $\mathbb{S}$. A key observation for our purposes is that the graph of λ can be also seen as a set of (directed) equations of the form $(g,f) = (f',g')$. In fact, if $\mathbb{T}$ and $\mathbb{S}$ are freely generated PROPs then $\mathbb{T}\,;\mathbb{S}$ also has a presentation by generators and equations: this is the same as the coproduct $\mathbb{T} + \mathbb{S}$, plus the equations encoded by λ.

As an example, we show how composing $\mathbb{C}^b$ and $\mathbb{M}^w$ yields the PROP of bialgebras. First observe that $\mathbb{C}^b \cong \mathbb{M}^{w\,op} \cong \mathbb{F}^{op}$. Then a distributive law $\lambda\colon \mathbb{M}^w\,;\mathbb{C}^b \Rightarrow \mathbb{C}^b\,;\mathbb{M}^w$ has type $\mathbb{F}\,;\mathbb{F}^{op} \Rightarrow \mathbb{F}^{op}\,;\mathbb{F}$, that is, it maps a pair $p \in \mathbb{F}[n,z]$, $q \in \mathbb{F}^{op}[z,m]$ to a pair $f \in \mathbb{F}^{op}[n,z]$, $g \in \mathbb{F}[z,m]$. This amounts to saying that λ maps *cospans* $n \xrightarrow{p} z \xleftarrow{q} m$ to *spans* $n \xleftarrow{f} r \xrightarrow{g} m$ in $\mathbb{F}$. Defining $n \xleftarrow{f} r \xrightarrow{g} m$ as the *pullback* of $n \xrightarrow{p} z \xleftarrow{q} m$ in $\mathbb{F}$ makes λ a distributive law [26]. The resulting PROP $\mathbb{C}^b\,;\mathbb{M}^w$ can be presented by operations and equations—those of $\mathbb{C}^b + \mathbb{M}^w$—together with those given by the graph of λ. One can thus obtain them from pullback squares in $\mathbb{F}$, for instance:



where the second diagram is obtained from the pullback by applying the isomorphisms $\mathbb{F} \cong \mathbb{M}^w$ and $\mathbb{F}^{op} \cong \mathbb{C}^b$. In fact, all the equations can be derived from just four pullbacks that yield the four equations (A9)-(A12) given above [26].

Therefore $\mathbb{C}^b\,;\mathbb{M}^w$ is the free PROP of (black-white) co/commutative bialgebras. One can consider the SMT of co/commutative bialgebras to be the theory of $\mathsf{Span}(\mathbb{F}) \cong \mathbb{F}^{op}\,;\mathbb{F}$ and, consequently, that each $c\colon n \to m$ in this PROP can be factorised as $c = c_1\,;c_2$, with $c_1 \in \mathbb{C}^b[n,z]$ and $c_2 \in \mathbb{M}^w[z,m]$ for some z.

4. Axiomatising $\mathsf{C}\overrightarrow{\mathsf{irc}}$: the Theory of $\mathsf{k}[x]$-Matrices

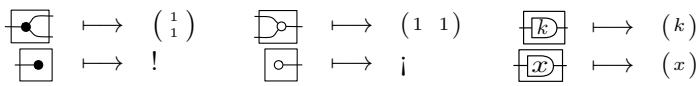
In this section we commence our investigation of the denotational semantics of Circ. We restrict to $\mathsf{C}\overrightarrow{\mathsf{irc}}$, to which we give a semantics in terms of matrices over $\mathsf{k}[x]$, the ring of polynomials with unknown x and values over k. Later, in §6, we will show that this is consistent with the intuitions given in Remark 2.1.

The semantic domain for $\mathsf{C}\overrightarrow{\mathsf{irc}}$ is the PROP $\mathsf{Mat}\,\mathsf{k}[x]$ where arrows $n \to m$ are $m \times n$ $\mathsf{k}[x]$-matrices, composition $A\,;B$ is matrix multiplication $B \times A$ and

the tensor $A \oplus B$ is defined as the matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. The symmetries are the rearrangements of the rows of the identity matrix. For instance $\sigma_{2,3}$ (see (4)) is:

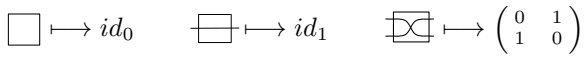
$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

The map $\overrightarrow{[\![\cdot]\!]} \colon \overrightarrow{\mathsf{Circ}} \to \mathsf{Mat}\,\mathsf{k}[x]$ is inductively defined as follows. For (1):



(5)

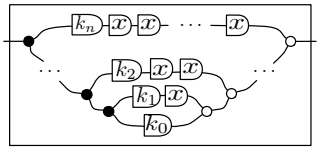
where $! \colon 0 \to 1$ and $¡ \colon 1 \to 0$ are given by initiality and finality of 0 in $\mathsf{Mat}\,\mathsf{k}[x]$. For (3):



$$c_1 \oplus c_2 \longmapsto \overrightarrow{[\![c_1]\!]} \oplus \overrightarrow{[\![c_2]\!]} \qquad c_1 \,;\, c_2 \longmapsto \overrightarrow{[\![c_1]\!]} \,;\, \overrightarrow{[\![c_2]\!]} \tag{6}$$

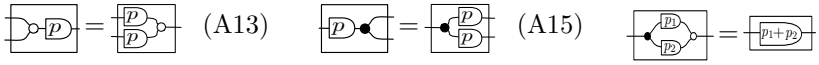
From (6), it is immediate that $\overrightarrow{[\![\cdot]\!]} \colon \overrightarrow{\mathsf{Circ}} \to \mathsf{Mat}\,\mathsf{k}[x]$ is a morphism of PROPs. Indeed $\overrightarrow{[\![\cdot]\!]}$ could also have been defined as the unique PROP morphism mapping the basic components as in (5). In the sequel, we will introduce several semantics maps and, to be concise, we will usually adopt this second formulation.

By definition, the semantics of any 1-to-1 circuit is a polynomial in $\mathsf{k}[x]$. Conversely, for any polynomial $p = k_0 + k_1 x + k_2 x^2 + \cdots + k_n x^n$, the following circuit, which hereafter we denote by $\boxed{p}$, has semantics (p).

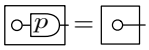
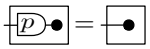
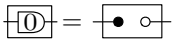


At this point, the connection between $\overrightarrow{\mathsf{Circ}}$ and the basic theories introduced in Section 3.1 should be more evident: the image of $\overrightarrow{\mathsf{Circ}}$ through the semantics $\overrightarrow{[\![\cdot]\!]}$ can be constructed as a quotient of $\mathbb{C}^b + \mathbb{K}[\mathbb{X}] + \mathbb{M}^w$.

Definition 4.1. The PROP $\mathbb{HA}$ is the quotient of $\overrightarrow{\mathsf{Circ}}$ by the equations of $\mathbb{M}^w$, $\mathbb{K}[\mathbb{X}]$ and $\mathbb{C}^b$ ((A1)-(A8)), the equations of bialgebras ((A9)-(A12)) and the following, where $p, p_1, p_2 \in \mathsf{k}[x]$.

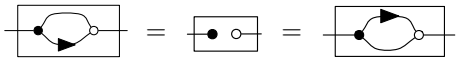


(A13) (A15) (A17)

 (A14)  (A16)  (A18)

We shall write $c \overset{\mathbb{HA}}{=} d$ when two circuits c, d of $\mathsf{Circ}^{\rightarrow}$ are equal as arrows of $\mathbb{HA}$.

Remark 4.2. *As hinted by its name,* $\mathbb{HA}$ *satisfies the axioms of* Hopf algebras *(see e.g. [28, 29]). Indeed, it inherits the bialgebra equations from* $\mathbb{C}^b\,;\mathbb{M}^w$ *and*  $: 1 \to 1$ *plays the role of the antipode — for which reason we fix notation* $:=$ *. The Hopf law holds by virtue of* (A7)*,* (A17) *and* (A18)*:*

 (Hopf)

It is straightforward to check that (A1)-(A18) are *sound* with respect to the semantics $[\![\cdot]\!]^{\rightarrow}$. For instance, both the left and the right hand side of (A10) have semantics $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. The following result is the key to proving completeness.

Lemma 4.3 (Factorisation of $\mathbb{HA}$)**.** *Any* $c \in \mathbb{HA}[n,m]$ *is equal to* $s\,;r\,;t \in \mathbb{HA}[n,m]$*, where* $s \in \mathbb{C}^b[n,z]$*,* $r \in \mathbb{K}[\mathbb{X}][z,z]$ *and* $t \in \mathbb{M}^w[z,m]$ *for some* $z \in \mathbb{N}$*.*

Proof. The reader is referred to [3, §3] for a proof. Since it sheds light on the modular nature of the axioms of $\mathbb{HA}$, we find instructive to sketch it below.

Following our discussion in §3.2, we can give an equivalent, modular, description of the PROP $\mathbb{HA}$, as the quotient of $\mathbb{C}^b + \mathbb{K}[\mathbb{X}] + \mathbb{M}^w$ by the equations (A9)-(A18). In fact, (A9)-(A16) can be seen as arising by the composition of PROPs $\mathbb{C}^b$, $\mathbb{K}[\mathbb{X}]$ and $\mathbb{M}^w$. The axioms (A13) and (A14) present a distributive law $\sigma\colon \mathbb{M}^w\,;\mathbb{K}[\mathbb{X}] \Rightarrow \mathbb{K}[\mathbb{X}]\,;\mathbb{M}^w$. Similarly, (A15) and (A16) present a distributive law $\tau\colon \mathbb{K}[\mathbb{X}]\,;\mathbb{C}^b \Rightarrow \mathbb{C}^b\,;\mathbb{K}[\mathbb{X}]$. By [30, Th.2.1], these laws, together with $\lambda\colon \mathbb{M}^w\,;\mathbb{C}^b \Rightarrow \mathbb{C}^b\,;\mathbb{M}^w$ which is presented by (A9)-(A12) (see §3.2), yield the composite $\mathbb{C}^b\,;\mathbb{K}[\mathbb{X}]\,;\mathbb{M}^w$. Now, $\mathbb{HA}$ is the quotient of $\mathbb{C}^b\,;\mathbb{K}[\mathbb{X}]\,;\mathbb{M}^w$ by (A18) and (A17), thus it enjoys the desired factorisation property. $\square$

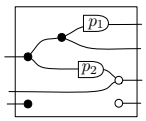
Lemma 4.3 suggests a canonical form for any circuit of $\mathsf{Circ}^{\rightarrow}$, which allows to easily read off the associated matrix.

Definition 4.4. A string diagram $c \in \mathbb{HA}[n,m]$ is in *matrix form* if (a) it is of shape $s\,;r\,;t \in \mathbb{HA}[n,m]$, with $s \in \mathbb{C}^b[n,z]$, $r \in \mathbb{K}[\mathbb{X}][z,z]$ and $t \in \mathbb{M}^w[z,m]$ for some $z \in \mathbb{N}$, (b) any port on the left boundary has exactly one connection with any port on the right boundary and (c) any such connection passes through exactly one scalar  . We say that there is a *k-path from i to j* if k is the scalar on the path from the ith port on the left to the jth port on the right, assuming a top-down enumeration.

When drawing matrix forms, for the sake of readability it will be often convenient to massage the above definition as follows: we typically omit to draw the scalar $k = 1$, by virtue of (A7), and omit the scalar $k = 0$, by (A18), leaving the ports in question disconnected.

Circuits in matrix forms have an intuitive representation as $\mathsf{k}[x]$-matrices, as shown by the following example.

Example 4.5. *Consider the circuit $c\colon 3 \to 4$ below and its representation as a 4×3 matrix M. Note $M_{ij} = p$ exactly when there is a p-path from j to i in c.*



$$\begin{pmatrix} p_1 & 0 & 0 \\ 1 & 0 & 0 \\ p_2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The choice of axioms in Definition 4.1 makes this interpretation a 1-1 correspondence between arrows of $\mathbb{HA}$ and $\mathsf{k}[x]$-matrices.

Proposition 4.6. *There is an isomorphisms of PROPs between $\mathbb{HA}$ and $\mathsf{Mat}\,\mathsf{k}[x]$.*

Proof. The map $\overrightarrow{[\![\cdot]\!]}\colon \overrightarrow{\mathsf{Circ}} \to \mathsf{Mat}\,\mathsf{k}[x]$ induces a morphism $\mathcal{S}\colon \mathbb{HA} \to \mathsf{Mat}\,\mathsf{k}[x]$ because it respects all the equations of $\mathbb{HA}$. The proof that $\mathcal{S}$ is an iso relies on showing that is full and faithful on circuits in matrix form—this is without loss of generality by Lemma 4.3. We refer to [3, Prop. 3.7] for the details. □

Corollary 4.7. *For all circuits c, d in $\overrightarrow{\mathsf{Circ}}$, $\overrightarrow{[\![c]\!]} = \overrightarrow{[\![d]\!]}$ iff $c \overset{\mathbb{HA}}{=} d$.*

The previous result can be conveniently exploited also for circuits in $\overleftarrow{\mathsf{Circ}}$. Indeed, $\overrightarrow{[\![\cdot]\!]}\colon \overrightarrow{\mathsf{Circ}} \to \mathsf{Mat}\,\mathsf{k}[x]$ induces the PROP morphism $\overrightarrow{[\![\cdot]\!]}^{op}$ between the opposite categories, that we hereafter denote by $\overleftarrow{[\![\cdot]\!]}\colon \overleftarrow{\mathsf{Circ}} \to \mathsf{Mat}\,\mathsf{k}[x]^{op}$. The PROP $\mathbb{HA}^{op}$, which is presented by the operations and equations of $\mathbb{HA}$ reflected about the y-axis, provides a sound and complete axiomatisation for $\overleftarrow{[\![\cdot]\!]}$.

Corollary 4.8. *For all circuits c, d in $\overleftarrow{\mathsf{Circ}}$, $\overleftarrow{[\![c]\!]} = \overleftarrow{[\![d]\!]}$ iff $c \overset{\mathbb{HA}^{op}}{=} d$.*

The reader may consult Appendix A for a reference card with the axioms of $\mathbb{HA}$ (Figure A.4) and of $\mathbb{HA}^{op}$ (Figure A.5).

The results of this section relied on the close connection between the algebra of matrices and the equational theory of bialgebras. Interestingly, matrices have also been used to reason about special Frobenius algebras, which are the other prominent equational theory that appears in our work: in [31] Kissinger shows that (finite) matrices with entries from a field of characteristic 0 are complete for *multigraph* categories, in which every object is equipped with a special (commutative) Frobenius algebra.

5. Axiomatising Circ: the Theory of Relational $\mathsf{k}(x)$-Subspaces

We now consider the task of giving a semantics to Circ. Recall that the semantics of a circuit in $\overrightarrow{\mathsf{Circ}}$ is a matrix, or in other words, a linear transformation. As explained in Remark 2.1, the intuition for circuits in $\overrightarrow{\mathsf{Circ}}$ is that the signal flows from left to right: left ports are inputs and right ports are outputs.

These traditional mores fail in Circ—indeed, only some circuits have a functional interpretation. Consider  $: 2 \to 0$: the component  accepts an arbitrary signal while  ensures that the signal is equal on the two ports. In other words, the circuit is a "bent wire" whose behaviour is relational: its ports are neither inputs nor outputs in any traditional sense. Indeed, the semantic domain for Circ is *linear relations* over $\mathsf{k}(x)$, the field of fractions of $\mathsf{k}[x]$.

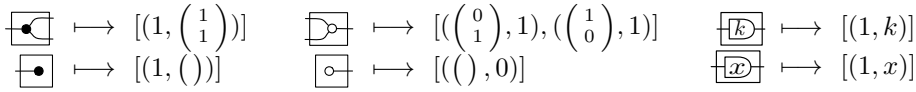
Definition 5.1. Let $\mathbb{SV}_{\mathsf{k}(x)}$ be the following PROP:

- arrows $n \to m$ are linear relations between $\mathsf{k}(x)^n$ and $\mathsf{k}(x)^m$, that is, (linear) subspaces of $\mathsf{k}(x)^n \times \mathsf{k}(x)^m$.
- composition is relational: for subspaces $G\colon n \to z$ and $H\colon z \to m$, their composition is the subspace

$$\{(u, w) \in \mathsf{k}(x)^n \times \mathsf{k}(x)^m \mid \exists v \in \mathsf{k}(x)^z.(u, v) \in G \wedge (v, w) \in H\}.$$

- The tensor product $\oplus$ on arrows is given by direct sum of spaces.
- The symmetries $n \to n$ are induced by bijections of finite sets, $\rho : n \to n$ is associated with the subspace generated by $\{(1_i, 1_{\rho i})\}_{i<n}$ where 1_k is the binary n-vector with 1 at the $(k+1)$-st coordinate and 0's elsewhere. For instance $\sigma_{1,1}\colon 2 \to 2$ is generated by $\{(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)), (\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right))\}$.

Let $[\mathbf{v}_1, \ldots, \mathbf{v}_n]$ denote the space generated by the vectors $\mathbf{v}_1 \ldots \mathbf{v}_n$ and $()$ the unique element of $\mathsf{k}(x)^0$. The semantic map $[\![\cdot]\!]\colon \mathsf{Circ} \to \mathbb{SV}_{\mathsf{k}(x)}$ is the only PROP morphism mapping the components in (1) as



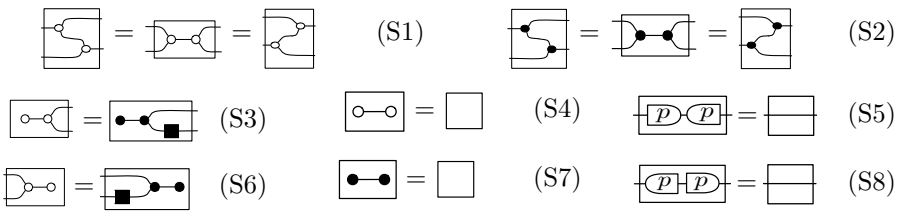
and for the components in (2) is symmetric, e.g.  is mapped to $[(() , 1)]$.

This semantics of Circ in term of subspaces is consistent with those of $\overrightarrow{\mathsf{Circ}}$ and $\overleftarrow{\mathsf{Circ}}$ in terms of matrices, in the sense that the diagram below commutes. The horizontal arrows in the bottom row are obtained by thinking of matrices as the (equivalent) linear maps between free $\mathsf{k}[x]$-modules: a linear map $f\colon \mathsf{k}[x]^n \to \mathsf{k}[x]^m$ is thus taken to its graph $\{(\mathbf{u}, f\mathbf{u}) \mid \mathbf{u} \in \mathsf{k}[x]^n\}$, considered as a (functional) subspace of $\mathsf{k}(x)^n \times \mathsf{k}(x)^m$. The commutativity of the diagram is straightforward: since $\overrightarrow{\mathsf{Circ}}$ and $\overleftarrow{\mathsf{Circ}}$ are free, it suffices to check the generators.

$$\begin{tikzcd}
\overrightarrow{\mathsf{Circ}} \arrow[r, hook] \arrow[d, "\overrightarrow{[\![\cdot]\!]}"'] & \mathsf{Circ} \arrow[d, "{[\![\cdot]\!]}"] & \overleftarrow{\mathsf{Circ}} \arrow[l, hook'] \arrow[d, "\overleftarrow{[\![\cdot]\!]}"] \\
\mathsf{Mat}\,\mathsf{k}[x] \arrow[r, hook] & \mathbb{SV}_{\mathsf{k}(x)} & \mathsf{Mat}\,\mathsf{k}[x]^{op} \arrow[l, hook']
\end{tikzcd}$$

Because $\mathsf{Circ} = \overrightarrow{\mathsf{Circ}} + \overleftarrow{\mathsf{Circ}}$, in order to obtain a sound and complete axiomatization for $[\![\cdot]\!]$, we can consider the quotient of $\mathbb{HA} + \mathbb{HA}^{op}$ by laws expressing the interactions between the two Hopf algebra structures. We call the resulting PROP of *interacting Hopf algebras* $\mathbb{IH}$.

Definition 5.2. The PROP $\mathbb{IH}$ is the quotient of Circ by the equations of $\mathbb{HA}$, $\mathbb{HA}^{op}$ and the following, where p ranges over $\mathsf{k}[x] \setminus \{0\}$.



The reader may find a reference card with all the axioms of $\mathbb{IH}$ in Appendix A. Equations (S1) and (S2) are known in the literature as *Frobenius laws* [5]. Interestingly, equations (S5) and (S8) reflect the fact that the domain $\mathsf{k}(x)$ of interpretation is a field: for any non-zero polynomial p, the circuit $\boxed{p}$ has inverse $\boxed{p}$ (see also the derived law (7)). The notation $\boxed{\blacksquare}$ replaces the antipodes $\boxed{\blacktriangleright}$ and $\boxed{\blacktriangleleft}$: they are equal as arrows in $\mathbb{IH}$ by virtue of (S5), (A8) and (A7). We write $c \overset{\mathbb{IH}}{=} d$ when c and d in Circ are equal as arrows in $\mathbb{IH}$.

The following result from [3] states that the axioms of $\mathbb{IH}$ characterise $\mathbb{SV}_{\mathsf{k}(x)}$.

Theorem 5.3. *There is an isomorphisms of PROPs between $\mathbb{IH}$ and $\mathbb{SV}_{\mathsf{k}(x)}$.*

The proof, sketched in §5.2, yields the following factorisation properties.

Proposition 5.4 (Factorisation of Circ)**.** *For all circuits c of Circ, there exist circuits c' in span form and c'' in cospan form such that $c \overset{\mathbb{IH}}{=} c' \overset{\mathbb{IH}}{=} c''$.*

Since all the axioms of $\mathbb{IH}$ are sound with respect to the semantics map $[\![\cdot]\!]$, from the above theorem immediately follows also their completeness.

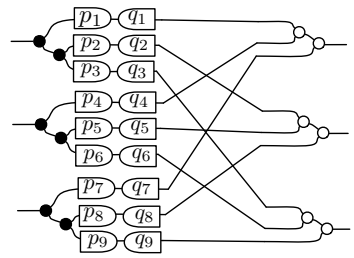
Corollary 5.5. *For all circuits c, d in Circ, $[\![c]\!] = [\![d]\!]$ iff $c \overset{\mathbb{IH}}{=} d$.*

It is useful for later reference to conclude with the following observation.

Remark 5.6. *As explained in § 4, there is a canonical way of representing any polynomial matrix $M \in \mathsf{Mat}\,\mathsf{k}[x][n,m]$ as a circuit $c \in \overrightarrow{\mathsf{Circ}}[n,m]$, which we called in* matrix form *— see Definition 4.4. We can use a similar form to*

represent matrices over $\mathsf{k}(x)$ *as circuits of* Circ. *Consider the following example:*

$$N = \begin{pmatrix} p_1/q_1 & p_4/q_4 & p_7/q_7 \\ p_2/q_2 & p_5/q_5 & p_8/q_8 \\ p_3/q_3 & p_6/q_6 & p_9/q_9 \end{pmatrix} \qquad d =$$



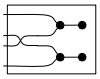
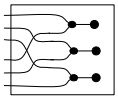
The circuit d *is a canonical representation of the matrix* N, *and indeed*

$$[\![d]\!] = \{(\mathbf{v}, N\mathbf{v}) \mid \mathbf{v} \in \mathsf{k}(x)^3\} = [(\mathbf{e}_i, A\mathbf{e}_i)]_{i \leq 3}$$

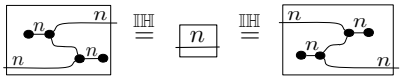
where $\{\mathbf{e}_i \mid i \leq 3\}$ *is the standard basis of* $\mathsf{k}(x)^3$.

5.1. The Structure of $\mathbb{IH}$*: Compact Closedness and Derived Laws*

For the developments of §7.1 it is useful to shed light on the self-dual compact closed structure of $\mathbb{IH}$. First, we define a sequence $\alpha_n \colon 2n \to 0$ of circuits:

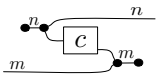
$\alpha_0 := id_0$ $\qquad \alpha_1 :=$  $\qquad \alpha_2 :=$  $\qquad \alpha_3 :=$  $\ldots$

Semantically, they all behave as bent wires: for instance, $[\![\alpha_1]\!] = \{\big((p,p),()\big) \mid p \in \mathsf{k}(x)\}$ and $[\![\alpha_2]\!] = \{\big((p,q,p,q),()\big) \mid p,q \in \mathsf{k}(x)\}$. One can define circuits from 0 to $2n$ symmetrically, starting from $\beta_2 :=$  $: 0 \to 2$. Now, let  be notation for β_n,  for α_n and for id_n. As shown in [3, §4], the βs and the αs form a self-dual compact closed structure on the category $\mathbb{IH}$, i.e.



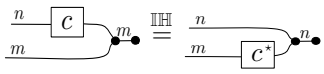
(CC1)

in $\mathbb{IH}$ for all $n \in N$. This yields a contravariant endofunctor $(\cdot)^\star$ on $\mathbb{IH}$ (see also [32, Rmk 2.1]): for any $c \colon n \to m$, the arrow $c^\star \colon m \to n$ is defined as follows.

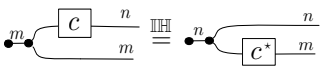


Using the equational theory of $\mathbb{IH}$, one can show (see [3, §4.1]) that $c^\star$ is just c reflected about the y-axis: for example,  $\overset{\mathbb{IH}}{=}$  and  $\overset{\mathbb{IH}}{=}$  .

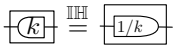
We list some useful derived laws in $\mathbb{IH}$ below. The proofs are in [3, §4].



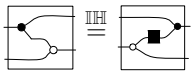
$\overset{\mathbb{IH}}{=}$ (CC2)



$\overset{\mathbb{IH}}{=}$ (CC3)



$\overset{\mathbb{IH}}{=}$ (7)



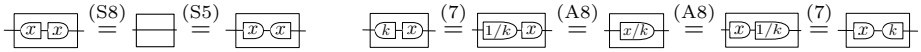
$\overset{\mathbb{IH}}{=}$ (8)

We also record the following lemma;  is the n-fold product of  .

Lemma 5.7. *For any $n, m \in \mathbb{N}$ and circuit $c \in \mathsf{Circ}[n, m]$,*

$\overset{\mathbb{IH}}{=}$ (9)

Proof. The proof is by induction on c. For the components in (1), the statement is given for  ,  ,  ,  ,  and  by (A14), (A13), (A16), (A15), (A8) and (A8) respectively. The derivations for  and  are:



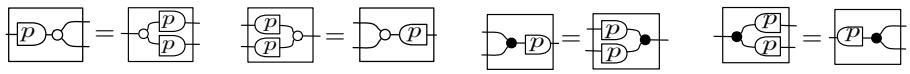
Similarly, one can easily check the statement for the remaining cases in (2) and (3). The inductive cases of parallel ($\oplus$) and sequential (;) composition of circuits are handled by simply applying the induction hypothesis. □

5.2. Soundness and Completeness of $\mathbb{IH}$: the Cube Construction

Theorem 5.3 and Proposition 5.4 follow immediately from more general results proved in [3, §6-10]. In this subsection we sketch the proof argument which is interesting in its own right, since it is a *modular* account of the theory of $\mathbb{IH}$. Its components are summarised by the cube diagram (⧈) below.

$$\begin{tikzcd}
 & \mathbb{HA} + \mathbb{HA}^{op} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"] & & \mathbb{IH}^w \arrow[dl] \arrow[dd, "\cong"] \\
\mathbb{IH}^b \arrow[rr] \arrow[dd, "\cong"] & & \mathbb{IH} \arrow[dd, dotted, "\cong"] & \\
 & \mathsf{Mat}\,\mathsf{k}[x] + \mathsf{Mat}\,\mathsf{k}[x]^{op} \arrow[rr] \arrow[dl] & & \mathsf{Span}(\mathsf{Mat}\,\mathsf{k}[x]) \arrow[dl] \\
\mathsf{Cospan}(\mathsf{Mat}\,\mathsf{k}[x]) \arrow[rr] & & \mathbb{SV}_{\mathsf{k}(x)} &
\end{tikzcd} \qquad (⧈)$$

The theory $\mathbb{IH}^w$ is presented by the equations of $\mathbb{IH}$ (Definition 5.2), but with the two leftmost axioms below replacing (S7) and (S8). Dually, $\mathbb{IH}^b$ is $\mathbb{IH}$ with the two rightmost axioms below replacing (S4) and (S5). As equational theories, $\mathbb{IH}^w$ and $\mathbb{IH}^b$ are weaker than $\mathbb{IH}$: all the equations below are derivable in $\mathbb{IH}$ [3].



In fact, $\mathbb{IH}^w$ and $\mathbb{IH}^b$ are the theories of (i.e. their free PROPs are isomorphic to) $\mathsf{Span}(\mathsf{Mat}\,\mathsf{k}[x])$ and $\mathsf{Cospan}(\mathsf{Mat}\,\mathsf{k}[x])$, respectively. First we focus on $\mathbb{IH}^w$ and $\mathsf{Span}(\mathsf{Mat}\,\mathsf{k}[x])$. Note that pullbacks in $\mathsf{Mat}\,\mathsf{k}[x]$ exist and are computed as in the category of sets in the equivalent category of free finitely generated $\mathsf{k}[x]$-modules, since $\mathsf{k}[x]$ is a principal ideal domain (PID).

Pullbacks give a distributive law of PROPs, and as we explained in §3.2, pullback diagrams "add equations" to the theory $\mathbb{HA} + \mathbb{HA}^{op}$. Indeed, for each equation of $\mathbb{IH}^w$ there is a corresponding "witnessing" pullback in $\mathsf{Mat}\,\mathsf{k}[x]$: this argument confirms the *soundness* of the theory of $\mathbb{IH}^w$ for $\mathsf{Span}(\mathsf{Mat}\,\mathsf{k}[x])$. The task of demonstrating the *completeness* of the axioms is more subtle: one has to prove that the axioms are sufficient for deriving *any* equation that arises from a pullback in $\mathsf{Mat}\,\mathsf{k}[x]$. The proof amounts to showing that linear algebraic manipulations on matrices that are performed when calculating the kernel of a linear transformation can be mimicked graphically in $\mathbb{IH}^w$.

Having constructed the isomorphism between $\mathbb{IH}^w$ and $\mathsf{Span}(\mathsf{Mat}\,\mathsf{k}[x])$, we can use the fact that the transpose operation on matrices induces a duality in $\mathsf{Mat}\,\mathsf{k}[x]$ to yield the isomorphism between $\mathbb{IH}^b$ and $\mathsf{Cospan}(\mathsf{Mat}\,\mathsf{k}[x])$.

Now let us again focus on the top face of (⧈). It is a pushout diagram in **PROP**: as only PROPs freely generated by SMTs are involved, this simply amounts to saying that the equational theory of $\mathbb{IH}$ can be presented as the union of the equational theories of $\mathbb{IH}^w$ and $\mathbb{IH}^b$. An appealing consequence of this construction is that $\mathbb{IH}$ inherits the factorisation properties of both $\mathbb{IH}^w$ and $\mathbb{IH}^b$. This gives us immediately Proposition 5.4.

The final ingredient in the proof Theorem 5.3 is showing that the bottom face of (⧈) is also a pushout in **PROP** — the morphisms appearing in this face will be detailed in diagram (⧈⧈) below, as they play a role in § 6. We would like to draw the reader's attention to the remarkable fact that subspaces over the *field of fractions* $\mathsf{k}(x)$ of $\mathsf{k}[x]$ arise from "glueing" spans and cospans of $\mathsf{k}[x]$-matrices. This fact holds for an arbitrary PID and its field of fractions: the proof can be found in [3, §9].

Summing up, the top and bottom faces of (⧈) are pushouts, and the three rear vertical morphisms are isomorphisms. The universal property of pushouts now ensures that the unique morphism from $\mathbb{IH}$ to $\mathbb{SV}_{\mathsf{k}(x)}$ is invertible.

6. Stream Semantics

With simple extensions of the semantics morphisms, we can interpret circuits of $\overrightarrow{\mathsf{Circ}}$ and Circ in terms of streams. First we need to recall some useful notions.

A formal Laurent series (fls) is a function $\sigma\colon \mathbb{Z} \to \mathsf{k}$ for which there exists $i \in \mathbb{Z}$ such that $\sigma(j) = 0$ for all $j < i$. The *codegree* of σ is the smallest $d \in \mathbb{Z}$ such that $\sigma(d) \neq 0$. We shall often write σ as $\ldots, \sigma(-1), \underline{\sigma(0)}, \sigma(1), \ldots$ with position 0 underlined, or as formal sum $\sum_{i=d}^{\infty} \sigma(i)x^i$. Using the latter notation,

the sum and product of fls $\sigma = \sum_{i=d}^{\infty} \sigma(i)x^i$ and $\tau = \sum_{i=e}^{\infty} \tau(i)x^i$ are given by:

$$\sigma + \tau = \sum_{i=min(d,e)}^{\infty} \big(\sigma(i) + \tau(i)\big)x^i \qquad \sigma \cdot \tau = \sum_{i=d+e}^{\infty} \Big(\sum_{k+j=i} \sigma(j) \cdot \tau(k)\Big)x^i \quad (10)$$

The units for $+$ and $\cdot$ are $\ldots 0, \underline{0}, 0 \ldots$ and $\ldots 0, \underline{1}, 0 \ldots$. Fls form a field $\mathsf{k}((x))$, where the inverse σ^{-1} for the fls σ with codegree d is given as follows.

$$\sigma^{-1}(i) = \begin{cases} 0 & \text{if } i < -d \\ \sigma(d)^{-1} & \text{if } i = -d \\ \frac{\sum_{i=1}^{n} \left(\sigma(d+i) \cdot \sigma^{-1}(-d+n-i)\right)}{-\sigma(d)} & \text{if } i = -s+n \text{ for } n > 0 \end{cases} \quad (11)$$

A *formal power series* (fps) is a fls with codegree $d \geq 0$. By (10), fps are closed under $+$ and $\cdot$, but not under inverse: it is immediate by (11) that σ^{-1} is a fps iff σ has codegree $d = 0$. Therefore fps form a ring which we denote by $\mathsf{k}[[x]]$.

We shall refer to both fps and fls as *streams*. Indeed, fls are sequences with an infinite future, but a finite past. Analogously to how a polynomial p can be seen as a fraction $\frac{p}{1}$, an fps σ can be interpreted as the fls $\ldots, 0, \underline{\sigma(0)}, \sigma(1), \sigma(2), \ldots$. A polynomial $p_0 + p_1 x + \cdots + p_n x^n$ can also be regarded as the fps $\sum_{i=0}^{\infty} p_i x^i$ with $p_i = 0$ for all $i > n$. Similarly, polynomial fractions can be regarded as fls: we define $\tilde{\cdot}\colon \mathsf{k}(x) \to \mathsf{k}((x))$ as the unique field morphism mapping $k \in \mathsf{k}$ to $\ldots 0, \underline{k}, 0 \ldots$ and the indeterminate x to $\ldots, 0, \underline{0}, 1, 0, \ldots$

Differently from polynomials, fractions can denote streams with possibly infinitely many non-zero values. For instance, (10) and (11) imply that $\frac{x}{1-x-x^2}$ is the Fibonacci series $\ldots, 0, \underline{0}, 1, 1, 2, 3, \ldots$. Moreover, while polynomials can be interpreted as fps, fractions need the full generality of fls: $\frac{1}{x}$ denotes $\ldots 0, 0, 1, \underline{0}, 0, \ldots$

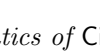
$$\begin{tikzcd}
\mathsf{k}[[x]] \arrow[rr, hook] & & \mathsf{k}((x)) \\
\mathsf{k}[x] \arrow[u, hook, "\hat{\cdot}"] \arrow[rr, hook] \arrow[urr, hook, bend right] & \mathsf{k}\langle x\rangle \arrow[ul, hook] \arrow[ur, hook] & \mathsf{k}(x) \arrow[u, hook, "\tilde{\cdot}"']
\end{tikzcd}$$

These translations are ring homomorphisms and are illustrated by the commutative diagram above. At the center, $\mathsf{k}\langle x\rangle$ is the ring of *rationals*, i.e., fractions $\frac{k_0+k_1x+k_2x^2\cdots+k_nx^n}{l_0+l_1x+l_2x^2\cdots+l_nx^n}$ where $l_0 \neq 0$. Differently from fractions, rationals denote only fps—in other words—bona fide streams that do not start "in the past". Indeed, since $l_0 \neq 0$, the inverse of $l_0 + l_1x + l_2x^2 \cdots + l_nx^n$ is, by (11), a fps. The streams denoted by $\mathsf{k}\langle x\rangle$ are known in literature as *rational streams* [33].

Hereafter, we shall often use polynomials and fractions to denote the corresponding streams. Moreover, $\mathsf{Mat}\,\mathsf{k}[[x]]$ and $\mathsf{Mat}\,\mathsf{k}\langle x\rangle$ denote the PROPs of matrices over $\mathsf{k}[[x]]$ and $\mathsf{k}\langle x\rangle$, defined analogously to $\mathsf{Mat}\,\mathsf{k}[x]$. Similarly, $\mathbb{SV}_{\mathsf{k}((x))}$ is the PROP of $\mathsf{k}((x))$ subspaces, defined as $\mathbb{SV}_{\mathsf{k}(x)}$.

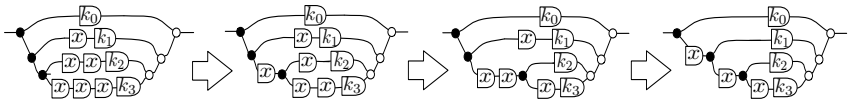
6.1. A stream semantics of $\overrightarrow{\mathsf{Circ}}$

The semantics $\overrightarrow{[\![\cdot]\!]}\colon \overrightarrow{\mathsf{Circ}} \to \mathsf{Mat}\,\mathsf{k}[x]$ of §4 allows us to regard the circuits in $\mathbb{HA}$ as *stream* transformers. Indeed, the interpretation of a polynomial in $\mathsf{k}[x]$ as fps in $\mathsf{k}[[x]]$ can be extended pointwise to a faithful PROP morphism $\hat{\cdot}\colon \mathsf{Mat}\,\mathsf{k}[x] \to \mathsf{Mat}\,\mathsf{k}[[x]]$. By taking $\overrightarrow{\langle\!\langle\cdot\rangle\!\rangle} = \overrightarrow{[\![\cdot]\!]}\,;\hat{\cdot}$, the semantics $\overrightarrow{\langle\!\langle c\rangle\!\rangle}$ of a circuit $c \in \overrightarrow{\mathsf{Circ}}[n,m]$ is a linear map of type $\mathsf{k}[[x]]^n \to \mathsf{k}[[x]]^m$.

Remark 6.1. *The semantics* $\overrightarrow{\langle\!\langle\cdot\rangle\!\rangle}$ *captures the operational intuition for* $\overrightarrow{\mathsf{Circ}}$ *given in Remark 2.1. The circuits carry individual elements of a* k*-stream, processing one after the other. Inputs arrive on the left and outputs are emitted on the right. For instance,* $\overrightarrow{\langle\!\langle$ $\rangle\!\rangle = (x)$ *maps every stream* $\sigma \in \mathsf{k}[[x]]$ *into the stream* $\sigma \cdot x$ *which, by* (10)*, is* $\underline{0}, \sigma(0), \sigma(1), \sigma(2), \ldots$ *Thus* *behaves as a* delay*. Instead, for* $k \in \mathsf{k}$*,* $\overrightarrow{\langle\!\langle$  $\rangle\!\rangle = (k)$ *maps* σ *to* $\sigma \cdot k = \underline{k\sigma(0)}, k\sigma(1), k\sigma(2), \ldots$ *Therefore*  *is an* amplifier*. For the remaining operations:*  *is an* adder*, its unit*  *emits the constant stream* $\underline{0}, 0, 0 \ldots$*,*  *is a* copier *and its counit*  *as the transformer taking any stream as input.*

One can readily check that this interpretation coincides with the semantics given in [9, §4.1]. Our approach has the advantage of making the circuits representation formal and allowing for equational reasoning, as shown for instance in Example 6.2 below. Indeed, since $\hat{\cdot}\colon \mathsf{Mat}\,\mathsf{k}[x] \to \mathsf{Mat}\,\mathsf{k}[[x]]$ is faithful, the axiomatization of $\mathbb{HA}$ is sound and complete also for $\overrightarrow{\langle\!\langle\cdot\rangle\!\rangle}$.

Example 6.2. *Consider the following derivation in the equational theory of* $\mathbb{HA}$*, where* (A15) *is used at each step.*



Any of the circuits above has stream semantics given by the matrix $(p) \in \mathsf{Mat}\,\mathsf{k}[[x]][1,1]$*, where* $p = \underline{k_0}, k_1, k_2, k_3, 0 \ldots$*. Along the lines of [9, Prop. 4.12], one can think of the derivation above as a procedure that reduces the total number of delays*  *appearing in the implementation of* $f\colon \sigma \mapsto \sigma \cdot p$*.*

6.2. A stream semantics of Circ

In §5, we gave a semantics to Circ in terms of subspaces of fraction of polynomials. In this section, we extend this semantics to subspaces of streams. While formal power series are enough to provide a stream semantics to $\overrightarrow{\mathsf{Circ}}$, for the whole of Circ one needs the full generality of Laurent series since, as we have discussed above, not all fractions of polynomials (e.g. $\frac{1}{x}$) denote fps.

The stream semantics is the unique PROP morphism $\langle\!\langle\cdot\rangle\!\rangle \colon \mathsf{Circ} \to \mathbb{SV}_{\mathsf{k}((x))}$ mapping the components in (1) as follows:

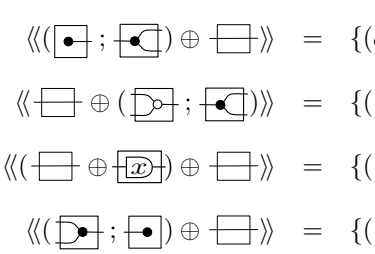
 $\mapsto$ $\{(\sigma, \begin{pmatrix}\sigma\\ \sigma\end{pmatrix}) \mid \sigma \in \mathsf{k}((x))\}$  $\mapsto$ $\{(\begin{pmatrix}\sigma\\ \tau\end{pmatrix}, \sigma+\tau) \mid \sigma, \tau \in \mathsf{k}((x))\}$

 $\mapsto$ $\{(\sigma, (\,)) \mid \sigma \in \mathsf{k}((x))\}$  $\mapsto$ $\{((\,), 0)\}$

 $\mapsto$ $\{(\sigma, k\cdot\sigma) \mid \sigma \in \mathsf{k}((x))\}$  $\mapsto$ $\{(\sigma, x\cdot\sigma) \mid \sigma \in \mathsf{k}((x))\}$

and symmetrically for the components in (2). Here 0, x and k denote streams.

Example 6.3. *In Example 2.2, we presented the circuit c_2 as the composition of four sequential chunks. Their stream semantics is displayed below.*



$$\begin{aligned}
\langle\!\langle(\square\,;\,\square)\oplus\square\rangle\!\rangle &= \{(\sigma_1, \begin{pmatrix}\tau_1\\ \tau_1\\ \sigma_1\end{pmatrix}) \mid \sigma_1, \tau_1 \in \mathsf{k}((x))\}\\
\langle\!\langle\square\oplus(\square\,;\,\square)\rangle\!\rangle &= \{(\begin{pmatrix}\tau_2\\ \sigma_2\\ \rho_2\end{pmatrix}, \begin{pmatrix}\tau_2\\ \sigma_2+\rho_2\\ \sigma_2+\rho_2\end{pmatrix}) \mid \sigma_2, \tau_2, \rho_2 \in \mathsf{k}((x))\}\\
\langle\!\langle(\square\oplus\square)\oplus\square\rangle\!\rangle &= \{(\begin{pmatrix}\tau_3\\ \sigma_3\\ \rho_3\end{pmatrix}, \begin{pmatrix}\tau_3\\ x\cdot\sigma_3\\ \rho_3\end{pmatrix}) \mid \sigma_3, \tau_3, \rho_3 \in \mathsf{k}((x))\}\\
\langle\!\langle(\square\,;\,\square)\oplus\square\rangle\!\rangle &= \{(\begin{pmatrix}\tau_4\\ \tau_4\\ \sigma_4\end{pmatrix}, \sigma_4) \mid \sigma_4, \tau_4 \in \mathsf{k}((x))\}
\end{aligned}$$

The composition in $\mathbb{SV}_{\mathsf{k}((x))}$ of the four linear relations above is

$$\{(\sigma_1, \sigma_4) \mid \textit{there exist } \sigma_2, \sigma_3, \tau_1, \ldots, \tau_4, \rho_2, \rho_3 \textit{ s.t. } \begin{cases}\tau_1 = \tau_2 = \sigma_2 = \tau_3 = \tau_4,\\ \sigma_2 + \rho_2 = \sigma_3,\ x\cdot\sigma_3 = \tau_4\\ \sigma_1 = \rho_2,\ \sigma_2+\rho_2 = \rho_3 = \sigma_4\end{cases}\}$$

By simple algebraic manipulations one can check that the above systems of equations has a unique solution given by $\sigma_4 = \frac{1}{1-x}\sigma_1$. Since $\langle\!\langle\cdot\rangle\!\rangle$ is a PROP morphism and c_2 is the composition of the four chunks above, we obtain

$$\langle\!\langle c_2\rangle\!\rangle = \{(\sigma_1, \frac{1}{1-x}\cdot\sigma_1) \mid \sigma_1 \in \mathsf{k}((x))\}.$$

This relation contains all pairs of streams that can occur on the left and on the right ports of c_2. For instance if $\underline{1}, 0, 0\ldots$ is on the left, $\underline{1}, 1, 1\ldots$ is on the right.

For the other circuit of Example 2.2, namely c_1, it is immediate to see that

$$\langle\!\langle c_1\rangle\!\rangle = \{((1-x)\cdot\sigma_1, \sigma_1) \mid \sigma_1 \in \mathsf{k}((x))\}$$

which is clearly the same subspace as $\langle\!\langle c_2\rangle\!\rangle$. In Example 8.7, we will prove the semantic equivalence of the two circuits by means of the equational theory of $\mathbb{IH}$. This is always possible since, as stated by the following theorem, the axiomatization of $\mathbb{IH}$ is sound and complete with respect to $\langle\!\langle\cdot\rangle\!\rangle$.

Theorem 6.4. *For all c, d in Circ, $c \overset{\mathbb{IH}}{=} d$ iff $\langle\!\langle c\rangle\!\rangle = \langle\!\langle d\rangle\!\rangle$.*

The argument relies on another "floor" ( ) below diagram ( ).

$$
\begin{tikzcd}
 & \mathsf{Mat}\,\mathsf{k}[x] + \mathsf{Mat}\,\mathsf{k}[x]^{op} \arrow[rr, "{[\kappa_1,\kappa_2]}"] \arrow[dl, "{[\iota_1,\iota_2]}"'] \arrow[dd, hook] & & \mathsf{Span}(\mathsf{Mat}\,\mathsf{k}[x]) \arrow[dl, "\Phi"'] \arrow[dd, hook, "\Theta"] \\
\mathsf{Cospan}(\mathsf{Mat}\,\mathsf{k}[x]) \arrow[rr, "\Psi"] \arrow[dd, hook, "\Upsilon"'] & & \mathbb{SV}_{\mathsf{k}(x)} \arrow[dd, dotted, "{[\cdot]}"] & \\
 & \mathsf{Mat}\,\mathsf{k}[[x]] + \mathsf{Mat}\,\mathsf{k}[[x]]^{op} \arrow[rr, "{[\kappa'_1,\kappa'_2]}"] \arrow[dl, "{[\iota'_1,\iota'_2]}"'] & & \mathsf{Span}(\mathsf{Mat}\,\mathsf{k}[[x]]) \arrow[dl, "\Phi'"'] \\
\mathsf{Cospan}(\mathsf{Mat}\,\mathsf{k}[[x]]) \arrow[rr, "\Psi'"] & & \mathbb{SV}_{\mathsf{k}((x))} &
\end{tikzcd}
\qquad (\;\;)
$$

We shall show that, just as for $[\![\cdot]\!]$ (Corollary 5.5), completeness of $\langle\!\langle\cdot\rangle\!\rangle$ can be derived by a universal categorical construction: in particular, $\langle\!\langle\cdot\rangle\!\rangle$ is the composition of $[\![\cdot]\!]$ with $[\cdot]$ in ( ). To this aim, we distill the components of ( ).

Top face. The top face is the bottom face of ( ). The map $[\kappa_1, \kappa_2]$ arises from:

$$
\begin{array}{ll}
\kappa_1 \colon \mathsf{Mat}\,\mathsf{k}[x] \to \mathsf{Span}(\mathsf{Mat}\,\mathsf{k}[x]) & \kappa_2 \colon \mathsf{Mat}\,\mathsf{k}[x]^{op} \to \mathsf{Span}(\mathsf{Mat}\,\mathsf{k}[x]) \\
A \colon n \to m \longmapsto (n \xleftarrow{id} n \xrightarrow{A} m) & B \colon n \to m \longmapsto (n \xleftarrow{B} m \xrightarrow{id} m)
\end{array}
$$

and, similarly, $[\iota_1, \iota_2]$ is the pairing of

$$
\begin{array}{ll}
\iota_1 \colon \mathsf{Mat}\,\mathsf{k}[x] \to \mathsf{Cospan}(\mathsf{Mat}\,\mathsf{k}[x]) & \iota_2 \colon \mathsf{Mat}\,\mathsf{k}[x]^{op} \to \mathsf{Cospan}(\mathsf{Mat}\,\mathsf{k}[x]) \\
A \colon n \to m \longmapsto (n \xrightarrow{A} m \xleftarrow{id} m) & B \colon n \to m \longmapsto (n \xrightarrow{id} n \xleftarrow{B} m).
\end{array}
$$

The morphism Φ maps $n \xleftarrow{V} z \xrightarrow{W} m$ to the linear relation

$$\{\, (\mathbf{u}, \mathbf{v}) \mid \mathbf{u} \in \mathsf{k}(x)^n,\ \mathbf{v} \in \mathsf{k}(x)^m,\ \exists \mathbf{w} \in \mathsf{k}(x)^z.\ \delta(V)\mathbf{w} = \mathbf{u} \wedge \delta(W)\mathbf{w} = \mathbf{v} \,\}$$

where $\delta \colon \mathsf{Mat}\,\mathsf{k}[x] \to \mathsf{Mat}\,\mathsf{k}(x)$ is the obvious embedding, and Ψ acts as follows:

$$n \xrightarrow{V} z \xleftarrow{W} m \quad \longmapsto \quad \{\, (\mathbf{u}, \mathbf{v}) \mid \mathbf{u} \in \mathsf{k}(x)^n,\ \mathbf{v} \in \mathsf{k}(x)^m,\ \delta(V)\mathbf{u} = \delta(W)\mathbf{v} \,\}.$$

Theorem 3 in [3] ensures that these maps are a pushout diagram in **PROP**.

Bottom face. The morphisms of the bottom face, $[\kappa'_1, \kappa'_2]$, $[\iota'_1, \iota'_2]$, Φ' and Ψ', are defined analogously. Since $\mathsf{k}[[x]]$ is a PID and $\mathsf{k}((x))$ is its field of fraction, by Theorem 3 of [3], the bottom face is also a pushout in **PROP**.

Vertical edges. The rear morphism follows from the embedding $\hat{\cdot} \colon \mathsf{Mat}\,\mathsf{k}[x] \to \mathsf{Mat}\,\mathsf{k}[[x]]$ described in §6.1. Θ maps a span $n \xleftarrow{V} z \xrightarrow{W} m$ to $n \xleftarrow{\widehat{V}} z \xrightarrow{\widehat{W}} m$. To verify that this is a morphism of PROPs, one needs to check the following.

Lemma 6.5. $\hat{\cdot} \colon \mathsf{Mat}\,\mathsf{k}[x] \to \mathsf{Mat}\,\mathsf{k}[[x]]$ *preserves pullbacks.*

Proof. See Appendix B. □

Similarly, the leftmost morphism Υ maps $n \xrightarrow{V} z \xleftarrow{W} m$ to $n \xrightarrow{\widehat{V}} z \xleftarrow{\widehat{W}} m$. Since $\mathsf{Mat}\,\mathsf{k}[x]$ and $\mathsf{Mat}\,\mathsf{k}[[x]]$ are both self-dual, it follows by Lemma 6.5 that $\hat{\cdot}$ also preserves pushouts and, therefore, Υ is a morphism of PROPs.

By definition, the left hand and rear faces commute. As a consequence, there exists $[\tilde{\cdot}] \colon \mathbb{SV}_{\mathsf{k}(x)} \to \mathbb{SV}_{\mathsf{k}((x))}$ given by the universal property of the top face of ($\substack{\square \\ \square}$). To give a concrete description of $[\tilde{\cdot}]$, observe that $\tilde{\cdot} \colon \mathsf{k}(x) \to \mathsf{k}((x))$ can be pointwise extended to matrices and sets of vectors. For a subspace H in $\mathbb{SV}_{\mathsf{k}(x)}$, let $[\tilde{H}]$ be the space in $\mathbb{SV}_{\mathsf{k}((x))}$ generated by the set of vectors $\tilde{H}$.

Lemma 6.6. *The morphism* $[\tilde{\cdot}] \colon \mathbb{SV}_{\mathsf{k}(x)} \to \mathbb{SV}_{\mathsf{k}((x))}$ *maps* H *in* $\mathbb{SV}_{\mathsf{k}(x)}$ *to* $[\widetilde{H}]$.

Proof. See Appendix B. □

Proposition 6.7. $\langle\!\langle \cdot \rangle\!\rangle = [\![\cdot]\!] \,;\, [\tilde{\cdot}]$.

Proof. Clear from definitions of $\langle\!\langle \cdot \rangle\!\rangle$ and $[\![\cdot]\!]$, and Lemma 6.6. □

By construction, the morphism $[\![\cdot]\!] \,;\, [\tilde{\cdot}]$ has the desired properties allowing to infer soundness and completeness of $\mathbb{IH}$ with respect to the stream semantics.

Proof of Theorem 6.4. Let c and d be in Circ. By Proposition 6.7, $\langle\!\langle c \rangle\!\rangle = \langle\!\langle d \rangle\!\rangle$ iff $[\widetilde{[\![c]\!]}] = [\widetilde{[\![d]\!]}]$. Now, $[\tilde{\cdot}]$ is given by the universal property in ($\substack{\square \\ \square}$): since all vertical maps of ($\substack{\square \\ \square}$) are faithful, also $[\tilde{\cdot}]$ is faithful. It follows that $[\widetilde{[\![c]\!]}] = [\widetilde{[\![d]\!]}]$ iff $[\![c]\!] = [\![d]\!]$ and, therefore, iff $c \overset{\mathbb{IH}}{=} d$ by Corollary 5.5. □

7. Axiomatising SF: the Theory of Rational Matrices

The relational semantics for Circ, developed in the previous sections, clearly also gives a denotation for circuits in its sub-PROP SF. However, as outlined in §2, we expect that signal flow graphs express *functional* behaviors. In this section we shall show that this is the case: our main result is that circuits in SF, up-to equality in $\mathbb{IH}$, *characterise* functional subspaces given by $\mathsf{k}\langle x \rangle$-matrices.

The correspondence between (orthodox) signal-flow diagrams and rational matrices is well-known (e.g. [9]): here we give a categorical, string-diagrammatic, account of this characterisation where notions of "input", "output" and direction of flow are derivative. The following is one direction of the correspondence.

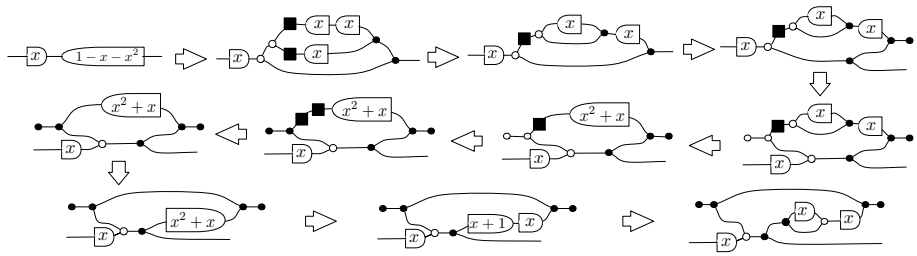
Proposition 7.1. *Suppose that* $c \in \mathsf{SF}[n, m]$. *Then* $[\![c]\!]$ *is the subspace* $[(\mathbf{e}_i, A\mathbf{e}_i)]_{i \leq n}$ *for some* $A \in \mathsf{Mat}\,\mathsf{k}\langle x \rangle[n, m]$, *where* $\{\mathbf{e}_i \mid i \leq n\}$ *is the standard basis of* $\mathsf{k}(x)^n$.

Proof. See Appendix B. □

Note that the converse does not hold: there are functional subspaces given by rational matrices that are in the image of circuits not in SF. In order to obtain full completeness (isomorphism) for $\mathsf{Mat}\,\mathsf{k}\langle x \rangle$, we are going to show that all such circuits are provably equivalent in $\mathbb{IH}$ to one in SF. The following example illustrates an instance of our general result.

Example 7.2. *The rational* $\frac{x}{1-x-x^2}$ *denoting the Fibonacci sequence can be succinctly represented as the circuit* $\boxed{x}\,;\boxed{1-x-x^2}$*, which is not in* SF*. Indeed, composing* $[\![\boxed{x}]\!] = [(1,x)]$ *with* $[\![\boxed{1-x-x^2}]\!] = [(1-x-x^2,1)]$ *yields the* $\mathsf{k}(x)$*-subspace* $[(1,\frac{x}{1-x-x^2})]$*. In terms of streams,* $\langle\!\langle \boxed{x}\,;\boxed{1-x-x^2} \rangle\!\rangle$ *is the* $\mathsf{k}((x))$*-subspace* $[(\ \underline{1},0,0,\dots\ ,\ \underline{0},1,1,2,3,5,\dots\ )]$*.*

The derivation in the equational theory of $\mathbb{IH}$ *below shows how we can "implement" the Fibonacci circuit, by transforming it into a circuit of* SF*.*



The strategy is to unfold $\boxed{1-x-x^2}$ *(using* (A17)^{op} *from* $\mathbb{HA}^{op}$*) and use the Frobenius axioms* (S2)-(S1) *to deform the circuit to obtain the feedback loop. Then the sub-circuit representing* x^2+x *is moved along*  *using* (CC2)*.*

In the Calculus of Signal Flow Diagrams II, we will explain formally in which sense the final circuit of the derivation can be thought as the implementation of the first one. At an intuitive level, this can be explained in terms of flows: in the first circuits it is not possible to assign a direction to the flow, while in the last one signal flows from left to right. Indeed, using the intuition of Remark 2.1 and the behaviour of  *,*  *as bent wires that merely forward signals from one port to the other, the reader can see that inputing the stream* $\underline{1},0,0,\dots$ *on the left yields the Fibonacci sequence* $\underline{0},1,1,2,3,5,\dots$ *as output on the right.*

In view of the above, we shall work with SF modulo $\mathbb{IH}$. Since morphisms of PROPs are identity-on-objects, we can simply take the image of SF in $\mathbb{IH}$.

Definition 7.3. $\mathbb{SF}$ is the sub-PROP of $\mathbb{IH}$ given by the image of

$$\mathsf{SF} \to \mathsf{Circ} \to \mathbb{IH}.$$

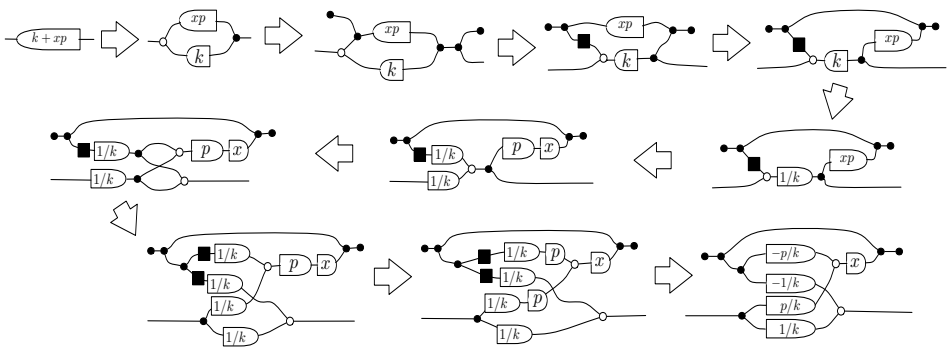
One can think of $\mathbb{SF}$ as consisting of all the circuits of Circ that are equivalent in $\mathbb{IH}$ to one of SF. We can now state the main theorem of this section.

Theorem 7.4. *There is an isomorphism of PROPs between* $\mathbb{SF}$ *and* $\mathsf{Mat}\,\mathsf{k}\langle x\rangle$*.*

The direction from circuits to matrices of Theorem 7.4 is already given by Proposition 7.1. The following statement takes care of the converse.

Proposition 7.5. *Suppose that* $A \in \mathsf{Mat}\,\mathsf{k}\langle x\rangle[n,m]$*. Then for any* $c \in \mathsf{Circ}[n,m]$ *such that* $[\![c]\!] = [(\mathbf{e}_i, A\mathbf{e}_i)]_{i\leq n}$ *there exists a circuit* $c' \in \mathsf{SF}[m,n]$ *such that* $c \overset{\mathbb{IH}}{=} c'$*.*

Proof. Let $\frac{1}{k+xp} \in \mathsf{k}\langle x\rangle$ be a rational, with $k \neq 0$ and $p \in \mathsf{k}[x]$. This can be seen as a 1×1 matrix of $\mathsf{Mat}\,\mathsf{k}\langle x\rangle$, yielding the subspace $[(1, \frac{1}{k+xp})]\colon 1 \to 1$. The following derivation shows that a circuit of Circ, whose semantics (via $[\![\cdot]\!]$) is the subspace $[(1, \frac{1}{k+xp})]$, is equal to one in SF. The sequence of applied equalities is: $(\mathrm{A17})^{op}$, (A4) + $(\mathrm{A4})^{op}$, (8) + (S2), (CC2), (7), (A13), (A10), (A15), (A13) + naturality of symmetry, (A8).



Now, fix a matrix $A \in \mathsf{Mat}\,\mathsf{k}\langle x\rangle[n,m]$ and the associated subspace $[(e_i, Ae_i)]_{i\leq n}$. Let $d \in \mathsf{Circ}[n,m]$ be the circuit in matrix form constructed as in Remark 5.6 whose $[\![\cdot]\!]$-semantics is the subspace $[(e_i, Ae_i)]_{i\leq n}$: each entry q of the matrix A — that is, a (rational) fraction $q = p_1/p_2 \in \mathsf{k}\langle x\rangle$ — is encoded as a component $\boxed{p_1}\,;\boxed{p_2}$ of d. By the observation above, we can put any such circuit $\boxed{p_2}$ in the form of a circuit of SF. Therefore, d is equal in $\mathbb{IH}$ to a circuit c where all components are in SF and, since SF is closed under $\oplus$ and $;$, then also c is a circuit of SF. □

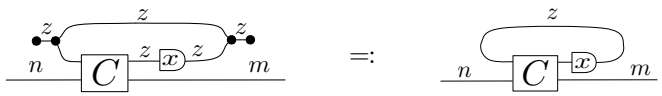
We can now prove of our characterisation result.

Proof of Theorem 7.4. There is an obvious embedding $\mathsf{Mat}\,\mathsf{k}\langle x\rangle \to \mathbb{SV}_{\mathsf{k}(x)}$ mapping $A \in \mathsf{Mat}\,\mathsf{k}\langle x\rangle[n,m]$ into the subspace $[(e_i, Ae_i)]_{i\leq n}$: the idea is to show that $\mathbb{SF}$ characterises its image. To do this, define $F\colon \mathbb{SF} \to \mathsf{Mat}\,\mathsf{k}\langle x\rangle$ as follows. By definition, an arrow f of $\mathbb{SF}$ is an $\mathbb{IH}$-equivalence class containing a circuit c of SF. By Proposition 7.1, $[\![c]\!] = [(e_i, Ae_i)]_{i\leq m}$ for some A in $\mathsf{Mat}\,\mathsf{k}\langle x\rangle$. We let F map f to A: Corollary 5.5 guarantees that F is well-defined and faithful. To see that F is full, let A be a matrix in $\mathsf{Mat}\,\mathsf{k}\langle x\rangle$. Because $[\![\cdot]\!]$ is full on $\mathbb{SV}_{\mathsf{k}(x)}$, there is a circuit c in Circ such that $[\![c]\!] = [(e_i, Ae_i)]_{i\leq m}$. By Proposition 7.5, there is also d in SF such that $c \overset{\mathbb{IH}}{=} d$ and $[\![c]\!] = [\![d]\!]$. We conclude that F is full and faithful and thus an isomorphism. □

7.1. A Trace Canonical Form for Circuits of SF

In this section we show that circuits of SF can always be put, using the equational theory of $\mathbb{IH}$, into a convenient shape: a core given by a circuit c of $\overrightarrow{\mathsf{Circ}}$ without delays, and an exterior part given by a "bundle" of feedback loops. We formally introduce this notion below.

Definition 7.6. For $n, m, z \in \mathbb{N}$, $c \in \mathsf{Circ}[z+n, z+m]$, the *$z$-feedback* $\mathsf{Tr}^z(c) \in \mathsf{Circ}[n, m]$ is the circuit below, for which we use the indicated shorthand notation:

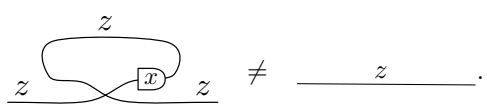


In particular, $\mathsf{Tr}^1(\cdot)$ coincides with the assignment $\mathsf{Tr}(\cdot)$ given in §2.2.

Proposition 7.7 (Trace form for SF)**.** *Let $\overrightarrow{\mathsf{Circ}}\backslash\mathsf{x}$ be the sub-PROP of $\overrightarrow{\mathsf{Circ}}$ whose circuits do not contain any delay* $\boxed{x}$*. For every circuit $d \in \mathsf{SF}[n, m]$, there are $z \in \mathbb{N}$ and $c\colon z+n \to z+m$ of $\overrightarrow{\mathsf{Circ}}\backslash\mathsf{x}$ such that $d \overset{\mathbb{IH}}{=} \mathsf{Tr}^z(c)$.*

The existence of this form is a folklore result in the theory of signal flow diagrams. Here we provide a novel proof that consists of showing that $\mathsf{Tr}^z(\cdot)_{z\in\mathbb{N}}$ is a *right trace* [1, §5.1] on the category $\mathbb{IH}$.

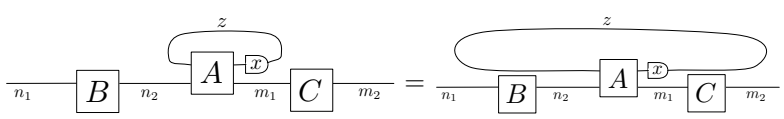
Remark 7.8. *Note that $\mathsf{Tr}^z(\cdot)_{z\in\mathbb{N}}$ is* not *the canonical trace induced by the compact closed structure of $\mathbb{IH}$ (§ 5.1), but a version "guarded" by a register. This makes our approach different from other works using traces to model recursion, like those of Stefanescu [34], Hasegawa [35] and Bloom and Ésik's axiomatisation of iteration theories [36, 37]. The traces considered there are not guarded by a register and indeed satisfy the "yanking" law [38], which is instead false for $\mathsf{Tr}^z(\cdot)_{z\in\mathbb{N}}$:*



Proposition 7.9. *The family $\mathsf{Tr}^z(\cdot)_{z\in\mathbb{N}}$ is a right trace on $\mathbb{IH}$.*

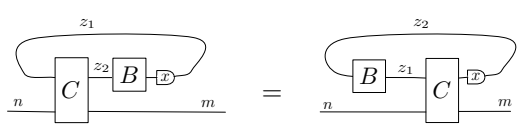
Proof. The axioms of (right) traced categories, as presented in [1, §5.1], are:

1. Tightening:



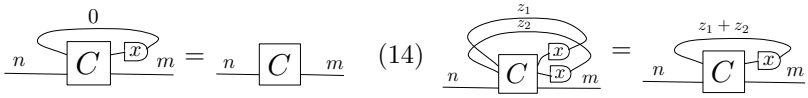
(12)

2. Sliding:

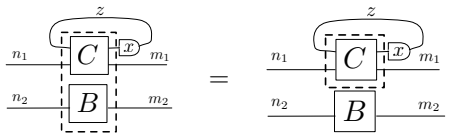


(13)

3. Vanishing:

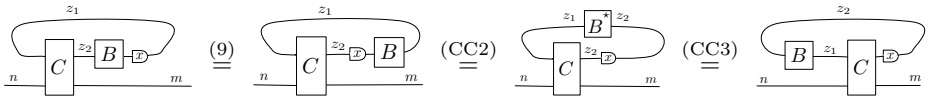


4. Strength:

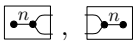


Tightening and strength hold for our definition of trace simply by laws of symmetric monoidal categories. Therefore we focus on sliding and vanishing.

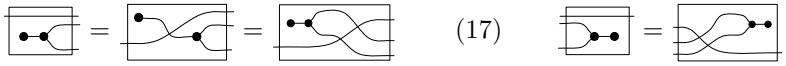
Sliding. The following derivation yields the sliding equation:



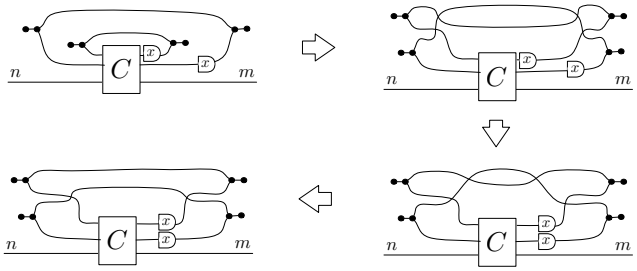
For the two last steps, observe that $B^{**} = B$ by definition of $(\cdot)^\star$ and (CC1).

Vanishing. Concerning vanishing, (14) holds because, by definition,  , and  are all equal to id_0 for $n = 0$. It remains to check (15). We provide the proof for $z_1, z_2 = 1$. The general case is handled (by induction) by the obvious generalisation of the same argument.

For this purpose, it will be useful to first introduce the following two equations, holding in Circ by naturality of symmetry.



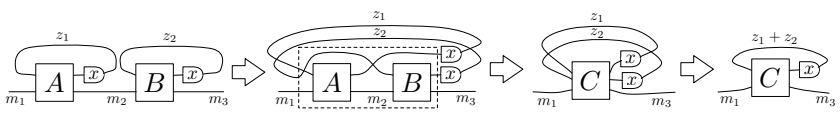
By definition, the first circuit below is $\mathsf{Tr}^1\mathsf{Tr}^1 c$ and the last is $\mathsf{Tr}^2 c$. The first step applies (17) and (18), the second and the third follow by axioms of symmetric monoidal categories.

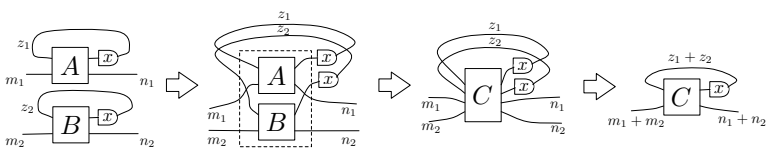


This concludes the proof of Proposition 7.9. □

We can now give the argument for Proposition 7.7.

Proof of Proposition 7.7 . The proof goes by induction on a circuit d of SF. If d is a component in (1) different from $\boxed{x}$, $d = \mathsf{Tr}^0(d)$. For $\boxed{x}$, it is easy to check that $\boxed{x} \overset{\mathbb{IH}}{=} \mathsf{Tr}^1(\boxed{\times})$. The second clause of the inductive definition of SF is the case in which $d = \mathsf{Tr}^1(c)$ for some circuit c of SF. By induction hypothesis $c \overset{\mathbb{IH}}{=} \mathsf{Tr}^z(c')$ for some c' in $\overrightarrow{\mathsf{Circ}}\backslash\mathsf{x}$ and thus, by (15), $d \overset{\mathbb{IH}}{=} \mathsf{Tr}^{1+z}(c')$. The remaining two cases are the ones in which d is given by sequential or parallel composition of circuits of SF—which are, by induction hypothesis, of the form described in the statement. The corresponding derivations, given below, use the properties of the trace; the circuit c is defined by the dotted square.





□

8. Realisability

In §7 we showed that, in the equational theory of $\mathbb{IH}$, restricting Circ to the syntax SF of signal flow graphs captures the rational behaviors in $\mathbb{SV}_{\mathsf{k}(x)}$. Moreover, the relations represented by SF give rise to particularly well-behaved functional relations under the stream semantics $\langle\!\langle\cdot\rangle\!\rangle \colon \mathsf{Circ} \to \mathbb{SV}_{\mathsf{k}((x))}$, since they do not actually require the full generality of Laurent series: any rational polynomial generates a fps, without the need for a "finite past." Indeed, these kind of stream transformers have been well-understood since at least the 1950s.

In the stream universe $\mathbb{SV}_{\mathsf{k}((x))}$, what can we say about circuits in Circ that do not have an equivalent circuit in SF? Do they define a more expressive family of signal flow circuits as stream transformers under the stream semantics?

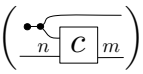
In this section we shall see that the answer to the last questions is NO, in fact, within the equational theory of $\mathbb{IH}$, Circ is nothing else but a "jumbled up" version of SF: more precisely, while every circuit in SF has inputs on the left and outputs on the right, for every circuit in Circ there is a way of partitioning its left and right ports into "inputs" and "outputs", in the sense that appropriate rewiring yields an $\mathbb{IH}$-equal circuit in SF. The main result of this section is the *realisability* theorem (Theorem 8.4) which guarantees that such an input-output partition exists—i.e. every circuit in Circ is a rewired circuit in SF. Note that such a partition is not unique, and this fact corresponds to

the physical intuition that in some circuits there is more than one way of orienting flow[3]. Moreover, we are able to crystallise what we consider to be the central methodological contribution of this paper: since it is only by forgetting the input-output distinction that the algebra $\mathbb{IH}$ of signal flow is revealed, and signal flow graphs can be given a compositional semantics, the notions of input and output cannot be considered as primitive; they are, rather, derived notions.

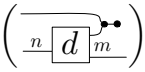
We begin by giving a precise definition of what we mean by "jumbling up" the wires of a circuit. First, for each $n, m \in \mathbb{N}$, we define circuits $\eta_n \colon n \to 1+1+n$ and $\epsilon_m \colon 1+1+m \to m$ in Circ as illustrated below.

$\eta_n :=$  $\qquad \epsilon_m :=$ 

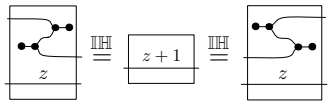
Next, we define the families of operators $\mathsf{L}_{n,m} \colon \mathsf{Circ}[n{+}1, m] \to \mathsf{Circ}[n, 1{+}m]$ and $\mathsf{R}_{n,m} \colon \mathsf{Circ}[n, 1{+}m] \to \mathsf{Circ}[1{+}n, m]$ as follows: for any circuit $c \in \mathsf{Circ}[n{+}1, m]$,

$$\mathsf{L}_{n,m}(c) = \eta_n \,;(id_1 \oplus c) \qquad \Big($$  $$\Big)$$

and, for any circuit $d \in \mathsf{Circ}[n, m+1]$

$$\mathsf{R}_{n,m}(d) = (id_1 \oplus d) \,;\epsilon_m. \qquad \Big($$  $$\Big)$$

Remark 8.1. *When considered as operations on $\mathbb{IH}$, $\mathsf{L}_{n,m}$ and $\mathsf{R}_{n,m}$ enjoy some interesting properties. Let $1 + - \colon \mathbb{IH} \to \mathbb{IH}$ be the functor acting on objects as $k \mapsto 1 + k$ and on arrows as $f \mapsto id_1 \oplus f$. This functor is self-adjoint: the unit and the counit are the η_n and ϵ_m defined as above. The fact that $\mathbb{IH}$ is a SMC implies naturality of η and ϵ. They satisfy the triangle equalities by* (CC1)*:*



(19)

The induced isomorphisms are $\mathsf{L}_{n,m}$, $\mathsf{R}_{n,m}$ defined as above. We can see $\mathsf{L}_{n,m}$ intuitively as "rewiring" the first port on the left to the right of the circuit. The fact that $\mathsf{L}_{n,m}$ and $\mathsf{R}_{n,m}$ are isomorphisms means, of course, that no information is lost – all such circuits can be "rewired" back to their original form.

Definition 8.2. A circuit $c_2 \in \mathsf{Circ}[n_2, m_2]$ is a *rewiring* of $c_1 \in \mathsf{Circ}[n_1, m_1]$ when c_2 can be obtained from c_1 by a combination of the following operations:

(i) application of $\mathsf{L}_{n,m}$, for some n and m,

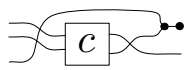
(ii) application of $\mathsf{R}_{n,m}$, for some n and m,

[3] In The Calculus of Signal Flow Diagrams II we shall explore the computational ramifications of the realisability theorem through its relationship with the operational semantics.

(iii) post-composition with a permutation,

(iv) pre-composition with a permutation.

Permutations are needed to rewire an arbitrary—i.e. not merely the first—port on each of the boundaries. For instance, they allow to rewire the second port on the right as the third on the left in the circuit $c\colon 2 \to 2$ below:



In light of Remark 8.1, "is a rewiring of" is an equivalence relation on the circuits of Circ under the equational theory of $\mathbb{IH}$: we shall say that circuits c and d are *rewiring-equivalent* when $c \overset{\mathbb{IH}}{=} d'$ for some rewiring d' of d. Moreover, at the semantics level, a rewiring denotes an isomorphisms between a subspace of type $\mathsf{k}(x)^n \times \mathsf{k}(x)^m$ and one of type $\mathsf{k}(x)^i \times \mathsf{k}(x)^j$ where $n+m = i+j$. For instance, for any circuit c, $[\![c]\!] \subseteq \mathsf{k}(x)^{n+1} \times \mathsf{k}(x)^m$ is isomorphic to $[\![\mathsf{L}_{n,m}(c)]\!] \subseteq \mathsf{k}(x)^n \times \mathsf{k}(x)^{m+1}$ as a subspace of $\mathsf{k}(x)^{n+m+1}$.

Lemma 8.3. *If c is a rewiring of d in Circ, then $[\![c]\!] \cong [\![d]\!]$ as vector spaces.*

Proof. It is enough to observe how $\mathsf{L}_{n,m}$, $\mathsf{R}_{n,m}$ and permutations affect the denoted subspaces:

(i) $\mathsf{L}_{n,m}$ induces an isomorphism $[\![c]\!] \to [\![\mathsf{L}_{n,m}(c)]\!]$ defined

$$(\begin{pmatrix} y \\ \mathbf{v} \end{pmatrix}, \mathbf{w}) \mapsto (\mathbf{v}, \begin{pmatrix} y \\ \mathbf{w} \end{pmatrix}).$$

(ii) $\mathsf{R}_{n,m}$ induces an isomorphism $[\![c]\!] \to [\![\mathsf{L}_{n,m}(c)]\!]$ defined

$$(\mathbf{v}, \begin{pmatrix} z \\ \mathbf{w} \end{pmatrix}) \mapsto (\begin{pmatrix} z \\ \mathbf{v} \end{pmatrix}, \mathbf{w}).$$

(iii) post-composition with a permutation σ induces an isomorphism $(\mathbf{v}, \mathbf{w}) \mapsto (\mathbf{v}, \mathbf{w}')$ with $\mathbf{w}'$ obtained from $\mathbf{w}$ by rearranging its rows according to σ.

(iv) pre-composition with a permutation σ induces an isomorphism $(\mathbf{v}, \mathbf{w}) \mapsto (\mathbf{v}', \mathbf{w})$ with $\mathbf{v}'$ obtained from $\mathbf{v}$ by rearranging its rows according to σ^{-1}.

□

We are now able to state the main result of this section.

Theorem 8.4 (Realisability)**.** *Every circuit in Circ is rewiring-equivalent to some circuit in SF.*

8.1. Proof of Theorem 8.4

We shall work with matrices (and the corresponding circuits) of a particular shape. We say that a matrix over $\mathsf{k}(x)$ is in *rational form* if all its entries are in fact rationals (in $\mathsf{k}\langle x\rangle$) and:

1. each non-zero row has an entry with value 1, called *pivot*.
2. if a column has a pivot entry, then the pivot is the only non-zero entry.

An example is given below, where $r_1, r_2, r_3 \in \mathsf{k}\langle x\rangle$.

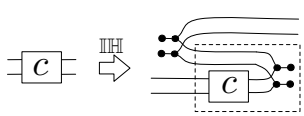
$$\begin{pmatrix} r_1 & 0 & 1 & 0 \\ r_2 & 1 & 0 & 0 \\ r_3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The following lemma is the final ingredient for the proof of Theorem 8.4—its proof, in Appendix B, is an easy exercise in linear algebra (see also Remark 8.6).

Lemma 8.5. *Every $\mathsf{k}(x)$ matrix is row equivalent to one in rational form.*

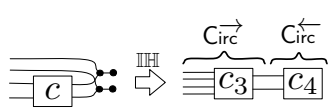
Proof of Theorem 8.4. Fix a circuit $c \in \mathsf{Circ}[n, m]$. In the following, we will sketch a recipe, using the equational theory of $\mathbb{IH}$, with which c is transformed into the rewiring of a circuit in SF. To improve readability, we shall draw any circuit as if both n and m were 2. It should be clear how our argument generalizes.

(i) First we transform c into the circuit c_1 on the right: the two are equal in $\mathbb{IH}$ by virtue of (CC1).



Let us call c_2 the circuit from $n+m$ to 0 delimited by the dotted square in the picture above. Since c_0 is obtained by rewiring c_2, it should be clear that, if c_2 can be rearranged as the rewiring of a circuit in SF, then so can c_0. Therefore, in the sequel we shift our focus to c_2.

(ii) Proposition 5.4 allows us to rewrite c_2 in cospan form, as the composition along a middle boundary z of c_3 and c_4 below, while preserving equality in $\mathbb{IH}$. By definition of cospan form, c_3 is an arrow of $\overrightarrow{\mathsf{Circ}}$, while c_4 is an arrow of $\overleftarrow{\mathsf{Circ}}$. For the sake of readability, we will draw z as if it were 2.

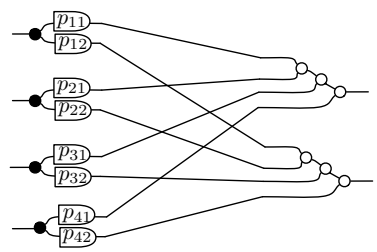


(iii) Since we are reasoning in $\mathbb{IH}$, all the equations of $\mathbb{HA}^{op}$ hold. Now, 0 is both initial and terminal in $\mathsf{Mat}\,\mathsf{k}[x]$; because $\mathbb{HA}^{op} \cong \mathsf{Mat}\,\mathsf{k}[x]^{op}$, this

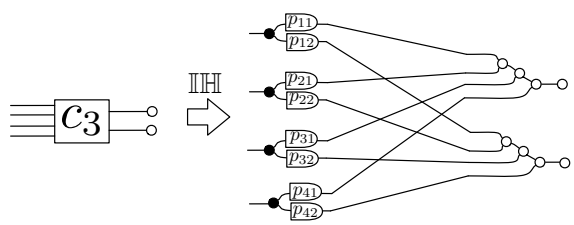
means that there is exactly one circuit of $\overleftarrow{\mathsf{Circ}}$, up to equality in $\mathbb{HA}^{op}$, from z to 0. It follows that c_4 and the z-fold $\oplus$-product of $\boxed{\multimap}$ (a circuit that we call c_5) are equal in $\mathbb{HA}^{op}$ — and thus in $\mathbb{IH}$. In particular $c_3 \,;\, c_4 \overset{\mathbb{IH}}{=} c_3 \,;\, c_5$:

(iv) Since c_3 is in $\overrightarrow{\mathsf{Circ}}$ we can use $\mathbb{HA}$ to reason about it. In particular, c_3 corresponds to a $z \times (m+n)$ matrix of polynomials M, because $\mathbb{HA} \cong \mathsf{Mat}\,\mathsf{k}[x]$. As discussed in § 4, there is a canonical way of representing M as a circuit c_6 of Circ in matrix form (Def. 4.4), below right:

$$M = \begin{pmatrix} p_{11} & p_{21} & p_{31} & p_{41} \\ p_{12} & p_{22} & p_{32} & p_{42} \end{pmatrix} \qquad c_6 =$$

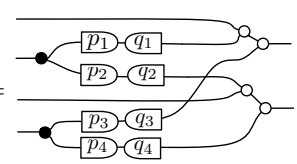


Since c_3 corresponds to M along the isomorphism $\mathbb{HA} \cong \mathsf{Mat}\,\mathsf{k}[x]$, it follows that $[\![c_3]\!]$ is $\{(\sigma, M \cdot \sigma \mid \sigma \in \mathsf{k}(x)^n\}$. Therefore $[\![c_3]\!] = [\![c_6]\!]$ meaning by Theorem 5.3 that $c_3 \overset{\mathbb{IH}}{=} c_6$. We can thus rewrite our circuit as follows:



(v) Using Lemma 8.5, we can then transform M into a matrix $\widehat{M}$ in rational form — for instance, the one on the left below. Since $\widehat{M}$ is a matrix over $\mathsf{k}(x)$, as observed in Remark 5.6, there is a canonical circuit c_7 of Circ, below right, representing it.

$$\widehat{M} = \begin{pmatrix} 1 & p_1/q_1 & 0 & p_3/q_3 \\ 0 & p_2/q_2 & 1 & p_4/q_4 \end{pmatrix} \qquad c_7 =$$



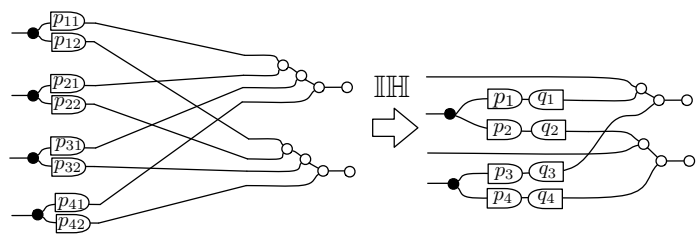
By definition of rational form, each non-zero row R in $\widehat{M}$ is associated with a pivot column C with the only non-zero value 1 at the intersection

of R and C. In order to graphically represent this property in c_7, we assume the following choice of pivots: the first and the third column for the first and second row respectively. Observe that an entry with value 0 corresponds to the circuit $\boxed{0}$, which in $\mathbb{IH}$ is equal to $\bullet\!\circ$: therefore we can avoid drawing the corresponding link in the circuit c_7.

We now claim that $c_6 \,;\, c_5 \overset{\mathbb{IH}}{=} c_7 \,;\, c_5$. By Theorem 5.3, to check this it suffices to show that $[\![c_6 \,;\, c_5]\!] = [\![c_7 \,;\, c_5]\!]$, that is:

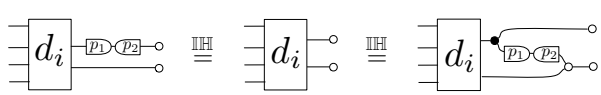
$$\{(\sigma, M \cdot \sigma \mid M \cdot \sigma = \mathbf{0}\} = \{(\sigma, \widehat{M} \cdot \sigma) \mid \widehat{M} \cdot \sigma = \mathbf{0}\}.$$

This is true because the two relations above describe the kernel of M and $\widehat{M}$ respectively, and $\widehat{M}$ is row-equivalent to M. It follows that we can rewrite $c_6 \,;\, c_5$ as $c_7 \,;\, c_5$, while preserving equality in $\mathbb{IH}$:

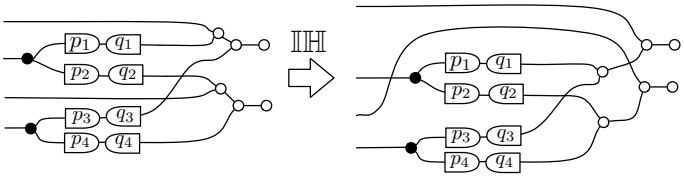


Remark 8.6. *One can argue in a more direct fashion by performing the linear algebraic manipulations involved in the proof of Lemma 8.5 graphically. Indeed, the row operations used to transform M into $\widehat{M}$ can be mimicked at the circuit level, using the equational theory of $\mathbb{IH}$. This procedure involves a sequence of row-equivalent matrices $M_0, M_1, \ldots, M_h$ represented by circuits $d_0, d_1, \ldots, d_h$, where $M_0 = M$, $d_0 = c_6$ and $M_h = \widehat{M}$, $d_h = c_7$. At each step, two kinds of operation can be applied to M_i in order to obtain M_{i+1}: the first is multiplying a row by an element $\frac{p_1}{p_2} \in \mathsf{k}(x)$, the second is replacing a row R_1 by $R_1 + \frac{p_1}{p_2} R_2$, where R_2 is another row. Bearing in mind that rows correspond to entries on the right boundary of d_i, the application of these two operations can be mimicked graphically as on the left and on the right below, respectively.*

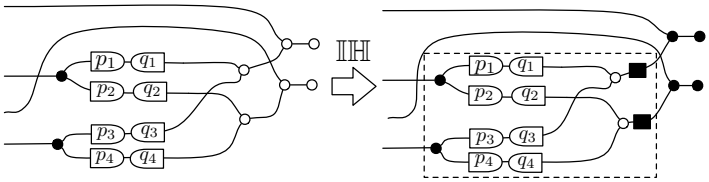
On the left, we represent the first row being multiplied by $\frac{p_1}{p_2}$. On the right, we have the second row being summed with the first one multiplied by $\frac{p_1}{p_2}$: the semantics of  *and*  *confirm our description. Since these are row operations, the resulting circuit d_{i+1} will still correspond to a matrix, namely M_{i+1}. An equational derivation can show that, modulo composition with c_5, the transformation of d_i into d_{i+1} is sound in $\mathbb{IH}$:*



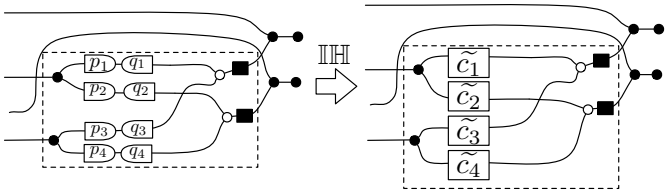
(vi) We now focus on circuit $c_7\,;c_5$. Our next step is to use associativity and commutativity of  to make one of the two legs of each component  be always attached to the pivot-wire of the corresponding row. Also, we use the axioms of SMCs and naturality of the symmetry  to push the pivot-wires towards the top of the circuit, as follows:



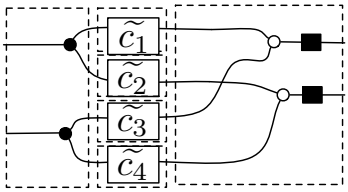
(vii) We can now remove the components of shape  by turning them into rewiring structure. This can be done by using axiom (S6) of $\mathbb{IH}$:



(viii) Let us call c_8 the rightmost circuit above: it is a rewiring of the circuit inscribed into the dotted square, which we call c_9. Since c_7 was constructed starting by a matrix in rational form, for all the components  in c_8, $\frac{p}{q}$ is a rational. Thus, using the fact that $\mathbb{SF} \cong \mathsf{Mat}\,\mathsf{k}\langle x\rangle$, we can rewrite in $\mathbb{IH}$ each such component as a circuit $\tilde{c}$ in SF:



Now, observe that c_9 can be seen as the composition of circuits in SF.



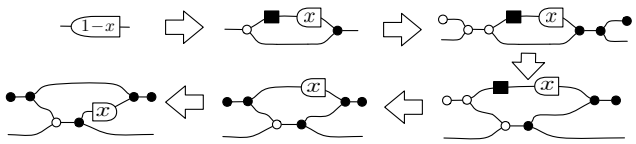
It follows that c_9 is also in SF and thus c_8 is the rewiring of a circuit in SF. Since c_8 was obtained by c_2 by only using rewriting steps allowed by the equational theory of $\mathbb{IH}$, the statement of the theorem follows. $\square$

As remarked previously, circuits in Circ are, in general, the rewiring of more than one signal flow graph. To illustrate this, we return to Example 2.2.

Example 8.7. *The circuit*  *is rewiring-equivalent to two different signal flow graphs, illustrated below. Intuitively, the choice depends on whether one considers signal to be flowing right-to-left or left-to-right.*

(20)

The equivalence holds by the following derivation in $\mathbb{IH}$*:*



Note that the last circuit above is just the rightmost in (20) *and the second above is rewiring equivalent to the left-hand in* (20)*, using the compact closed structure of* $\mathbb{IH}$ *(see §5). The derivation also shows, by Corollary 6.4, that the two circuits of Example 2.2 indeed have the same semantics.*

We conclude with some interesting observations stemming from Theorem 8.4.

Proposition 8.8. *Let* c *be a circuit in* Circ *and suppose that* $c' \colon n \to m$ *is a rewiring-equivalent circuit in* SF*. Then the dimension of* $[\![c]\!]$ *is* n*.*

Proof. By Lemma 8.3, $[\![c]\!] \cong [\![c']\!]$ as vector spaces. Since c' is in SF, $[\![c']\!]$ is a functional subspace by Theorem 7.4, whence its dimension is n. □

Corollary 8.9. *Let* c *be a circuit in* Circ *and suppose that* $c_1 \colon n_1 \to m_1$ *and* $c_2 \colon n_2 \to m_2$*, in* SF*, are rewiring-equivalent to* c*. Then* $n_1 = n_2$ *and* $m_1 = m_2$*.*

As a consequence of Theorem 8.4, we can transform any circuit into one where the direction of the flow, inputs and outputs are determined. Intuitively for circuits in SF, the signal flows from the input ports on the left to the output ports on the right. The rewiring just exchanges the positions of some ports and therefore in a circuit which is the rewiring of an orthodox signal flow graph it is always possible to identify inputs, outputs and the direction of the flow.

Example 8.7 shows that a circuit $c \in$ Circ may be rewiring equivalent to several circuits in SF, allowing for different flow orientations. However, by Corollary 8.9, the number of inputs is constant and coincides with the dimension of $[\![c]\!]$.

The operational intuitions will be made formal in the The Calculus of Signal Flow Diagrams II, where we will show that to effectively execute our circuits as state machines, one actually needs to identify the direction of the flow.

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Appendix A. Axiomatisation of Circ

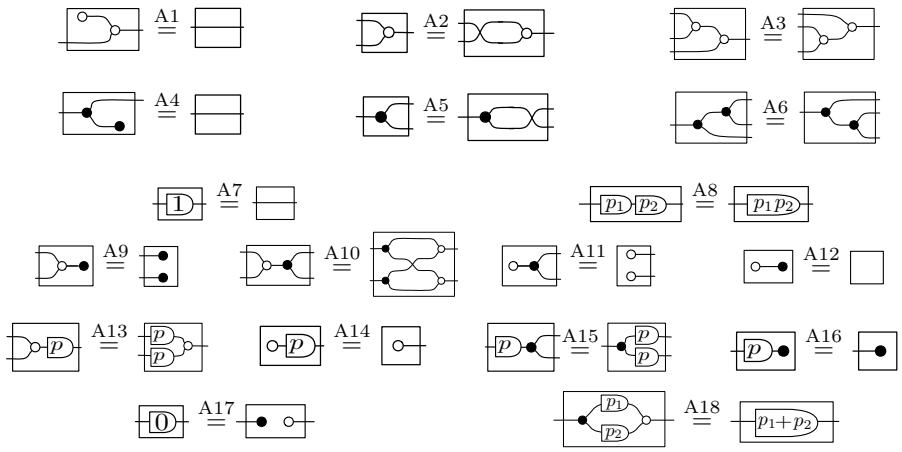

Figure A.4: Axioms of $\mathbb{HA}$, describing the interaction of the generators in (1).

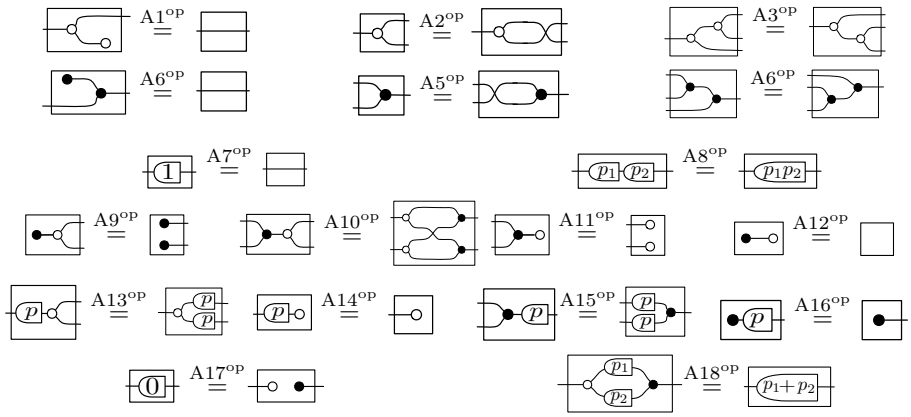

Figure A.5: Axioms of $\mathbb{HA}^{op}$, describing the interaction of the generators in (2).

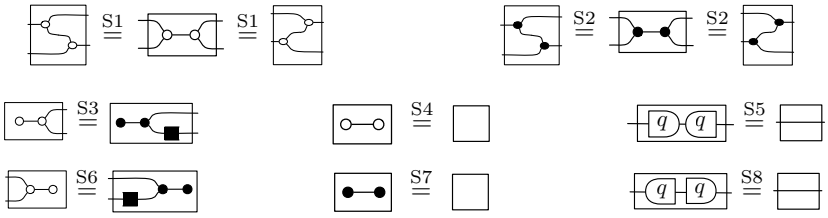

Figure A.6: $\mathbb{IH}$ is presented by the axioms of $\mathbb{HA}$, $\mathbb{HA}^{op}$ and the equations S1-S8 above, describing the interaction of generators in (1) with those in (2). q ranges over $\mathsf{k}[x] \setminus \{0\}$.

Appendix B. Omitted Proofs of Sections 6, 7 and 8

Proof of Lemma 6.5. First observe that the PROP $\mathsf{Mat}\,\mathsf{k}[x]$ is equivalent to the category $\mathsf{FMod}\,\mathsf{k}[x]$ of finite-dimensional free $\mathsf{k}[x]$-modules. Now, because $\mathsf{k}[x]$ is a principal ideal domain (PID), submodules of free modules are also free: thus pullbacks in $\mathsf{FMod}\,\mathsf{k}[x]$ can be calculated as in the abelian category $\mathsf{Mod}\,\mathsf{k}[x]$ of $\mathsf{k}[x]$-modules, where they are calculated as in the category of sets.

Now the diagram below left is a pullback in $\mathsf{Mat}\,\mathsf{k}[x]$ if and only if, in the diagram in $\mathsf{Mod}\,\mathsf{k}[x]$ below right, we have

$$C = \mathrm{Ker}(A|-B)\,;\pi_1 \quad \text{and} \quad D = \mathrm{Ker}(A|-B)\,;\pi_2$$

where $A|-B\colon \mathsf{k}[x]^n \oplus \mathsf{k}[x]^m \to \mathsf{k}[x]^z$ is the copairing of $A\colon \mathsf{k}[x]^n \to \mathsf{k}[x]^z$ and $-B\colon \mathsf{k}[x]^m \to \mathsf{k}[x]^z$ and $\mathrm{Ker}(A|-B)$ is its kernel.

$$\begin{tikzcd}
 & r \arrow[dl, "C"'] \arrow[dr, "D"] & \\
n \arrow[dr, "A"'] & & m \arrow[dl, "B"] \\
 & z &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathsf{k}[x]^r \arrow[dl, "C"'] \arrow[d, "\mathrm{Ker}(A|-B)"] \arrow[dr, "D"] & \\
\mathsf{k}[x]^n \arrow[dr, "A"'] & \mathsf{k}[x]^n \oplus \mathsf{k}[x]^m \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & \mathsf{k}[x]^m \arrow[dl, "B"] \\
 & \mathsf{k}[x]^z &
\end{tikzcd}$$

The same holds for pullbacks in $\mathsf{Mat}\,\mathsf{k}[[x]]$, since $\mathsf{k}[[x]]$ is also a PID. Therefore, our proof reduces to check that, for arbitrary M in $\mathsf{Mat}\,\mathsf{k}[x]$, $\mathrm{Ker}(\widehat{M}) = \widehat{\mathrm{Ker}(M)}$.

For an arbitrary PID, every matrix M can be decomposed as $H = MU$ where U is an invertible matrix and H is a matrix in Hermite Normal Form (HNF), a generalization of Column Echelon Form to the setting of PIDs (see e.g. [39, Def. 2.4.2]). A crucial aspect of HNF is that the first r columns of H, for some natural number r, must have all entries 0. Then, $\mathrm{Ker}(M)$ is given exactly by the first r columns of U (see e.g. [39, Prop. 2.4.9] and [3, Prop. 2]).

Now, given a matrix M in $\mathsf{Mat}\,\mathsf{k}[x]$, its decomposition as $H = MU$ can be computed by iterating elementary column operations, expressed by the invertible matrix U. Exactly the same operations can be performed in $\mathsf{Mat}\,\mathsf{k}[[x]]$ on the matrix $\widehat{M}$. In this way, we decompose $\widehat{M}$ as $\widehat{H} = \widehat{M}\widehat{U}$. By definition, in order to check that a matrix is in HNF, it suffices to verify the position of the 0-entries. The embedding $\widehat{\cdot}$ preserves 0: therefore, since MU is in HNF then also $\widehat{M}\widehat{U}$ must be in HNF. To conclude, let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be the initial columns of U with all entries 0, yielding $\mathrm{Ker}(M)$. Since $\widehat{M}\widehat{U}$ is in HNF, the same vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$, now considered as the first r columns of $\widehat{U}$, yield the matrix $\mathrm{Ker}(\widehat{M})$. Therefore $\widehat{\mathrm{Ker}(M)} = \mathrm{Ker}(\widehat{M})$. □

For the next proof, it is useful to first fix some notation. The embeddings between $\mathsf{k}[x]$, $\mathsf{k}[[x]]$ $\mathsf{k}(x)$ and $\mathsf{k}((x))$, defined in §6, lift to the faithful morphisms

of the corresponding PROPs of matrices, as summarised below.

$$\begin{array}{ccc} \mathsf{Mat}\,\mathsf{k}[[x]] & \xhookrightarrow{\nu} & \mathsf{Mat}\,\mathsf{k}((x)) \\ \hat{\cdot}\uparrow & & \uparrow\tilde{\cdot} \\ \mathsf{Mat}\,\mathsf{k}[x] & \xhookrightarrow[\delta]{} & \mathsf{Mat}\,\mathsf{k}(x) \end{array} \tag{B.1}$$

Proof of Lemma 6.6. By Lemma 17 in [3], for every $H \in \mathbb{SV}_{\mathsf{k}(x)}[n,m]$ there exists a span $n \xleftarrow{V} k \xrightarrow{W} m$ in $\mathsf{Mat}\,\mathsf{k}[x]$ such that $\Phi(n \xleftarrow{V} k \xrightarrow{W} m) = H$, i.e.,

$$H = \{\, (\mathbf{u},\mathbf{v}) \mid \mathbf{u} \in \mathsf{k}(x)^n,\ \mathbf{v} \in \mathsf{k}(x)^m,\ \exists \mathbf{w} \in \mathsf{k}(x)^k.\ \delta(V)\mathbf{w} = \mathbf{u} \wedge \delta(W)\mathbf{w} = \mathbf{v} \,\}.$$

For $1 \le i \le k$, let $\mathbf{v}_i \in \mathsf{k}[x]^n$ and $\mathbf{w}_i \in \mathsf{k}[x]^m$ be the i-th column vectors of V and W, respectively. Then, $\{(\delta(\mathbf{v}_i), \delta(\mathbf{w}_i)) \mid 1 \le i \le k\}$ spans H.

Since $[\tilde{\cdot}]$ makes the rightmost front face of ($\boxed{\Phi}$) commute, it maps H into $\Phi' \circ \Theta(H)$ that is

$$\{\, (\mathbf{u},\mathbf{v}) \mid \mathbf{u} \in \mathsf{k}((x))^n,\ \mathbf{v} \in \mathsf{k}((x))^m,\ \exists \mathbf{w} \in \mathsf{k}((x))^k.\ \nu(\widehat{V})\mathbf{w} = \mathbf{u} \wedge \nu(\widehat{W})\mathbf{w} = \mathbf{v} \,\}$$

which, by (B.1), is

$$\{\, (\mathbf{u},\mathbf{v}) \mid \mathbf{u} \in \mathsf{k}((x))^n,\ \mathbf{v} \in \mathsf{k}((x))^m,\ \exists \mathbf{w} \in \mathsf{k}((x))^k.\ \widetilde{\delta(V)}\mathbf{w} = \mathbf{u} \wedge \widetilde{\delta(W)}\mathbf{w} = \mathbf{v} \,\}.$$

This space is spanned by $\{(\widetilde{\delta(\mathbf{v}_i)}, \widetilde{\delta(\mathbf{w}_i)}) \mid 1 \le i \le k\}$. This set is obtained by embedding via $\tilde{\cdot}$ the generators of H into $\mathsf{k}((x))$. Therefore $[\tilde{\cdot}](H) = [\widetilde{H}]$. $\square$

Proof of Proposition 7.1. The proof is by structural induction on c, following the inductive definition of SF in §2.2. If c is in $\overrightarrow{\mathsf{Circ}}$ then $[\![c]\!] = [(e_i, Ae_i)]_{i \le n}$ for some matrix $A \in \mathsf{Mat}\,\mathsf{k}[x][n,m]$ and clearly any (ordinary) polynomial is rational.

Inductively, suppose that $[\![c\colon n+1 \to m+1]\!]$ is $[(e_i, Ae_i)]_{i \le n+1}$ for some $A \in \mathsf{Mat}\,\mathsf{k}\langle x\rangle[n+1, m+1]$. We need to show that $\mathsf{Tr}(c)$ is $[(e_i, A'e_i)]_{i \le n}$ for some $A' \in \mathsf{Mat}\,\mathsf{k}\langle x\rangle[n,m]$.

For this purpose, suppose that $\sigma = \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_{n+1} \end{pmatrix}$ and $\tau = \begin{pmatrix} \tau_1 \\ \vdots \\ \tau_{m+1} \end{pmatrix}$ are $\mathsf{k}(x)$-vectors such that $A\sigma = \tau$. This means that

$$\begin{array}{rcl} \tau_1 & = & A_{1,1}\sigma_1 + A_{1,2}\sigma_2 + \cdots + A_{1,n+1}\sigma_{n+1} \\ \vdots & & \\ \tau_i & = & A_{i,1}\sigma_1 + A_{i,2}\sigma_2 + \cdots + A_{i,n+1}\sigma_{n+1} \\ \vdots & & \\ \tau_m & = & A_{m,1}\sigma_1 + A_{m,2}\sigma_2 + \cdots + A_{m+1,n+1}\sigma_{n+1} \end{array}$$

The semantics of $\mathsf{Tr}(c)$ is the subspace corresponding to the solution of the above system of equations plus

$$\sigma_1 = x \cdot \tau_1.$$

By replacing σ_1 with $x \cdot \tau_1$ in the first equation, one can deduce that $\tau_1(1 - A_{1,1} \cdot x) = \sum_{j=2}^{n+1} A_{1,j}\sigma_j$. Note that $1 - A_{1,1} \cdot x \neq 0$ since, by assumption, $A_{1,1} \neq \frac{1}{x}$. Therefore we can safely conclude that

$$\tau_1 = \sum_{j=2}^{n+1} \left(\frac{A_{1,j}}{1 - A_{1,1} \cdot x} \right) \sigma_j$$

We can now replace σ_1 by $x \cdot \sum_{j=2}^{n+1} \left(\frac{A_{1,j}}{1 - A_{1,1} \cdot x} \right) \sigma_j$ in the above system of equations and obtain

$$\tau_i = A_{i,1} x \cdot \sum_{j=2}^{n+1} \left(\frac{A_{1,j}}{1 - A_{1,1} \cdot x} \right) \sigma_j + \sum_{j=2}^{n+1} A_{i,j}\sigma_j$$

for all $2 \leq i \leq m+1$. We thus have m equations with n variables (namely σ_j for $2 \leq j \leq n+1$). These form a matrix A' with m columns and n rows. In order to conclude, we have to show that all the entries of this matrix are rationals.

Since $A_{1,1}$ is a rational we can write it as $\frac{p}{k+q \cdot x}$ for some polynomials p, q and scalar $k \neq 0$. So $1 - A_{1,1} \cdot x = \frac{k+(q-p)\cdot x}{k+q \cdot x}$ and $\frac{1}{1-A_{1,1}\cdot x} = \frac{k+q\cdot x}{k+(q-p)\cdot x}$ which is a rational since $k \neq 0$. Since rationals form a ring, i.e., they are closed under $+$ and $\cdot$, all the entries of A' are rationals.

The remaining inductive cases are the ones in which $c = c_1 \,;\, c_2$ and $c = c_1 \oplus c_2$ for circuits c_1, c_2 of SF. The statement is easily verified by functoriality of $[\![\cdot]\!]$ and definition of $\oplus$ and $\,;\,$ in $\mathbb{SV}_{\mathsf{k}(x)}$. $\square$

Proof of Lemma 8.5. We show a procedure similar to Gaussian elimination that, using elementary row operations, transforms $n \times m$ matrices to rational form.

First, we set all the entries of the first row to be polynomials $p_1, \ldots, p_m$ (simply by multiplying this row by the product of all denominators). Like for formal Laurent series, we define the *codegree* of a polynomial $k_0 + k_1 x + \cdots + k_z x^z$ to be the smallest $a_i \neq 0$; for instance $1+x$ has codegree 0 and $x+x^2$ has codegree 1. Amongst $p_1, \ldots, p_m$, we pick p_{κ_1} with minimal codegree and we multiply the first row by $\frac{1}{p_{\kappa_1}}$. In the resulting row, all the entries are rationals, since they are fractions $\frac{p_i}{p_{\kappa_1}}$ where the denominator has codegree smaller or equal than the nominator. Moreover in the κ_1-th position there is 1. We call κ_1 the *pivot* of the first row and this sub-procedure *rationalization* of a row.

Second, we bring to 0 all the entries below the first pivot. Like in Gaussian elimination, this can be done by simply adding to each row a scalar multiple of the first one. This second sub-procedure is the *downward substitution* of a pivot.

Rationalization and downward substitution can be iteratively applied to all the (non zero) rows in the matrix so to obtain a novel matrix where (a) all the entries are rationals, (b) each (non zero) row has a pivot with coefficient 1 and (c) all the entries *below* a pivot are 0. For instance, one can obtain a matrix as

the following.

$$\begin{pmatrix} r_1 & r_2 & 1 & r_3 \\ r_4 & 1 & 0 & r_5 \\ r_6 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

For having a matrix in rational form, we only need to transform our matrix such that all entries *above* the chosen pivot entries are 0. We start from the last (non zero) row, which we call s with pivot κ_s. Like for downward substitution, we can add to each row above s a scalar multiple of s, but we have to do it carefully, by checking that the resulting rows are in the good shape. Take a (non-zero) row j above s, and call $r_{j\kappa_s}$ the κ_s-th entry of such row. By virtue of (a), $r_{j\kappa_s}$ is a rational. By adding to the row j, the row s multiplied by $-r_{j\kappa_s}$ we obtain a new row where (d) the κ_s-th entry is 0; (e) the entry at the the pivot κ_j is 1 since, by (c), in s the κ_j-th entry is 0; (f) all the entries of the row are rationals, since they are obtained by additions and multiplications of rationals (and rationals form a ring). We can repeat this for all the pivots κ_u and for all the rows above u and we will eventually obtain a matrix where by (f) all the entries are rationals, each row has, by (e), a pivot with entry 1 and all the entries above and below a pivot are 0 by (c) and (d). $\square$