REFINEMENTS IN BOUNDARY COMPLEXES OF POLYTOPES

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Abstract

A complex $\mathcal{K}$ is said to be a refinement of a complex $\mathcal{L}$ if there exists a homeomorphism ψ: set $\mathcal{K} \longrightarrow$ set $\mathcal{L}$ such that for each face L of $\mathcal{L}$, $\psi^{-1}(L)$ is a union of faces of $\mathcal{K}$. A face K of $\mathcal{K}$ is said to be principal if $\psi(K)$ is a face of $\mathcal{L}$.

Some results concerning 3-polytopes are shown not to extend to higher dimensions. For $d \geq 4$, there exist simple d-polytopes with d+8 facets whose boundary complex cannot be expressed as a refinement of the boundary complex of a d-polytope with d+7 facets. For $d \geq 4$, there exist simple d-polytopes whose graphs do not contain refinements of the complete graph on d+1 vertices, if three particular vertices are preassigned as principal. A conjecture of Grunbaum is answered in the negative by constructing, for $d \geq 4$, simple d-polytopes P with d+4 facets in which two particular vertices may not be preassigned as principal if the boundary complex of P is expressed as a refinement of the boundary complex of a d-simplex; for $d \geq 6$, non-simple d-polytopes with d+3 facets having the same property are constructed.

The main positive result is that the boundary complex of a d-polytope with d+2 facets, (d+3 facets if $d = 4,5$), may be expressed as a refinement of the boundary complex of the d-simplex with any two preassigned vertices principal.

Several conjectures are made, among them the following generalization of Balinski's theorem on the d-connectedness of the graph of a d-polytope. If $d_1+\dots+d_k = d$, $d_i \in \mathbb{N}$, then between any two vertices of a d-polytope exist strong chains of d_i-faces, $i = 1,\dots,k$, disjoint except for the chosen vertices.

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Introduction

The combinatorial structure of 3-polytopes is quite well understood, thanks to Steinitz's theorem, which says that a graph is isomorphic to the graph of a 3-polytope if and only if it is planar and 3-connected. However, for $d \geq 4$, no comparable theorem exists; to characterize the boundary complexes of d-polytopes combinatorially would seem to be a very difficult task.

Even in four dimensions, we have no idea what the necessary and sufficient conditions analogous to "planar" and "3-connected" might be. In three dimensions, the combinatorial type of a polytope is determined by its graph. In higher dimensions, combinatorially distinct polytopes may have isomorphic graphs, and indeed, a polytope and a triangulation of the sphere not combinatorially isomorphic to a polytope may have isomorphic graphs (Altshuler and Steinberg (1973)). More than just the graph must be considered.

In order to find necessary conditions for d-polytopes, $d \geq 4$, we naturally seek to generalize properties of 3-polytopes. Let P be a 3-polytope, and x_1 and x_2 vertices of P. By the 3-connectedness of P, there exist three paths Γ_1, Γ_2 and Γ_3 in the graph G(P) of P from x_1 to x_2, pairwise disjoint except for their endpoints. We may assume that these paths are arcs. At least two of these arcs are not single edges, so we have vertices, say $x_3 \in \Gamma_1 \setminus \{x_1, x_2\}$, $x_4 \in \Gamma_2 \setminus \{x_1, x_2\}$. Since the graph of P is 3-connected, there exists an arc Γ_4 from x_3 to x_4 avoiding x_1 and x_2. The arc Γ_4 contains a subarc Γ_5 meeting Γ_1 in a single vertex x_5 and $\Gamma_2 \cup \Gamma_3$ in a single vertex x_6, where possibly $x_5 = x_3$ or $x_6 = x_4$. Suppose that $x_6 \in \Gamma_2$. Then the set $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_5$ is a homeomorphic image of the complete graph on four vertices K_4. The edges of K_4 correspond to arcs in Γ which are unions of edges of G(P). The vertices x_1, x_2, x_5, x_6 corresponding to vertices of K_4 under the homeomorphism are called principal vertices. We have just shown that in the graph of a 3-polytope, we may find a refinement of K_4 in which any two preassigned vertices are principal. Gallivan (1974) has shown that if three vertices are preassigned, a refinement of K_4 can be found in G(P) for which these three vertices are principal. In Chapter 5, we show that this result fails to generalize by finding d-polytopes, $d \geq 4$, for which at most two vertices may be preassigned in a refinement of the complete graph K_{d+1} on d+1 vertices.

We may attempt to generalize the 3-dimensional result in another way. It may be shown that given a refinement of K_4 in G(P), there exists a homeomorphism Ψ: bd P $\rightarrow$ bd T^3 from the boundary of P to the boundary of the 3-simplex T^3 such that the refinement of K_4 corresponds to $\Psi^{-1}(G(T^3))$. The map Ψ has the property that for any face F of T^3, $\Psi^{-1}(F)$ is a union of faces of P. This is the defining property of a refinement map. Gallivan's result states that three principal vertices may be preassigned if the boundary complex $\mathcal{B}(P)$ of P is expressed as a refinement of $\mathcal{B}(T^3)$. Grünbaum (1965; 1967, p. 200) has shown that for a d-polytope P, $\mathcal{B}(P)$ is a refinement of $\mathcal{B}(T^d)$ and that one vertex may be preassigned as principal. In view of the counterexample above, at most two vertices may be preassigned for $d \geq 4$. It is tempting to conjecture that two vertices may always be preassigned as principal, as this is a common generalization of Grünbaum's result and Balinski's theorem on the d-connectedness of d-polytopes. However, in Chapter 6, we describe d-polytopes for $d \geq 4$ for which not even two vertices may be preassigned as principal.

The first of these counterexamples was discovered in an attempt to prove the conjecture using the technique of facet splitting, which we now describe briefly. Given a 3-connected planar graph G embedded in the plane, we may draw an arc across one of the cells of G, dividing this cell into two cells, and yielding a new graph G'. It is clear that G' is planar, and not hard to show that G' is 3-connected. Therefore G and G' are isomorphic to graphs of 3-polytopes P and P' respectively. We say P' is constructed from P by facet splitting. The inverse operation is called deleting an edge of P'. The analogous operation in higher dimensions is deleting a (d-2)-face. It may be shown that every 3-connected planar graph except K_4 contains an edge which may be deleted to yield a 3-connected planar graph.

Thus, every 3-polytope may be constructed from a simplex by a sequence of facet splittings.

In higher dimensions, Barnette (1975) has remarked that the cyclic d-polytope with d+2 vertices may not be constructed by facet splitting. We might hope that simple polytopes, at least, might be constructed by facet splitting, but in Chapter 7 we construct a simple d-polytope which may not be so constructed.

We conclude that if these properties of 3-polytopes are to extend to higher-dimensional polytopes, a more subtle generalization is required.

Chapter 1. DEFINITIONS AND TERMINOLOGY

In this chapter we give some elementary algebraic and topological definitions and outline briefly the terminology of convex polytopes in two and three dimensions. Grünbaum's Convex Polytopes (1967) is our main general reference and provides a comprehensive introduction. McMullen and Shephard (1971) also develop the theory from first principles, sometimes from a geometric rather than an algebraic point of view. Proofs of most of the assertions we make in this chapter may be found in either of these books.

1. Basic definitions

Throughout we shall be dealing with d-dimensional real Euclidean space E^d, where d is a non-negative integer. The reals we denote by $\mathbb{R}$, the positive integers by $\mathbb{N}$. The scalar product of $x \in E^d$ with coordinates $(\alpha_1, \ldots, \alpha_d)$ and $y \in E^d$ with coordinates $(\beta_1, \ldots, \beta_d)$ is defined to be

$$\langle x, y\rangle = \alpha_1\beta_1 + \ldots + \alpha_d\beta_d.$$

The linear hull or linear span of $X \subseteq E^d$ is defined to be

$$\text{lin } X = \{\sum_{i=1}^{n} \lambda_i x_i \mid n \in \mathbb{N},\ x_i \in X,\ \lambda_i \in \mathbb{R}\}.$$

Sets of this form are the subspaces of E^d. A set $X \subseteq E^d$ is said to be in linearly general position if every subset of k points of X, $k \leq d$, spans a k-dimensional subspace of E^d.

The affine hull or affine span of $X \subseteq E^d$ is defined to be

$$\text{aff } X = \{\sum_{i=1}^{n} \lambda_i x_i \mid n \in \mathbb{N},\ x_i \in X,\ \lambda_i \in \mathbb{R},\ \sum_{i=1}^{n}\lambda_i = 1\}.$$

These sets are the affine subspaces or flats of the space E^d.

Each flat M is a translate of a unique subspace L. The dimension dim M of M is defined to be the dimension of L. For any set $X \subseteq E^d$, the dimension is defined to be dim X = dim aff X. Conventionally, dim $\emptyset = -1$. If X is a linear subspace or flat, X = aff X and our definitions are consistent. We shall often refer to an "object of dimension k" as a "k-object", e.g., k-flat.

A set $X \subseteq E^d$ is said to be in affinely general position if each subset of $k+1$ points, $k \leq d$, spans a k-flat affinely.

Two subspaces of E^d have the origin in common, but two flats may be disjoint. In general, if $M, L \subseteq E^d$ are flats, either

$$M \cap L = \emptyset \quad \text{or}$$

$$\dim M \cap L = \dim M + \dim L - d.$$

Two flats are said to be parallel if one lies in a translate of the other.

The positive hull of $X \subseteq E^d$ is defined to be

$$\text{pos } X = \Big\{ \sum_{i=1}^{n} \lambda_i x_i \;\Big|\; n \in \mathbb{N},\ x_i \in X,\ \lambda_i \in \mathbb{R},\ \lambda_i \geq 0 \Big\}.$$

We now come to convexity. The convex hull of $X \subseteq E^d$ is defined to be

$$\text{conv } X = \Big\{ \sum_{i=1}^{n} \lambda_i x_i \;\Big|\; n \in \mathbb{N},\ x_i \in X,\ \lambda_i \in \mathbb{R},\ \lambda_i \geq 0,\ \sum_{i=1}^{n} \lambda_i = 1 \Big\}.$$

We often denote the closed line segment conv $\{x_1, x_2\}$, $x_1, x_2 \in E^d$, by $[x_1, x_2]$, extending the usual notation for closed intervals in $\mathbb{R}$.

A set $C \subseteq E^d$ is said to be convex if $x_1, x_2 \in C$ implies $[x_1, x_2] \subseteq C$. The convex hull of X is the unique minimal convex set containing X. It is immediate from the definition that the intersection of convex sets is convex, and so we may also define conv X as the intersection of all convex sets containing X.

The convex hull of a finite set of points is called a convex polytope. For brevity we shall refer to convex polytopes simply as polytopes. The set of all d-polytopes in E^d will be written $\mathcal{P}^d$.

A 0-polytope is a single point. A 1-polytope is a line segment. The 2-polytopes are the convex polygons. Among the 3-polytopes are the Platonic solids - the tetrahedron, cube, octahedron, icosahedron and dodecahedron.

The most elementary d-polytope is the d-simplex T^d, defined to be the convex hull of an affinely independent set of $d+1$ points.

The topology we shall use is the usual topology of E^d as a metric space induced by the Euclidean metric

$$\rho(x,y) = \langle x-y,\ x-y\rangle^{1/2}.$$

With respect to this topology, the <u>closure</u>, <u>interior</u> and <u>boundary</u> of $X \subseteq E^d$ are denoted by $\mathrm{cl}\ X$, $\mathrm{int}\ X$ and $\mathrm{bd}\ X$. For any $\epsilon > 0$, $x \in E^d$, the (closed) <u>d-ball</u> with radius ϵ and centre x is

$$B_\epsilon(x) = \{y \in E^d \mid \rho(x,y) \leq \epsilon\};$$

the corresponding <u>open d-ball</u> is

$$\mathrm{int}\ B_\epsilon(x) = \{y \in E^d \mid \rho(x,y) < \epsilon\}$$

and the corresponding <u>(d-1)-sphere</u> is

$$S_\epsilon(x) = \{y \in E^d \mid \rho(x,y) = \epsilon\}.$$

A <u>d-cell</u> is any set homeomorphic to a d-ball. We will often be careless and refer to a "set homeomorphic to a (d-1)-sphere" simply as a (d-1)-sphere. The d-cell and (d-1)-sphere, considered as topological spaces, are denoted B^d and S^{d-1}. Note that the topological dimension of, for instance, a d-cell, need not be the same as the dimension of its affine hull.

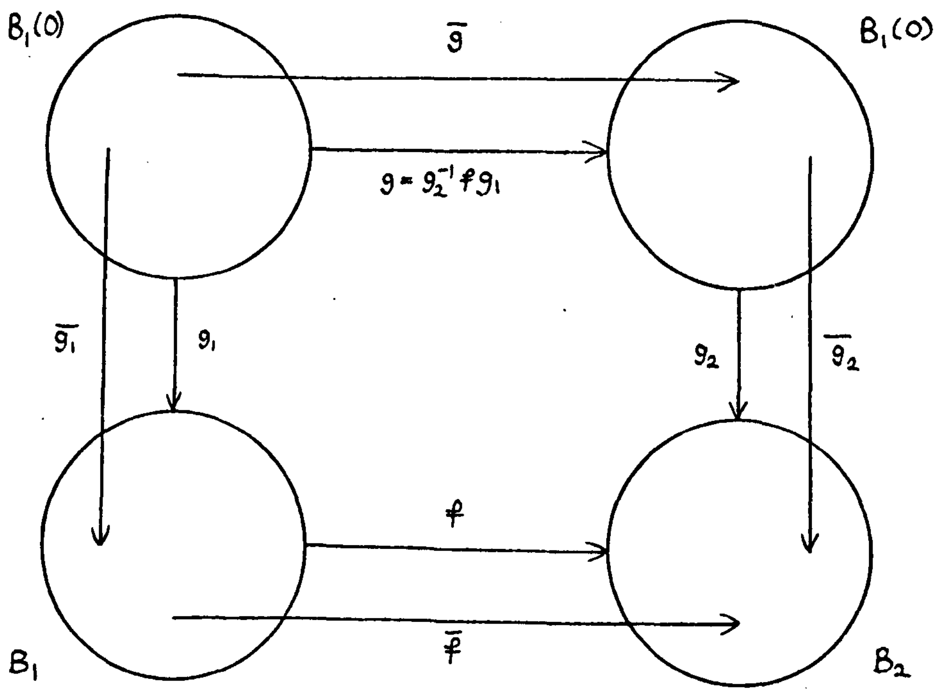
If $M \subseteq E^d$ is a k-flat, the <u>relative topology</u> on M is that induced by the topology on E^d, or, equivalently, that induced by the Euclidean metric. With respect to this topology, the <u>relative interior</u> and <u>relative boundary</u> of $X \subseteq M$ are written $\mathrm{relint}\ X$ and $\mathrm{relbd}\ X$.

The importance of the relative topology for the study of convex sets stems from the fact that if $C \subseteq E^d$ is a k-dimensional compact convex set, $0 \leq k \leq d$, then C is a k-cell. This implies that $\mathrm{relint}\ C \neq \emptyset$.

We will often be defining functions of the following sort. Let $C \subseteq E^d$ be compact, convex and d-dimensional and let $a \in \mathrm{int}\ C$. Let $\epsilon \geq 0$ be sufficiently small that $B_\epsilon(a) \subseteq \mathrm{int}\ C$. Then each $x \in S_\epsilon(a)$ determines a ray emanating from a, containing x and meeting $\mathrm{bd}\ C$ in a unique point $f(x)$. The function $F: S_\epsilon(a) \rightarrow \mathrm{bd}\ C$ is an example of <u>radial projection</u> with centre of projection a. In general, we may define a radial projection $f: X \to Y$ with centre y for any $X, Y \subseteq E^d$ such that $y \notin X$ and such that every ray R from y that meets X meets Y in a single point, which we define to be $f(R \cap X)$. Ordinary projection into a (d-1) flat of E^d may be thought of as a radial projection whose centre is a point at infinity. A radial projection is linear if and only if X and Y lie in parallel (d-1)-flats of E^d.

The radial projection $f: S_\epsilon(a) \to \text{bd } C$ is a homeomorphism between (d-1)-spheres. These spheres bound d-balls $B_\epsilon(a)$ and C respectively. The homeomorphism f extends to a homeomorphism $\bar{f}: B_\epsilon(a) \to C$ via the following construction - one we will use many times. Define $\bar{f}(x) = f(x)$ for $x \in S_\epsilon(a)$ and define $\bar{f}(a) = a$. For each point $z \in B_\epsilon(a) \setminus \{a\}$ there exist unique $\lambda \in (0,1]$, $x \in S_\epsilon(a)$ such that

$$z = (1-\lambda)a + \lambda x.$$



<u>Fig. 1.1</u>

Define

$$\overline{f}(z) = (1-\lambda)a + \lambda f(x).$$

Essentially we define $\overline{f}$ on the boundary and at one interior point and extend linearly in each ray emanating from the centre.

Let B_1 and B_2 be d-cells with boundary S_1 and S_2 respectively and suppose $f: S_1 \to S_2$ is a homeomorphism. We can use the idea of the previous paragraph to extend f to a homeomorphism of B_1 to B_2. Since B_i is a d-cell there exists a homeomorphism $\overline{g}_i: B_1(0) \to B_i$ and a homeomorphism $g_i: S_1(0) \to S_i$, where $g_i = \overline{g}_i|_{S_1(0)}$. Then $g = (g_2)^{-1} f g_1$ is a homeomorphism of $S_1(0)$ onto itself. Each point $z \in B_1(0)$ has a unique representation

$$z = (1-\lambda)0 + \lambda x = \lambda x, \quad \lambda \in (0,1], \; x \in S_1(0).$$

Hence g extends to a homeomorphism $\overline{g}: B_1(0) \to B_1(0)$ if we define

$$\overline{g}(z) = \lambda g(x).$$

Then $\overline{f} = \overline{g}_2 \overline{g} (\overline{g}_1)^{-1}$ is the desired homeomorphism of B_1 onto B_2 extending f.

2. <u>Support and separation</u>

We define a hyperplane H in E^d to be a set of the form

$$\{x \in E^d \mid \langle x,u \rangle = \alpha\},$$

for some $u \in E^d \setminus \{0\}, \alpha \in \mathbb{R}$.

The vector u is a <u>normal</u> to H. Any non-zero multiple of u is also a normal to H. In E^d the hyperplanes are precisely the (d-1)-flats, so in E^1, E^2 and E^3 the hyperplanes are the points, lines and planes respectively.

Each hyperplane H bounds two closed half-spaces

$$H^+ = \{x \in E^d \mid \langle x,u \rangle \geq a\}$$
$$H^- = \{x \in E^d \mid \langle x,u \rangle \leq a\}$$

and two open halfspaces

$$\text{int } H^+ = \{x \in E^d \mid \langle x,u \rangle > a\}$$

$$\text{int } H^- = \{x \in E^d \mid \langle x,u\rangle < a\}.$$

The vector u is an <u>outer normal</u> to H^- and an <u>inner normal</u> to H^+.

A hyperplane H is said to <u>support</u> a set C if $H \cap \text{cl } C \neq \emptyset$ and C lies in H^+ or H^-. If C is already closed, the first requirement becomes simply $H \cap C \neq \emptyset$ and H is said to support C in $C \cap H$. Among all closed subsets of E^d, convex sets may be characterized as those sets C such that for every $x \in \text{bd } C$, there exists a hyperplane supporting C and containing x.

Closely related to support is separation. Two sets X and Y in E^d are said to be <u>separated</u> by a hyperplane H if X lies in one of the closed halfspaces bounded by H and Y lies in the other, and <u>strictly separated</u> if X lies in one of the open halfspaces and Y lies in the other. It may be shown that a closed convex set and a disjoint compact convex set may be strictly separated. In particular, if C is compact and convex and $x \notin C$, there is a closed halfspace containing C but not x; hence, C is the intersection of the closed halfspaces containing C. Clearly C is also the intersection of the closed halfspaces bounded by supporting hyperplanes.

3. <u>Faces</u>

We can now define the important notion of "face". For general convex sets, "face" may have one of two meanings, but for polytopes the two meanings coincide and we need not concern ourselves with the difference. We use the following definition. The <u>faces</u> of a polytope $P \subseteq E^d$ are the sets of the form $H \cap P$, where H is a hyperplane supporting P, together with $\emptyset$ and P itself. The faces $\emptyset$ and P are called the <u>improper</u> faces; all other faces are <u>proper</u>. Note that the definition does not require $\dim P = d$.

Each face F of a polytope is again a polytope. If M is a flat containing F, we may apply the definition of face in the flat M rather than E^d, to find that the faces of F are precisely the faces of P contained in F. This means that the facial structure of P is intrinsic to P and not dependent on the dimension of the space containing P. Another way of saying this is that a face of a face is again a face.

If F_1 and F_2 are faces of P, then $F_1 \cap F_2$ is a face of P.

A face of dimension 0, 1 or d-1 of a d-polytope P is called a <u>vertex</u>, <u>edge</u>, or <u>facet</u> of P, respectively. The set of vertices of P is denoted by vert P. It may be shown that a polytope is the convex hull of its vertices.

Recall that a d-simplex T^d is the convex hull of d+1 affinely independent points, say $X = \{x_1,\dots,x_{d+1}\}$. Each point x_i is a vertex of T^d, and furthermore for each $Y \subseteq X$, conv Y is a face of T^d. Since Y is also affinely independent, each face of T^d is a simplex.

An important special class of d-polytopes is the class of <u>simplicial</u> d-polytopes, consisting of those d-polytopes all of whose facets are simplices. Equivalently, since every face of a simplex is again a simplex, all the proper faces are simplices. Since any set of d points spanning a (d-1)-flat must be affinely independent, another equivalent statement is that all the facets have exactly d vertices.

The set of all faces of P is denoted by $\mathcal{F}(p)$; the set of all faces other than P is denoted by $\mathcal{B}(p)$, the <u>boundary complex</u> of P. It is clear that $P = \cup\mathcal{F}(P)$, and it may be shown that relbd $P = \cup\mathcal{B}(P)$.

It may also be shown that for each $x \in P$, there exists a unique face $F \in \mathcal{F}(P)$ such that $x \in$ relint F. The face F has the property that if $x \in \text{bd}\, P$, H supports P and $x \in H$, then $F \subseteq H$. Of course, if $x \in \text{relint}\, P$, then $F = P$.

We define the <u>k-skeleton</u> of P to be

$$\text{skel}_k P = \{F \in \mathcal{C}(P) \mid \dim F \leq k\}.$$

Thus the 0-skeleton consists of vert $P \cup \{\emptyset\}$, and the 1-skeleton consists of the edges and vertices of P (together with $\emptyset$). We shall usually call the 1-skeleton the <u>graph</u> of P, denoted $G(P)$.

A <u>graph</u> G may be defined abstractly, as a set $\mathcal{V}$, the set of <u>vertices</u> of G, together with a set

$$\mathcal{E} = \{\{v_i, v_j\} \mid v_i, v_j \in \mathcal{V},\ v_i \neq v_j\}$$

called the set of <u>edges</u> of G. Although an edge of a graph defined in this way is not the same as an edge of a polytope, as an edge of a polytope is the convex hull of its two vertices, the difference is not essential and hence we will not make a distinction.

We remarked above that a face of a face is a face. This fact implies that the relation "is a face of" is transitive and hence $\mathcal{C}(P)$ is partially ordered by this relation. This partial ordering is the same as that induced on $\mathcal{C}(P)$ by set inclusion. Also, if $F_1, F_2 \in \mathcal{C}(P)$, then $F_1 \cap F_2 \in \mathcal{C}(P)$ and is the unique largest face of P contained in F_1 and F_2. We shall see in a moment that there is a unique smallest face containing F_1 and F_2. In other words, with respect to the partial order induced by inclusion, each pair of faces has a greatest lower bound and a least upper bound. A partially ordered set with this property is a <u>complete lattice</u>. The set $\mathcal{C}(P)$ endowed with this natural lattice structure is called the <u>facial lattice</u> of P.

Two polytopes P and Q are said to be <u>combinatorially isomorphic</u> or <u>of the same combinatorial type</u> if $\mathcal{C}(P)$ and $\mathcal{C}(Q)$ are isomorphic as lattices - that is, there is a bijection $\phi: \mathcal{C}(P) \rightarrow \mathcal{C}(Q)$ such that for $F_1, F_2 \in \mathcal{C}(P)$, $F_1 \subseteq F_2$ if and only if $\phi(F_1) \subseteq \phi(F_2)$. We write $P \approx Q$. This relation is easily seen to be an equivalence relation and hence we may speak of the <u>combinatorial equivalence class</u> or <u>combinatorial isomorphism class</u> of P. It is not hard to see that the combinatorial type of P is determined by the subsets of vert P which are of the form vert F, where F is a facet of P.

Combinatorial isomorphism is central to our enquiry, since our results will depend only upon properties of the facial lattice and not upon the metric structure.

A set $\{F_0, \ldots, F_{d-1}\}$ of proper faces of $P \in \mathcal{P}^d$ such that F_i is an i-face and $F_0 \subseteq \ldots \subseteq F_{d-1}$ is called a <u>tower of faces</u>. Any set $\{G_1, \ldots, G_k\}$ of proper faces of P such that $G_1 \subsetneq \ldots \subsetneq G_k$ may be extended to a tower of faces.

Let x be a vertex of $P \in \mathcal{P}^d$, and let H be a hyperplane strictly separating x from the other vertices of P. Then $H \cap P$ is a (d-1)-polytope which we call a <u>vertex figure</u> of P at x and which we denote by P/x. The k-faces of $H \cap P$ are precisely the sets of the form $F \cap H$ where F is a (k+1)-face of P and $x \in F$. Thus the facial lattice of P/x is isomorphic to the sublattice of $\mathcal{T}(P)$

$$\{F \in \mathcal{T}(P) \mid x \in F \subseteq P\}.$$

More generally, it may be shown (McMullen and Shephard (1971)), that if F_1, F_2 are faces of P, $F_1 \subseteq F_2$, then

$$\{F \in \mathcal{T}(P) \mid F_1 \subseteq F \subseteq F_2\}$$

is the facial lattice of a polytope, denoted F_2/F_1.

4. <u>Duality</u>

If $\mathcal{L}$ is a lattice, the <u>dual lattice</u> $\mathcal{L}^*$ is defined by reversing the partial order as follows. Let $f: \mathcal{L} \to \mathcal{L}^*$ be a bijection. The partial order on $\mathcal{L}^*$ is defined by $f(a) \leq f(b)$ if and only if $b \leq a$. Observe that $\mathcal{L}$ and $\mathcal{L}^{**}$ are isomorphic lattices.

Polytopes P and Q are said to be <u>dual</u> if their facial lattices are dual. It is immediate that the polytopes dual to P form a combinatorial equivalence class. For example, the cube and regular octahedron are dual, as are the icosahedron and dodecahedron. The tetrahedron is dual to itself; a polygon with n sides is also dual to itself.

Every polytope has a dual. To construct a polytope dual to $P \in \mathcal{P}^d$, assume $0 \in \operatorname{int} P$. Then the <u>polar set</u> P^* of P is defined to be

$$P^* = \{x \in E^d \mid \langle x,y\rangle \leqslant 1,\ y \in P\}.$$

It may be shown that P^* is dual to P.

The face of P^* corresponding to the proper face F of P under duality is the face

$$\hat{F} = \{x \in P^* \mid \langle x,y\rangle = 1,\ y \in F\}.$$

For the improper faces P and $\emptyset$, $\hat{P} = \emptyset$ and $\hat{\emptyset} = P^*$. It may be shown that the faces of P^* are precisely the sets $\hat{F}$ and that $\hat{F}_1 \subseteq \hat{F}_2$ if and only if $F_2 \subseteq F_1$, which of course is exactly what the definition of duality requires.

Under duality, a k-face corresponds to a (d-k-1)-face. In particular, a vertex of P corresponds to a facet of P^* and vice versa.

Corresponding to the vertex $y_1 \in P$ is the facet of P^*

$$\hat{y}_1 = \{x \in P^* \mid \langle x,y_1\rangle = 1\}.$$

It follows easily from the definitions of duality and of vertex figure that $\hat{y}_1$ is a (d-1)-polytope dual to P/y_1. In general, for any proper face F of P, $P/F \approx \hat{F}^*$.

Therefore in the special case where P is simplicial, P^* will be a polytope whose vertex figures are simplices. Such a polytope is called a <u>simple</u> polytope. At each vertex x of P^*, the vertex figure is a (d-1)-simplex with d facets. There is a bijection between facets of the vertex figure and facets of P^* meeting x, so precisely d facets of P^* contain x. Another definition of simple d-polytope is the requirement that each vertex lie in exactly d facets or, equivalently, for $k = 0,\ldots,d-1$, that each k-face lie in exactly $d-k$ facets. This statement is merely the dual of the definition of a simplicial d-polytope P as a polytope in which each k-face other than P itself contains exactly $k+1$ vertices. It is clear that every face of a simple polytope is simple.

A <u>simple vertex</u> of a d-polytope is defined to be a vertex contained in exactly d facets.

We defined a polytope as the convex hull of a finite set of points and found that a polytope was in fact the convex hull of a particular minimal set of points, namely its vertices. From the polar construction we may deduce that if vert $P = \{y_1,\ldots,y_n\}$, then

$$P^* = \{x \in E^d \mid \langle x,y_i\rangle \leq 1,\ i = 1,\ldots,n\}.$$

So, the expression of P as the convex hull of n vertices leads to an expression of P^* as an intersection of n closed halfspaces. The bounding hyperplanes have equation

$$\langle x,y_i\rangle = 1,\ i = 1,\ldots,n.$$

But the facet $\hat{y}_i$ of P^* lies in, and in fact spans, the hyperplane with equation $\langle x,y_i\rangle = 1$.

Since $P = (P^*)^*$, any polytope may be regarded as a polar set, so we may choose to regard P as the convex hull of a finite set of points, its vertices, or as the intersection of a finite set of closed halfspaces, determined by the facets of P. An expression of a non-empty face F of P as the convex hull of the vertices contained in F corresponds under duality to an expression of $\hat{F}$ as the intersection of facets containing $\hat{F}$. Hence in a given polytope, we may regard the faces as convex hulls of vertices or as intersections of facets.

We will often find it more convenient or more intuitive to prove certain results by considering the dual of a polytope rather than the polytope itself.

5. Schlegel diagrams

A Schlegel diagram of $P \in \mathcal{P}^d$ is a method of representing P in E^{d-1}, defined as follows. Let $F_1, \ldots, F_n$ be the facets of P. Then P may be represented as

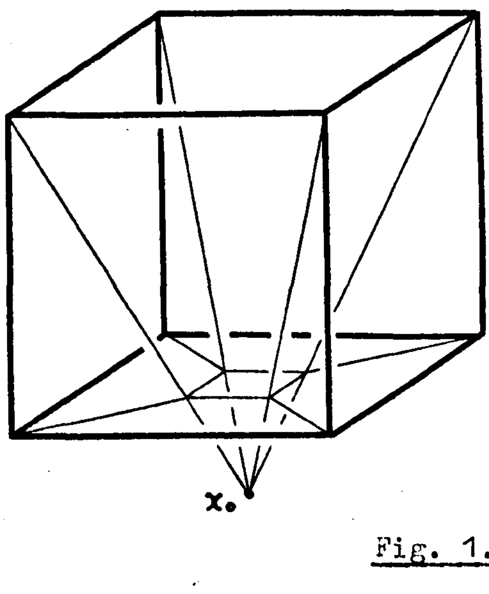
$$P = H_1^- \cap \ldots \cap H_n^-$$

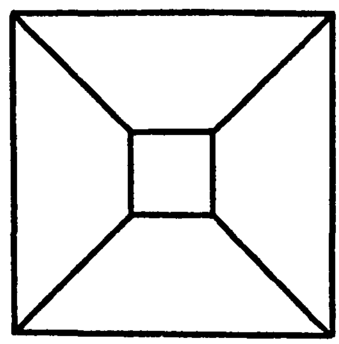
where H_i^- is a closed halfspace containing F_i in its boundary.

Let

$$x_0 \in \operatorname{int}(H_2^- \cap \ldots \cap H_n^- \setminus P).$$

Let $S = F_2 \cup \ldots \cup F_n$. Radial projection $f: S \to F$, with centre





Schlegel diagram

Fig. 1.2

x_0, is a homeomorphism of S onto F_1. The set

$$\{f(F) \mid F \in \mathcal{B}(P) \setminus F_1\}$$

is a Schlegel diagram of P, based on F_1. (See Fig. 1.2.) The sets $f(F_2), \ldots, f(F_n)$ are (d-1)-polytopes packed together to form $f(F_1)$. We will be using Schlegel diagrams as "pictures" of 3- and 4-dimensional polytopes; that is, as visual aids rather than theoretical tools.

6. Transformations

Here we discuss some transformations on E^d which preserve the combinatorial type of polytopes in $\mathcal{P}^d$. We will regard $x \in E^d$ as a row vector - that is, a $1 \times d$ matrix.

A <u>linear transformation</u> $f: E^d \to E^d$ may be written

$$f(x) = xA$$

where A is a $d \times d$ matrix. If A is a non-singular matrix, f is a non-singular linear transformation. In this case, if $P \in \mathcal{P}^d$ and H supports P in F, it is clear that $f(H)$ is a hyperplane supporting $f(P)$ in $f(F)$. Hence $P \approx f(P)$. If $X, Y \subseteq E^d$, and there exists a non-singular linear transformation f such that $Y = f(X)$, X and Y are said to be <u>linearly equivalent</u>.

Similarly, an <u>affine transformation</u> $f: E^d \to E^d$ may be written

$$f(x) = xA + b,$$

where A is a $d \times d$ matrix and $b \in E^d$. If $b = 0$, f is linear. The transformation is non-singular if A is non-singular. <u>Affine equivalence</u> of sets of E^d is defined in analogy to linear equivalence and again it is clear that affinely equivalent polytopes are combinatorially equivalent.

A <u>projective transformation</u> f of E^d is defined by

$$f(x) = \frac{xA + b}{\langle x, c\rangle + \delta}$$

where A is a $d \times d$ matrix, $b, c \in E^d$ and $\delta \in \mathbb{R}$ such that $(c, \delta) \neq (0, 0)$.

If $c = 0$, f is affine. Otherwise, f is not defined for the hyperplane

$$H = \{x \in E^d \mid \langle x, c\rangle + \delta = 0\},$$

and the domain of f is $E^d \setminus H$. We say that H is the <u>hyperplane sent to infinity</u> by f. If $X \subseteq E^d$, f is said to be <u>permissible</u> for X if $H \cap X = \emptyset$. If the $(d+1) \times (d+1)$ matrix $\begin{pmatrix} A & c^T \\ b & \delta \end{pmatrix}$ (where c^T

denotes the transpose of c) is non-singular, f is non-singular. In this case, f^{-1} exists and is also a projective transformation. Two subsets $X, Y \subseteq E^d$ are said to be <u>projectively equivalent</u> if there exists a non-singular projective transformation f permissible for X such that $f(X) = Y$.

Using the model of projective d-space in which points of the projective space correspond to lines in E^{d+1} containing the origin, it may be shown that a non-singular projective transformation f maps k-flats onto k-flats. Hence, if $P \in \mathcal{P}^d$, H is a hyperplane supporting P in F and f is a non-singular projective transformation of E^d permissible for P, then f(H) supports f(P) in f(F) and so $P \approx f(P)$.

Linear and affine transformations preserve parallelism; projective transformations do not. For instance, let $H_1, H_2 \subseteq E^d$ be hyperplanes. If $H_1 \cap H_2 = \emptyset$, H_1 and H_2 are parallel; if $H_1 \cap H_2 \neq \emptyset$, $H_1 \cap H_2$ is a (d-2)-flat. In the latter case, let H be a hyperplane containing $H_1 \cap H_2$ but not H_1 or H_2. Then if g is a non-singular projective transformation taking H to infinity, $g(H_1)$ and $g(H_2)$ are parallel hyperplanes.

7. <u>Gale diagrams</u>

Let X be an ordered set of n points spanning E^d affinely. We write X as an n-tuple $(x_1, \ldots, x_n)$. A Gale transform of X is an n-tuple $\overline{X} = (\overline{x}_1, \ldots, \overline{x}_n)$ of points of E^{n-d-1} which is defined in such a way that certain properties of X correspond to certain other properties of $\overline{X}$. In particular, if X is the set of vertices of $P \in \mathcal{P}^d$, the combinatorial type of P may be determined from $\overline{X}$. If P has a small number of vertices, the Gale transform lies in a low-dimensional space, and hence it may be very much easier to describe P indirectly through a Gale transform rather than directly.

We first define the Gale transform, or affine transform, of X. An affine dependence of X is an n-tuple $(\alpha_1,\ldots,\alpha_n) \in E^n$ such that

$$\alpha_1 x_1 + \ldots + \alpha_n x_n = 0,$$
$$\alpha_1 + \ldots + \alpha_n = 0.$$

The affine dependences of X form a linear subspace A of E^n. Since $\operatorname{aff} X = E^d$, $\dim A = n-d-1$.

Let

$$\{(\alpha_{i1},\ldots,\alpha_{in}) \mid i = 1,\ldots,n-d-1\}$$

be a linear basis for A and define for $j = 1,\ldots,n$,

$$\overline{x}_j = (\alpha_{1j},\ldots,\alpha_{(n-d-1)j})$$

Then the Gale transform (or affine transform) of X is defined to be the ordered set

$$\overline{X} = (\overline{x}_1,\ldots,\overline{x}_n).$$

Since the rank of the matrix (α_{ij}) is $n-d-1$, $\operatorname{lin} \overline{X} = E^{n-d-1}$.

To summarize what we have done, let B be the $n \times n$ matrix

$$\begin{pmatrix} x_1 & 1 & \overline{x}_1 \\ \vdots & \vdots & \vdots \\ x_n & 1 & \overline{x}_n \end{pmatrix}.$$

The matrix B is non-singular and has the property that each of the first $d+1$ columns is orthogonal to each of the last $n-d-1$ columns. For any matrix with the above form and these two properties, X and $\overline{X}$ stand in the relation of set and Gale transform.

From the matrix formulation follow immediately some properties of the transform. Firstly, the centroid of $\overline{X}$ - the point $\frac{1}{n}(\overline{x}_1 + \ldots + \overline{x}_n)$ - is necessarily 0. If the centroid of X is 0, the relations between X and $\overline{X}$ are symmetrical and hence X is a Gale transform of $\overline{X}$. In any case, $\overline{X}$ is a transform of any subset of E^d affinely equivalent to X, and any subset of E^{n-d-1} linearly equivalent to $\overline{X}$ is a transform of X.

We now give without proof properties of the transform relevant to polytopes. A point $x_i \in X$ lies in $\operatorname{conv}(X \setminus \{x_i\})$ if and only if there exists an open halfspace in E^{n-d-1} with 0 on its boundary and containing exactly one point of $\overline{X}$, the point $\overline{x}_i$. Since $X = \operatorname{vert} P$ for $P \in \mathcal{P}^d$ if and only if no point of X is in the convex hull of the remaining points, a necessary and sufficient condition for $X = \operatorname{vert} P$ is that every open halfspace with 0 on its boundary contain at least two points of $\overline{X}$. We call this the <u>diagram condition</u> (for polytopes).

For us the most important property of transforms is the following, characterizing the faces of a polytope $P \in \mathcal{P}^d$ such that $X = \operatorname{vert} P$ in terms of the transform $\overline{X}$. For $Y \subseteq X$, write

$$\overline{Y} = \{\overline{x}_i \in \overline{X} \mid x_i \in Y\}.$$

A <u>coface</u> of P is a subset of its vertices of the form

$$Y = \{x_i \in X \mid x_i \notin F\},$$

where $F \in \mathcal{B}(P)$. Then Y is a coface of P if and only if $0 \in \operatorname{relint} \operatorname{conv} \overline{Y}$.

This characterization enables us to determine the combinatorial type of P from a transform of vert P.

If $\overline{X} = \{\overline{x}_1, \ldots, \overline{x}_n\} \subseteq E^{n-d-1}$ is a transform of vert P, and $\overline{X}' = \{\overline{x}_1', \ldots, \overline{x}_n'\} \subseteq E^{n-d-1}$ has the property that for all $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$,

$$0 \in \operatorname{relint} \operatorname{conv}\{\overline{x}_{i_1}', \ldots, \overline{x}_{i_k}'\},$$

if and only if

$$0 \in \operatorname{relint} \operatorname{conv}\{\overline{x}_{i_1}, \ldots, \overline{x}_{i_k}\},$$

then $\overline{X}'$ is said to be <u>isomorphic</u> to $\overline{X}$. Any set isomorphic to $\overline{X}$ is called a <u>Gale diagram</u> of P. Any pair of Gale diagrams isomorphic to the same transform are themselves isomorphic.

It is immediate from these definitions that given two polytopes and a Gale diagram for each, the polytopes are of the same combinatorial type if and only if their Gale diagrams are isomorphic.

If $\overline{X}' = \{\overline{x}_1', \ldots, \overline{x}_n'\}$ is a Gale diagram, it is easily seen that if $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, $\alpha_1, \ldots, \alpha_n > 0$, then $\overline{X}'' = \{\alpha_1\overline{x}_1', \ldots, \alpha_n\overline{x}_n'\}$ is isomorphic to $\overline{X}'$.

The scalars α_i may be chosen so that all non-zero points of $\overline{X}''$ lie at distance 1 from 0. In other words, we may choose $\overline{X}''$ to consist of a possibly empty set of points at 0 and a subset of the unit sphere $S_1(0)$. A set derived from a Gale transform in this way is called a standard Gale diagram, usually denoted $\overline{X}$. Some polytopes and their Gale diagrams appear in Fig. 1.3. All of these diagrams except that of the triangular bipyramid have centroid 0 and therefore are in fact affine transforms.

A Gale diagram $\overline{X}'$ need not have centroid 0, but since $\overline{X}'$ is isomorphic to a transform, $0 \in \operatorname{relint} \operatorname{conv} \overline{X}'$. It is not hard to see that suitable choice of $\alpha_1, \ldots, \alpha_n$ makes the centroid of $\overline{X}''$ 0, and hence $\overline{X}''$ is a transform.

Conventionally, 2-dimensional Gale diagrams, representing d-polytopes with d+3 vertices are drawn as in Fig. 1.3. If we do not label the points of the diagram, we indicate the number of points coincident with each end of a diameter or with 0, if that number - the multiplicity - is greater than 1.

A d-simplex has d+1 vertices and hence its Gale diagram lies in E^0 and must therefore consist of d+1 points coincident with 0.

Let Y be the coface of a face F of P. The dimension of F may be determined from $\overline{X}$ (either diagram or transform) by the following formula:

$$\dim F = d + \dim \overline{Y} - \operatorname{card} \overline{Y},$$

where $\operatorname{card} \overline{Y}$ denotes the number of elements in $\overline{Y}$. In particular, F is a facet if and only if

$$\dim \overline{Y} = \operatorname{card} \overline{Y} - 1,$$

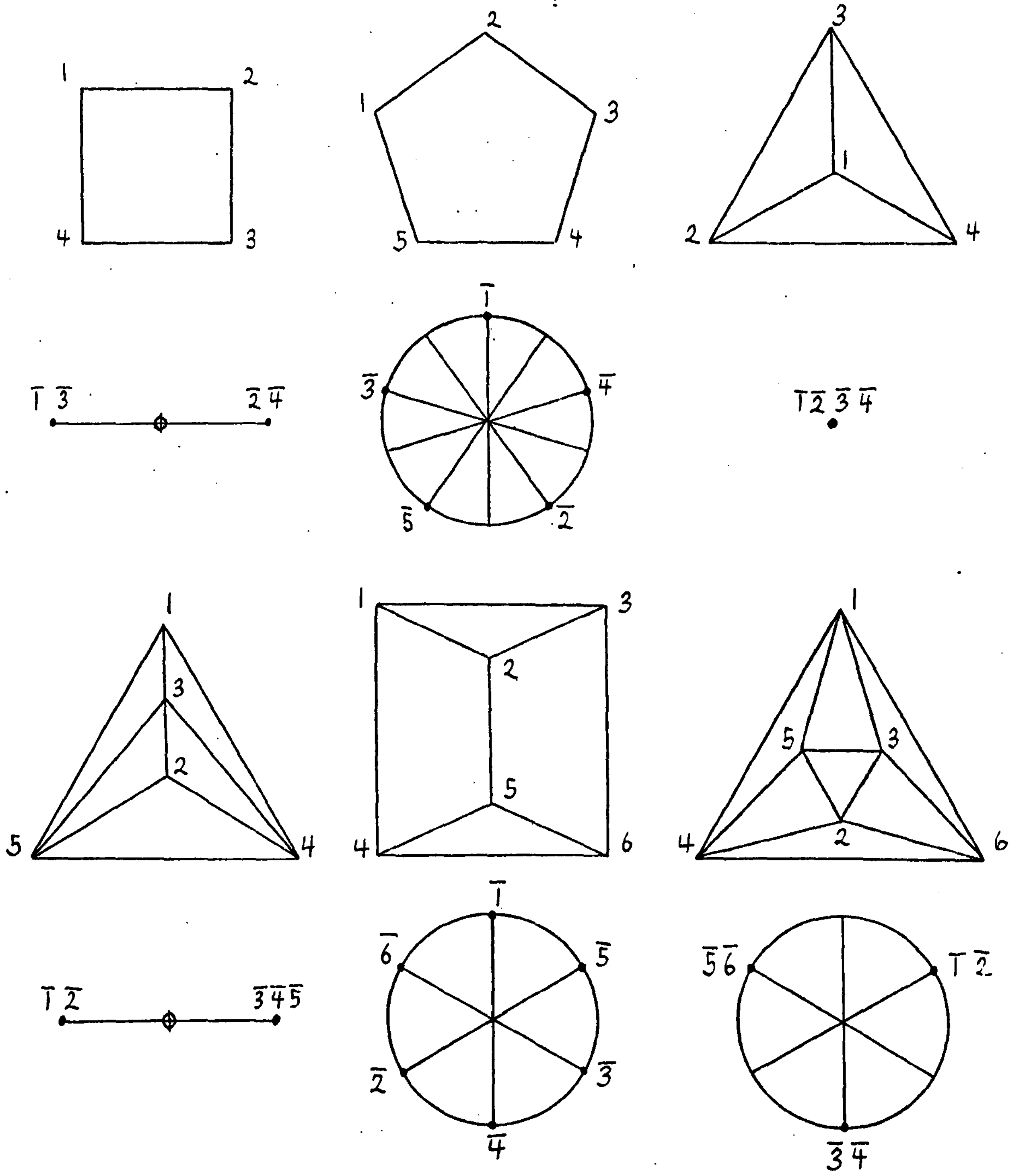


Fig. 1.3

which is true if and only if $\overline{Y}$ is the set of vertices of a non-degenerate simplex with O in its relative interior.

The simplicial faces of P may be characterized using the above formula, although we omit the proof. The face F is simplicial if and only if for every hyperplane in E^{n-d-1} through O, $O \notin \text{relint conv}\,(\overline{Y} \cap H)$.

Suppose $\overline{X}$ fails to satisfy the diagram condition for polytopes. Then conv X will be a polytope in $\mathcal{P}^d$, but some points of X will not be vertices of P. We may eliminate these <u>redundant</u> points from the diagram by the following procedure. Suppose $x_i \in \operatorname{conv}(X\setminus\{x_i\})$. Then as we have seen there is an open halfspace of E^{n-d-1} with 0 on its boundary meeting $\overline{X}$ in the single point $\overline{x}_i$. Let H be a hyperplane orthogonal to $\operatorname{lin} \overline{x}_i$ and let π:aff $X \rightarrow H$ be orthogonal projection onto H. Then $\pi(\overline{X}\setminus\{\overline{x}_i\})$ is a Gale diagram in H, but not necessarily a transform. Repetition of this procedure eventually eliminates all redundant points and yields a diagram of P.

If $\overline{X}$ is the Gale diagram of $P\in\mathcal{P}^d$, a diagram of the vertex figure P/x_i is constructed by removing $\overline{x}_i$ from $\overline{X}$ and eliminating redundant points as described in the previous paragraph. Redundant points correspond to the vertices of P not joined by an edge to x_i.

The combinatorial type of a polytope P^* dual to P may also be determined from the Gale diagram. Under duality, to each vertex x_i of P corresponds a facet F_i of P^*, and to the set of all vertices contained in a given face F of P corresponds the set of all facets of P^* containing the face $\hat{F}$ which corresponds to F. Hence if

$$F = \operatorname{conv}\{x_i \in X \mid i \in I\},$$

then

$$\hat{F} = \bigcap\{F_i \mid i \in I\}.$$

For each $\hat{F}\in \mathcal{B}(P^*)$ we define the <u>complete set of facets</u> associated with $\hat{F}$ to be $\{F_i \mid \hat{F} \subseteq F_i\}$. (It may happen that $\hat{F}$ is the intersection of some proper subset of the complete set, but we will not concern ourselves with such partial sets.)

Corresponding to X, let X^* be the ordered set $(F_1,\ldots,F_n)$ and let $\overline{X}^* = (\overline{F}_1,\ldots,\overline{F}_n)$, where $\overline{x}_i = \overline{F}_i$. We will call $\overline{X}^*$ a <u>dual Gale diagram</u> for P^*. A <u>coface</u> of a face F of P will be, in the

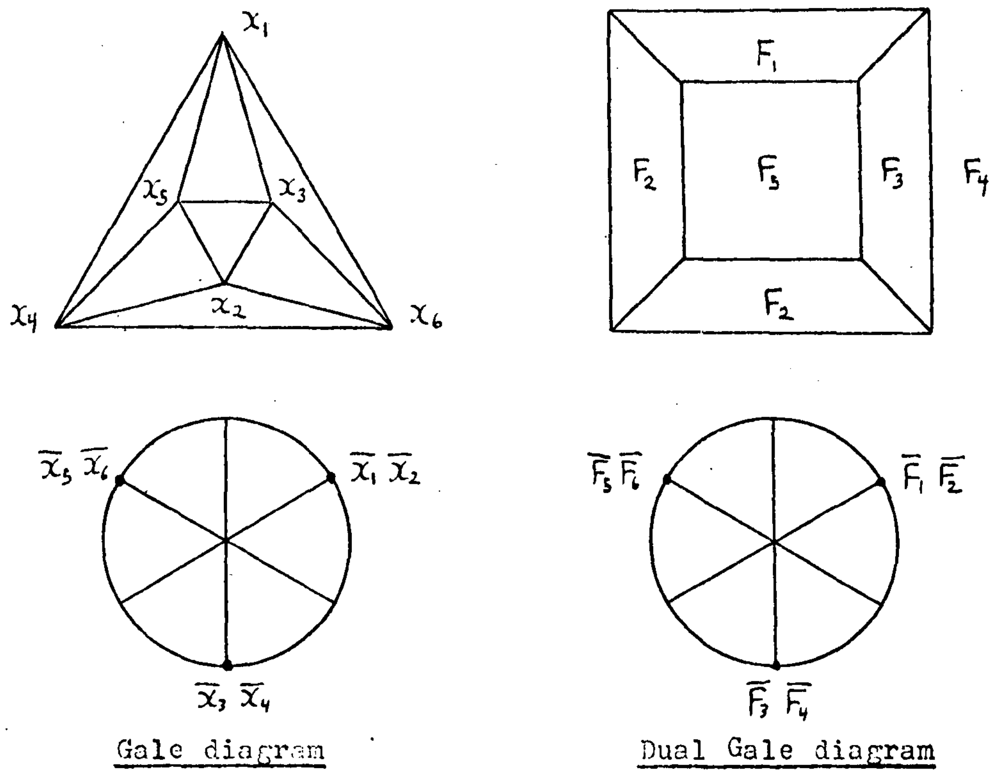


Fig. 1.4

dual context, the set of facets of P^* not containing $\hat{F}$. A <u>redundant</u> facet for $\hat{F}$ is a facet meeting $\hat{F}$ in a face which is not a facet of $\hat{F}$.

The characterization of the faces of P^* in terms of the dual Gale diagram is that Y is the coface of a face of P if and only if $0 \in \operatorname{relint} \operatorname{conv} \overline{Y}$. See Fig. 1.4 for an illustration.

Under duality, the vertex figure P/x_i corresponds to the facet F_i. The construction of a dual diagram for F_i is particularly easy from the dual diagram of P^*. Simply remove the point $\overline{F}_i$ from $\overline{X}^*$, and eliminate redundant points from $\overline{X}^* \setminus \{F_i\}$. The resulting set will be a dual diagram of F_i.

For dual diagrams, we have the followeng formula:

$$\dim \hat{F} = \operatorname{card} \overline{Y} - \dim \overline{Y} - 1,$$

where **Y** is the coface of a face $\hat{F}$.

Dual diagrams make it easy to construct polytopes which have a face of a certain combinatorial type. For, if

$\overline{X}^* = (\overline{F}_1,\ldots,\overline{F}_n) \subseteq E^{n-d-1}$ is a dual diagram of a d-polytope P^* with n facets, then adding another point $\overline{F}_{n+1} \in E^{n-d-1}$ yields the set $\overline{Z}^* = (\overline{F}_1,\ldots,\overline{F}_{n+1})$, which clearly satisfies the diagram condition and is therefore the dual diagram of a (d+1)-polytope Q^* with n+1 facets. By the previous paragraph, P^* is a facet of Q^*. If we add k new points to $\overline{X}^*$, the result will be a dual Gale diagram of a (d+k)-polytope with n+k facets containing P^* as a d-face.

A full treatment of dual diagrams appears in McMullen (1973).

Chapter 2. CONSTRUCTION OF POLYTOPES

In this chapter we describe several methods of constructing d-polytopes. Some of these methods use lower-dimensional polytopes as building-blocks; others - such as the "beneath-beyond" construction - are essentially d-dimensional.

1. Cyclic polytope

This is a most important example which we shall frequently be seeing.

The moment curve M in E^d is defined parametrically by

$$f(t) = (t, t^2, \ldots, t^d), \quad t \in \mathbb{R}.$$

The cyclic polytope $C(n,d)$ is the convex hull of n distinct points of M. The fact that no hyperplane meets M in more than d points implies that $C(n,d)$ is simplicial.

Supposing the vertices of $C(n,d)$ to be

$$X = \{f(t_1), \ldots, f(t_n)\}, \quad t_1 < \ldots < t_n,$$

the facets are determined by Gale's evenness condition: the sets

$$Z = \{f(t_{i_1}), \ldots, f(t_{i_d})\}$$

which determine facets of P are precisely those sets of d points such that if $f(t_{k_1}), f(t_{k_2}) \notin Z$, $k_1 < k_2$, then there are an even number of $\{i_1, \ldots, i_d\}$ between k_1 and k_2. This criterion implies that every point of X is in some facet and so vert $C(n,d) = X$. Furthermore, the convex hull of any n distinct points of M will be a polytope of the same combinatorial type as $C(n,d)$.

Of course, $C(d+1,d) \approx T^d$.

If $j = [\frac{1}{2}d]$, $k = [\frac{1}{2}(d+1)]$, (where $[a]$ denotes the greatest integer less than or equal to a), and $T^k, T^j \subseteq E^d$ are simplices such that $T^k \cap T^j$ is a single point in the relative interior of both, then $C(d+2,d) \approx \mathrm{conv}(T^k \cup T^j)$.

If d is even, $C(n,d)$ has more symmetry than the construction suggests. The indices $\{1, \ldots, n\}$ may be imagined to label the vertices of a regular n-gon (polygon with n vertices). Each symmetry of the n-gon defines a permutation of the vertices of $C(n,d)$ which induces a combinatorial isomorphism of $C(n,d)$ with itself. For $n \geq d+3$, these are the only combinatorial symmetries of $C(n,d)$.

If d is odd, $C(n,d)$ may be constructed from $C(n-1,d-1)$. Assume $C(n-1,d-1)$ lies in a hyperplane H, and $x_1, x_2 \in E^d$ are strictly separated by H such that a vertex y of $C(n-1,d-1)$ lies in $\operatorname{relint}[x_1,x_2]$. Then

$$C(n,d) \approx \operatorname{conv}(C(n-1,d-1) \cup \{x_1,x_2\}).$$

Transposing x_1 and x_2 yields a symmetry of $C(n,d)$. There is a symmetry of $C(n,d)$ fixing x_1 and x_2. These two symmetries generate the symmetry group of $C(n,d)$, which is of order 4.

An interesting property of the cyclic polytope is that every set of $[\frac{1}{2}d]$ vertices determines a face. A polytope such that every set of k vertices determines a face is said to be <u>k-neighbourly</u>. If may be shown that a polytope which is not a simplex is at most $[\frac{1}{2}d]$-neighbourly. Hence cyclic polytopes attain maximal neighbourliness. Any such $[\frac{1}{2}d]$-neighbourly polytope is called <u>neighbourly</u>. In even dimensions, neighbourly polytopes are necessarily simplicial - not so in odd dimensions. Clearly every polytope is 1-neighbourly. The first non-trivial case is $d = 4$, where the cyclic polytopes have the property that every pair of vertices determines an edge. Their duals are simple polytopes for which each pair of facets meets. The cyclic polytopes are by no means the only neighbourly polytopes - see Altshuler and Steinberg (1973) and (1976) - but they form the only so far recognizable family.

We now describe methods of constructing new polytopes from old.

2. Pyramid

Let $P \subseteq E^m$ be a d-polytope, $d < m$, and let $x \in E^m$, $x \notin \operatorname{aff} P$. The pyramid with base P and apex x is defined to be

$$Q = \operatorname{conv}(P \cup \{x\}).$$

We also say that Q is a pyramid over P. The faces of Q are the faces of P, together with faces of the form $\operatorname{conv}(F \cup \{x\})$ where F is a face of P. The pyramid has dimension d+1 and has one more vertex and one more facet than P. The dual to Q is a pyramid over the dual of P.

It may be that the base of Q is itself a pyramid with base R, say. Then Q is said to be a 2-fold pyramid over R. In general a k-fold d-pyramid is obtained by applying the pyramid construction k times over a (d-k)-polytope. A d-simplex is a d-fold pyramid over a point.

In the Gale diagram of a pyramid Q with apex x_1, the remaining vertices determine a facet. Hence $0 \in \operatorname{relint} \operatorname{conv} \{\bar{x}_1\} = \{\bar{x}_1\}$. Thus a polytope is a pyramid with apex x_1 if and only if in the Gale diagram $\bar{x}_1$ is coincident with 0. In particular a d-simplex is a pyramid with respect to any vertex, reflecting the fact that the Gale diagram consists of d+1 points coincident with 0.

3. Cartesian product

For $i = 1,2$, let $P_i \subseteq E^{m_i}$ be a d_i-polytope. The Cartesian product of P_1 and P_2 is defined as

$$P_1 \times P_2 = \{(x_1,x_2) \mid x_1 \in P_1, x_2 \in P_2\}.$$

Let $Q = P_1 \times P_2$. It may be shown that Q is a polytope. The k-faces of Q are precisely $\emptyset$, if $k = -1$, and the sets of the form $F_1 \times F_2$, where F_i is a k_i-face of P_i, $i = 1,2$, and $k_1+k_2 = k$. In particular $\dim Q = d_1+d_2$; a vertex of Q is a product of a vertex from each factor; and a facet of Q is a product of one factor by a facet of the other.

The product of two polytopes is simple if and only if both factors are simple.

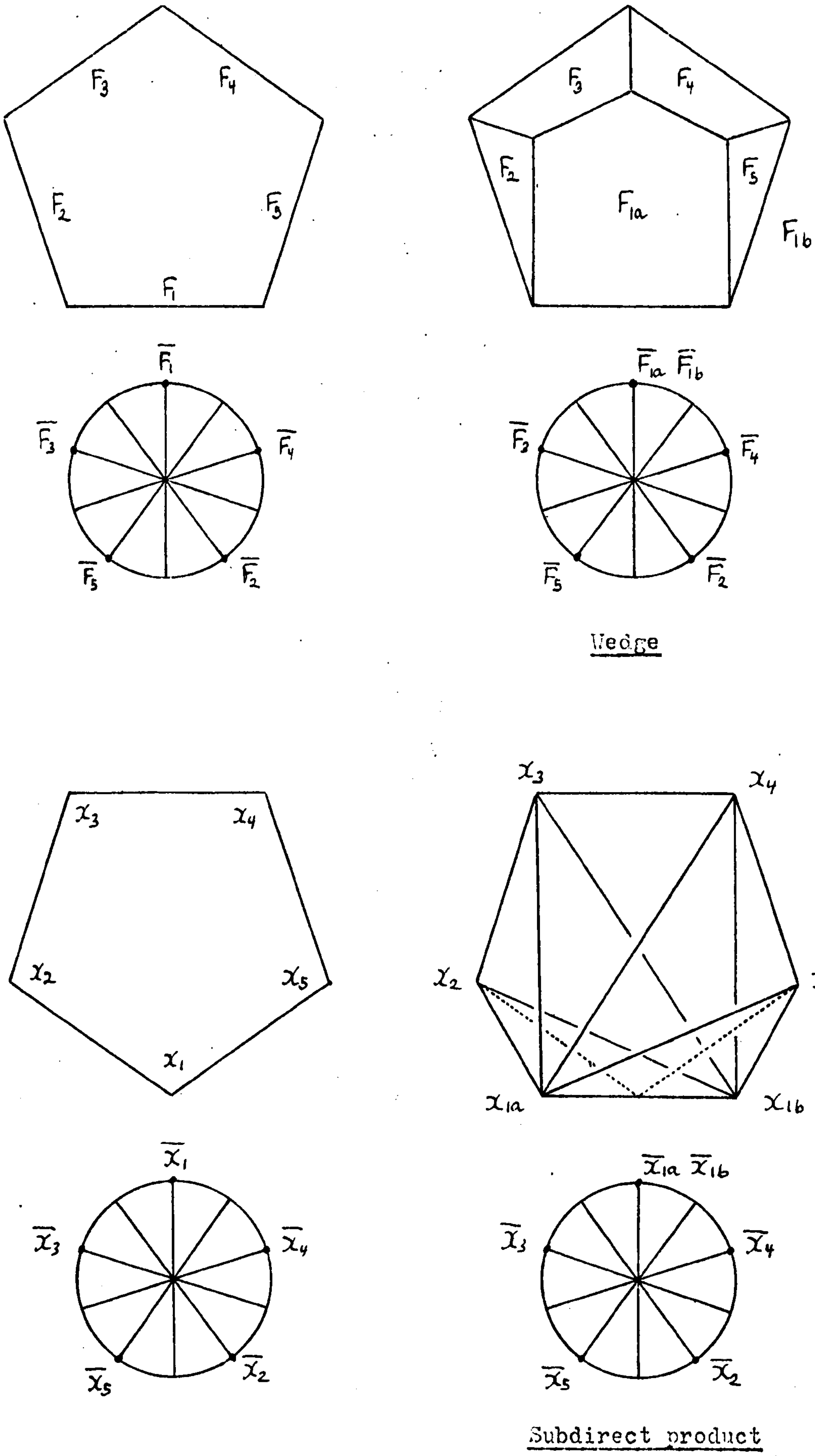
In the special case where one of the factors, say P_2, is a line segment, Q is a <u>prism</u> over P_1. If $P_2 = [0,1]$, the sets $P_1 \times \{0\}$ and $P_1 \times \{1\}$ are facets of Q isometric to P_1, called the <u>lower</u> and <u>upper bases</u> of Q. The other facets of Q are prisms over facets of P_1. Hence Q has two more facets and twice as many vertices as P_1.

We also remark - although we shall not need this fact - that the polytope $\mathrm{conv}(P_1^* \cup P_2^*)$ is dual to Q, where P_1^* and P_2^* are dual to P_1 and P_2, and $\mathrm{aff}\, P_1^* \cap \mathrm{aff}\, P_2^*$ is a single point in the relative interior of both P_1^* and P_2^*. In particular, C(d+2,d) is dual to a product of two simplices.

4. <u>Wedge and subdirect product</u>

Let $P \subseteq E^d$ be a d-polytope and F a proper face of P. Let C be the product of P and the halfline $[0,\infty)$ and suppose $C \subseteq E^{d+1}$. Let H be a hyperplane meeting int C such that $H \cap P = F$. One of the closed halfspaces, say H^+, contains P. Define the <u>wedge</u> Q over P with <u>foot</u> F to be $C \cap H^+$. This definition follows Klee and Walkup (1967). This construction yields a polytope combinatorially isomorphic to a prism if we ignore the stipulation that F be a proper face of P and let $H \cap P = \emptyset$. In any case, $P \times \{0\}$ and $H \cap P$ are combinatorially isomorphic to P and are called the <u>upper</u> and <u>lower bases</u>. A k-face of Q is either a k-face of one of the bases, or the wedge over a (k-1)-face G of P with foot $G \cap F$, or a face K of Q intersecting both bases but not F; in the last case K is combinatorially isomorphic to $(K \cap P) \times [0,1]$.

The wedge Q is a (d+1)-polytope with one more facet than P



F_1 F_2 F_3 F_4 F_5
F_{1a} F_{1b}
$\bar{F}_1$ $\bar{F}_2$ $\bar{F}_3$ $\bar{F}_4$ $\bar{F}_5$
$\bar{F}_{1a}$ $\bar{F}_{1b}$
Wedge
x_1 x_2 x_3 x_4 x_5
x_{1a} x_{1b}
$\bar{x}_1$ $\bar{x}_2$ $\bar{x}_3$ $\bar{x}_4$ $\bar{x}_5$
$\bar{x}_{1a}$ $\bar{x}_{1b}$
Subdirect product

Fig. 2.1

if F is a facet of P, and 2 more facets otherwise. Also, Q is simple if and only if P is simple and F is empty or a facet of P.

To describe the dual of the wedge construction, let P^* be dual to P, and $\hat{F}$ the face of P^* corresponding to F under duality. Let $E = \mathrm{conv}\{x_1, x_2\}$ be a line segment such that $\mathrm{aff}\, E \cap \mathrm{aff}\, P^*$ is a single point in the relative interior of both E and F. Then $Q^* = \mathrm{conv}(P^* \cup E)$ is dual to the wedge over P with foot F. This construction is a special case of the <u>subdirect product</u> of two polytopes - in this case P^* and E - described by McMullen (1976).

We will be especially interested in the case where F is a facet and $\hat{F}$ a vertex. The Gale diagram of Q^* is then easily found from the Gale diagram of P^*: the point $\overline{x}$ in the diagram of P^* corresponding to $\hat{F}$ is replaced by a pair of points corresponding to the vertices x_1, x_2 of Q^*, as in Fig. 2.1a.

The dual diagram of Q is obtained simply by doubling the point $\overline{F}$ of the dual diagram of P, as in Fig. 2.1b.

Thus, presented with a dual diagram $\{\overline{F}_1, \ldots, \overline{F}_{n+1}\}$ of Q, in which $\overline{F}_n$ and $\overline{F}_{n+1}$ coincide, and such that $\{\overline{F}_1, \ldots, \overline{F}_n\}$ is the dual diagram of a polytope P, we may conclude that Q is a wedge over P with foot F_n.

5. <u>Beneath and beyond</u>

The inductive constructions so far have produced new polytopes of higher dimension from ones of lower dimension. Furthermore, each of these constructions gives the same result combinatorially regardless of which particular representatives of the combinatorial equivalence classes we choose, as, for instance, the base of a pyramid or the factors of a Cartesian product. We now describe a method of constructing from a polytope P a polytope Q of the same dimension but with one more vertex. Which combinatorial types

Q may have depends not only on the type of P, but on the particular representative chosen.

Let P be a d-polytope, the intersection of n closed halfspaces

$$H_i^- = \{y \in E^d \mid \langle y, v_i \rangle \leqslant 1\}, \quad i = 1, \ldots, n,$$

such that each bounding hyperplane H_i meets P in a facet F_i. Let

$$x \in E^d \setminus (P \cup H_1 \cup \ldots \cup H_n).$$

Then x is said to be <u>beneath</u> those facets F_i for which $\langle x, v_i \rangle < 1$ and <u>beyond</u> those facets for which $\langle x, v_i \rangle > 1$. An equivalent statement of the definition may be made in terms of points

$$y_i \in \text{relint } F_i, \quad i = 1, \ldots, n.$$

Then x is beneath those facets for which $[x, y_i] \cap \text{int } P \neq \emptyset$, and beyond those facets F_i for which $[x, y_i] \cap P = \{y_i\}$.

Let $Q = \text{conv}(P \cup \{x\})$. Each facet that x is beneath will be a facet of Q. The other facets of P will be destroyed and be replaced by facets containing x. In general, a face F of P is a face of Q if and only if x is beneath some facet of P containing F. Thus the proper faces of Q are of two types. Firstly, faces contained in at least one facet that x is beneath. Secondly, faces of the form $\text{conv}(F \cup \{x\})$, where F lies in at least one facet that x is beneath and one facet that x is beyond. The combinatorial type of Q is determined by two considerations - the combinatorial type of P, and the collection of facets that x is beyond.

In the simplest case, x is beyond exactly one facet, say F_1, of P. Then Q is P modified by the addition of a shallow pyramid over F_1 with apex x. If F is any proper face of P, we can find a point beyond every facet containing F and beneath every other facet. For, choose $z \in \text{int } P$, $y \in \text{relint } F$ and $x \in E^d \setminus P$ such that $y \in \text{relint } [x, z]$. Then x is beyond every facet containing F, and if x is chosen sufficiently near y, beneath all the others.

In general, if F_1 and F are faces of P with F_1 a proper face of F, we can find a point beyond exactly those facets containing F_1 but not F, as follows. Let x be as just defined, and let $w \in \text{relint } F_1$. A point v such that $w \in \text{relint } [v,x]$ and sufficiently near w will be beyond precisely the specified facets.

The construction just described is a special case. In the most general case of adding a vertex, x may lie in the affine hull of a proper face of P. We will not consider this situation in detail.

These constructions are probably easier to visualize dually. The new vertex x corresponds to a closed halfspace H^- truncating the polar P* of P to yield the polar Q* of Q. Since x is not in the affine hull of any facet of P, $H \cap \text{vert } Q^* = \emptyset$. We have $Q^* = P^* \cap H^-$. Adding x beyond a single facet of P corresponds to truncating the corresponding vertex of P*. Adding x beyond exactly those facets containing a given k-face corresponds to truncating the corresponding (d-k-1)-face of P*. Adding x beyond the facets containing F_1 but not F corresponds to truncating precisely the vertices of P* in $\hat{F}_1$ but not $\hat{F}$ by a single hyperplane.

One observation is certainly easier in the dual situation. The hyperplane H divides P^* into two polytopes, one of which we take to be Q*. To what does $R^* = P^* \cap H^+$ correspond? Since R* contains exactly the vertices of P* which are not vertices of Q*, R corresponds to a polytope $\text{conv}(P \cup \{x'\})$, where x' is beneath a facet if and only if x is beyond that facet. Such a point may not exist for a particular P, but we can always find a projectively equivalent P' for which the desired point does exist, by letting M be a hyperplane strictly separating x and P and letting f be a projective equivalence sending M to infinity. Then $x' = f(x)$ and $P' = f(P)$ satisfy our requirements.

Given P with facets $\{F_1,\ldots,F_n\}$ and a subset $\{F_1,\ldots,F_k\}$, we may ask whether or not a point exists which is beyond exactly this subset of facets. In view of the previous paragraph, a more natural question is whether or not there exists a projectively equivalent P' and a point x' such that x' is beyond (or, equivalently, again by the previous paragraph, beneath) exactly $\{F_1', \ldots, F_k'\}$. In the dual situation we are asking whether the vertices $\{x_1, \ldots, x_k\}$ of P^* corresponding to $\{F_1, \ldots, F_k\}$ can be strictly separated from the remaining vertices by a hyperplane. In other words, we wish to know whether or not there exist $\alpha \in \mathbb{R}$, $u \in E^d$ such that

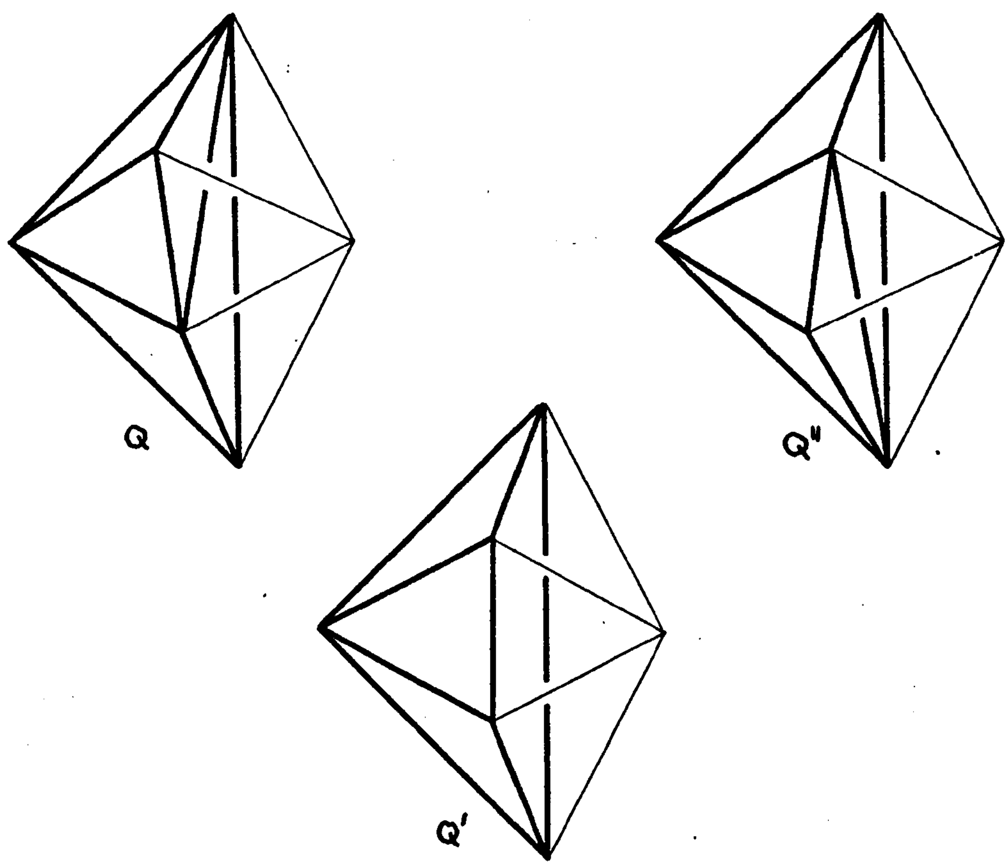


Fig. 2.2

$$\langle x_i, u\rangle > \alpha, \quad i = 1, \ldots, k$$

$$\langle x_i, u\rangle < \alpha, \quad i = k+1, \ldots, n.$$

This may be shown to be essentially a linear programming problem and hence an algorithm for deciding the question exists.

If no projectively equivalent polytope exists, we may still ask whether or not there exists a polytope P'' such that $P \approx P''$ and such that a point x'' may be found beyond exactly $\{F_1'', \ldots, F_k''\}$. We know of no way to decide this question in general.

Instead of adding a vertex, a vertex may be removed. Given $Q \in \mathcal{P}^d$, vert $Q = \{x_1, \ldots, x_n\}$, let $P = \operatorname{conv}\{x_1, \ldots, x_{n-1}\}$. Either dim $P = d-1$, in which case Q is a pyramid with apex x_n and base P, or dim $P = d$. In the latter case, Q is obtained by adding a vertex to P as described above. If $Q \approx Q'$, $Q' = \operatorname{conv}\{x_1', \ldots, x_n'\}$ and $P' = \operatorname{conv}\{x_1', \ldots, x_{n-1}'\}$, then it does not follow that $P \approx P'$. In Fig. 2.2, Q, Q' and Q'' are of the combinatorial type of the regular octahedron, but P, P' and P'' are not all of the same combinatorial type. Even though $P \approx P''$, the isomorphism is not induced by the natural correspondence of the vertices.

We conclude that the result of adding or removing a vertex depends not only upon the combinatorial type but upon the particular representative, in contrast to the construction of Cartesian product, for instance, which depends only upon the combinatorial type of the factors.

Chapter 3. COMPLEXES

1. Geometric cell complexes

A geometric cell complex is a finite collection $\mathcal{K}$ of polytopes in E^d, called the faces of $\mathcal{K}$, satisfying the following conditions.

1. If $K \in \mathcal{K}$, and $J \in \mathcal{C}(K)$, then $J \in \mathcal{K}$.
2. If $J, K \in \mathcal{K}$ then $J \cap K$ is a face of both J and K. (Note that $\emptyset \in \mathcal{K}$.)

The underlying set of $\mathcal{K}$ is the union of faces of $\mathcal{K}$ and is written set $\mathcal{K}$ or $\cup\mathcal{K}$.

The sets $\mathcal{B}(P)$ and $\mathcal{C}(P)$ associated with a polytope P are geometric cell complexes. The set $\mathcal{B}(P)$ is usually called the boundary complex of P.

A subcomplex of $\mathcal{K}$ is a subset of $\mathcal{K}$ which is itself a complex. If $\mathcal{L} \subseteq \mathcal{K}$, in general $\mathcal{L}$ is not a subcomplex, but

$$\{F \mid \text{There exists } C \in \mathcal{L} \text{ such that } F \in \mathcal{C}(C)\}.$$

is - the subcomplex generated by $\mathcal{L}$.

In a simplicial complex, all the faces are simplices.

Two complexes $\mathcal{K}_1$ and $\mathcal{K}_2$ are said to be isomorphic via an isomorphism $f: \mathcal{K}_1 \to \mathcal{K}_2$ if f is a bijection with the property that for all $K_1, K_2 \in \mathcal{K}_1$,

$$f(K_1 \cap K_2) = f(K_1) \cap f(K_2).$$

A complex $\mathcal{K}$ is said to be a refinement of a complex $\mathcal{L}$ if there exists a homeomorphism $f: \text{set}\,\mathcal{K} \to \text{set}\,\mathcal{L}$ such that for each $L \in \mathcal{L}$, $f^{-1}(L)$ is a union of faces of $\mathcal{K}$. Sets of the form $f^{-1}(L)$, $L \in \mathcal{L}$, are the pseudofaces of the refinement. If $f^{-1}(L)$ is a face of $\mathcal{K}$, $f^{-1}(L)$ is called a principal face of the refinement. Thus a face of $\mathcal{K}$ which is also a pseudoface of the refinement is principal. The homeomorphism f is called the refinement map.

It is clear that the composition of refinement maps is again a refinement map, so if $\mathcal{K}_1$ is a refinement of $\mathcal{K}_2$, and $\mathcal{K}_2$ is a refinement of $\mathcal{K}_3$, then $\mathcal{K}_1$ is a refinement of $\mathcal{K}_3$.

It also follows from the definition that if $\mathcal{K}_1$ is a refinement of $\mathcal{K}_2$, and $\mathcal{K}_2$ is a refinement of $\mathcal{K}_1$, then all faces are principal and $\mathcal{K}_1$ and $\mathcal{K}_2$ are isomorphic complexes.

In Fig. 3.1 we illustrate the fact that a cube is a refinement of a triangular prism, and that a triangular prism is a refinement of a simplex.

Using the fact that a homeomorphism between the boundaries of two d-cells extends to a homeomorphism between the cells themselves, it is trivial that for $P, Q \in \mathcal{P}^d$, $\mathcal{B}(P)$ is a refinement of $\mathcal{B}(Q)$ if and only if $\mathcal{C}(P)$ is a refinement of $\mathcal{C}(Q)$.

A finite graph G may be embedded in E^3 in such a way that each edge is a line segment, and the intersection of any pair of edges is either empty or an endpoint of both. Thus G may be regarded as a complex, and we may speak of one graph being a refinement of another. In Fig. 3.2 we show that the graph of a

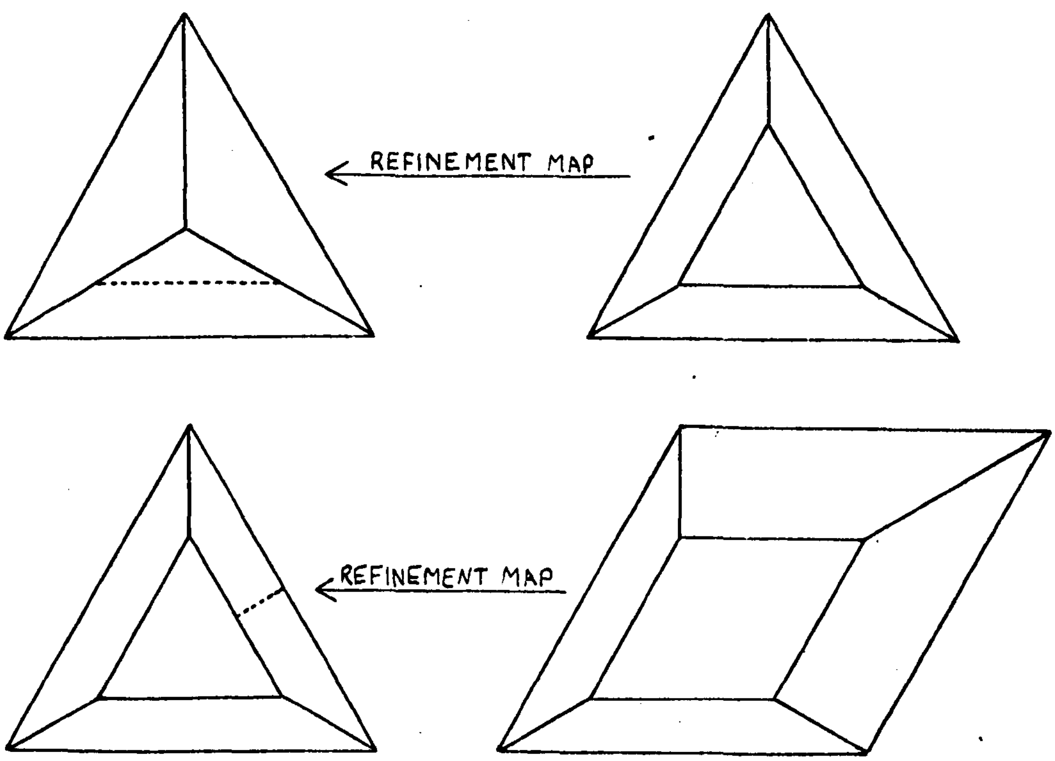


Fig. 3.1

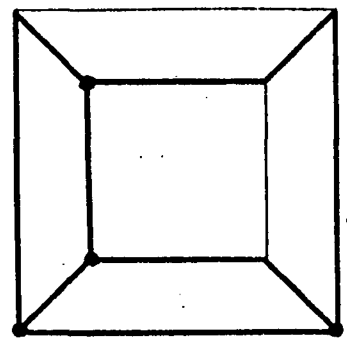


Fig. 3.2

3-cube contains a refinement of the graph of a 3-simplex. Principal vertices are emphasized.

2. Generalized combinatorial cells

For some purposes, we will wish to consider objects which "look like" the facial complexes of polytopes. The boundary of a polytope is a union of k-polytopes, $0 \leq k \leq d-1$, fitting together in a nice way. In particular, each k-polytope is a k-cell. Barnette (1975) makes precise the idea of "fitting k-cells together nicely" in order to define the generalized combinatorial cell (abbreviated to gcc).

We proceed by induction. A (-1)-gcc is $\emptyset$ and a 0-gcc is a point. A d-gcc C is a d-cell whose boundary is the union of a finite collection $\mathcal{B}_C(C)$ of k-gcc's called the faces of C, $-1 \leq k \leq d-1$, satisfying the following conditions.

1. If F is a face of C and G is a face of F, then G is a face of C.
2. If F,G are faces of C, then $F \cap G$ is a face of F and G. (Note that $\emptyset$ is always a face of C.)

Terminology for gcc's is defined in analogy with polytopes.

For instance, the (d-1)-faces, 1-faces and 0-faces of C are called the <u>facets</u>, <u>edges</u> and <u>vertices</u>. One difference between gcc's and polytopes is that a given d-cell may have more than one gcc structure defined upon it. Hence we use subscripts to distinguish $\mathcal{B}_C(C)$ and $\mathcal{B}_D(D)$, if $C = D$.

The <u>facial lattice</u> of a d-gcc C is $\mathcal{L}_C(C) = \mathcal{B}_C(C) \cup \{C\}$. Using the facial lattice of C, the <u>vertex figure</u> of C at a vertex x may be defined as a (d-1)-gcc whose facial lattice is isomorphic to

$$\{F \in \mathcal{L}_C(C) \mid x \in F\},$$

if such a d-gcc exists.

Two gcc's are said to be <u>combinatorially isomorphic</u> if their lattices are isomorphic.

In a <u>simple</u> d-gcc, every vertex lies in exactly d facets.

It is clear that every 1-gcc is isomorphic to a line segment and every 2-gcc is isomorphic to a polygon. Steinitz's theorem may be used to prove that every 3-gcc is isomorphic to a 3-polytope. However, an example of Barnette and Wegner (1971) shows that there exist 4-gcc's which are not isomorphic to any 4-polytope.

A d-gcc in which all the facets are isomorphic to d-simplices is a <u>simplicial</u> d-gcc or a <u>triangulation of the (d-1)-sphere</u>. Klee (1975) has defined (d-1)-pseudomanifolds, which are (essentially) triangulations of connected manifolds without boundary.

A simplicial complex whose underlying set is homeomorphic to a (d-1)-sphere is called a <u>combinatorial (d-1)-sphere</u> and is in particular a triangulation of the (d-1)-sphere and the boundary of a d-gcc.

<u>Theorem 3.1:</u> If $g: \mathcal{L}_{C_1}(C_1) \to \mathcal{L}_{C_2}(C_2)$ is an isomorphism between d-gcc's C_1 and C_2, there exists a homeomorphism $f: C_1 \to C_2$ such

that if F_1 is a face of C_1 and F_2 is a face of C_2 such that $g(F_1) = F_2$, then $f(F_1) = F_2$.

Proof: Let $C_i(k)$ be the union of k-faces of C_i, $i = 1,2$; $k = 0,\dots,d$. We proceed by induction on k to define homeomorphisms of the desired sort $f_k: C_1(k) \to C_2(k)$, such that f_k extends f_{k-1}, $k = 1,\dots,d$. If $k = 0$, the isomorphism g gives a bijection between vert $C_1 = C_1(0)$ and vert $C_2 = C_2(0)$, so we may define f_0 to be g restricted to vert C_1. Assume by induction that $k \leq d$ and that f_{k-1} has been defined. Consider a k-face K_1 of C_1, corresponding to a k-face $K_2 = g(K_1)$ of C_2. The relative boundary of K_i is the union of (k-1)-cells in K_i, $i = 1,2$. Hence

$$K_i \cap C_i(k-1) = \text{relbd } K_i.$$

Hence f_{k-1} induces a homeomorphism between relbd K_1 and relbd K_2. As we have seen in Chapter 1, a homeomorphism between the boundaries of two k-cells extends to a homeomorphism between the two k-cells themselves. We can thus extend f_{k-1} to $K_1 \cup C_1(k-1)$. Since the relative interiors of different k-cells of C_1(and C_2) are disjoint, f_{k-1} extends to each k-cell in turn, thus defining $f_k: C_1(k) \to C_2(k)$ as desired. Then $f = f_d$ is the required homeomorphism. This concludes the proof.

Corollary: If $P_1, P_2 \in \mathcal{P}^d$, and P_1 and P_2 are isomorphic via an isomorphism $g: \mathcal{C}(P_1) \to \mathcal{C}(P_2)$, then there is a homeomorphism $h: P_1 \to P_2$ such that for $F \in \mathcal{C}(P_1)$, $g(F) = h(F)$. This homeomorphism and its inverse are refinement maps, and $\mathcal{C}(P_1)$ ($\mathcal{B}(P_1)$) is a refinement of $\mathcal{C}(P_2)$ ($\mathcal{B}(P_2)$) and vice versa.

Refinement maps for gcc's are defined in exactly the same way as for cell complexes. The gcc's we shall consider will usually be defined in the following wey. The gcc C will be a polytope P, and the facets of C will be unions of facets of P. For instance, suppose $P \in \mathcal{P}^d$ is a refinement of T^d via the refinement map $f: P \to T^d$. Denoting the facets of T^d by $F_1, \ldots, F_{d+1}$, by the definition of f as a refinement map, $f^{-1}(F_i)$ is a union of faces of P. In fact since f is a homeomorphism, f preserves dimension and hence $f^{-1}(F_i)$ is a union of facets of P. Let $C_i = f^{-1}(F_i)$, $i = 1, \ldots, d+1$. Then P is a d-gcc C with facets $C_1, \ldots, C_{d+1}$. The faces of C are sets of the form

$$C_{i_1} \cap \ldots \cap C_{i_k}, \quad 1 \leq k \leq d+1.$$

The map f induces an isomorphism of C and T^d, since

$$f(C_{i_1} \cap \ldots \cap C_{i_k}) = F_{i_1} \cap \ldots \cap F_{i_k}.$$

We remark that a d-gcc C is determined by its facets, since the faces of C are precisely the intersections of subsets of the set of facets.

<u>Theorem 3.2</u>: a. A d-gcc has at least d+1 facets.

b. A d-gcc with exactly d+1 facets is isomorphic to a d-simplex.

c. Let C be a d-cell and $C_1, \ldots, C_{d+1} \subseteq \text{bd } C$ such that

i. $\text{bd } C = C_1 \cup \ldots \cup C_{d+1}$;

ii. $C_{i_1} \cap \ldots \cap C_{i_k}$ is a (d-k)-cell, $k = 1, \ldots, d+1$. (The (-1)-cell is $\emptyset$.)

Then $C_1, \ldots, C_{d+1}$ are the facets of a d-gcc isomorphic to a d-simplex.

<u>Proof</u>: a. We first prove that if C is a d-gcc and C_1 is a facet of C, for each $x \in \text{relbd } C_1$, there exists a facet of C other than C_1 containing x. For, since $x \in \text{relbd } C_1$, a neighbourhood of x must contain points of $(\text{bd } C) \setminus C_1$. Since there are only finitely many other facets, each of which is closed, some facet other than C_1 contains x.

If C_2 is a facet of C_1, and $x \in \text{relint } C_2$, a facet containing x contains C_2.

We can now prove the first part of the theorem by an easy induction. The first part is clearly true if $d = 0,1$. Suppose each $(d-1)$-gcc has at least d facets. If C is a d-gcc and C_1 is a facet of C, then C_1 is a $(d-1)$-gcc with at least d facets. Each facet of C_1 lies in a facet of C, each of which is different, so we have at least d more facets of C. Together with C_1, there are at least $d+1$, establishing the first part.

b. The second part is true for $d = 0,1$. Assume that the second part is true for $d-1$ and let C be a d-gcc with $d+1$ facets $C_1,\ldots,C_{d+1}$. Each facet has at least d facets by the first part and since it is clear that each facet can have no more than d facets, each facet is isomorphic to a simplex by induction. Hence, for $i \leq k \leq d+1$, $C_{i_1} \cap \ldots \cap C_{i_k}$ is a face of C. Therefore, if we denote the facets of T^d by $F_1,\ldots,F_{d+1}$, the correspondence

$$C_{i_1} \cap \ldots \cap C_{i_k} \longrightarrow F_{i_1} \cap \ldots \cap F_{i_k}$$

yields a bijection between the faces of C and $\mathcal{B}(T^d)$. This bijection is an isomorphism. The second part is now established.

c. The third part is true if $d = 1$. Assume the third part for $d-1$.

To prove the theorem for d, we show that

$$\mathcal{T}(C) = \{C_{i_1} \cap \ldots \cap C_{i_k} \mid \{i_1,\ldots,i_k\} \subseteq \{1,\ldots,d+1\}\}$$

is the set of faces of a d-gcc.

The elements of $\mathcal{T}(C)$ satisfy the conditions 1 and 2 of gcc's. It suffices to show that $C_1,\ldots,C_{d+1}$ are gcc's.

Consider C_1. We show that $C_1 \cap C_i$, $i = 2,\ldots,d+1$ are the facets of C_1 as a $(d-1)$-gcc. Since $C_1 \cap C_i$ is a $(d-2)$-cell by (ii), and $C_1, C_i \subseteq \operatorname{bd} C$, $C_i \cap \operatorname{relint} C_1 = \emptyset$, otherwise $C_1 \cap C_i$ contains a small open $(d-1)$-cell. Hence $C_1 \cap C_i \subseteq \operatorname{relbd} C_i$, $i = 2,\ldots,d+1$.

Every point in relbd C_1 lies in at least one of the facets C_i, $i = 2,\ldots,d+1$, since bd $C = C_1 \cup \ldots \cup C_{d+1}$. Hence

$$\operatorname{relbd} C = (C_1 \cap C_2) \cup \ldots \cup (C_1 \cap C_{d+1}).$$

By (ii),

$$\begin{aligned}&(C_1 \cap C_{i_1}) \cap \ldots \cap (C_1 \cap C_{i_k})\\ &= C_1 \cap C_{i_1} \cap \ldots \cap C_{i_k}\end{aligned}$$

is a (d-1-k)-cell. By the inductive assumption, C_1 is a (d-1)-gcc, and identical arguments show that each of $C_1,\ldots,C_{d+1}$ is a gcc. Hence C is a d-gcc. By (b), C is isomorphic to a d-simplex, concluding the proof.

3. CW-complexes and suspensions

For some purposes we need to define structures, called CW-complexes, even more general than gcc's or cell complexes. These objects were first defined by Whitehead (1949) in the context of algebraic topology. Intuitively, a CW-complex is a topological space formed as a union of k-cells such that each point lies in the relative interior of exactly one cell and such that the intersection of any pair of k-cells is a union of j-cells, $j < k$.

To make this precise, we need a preliminary definition. A topological space X^* is obtained by <u>adjoining an n-cell</u> to a topological space X if $X^* \setminus X$ is homeomorphic to an open n-cell and there exists a continuous function $f: B^n \to X^*$ which maps int B^n homeomorphically onto $X^* \setminus X$ and maps bd B^n into X.

A structure of <u>finite CW-complex</u> is defined on a Hausdorff space X by the prescription of an ascending sequence

$$X^0 \subseteq X^1 \subseteq X^2 \subseteq \ldots \subseteq X^m$$

of closed subspaces of X satisfying the following conditions.

a. X^0 is a finite set of points.

b. For $n > 0$, X^n is obtained from X^{n-1} by adjoining a finite number of n-cells.

c. $X = X^0 \cup \ldots \cup X^n$.

For us, X will either be a subset of E^d for some d or else homeomorphic to such a subset, so X has the relative topology induced by the topology of E^d. The empty set, together with the points of X^0 and the n-cells adjoined at each stage form the set of faces of X. The faces form a partially ordered set.

The intersection of faces need not be a face; this is the main distinction between a complex and a CW-complex.

Two CW-complexes X and Y are said to be isomorphic if there exists a bijection preserving dimension and set inclusion between the faces of X and the faces of Y.

It is clear that every gcc and every cell complex may be regarded as a CW-complex. The definition of refinement extends in the obvious way to CW-complexes: a CW-complex X is a refinement of a CW-complex Y if there exists a homeomorphism $h: X \to Y$ such that for every face C of Y, $h^{-1}(C)$ is a union of faces of X. A principal face of the refinement is a face B of X such that h(B) is a face of Y.

We now give examples, anticipating section 4.2. Let $\Gamma_1, \ldots, \Gamma_d$ be arcs joining two points x_1 and x_2, $x_1 \neq x_2$, such that $\Gamma_i \cap \Gamma_j = \{x_1, x_2\}$, if $i \neq j$. Then $\Gamma_1 \cup \ldots \cup \Gamma_d$ is a CW-complex $\Gamma(d)$ whose faces are $\emptyset$, x_1, x_2, $\Gamma_1, \ldots, \Gamma_d$.

Let $P \in \mathcal{P}^3$, $x_1, x_2 \in \text{vert } P$, $x_1 \neq x_2$. By theorem 4.1, between x_1 and x_2 run three arcs Γ_1, Γ_2, Γ_3 meeting pairwise in $\{x_1, x_2\}$. The set $(\text{bd } P) \setminus (\Gamma_1 \cup \Gamma_2 \cup \Gamma_3)$ has three components whose closures we call C_1, C_2 and C_3. Then bd P has the structure of a CW-complex with faces $\emptyset$, x_1, x_2, Γ_1, Γ_2, Γ_3, C_1, C_2, C_3.

In the above examples the points x_1 and x_2 played special roles. We now define the suspended complex, of which the above examples are particular cases. This definition may be made in two ways - an abstract way and a more concrete way which assumes that the given complex is embedded in E^d for some d. The suspended complex $S(C)$ over a CW-complex C with faces $\mathcal{T}$ is defined abstractly as a topological space homeomorphic to the space derived from $C \times [0,1]$ by identifying $C \times \{0\}$ and $C \times \{1\}$ to points y_1 and y_2, called the suspension points. The faces of $S(C)$ are $\emptyset$, y_1 and y_2, and sets of the form $F \times [0,1]$, for $F \in \mathcal{T}$, where $F \times \{0\}$ and $F \times \{1\}$ are identified to y_1 and y_2 respectively. Note that if $F \in \mathcal{T}$, F is not a face of $S(C)$, but $S(F)$ is a face of $S(C)$.

If C is embedded in E^d, we may define $S(C)$ more concretely as follows. Let E^d be embedded as a subspace of E^{d+1} and let $y_1, y_2 \in E^{d+1}$ such that y_1 and y_2 are strictly separated by E^d. Then

$$S(C) = \{\lambda y_i + (1-\lambda)x \mid i \in \{1,2\},\ \lambda \in [0,1],\ x \in C\},$$

or any homeomorphic space, is the suspended complex over C. For example, the CW-complex $\Gamma(d)$ is the suspended complex over a set of d points, as in Fig. 3.3a.

As illustrated in Fig. 3.3b, the CW-complex determined by

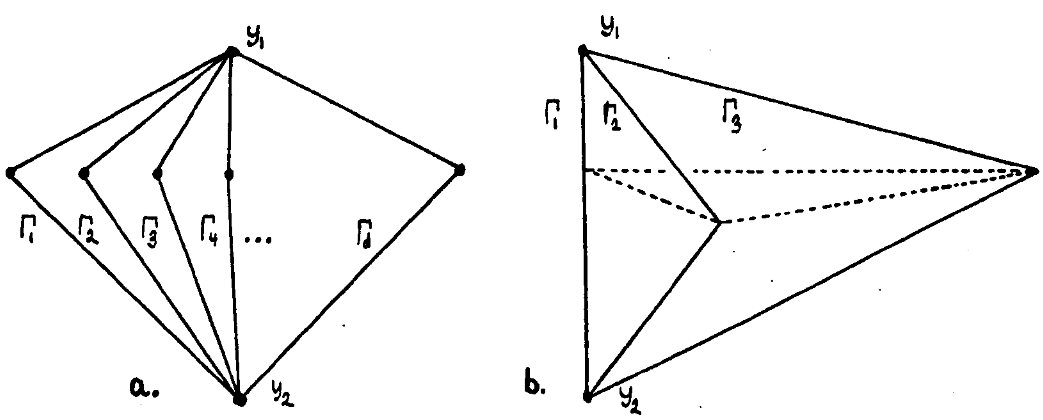


Fig. 3.3

three disjoint arcs joining two vertices of a 3-polytope is homeomorphic to $S(\mathrm{bd}\ T^2)$, where $\mathrm{bd}\ T^2$ has the usual facial structure. We assume hereafter that suspended complexes, for instance $S(\mathrm{bd}\ T^d)$, have the natural facial structure induced by $\mathcal{B}(T^d)$.

We may generalize this construction to get the following result.

Theorem 3.3. The complex $\mathcal{B}(T^d)$ is a refinement of $S(\mathrm{bd}\ T^{d-1})$, in which any pair of vertices of T^d may be chosen to correspond to the suspension points.

Proof: Let $x_1, x_2 \in \mathrm{vert}\ T^d$. The basic idea is to let $S(T^{d-1}) = T^d$. We use the second definition of suspended complex. The vertices of T^d other than x_1 and x_2 determine a (d-2)-face F. The affine hull of F and the midpoint of $[x_1, x_2]$ is a hyperplane H strictly separating x_1 and x_2. Now, with x_1 and x_2 as suspension points, $S(T^{d-1}) = S(T^d \cap H) = T^d$. The faces of $S(T^{d-1})$ are of three sorts.

1. $\emptyset$, x_1, x_2.
2. Faces of T^d containing x_1 and x_2.
3. Unions of pairs F_1, F_2 of k-faces of T^d, $1 \leq k \leq d-1$, such that $x_1 \in F_1$, $x_2 \in F_2$, $x_2 \notin F_1$, $x_1 \notin F_2$ and $F_1 \cap F_2$ is a (k-1)-face of T^d.

The fact that $\mathcal{B}(T^d)$ is a refinement of $S(\mathrm{bd}\ T^{d-1})$ is trivial, because the faces of $S(\mathrm{bd}\ T^{d-1})$ are unions of faces of T^d. Hence the identity map from $\mathrm{bd}\ T^d$ to $S(\mathrm{bd}\ T^{d-1})$ is a refinement map. This concludes the proof.

Chapter 4. SOME KNOWN RESULTS CONCERNING REFINEMENTS

1. Introduction

The definition of a polytope as the convex hull of a finite set of points or as the intersection of a finite set of closed halfspaces - provided that the intersection is bounded - could hardly be more straightforward. Yet, to characterize the possible combinatorial types of polytopes is a difficult task even in three dimensions, the first non-trivial case. For higher dimensions, the problem is unsolved.

Call a complex polytopal if it is combinatorially equivalent to the boundary complex of a polytope. It is easy to see that all 1-gcc's and 2-gcc's are polytopal. We will see below that Steinitz's theorem (theorem 4.2) implies that every 3-gcc is polytopal.

However, for $d \geq 4$, a d-gcc is not necessarily polytopal. Grünbaum and Sreedharan (1967) describe a non-polytopal triangulation of the 4-sphere with 8 vertices. Barnette and Wegner (1971) discuss the dual of this triangulation, which yields a non-polytopal simple 4-gcc with 8 facets. Mani (1972) constructs for $d \geq 4$ a non-polytopal simplicial complex $\mathcal{K}_d$ with d+4 vertices homeomorphic to a (d-1)-sphere. This complex is a triangulation of the (d-4)-fold pyramid over Grünbaum and Sreedharan's example. It may be shown that the duals to Mani's examples exist, and provide non-polytopal simple d-gcc's with d+4 facets.

We may ask the question: What are necessary and sufficient conditions for a d-gcc to be polytopal? It is clearly necessary that a d-gcc imitate the local behaviour of a d-polytope. This local behaviour may be described in terms of certain complexes associated with a face F of a complex $\mathcal{K}$, namely the star

$$st(F;\mathcal{K}) = \{G \in \mathcal{K} \mid F \leq G\},$$

the antistar

$$\text{ast}\,(F;\mathcal{K}) = \{G \in \mathcal{K} \mid F \cap G = \emptyset\}$$

and the linked complex or link

$$\text{link}(F;\mathcal{K}) = \text{st}(F;\mathcal{K}) \cap \text{ast}(F;\mathcal{K}).$$

In a d-polytopal gcc C, the following conditions must hold for every proper face F:

1. F is polytopal;
2. setstar$(F;\mathcal{B}(C))$ is a (d-1)-cell and
3. if F is a vertex,
   a. link$(F;\mathcal{B}(C))$ is (d-1)-polytopal,
   b. the vertex figure of P at F exists and is (d-1)-polytopal and
   c. setast$(F;\mathcal{B}(C))$ is a (d-1)-cell.

(It is not claimed that this list is exhaustive.) Furthermore C must have a dual, also satisfying these conditions.

Mani's examples $\mathcal{K}_d$ just mentioned satisfy 1, 2 and 3. Barnette and Wegner show that $\mathcal{K}_4^*$ also satisfies 1, 2 and 3 but is not polytopal. Hence conditions 1, 2 and 3 fail to distinguish polytopal and non-polytopal gcc's.

We would like to find additional properties of a combinatorial nature of the boundary complex of a polytope, and, in particular, properties peculiar to polytopes as opposed to gcc's. Grünbaum (1967, Ch. 11) discusses these and related ideas at much greater length.

We will consider this problem from the point of view of refinements. We find, however, that certain properties of 2- and 3-polytopes fail to generalize to higher-dimensional polytopes.

2. Connectivity of graphs of polytopes

A path of length k in a graph G from a vertex x_1 to a vertex x_{k+1} is a union of edges $[x_i, x_{i+1}]$, $i = 1,\ldots,k$. An arc is a path for which $x_i \neq x_j$ if $i \neq j$. It is well-known that every path from x_1 to x_{k+1} contains an arc from x_1 to x_{k+1}. A graph G is said to be n-connected if between any pair $\{x_1, x_2\}$ of distinct vertices there exist n paths $\Gamma_1,\ldots,\Gamma_n$ such that

$$\Gamma_i \cap \Gamma_j = \{x_1, x_2\},\ i \neq j.$$

Such paths are customarily called disjoint paths from x_1 to x_2, although this is not the usual set-theoretic meaning of "disjoint".

Theorem 4.1 (Balinski (1961)): The graph of a d-polytope is d-connected.

Thus between any pair of vertices x_1 and x_2 of a d-polytope run d disjoint paths. Only d edges meet a given vertex of a simple

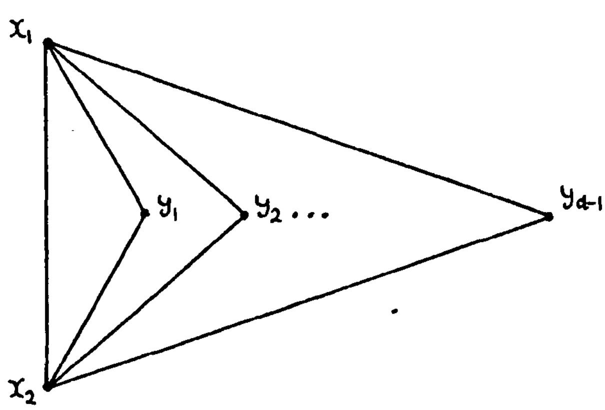


Fig. 4.1

d-polytope, so this result is best possible. Theorem 4.1 implies that x_1 and x_2 are joined by d disjoint arcs, say $\Gamma_1,\ldots,\Gamma_d$, of which at most one is of length 1 and the rest are of length at least 2. These arcs therefore form a subgraph H of the graph G of the d-polytope P which is a refinement of the graph H(d) shown in Fig. 4.1. The set $\Gamma_1 \cup \ldots \cup \Gamma_d$ is the CW-complex $\Gamma(d)$ defined in section 3.3. The graph $H(d)$ is a refinement of $\Gamma(d)$.

It is clear that a graph K is n-connected if and only if for any pair of distinct vertices x_1 and x_2 there exists a refinement of $\Gamma(n)$ (or $H(n)$) in K for which x_1 and x_2 are principal vertices. Theorem 4.1 is therefore equivalent to the statement that for $P \in \mathcal{P}^d$ and $x_1, x_2 \in \text{vert } P$, $x_1 \neq x_2$, there exists a refinement of $\Gamma(d)$ in the graph of P for which x_1 and x_2 are principal. Or, again equivalently, we may say that $\mathcal{B}(P)$ contains a subcomplex which is a refinement of $\Gamma(d)$ with x_1 and x_2 principal.

3. Refinements in the boundary complexes of 2-polytopes and 3-polytopes

The boundary of a convex polygon P is a simple circuit of at least three edges and hence may be regarded as a refinement of the boundary complex of a triangle. Any three vertices of P may be chosen as principal. Also, the boundary of a polygon with n+1 edges is a refinement of the boundary of a polygon with n edges.

Boundary complexes of 3-polytopes may be characterized in terms of refinements in a considerably more complicated fashion.

It is easy to see that the combinatorial type of a 3-polytope P is determined by its graph G(P). For, G(P) may be embedded in a 2-sphere S^2 by a suitable radial projection π. The components of $S^2 \setminus \pi(G(P))$ correspond to the facets of P. If ς: $G(P) \to S^2$ is another embedding, there will be a homeomorphism $h: S^2 \to S^2$ such that $h \cdot \varsigma = \pi$, and so any two 3-polytopes with the same graph will be combinatorially isomorphic. A graph isomorphic to the graph of a d-polytope is said to be <u>d-polytopal</u>.

<u>Theorem 4.2 (Steinitz (1922))</u>: A graph is 3-polytopal if and only if it is planar and 3-connected.

Given $P \in \mathcal{P}^3$, a Schlegel diagram of P provides an embedding of G(P) in E^2, so G(P) is planar. By theorem 4.1, G(P) is 3-connected. The given conditions are therefore necessary.

The other direction is difficult. The reader is referred to Grünbaum (1967) or, for two proofs more in the spirit of the present work, Barnette and Grünbaum (1969).

The complete graph K_n has n vertices, each pair of which determine an edge. The complete bipartite graph $K_{m,n}$ has a block of m vertices and a block of n vertices. Two vertices determine an edge if and only if they lie in different blocks. By a theorem of Kuratowski, a graph K is planar if and only if K contains no subgraph which is a refinement or the complete graph K_5 or the complete bipartite graph $K_{3,3}$. Thus we may restate theorem 4.2 in terms of refinements as follows: a graph K is 3-polytopal if and only if

1. K contains no refinement of K_5 or $K_{3,3}$ and
2. for every two vertices x_1, x_2, K contains a refinement of $\Gamma(3)$ for which x_1 and x_2 are principal.

4. Polytopes are refinements of simplices

Two- and three-dimensional polytopes are determined up to combinatorial type by their graphs. This is not the case in higher dimensions. For instance, the 1-skeleton of a neighbourly 4-polytope with n vertices is isomorphic to K_n. Grünbaum (1967, p. 124) shows that there exist combinatorially distinct neighbourly 4-polytopes with 8 vertices. In general, for $d \geq 3$ there exist non-isomorphic d-polytopes with isomorphic ($[\frac{1}{2}d] - 1$)-skeletons. As well, Altshuler and Steinberg (1973) construct a triangulation of the 3-sphere whose 1-skeleton is isomorphic to K_9 but which is not polytopal.

Hence to distinguish polytopes from complexes that look like polytopes we must consider refinements of higher dimension. In particular we will investigate how $\mathcal{B}(P)$ itself may be expressed as a refinement. The most important result in this direction is that the boundary complex of a d-polytope is a refinement of the boundary complex of the d-simplex T^d.

Theorem 4.3 (Grünbaum (1965; 1967, p. 200): Let $P \in \mathcal{P}^d$ and let $x_o = F_o \subseteq \ldots \subseteq F_{d-1}$ be a tower of faces of P such that $i = \dim F_i$. Then $\mathcal{B}(P)$ may be expressed as a refinement of $\mathcal{B}(T^d)$ in such a way that each face of the tower is a principal face of the refinement.

To prove this theorem we need a lemma.

Lemma 4.1: Let $Q, Q' \in \mathcal{P}^{d-1}$ and let P, P' be pyramids over Q and Q' respectively with apices x_o and x_o'. Then a refinement map $\psi: Q \to Q'$ may be extended to a refinement map $\bar{\psi}: P \to P'$. The principal faces of the refinement $\bar{\psi}$ are the faces of P of the form F or $\operatorname{conv}(F \cup \{x_o\})$, where F is a face of Q which is a principal face of the refinement ψ.

Proof: Each point $y \in P$ has a representation

$$y = \lambda(y) + (1 - \lambda(y))x_o, \quad 0 \leq \lambda(y) \leq 1, \quad f(y) \in Q,$$

where f(y) is uniquely defined unless $y = x_o$, in which case $\lambda(y) = 0$ and f(y) is undefined. Then $\bar{\psi}(y)$ is defined to be

$$\bar{\psi}(y) = \lambda(y)\psi(f(y)) + (1 - \lambda(y))x_o'.$$

The map $\bar{\psi}$ is easily checked to be a homeomorphism extending ψ. Clearly $\bar{\psi}(x_o) = x'_o$. It is equally clear that the principal faces of ψ are of the types described. The lemma is now proved.

Proof of theorem 4.3: We use induction on d. For $d = 1$, P is a simplex and the theorem holds. Assume that the theorem holds for d-1.

Let H be a hyperplane strictly separating x_o from the remaining vertices of P and let H^- be the closed halfspace bounded by H and containing x_o. Then $P \cap H$ is a $(d-1)$-polytope and $P \cap H^-$ is a pyramid over $P \cap H$ with apex x_o. The faces

$$F_1 \cap H \subseteq \dots \subseteq F_{d-1} \cap H$$

form a tower of faces of $P \cap H$.

The first step is to show that there exists a refinement map $\phi: P \to P \cap H^-$ such that $\phi(F_i) = F_i \cap H^-$, $i = 0, \dots, d-1$. For each $x \in P \setminus \{x_o\}$ the line $\operatorname{aff}\{x, x_o\}$ meets $\operatorname{setast}(x_o; \mathcal{B}(P))$ in a unique point $f(x)$ and H in a unique point $g(x)$. Therefore we have a unique $\lambda(x) \in (0,1]$ and a representation

$$x = \lambda(x)f(x) + (1 - \lambda(x))x_o.$$

Now define

$$\phi(x) = \lambda(x)g(x) + (1 - \lambda(x))x_o, \quad x \neq x_o$$

$$\phi(x_o) = x_o.$$

This is the usual sort of linear extension of a radial projection. It is easy to check that $\phi: P \to P \cap H^-$ is a refinement map and that $F_o, \dots, F_{d-1}$ are principal faces.

The second step is to show that $\mathcal{T}(P \cap H^-)$ is a refinement of $\mathcal{T}(T^d)$. By induction, there is a refinement map $\psi: P \cap H \to T^{d-1}$ for which the faces $F_1 \cap H, \dots, F_{d-1} \cap H$ are principal.

By lemma 4.1, since T^d is a pyramid over T^{d-1}, ψ extends to a refinement map $\bar{\psi}: P \cap H^- \to T^d$ for which $\operatorname{conv}((F_i \cap H) \cup \{x_o\})$ is principal for $i = 1, \dots, d-1$. Since $\operatorname{conv}((F_i \cap H) \cup \{x_o\}) = F_i \cap H^-$, and $x_o = F_o$ is principal, we conclude that $F_o \cap H^-, \dots, F_{d-1} \cap H^-$ are principal faces.

Now, $\bar{\psi} \cdot \phi: P \to T^d$ is the desired refinement map for which $F_o, \dots, F_{d-1}$ are principal. This concludes the proof of the theorem.

For $d \geq 4$, it is not known whether or not all d-gcc's or triangulations of the (d-1)-sphere are refinements of simplices.

In particular, theorem 4.3 tells us that any vertex of P may be preassigned as a principal vertex of a refinement of $\mathcal{B}(T^d)$. As we have already remarked, if $d = 2$, three vertices may be preassigned. For $d = 3$, we have the following result.

Theorem 4.4 (Gallivan (1974)): If x_1, x_2, x_3 are vertices of $P \in \mathcal{P}^3$, then $\mathcal{B}(P)$ may be expressed as a refinement of $\mathcal{B}(T^3)$ for which x_1, x_2 and x_3 are principal.

Simple examples show that this result is best possible. For instance, we cannot choose the four vertices of a square face of a triangular prism to be principal.

Theorem 4.4 is proved by showing that the graph of P contains a refinement C of K_4 for which x_1, x_2 and x_3 are principal. Then $\operatorname{bd} P \setminus C$ has four connected components whose closures $C_1, \ldots, C_4$ are 2-cells. It is clear that $C_1, \ldots, C_4$ are the facets of a 3-gcc which is combinatorially a 3-simplex.

Chapter 5. REFINEMENTS OF COMPLETE GRAPHS

In the previous chapter, we saw that if three vertices of $P \in \mathcal{P}^d$, $d = 2,3$, are preassigned, there exists a refinement of K_{d+1} in $G(P)$ with these three vertices principal. We now show that this result cannot be extended to higher dimensions.

<u>Proposition 5.1</u>: There exists a set of three vertices of $T^p \times T^q \in \mathcal{P}^{p+q}$, $p,q \geq 2$, which are not all principal for any refinement of K_{p+q+1} in $G(T^p \times T^q)$.

<u>Proof</u>: Let

$$\operatorname{vert} T^p = \{x_1,\ldots,x_{p+1}\},$$

$$\operatorname{vert} T^q = \{y_1,\ldots,y_{q+1}\}.$$

Let $P = T^p \times T^q$. Then

$$\operatorname{vert} P = \{(x_i,y_j) \mid i = 1,\ldots,p+1;\ j = 1,\ldots,q+1\}.$$

The vertices (x_i, y_j) and $(x_{i'},y_{j'})$ determine an edge of P if and only if $i = i'$ or $j = j'$.

Hence we may represent $G(P)$ as a $(p+1) \times (q+1)$ rectangle of vertices such that a pair of vertices determines an edge if and only if both vertices lie in the same row or the same column.

Let us choose $z_1 = (x_1,y_1)$ and $z_2 = (x_2,y_2)$ as principal vertices. We will show that there are exactly two refinements of K_{d+1}, where $d = p+q$, in $G(P)$ with z_1 and z_2 principal, and that neither contains $z_3 = (x_3,y_3)$ as a principal vertex. Hence for no refinement of K_{d+1} are z_1, z_2 and z_3 all principal.

Call the desired refinement with z_1 and z_2 principal C, a subgraph of $G(P)$. Since both T^p and T^q are simple, P is simple and hence exactly d edges meet at each vertex. At each principal vertex of C, d edges meet. Hence C contains every edge meeting z_1 or z_2. (See Fig. 5.1a.) Let C_0 be the subgraph of C consisting of these edges.

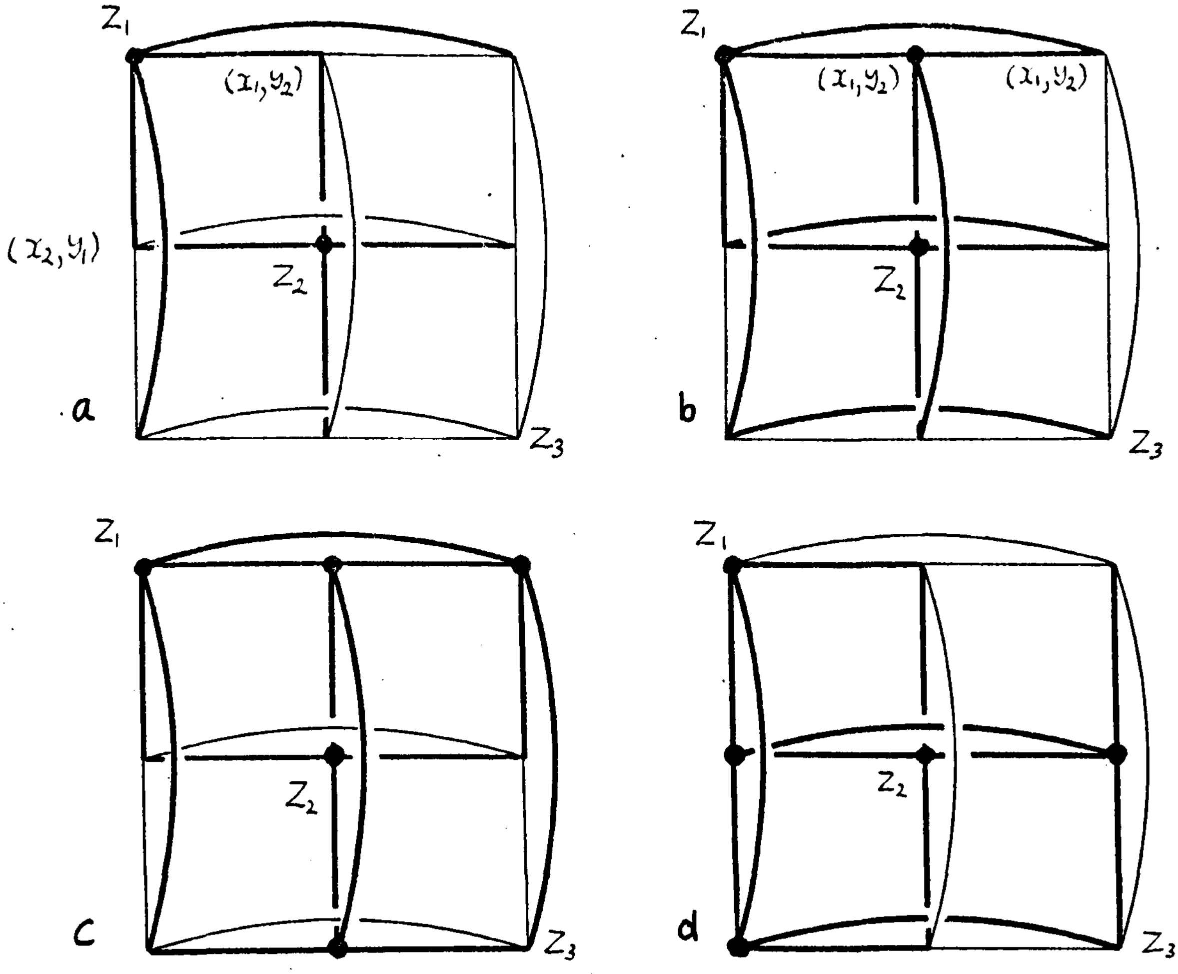


Fig. 5.1

In C_o two paths join z_1 and z_2. At most one of these paths corresponds to an edge of K_{d+1}. Hence one of these paths corresponds to a path in K_{d+1} containing some vertex of K_{d+1} other than one corresponding to z_1 or z_2. Therefore either (x_1,y_2) or (x_2,y_1) is principal.

Suppose (x_1,y_2) is principal. Let C_1 consist of all edges meeting z_1, z_2 or (x_1,y_2). (See Fig. 5.1b) Now, z_1 and (x_1,y_2) are joined by an edge of C_1, which must correspond to an edge of K_{d+1}. Hence every other path in C_1 joining z_1 and (x_1,y_2) must contain a principal vertex. Hence the vertices $(x_1,y_3),\ldots,(x_1,y_{q+1})$ are principal. In other words, all $q+1$ of the vertices in the x_1 row are principal because two of them are.

An identical argument shows that all $p+1$ vertices of the

y_2 column are principal. We have now found

$$(p+1) + (q+1) - 1 = d+1$$

principal vertices; that is, all of them. The graph C consisting of all of the edges meeting this set of principal vertices is easily seen to be the desired refinement of K_{d+1}. (See Fig. 5.1c.)

For, any two principal vertices in the same row (or column) are joined by an edge of C. Any pair, say (x_1,y_j) and (x_i,y_2), not in the same row (or column) are joined by a path of two edges of C passing through the intermediate vertex (x_i,y_j). This intermediate vertex is used only by one pair (x_1,y_j), (x_i,y_2). Hence C is homeomorphic to K_{d+1}.

If (x_2,y_1) is chosen as a principal vertex, we get a second alternative, illustrated in Fig. 5.1d, with principal vertices lying in the x_2 row and y_1 column. The two cases are distinct, but symmetrical. In neither case is z_3 principal.

In the remaining case, both (x_2,y_1) and (x_1,y_2) are principal. But then all the vertices of the x_1 and x_2 rows and the y_1 and y_2 columns must be principal, a total of

$$2(p+1) + 2(q+1) - 4 = 2d - 2 > d+1,$$

since $d \geqslant 4$. This contradiction shows that the two refinements already found are the only ones, and concludes the proof.

We are left with the case of two preassigned principal vertices, and $d \geqslant 4$. The following conjecture is apparently due to Larman and Mani (1970). They prove the conjecture in the simplicial case.

<u>Conjecture 5.1</u>: Let $P \in \mathcal{P}^d$ and let $x_1, x_2 \in$ vert P. Then G(P) contains a refinement of K_{d+1} for which x_1 and x_2 are principal vertices.

For $d \geqslant 4$, the answer is unknown.

The conjecture is a strengthening of Balinski's theorem. Let y_1, y_2 be distinct vertices of K_{d+1} and let $H(d)$ be the subgraph consisting of all edges meeting $\{y_1, y_2\}$. We saw in section 4.2 that Balinski's theorem is equivalent to the statement that in the graph of $P \in \mathcal{P}^d$ we may preassign any pair of vertices x_1, x_2 to correspond to y_1 and y_2 in a refinement of $H(d)$ in $G(P)$. The conjecture states that we can find a refinement of $H(d)$ which we can extend to get a refinement of K_{d+1}.

Chapter 6. REFINEMENTS WITH TWO PREASSIGNED VERTICES

1. Introduction

Theorem 4.3 states that given $P \in \mathcal{P}^d$ and a tower of faces $F_o \subseteq \ldots \subseteq F_{d-1}$ of P we may express $\mathcal{B}(P)$ as a refinement of $\mathcal{B}(T^d)$ such that every face of the tower is principal. Theorem 4.1 may be interpreted as asserting that given $P \in \mathcal{P}^d$ and $x_1, x_2 \in \text{vert } P$ we may find a refinement of the CW-complex H(d) for which x_1 and x_2 are principal. Grünbaum (1967, Ex. 11.1.4) makes the following conjecture, which is a common generalization of the two theorems just mentioned: with the notation above, if $x_1 = F_o$ and $x_2 \notin F_{d-1}$, there exists a piecewise affine refinement map from P onto T^d for which $F_o, \ldots, F_{d-1}$ and x_2 are principal faces.

The conjecture is clearly true if $d = 1,2$, but we will now show that it is false if $d \geq 3$. A counterexample Q is the prism over a (d-1)-simplex. Let

$$\{x_{10}, \ldots, x_{d0}\},$$
$$\{x_{11}, \ldots, x_{d1}\}$$

be the vertices of the lower and upper bases respectively, labelled so that $[x_{i0}, x_{i1}]$ is an edge of Q for $i = 1, \ldots, d$. Let

$$F_o = x_{10},$$
$$F_1 = [x_{10}, x_{20}],$$

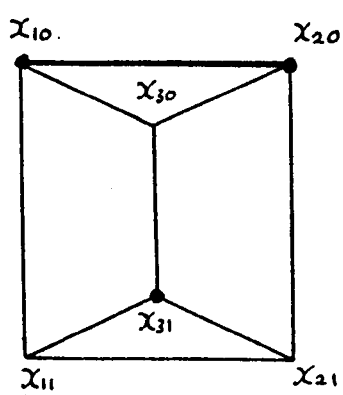


Fig. 6.1

$F_2 = \mathrm{conv}\{x_{10}, x_{20}, x_{11}, x_{21}\}$,

$x_2 = x_{31}$.

(See Fig. 6.1.) If the desired refinement $\Psi: Q \to T^d$ exists, there will be three arcs $\Gamma_1, \Gamma_2, \Gamma_3$ from x_{31} to F_2 corresponding to edges of T^d; that is, $\Psi(\Gamma_i)$ is an edge of T^d. Each of these arcs meets F_2 in a principal vertex.

Since Q is simple, each of the d edges containing x_{31} lies in an arc corresponding to an edge of T^d. Since $x_{31} \notin F_2$, $x_{11}, x_{21} \in F_2$, two of these edges of T^d correspond to the paths in Q consisting of the single edges $[x_{31}, x_{11}]$ and $[x_{31}, x_{21}]$. Hence we have say $\Gamma_1 = [x_{31}, x_{11}]$ and $\Gamma_2 = [x_{31}, x_{21}]$. But then x_{11} and x_{21} are principal vertices, lying in F_2. Since x_{10} and x_{20} are preassigned as principal vertices, we have four principal vertices in F_2 - a contradiction, since F_2 is a principal 2-face and hence contains exactly three principal vertices. Hence the desired refinement fails to exist for Q.

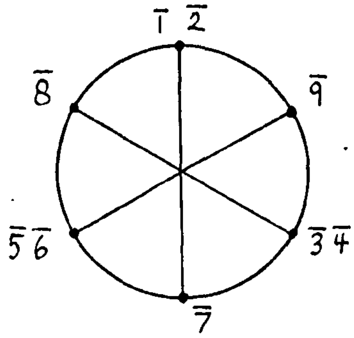
2. Refinements of simplices with two preassigned principal vertices: counterexamples

A weaker version of Grünbaum's conjecture is much more difficult to settle.

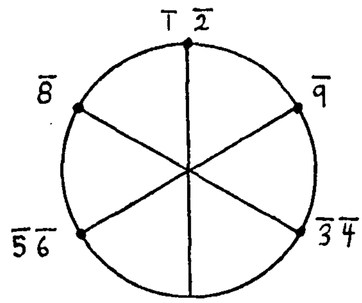
Question 6.1: Can the boundary complex of a d-polytope be expressed as a refinement of the boundary complex of the d-simplex with two preassigned principal vertices?

In this section, we describe two polytopes for which the answer is: No.

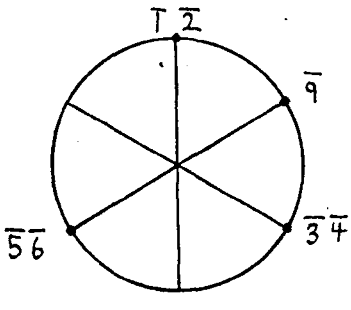
Theorem 6.1: For $d \geq 6$, there exist $P \in \mathcal{P}^d$ with d+3 facets and $x_1, x_2 \in \mathrm{vert}\, P$ such that $\mathcal{B}(P)$ cannot be expressed as a refinement of $\mathcal{B}(T^d)$ in such a way that x_1 and x_2 are principal vertices.



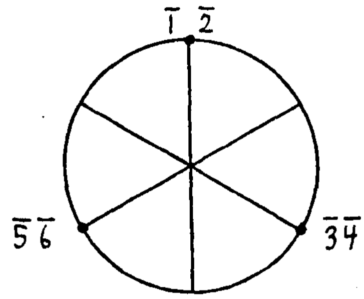
a. P



b. The facet 7.



c. The 4-face 78.



d. The 3-face 789 = F.

Fig. 6.2

Proof: For $d = 6$, let P be the polytope with facets 1,...,9 with the dual Gale diagram $\bar{X}^* = \{\bar{1},\ldots,\bar{9}\}$ of Fig. 6.2a. For brevity, we shall denote faces, for instance, $i \cap j \cap k$, expressed as intersections of facets, as ijk.

Let F be the 3-face 789. Removing the points $\bar{7}$, $\bar{8}$ and $\bar{9}$ in turn from $\bar{X}^*$, at each stage the remaining points satisfy the diagram condition for polytopes. Hence the dual Gale diagram of F is that of Fig. 6.2d, and F is a cube with facets 1789,...,6789, illustrated in Fig 6.3a. The two special vertices will be diamet-

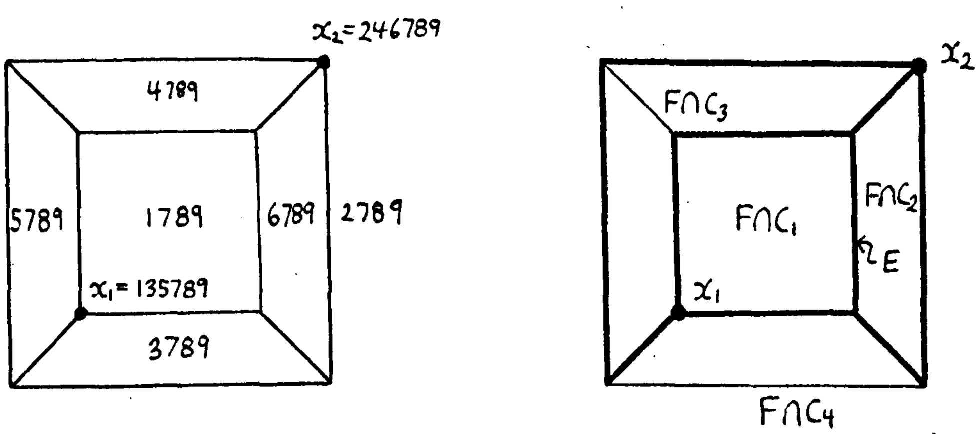


Fig. 6.3

rically opposite vertices of F: $x_1 = 135789$, $x_2 = 246789$. Observe that each special vertex is simple in both F and P.

We assume that a refinement map $\Psi: P \to T^6$ exists for which x_1 and x_2 are principal and obtain a contradiction. Let $F_1,\ldots,F_7$ be the facets of T^d and let $C_i = \Psi^{-1}(F_i)$, $i = 1,\ldots,7$. By definition of refinement map, each pseudofacet C_i is a 5-cell and is the union of one or more facets of P. The intersection of k different pseudofacets is a (6-k)-cell. The principal faces of the refinement are precisely those faces of P which may be expressed as an intersection of pseudofacets - that is, those faces which are pseudofaces.

We claim that 7, 8 and 9 are principal faces, and hence $F = 789$ is also a principal face. To see this, observe that exactly six facets of P contain x_j, $j = 1,2$, and six pseudofacets contain x_j. Hence each of the facets containing x_j belongs to a different pseudofacet. Since the facet 7 contains F and hence x_1 and x_2, and since every facet of P contains either x_1 or x_2, the pseudofacet containing 7 cannot contain any other facet. Therefore 7, and similarly 8 and 9, are pseudofacets, say C_5, C_6, C_7, and $F = C_5 \cap C_6 \cap C_7$ is a principal face.

The map $\Psi|_F: F \to T^3$ is therefore a refinement map, for which the pseudofacets of F are $F \cap C_1,\ldots,F \cap C_4$. But we know that there is essentially one way of expressing F as a refinement of a simplex with x_1 and x_2 principal, that shown in Fig. 6.3b. The other refinements are obtained from this one by applying a symmetry of the cube leaving x_1 and x_2 fixed. The unique pseudo-edge E not meeting x_1 or x_2 is always a principal edge. We may label the C_i's so that $E = F \cap C_1 \cap C_2$. The choice of E determines the refinement in F. Since the pseudofacets of Ψ are determined

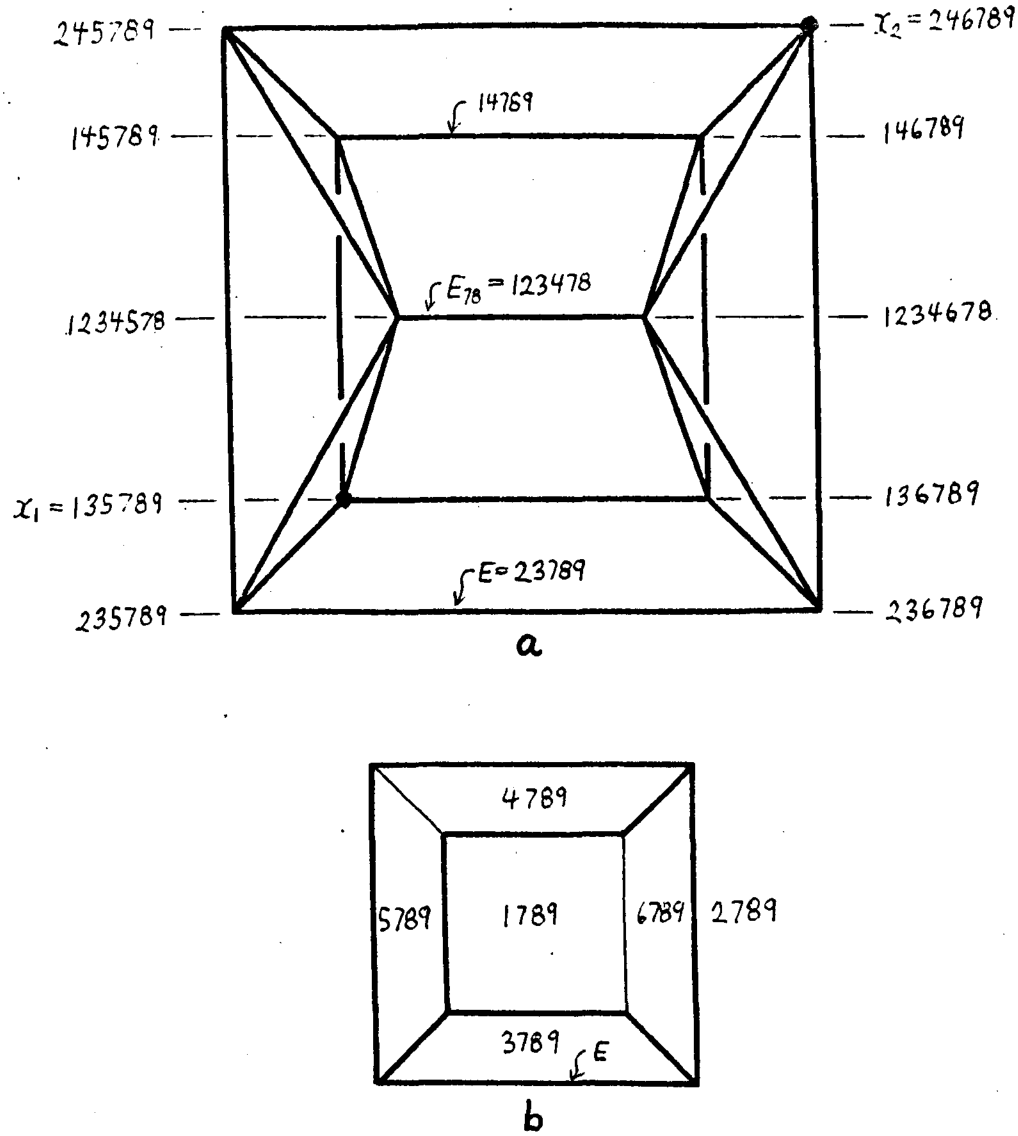


Fig. 6.4

by the pseudofacets of $\Psi|_F$, the choice of E determines the refinement in P.

The final step is to show that no choice of E induces a suitable refinement in all three of the 4-faces 78, 89 and 79. Consider the 4-face 78 whose Schlegel diagram based on F is shown in Fig. 6.4a.

The 4-face 78 contains precisely two vertices, namely 1234578 and 1234678, not in F, which determine an edge E_{78} = 123478. Suppose that E were the edge 23789, inducing the refinement of F indicated in Fig. 6.4b. In the Schlegel diagram of 78, E and E_{78} lie in parallel lines, and will therefore be called parallel edges. We have pseudofacets

$C_1 = 3,$

$C_2 = 2,$

$C_3 = 1 \cup 6,$

$C_4 = 4 \cup 5.$

We now compute, using the dual Gale diagram of P to determine complete sets of facets for the various faces,

$$\begin{aligned}
& C_1 \cap C_2 \cap C_3 \cap C_4 \cap C_5 \cap C_6 \\
&= 3 \cap 2 \cap (1 \cup 6) \cap (4 \cup 5) \cap 7 \cap 8 \\
&= 2378 \cap (1 \cup 6) \cap (4 \cup 5) \\
&= 2378 \cap (14 \cup 15 \cup 46 \cup 56) \\
&= 123478 \cup 123578 \cup 234678 \cup 235678 \\
&= 123478 \cup 1234578 \cup 1234678 \cup \emptyset \\
&= 123478 \\
&= E_{78}.
\end{aligned}$$

But the intersection of six pseudofacets must be a vertex, not an edge. This contradiction shows that we cannot choose E = 23789. A symmetrical argument shows that E cannot be 14789. Hence in 78 we cannot choose E parallel to E_{78}.

A symmetrical argument applied to 89 and 79 proves that E cannot be parallel to E_{79} or E_{89}. But every edge of F is parallel to one of E_{78}, E_{79} or E_{89}. Hence the desired refinement in P fails to exist.

To construct examples for $d > 6$, suppose that $Q \in \mathcal{P}^d$, $d > 6$, has d+3 facets, and a 6-face isomorphic to P. Call this 6-face P. Let the special vertices x_1 and x_2 of Q be the special vertices of P. There are three facets of P not containing x_1, and hence three facets of Q not containing x_1. Hence x_1 lies in at most d facets, and so is simple. Similarly, x_2 is simple.

Assume $\Psi: Q \to T^d$ is a refinement map for which x_1 and x_2 are principal. Each facet of the refinement containing x_1 lies in a different pseudofacet, since x_1 is simple, and similarly for x_2.

There are three facets containing x_1 but not x_2 and three facets containing x_2 but not x_1. Since P is a 6-face, at least d-6 facets of Q contain P. All the facets of Q are accounted for, so every facet of Q meets $\{x_1, x_2\}$.

Therefore every facet F containing P must be a pseudofacet of the refinement, and hence a principal face. The intersection of principal faces is a principal face, so P is principal. But then $\Psi|_P: P \to T^6$ is the forbidden refinement map on P. This contradiction establishes that Ψ cannot exist.

A particular choice for Q is the (d-6)-fold pyramid over F. Other examples may be constructed, including (d-6)-fold wedges over P, by adding d-6 points to the dual diagram of P, as described in section 1.7.

This concludes the proof.

The 6-dimensional counterexample P of theorem 6.1 may be described directly in terms of wedges, using the remarks on the dual diagrams of wedges in section 2.4. Let R be a triangular bipyramid, let $y, z \in \text{vert } R$ be the unique pair of vertices of R not joined by an edge, and let G_1, G_2 and G_3 be the three facets of R containing y. Let R_1 be the wedge over R with foot G_1. Two of the facets of R_1 are wedges over G_2 and G_3. Let R_2 be the wedge over R_1 whose foot is the wedge over G_2. Now one of the facets of R_2 is a 2-fold wedge over G_3. Then P is isomorphic to the wedge R_3 over R_2 whose foot is this 2-fold wedge.

The cubical face F of P arises from z in the following way.

Since $z \notin G_1$, R_1 contains a face F_1 which is a prism over z; that is, a line segment. Since F_1 does not meet the wedge over G_2, R_2 contains a face F_2 which is a prism over F_1; that is, a square. Similarly, $R_3 = P$ contains a prism over F_2; this is the cubical face F.

Theorem 6.2: For $d \geq 4$, there exist simple d-polytopes P_d with d+4 facets and $x_1, x_2 \in \text{vert } P_d$ such that $\mathcal{B}(P_d)$ cannot be expressed as a refinement of $S(\mathcal{B}(T^{d-1}))$ in such a way that x_1 and x_2 are principal vertices.

Proof: The polytope P_d is the dual of the convex hull of the cyclic polytope $C(d+3,d)$ with a certain (d+4)th point.

In order to show that the desired refinement fails to exist, we show firstly that P_4 is embedded as a 4-face of P_d in such a way that the special vertices of P_4 coincide with the special vertices of P_d. Secondly, we prove that the refinement fails to exist for P_4. Finally, we prove that the desired refinement on P_d must have P_4 as a principal face, and hence induce the refinement on P_4. This contradiction will conclude the proof.

First, we describe P_d. The construction for even d is slightly different from that for odd d, but both cases can be done simultaneously if we use appropriate notation. Let

$$\text{vert } C(d+3,d) = \{1,\ldots,d+3\},$$

arranged in the natural order on the moment curve. Let $n = [\tfrac{1}{2}d]$ and write $i' = 2n+1+i$. Call the new vertex 0.

The new vertex will be above all but one facet Y containing a given (d-3)-face Z, and below every other facet. We showed in section 2.5 that such a point exists for a polytope projectively equivalent to $C(d+3,d)$.

For brevity, we will refer to faces by listing the vertices without punctuation. Let

$$Z = 67\ldots(d+3),$$

$$Y = 4567\ldots(d+3).$$

Gale's evenness condition implies that Y is indeed a facet of $C(d+3,d)$. The dual P_d^* of P_d will be $\mathrm{conv}(C(d+3,d)\cup\{0\})$, where 0 lies above every facet of $C(d+3,d)$ containing Z except Y, and below every other facet. If F is a face of a polytope Q*, denote by $\hat{F}$ the corresponding face in Q. The special vertices $x_1 = \hat{V}$, $x_2 = \hat{W}$ in P_d correspond to facets V,W of P_d^*. Define

$$V = \begin{cases} 12346\ldots(2n+1), & d = 2n \\ 12346\ldots(2n+1)3', & d = 2n+1 \end{cases}$$

$$W = \begin{cases} 56\ldots(2n+1)1'2'0, & d = 2n \\ 56\ldots(2n+1)1'2'3'0, & d = 2n+1. \end{cases}$$

Interpret $6\ldots(2n+1)$ as $\emptyset$ for $n = 2$.

In a moment we will see that V,W actually are facets of P_d^*. In any event, let

$$X = V\cap W = \begin{cases} 6\ldots(2n+1), & d = 2n \\ 6\ldots(2n+1)3', & d = 2n+1. \end{cases}$$

Then

$$V = 1234X,$$

$$W = 51'2'0X.$$

The evenness condition implies that $45\ldots(d+3)$ is a facet of $C(d+3,d)$, and since this facet is a simplex, X is a face of $C(d+3,d)$. Note that if $d = 4$, then $X = \emptyset$, and therefore $\hat{X} = P_4$. In general, X is a $(d-5)$face (the (-1)-face is $\emptyset$) and $\hat{X}$ is a 4-face.

In order to show that X is a face of P_d^*, recall from section 2.5 that a face K of $C(d+3,d)$ is not a face of P_d^* if and only if 0 lies above every facet of $C(d+3,d)$ containing K. Let us call such a face of $C(d+3,d)$ a face <u>hidden</u> by 0.

Since every hidden facet contains Z, if K is hidden, then every facet containing K contains Z; that is, Z is a face of K. But Z is not a face of X, so X is a face of P_d^*.

The evenness condition implies that Z belongs to exactly five facets of C(d+3,d). The four hidden facets are

$$156\ldots(d+3) = 151'2'X$$

$$126\ldots(d+3) = 121'2'X$$

$$236\ldots(d+3) = 231'2'X$$

$$346\ldots(d+3) = 341'2'X$$

and hence the hidden (d-2)-faces are

$$151'2'X \cap 121'2'X = 11'2'X$$

$$121'2'X \cap 231'2'X = 21'2'X$$

$$231'2'X \cap 341'2'X = 31'2'X.$$

Any intersection of three or more hidden facets is simply Z, which is not hidden; therefore, there are no other hidden faces.

The facets of P_d^* are of two types: firstly, facets of C(d+3,d) not hidden by O and, secondly, facets determined by O and a (d-2)-face which is the intersection of a hidden facet with a non-hidden facet; that is, a (d-2)-face of a hidden facet which is not itself hidden.

We are interested only in facets containing X. The reason for this is that $P_d^*/X \approx P_4^*$. In order to show that P_4 is embedded in P_d as a 4-face, we need only show that $P_d^*/X \approx P_4^*/X$, since

$$P_4^*/X = P_4^*/\emptyset \approx P_4^*.$$

In Table 6.1 are listed the quadruples abcd such that abcdX is a facet of P_d^*. These quadruples denote the 3-faces of link$(X;P_d^*)$. Since P_d^* is simplicial,

$$\text{link}(X;P_d^*) \approx P_d^*/X \approx (X)^*.$$

Each of these quadruples is denoted by a capital letter. These

Vertices of P_4; facets of P_4^*			
A.	1234	K.	151'0
B.	1232'	L.	152'0
C.	1245	M.	51'2'0
D.	1251'	N.	121'0
E.	1342'	O.	122'0
F.	1452'	P.	231'0
G.	2345	Q.	232'0
H.	2351'	R.	341'0
I.	3451'	S.	342'0
J.	451'2'	T.	41'2'0

Facets of P_4; vertices of P_4^*	
1.	A B C D E F K L N O
2.	A B C D G H N O P Q
3.	A B E G H I P Q R S
4.	A C E F G I J R S T
5.	C D F G H I J K L M
1'.	D H I J K M N P R T
2'.	B E F J L M O Q S T
0.	K L M N O P Q R S T

	1	2	3	4	5	1'	2'	0
1	*	6	3	4	5	3	5	4
2	6	*	6	3	4	4	3	4
3	3	6	*	6	3	4	4	4
4	4	3	6	*	5	3	5	3
5	5	4	3	5	*	6	4	3
1'	3	4	4	3	6	*	3	6
2'	5	3	4	5	4	3	*	6
0	4	4	4	3	3	6	6	*

(*=10)

Edge-valence matrix of P_4^*

Table 6.1

letters label the vertices of $\hat{X}$ (or facets of the dual) and the numbers label the facets of $\hat{X}$ (or vertices of the dual).

The facets of P_d^* containing X are determined by Gale's evenness condition; the remaining facets contain 0 and are computed from the list of hidden facets and (d-2)-faces as described above.

The special vertices of P_d correspond to 1234X and 51'2'0X. In particular, the special vertices of P_4 correspond to 1234 and 51'2'0. If we regard P_4 as a face of P_d, so $P_4 = \hat{X}$, the special vertices of P_4 and P_d coincide and we may denote them by $x_1 = A$ and $x_2 = M$.

The edge-valence matrix of Table 6.1 is a concept of Altshuler and Steinberg (1973). The (ij) th entry is the number of facets containing the face ij. Since all the entries are non-zero, P_4^* is a neighbourly 4-polytope. Grünbaum (1967, p. 124) constructs this polytope as an example of a non-cyclic neighbourly 4-polytope.

Our second task is to prove that $\mathcal{B}(P_4)$ is not a refinement of the CW-complex $S(\mathcal{B}(T^3))$ for which A and M are principal vertices. Note that $S(\mathcal{B}(T^3))$ has only two vertices - the suspension points y_1 and y_2. We assume that a refinement map $\Psi: P_4 \to S(T^3)$, such that $\Psi(A,M) = \{y_1, y_2\}$, exists and find a contradiction.

From now on, A,B,C,...,T denote vertices of P_4 and 1,2,...,2',0 denote facets.

In $S(\mathcal{B}(T^{d-1}))$ for each facet there is a unique edge meeting the facet in just $\{y_1, y_2\}$. Hence in $\mathcal{B}(P_4)$ for each pseudofacet there is a pseudoedge meeting the pseudofacet in just $\{x_1, x_2\}$. This pseudoedge is an arc. Hence in P_4 the candidates for pseudofacets are the unions of facets admitting disjoint paths from A to M. Since there are four pseudofacets, each of which contains A and M, each pseudofacet in P_4 contains exactly one facet meeting

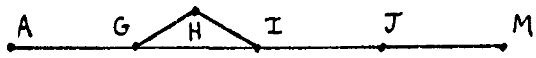
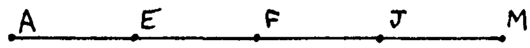
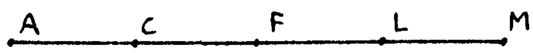
Facet pair $a \cup b$	Vertices not in $a \cup b$	The subgraph of $G(P_4)$ spanned by A, M and the vertices not in $a \cup b$
$1 \cup 5$	P Q R S T	
$1 \cup 1'$	G Q S	
$1 \cup 2'$	G H I P R	
$1 \cup 0$	G H I J	
$2 \cup 5$	E R S T	
$2 \cup 1'$	E F L	
$2 \cup 2'$	I K R	
$2 \cup 0$	E F J	
$3 \cup 5$	N O T	
$3 \cup 1'$	C F L	
$3 \cup 2'$	C D N K	
$3 \cup 0$	C F J	
$4 \cup 5$	B N O P Q	
$4 \cup 1'$	B L O Q	
$4 \cup 2'$	D H K N P	
$4 \cup 0$	B D H	

Table 6.2

A and one facet meeting M. Since no facet contains both A and M, and there are only eight facets, each pseudofacet in P_4 is the union of an element of $\{1,2,3,4\}$ with an element of $\{5,1',2',0\}$. Because P_4 is simple and dual to a neighbourly polytope, each pair of facets meets in a 2-face and hence the union of any pair of facets is a 3-cell.

In Table 6.2, for each pair of facets $a \cup b$, $a \in \{1,2,3,4\}$, $b \in \{5,1',2',0\}$, we give the subgraph of the graph of

P_4 generated by A, M and the vertices of P_4 not in $a \cup b$, if A and M are connected in that subgraph. From Table 6.2 we can read off the pairs of facets $a \cup b$ admitting a disjoint path from A to M.

Table 6.2 shows that $1 \cup 0$ and $4 \cup 1'$ must be two of the pseudofacets of P_4. The other two pseudofacets consist of an element of $\{2,3\}$ with an element of $\{5,2'\}$. Hence the remaining pseudofacets are $2 \cup 5$ and $3 \cup 2'$.

We now show that these pseudofacets cannot come from a refinement of $S(\mathcal{B}(T^3))$. For, let $F_1,\ldots,F_4$ be the facets of T^3, labelled arbitrarily. Since $F_1 \cap \ldots \cap F_4 = \emptyset$, in $S(\mathcal{B}(T^3))$,

$$S(F_1) \cap \ldots \cap S(F_4) = \{y_1,y_2\}.$$

Hence we would have

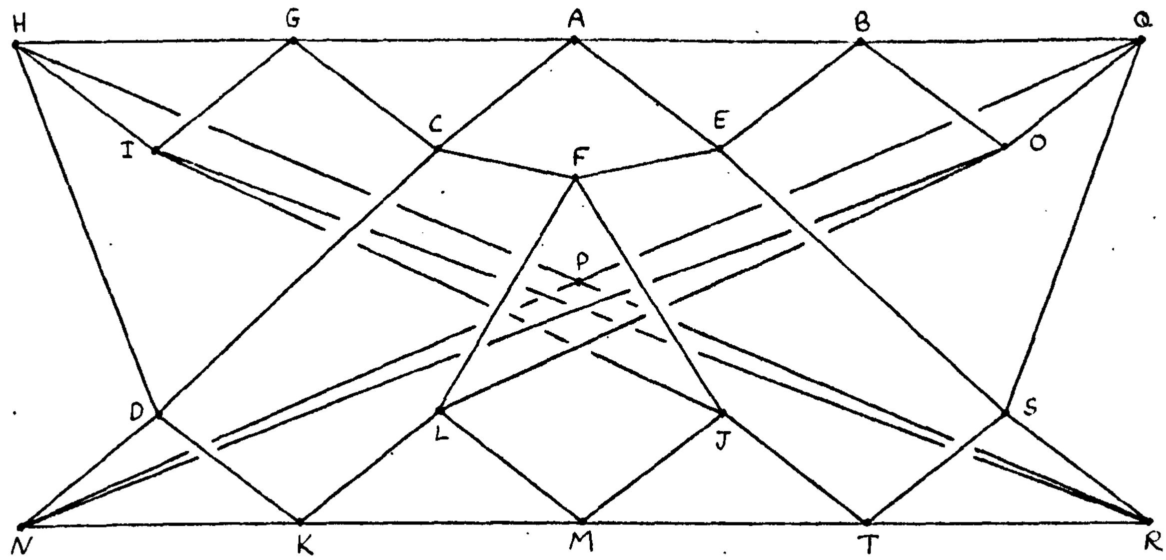
$$(1 \cup 0) \cap (2 \cup 5) \cap (3 \cup 2') \cap (4 \cup 1') = \{A,M\}.$$

But

$$\begin{aligned}&(1 \cup 0) \cap (2 \cup 5) \cap (3 \cup 2') \cap (4 \cup 1')\\ &= \operatorname{vert} P_4 \setminus (\{G,H,I,J\} \cup \{E,R,S,T\} \cup \{C,D,N,K\} \cup \{B,L,O,Q\})\\ &= \{A,F,M,P\}.\end{aligned}$$

Therefore our only candidates are not pseudofacets in P_4, establishing that $\mathcal{B}(P_4)$ is not a refinement of $S(\mathcal{B}(T^3))$.

It follows easily that $\mathcal{B}(P_d)$ is not a refinement of $S(\mathcal{B}(T^{d-1}))$, with x_1 and x_2 principal. For, suppose the contrary and there exists a refinement map $\Psi: P_d \rightarrow S(T^{d-1})$. Each of x_1 and x_2 lies in exactly d facets of P_d, and exactly d pseudofacets. Hence each facet containing x_1 (or x_2) belongs to a different pseudofacet. Every facet of P_d contains either x_1 or x_2, so we conclude that the facets containing both x_1 and x_2 are pseudofacets and hence principal faces. Since the smallest face containing x_1 and x_2 is P_4, every facet containing P_4 is principal. Hence P_4 itself



The graph of P_4

Fig. 6.5

is a principal face. But then $\Psi|_{P_4}$ is a refinement map of P_4 onto a 4-face of $S(T^{d-1})$, which is of the form $S(T^3)$. We have already shown that no such map exists. This contradiction concludes the proof.

Corollary 6.1: For $d \geq 4$, there exist simple d-polytopes P_d with d+4 facets and $x_1, x_2 \in$ vert P_d such that $\mathcal{B}(P_d)$ cannot be expressed as a refinement of $\mathcal{B}(T^d)$ in such a way that x_1 and x_2 are principal vertices.

The counterexample of theorem 6.3 may be modified in a trivial way to yield the following theorem.

Theorem 6.4: For $d \geq 4$, there exists a polytope Q_d such that for each pair i,j, $0 \leq i,j \leq d-1$, there exist faces F_i, F_j such that $\dim F_i = i$, $\dim F_j = j$, and for no representation of $\mathcal{B}(Q_d)$ as a refinement of $\mathcal{B}(T^d)$ are both F_i and F_j principal faces.

Proof: Construct Q_4 by truncating P_4 at A and at M to produce two new simplicial facets G_1 and G_2. We claim that no refinement of Q_4 can have principal faces in both G_1 and G_2. For suppose that $\Psi: Q_4 \to T^4$ yields such a refinement. Both G_1 and G_2 cannot be be principal, as any pair of pseudofacets meet.

If G_1 is a pseudofacet, but G_2 is not, the d pseudofacets of Q_4 other than G_1 define a refinement of $S(\mathcal{B}(T^3))$ in P_4 for which A and M are principal, a contradiction.

Hence neither G_1 nor G_2 are principal. But if G_1 and G_2 both contain principal faces, both G_1 and G_2 contain principal vertices. Hence at least d pseudofacets meet each of G_1 and G_2. If d+1 pseudofacets meet G_1, say, then G_1 must be one of them and is principal. Therefore exactly d pseudofacets meet each of G_1 and G_2. Equivalently, exactly one pseudofacet, say C_d, meets G_1 but not G_2, and exactly one pseudofacet, say C_{d+1}, meets G_2 but not G_1. Let the other pseudofacets be $C_1,\ldots,C_{d-1}$. Then $C_1,\ldots,C_{d-1},C_d \cup C_{d+1}$ defines a refinement of $S(\mathcal{B}(T^3))$ on Q_4 which implies a refinement of $S(\mathcal{B}(T^3))$ in P_4 with A and M principal, a contradiction.

We may therefore choose $F_i \subseteq G_1$, $F_j \leq G_2$ arbitrarily to satisfy the theorem.

Let Q_d be a (d-4)-fold pyramid over Q_4, $d \geqslant 4$. Assume by induction that Q_{d-1} is not a refinement of T^{d-1} with principal vertices in both G_1 and G_2. Then Q_d is a pyramid over Q_{d-1} with apex x, say. Let us assume Q_d is a refinement of T^d with principal vertices in both G_1 and G_2, and derive a contradiction. Only one facet, namely Q_{d-1}, of Q_d does not contain x. Hence x lies in at least d pseudofacets of the refinement. Therefore x lies in exactly d pseudofacets and is principal. But then Q_{d-1} is also principal, and hence the forbidden refinement is induced in Q_{d-1}.

Therefore F_i and F_j satisfy the theorem if F_i and F_j meet G_1 and G_2 respectively, and are simplicial. For, all the vertices of F_i and F_j will be principal, and at least one lies in G_1 and one in G_2. A suitable choice may be always be made by taking F_i

and F_j to be pyramids over faces of G_1 and G_2, respectively. This concludes the proof.

An i-face and a j-face of a d-simplex meet in a face of dimension at least $i+j-d$. Hence if K_i and K_j are an i-face and a j-face of a d-polytope Q, such that

$$\dim(K_i \cap K_j) < i + j - d,$$

both K_i and K_j cannot be principal if $\mathcal{B}(Q)$ is expressed as a refinement of $\mathcal{B}(T^d)$. Part of theorem 6.4 is easily satisfied.

To sum up, for $d \geqslant 4$, we have examples where pairs of faces, neither of which is a face of the other, cannot both be principal. Hence theorem 4.3, stating that any tower of faces may be chosen as principal, is, in a sense, the best possible.

3. Refinements with two principal vertices: positive results

For small numbers of facets, the desired refinements do exist.

Theorem 6.5: If $P \in \mathcal{P}^d$, $x_1, x_2 \in$ vert P, and P has d+2 facets, then there exists a refinement map $\psi: P \to T^d$ such that x_1 and x_2 are principal vertices.

Proof: We proceed by induction on d. The case $d = 1$ is trivial. Assume the result for d-1 and consider $P \in \mathcal{P}^d$. Let F be the smallest face containing x_1 and x_2. There are at most two facets not containing x_1, and at most two not containing x_2. Hence there are at most four facets not containing F, and so F has at most four facets.

Therefore F is either a simplex or a quadrilateral, and either x_1 and x_2 are joined by an edge, or there exists a third vertex x_3 joined by an edge to x_1 and x_2. In the first case, theorem 4.3 yields the desired result.

In the second case, consider the vertex figure P/x_3. At least one facet of P does not contain x_3, so P/x_3 is a (d-1)-polytope with at most d+1 facets. By induction, we may choose a refinement in P/x_3 with $[x_1,x_3]/x_3$, $[x_2,x_3]/x_3$ as principal vertices. Using lemma 4.1 and imitating the proof of theorem 4.3 we obtain the desired result.

The remaining cases are $d = 4$ and $d = 5$, with d+3 facets. Refinements do exist, but we postpone the proof until the next chapter.

Rather than consider restricted classes of polytopes, we will be less ambitious in our choice of complex to embed in the boundary of a d-polytope with two preassigned principal vertices. The following theorem merely states well-known facts concerning polytopes in the language of refinements.

Theorem 6.6: If $\mathcal{A}$ is one of the following pseudocomplexes, then for any pair of vertices x_1, x_2 of a d-polytope P, $\mathcal{B}(P)$ contains a refinement of $S(\mathcal{A})$ for which x_1 and x_2 are principal (that is, correspond to the suspension points).

a. A set of d distinct points.

b. A complex satisfying the theorem for d-1.

c. $S^{(d-1)}(\mathcal{B}(T^1))$ (where $S^{(n)}$ denotes taking suspensions n times).

d. If $d = 3$, $S(\mathcal{B}(T^2))$.

Proof: a. The CW-complex $S(\mathcal{A})$ corresponds to a set of d disjoint arcs from x_1 to x_2 and hence part a follows from the d-connectedness of G(P).

b. Let H_i be hyperplanes such that $P \cap H_i = \{x_i\}$, $i = 1,2$, and assume $H_1 \cap H_2 \neq \emptyset$, as we may easily do. Choose

$$a \in H_1 \cap H_2 \setminus \bigcup\{\text{aff } F \mid F \text{ is a facet of } P.\}.$$

Let H be a hyperplane strictly separating a and P and let $\pi: P \to H$

be radial projection with centre a. Then πP is a (d-1)-polytope, and $\pi x_1, \pi x_2 \in$ vert πP. Let

$$\mathcal{P}_1 = \{F \mid F \in \mathcal{T}(G),\ G \text{ is a facet of } P,\ a \text{ is beyond } G\},$$

$$\mathcal{P}_2 = \{F \mid F \in \mathcal{T}(G),\ G \text{ is a facet of } P,\ a \text{ is beneath } G\},$$

and let $P_i = \cup \mathcal{P}_i$.

It is well-known that $\pi|_{P_i}: P_i \to \pi P$ is a refinement map and that

$$\pi|_{P_1 \cap P_2}: P_1 \cap P_2 \to \text{bd}\ \pi P$$

defines a combinatorial isomorphism of $\mathcal{P}_1 \cap \mathcal{P}_2$ and $\mathcal{B}(\pi P)$. Hence $\mathcal{B}(P)$ contains an isomorphic copy of $\mathcal{B}(\pi P)$ which contains x_1 and x_2. Hence for any refinement in $\mathcal{B}(\pi P)$ there is an isomorphic refinement in $\mathcal{B}(P)$.

c. An easy inductive proof shows that $S^{(d-1)}(\mathcal{B}(T^1))$ is homeomorphic to a (d-1)-sphere and that it is the union of two (d-1)-balls intersecting in a (d-2)-sphere containing the suspension points. Observe that $\mathcal{B}(T^1)$ itself is simply a set of two distinct points.

Using the notation of part b, in bd P, P_1 and P_2 are (d-1)-balls meeting in the (d-2)-sphere $P_1 \cap P_2$. Since $\mathcal{P}_1 \cap \mathcal{P}_2$ is combinatorially isomorphic to the boundary complex of a (d-1)-polytope, $\mathcal{P}_1 \cap \mathcal{P}_2$ is a refinement of $S^{(d-2)}(\mathcal{B}(T^1))$ by induction, with refinement map

$$\psi: P_1 \cap P_2 \to S^{(d-2)}(\mathcal{B}(T^1)),$$

for which x_1 and x_2 are principal.

Extending ψ, defined on bd P_1 and bd P_2 to a homeomorphism

$$\bar{\psi}: P_1 \cup P_2 \to S^{(d-1)}(\mathcal{B}(T^1))$$

yields the desired refinement.

d. Let $\Gamma_1, \Gamma_2, \Gamma_3$ be three disjoint arcs from x_1 to x_2 and let D_1, D_2, D_3 be the three connected components of bd $P \setminus (\Gamma_1 \cup \Gamma_2 \cup \Gamma_3)$.

The D_i are open 2-cells and $C_i = \mathrm{cl}\, D_i$ are 2-cells. Then C_1, C_2, C_3 are the pseudofacets of a refinement of $S(\mathcal{B}(T^2))$ in $\mathcal{B}(P)$ for which x_1 and x_2 are principal vertices.

This concludes the proof of the theorem.

4. Conjectures

Parts c and d of theorem 6.6 motivate the following conjecture.

Conjecture 6.1: For $P \in \mathcal{P}^d$, $d \geq 2$, for any preassignment of principal vertices there exists a refinement map

$$\psi: P \to S^{(d-2)}(T^2).$$

For $d = 2$, $S^{(0)}(T^2) = T^2$, and there are three principal vertices. We have shown that three vertices may be preassigned as principal for a refinement map $\psi: P \to T^2$, so the conjecture is true in this case.

For $d \geq 3$, there are two principal vertices to preassign. Theorem 6.6d states that the conjecture is true for $d = 3$. For $d \geq 4$, nothing is known.

Conjecture 6.2: For $d \geq 3$, there exists $Q_d \in \mathcal{P}^d$ such that the principal vertices may not be preassigned arbitrarily in a refinement map

$$\psi: Q_d \to S^{(d-3)}(T^3).$$

For $d = 3$, $S^{(0)}(T^3) = T^3$, and so there are four principal vertices. As we have seen, the triangular prism is a suitable Q_3.

For $d \geq 4$, there are two principal vertices. The counterexample P_4 of theorem 6.2 is a suitable Q_4. For $d \geq 5$, nothing is known.

The following conjecture is a natural generalization of Balinski's theorem concerning the d-connectedness of polytopes. For $P \in \mathcal{P}^d$, $x_1, x_2 \in \mathrm{vert}\, P$, a strong j-chain of faces joining x_1 and x_2

is a sequence $F_1, \ldots, F_n$ of j-faces of P such that $x_1 \in F_1$, $x_2 \in F_n$ and $\dim F_i \cap F_{i+1} = j-1$, $i = 1, \ldots, n-1$.

<u>Conjecture 6.3</u>: For $P \in \mathcal{P}^d$, $x_1, x_2 \in \text{vert } P$ and $d_1, \ldots, d_k \in \mathbb{N}$ such that $d_1 + \ldots + d_k = d$, there exists a set of chains C_j such that C_j is a strong d_j-chain from x_1 to x_2, and $C_j \cap C_l = \{x_1, x_2\}$ if $j \neq l$.

If $d_1 = \ldots = d_d = 1$, the conjecture asserts that x_1 and x_2 are joined by d disjoint paths, a statement we know to be true. If $d = 3$, any pair of vertices are joined by a path and a disjoint facet chain, so the conjecture is true in this case also. However, for $d \geq 4$, nothing is known, even for the special case of a path and a disjoint facet chain.

If P is expressed as a refinement of $S(\mathcal{B}(T^{d-1}))$, with x_1 and x_2 as principal vertices, there exist disjoint d_j-pseudofaces. These pseudofaces are suspensions of a set of disjoint (in the usual set-theoretic sense) (d_j-1)-faces of T^{d-1}. It may be shown that each d_j-pseudofacet contains a strong d_j-chain from x_1 to x_2. Hence the conjecture is true for P.

Chapter 7. SPLICING AND SPLITTING FACETS

1. Definitions and preliminaries

Let C be a d-gcc with facets $F_1, \ldots, F_{n+1}$ and suppose that $F_n \cap F_{n+1}$ is a (d-2)-face. Then $F_n \cup F_{n+1}$ is a (d-1)-cell. Let $G_i = F_i$, $i = 1, \ldots, n-1$, and $G_n = F_n \cup F_{n+1}$. If $G_1, \ldots, G_n$ are the facets of a d-gcc D, we say that D is obtained from C by <u>splicing</u> the facets F_n and F_{n+1}, or by <u>deleting</u> the (d-2)-face $F_n \cap F_{n+1}$.

Fig. 7.1 depicts a cube C with its usual facial structure as a polytope, in which the facets F_5 and F_6 may be spliced to yield a 3-gcc D isomorphic to a triangular prism. In D, G_4 and G_5 may be spliced to produce a 3-gcc E, isomorphic to a simplex. Any triangular and any square face of D may be spliced, but no pair of square facets may be spliced. For, $G_3 \cap (G_2 \cup G_5)$ is not connected and hence cannot be a cell, but the definition of gcc requires that the intersection of faces be a face, and all non-

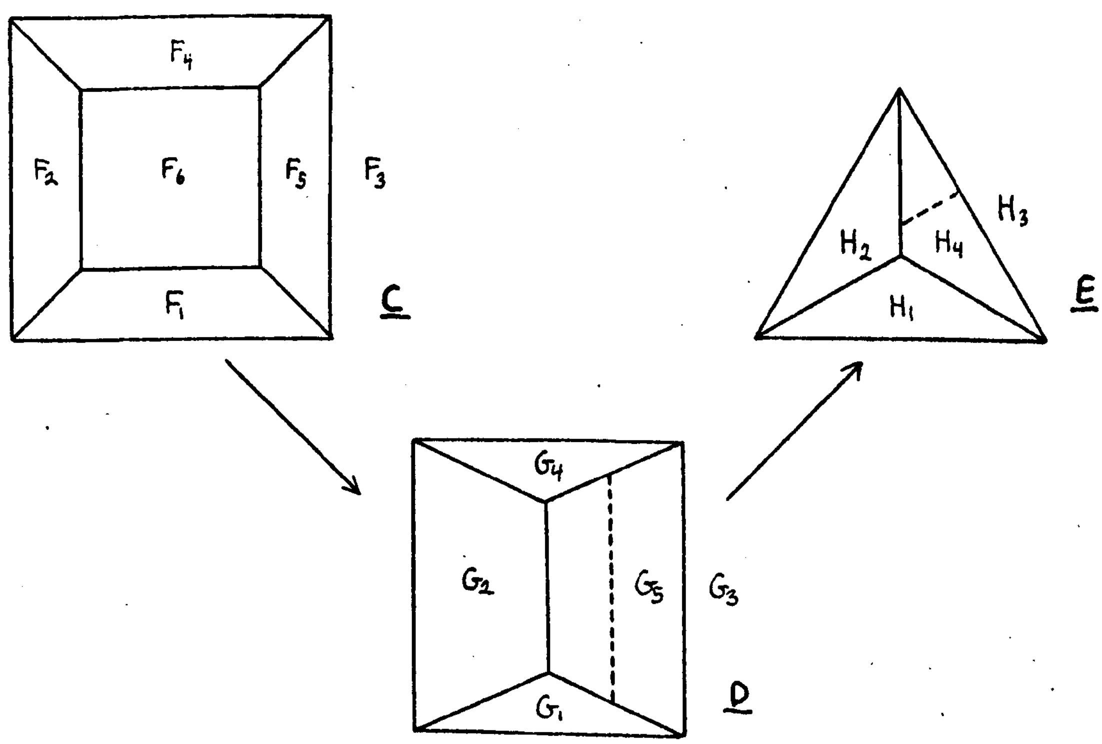


Fig. 7.1

empty faces are cells. Therefore G_2 and G_5 may not be spliced.

We conclude that for $d \geq 3$, a d-gcc may contain (d-2)-faces which may not be deleted. It is clear that any vertex of a 2-gcc which is not a simplex may be deleted.

Although we will be concerned primarily with splicing pairs of facets, we remark that the above definition may be extended to splice any number of facets of C whose union is a (d-1)-cell. For instance, for $P \in \mathcal{P}^d$, $x \in \text{vert } P$, the facets of $\text{ast}(x;\mathcal{B}(P))$ may be spliced to yield a gcc isomorphic to the pyramid over the vertex figure at x. By splicing facets, we shall mean splicing pairs of facets unless otherwise indicated.

The main object of this chapter is to prove that for $d = 4,5$, d-polytopes with d+3 facets are refinements of the d-simplex with two preassigned principal vertices. We shall need several lemmas.

Lemma 7.1: Let C be a d-gcc with facets $F_1,\ldots,F_{n+1}$ such that $F_m \cup \ldots \cup F_{n+1}$ is a (d-1)-cell. Then $F_m,\ldots,F_{n+1}$ may be spliced if and only if for all

$$K = \cap\mathcal{F}, \quad \mathcal{F} \subseteq \{F_1,\ldots,F_{m-1}\},$$

the set

$$K \cap (F_m \cup \ldots \cup F_{n+1})$$

is a face of K or is a cell which is a union of facets of K.

Proof: Let $G_i = F_i$, $i = 1,\ldots,m-1$, and $G_m = F_m \cup \ldots \cup F_{n+1}$.

To show that the given condition is necessary, suppose that for some $\mathcal{F} \subseteq \{F_1,\ldots,F_{m-1}\}$ the set $B = K \cap (F_m \cup \ldots \cup F_{n+1})$ is neither a face of K nor a union of facets of K, but $G_1,\ldots,G_m$ are the facets of a d-gcc D. Then B is a face in D and hence a cell. Since B is not a union of facets in C of K, B contains no facets in C of K. Hence each facet in C of K is of the form $K \cap F_j$, $j = 1,\ldots,m-1$ (although a set of this form need not be a facet of K). Therefore each facet in C of K, $K \cap F_j = K \cap G_j$, is a facet in D of K. Since B is a face of D, B is an intersection of facets in D of K. But this is a contradiction, since B would then be an intersection of facets in C of K; that is, a face in C of K. This establishes the result in one direction.

To show that the given condition is sufficient, suppose it holds. Define D to be the set C with boundary structure

$$\mathcal{B}_D(D) = \{G_{i_1} \cap \ldots \cap G_{i_k} \mid \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, m\}\}.$$

We claim that D is a gcc.

A boundary structure is induced on each face $F \in \mathcal{B}_D(D)$ by

$$\mathcal{B}_D(F) = \{F \cap G \mid G \in \mathcal{B}_D(D),\ F \nsubseteq G\}.$$

It is clear that if $F \in \mathcal{B}_D(D)$ and $G \in \mathcal{B}_D(F)$, then $G \in \mathcal{B}_D(D)$. Also, if $F, G \in \mathcal{B}_D(D)$, then $F \cap G \in \mathcal{B}_D(D)$.

It remains to show that each $F \in \mathcal{B}_D(D)$ is a gcc. Assume by induction that the result holds for k-gcc's, $k \leq d-1$. The result is clearly true for 2-gcc's.

In the first case, $F \in \mathcal{B}_D(D)$, $F \nsubseteq G_m$. Then F is of the form

$$F = \cap \mathcal{F},\ \emptyset \neq \mathcal{F},\ \mathcal{F} \subseteq \{F_1, \ldots, F_{m-1}\}.$$

Either $F \cap G_m$ is a face in C of F, in which case $\mathcal{B}_D(F) = \mathcal{B}_C(F)$ and F is a gcc, or $F \cap G_m$ is a cell and a union of facets of F. In the latter case, we show by induction that the facets of F in $F \cap G_m$ may be spliced. The remaining facets of F are of the form $F \cap F_i$, $i \in \{1, \ldots, m-1\}$, although not all such sets are facets of F. Let J be a face of F which is an intersection of some of these remaining facets. Then $J = \cap \mathcal{A}$, $\mathcal{A} \subseteq \{F_1, \ldots, F_{m-1}\}$, and hence $J \cap (G_m \cap F) = J \cap G_m$ is a face of J or a cell which is a union of facets of J, by the hypothesis of the lemma. By the inductive assumption, F is a gcc.

In the second case, if $F \in \mathcal{B}_D(D)$, $F \subseteq G_m$, $F \neq G_m$, then since

$F \neq G_m$, for some $i \in \{1,\ldots,m-1\}$, $F \subseteq F_i = G_i$. Therefore $F \in \mathcal{B}_D(G_i)$, and since G_i is a gcc by the first case, F is a d-gcc.

The third case is G_m itself. But G_m is a (d-1)-cell by assumption, and by the first two cases all its faces are gcc's. Hence G_m is a gcc.

We have now shown that the elements of $\mathcal{B}_D(D)$ are gcc's. This concludes the proof.

Lemma 7.2: Let $\overline{X}^* = \{\overline{F}_1,\ldots,\overline{F}_n\}$ be a dual diagram of $P \in \mathcal{P}^d$. Then F_1 and F_2 may be spliced only if for all $\overline{Y} \subseteq \overline{X}^*$ such that

$$0 \in \operatorname{relint} \operatorname{conv} \overline{Y},$$

$$0 \in \operatorname{conv}(\overline{Y} \setminus \{\overline{F}_1\}),$$

$$0 \in \operatorname{conv}(\overline{Y} \setminus \{\overline{F}_2\}),$$

it is true that

$$0 \in \operatorname{conv}(\overline{Y} \setminus \{\overline{F}_1,\overline{F}_2\}).$$

Proof: Assume F_1 and F_2 may be spliced to form a gcc C. Then each face K of P, $K \nsubseteq F_1$, $K \nsubseteq F_2$, is a face of C. Hence $K \cap (F_1 \cup F_2)$ is either empty, or a cell. Since a cell is connected, if $K \cap F_1 \neq \emptyset$ and $K \cap F_2 \neq \emptyset$, then $K \cap F_1 \cap F_2 \neq \emptyset$.

Translating this statement into dual diagram terms gives us the lemma. Let Y be the coface of K. Then K is a face, so $0 \in \operatorname{relint} \operatorname{conv} \overline{Y}$. If $K \cap F_1 \neq \emptyset$, then $K \cap F_1$ is a face of P and hence for some $\overline{Y}_1 \subseteq \overline{Y} \setminus \{\overline{F}_1\}$, $0 \in \operatorname{relint} \operatorname{conv} \overline{Y}_1$. Conversely, if $0 \in \operatorname{conv}(\overline{Y} \setminus \{\overline{F}_1\})$, there is a unique face R_1 of the polytope $\operatorname{conv}(\overline{Y} \setminus \{\overline{F}_1\})$ containing 0 in its relative interior. Choosing $\overline{Y}_1 = R_1 \cap (\overline{Y} \setminus \{F_1\})$, we see that Y_1 is the coface of the face $K \cap F_1$. Similarly, $K \cap F_2 \neq \emptyset$ if and only if $0 \in \operatorname{conv}(\overline{Y} \setminus \{\overline{F}_2\})$ and $K \cap F_1 \cap F_2 \neq \emptyset$ if and only if $0 \in \operatorname{conv}(\overline{Y} \setminus \{\overline{F}_1,\overline{F}_2\})$. The lemma is now proved.

Lemma 7.3: Let $\overline{X}^* = \{\overline{F}_1,\dots,\overline{F}_{d+3}\}$ be a contracted dual Gale diagram in E^2 of $P\in\mathcal{P}^d$. Then F_1 and F_2 may be spliced if and only if either $\overline{F}_1$ and $\overline{F}_2$ are diametrically opposite, or, $\overline{F}_1$ and $\overline{F}_2$ lie on adjacent diameters, and $\overline{F}_1$ is adjacent to an empty end of the diameter of $\overline{F}_2$ and $\overline{F}_2$ is adjacent to an empty end of the diameter of $\overline{F}_1$.

Proof: To show that the given condition is sufficient, let K be a face of P contained in at least one facet of P other than F_1 or F_2. By lemma 7.1, F_1 and F_2 may be spliced if $K\cap(F_1\cup F_2)$ is a face of K or a cell which is a union of facets of K. We may assume without loss of generality that P is not a pyramid, and hence no point of $\overline{X}^*$ lies at the origin, for it is easy to see that facet splicings in a d-pyramid are precisely those induced by splicings of (d-2)-faces in the base.

Let Y, Y_1, Y_2, Y_{12} be the cofaces of K, $K\cap F_1$, $K\cap F_2$, $K\cap F_1\cap F_2$. As in the proof of lemma 7.2, $\overline{Y}_1$, $\overline{Y}_2$ and $\overline{Y}_{12}$ may be computed from $\overline{Y}$ as follows. Since $K\cap F_1\subseteq F_1$, $\overline{F}_1\notin\overline{Y}_1$. Let

$$S_1 = \operatorname{conv}(\overline{Y}\setminus\{\overline{F}_1\})$$

and let R_1 be the face of S_1 containing 0 in its relative interior if $0\in S_1$, and $\emptyset$ otherwise. Then

$$\overline{Y}_1 = R_1\cap(\overline{Y}\setminus\{\overline{F}_1\}).$$

Similarly, S_2 and R_2, and S_{12} and R_{12}, are defined with respect to $K\cap F_2$ and $K\cap F_1\cap F_2$, respectively.

We now prove that the conditions of the lemma imply that if $0\in S_1$ and $0\in S_2$, then $0\in S_{12}$. (This is the necessary condition imposed by lemma 7.2.)

In the first case, assume F_1 and F_2 lie on adjacent diameters as described in the hypothesis of the lemma and assume $0\in S_1$, $0\in S_2$, but $0\notin S_{12}$. Since

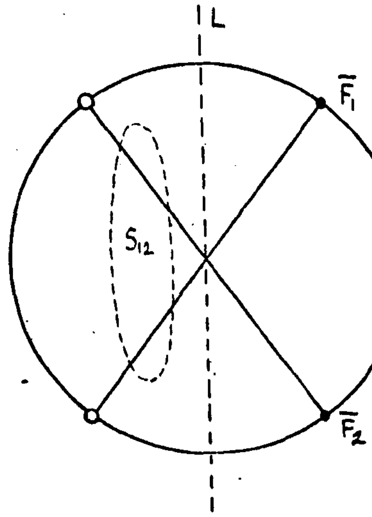


Fig. 7.2

$$0 \in S_1 = \operatorname{conv}(S_{12} \cup \{\overline{F}_2\}),$$

we have

$$S_{12} \cap \operatorname{pos}\{-\overline{F}_2\} \neq \emptyset;$$

since

$$0 \in S_2 = \operatorname{conv}(S_{12} \cup \{\overline{F}_1\}),$$

we have

$$S_{12} \cap \operatorname{pos}\{-\overline{F}_1\} \neq \emptyset.$$

Because S_{12} is compact and convex, and pos $\{\overline{F}_1, \overline{F}_2\}$ is closed and convex, there exists a line L' strictly separating these two sets. A translate of L' supports pos $\{\overline{F}_1, \overline{F}_2\}$ in 0; call this translate L. The choice of $\overline{F}_1$ and $\overline{F}_2$ now assures us that all the points of $\overline{X}^*$ on the same side of L as S_{12} lie in relint pos $\{-\overline{F}_1, -\overline{F}_2\}$ and hence

$$S_{12} \subseteq \operatorname{relint} \operatorname{pos}\{-\overline{F}_1, -\overline{F}_2\}.$$

But

$$S_{12} \cap \operatorname{pos}\{-F_1\} \neq \emptyset.$$

This contradiction establishes that if $0 \in S_1$ and $0 \in S_2$, then $0 \in S_{12}$.

In the second case, $\overline{F}_1$ and $\overline{F}_2$ are diametrically opposite.

The proof is similar to that of the first case. We do not need the second case in the sequel, so we will not go into details.

The remainder of the proof consists of showing that in two dimensions the condition that $0 \in S_1$ and $0 \in S_2$ imply $0 \in S_{12}$ is sufficient as well as necessary for facet splicing.

If $K \cap F_1 \subseteq F_2$ or $K \cap F_2 \subseteq F_1$, then $K \cap (F_1 \cup F_2) \subseteq K \cap F_2$ or $K \cap (F_1 \cup F_2) \in K \cap F_1$, and hence $K \cap (F_1 \cup F_2)$ is a face of K.

Assume therefore that $K \cap (F_1 \cup F_2) \not\subseteq F_1$, $K \cap (F_1 \cup F_2) \not\subseteq F_2$. We must prove that $K \cap (F_1 \cup F_2)$ is a cell which is a union of facets of K. Equivalently, we require, if $\dim K = k$, that

$$\dim K \cap F_1 = \dim K \cap F_2 = k-1,$$

$$\dim K \cap F_1 \cap F_2 = k-2.$$

Using the formula relating the dimension of a face and the dimension of its coface in the dual diagram,

$$\text{card } \overline{Y}_1 - \dim \text{lin } \overline{Y}_1 - 1 = \text{card } \overline{Y}_2 - \dim \text{lin } \overline{Y}_2 - 1 = k-1,$$

$$\text{card } \overline{Y}_{12} - \dim \text{lin } \overline{Y}_{12} - 1 = k-2.$$

Hence

$$\begin{aligned} k &= \text{card } \overline{Y}_1 - \dim \text{lin } \overline{Y}_1 \\ &= \text{card } \overline{Y}_2 - \dim \text{lin } \overline{Y}_2 \\ &= \text{card } \overline{Y}_{12} - \dim \text{lin } \overline{Y}_{12} + 1 \end{aligned}$$

is what we need to prove.

Since $S_{12} \neq \emptyset$, there are three cases: $\dim \text{lin } \overline{Y}_{12} = 0, 1, 2$.

Since no points of $\overline{X}^*$ lie at 0, $\dim \text{lin } \overline{Y}_{12} = 0$ is impossible.

If $\dim \text{lin } \overline{Y}_{12} = 1$, the diameter of $\overline{Y}_{12}$ contains points of $\overline{Y}_{12}$ at both ends. Choice of $\overline{F}_1$ and $\overline{F}_2$ dictates that $\overline{F}_1$ and $\overline{F}_2$ are strictly separated by this diameter. But since $\dim \text{lin } \overline{Y}_{12} = 1$, one of the open halfspaces determined by this diameter contains at most one point of $\overline{Y}$. This point is one of

$\overline{F}_1$ or $\overline{F}_2$, say $\overline{F}_1$. But then $\overline{F}_2 \notin \overline{Y}_1$, so $K \cap F_1 \subseteq F_2$, a contradiction. Hence for $i = 1,2$,

$$\text{card } \overline{Y}_i - \dim \text{lin } \overline{Y}_i = \text{card } \overline{Y} - 1 - \dim \text{lin } \overline{Y} = k,$$

$$\text{card } \overline{Y}_{12} - \dim \text{lin } \overline{Y}_{12} = \text{card } \overline{Y} - 2 - \dim \text{lin } \overline{Y} = k-1,$$

as required. This concludes the first case.

In the second case, $\overline{F}_1$ and $\overline{F}_2$ are diametrically opposite. The proof is the same as in the first case, except when $\dim \text{lin } \overline{Y}_{12} = 1$. It is then possible for $\overline{F}_1$ and $\overline{F}_2$ to lie on the same diameter as $\overline{Y}_{12}$. But then

$$\overline{Y}_1 = \overline{Y} \setminus \{\overline{F}_1\},$$

$$\overline{Y}_2 = \overline{Y} \setminus \{\overline{F}_2\},$$

$$\overline{Y}_{12} = \overline{Y} \setminus \{\overline{F}_1, \overline{F}_2\},$$

so the dimensions of $K \cap F_1$, $K \cap F_2$ and $K \cap F_1 \cap F_2$ are as required. We have now shown that if $\overline{F}_1$ and $\overline{F}_2$ satisfy the conditions of the lemma, then F_1 and F_2 may be spliced.

In the other direction, we now show that if $\overline{F}_1$ and $\overline{F}_2$ fail to satisfy the conditions of the lemma, we can find $\overline{Y} \subseteq \overline{X}^*$ such that

$$0 \in \text{conv}(\overline{Y} \setminus \{\overline{F}_1\}),$$

$$0 \in \text{conv}(\overline{Y} \setminus \{\overline{F}_2\}),$$

but

$$0 \notin \text{conv}(\overline{Y} \setminus \{\overline{F}_1, \overline{F}_2\}).$$

Then by lemma 7.2, F_1 and F_2 may not be spliced.

We assume that $\overline{X}^*$ is contracted; that is, no two diameters have adjacent empty ends. Again, we assume no points of $\overline{X}^*$ lie at 0. There are several cases to consider. Assume that $\overline{F}_1$ and $\overline{F}_2$ do not satisfy the conditions of the lemma.

In the first case, $\overline{F}_1$ and $\overline{F}_2$ occupy the same diameter. Since they are not at opposite ends, they must coincide. If there exists a point $\overline{F}_3$ opposite to $\overline{F}_1$ and $\overline{F}_2$, let

$$\overline{Y} = \{\overline{F}_1, \overline{F}_2, \overline{F}_3\}.$$

If the end opposite $\overline{F}_1$ and $\overline{F}_2$ is empty, the two diameters adjacent to this empty end are occupied, since $\overline{X}^*$ is contracted, by a point $\overline{F}_3$ on one side and $\overline{F}_4$ on the other. Let

$$\overline{Y} = \{\overline{F}_1, \overline{F}_2, \overline{F}_3, \overline{F}_4\}.$$

In the second case, $\overline{F}_1$ and $\overline{F}_2$ occupy adjacent diameters, and are adjacent to each other. If both of the opposite ends of these diameters are occupied by points $\overline{F}_3$ and $\overline{F}_4$, say, let

$$\overline{Y} = \{\overline{F}_1, \overline{F}_2, \overline{F}_3, \overline{F}_4\}.$$

If the end opposite $\overline{F}_1$ is occupied by $\overline{F}_3$, say, and the end opposite $\overline{F}_2$ is empty, let $\overline{F}_4$ be adjacent to this empty end, but not coincident with $\overline{F}_3$. Let

$$\overline{Y} = \{\overline{F}_1, \overline{F}_2, \overline{F}_3, \overline{F}_4\}.$$

Both ends opposite to $\overline{F}_1$ and $\overline{F}_2$ cannot be empty, because if this were the case and $\overline{F}_1$ and $\overline{F}_2$ were adjacent, $\overline{X}^*$ would not be contracted; if $\overline{F}_1$ and $\overline{F}_2$ were not adjacent, $\overline{F}_1$ and $\overline{F}_2$ would satisfy the conditions of the lemma.

In the third case, the diameters of $\overline{F}_1$ and $\overline{F}_2$ are not adjacent. Again, if both opposite ends are occupied by $\overline{F}_3$ and $\overline{F}_4$, say, let

$$\overline{Y} = \{\overline{F}_1, \overline{F}_2, \overline{F}_3, \overline{F}_4\}.$$

If the end opposite $\overline{F}_1$, say, is empty, there exists a point $\overline{F}_3$ adjacent to this empty end such that

$$0 \in \text{relint conv}\,\{\overline{F}_1, \overline{F}_2, \overline{F}_3\}.$$

Let

$$\overline{Y} = \{\overline{F}_1, \overline{F}_2, \overline{F}_3\}.$$

There are no other possibilities. Thus either $\overline{F}_1$ and $\overline{F}_2$ satisfy lemma 7.3 or they fail lemma 7.2. This concludes the proof.

We will see (theorem 7.1) that a d-gcc with d+2 facets is polytopal. Hence splicing a pair of facets of a d-polytope with $d+3$ facets yields a polytopal d-gcc. In general, for $d \geq 4$, it is not known whether splicing a pair of facets of a polytopal d-gcc yields a polytopal d-gcc, although it may be conjectured that this is the case.

Given a d-gcc, it may be possible to use lemma 7.1 to construct a d-gcc with fewer facets. For instance, if x is a vertex of $P \in \mathcal{P}^d$, the facets not containing x may be spliced to yield a d-gcc isomorphic to the pyramid over the vertex figure at x.

The inverse of splicing a pair of facets is called facet splitting. Barnette (1975) defines facet splitting for gcc's directly in the following way.

Let C be a d-gcc. A (d-1)-cell K is said to separate C if $C \setminus K$ has two components C_1 and C_2 such that $C_1 \cup K$ and $C_2 \cup K$ are d-cells. The cell K is said to split C provided

1. relint $K \subseteq$ relint C, relbd $K \subseteq$ relbd C,
2. K separates C,
3. for every facet F of C, $K \cap F$ is a cell (possibly $\emptyset$),
4. if K meets a face F of C, then either $F \subseteq K$, $F \cap K$ is a face of F or $K \cap F$ separates F.

If K splits C, then the d-cells $C_i \cup K$ are d-gcc's if the faces of $C_i \cup K$ are defined to be sets of the form

$(C_i \cup K) \cap F$, $F \in \mathcal{C}_C(C)$,

$K \cap F$, $F \in \mathcal{C}_C(C)$.

To define facet splitting, let D be a d-gcc with facets $G_1, \ldots, G_n$. We may split the facet G_n by a (d-2)-cell K to yield two new (d-1)-gcc's F_n and F_{n+1}. If the (d-1)-cells $G_1, \ldots, G_{n-1}, F_n, F_{n+1}$ define a new gcc structure B, we say that B, or any gcc isomorphic to B, is constructed from C by splitting

<u>the facet</u> G_n.

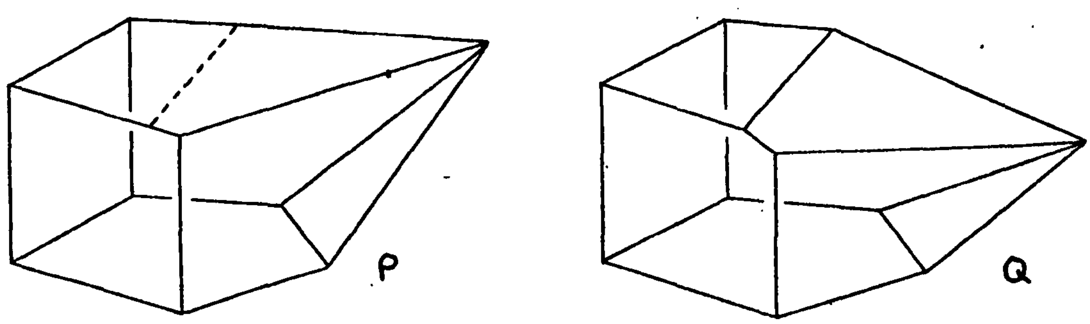
In general, there will be more than one way to split a particular facet depending on the choice of splitting face, in contrast to facet splicing, where the result is uniquely determined by the choice of the facets to be spliced.

Barnette and Grünbaum (1969) discuss facet splitting for 3-polytopes as a method of proof of Steinitz's theorem. If G is a facet of $P \in \mathcal{P}^3$ and E is a line segment meeting relint G, both of whose endpoints lie in relbd G, then it is easy to see that E splits G into 2-cells G_1 and G_2, and that the resulting complex is a gcc.

Barnette and Grünbaum prove that C is polytopal by showing that a polytope Q realizing C may be constructed by rotating G_2 slightly about the edge E and moving the other faces of P suitably, as in Fig. 7.3.

Furthermore, every $Q \in \mathcal{P}^3$ other than the simplex may be constructed from the simplex by a sequence of facet splittings. Equivalently, Q contains a pair of facets which may be spliced.

For $d \geq 4$, splitting a facet of a d-gcc need not yield a d-gcc. Facet splittings yielding gcc's are characterized in the following lemma. A d-gcc C is said to be <u>simple</u> at the k-face G provided



<u>Fig. 7.3</u>

F lies in exactly d-k facets of C.

Lemma 7.3 (Barnette (1975)): Let C be a d-gcc and G a facet of C split by a (d-2)-cell K. The facet splitting yields a d-gcc if and only if every face F which is split by $F \cap K$ is simple in C.

We shall see in section 7.4 that for $d \geq 4$ there exist d-polytopes - even simple d-polytopes - which cannot be constructed by facet splitting.

2. Polytopes and facet splitting

Theorem 3.2 stated that a d-gcc with d+1 facets is isomorphic to a d-simplex and hence is polytopal.

Theorem 7.1: A d-gcc with d+2 facets is polytopal.

Proof: Let C be a d-gcc with d+2 facets. There are two cases.

In the first case, C contains a vertex x lying in d+1 facets. The remaining facet G is a (d-1)-gcc with d+1 facets. By induction, assume that G is polytopal. Therefore C is isomorphic to the pyramid over G and is polytopal.

In the second case, each vertex of C is simple. Let x be any vertex of C and let G_1 and G_2 be the two facets of C not meeting x. Barnette (1975) proves that $G_1 \cup G_2$ is a (d-1)-cell and that a d-gcc isomorphic to C may be constructed by splitting a facet of the d-simplex.

We will now prove that splitting a facet of the simplex T^d yields a polytopal d-gcc. Let the facets of T^d be $F_0, \ldots, F_d$ and let K be a (d-2)-cell splitting F_0 into (d-1)-cells G_1 and G_2, yielding a d-gcc D with facets $F_1, \ldots, F_d, G_1, G_2$. For $d = 2$, the theorem is true, so assume $d \geq 3$. Let

$$X_1 = (\operatorname{vert} T^d \cap G_1) \setminus K,$$

$X_2 = (\text{vert } T^d \cap G_2) \setminus K,$

$X_0 = K \cap \text{vert } T^d,$

$X = X_0 \cup X_1 \cup X_2.$

Let H be a hyperplane containing X_0 and strictly separating X_1 and X_2, and let $L = H \cap F_0$. The (d-2)-cell L splits F_0 into two (d-2)-cells H_1 and H_2. The set T^d becomes a d-gcc B with facets $F_1, \ldots, F_d$, H_1, H_2. We claim that B is isomorphic to D.

The combinatorial type of a gcc is determined by knowledge of which vertices belong to which facets. We prove that $B \approx D$ by exhibiting a bijection ψ: vert $B \rightarrow$ vert D such that

$$\psi(\text{vert}_B\, F_i) = \text{vert}_D\, F_i, \ i = 1, \ldots, d;$$

$$\psi(\text{vert}_B\, H_i) = \text{vert}_D\, G_i, \ i = 1,2.$$

Define ψ as the identity on vert T^d. The remaining vertices of B are of the form $L \cap \text{relint } E$ for an edge $E = \text{conv}\{x_1, x_2\}$ of T^d such that $x_1 \in X_1$, $x_2 \in X_2$. Every such edge determines a vertex of B. On the other hand, K also meets relint E, since x_1 and x_2 are not connected in $F_0 \setminus K$. Since $K \cap E$ separates E, $K \cap E$ is a 0-cell; that is, a point. Moreover, for each edge E not of the above form, E is not split by K nor do we have $E \subseteq K$, so $K \cap \text{relint } E = \emptyset$.

We conclude that for an edge of T^d, $K \cap \text{relint } E \neq \emptyset$ if and only if $L \cap \text{relint } E \neq \emptyset$. Hence $\psi(E \cap L) = E \cap K$ defines a bijection for the remaining vertices of B. It is clear that ψ is the desired bijection.

Hence B and D are isomorphic.

In order to show that B is polytopal, let H_0 be a hyperplane containing L and strictly separating vert $T^d \setminus (X_0 \cup X_2)$ from X_2 and let H_0^- be the closed halfspace it bounds containing X_1. Then $P = T^d \cap H_0^-$ is a d-polytope with d+2 facets whose boundary complex is isomorphic to B. (Intuitively, P is constructed by rotating the facet G_2 of B slightly about the face L.)

This concludes the proof.

It is not known whether every d-gcc C with d+3 facets is polytopal. Mani (1972) has shown that every triangulation of the (d-1)-sphere with d+3 facets is polytopal. By the above lemma, splicing a pair of facets of C yields a polytopal d-gcc. In particular, if C itself is polytopal, facet splicing yields a polytopal d-gcc.

Conjecture 7.1: Splicing a pair of facets of a polytopal d-gcc C yields a polytopal d-gcc.

The conjecture is clearly true if $d = 2$. Steinitz's theorem implies that the conjecture is true for $d = 3$. For $d \geq 4$, if C has d+4 or more facets, nothing is known.

3. Refinements

In this section we will finally settle the case of refinements of the simplex with two preassigned principal vertices for d-polytopes with d+3 facets, $d = 4,5$.

Lemma 7.4: If C is a d-gcc obtained from a d-gcc D by deleting a (d-2)-face K, common to facets F_1 and F_2, then D is a refinement of C, and the non-principal faces are precisely those faces F which lie in F_1 or F_2 and contain a face of K which is simple in D.

Proof: By theorem 3.1, the d-gcc's C and D may be assumed to have the same underlying set. Then the identity map from C to D yields the desired refinement map.

Theorem 7.2: For $d = 4,5$, $P \in \mathcal{P}^d$, $x_1, x_2 \in \text{vert } P$, if P has d+3 facets, there exists a refinement map $\Psi: P \to T^d$ such that x_1 and x_2 are principal vertices.

Proof: The proof is in several steps.

The first step is to examine the case in which there exists a third vertex x_3 joined by an edge to x_1 and x_2. In this case, if the theorem is true for d-1, we may use the argument of theorem 6.5 to establish the result for d.

Let F be the smallest face of P containing both x_1 and x_2. There are at most three facets containing x_1 but not x_2, and three containing x_2 but not x_1. The remaining facets contain F. Hence F has at most six facets.

We may choose d facets meeting x_1 such that the d outer normals are linearly independent. The face F lies in at least d-3 of these facets, and, since their facet normals are linearly independent, the intersection of these d-3 facets is 3-dimensional. Hence $\dim F \leq 3$.

If $\dim F = 1$, F is the edge $[x_1, x_2]$, and the theorem follows from theorem 4.3.

If $\dim F = 2$, the third vertex x_3 exists, or $[x_1, x_2]$ is an edge, unless F is a hexagon. To see that F cannot be a hexagon, suppose the contrary and observe that since F has six facets, each corresponding to a facet of P not containing F, there are three facets containing x_1 but not x_2, and vice versa. Hence both x_1 and x_2 are simple vertices, and F is the intersection of d-3 facets. Hence $\dim F = 3$, the desired contradiction.

If $\dim F = 3$ and F has no more than five facets, then either x_3 exists, or $[x_1, x_2]$ is an edge.

For, if F has a 5-valent vertex, F is a pyramid, and x_3 exists.

If F has a 4-valent vertex, F is constructed by splitting the base of a square pyramid by an edge E. The possibilities for F are shown in Fig. 7.4. If the distance (minimum path length)

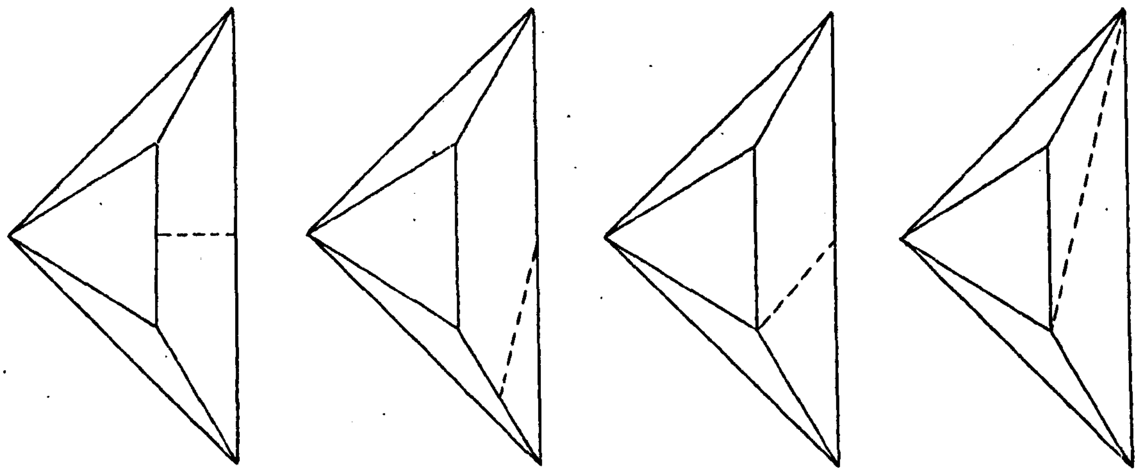


Fig. 7.4

between x_1 and x_2 is to be greater than 2, at least one of x_1, x_2 must be an endpoint of E. But each endpoint of E is at distance no greater that 2 from every other vertex, establishing the claim.

The remaining case is F simple with six facets. There is only one simple 3-polytope with five facets, the triangular prism,

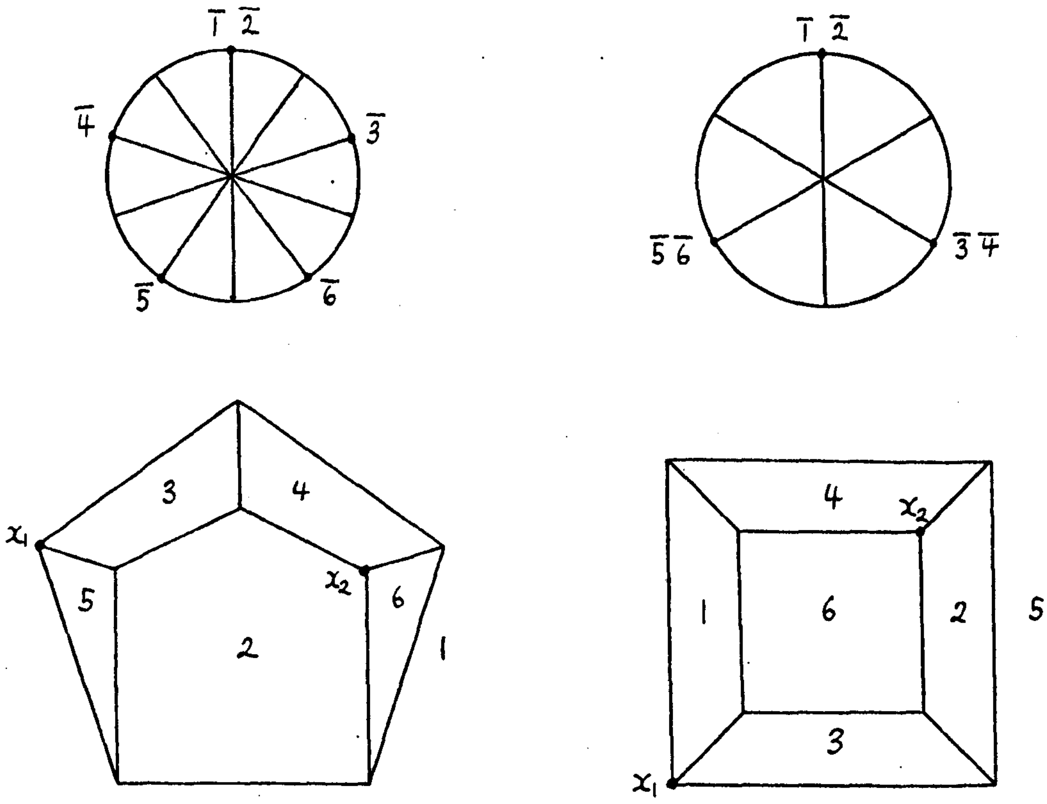


Fig. 7.5

We find that there are only two different simple 3-polytopes F with six facets - the cube and the pentagonal wedge - and, modulo the combinatorial symmetry of F, one way in each to choose a pair of vertices at distance 3. These polytopes and their dual Gale diagrams appear in Fig. 7.5.

The second step is to consider the special case where F is a cube or a pentagonal wedge. In this case, for each vertex of F there are three facets of P not containing it, and hence at most d containing it. Therefore every vertex of F is simple in P.

Each facet of P contains either x_1 or x_2. Hence each facet containing x_1 lies in a different pseudofacet if a refinement exists, and similarly for x_2. Each facet containing x_1 and x_2 is principal, and so the d-3 facets containing F are principal. An intersection of principal faces is principal; hence F itself is principal.

Therefore $\Psi|_F: F \to T^3$ is a refinement map, and so the refinement on P induces a refinement on F.

We wish to find a (d-2)-face K of P, with $x_1, x_2 \notin K$, such that K may be deleted. The resulting d-gcc Q will have d+2 facets and by theorem 7.1 will be polytopal. By lemma 7.4, $\mathcal{B}(P)$ is a refinement of $\mathcal{B}(Q)$ and since $x_1, x_2 \notin K$, x_1 and x_2 are principal faces. Theorem 6.5 then implies that $\mathcal{B}(Q)$ is a refinement of $\mathcal{B}(T^d)$ with x_1 and x_2 principal, and hence $\mathcal{B}(P)$ is a refinement of $\mathcal{B}(T^d)$ with x_1 and x_2 principal.

Label the facets of P 1, 2,..., d+3 so that

$$x_1 \in \{1, 3, 5, 7, 8, \ldots, d+3\},$$
$$x_2 \in \{2, 4, 6, 7, 8, \ldots, d+3\}.$$

The (d-2)-face K must be deleted by splicing one of $\{1, 3, 5\}$ with one of $\{2, 4, 6\}$. Call such a splicing <u>admissible</u>. If we delete any other (d-2)-face, one or both of the special vertices will disappear, and we will be unable to carry out the induction.

Lemma 7.3 established a criterion on the dual Gale diagram $\overline{X}^*$ of P determining pairs of facets which may be spliced: if $\overline{i}$ and $\overline{j}$ are points of the dual Gale diagram such that $\overline{j}$ is adjacent to an empty end of the diameter containing $\overline{i}$, and $\overline{i}$ is adjacent to an empty end of the diameter containing $\overline{j}$, then i and j may be spliced. Call this the splicing criterion.

Let $d = 4$. The dual Gale diagram of P is constructed by adding a point $\overline{7}$ to the dual diagram of F. We must be careful to consider all the possible dual diagrams of F, which will of course be isomorphic, but will differ in how a seventh point may be added. We will now show that regardless of how $\overline{7}$ is chosen, an admissible splice is always possible.

In the first case, let F be a cube, with a dual Gale diagram $\overline{X}^*$ isomorphic to that of Fig 7.6a. Let $B_1 = \text{pos}\{\overline{1},\overline{2}\}$, $B_2 = \text{pos}\{\overline{3},\overline{4}\}$, $B_3 = \text{pos}\{\overline{5},\overline{6}\}$, and let W_1, W_2 and W_3 be the three components of $E^2 \setminus (B_1 \cup B_2 \cup B_3)$ labelled so that

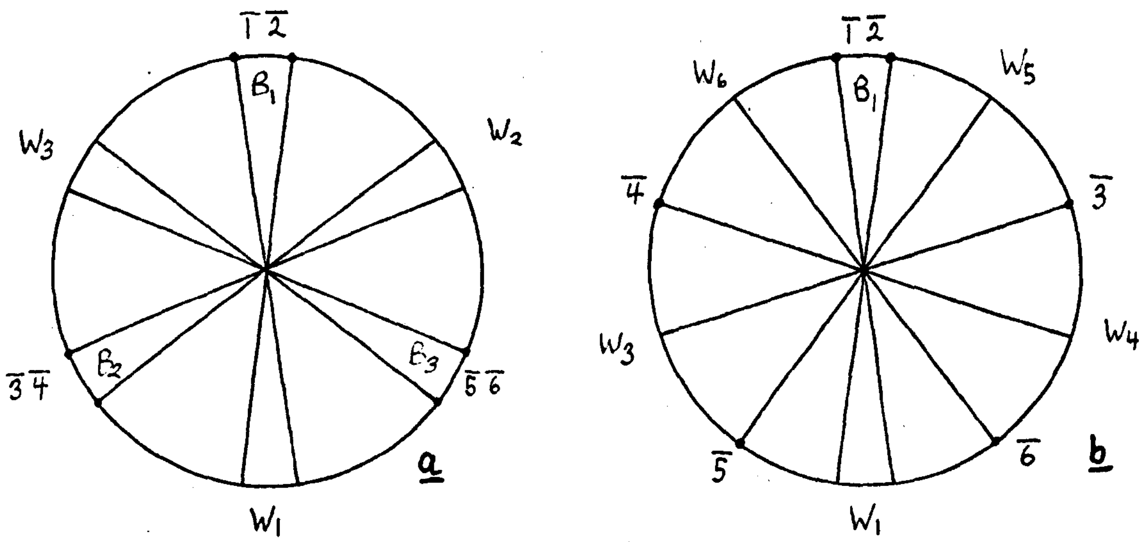


Fig. 7.6

$B_i \cap \mathrm{cl}\, W_i = 0$. Let $B = B_1 \cup B_2 \cup B_3$, $W = W_1 \cup W_2 \cup W_3$. In F itself, the 1-faces 14, 16, 23, 34, 25, 45 may be deleted admissibly. If $\overline{7}$ is to interfere with a splicing of facets a and b, where $\overline{a} \in B_i$, $\overline{b} \in B_j$, then $\overline{7} \in W_i \cup W_j$. Otherwise, in a dual diagram isomorphic to $\overline{X}^* \cup \overline{7}$, $\overline{a}$ and $\overline{b}$ satisfy the splicing criterion and hence may be spliced. Hence, if $\overline{7} \in W_1$, then there exist $\overline{a} \in B_1$, $\overline{b} \in B_2$, such that $a \cup b$ is an admissible splice. By symmetry, any choice of $\overline{7}$ yields an admissible splice.

In the second case, F is a pentagonal wedge with dual Gale diagram $\overline{X}^*$ isomorphic to that of Fig. 7.6b. Let

$$B_1 = \mathrm{pos}\,\{\overline{1},\overline{2}\},$$

$$B_i = \mathrm{pos}\,\{\overline{i}\},\ i = 3,\ 4,\ 5,\ 6,$$

$$B = B_1 \cup B_3 \cup B_4 \cup B_5 \cup B_6.$$

Label the five components of $E^2 \setminus B$ as W_i, $i = 1,3,4,5,6$, such that $-\overline{i} \in W_i$. Let $W = W_1 \cup W_3 \cup W_4 \cup W_5 \cup W_6$. The admissible splices of F are $1 \cup 6$, $2 \cup 5$, $3 \cup 4$. To interfere with the splice $3 \cup 4$, we must have $\overline{7} \in W_3 \cup W_4$. To interfere with one of the splices $2 \cup 5$ or $1 \cup 6$, we must have $\overline{7} \in W_1 \cup W_5 \cup W_6$. Hence regardless of the choice of $\overline{7}$, there exists an admissible splice for P.

Hence an admissible splice always exists and P may be expressed as a refinement in the desired fashion.

For $d = 5$, since the result is true for $d = 4$, if there exists $x_3 \in \mathrm{vert}\, P$ joined by an edge to x_1 and x_2, then the argument of theorem 6.5 yields the desired refinement in P.

If no such vertex exists, once again the smallest face F containing x_1 and x_2 is a cube or a pentagonal wedge, and the dual diagram of P is obtained by adding two points $\overline{7}$ and $\overline{8}$ to the dual diagram $\overline{X}^*$ of F. We now show that, except for a few cases, there is an admissible splicing for P.

In the first case, F is a cube. Using the notation defined when discussing $d = 4$, we see that $\overline{7}$ and $\overline{8}$ interfere with splices only if they lie in W. If $\overline{7}, \overline{8} \in W_1$, say, then $3 \cup 6$ and $4 \cup 5$ are admissible splices. Using the symmetry of $\overline{X}^*$, we may assume without loss of generality that $\overline{7} \in W_2$, $\overline{8} \in W_3$.

We now determine the dual diagrams $\overline{Z}^*$ which do not yield admissible splicings. Since $\overline{7}, \overline{8} \notin W_1$, we may assume $\overline{1}$ and $\overline{2}$ are coincident. We may also assume that the diameter of $\overline{1}$ and $\overline{2}$ is adjacent to the diameters of $\overline{7}$ and $\overline{8}$. Otherwise an empty end of a diameter containing one of $\overline{3}$, $\overline{4}$, $\overline{5}$, $\overline{6}$ intervenes, say $\overline{3}$, and then $\overline{3}$ and $\overline{2}$ satisfy the splicing condition.

Having assumed that $\overline{1}$ and $\overline{2}$ are coincident and that $\overline{7}$ and $\overline{8}$ are adjacent to $\overline{1}$ and $\overline{2}$, the remaining freedom of choice involves the placement of $\overline{3}$, $\overline{4}$, $\overline{5}$, $\overline{6}$. One, both or neither of $\overline{3}$ and $\overline{4}$ may lie on the diameter of $\overline{7}$; one, both or neither of $\overline{5}$ and $\overline{6}$ may lie on the diameter of $\overline{8}$. The choices which yield no admissible facet splitting are 1, 2 and 3 of Fig. 7.7. Observe that the transpositions $(1\ 2)$, $(3\ 4)$ and $(5\ 6)$ are symmetries of the cube and hence relabelling $\{1, 3, 5\}$ as $\{2, 4, 6\}$ merely relabels x_1 as x_2 and x_2 as x_1.

In the second case, we carry out the same procedure for F a pentagonal wedge. Use the same notation as for $d = 4$. To interfere with the splice $3 \cup 4$, one of $\overline{7}$ or $\overline{8}$, say $\overline{7}$, must lie in $W_3 \cup W_4$; by symmetry, we may assume $\overline{7} \in W_3$. To interfere with $2 \cup 5$ or $1 \cup 6$, $\overline{8}$ must then lie in $W_1 \cup W_5 \cup W_6$. The three possibilities are therefore:

$\overline{7} \in W_3$, $\overline{8} \in W_1$; no admissible splices are possible;

$\overline{7} \in W_3$, $\overline{8} \in W_5$: $1 \cup 6$ may be spliced;

$\overline{7} \in W_3$, $\overline{8} \in W_6$: $2 \cup 5$ may be spliced.

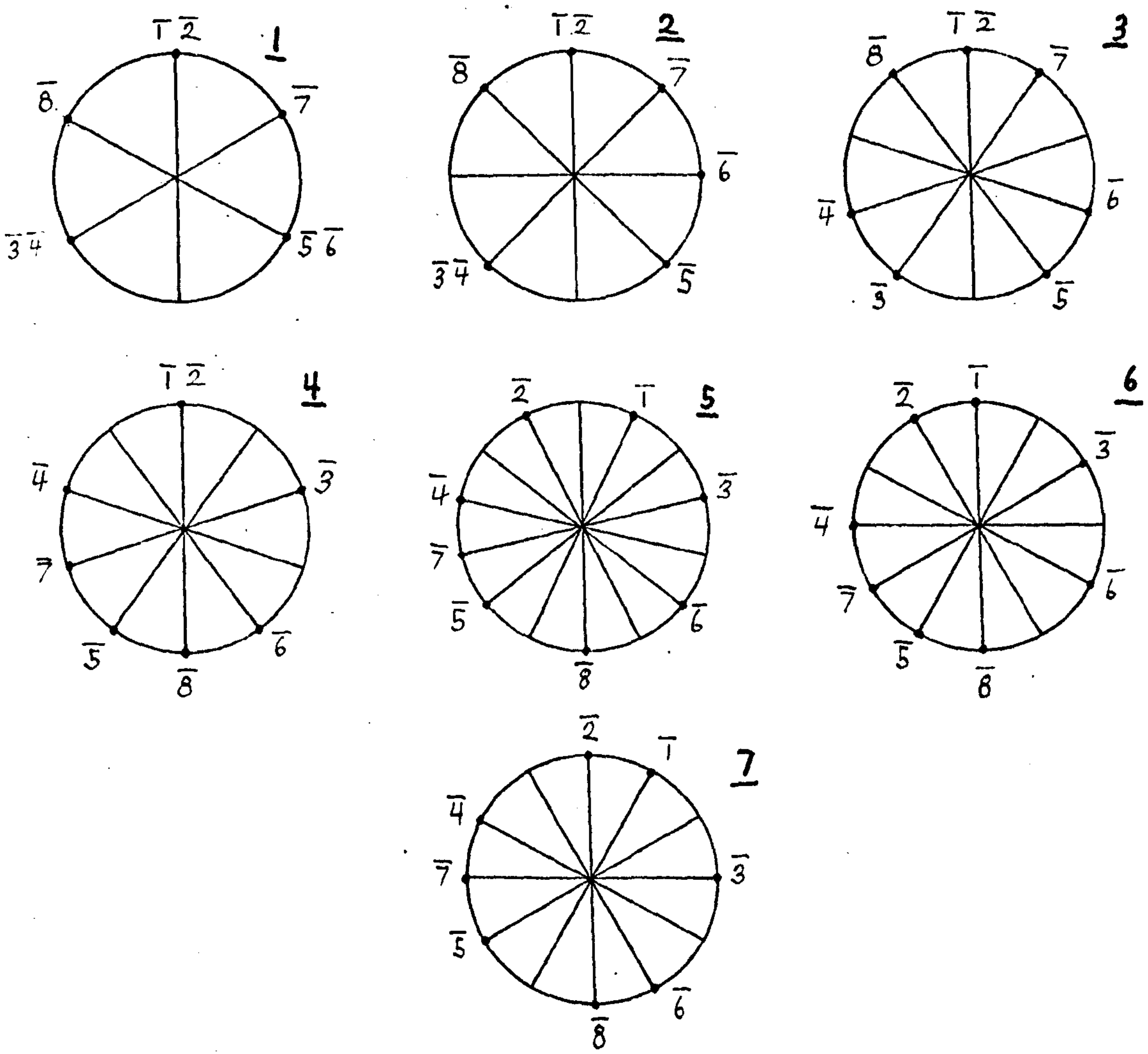


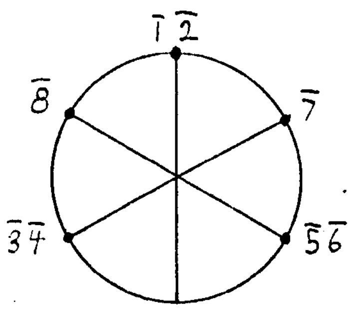
Fig. 7.7

Assume $\overline{7} \in W_3$, $\overline{8} \in W_1$, and no admissible splices exist for P. To interfere with the splice $3 \cup 4$, $\overline{7}$ and $\overline{3}$ must lie at opposite ends of the same diameter. If $\overline{1}$ and $\overline{2}$ occupy the same diameter, $\overline{8}$ must lie at the opposite end. On the other hand, if $\overline{1}$ and $\overline{2}$ are not coincident, and the diameter of $\overline{1}$ is adjacent to the diameter of $\overline{6}$, then the diameter of $\overline{2}$ must be adjacent to that of $\overline{5}$, and no choice of $\overline{8}$ interferes with $2 \cup 5$ and $1 \cup 6$. Hence if $\overline{1}$ and $\overline{2}$ are not coincident, the diameter of $\overline{1}$ is adjacent to the diameter of $\overline{5}$, and the diameter of $\overline{2}$ is adjacent to the diameter of $\overline{6}$. In this case there are three choices of $\overline{8}$ interfering with all remaining splices. The four examples with no admissible splice are 4, 5, 6 and 7 of Fig. 7.7.

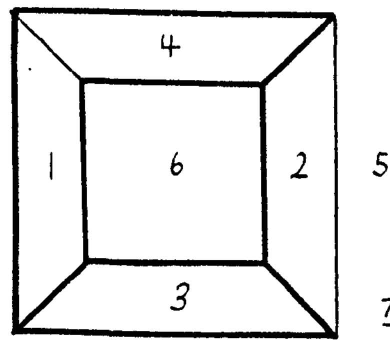
We have now proved the theorem for all but the seven 5-polytopes of Fig. 7.7. The third and final step is to show that each of these seven satisfies the theorem. None of these examples contains a vertex x_3 joined by an edge to x_1 and x_2.

We will choose pseudofacets for each example and show that the intersection of any k pseudofacets is a (5-k)-cell, and hence by theorem 3.2, the 5-gcc C defined by these pseudofacets is isomorphic to T^5. Any homeomorphism $\Psi: C \longrightarrow T^5$ taking pseudofacets onto facets of T^5 will be the desired refinement map.

1. Pseudofacets: $1\cup 4, 2\cup 5, 3, 6, 7, 8$.



i. The refinement induced in 78.



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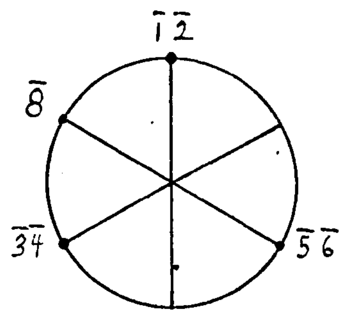
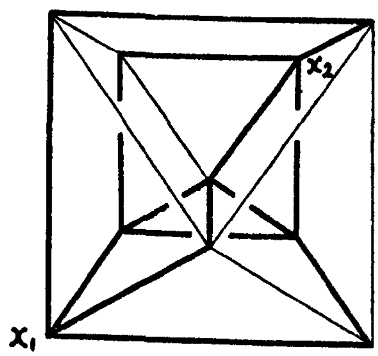
ii. The refinement induced in 7. The Schlegel diagram is based on 78.





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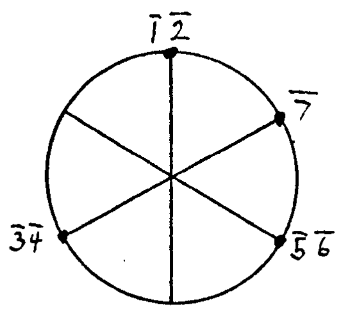
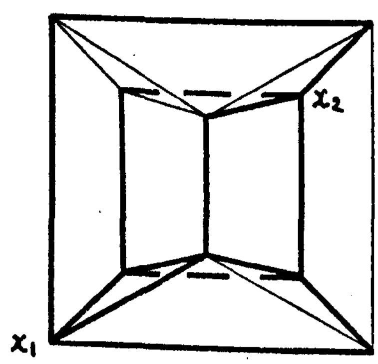
iii. The refinement induced in 8. The Schlegel diagram is based on 78.





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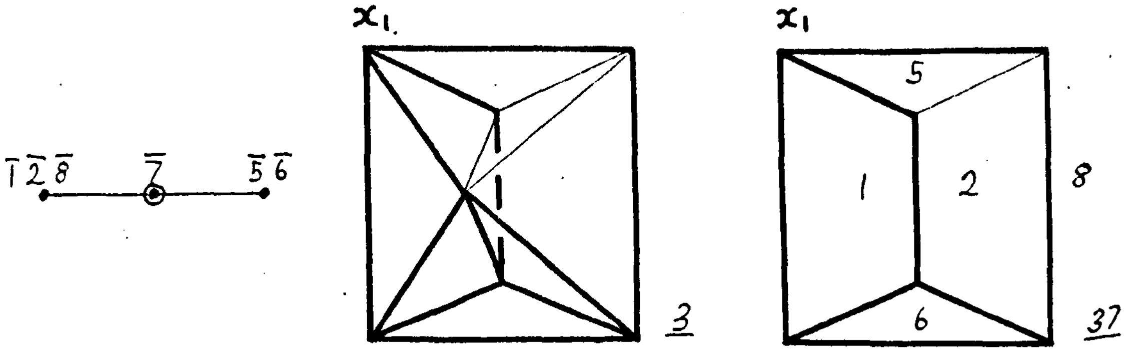
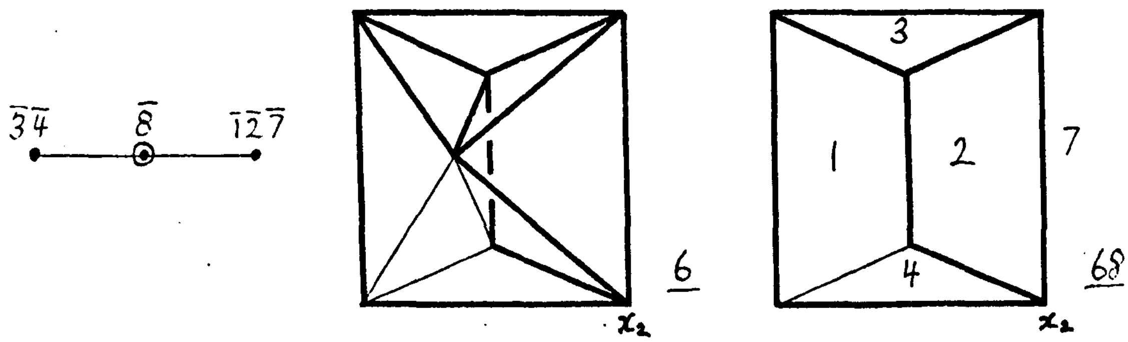
iv. The refinement induced in 3. The Schlegel diagram is based on 37.



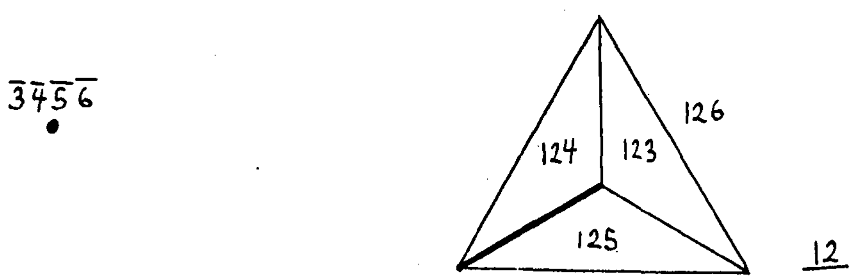
v. The refinement induced in 6. The Schlegel diagram is based on 68.



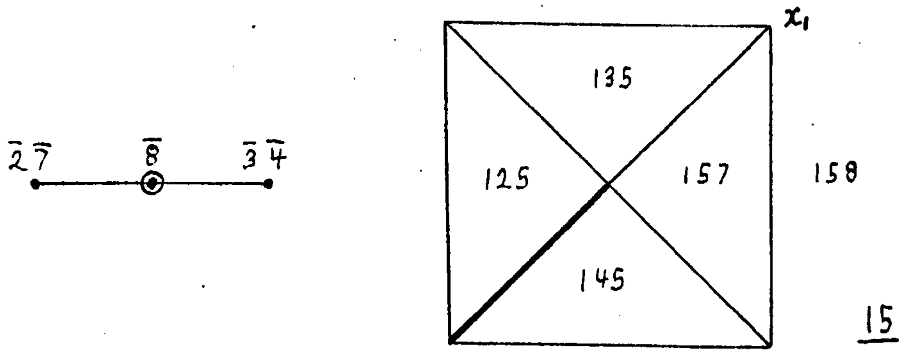
vi. The only intersection of pseudofacets we have not yet checked is $(1\cup 4)\cap(2\cup 5)$. All other intersections of more than one pseudofacet lie in one of the facets 3,6,7,8, which we have already checked. We compute

$$(1\cup 4)\cap(2\cup 5) = 12\cup 15\cup 24\cup 45.$$

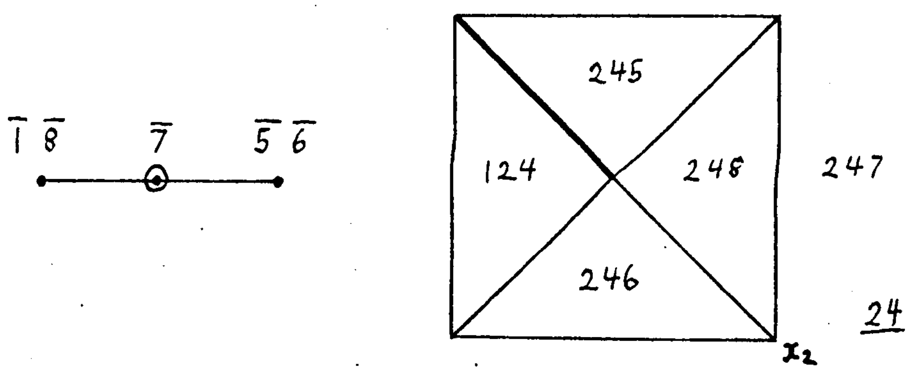
a. Compute 12.



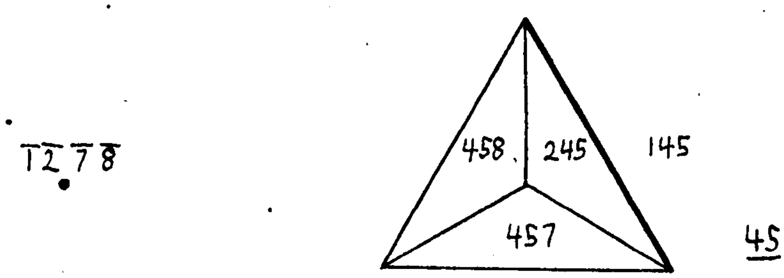
b. Compute 15.



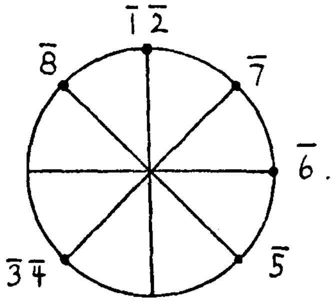
c. Compute 24.



d. Compute 45.

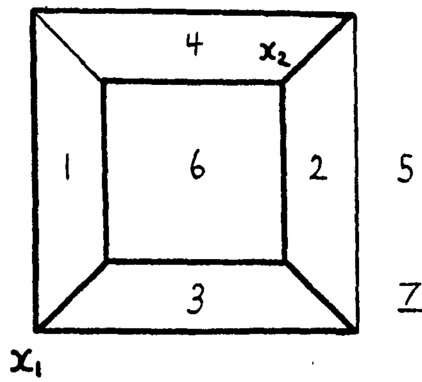


These 4 3-faces share an edge 1245, drawn boldly. Since $12 \cap 15 = 125$, $15 \cap 45 = 145$, $45 \cap 24 = 245$, $24 \cap 12 = 124$, the edge 1245 is in the relative interior of $(1 \cup 4) \cap (2 \cup 5)$. Hence $(1 \cup 4) \cap (2 \cup 5)$ is a 3-cell as required.

2. 

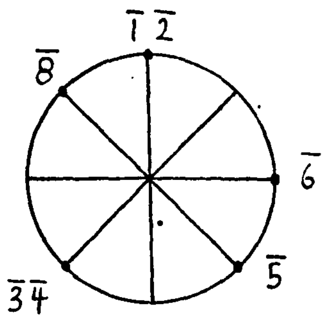
Pseudofacets: $1 \cup 4$, $2 \cup 5$, 3, 6, 7, 8.

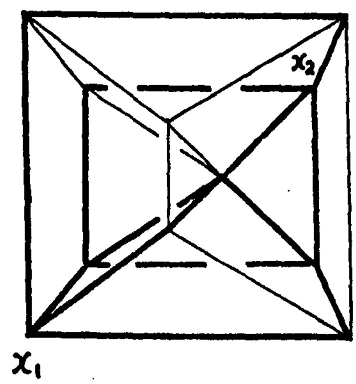
i. The refinement induced in $F = 78$.



78

ii. The refinement induced in 7. The Schlegel diagram is based on 78.





7

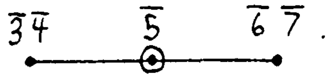
iii. The refinement induced in 8 is isomorphic to 1.iii.

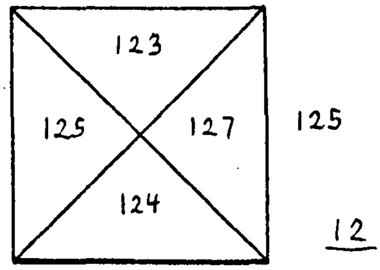
iv. The refinement induced in 3 is isomorphic to 1.iv.

v. The refinement induced in 6 is isomorphic to 1.v.

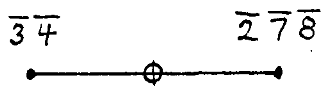
vi. To prove that $(1 \cup 4) \cap (2 \cup 5)$ is a 3-cell, compute $12 \cup 15 \cup 24 \cup 45$.

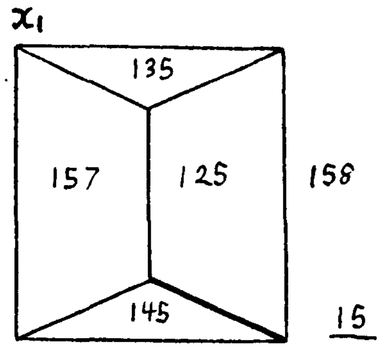
a. Compute 12.



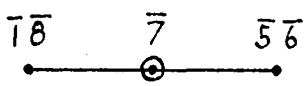


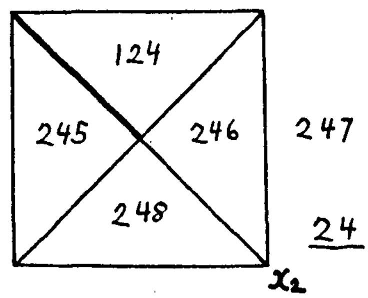
b. Compute 15.



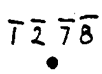


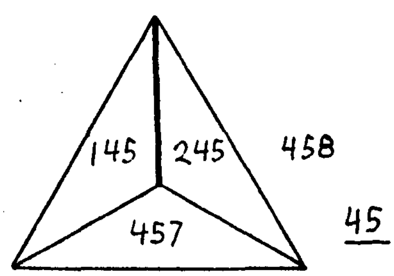
c. Compute 24.





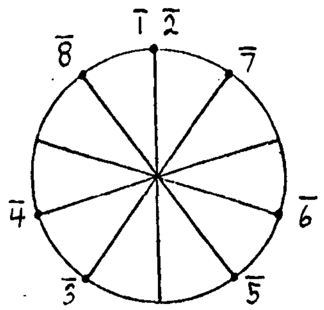
d. Compute 45.



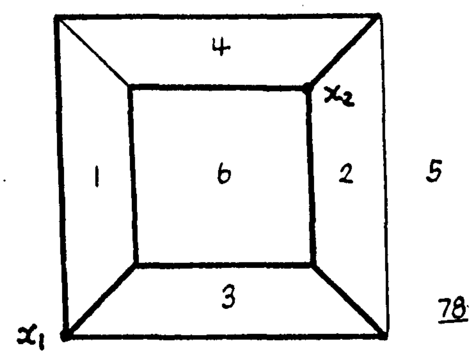


The 4 3-faces fit neatly around the edge 1245 to form a 3-cell.

3. Pseudofacets: $1 \cup 4$, $2 \cup 5$, 3, 6, 7, 8.

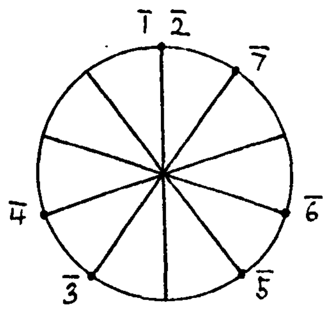


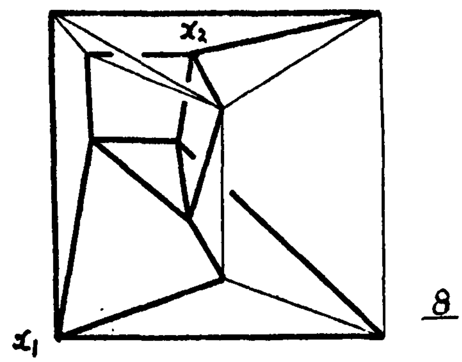
i. The refinement induced in $F = 78$.



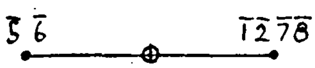
ii. The refinement induced in 7 is isomorphic to 2.ii.

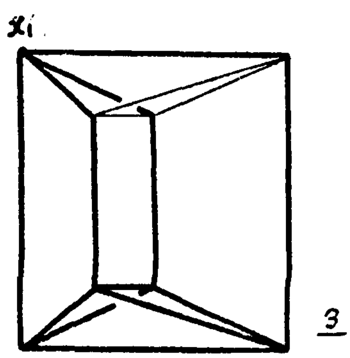
iii. The refinement induced in 8. The Schlegel diagram is based on 78.

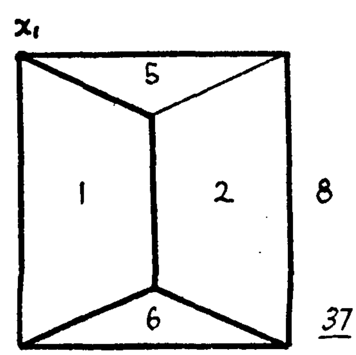




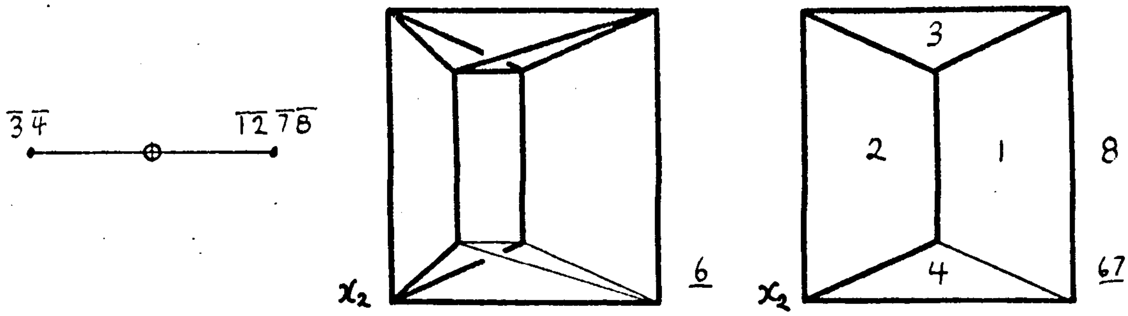
iv. The refinement induced in 3. The Schlegel diagram is based on 37.





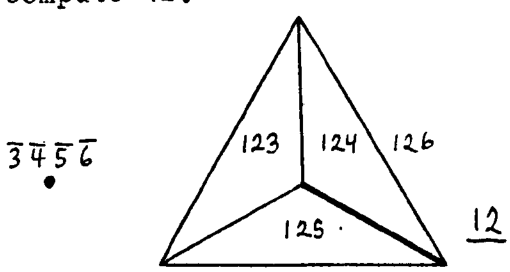


v. The refinement induced in 6. The Schlegel diagram is based on 67.

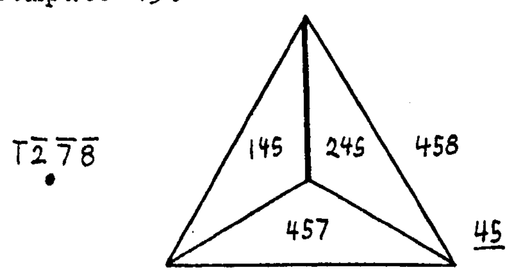


vi. To show that $(1 \cup 4) \cap (2 \cup 5)$ is a 3-cell, compute $12 \cup 15 \cup 24 \cup 45$.

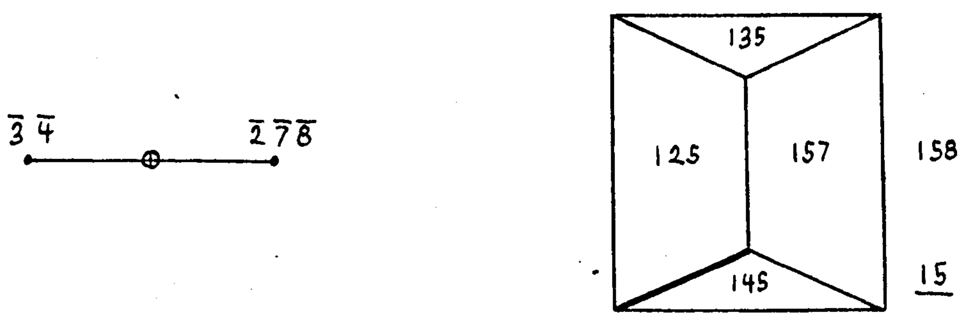
a. Compute 12.



b. Compute 45.



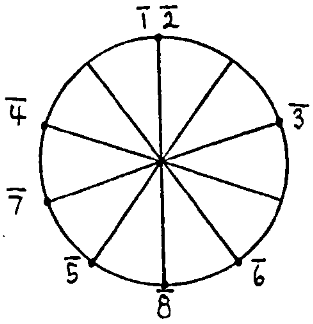
c. Compute 15.



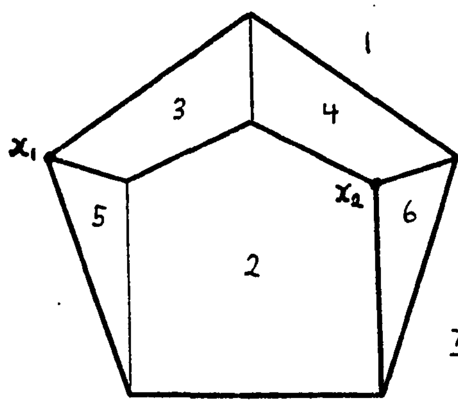
d. Compute 24.

The four 3-faces fit nicely around the edge 1245 to form a 3-cell.

4. Pseudofacets: $2 \cup 5$, $3 \cup 4$, 1, 6, 7, 8.

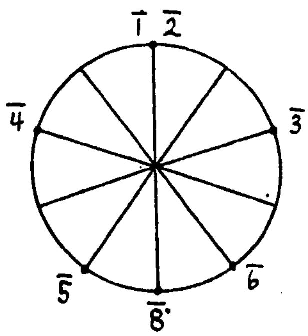


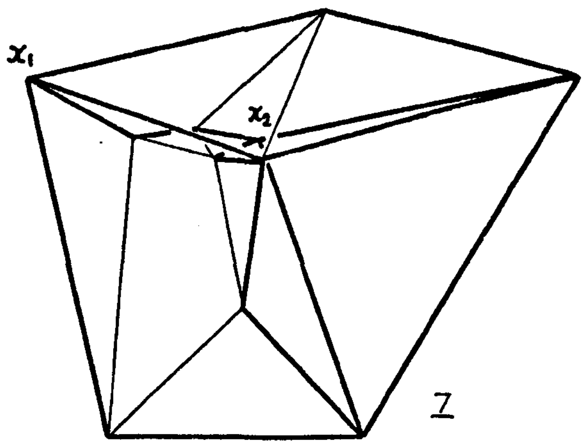
i. The refinement induced in F = 78.



78

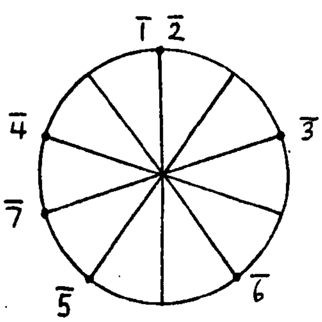
ii. The refinement induced in 7. The Schlegel diagram is based on 78.

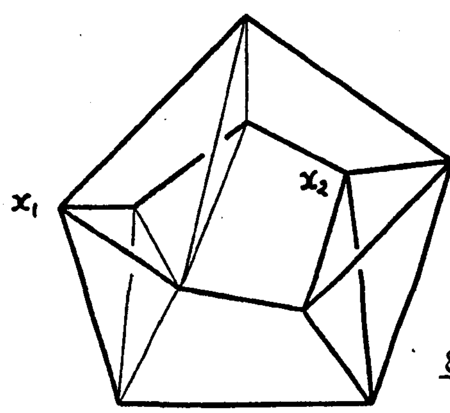




7

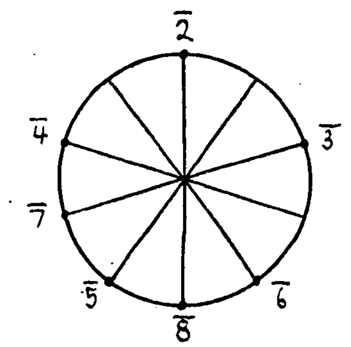
iii. The refinement induced in 8. The Schlegel diagram is based on 78.

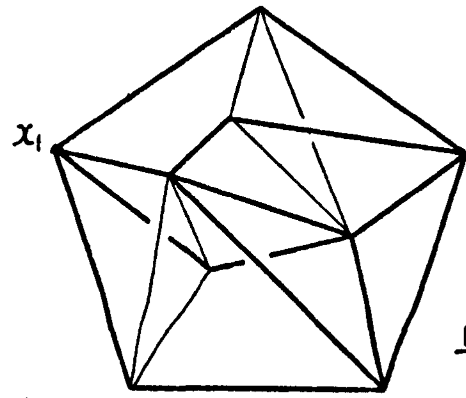




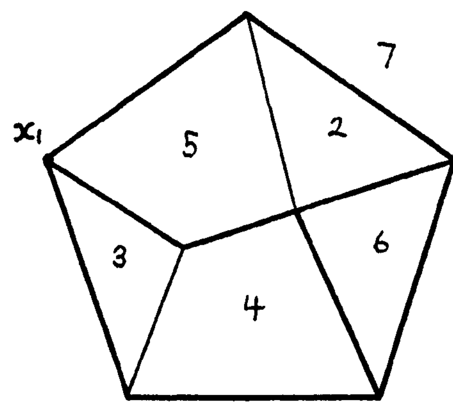
8

iv. The refinement induced in 1. The Schlegel diagram is based on 18.



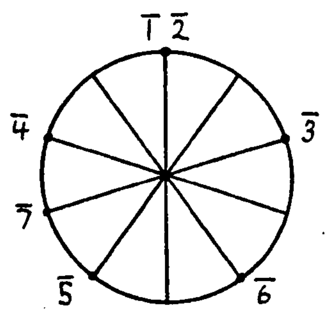


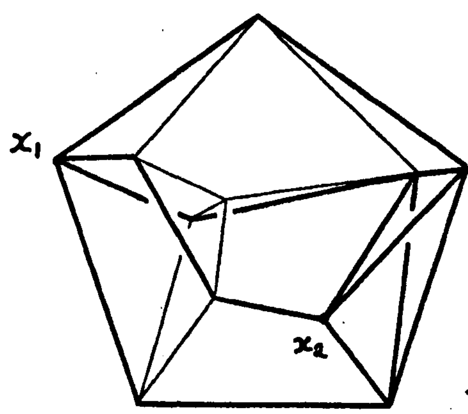
1



18

v. The refinement induced in 8. The Schlegel diagram is based on 18.

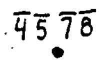


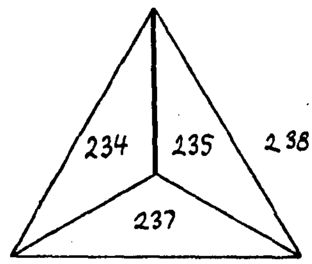


8

vi. To prove that $(2 \cup 5) \cap (3 \cup 4)$ is a 3-cell, compute $23 \cup 24 \cup 35 \cup 45$.

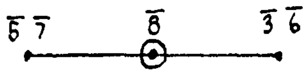
a. Compute 23.

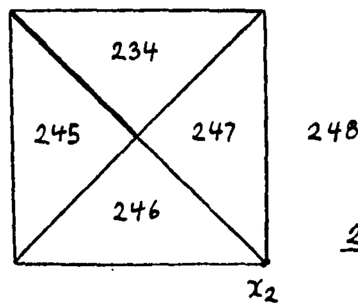




<u>23</u>

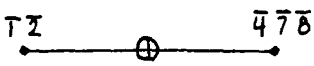
b. Compute 24.

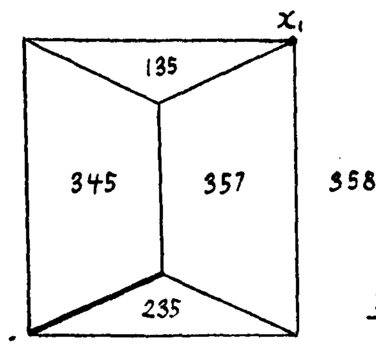




<u>24</u>

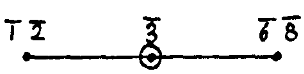
c. Compute 35.

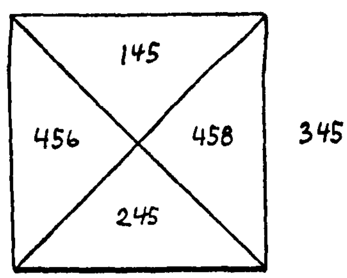




<u>35</u>

d. Compute 45.

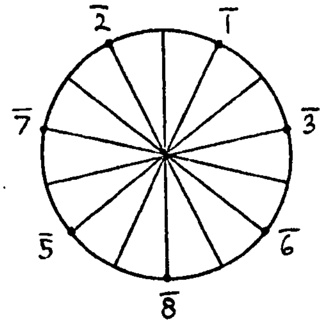




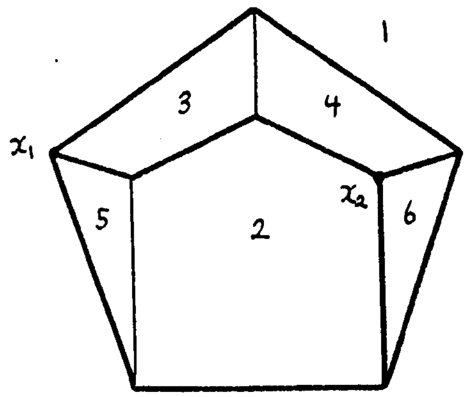
<u>45</u>

The four 3-cells fit nicely around the edge 2345 to form a 3-cell.

5. Pseudofacets: $2 \cup 5$, $3 \cup 4$, 1, 6, 7, 8.

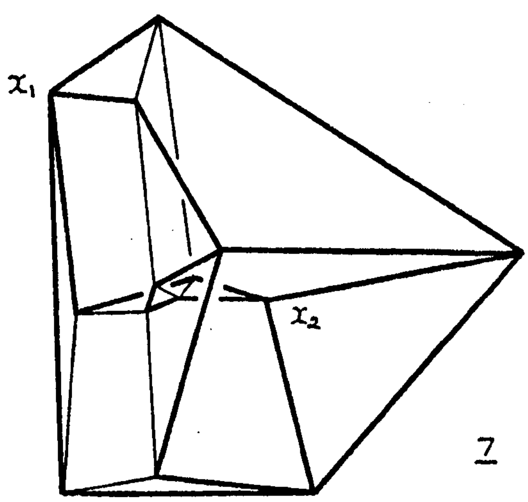


i. The refinement induced in F = 78.



78

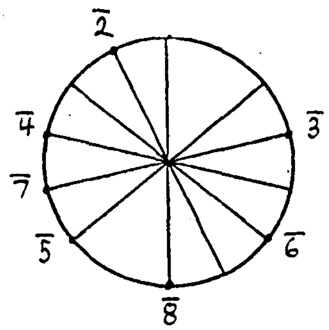
ii. The refinement induced in 7. The Schlegel diagram is based on 78.

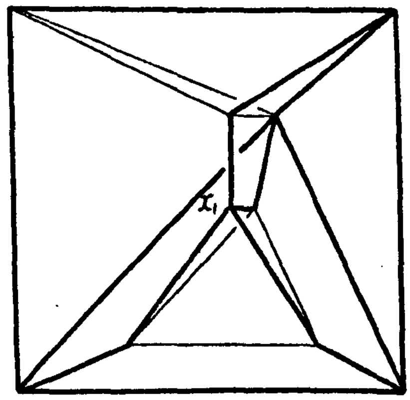


7

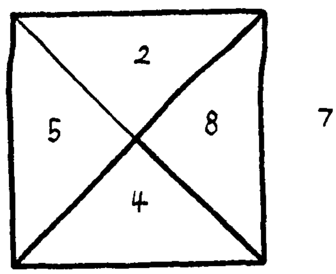
iii. The refinement induced in 8 is isomorphic to 4.iii.

iv. The refinement induced in 1. The Schlegel diagram is based on 16. The tetrahedral facet disjoint from 16 is 13.



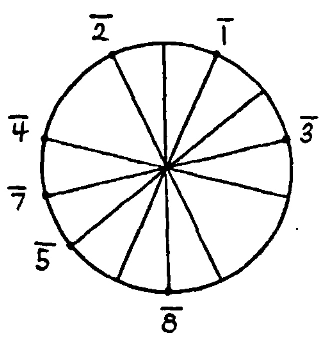


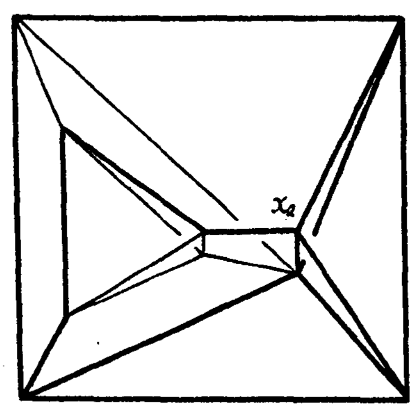
<u>1</u>



<u>16</u>

v. The refinement induced in 6. The Schlegel diagram is based on 16. The tetrahedral facet disjoint from 16 is 36.

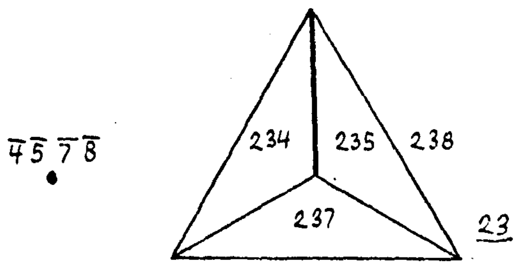




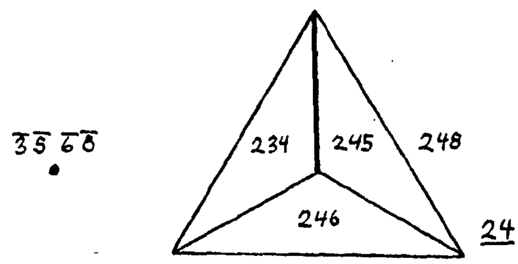
<u>6</u>

vi. To show that $(2\cup 5)\cap(3\cup 4)$ is a 3-cell, compute $23\cup 24\cup 35\cup 45$.

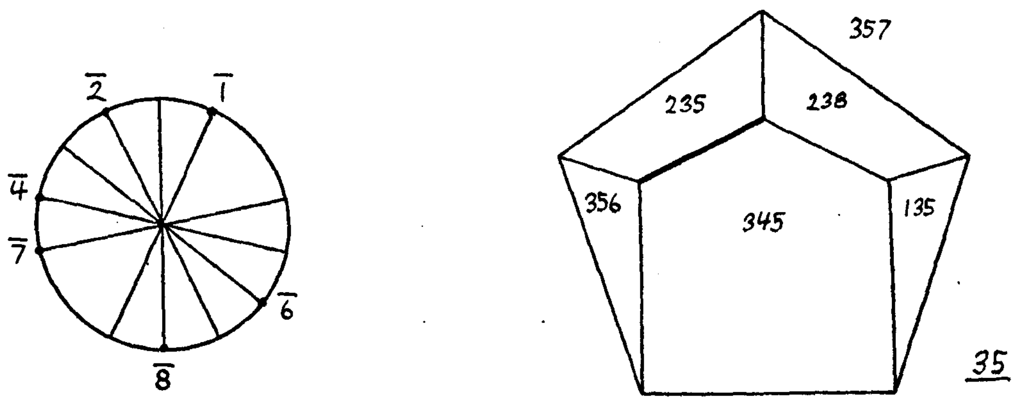
a. Compute 23.



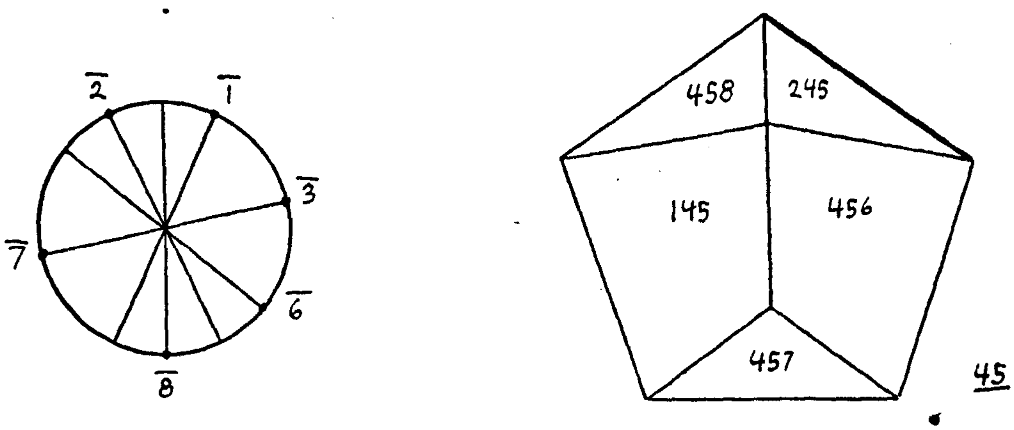
b. Compute 24.



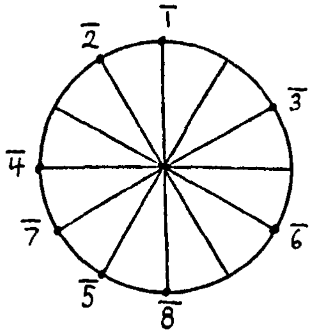
c. Compute 35.



d. Compute 45.

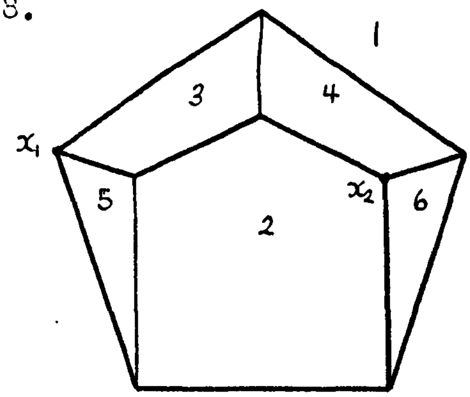


These four 3-faces fit nicely around the edge 2345 to yield a 3-cell.

6. 

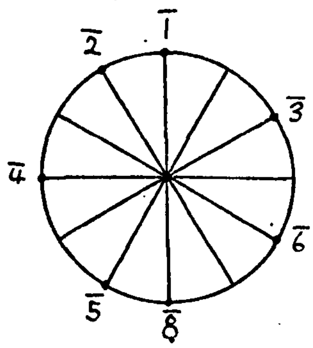
Pseudofacets: $2\cup 5$, $3\cup 4$, 1, 6, 7, 8.

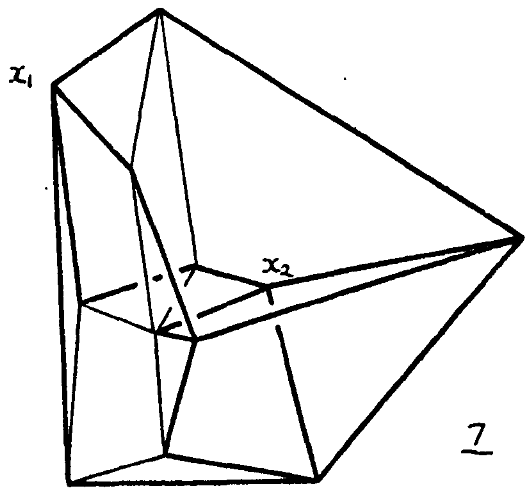
i. The refinement induced in F = 78.



78

ii. The refinement induced in 7. The Schlegel diagram is based on 78.



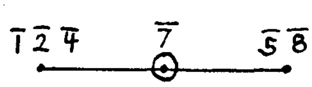


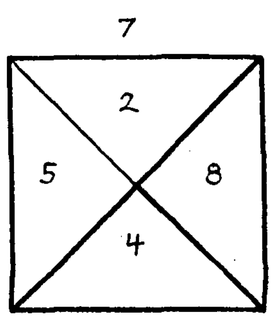
7

iii. The refinement induced in 8 is isomorphic to 4.iii.

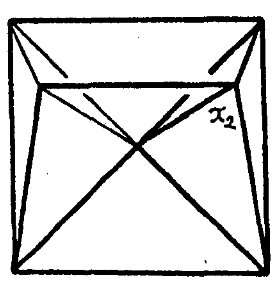
iv. The refinement induced in 1 is isomorphic to 5.iv.

v. The refinement induced in 6. The Schlegel diagram is based on 16.





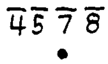
16

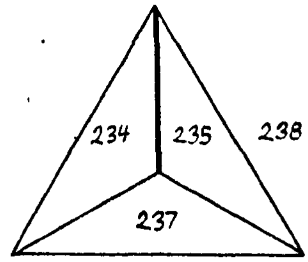


6

vi. To show that $(2\cup 5)\cap(3\cup 4)$ is a 3-cell, compute $23\cup 24\cup 35\cup 45$.

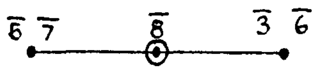
a. Compute 23.

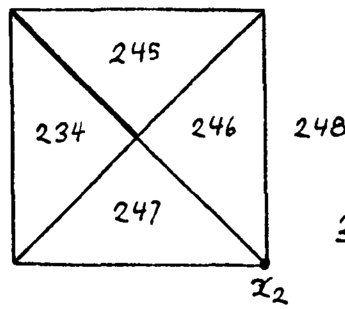




23

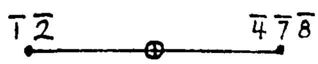
b. Compute 24.

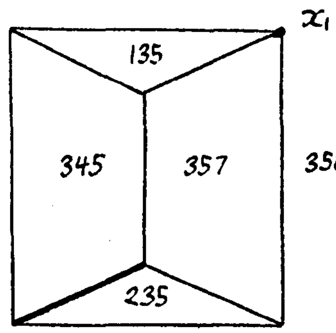




24

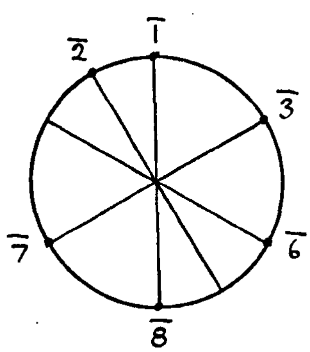
c. Compute 35.

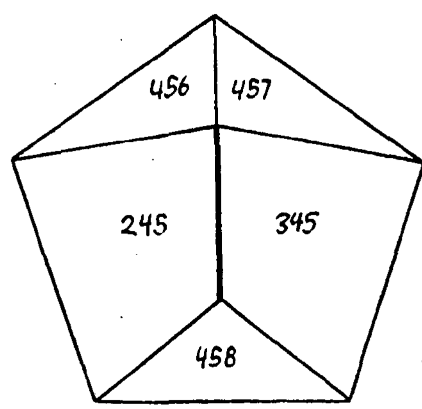




35

d. Compute 45.

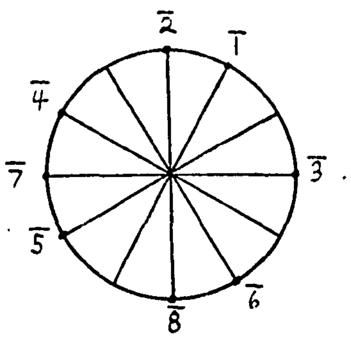




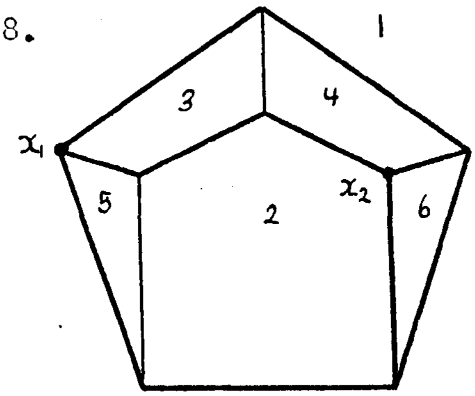
45

The four 3-faces fit nicely around the edge 2345 to form a 3-cell.

7. Pseudofacets: $2\cup5$, $3\cup4$, 1, 6, 7, 8.



i. The refinement induced in F = 78.



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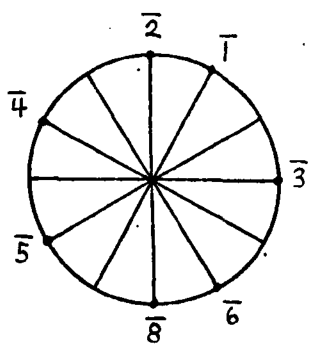
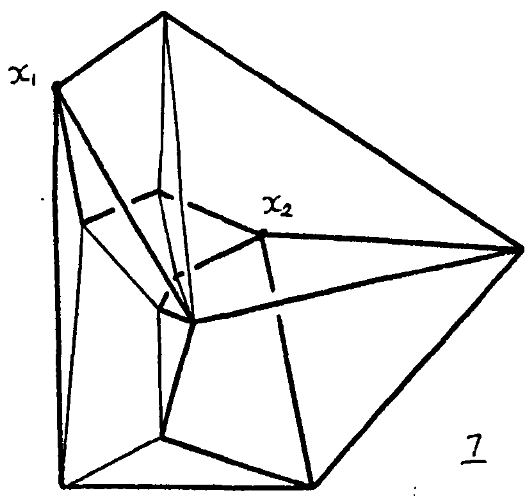
ii. The refinement induced in 7. The Schlegel diagram is based on 78.



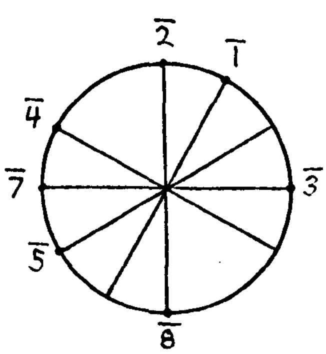


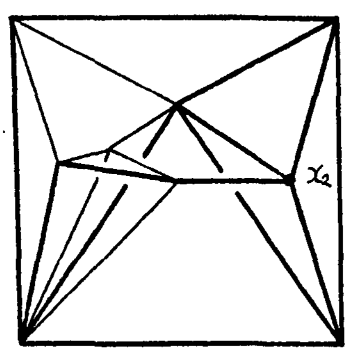
7

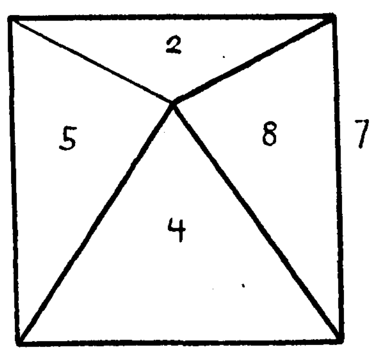
iii. The refinement induced in 8 is isomorphic to 4,iii.

iv. The refinement induced in 1 is isomorphic to 4.iv.

v. The refinement induced in 6. The Schlegel diagram is based on 16. The tetrahedral facet not meeting 16 in a 2-face is 36.

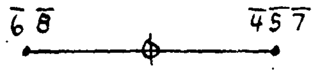


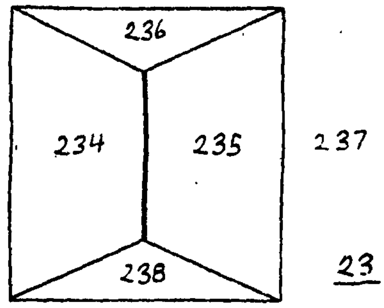




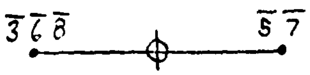
vi. To prove $(2 \cup 5) \cap (3 \cup 4)$ is a 3-cell, compute $23 \cup 24 \cup 35 \cup 45$.

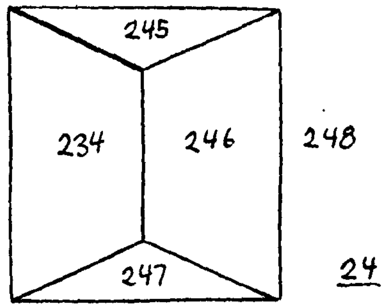
a. Compute 23.



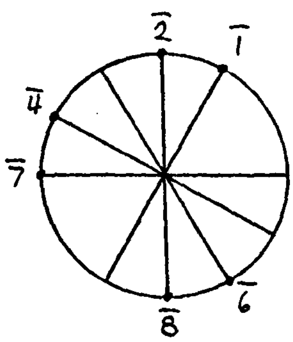


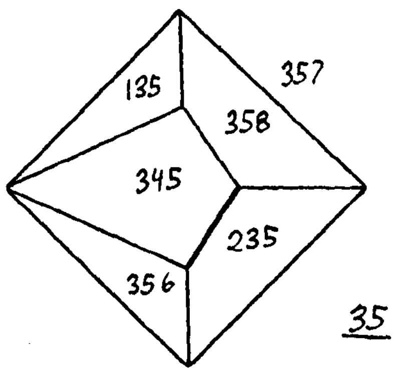
b. Compute 24.



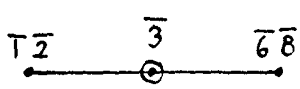


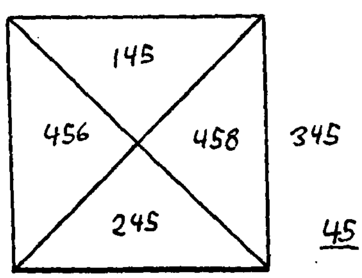
c. Compute 35.





d. Compute 45.





The four 3-faces fit nicely around the edge 2345 to form a 3-cell.

Since all the exceptional cases possess the desired refinement, the theorem is now proved.

4. A counterexample

For $d \leq 3$, every polytope P except the simplex may be constructed by splitting a facet F_o of a polytope Q with one less facet. The two facets of P produced by splitting F_o may be spliced to obtain a gcc isomorphic to Q.

For $d \geq 4$, there are d-polytopes other than the simplex which cannot be constructed by facet splitting. Following Barnette (1975), we remark that no 2-neighbourly d-polytope, $d \geq 4$, with d+2 or more vertices can be constructed in this way. If such a construction were possible, P would contain a pair of facets F_1 and F_2 meeting in a (d-2)-face F, and since P is 2-neighbourly, an edge E joins a vertex of $F_1 \setminus F$ with a vertex of $F_2 \setminus F$. But then splicing F_1 and F_2 yields a d-gcc in which the edge E meets a facet $F_1 \cup F_2$ in two points: a contradiction.

In the same paper, Barnette constructs a simple 4-gcc with 10 facets which cannot be constructed by facet splitting. This gcc is not polytopal, however, and we may ask whether or not all simple polytopes may be constructed by facet splitting.

In this section we describe simple polytopes which cannot be so constructed.

Theorem 7.3: There exist simple d-polytopes P_d with d+8 facets which cannot be constructed by facet splitting, if $d \geq 4$.

Proof: We first describe P_4. The polytope P_4 will have the property that for every pair F_1, F_2 of facets which meet, there will be a facet F_3 such that $F_3 \cap F_1 \neq \emptyset$, $F_3 \cap F_2 \neq \emptyset$ and $F_3 \cap F_1 \cap F_2 = \emptyset$. This property implies that F_1 and F_2 cannot be spliced, for $F_3 \cap (F_1 \cup F_2)$ is not connected and is therefore not a cell as required by lemma 7.1.

The dual P_4^* will be a simplicial polytope with the property that for every pair of vertices x_1 and x_2 joined by an edge there exists a vertex x_3 such that $[x_1,x_3]$ and $[x_2,x_3]$ are edges but $\operatorname{conv}\{x_1,x_2,x_3\}$ is not a 2-face. We shall say that $[x_1,x_2]$ is an edge in a missing 2-face. (In general a missing k-face of a polytope P is a simplex K such that relint $K \subseteq$ relint P and relbd $K \subseteq$ relbd P.)

To construct P_4^*, we use the beneath-beyond construction to add three new vertices a, b and c to the cyclic polytope C(9,4) with vertices 1,...,9 . Each vertex i lies beyond exactly the collection of facets $\mathcal{C}_i$ of C(9,4), i = a,b,c. The sets $\mathcal{C}_a$, $\mathcal{C}_b$ and $\mathcal{C}_c$ are pairwise disjoint so that the three new vertices may be added independently of each other.

Each set $\mathcal{C}_i$ consists of a facet G_i together with the four facets meeting G_i in a 2-face.

We now show that if P is a 4-polytope with facets $F_0,\ldots,F_n$ and $\mathcal{C}$ consists of a facet F_0 and its four immediate neighbours F_1,F_2,F_3,F_4, there exists a polytope P' projectively equivalent to P and a point x beyond precisely the facets of $\mathcal{C}'$, the set of facets of P' corresponding to $\mathcal{C}$.

Let

$$H_i = \operatorname{aff} F_i,\ i = 0,1,2,3,4,$$

and let H_i^- be the closed halfspace bounded by H_i and containing P. Let $y = H_1 \cap H_2 \cap H_3 \cap H_4$ and suppose that y exists and is beyond F_0, applying if necessary a non-singular projective transformation to P. A hyperplane H separating y and F_0 and not containing y is easily seen to satisfy $H \cap P = H \cap F_0$. Hence H is not of the form aff F_j for some facet F_j of P. A hyperplane H containing y meets F_0 in the set

$$F_{i_1} \cap \ldots \cap F_{i_k},\quad \{i_1,\ldots,i_k\} \subseteq \{1,\ldots,n\},$$

if and only if H meets the cone

$$H_{i_1}^- \cap \ldots \cap H_{i_k}^-$$

in the face

$$H_{i_1} \cap \ldots \cap H_{i_k}.$$

Hence H meets P in the face

$$P \cap (H_{i_1} \cap \ldots \cap H_{i_k}),$$

which is either one of $F_1, \ldots, F_4$ or a face of dimension less than 3, and again H is not of the form aff F_j for some facet other that $F_1, \ldots, F_4$. Therefore y is beneath every facet of P except $F_0, \ldots, F_4$.

Let $z \in$ relint F_0 and x a point such that $y \in$ relint $[x,z]$ sufficiently close to y that x is also beneath every facet of P except $F_0, \ldots, F_4$. It is clear that x is beyond $F_0, \ldots, F_4$.

Thus, starting with C(9,4), the three points a, b and c may be added one at a time, applying a projective equivalence if necessary at each stage, to yield P_4^*.

The faces of C(9,4) eliminated by adding the point i are the facets of $\mathcal{C}_i$ and the four 2-faces of G_i. Hence all the edges of C(9,4) are also edges of P_4^*.

We now show that for $i \in \{a,b,c\}$, each edge containing i lies in a missing 2-face. Suppose vert $G_i = \{p,q,r,s\}$. Each of the four facets of $\mathcal{C}_i$ other than G_i contains exactly one vertex not in G_i. Label these vertices p', q', r', s' so that j and j' are not joined by an edge in C_i = set $\mathcal{C}_i$, j = p,q,r,s. (See Fig. 7.8.)

Since C(9,4) is neighbourly and each vertex in C_i is a vertex of C(9,4), each pair of elements of $\{p,q,r,s,p',q',r',s'\}$ determines an edge. Hence i, p and p' are three vertices of a triangular circuit, but since $[p,p']$ is not an edge of C_i,

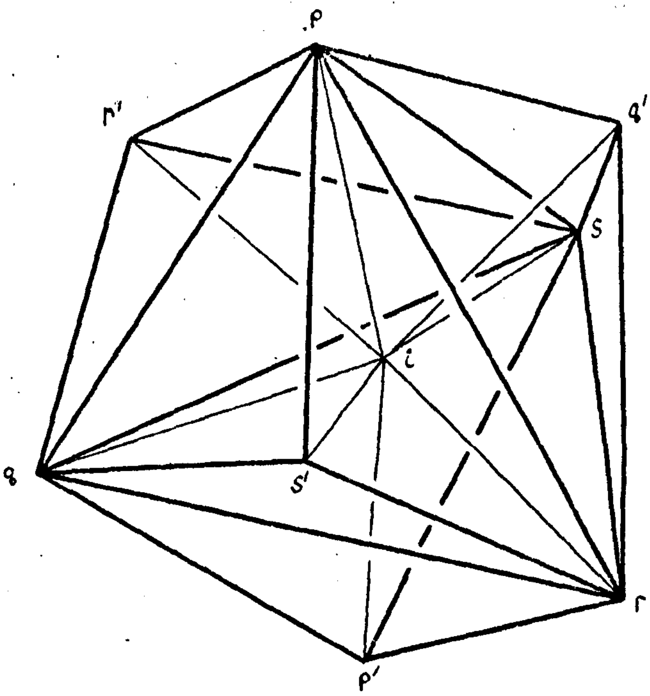


Fig. 7.8

conv $\{i,p,p'\}$ is not a 2-face. Hence conv $\{i,p,p'\}$ is a missing 2-face. Similarly, $\{i,q,q'\}$, $\{i,r,r'\}$, $\{i,s,s'\}$ determine missing 2-faces. Hence every edge containing i lies in a missing 2-face.

Some of the edges of C(9,4) do not lie in missing 2-faces. Ww now show how to choose the τ_i's so as to involve these edges in missing 2-faces.

It is clear from Fig. 7.8 that any pair of vertices from $\{p,q,r,s\}$ or from $\{p',q',r',s'\}$ lie in a missing 2-face. For instance, $\{p,q,r\}$, $\{p',q',i\}$ determine missing 2-faces.

In a neighbourly 4-polytope with n vertices, any three vertices x_1,x_2,x_3 determine a triangular circuit, and hence the edges lying in no missing 2-face lie in n-2 2-faces, or, equivalently, n-2 facets. Table 7.1 is a combinatorial description of C(9,4), following Altshuler and Steinberg (1973), who label C(9,4) as N_1^9. The vertices of N_1^9 lie on the moment curve in

Facets of N_1^9

A.	2345	J.	3478	S.	1679
B.	2356	K.	1458	T.	2679
C.	2367	L.	4578	U.	1569
D.	3467	M.	1568	V.	2569
E.	3456	N.	1678	W.	1459
F.	4567	O.	5678	X.	2459
G.	1238	P.	1789	Y.	1349
H.	2378	Q.	2789	Z.	2349
I.	1348	R.	1289	$.	1239

Facets of the dual $(N_1^9)^*$

1. G I K M N P R S U W Y $
2. A B C G H Q R T V X Z $
3. A B C D E G H I J Y Z $
4. A D E F I J K L W X Y Z
5. A B E F K L M O U V W X
6. B C D E F M N O S T U V
7. C D F H J L N O P Q S T
8. G H I J K L M N O P Q R
9. P Q R S T U V W X Y Z $

Table 7.1

	1	2	3	4	5	6	7	8	9
1	*	3	4	4	4	4	3	7	7
2	3	*	7	3	4	4	4	4	7
3	4	7	*	7	3	4	4	4	3
4	4	3	7	*	7	3	4	4	4
5	4	4	3	7	*	7	3	4	4
6	4	4	4	3	7	*	7	3	4
7	3	4	4	4	3	7	*	7	4
8	7	4	4	4	4	3	7	*	3
9	7	7	3	4	4	4	4	3	*

(* = 12)

The edge-valence matrix of N_1^9

The 7-valent edges are 19, 92, 23, 34, 45, 56, 67, 78, 81.

	G_i	The four neighbours of G_i	p q r s	p' q' r' s'	7-valent edges of N_1^9 in missing 2-faces of P_4^*
$\mathcal{C}_a$	C 2367	B 2356 D 3467 H 2378 T 2679	2 3 6 7	4 9 8 5	23, 67, 45
$\mathcal{C}_b$	I 1348	G 1238 J 3478 K 1458 Y 1349	1 3 4 8	7 5 2 9	81, 34, 92
$\mathcal{C}_c$	U 1569	M 1568 S 1679 V 2569 W 1459	1 5 6 9	2 7 4 8	19, 56, 78

Table 7.1

the order 34567819.

Choosing

$$\mathcal{C}_a = \{C, B, D, H, T\},$$

$$\mathcal{C}_b = \{I, G, J, K, Y\},$$

$$\mathcal{C}_c = \{U, M, S, V, W\}$$

yields P_4^* in which every edge lies in a missing 2-face. Then P_4 is a simple 4-polytope with 12 facets which cannot be constructed by facet splitting.

Higher-dimensional counterexamples are constructed inductively. Suppose $d \geq 5$ and that we have already constructed P_{d-1}. Let F be a facet of P_{d-1} and let P_d be the wedge with foot F over P_{d-1}. Clearly P_d is a simple d-polytope with $d+8$ facets.

Let P_o, P_1 be the upper and lower bases of P_d and assume F_1 and F_2 are a pair of facets which may be spliced. There are four cases to consider.

Firstly, if neither F_1 nor F_2 is a base of P_d, splicing F_1 and F_2 induces a splicing in P_o and P_1. Since P_o and P_1 are combinatorially equivalent to P_{d-1}, and P_{d-1} admits no splicing, this is a contradiction.

Secondly, if $F_1 = P_o$, $F_2 = P_1$, since there is a vertex in P_{d-1} not in F, wedging yields an edge of P_d meeting P_o and P_1 but not F. Hence F_1 and F_2 may not be spliced.

Hence one of F_1, F_2 is a base, say $F_1 = P_1$, and the other is not.

Thirdly, if $F_1 = P_1$ and $F_2 \cap F = \emptyset$, the facet F_2 is a prism with a base in each of P_o and P_1. Hence

$$\begin{aligned} & P_o \cap (F_1 \cup F_2) \\ &= P_o \cap (P_1 \cup F_2) \\ &= (P_o \cap P_1) \cup (P_o \cap F_2) \\ &= F \cup (P_o \cap F_2), \end{aligned}$$

Polytope	$\mathcal{C}_a$	$\mathcal{C}_b$	$\mathcal{C}_c$	
N_1^9	C B D H T,	I G J K Y,	U M S V W	or
	J D H I L,	T C Q S V,	W K U X Y	or
	H C G J Q,	K I L M W,	V B T U X	
N_3^9	H C G J Y,	K I L M V,	S B P T U	
N_{10}^9	A B C G V,	K J L M Y,	S D P T U	
N_{20}^9	F C E I R,	L J K O T,	W B P X Y	

Table 7.2

a disconnected set and therefore not a cell, implying that F_1 and F_2 may not be spliced.

Fourthly, if $F_1 = P_1$ and $F_2 \cap F \neq \emptyset$, then in P_o, $F_1 \cap P_o$ and $F_2 \cap P_o$ are (d-2)-faces meeting on the (d-3)-face $F_1 \cap F_2 \cap F$. Hence splicing F_1 and F_2 induces a splicing of two facets of P_o, which is impossible.

We conclude that no pair of facets of P_d may be spliced and so P_d cannot be constructed by facet splitting.

This concludes the proof.

Using Altschuler and Steinberg's (1973) list of neighbourly 4-polytopes with 9 vertices, the technique of the proof above yields three more possible P_4's. See Table 7.2.

Barnette's counterexample was constructed by adding a vertex to a neighbourly triangulation of the 3-sphere described by Altschuler and Steinberg (1974).

Altschuler (1976) describes a neighbourly triangulation of the 3-sphere with 10 vertices such that every edge belongs to a missing 2-face. This triangulation is not known to be polytopal, although it satisfies all the known conditions.

Barnette's counterexample has a neighbourly dual and therefore neither it nor its dual may be constructed by facet splitting. It would be interesting to find a polytope with this property. A simple polytope not constructible by facet splitting whose dual is neighbourly would do.

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