Caracciolo et al. Reply: As Patrasciou and Seiler [1] note, there are two very different limits that can be taken in a two-dimensional $\sigma$ model: (a) $\beta \to \infty$ at fixed $L < \infty$, or (b) $\beta \to \infty$ and $L \to \infty$ such that the ratio $x \equiv \xi(\beta, L)/L$ is held fixed. Limit (b) is the one relevant to finite-size scaling, while perturbation theory is clearly valid in limit (a). The deep question is whether the perturbation theory derived from the study of limit (a) is also correct in the double limit obtained by first taking limit (b) and then taking $x \to \infty$. The conventional wisdom says yes: indeed, this or a similar interchange of limits underlies the conventional derivations of asymptotic freedom. Patrasciou and Seiler say no: they suspect that asymptotic freedom is false [2]. At present, no rigorous proof is available to settle this question one way or the other.

Our analysis [3] of our Monte Carlo data is based on finite-size scaling [4–6], i.e., limit (b). Thus, at each fixed $x \equiv \xi(\beta, L)/L$, we ask whether the ratios $O(\beta, 2L)/O(\beta, L)$ have a good limit as $L \to \infty$, and we attempt to evaluate this limit numerically in the usual way: namely, we evaluate the ratios over a wide range of $L$ (from 32 to 256), and we ask whether these ratios appear to be converging to a limit as $L$ grows. We find, in fact, that the ratios are constant within error bars for $L \geq 64–128$ (depending on the value of $x$). Of course, it is conceivable that this apparent limiting value is a deception—i.e., a “false plateau”—and that at much larger values of $L$ the ratio will change dramatically. We acknowledge as much in the penultimate paragraph of our Letter. This caveat is not special to our work, but is inherent in any numerical work which attempts to evaluate a limit (here $L \to \infty$) by taking the relevant parameter almost to the limit (here $L$ large but finite).

In any case, there is no evidence that this perverse scenario in fact occurs. The corrections to scaling in our data are very weak—less than 2% even at $L = 32$, and a fraction of a percent or smaller for $L \geq 64–128$—and are perfectly consistent with a behavior of the form

$$O(\beta, 2L)/O(\beta, L) = F_O(x) + G_O(x)/L^\lambda + \cdots,$$

where the correction term $G_O$ is negative for $0.3 \leq x \leq 0.7$ and is perhaps slightly positive for $x \approx 0.7$. If all hell breaks loose for larger $L$—as the Patrasciou-Seiler scenario would require—we certainly see no hint of it at $L \leq 256$.

Patrasciou-Seiler also note that our Monte Carlo data at $x \approx 0.7$ agree well with the two-loop perturbative prediction, shown as a dotted curve in Fig. 2 of [3]. But this does not mean that we are assuming asymptotic scaling (whether explicitly or implicitly). Quite the contrary: our data at $x \approx 0.7$ constitute a (weak) test of asymptotic scaling. The same point ($\beta, L$) may well lie within the range of validity (to some given accuracy) of two distinct expansions. The fact that our data points at large $x$ are consistent with finite-volume perturbation theory [limit (a)] does not constitute evidence against their also being consistent with nonperturbative finite-size scaling [limit (b)].

Of course, since our Monte Carlo data for $F_O(x)$ at $x \approx 0.7$ do in fact agree closely with the two-loop perturbative formula (to within about 1%), and our data for $O(\beta, L)$ also agree well with the fixed-$L$ perturbation expansion (to within a few percent), it is then inevitable that our extrapolated values $\xi_\infty(\beta)$ at the largest values of $\beta$ will be consistent with asymptotic scaling, in the sense that $\xi_\infty(\beta)/(e^{2\pi^2\beta/(N-2)} \beta^{-1/(N-2)})$ will be roughly constant. However, it is by no means inevitable that this constant value will agree with the Hasenfratz-Maggiore-Niedermayer prediction to within 4%. It seems to us that this apparent coincidence is significant evidence in favor of the asymptotic-freedom picture.

Finally, Patrasciou and Seiler [7] have found an unusual boundary condition for which the $L \to \infty$ limit of the perturbative coefficients disagrees with those obtained from the same limit in periodic boundary conditions.

Received 27 November 1995

PACS numbers: 11.10.Hi, 05.70.Jk, 11.15.B+, 11.15.Ha