The Geometry and Topology of Stable Coisotropic Submanifolds

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I, Tobias Sodoge confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

Für meine Eltern, denen ich alles verdanke.

(*To my parents, to whom I owe everything.*)

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Abstract

In this thesis I study the geometry and topology of *coisotropic* submanifolds of *symplectic* manifolds. In particular of *stable* and of *fibred* coisotropic submanifolds. I prove that the *symplectic quotient* B of a stable, fibred coisotropic submanifold C is *geometrically uniruled* if one imposes natural geometric assumptions on C. The proof has four main steps. I first assign a Lagrangian graph L_C and a stable hypersurface H_C to C, which both capture aspects of the geometry and topology of C. Second, I adapt and apply *Floer theoretic* methods to L_C to establish existence of holomorphic discs with boundary on L_C. I then *stretch the neck* around H_C and apply techniques from *symplectic field theory* to obtain more information about these holomorphic discs. Finally, I derive that this implies existence of a non-constant holomorphic sphere through any given point in B by glueing a holomorphic to an antiholomorphic disc along their common boundary and a simple argument.

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Chapter 1

Introduction

In this thesis I study the geometry and topology of *coisotropic* submanifolds of *symplectic manifolds*. In this introduction I present in

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Section 1.3 An overview of previous research and results on coisotropic submanifolds.

Section 1.4 An outline of the thesis.

1.1 Coisotropic submanifolds

A *symplectic structure* on a smooth manifold W is a closed, non-degenerate 2-form $\omega \in \Omega^2(W)$, where $\Omega^l(W)$ denotes the space of smooth sections of the l-th exterior power of the cotangent bundle of W for a non-negative integer l. A symplectic manifold of dimension $2n$ is a pair (W, ω) consisting of a smooth $2n$-dimensional manifold W and a symplectic structure ω. The *symplectic complement* of a submanifold N of (W, ω) at a point $x \in N$ is defined by

$$T_xN^\omega = \{v \in T_xW \mid \omega(v, w) = 0 \text{ for all } w \in T_xN\},$$

where T_xN denotes the tangent space of N at x. A submanifold C of (W, ω) is *coisotropic* if

$$T_xC^\omega \subseteq T_xC \quad \text{for all } x \text{ in } C.$$

By the non-degeneracy of the 2-form ω, the dimension of T_xC^ω agrees with the codimension of T_xC in T_xW and is thus an integer k between 0 and n.

It follows from Lemma 5.4.1 in [MS17] that a coisotropic submanifold C is a foliated manifold $(C, \mathcal{F})$. I explain in Section 2.1, that coisotropic submanifolds form the naturally interesting foliated spaces in symplectic manifolds. The foliation $\mathcal{F}$ is called the *characteristic foliation* of C. Given a point x in a codimension k coisotropic submanifold C, F_x,the leaf through x, is tangent to T_xC^ω and thus k-dimensional. I prove this in Lemma 2.1.

Coisotropics form a broad class of submanifolds of symplectic manifolds. The manifold W itself is coisotropic since T_qW^ω is trivial at every point $q \in W$ by the non-degeneracy of ω. Thus the leaves of the foliation are the points of W. Again by the non-degeneracy of ω, every hypersurface H in a symplectic manifold is coisotropic. Hence the characteristic foliation of a hypersurface is one dimensional. By definition, Lagrangian submanifolds L are coisotropic submanifolds of maximal codimension n, i.e. $T_xL^\omega = T_xL$. A Lagrangian L is foliated by a single leaf, namely L itself.

Conisder a coisotropic $(C, \mathcal{F})$ as a foliated manifold. At each point $x \in C$ one can form the quotient T_xC/T_xC^ω. This $2n - 2k$ dimensional quotient of vector spaces naturally inherits a symplectic structure, since the restriction of ω to C at each x in C vanishes along the symplectic complement, which agrees with the tangent space to the leaf F_x at x. One is tempted to form the quotient $C/\mathcal{F}$ by identifying all points on the same leaf on all of C. However, the quotient $C/\mathcal{F}$ will often fail to be Hausdorff (see Example 2.6). If the quotient is a smooth Hausdorff manifold, write $B = C/\mathcal{F}$ and call B the *symplectic quotient* of C.

The notation, which I use for the symplectic quotient originates in the case where the leaves F of the characteristic foliation $\mathcal{F}$ fit together to form a smooth fibre bundle $F \to C \to B$ with C as the total space of the fibre bundle. In this case the base B of the fibre bundle carries a smooth symplectic structure ω_B which is induced by ω. For a proof of this fact see Proposition 5.4.5. in [MS17]. A famous instance where this structure arises is the following

Example 1.1 (The generalised Hopf fibration).
Consider $\mathbb{R}^{2n}$ with its standard symplectic structure ω_0 and its standard complex structure J_0. The generalised Hopf fibration

$$S^1 \to S^{2n-1} \to \mathbb{CP}^{n-1},$$

is a fibre bundle with the $2n-1$ dimensional unit sphere S^{2n-1} as its total space. The symplectic quotient of the coisotropic $S^{2n-1} \subset (\mathbb{R}^{2n}, \omega_0)$ is the complex projective space $\mathbb{CP}^{n-1}$ of real dimension $2n-2$.

By considering the cartesian product of k copies of the Hopf fibration one obtains a fibre bundle with total space the coisotropic $C = S^{2n-1} \times \cdots \times S^{2n-1}$ of codimension k in $\mathbb{R}^{2n} \times \cdots \times \mathbb{R}^{2n} \cong \mathbb{R}^{2kn}$. The fibres are k-dimensional tori $\mathbb{T}^k = S^1 \times \cdots \times S^1$ and the symplectic quotient B, a product of k projective spaces $\mathbb{CP}^{n-1} \times \cdots \times \mathbb{CP}^{n-1}$, has dimension $2k(n-1)$.

The process by which one obtains the symplectic quotient is often called *symplectic reduction*. This terminology has its origin in the context of Hamiltonian group actions, which were presented by Marsden and Weinstein as "a unified framework for the construction of symplectic manifolds from systems with symmetries" in [MW74]. More precisely Marsden and Weinstein consider a free and proper Hamiltonian group action G on symplectic manifold W. By Theorem 1 of [MW74], a regular level set of the associated moment map is a coisotropic submanifold C of W, the leaf F_x is the orbit of $x \in C$ under the action of G, and the quotient B, usually denoted by $W /\!\!/ G$ in this context, is a smooth symplectic manifold. In tribute to the authors of [MW74] the symplectic quotient $W /\!\!/ G$ is called the *Marsden-Weinstein quotient*. Here is a well-known example of symplectic reduction:

Example 1.2 (The complex Grassmanian)**.**
Consider, for $k \leq n$, the space $hom(\mathbb{C}^k, \mathbb{C}^n)$ of homomorphisms of $\mathbb{C}^k$ into $\mathbb{C}^n$. Identify this space with the space of n by k complex matrices $\mathbb{C}^{n\times k}$ and equip it with the Hermitian inner product $\mathrm{tr}(A^*B)$, where $\mathrm{tr}(\cdot)$ denotes the trace operator, A^* denotes the conjugate transpose of the matrix $A \in \mathbb{C}^{n\times k}$ and $B \in \mathbb{C}^{n\times k}$. Then a symplectic form on $\mathbb{C}^{n\times k}$ is given by

$$\omega_{\mathrm{tr}}(A, B) := \mathrm{Im}(\mathrm{tr}(A^*B)),$$

where $\mathrm{Im}(\cdot)$ denotes the imaginary part of a complex number. It is a standard fact (see for example Exercise 5.43 of [MS17]) that the action of the unitary group $U(k)$ on $\mathbb{C}^{n\times k}$ by right multiplication is Hamiltonian with moment map

$$\mu(A) = \frac{1}{2i}A^*A.$$

The level set $\mu^{-1}(\frac{1}{2i}\mathbb{1})$ is a coisotropic submanifold of $(\mathbb{C}^{n\times k}, \omega_{\mathrm{tr}})$ by the theorem of Marsden-Weinstein above. The coisotropic submanifold C is diffeomorphic to the space of unitary k-frames, also called the Stiefel manifold, $S(k, n, \mathbb{C})$. The Marsden-Weinstein quotient

$$B = \mu^{-1}(\frac{1}{2i}\mathbb{1}) /\!\!/ U(k)$$

of C is diffeomorphic to the complex Grassmannian, $G(k, n, \mathbb{C})$.

We have just seen in Example 1.2 above how coisotropic submanifolds arise naturally in symplectic topology as the level sets of moment maps. In algebraic geometry, coisotropic submanifolds arise naturally in the context of normal crossing divisors. For concreteness I base the following exposition on [Rua02], where Ruan considers the following situation:

Let $D = \cup_{i=1}^{n} D_i$ be a normal crossing divisor, where each D_i is a smooth divisor in a $2n$-dimensional Kähler manifold (X, ω_g), and ω_g denotes the Kähler form. For each index set $I \in \mathcal{I}$, where $\mathcal{I}$ denotes the collection of index sets of $\{1, \ldots, n\}$, set

$$D_I = \cap_{i \in I} D_i,$$

when the intersection is non-empty. Then D_I is a Kähler submanifold of X of real dimension $2n - 2|I|$. Denoting by U_i a tubular neighbourhood of D_i,

$$U_I = \cap_{i=1}^{n} U_i$$

defines a tubular neighbourhood of D_I.

By Lemma 7.2 of [Rua02] for each x in D_I there exist holomorphic coordinates (w_x^I, z_x^I) in a neighourhood of x, such that, near x one has $D_I = \{z_x^I = 0\}$ and w_x^I are holomorphic coordinates on D_I. These coordinates vary smoothly with x. In Proposition 7.1 of [Rua02] Ruan shows that these charts give rise to fibrations $\pi_I : U_I \to D_I$ with holomorphic fibres for each index set I, which vary smoothly with $x \in D_I$ and which are compatible with the obvious stratification of D. The holomorphic coordinates on each fibre $\pi_I^{-1}(x)$ are given by the z_x^I. These determine a rank $|I|$ real torus $\mathbb{T}^{|I|}$ action on each fibre, which varies smoothly with x and thus gives rise to a smooth, real $\mathbb{T}^{|I|}$ action on U_I.

By Theorem 7.2 of [Rua02] the Kähler form ω_g can be perturbed such that all different components of D intersect orthogonally with respect to ω_g. Ruan calls such a metric a global toroidal metric for (X, D). Moreover, Theorem 7.2 asserts that ω_g can be made flat on each fibre of $\pi_I : U_I \to D_I$ for every index set I and that, by possibly shrinking the collection $\{U_I\}$, the projections $\{\pi_I\}$ can be made compatible with the stratification of D in the sense of Proposition 7.1 of [Rua02].

Call such a system of neighbourhoods $\{\pi_I, U_I,\}_{I \in \mathcal{I}}$ with holomorphic coordinates (w_x^I, z_x^I) which are compatible with the stratification of D and equipped with a a global toroidal metric ω_g^{flat}, which is flat on each fibre, a *global toroidal structure* $\mathcal{T}$ for (X, D, ω_g).

To see how coisotropic submanifolds come into play in this context fix a normal crossing divisor D in a Kähler manifold (X, ω_g) and a global toroidal structure $\mathcal{T}$ for it. Set

$$H_i^I(w_x^I, z_x^I) = \frac{1}{2}|z_x^{I,i}|^2$$

for the norm $|\cdot|$ induced by the Hermitian inner product $||\cdot||$ on each fibre. This defines an $|I|$-tuple of Hamiltonians on U_I for a smooth $\mathbb{T}^{|I|}$-action on U_I determined by the z_x^I. Choose level sets $C_I^i := (H_i^I)^{-1}(r_i^I)$ for $r_i^I \in \mathbb{R} \setminus \{0\}$ and $i \in I$. Then for each set I, the submanifold $C_I := \bigcap_{i \in I} C_I^i$ is coisotropic in $(U_I, \omega_g^{\text{flat}})$ and is the total space of the fibre bundle

$$\mathbb{T}^{|I|} \to C_I \to D_I.$$

Since D_I is Kähler and therefore symplectic , and $\mathbb{T}^{|I|}(x)$ is Lagrangian in $\pi_I^{-1}(x)$, the total space C_I of the torus bundle is coisotropic and of real codimension $|I|$. Thus one may view D_I as the symplectic quotient of C_I. The process of symplectic reduction corresponds to collapsing the torus fibres. Notice that Ruan's construction provides a very rich family of examples of coisotropics. For every normal crossing divisor D in a Kähler manifold (X, ω_g) as above there exists a family of coisotropics $\{C_I\}_{I \in \mathcal{I}}$. For a given index set I, the real codimension of a coisotropic submanifold C_I is $|I|$, and thus for each integer $|I|$ between 0 and n there exist n choose $|I|$ coisotropic submanifolds. Below is an illustration of Ruan's construction in a very simple, yet illuminating case:

Example 1.3 (Complete intersections in $\mathbb{CP}^3$).
Consider the toric manifold $\mathbb{CP}^3$ with the Fubini study form ω_{FS}. Recall that the

action of the complex torus $(\mathbb{C}^*)^3$ on $\mathbb{CP}^3$ is given by

$$
\begin{aligned}
(\mathbb{C}^*)^3 \times \mathbb{CP}^3 &\to \mathbb{CP}^3 \\
(\lambda_1, \lambda_2, \lambda_3) &\mapsto [z_0 : \lambda_1 z_1 : \lambda_2 z_2 : \lambda_3 z_3]
\end{aligned}
$$

Define divisors D_2 and D_3 as follows:

$$
\begin{aligned}
D_2 &= \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \mid z_2 = 0\} \cong \mathbb{CP}^2 \\
D_3 &= \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \mid z_3 = 0\} \cong \mathbb{CP}^2.
\end{aligned}
$$

Then $D = D_2 \cup D_3$ is a normal crossing divisor, and

$$
D_{\{2,3\}} = \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \mid z_2 = z_3 = 0\} \cong \mathbb{CP}^1
$$

is Kähler. Choose the chart

$$
V_1 = \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \mid z_0 \neq 0\}.
$$

The holomorphic toroidal coordinates on a neighbourhood of a point $x = [1 : z_1 : 0 : 0]$ in $D_{2,3}$ can in this case be constructed explicitly by defining

$$
\begin{aligned}
\phi : \mathbb{C}^3 &\to U_{\{2,3\}} \\
(w_1, w_2, w_3) &\mapsto [1 : w_1 : w_2 : w_3].
\end{aligned}
$$

Notice that $U_{\{2,3\}} \cap D_2 = \{[1 : z_1 : 0 : z_3]$ gives a holomorphic coordinate chart for a neighbourhood of D_2 in its normal bundle. Similarly $U_{\{2,3\}} \cap D_3 = \{[1 : z_1 : z_2 : 0]$ and thus the neighbourhood $U_{\{2,3\}}$ is compatible with the stratification of D. In polar coordinates one has $(w_2, w_3) = (r_2 e^{i\theta_2}, r_3 e^{i\theta_3})$ for $r_2, r_3 \in \mathbb{R}$ and θ_2 and $\theta_3 \in [0, 2\pi]$. Define Hamiltonians

$$
H_i = \frac{1}{2}|w_i|^2 = \frac{1}{2}r_i^2 \quad \text{for } i = 2, 3.
$$

Then for some fixed $r, r' \in \mathbb{R} \setminus \{0\}$ a coisotropic $C_{\{2,3\}} = C_{\{2,3\}}(r, r') \subset (U_{\{2,3\}}, \omega_{FS}$, is given by

$$C_{\{2,3\}} = [1 : w_1 : re^{i\theta_2} : r'e^{i\theta_3}].$$

Thus $C_{\{2,3\}}$ is a fibre product $S^3 \times_{\mathbb{CP}^1} S^3$ of two Hopf fibrations and in particular a codimension 2 coisotropic in $\mathbb{CP}^3$. Its preimage in $S^7 \subset \mathbb{C}^4$ is a five dimensional coisotropic and the symplectic quotient, $D_{\{2,3\}}$ of $C_{\{2,3\}}$ is a copy of $\mathbb{CP}^1$.

Example 1.3 above can be generalised in various ways. For instance to

Example 1.4 (Divisors in $\mathbb{CP}^n$).
Again, consider $\mathbb{CP}^n$ with the Fubini-study form ω_{FS} and a complete intersection D_k defined by k homogenous equations of degrees $d_1, \dots, d_k$ such that D_k is a normal crossing divisor. Then by Ruan's construction one obtains a coisotropic C_k which is a real torus bundle over D_k. If $d_1 + \cdots + d_k \leq n$, V is a Fano variety, by the following equality for first Chern class $c_1(D_k)$:

$$c_1(D_k) = ((n+1) - (d_1 + \cdots + d_k))\,[\omega_{FS}]. \tag{1.1}$$

Note that if one choose $d_1 = 1$, then $C_k \subset \mathbb{C}^{n-1}$ and is therefore displaceable.

Notice that Examples 1.1, 1.2, 1.3 and 1.4 all share a common characteristic: the symplectic quotients $\mathbb{CP}^n$, $G(k, n, \mathbb{C})$ or a Fano variety are *uniruled*.

In the Fano case this follows from Mori's bend and break agruments, see [Mor79]. The precise definition of "uniruledness" depends on the setting. Roughly speaking a space is uniruled if, given a point constraint, there exists a non-constant sphere meeting this point constraint. For a precise definition in the symplectic setting see

[Voi08], and in the algebraic geometry setting see Chapter 4 of [Deb01]. For the purposes of this thesis I introduce the notion of *geometrical uniruledness* below.

Definition 1.5 (geometrically uniruled)**.**
The symplectic quotient (B, ω_B) of a fibred coisotropic submanifolds C is geometrically uniruled if B has the following property:

Given any point $b \in B$, for every ω_B- compatible almost complex structure J_B on B, there exists a non-constant J_B-holomorphic sphere

$$v : (\mathbb{C} \cup \{\infty\}, i) \to (B, J_B)$$

passing through b.

Also notice that the leaves of the characteristic foliation $\mathcal{F}$ of C are tori in Example 1.1, 1.3 and 1.4 but not in Example 1.2. With these examples in mind, the following question seems natural:

Question 1.
If one imposes natural geometric assumptions (like displaceability) on a coisotropic submanifold C of a symplectic manifold (W, ω), what are the consequences for the symplectic quotient B of C?

The main result of this thesis is a first answer to this question. I state it, explain how it is related to the above examples and present an outline of its proof in the next section.

1.2 Statement of the main result and summary of the proof

Theorem 1.6.

Let C be a closed, codimension k, coisotropic submanifold of a symplectic manifold (W, ω). If C is fibred and stable, then C is the total space of a torus fibre bundle

$$\mathbb{T}^k \to C \to B$$

over its symplectic quotient (B, ω_B). Assume that C is monotone and has minimal Maslov number at least three. If C is displaceable, then the symplectic quotient (B, ω_B) is geometrically uniruled.

Some remarks on assumptions and assertions of the theorem are in order:

By the nature of Question 1 some assumptions on the coisotropic are indispensable even to make sense of the question. Recall from Section 1.1 that the symplectic quotient B is not necessarily Hausdorff. Therefore, unless one develops a theory for non-Hausdorff symplectic manifolds, one needs to make an assumption that ensures the B is smooth symplectic manifold. I now briefly explain the assumptions of Theorem 1.6 above.

Fibredness: A coisotropic submanifold C is called *fibred* if the leaves of the isotropic foliation $\mathcal{F}$ are closed submanifolds of C and the holonomy of each leach is trivial. (See Definition 2.7). I introduce fibred coisotropic submanifolds in Section 2.2. In particular, I show in Proposition 2.8 that a fibred coisotropic submanifold C is the total space of a fibre bundle $F \to C \to B$ over its symplectic quotient (B, ω_B) as it is the case in Examples 1.1, 1.2, 1.3 and 1.4 above. Thus the leaves of the characteristic foliation $\mathcal{F}$ of a fibreed coisotropic are the fibres of the fibration and thus all diffeomorphic. In particular, their geometry cannot change drastically under arbitrarily small perturbations (see again Example 2.6).

The notions of *monotonicity*, the *minimal Maslov number* and *displaceability* of a coisotropic submanifold are introduced and explained in detail in Section 3.1.1. I give a quick explanation of these assumptions below.

Monotonicity: I define *monotonicity* of C in Definition 3.4 as monotonicity of a Lagrangian submanifold which is called the *Lagrangian graph* L_C. This Lagrangian is the fibre product $C \times_B C$ in the symplectic manifold $(W \times W, -\omega \times \omega)$. For the definition of L_C as a set see Equation 1.2 below. I introduce L_C in detail in Section 3.1. A Lagrangian submanifold is *monotone* if the *symplectic energy* of a holomorphic disc with boundary on the Lagrangian is positively proportional to the *Maslov index* of the disc by a fixed constant independent of the disc.

Note that monotonicity of L_C as a Lagrangian, despite implying monotonicity of W, is not the same as monotonicity of the symplectic quotient B. Nonetheless, Example 1.4 aims to give some intuition why the monotonicity assumption is necessary. If the complete intersection is Fano and therefore monotone it is uniruled by Mori's bend and break arguments [Mor79]. However, it is not difficult to construct a non-uniruled complete intersection D_k: choose k equations such that the sum of the degrees $d_1 + \cdots + d_k$ is at least $n+1$. By Equation (1.1), D_k is now either Calabi-Yau or of general type and therefore not necessarily uniruled. Since D_k is a normal crossing divisor the coisotropic C_k can still be constructed using Ruan's method described above. Thus C_k is a torus bundle over a symplectic quotient D_k, which is not necessarily uniruled.

On the technical side of things, the monotonicity assumption makes the *pearl complex* machinery of Biran-Cornea (see [BC07]) available, which is the formulation of Floer theory used in the proof of Theorem 1.6.

Minimal Maslov number: The *minimal Maslov number* of a coisotropic submanifold is defined as the minimal Maslov number of the associated Lagrangian graph L_C (see Definition 3.5). I recall the definition of the *minimal Maslov number* of a Lagrangian in Equation 3.3 and compute it for the Lagrangian graph of the generalised Hopf fibration, Example 1.1 above, in Example 3.6. Roughly speaking the Maslov index of a disc with boundary on a Lagrangian measures the rotation of the Lagrangian tangent planes around the boundary of the disc. The minimal Maslov number of a monotone Lagrangian L is then the minimal Maslov index of all discs with boundary on L. It is positive if such a disc exists and set to "∞" otherwise.

The assumption on the minimal Maslov number is necessary. To see this, assume the theorem holds and that the codimension of C is n, so that C is a Lagrangian torus $\mathbb{T}^n$. Now there certainly cannot exists any non-constant holomorphic sphere in the symplectic quotient B, which is a point. By definition, L_C is diffeomorphic to $\mathbb{T}^{2n}$, and therefore cannot be a monotone, displaceable Lagrangian torus of minimal Maslov number at least three. Thus, assuming L_C is monotone and displaceable, the assumption that N_{L_C} is at least three must fail. If L_C is orientable, it follows that the minimal Maslov number N_{L_C} is equal to two, since it is strictly positive by the displaceability assumption. This is in accordance with the Audin conjecture for monotone tori being true, see for example [Dam12]. In fact, the assumption that N_{L_C} is at least three is crucial in the proof of Theorem 1.7 below. I would like to point out that I discovered that part of the proof of Theorem 1.7 is similar to Buhovsky's proof of the Audin conjecture for monotone tori in [Buh10] after proving the theorem.

It is worth pointing out that Ziltener defines a Maslov index for coisotropic submanifolds in [Zil09], which agrees with the definition put forward in [Gin11]. Oh also defines a coisotropic Maslov index in [Oh03]. It would have been also possible to phrase our assumption as a requirement on the coisotropic Maslov index and it would be interesting to relate the Maslov index of L_C to the coisotropic Maslov index.

Displaceability: A submanifold N is *Hamiltonian displaceable* if there exists a Hamiltonian symplectomorphism ψ such that $\psi(N) \cap N = \emptyset$. I recall the notion of (Hamiltonian) displaceability in Definition 3.7 and begin to explore the relation of displaceability of C and of L_C in Section 3.1.2. The proof presented in this thesis relies on the property of L_C to bound non-trivial holomorphic discs. Assuming the existence of such discs is, despite being sufficient, somewhat artificial. A more natural geometric condition which implies the existence of many non-constant holomorphic discs with boundary on L_C is displaceability of C. Notice that it would also be sufficient to assume displaceability of the Lagrangian L_C, which is implied by displaceability of C (see Lemma 3.11). For the proof, the most important consequence of displaceability of C is the vanishing of Floer homology of L_C, which I prove in Lemma 4.23. Notice that the coisotropic submanifolds in Examples 1.1, 1.2 the subexample of Example 1.4 obtained by choosing $d_1 = 1$, are submanifolds of $\mathbb{C}^n$,$\mathbb{C}^{kn}$ and $\mathbb{C}^{n-1}$ respectively, where every submanifold is displaceable. Generally speaking, if one considers coisotropics in $W = \mathbb{R}^{2N}$ for some N one can drop the displaceability assumption.

Stability: I examine the stability condition on the coisotropic C, which was introduced by Bolle in [Bol98], in detail in Section 2.3. A *stable* coisotropic submanifold C (see Definitions 2.11 and 2.13) is the straightforward generalisation of a stable hypersurface (see Lemma 2.1 in [CV10] for the definition) to higher codimension. A stable coisotropic submanifold of codimension 1 is a stable hypersurface. Roughly speaking stability means in this context that the characteristic foliation $\mathcal{F}$ of C remains unchanged under small perturbations in the normal directions of the coisotropic.

Imposing the stability condition on C has several important consequences. First, it implies that C has a trivial normal bundle and second that there exists a model neighbourhood U of C, which is symplectomorphic to $B^k_{\epsilon_0} \times C$, where $B^k_{\epsilon_0}$ denotes the ball of radius ϵ_0 in $\mathbb{R}^k$, and k is the codimension of C in W. The coisotropic submanifold C is embedded as $\{0\} \times C$ in this neighbourhood and the symplectic form is given explicitly by 2.6. Moreover the characteristic foliation $\mathcal{F}_p$ of $\{p\} \times C$ is conjugate via a family of diffeomorphisms smoothly depending on the coordinate p in B^k_ϵ to the foliation $\mathcal{F}$ of $\{0\} \times C$. I prove existence of such a neighbourhood in Lemma 2.18 and call it a *Bolle neighbourhood* in tribute to Bolle who established its existence in Section 5 of [Bol98]. By the Arnold-Liouville Theorem (see Section 10 of [Arn89]), a stable Lagrangian is necessarily a torus. As a straightforward application of this theorem, I prove in Proposition 2.22 that the closed leaves of a stable coisotropic are k-dimensional tori. In particular, the assertion of the theorem that fibred, stable coisotropics are the total spaces of torus bundles follows from this proposition.

As one might expect, Examples 1.1, 1.3 and 1.4 are stable coisotropic submanifolds. I explain this in more detail in Section 2.3. Example 1.2 illustrates that the stability condition is not necessary, since $U(k)$ is not a torus, but the symplectic quotient $G(k, n, \mathbb{C})$ is uniruled. If one chooses a maximal torus in $U(k)$ and considers the action of this maximal torus, one obtains the partial flag variety as the symplectic quotient, which is also uniruled. I spell this out in Example 2.25 in Section 2.3. Theorem 7.5 of Usher's paper [Ush11] also suggests that one could hope to relax the stability assumption on C, to the assumption that the there exists a Riemannian metric which renders the leaves totally geodesic. The proof presented in this thesis however relies heavily on the stability assumption.

Before giving the summary of the proof of Theorem 1.6, I would like to remark that the assertion of the theorem is different from other results on uniruling in the following sense: Being *symplectically uniruled* means that there exists a non-vanishing Gromov-Witten invariant. Proving this for B would imply the geometric statement about the existence of non-trivial holomorphic spheres through any given point in B from the assertion of Theorem 1.6. In this thesis, I derive the geometric statement directly, hence the term *geometrically uniruled*. It would be very interesting to compute the Gromov-Witten invariants of B and to relate them to the Fukaya-Floer alebgra of the Lagrangian L_C.

1.3 Summary of the proof of the main theorem

The main obstruction to answering questions like Question 1 is the lack of mathematical machinery which is tailored to study coisotropic submanifolds. I therefore chose the strategy below for the proof:

(I) Assign a Lagrangian submanifold L_C and a hypersurface H_C to C. Both L_C and H_C capture some relevant parts of the geometry and topology of C but both have the advantage of belonging to classes of submanifolds of symplectic manifolds for which more mathematical machinery is available.

(II) Adapt existing theories for Lagrangians and hypersurfaces to L_C and H_C in order to make the theories incorporate the structures of C which are captured in L_C and H_C.

(III) Apply standard techniques to L_C and H_C and thereby extract information about C.

More concretely, Chapter 3 is dedicated to accomplishing (I): I show in Section 3.1 how to assign a *Lagrangian graph* L_C to a given fibred coisotropic submanifold $C \subset W$. The Lagrangian graph is defined as a subset of the *twisted product*, $(W \times W, -\omega \times \omega)$, of (W, ω) by

$$L_C = \{(x, y) \in C \times C \mid \pi_B(x) = \pi_B(y)\}. \tag{1.2}$$

I prove in Lemma 3.2 that L_C is a Lagrangian submanifold of $(W \times W, -\omega \times \omega)$. In a nutshell this follows from the fact that ω vanishes along the leaves and that one uses opposite signs in both factors. More abstractly, L_C can be described as a fibre product $C \times_B C$ of C with itself over B (see Definition 3.1). Also observe that L_C is a special case of a Lagrangian correspondence. The most important feature of this assignment is that L_C inherits a fibre bundle structure from the fibred coisotropic C. I demonstrate this in Lemma 3.2. Moreover, I explain in Section 3.1.2 how the self intersection theory of L_C as a monotone Lagrangian, which can be studied via Lagrangian Floer theory, is related to the self intersection theory of C. Thus, as a consequence of assigning L_C to C, Lagrangian Floer theory and its algebraic machinery become available to study fibred coisotropic submanifolds C.

In Section 3.2 I assign a stable hypersurface H_C to C. More precisely, I show that for every $\epsilon < \epsilon_0$ the hypersurface

$$H_{C,\epsilon} := S^{k-1}_\epsilon \times C$$

contained in the Bolle neighbourhood of a stable coisotropic submanifold C is a stable hypersurface. The most important feature of this assignment is that the Reeb dynamics of H_C correspond to the *generalised Reeb dynamics* on the (isotropic) leaves F of the coisotropic submanifold C in an appropriate sense (see Definition 3.16 for details). The Reeb dynamics on stable hypersurfaces are intimately related to the asymptotics of holomorphic curves in the symplectisations of stable hypersurfaces (see for example [Hof93] or [Abb14]). Roughly speaking, by studying C through H_C, one extends this relation to stable coisotropic submanifolds. Thus assigning H_C to a stable coisotropic C makes techniques from symplectic field theory (see [EGH00]), and in particular *neck stretching* available to study stable coisotropic submanifolds.

With these assignments in place, Theorem 1.6 is proved in three main steps. The first two steps are proving Theorem 1.7, stated below, in Chapter 4 and Theorem 1.8, stated further below, in Chapter 5. The proofs of these theorems are subdivided into adapting theories to L_C and H_C respectively, i.e. (II) and then applying standard techniques, i.e. (III). The last step in the proof of Theorem 1.6 is a simple argument which I present in Chapter 6.

Theorem 1.7.
Let C be a fibred, stable coisotropic submanifold of a symplectic manifold (W, ω). Assume that the Lagrangian graph L_C in the product $(W \times W, -\omega \times \omega)$ is monotone and has minimal Maslov number N_{L_C} at least three. Let b be any point in the symplectic quotient B of C.

If L_C is displaceable, then there exist:

(M) *An almost fibred Morse function f on L_C such that the unique global minimum x of f on L_C is contained in $f_B^{-1}(0)$ and projects to $(b, b) \in \Delta B$ the diagonal in $B \times B$.*

(E) *A constant $E_0 > 0$, such that for all ω-compatible almost complex structures J on W, there exists at least one pearly trajectory P of energy at most E_0 and with the following property:*

(P) *The pearly trajectory P connects a critical point y of f contained in $f_B^{-1}([1, \infty))$ to the minimum x of f.*

An *almost fibred Morse function* f is a Morse function, which is constructed by lifting a Morse function f_B from B to L_C and perturbing it by a small Morse function f_F on a typical fibre (see Definition 4.6 for details). I recall the definition of a *pearly trajectory* in Definitions 4.12 and 4.14 respectively. Roughly speaking, a pearly trajectory is a configuration of holomorphic discs which are arranged along gradient flow lines of Morse functions on L_C. Pearly trajectories play a key role in defining the algebraic structures on the *pearl complex* in [BC07], where this complex is used to define the Lagrangian quantum cohomology ring of a monotone Lagrangian L. This cohomology theory is isomorphic to the self-Floer cohomology of L via the PSS map. The *energy of a pearly trajectory* is the symplectic area associated to the homology class of the pearly trajectory (see Definition 4.16) and ΔB denotes the diagonal in $B \times B$.

To prove Theorem 1.7 above I first I adapt the construction of the pearl complex to make it incorporate the fibre bundle structure of L_C (and thus of C) in Sections 4.2 and 4.3. As a result of the adapation, the algebraic structures defined on the pearl complex "see" information contained in the fibred coisotropic C. Then, in Section 4.4, I apply the algebraic structures defined on this *almost fibred pearl complex* to carry out the proof of Theorem 1.7, which I now describe briefly.

The displaceability of L_C implies the existence of at least one pearly trajectory P ending in the unique minimum x of an almost fibred Morse function f on L_C. I prove this in Lemma 4.23. By the assumption that C is fibred, L_C is a torus fibration (see Proposition 3.2). In particular the fibre over the minimum x is a $2k$ dimensional torus $\mathbb{T}^{2k}_x = f_B^{-1}(0)$. The set of critical points of f generates the pearl complex. It can be partitioned into the set of critical points contained in $f_B^{-1}(0)$ and in $f_B^{-1}([1, \infty))$ by an appropriate choice of f_B.

To prove Theorem 1.7 one needs to eliminate the possibility that all pearly trajectories P ending in the minimum x are contained entirely in the torus fibre, $\mathbb{T}^{2k}_x$, over the minimum. First of all notice that if the minimal Maslov number is at least $2k+2$, this is impossible as the pearly differential then counts pearly trajectories which connect critical points of index difference at least $2k+1$. Thus no pearly trajectory ending in the minimum can emanate from a critical point in the fibre $\mathbb{T}^{2k}_x$ and the theorem follows.

To achieve the improvement that the assumption of Minimal Maslov number at least three suffices one argues as follows: If there exists a pearly trajectory, beginning in $f_B^{-1}([1,\infty))$, the Theorem follows. If not, all pearly trajectories ending in the minimum emanate from critical points y of f which are contained in $\mathbb{T}^{2k}_x = f_B^{-1}(0)$. Every such critical point y can be generated as sums of linear combinations of Morse cup products $x_1 \star_0 \cdots \star_0 x_K$ of index 1 critical points $x_1, \dots, x_K$ in the fibre $\mathbb{T}^{2k}_x$ (see Definition 4.22). One then analyses pearly trajectories arising from linear combinations of quantum deformations $x_1 \star \cdots \star x_K$ of the Morse cup products of the x_i, which were used to generate y. The assumption that the minimal Maslov number is at least three then implies that there exists a pearly trajectory which emanates from a critical point y' contained in $f_B^{-1}[1,\infty)$ and satisfies the properties stated in the assertion of Theorem 1.7 (see Porposition 4.24). A more detailed outline of the proof is given at the beginning of Chapter 4.

Observe that Theorem 1.7 asserts nothing about the the holomorphic discs $u : (D, \partial D) \to (W, L_C)$ which contribute to the pearly trajectory P. For example, the interior of these holomorphic discs is not necessarily contained in L_C or even in a neighbourhood of L_C and therefore, a priori, cannot be projected to B. In order to obtain more information about the holomorphic discs contributing to P, I adapt and apply techniques from symplectic field theory. More precisely, I prove Theorem 1.8 below.

Theorem 1.8.
Let C be a fibred, stable coisotropic submanifold of a symplectic manifold (W, ω). Assume that the Lagrangian graph L_C in the product $(W \times W, -\omega \times \omega)$ is monotone and has minimal Maslov number N_{L_C} at least three. Let b be any point in the symplectic quotient B of C.

If L_C is displaceable, then there exist:

(M) *An almost fibred Morse function f on L_C such that the unique global minimum x of*

f on L_C is contained in $f_B^{-1}(0)$ and projects to $(b,b) \in \Delta B$.

(*E*) *A constant $E_0 > 0$, such that for all ω_B-compatible almost complex structures J_B on B, there exists at least one punctured pearly trajectory pP of energy at most E_0 and with the following properties:*

(*pP1*) *The punctured pearly trajectory pP connects a critical point y of f contained in $f_B^{-1}([1,\infty)$ to the minimum x of f.*

(*pP2*) *The punctured pearly trajectory pP contains at least one punctured, non-trivial holomorphic curve*

$$\tilde{u} : (S, \partial S, j) \to (\tilde{W}_C \times \tilde{W}_C, L_C, -\tilde{J}_C \times \tilde{J}_C)$$

with the following properties:

(*S1*) *The intersection $\tilde{u}(\partial S) \cap f_B^{-1}(0)$ and the intersection $\tilde{u}(\partial S) \cap f_B^{-1}((0,\infty))$ are both non-empty.*

(*S2*) *If $\tilde{u}$ is unbounded near an interior puncture, then $\tilde{u}$ is asymptotic to a cylinder over a generalised Reeb orbit on C when approaching the puncture.*

(*S3*) *All other boundary and interior punctures of $\tilde{u}$ are removable.*

Here $(S, \partial S)$ denotes a nodal, stable connected Riemann surface with nonempty boundary of genus zero.

A *punctured pearly trajectory* is a pearly trajectory in which the domains of the contributing holomorphic discs are allowed to degenerate to *nodal*, connected, *stable*, genus zero Riemann surfaces (see Definitions 5.10, 5.11 and 5.12) with nonempty boundary. The manifold $\tilde{W}_C$ is the *symplectic cobordism* (see Definition 5.4) obtained as the *symplectic completion* of the Bolle neighbourhood of C and diffeomorphic to $\mathbb{R}^k \times C$. The almost complex structure $\tilde{J}_C$ is the limit of a sequence $(J_{\mathcal{S}}^{\tau})_{\tau \geq 0}$ of almost complex structures used in a *neck-stretching procedure* on $\tilde{W}_C$. These almost complex structures $J_{\mathcal{S}}^{\tau}$ are *adjusted* to the stable coisotropic $(C, \mathcal{S})$(see Definition

5.5). In particular, they render the projections to B, and the symplectic quotient of H_C holomorphic and are radially invariant in $\mathbb{R}^k$ (see Sections 5.3 and 5.5.3 for details).

Roughly speaking the proof of Theorem 5.1 is a translation of the ideas of the compactness proof in symplectic field theory from [Bou+03] to the present setting.

Recall that, as a consequence of the stability requirement on C, there exists a Bolle neighbourhood U of C symplectomorphic via a map ψ to $B^k_{\epsilon_0} \times C$. The symplectic form on U is given explicitly by Equation 2.6. By looking at C from a Hamiltonian group action perspective, the boundary of U can be identified with the stable hypersurface H_C (see Section 3.2.2). By construction, there is a one to one correspondence of the set of generalised Reeb trajectories $\mathcal{G}$ on C and the set of Reeb trajectories $\mathcal{R}$ on H_C (see Proposition 3.25). The coisotropic submanifold C gets embedded into U as $\{0\} \times C$. Thus one can interpret H_C as a stable hypersurface separating W into symplectic cobordisms (see Definition 5.4).

It is a common technique in symplectic and contact topology to "stretch the neck" around a stable hypersurface H in order to obtain information about holomorphic curves in the manifold W (see for example [EGH00], [Bou+03], [CM05] and the references therein). "The neck" refers to a neighbourhood diffeomorphic to $(-\epsilon, \epsilon) \times H$, which gets "stretched" to $\mathbb{R} \times H$. Stretching the neck is also called "splitting" as it results in disjoint, non-compact symplectic cobordisms. In the present case these disjoint components are $\tilde{W}_C \cong \mathbb{R}^k \times C$, the *symplectic completion* of the Bolle neighbourhood U, $\tilde{W}_H \cong \mathbb{R} \times H_C$, called the *symplectization of* H_C and $\tilde{W}_R$, the symplectic completion of $W \setminus U$. As a result of splitting, a J-holomorphic curve $u : S \to W$ with domain a Riemann surface S which satisfies certain assumptions, defines (see again [Bou+03]), a punctured $\tilde{J}_C$-holomorphic curve $\tilde{u}_C$ in $\tilde{W}_C$, where the domain of $\tilde{u}_C$ is a nodal Riemann surface.

As alluded to above, the almost complex structure $\tilde{J}_S$ is a limit of a sequence of almost complex structures J^τ_S for $\tau > 0 \in \mathbb{R}$ which are translationally invariant on the longer and longer necks $(-\tau, \tau) \times H$. These specific families of almost complex structures play a key role in obtaining more information about the holomorphic curves via splitting the manifold.

I show in Section 5.3 how to construct such a family of almost complex structures $J_{\mathcal{S}}$. The correspondence of the generalised Reeb trajectories $\mathcal{G}$ on C and the Reeb trajectories $\mathcal{R}$ on H_C, now implies that if the $\mathbb{R}^k$ component of $\tilde{u}_C$ is unbounded near a puncture, $\tilde{u}_C$ is asymptotic to a cylinder over a generalised Reeb orbit on C. I explain this in Proposition 5.14. The main assumptions on the holomorphic curve u which are needed to ensure this behaviour are finiteness of energy (see Section 5.5.2), and that, if the domain of u has non-empty boundary ∂S, u maps the boundary to a Lagrangian submanifold L of W i.e. $u(\partial S) \subset L$.

Set

$$W^- \times W^+ := (W \times W, -\omega \times \omega, -J \times J).$$

To prove Theorem 5.1 one uses this apparatus as follows: Theorem 1.7 implies that there exists a pearly trajectory which, by definition of a pearly trajectory, contains at least one non-trivial $(-J \times J)$-holomorphic disc

$$u = (u^-, u^+) : (D, \partial D) \longrightarrow (W^- \times W^+, L_C)$$

The component u^- mapping to the first factor of $W \times W$ satisfies $u^-(\partial D) \subset C$. If the codimension of C is not n, C is not Lagrangian, and thus the results from [Bou+03] do not apply directly to u^- and likewise do not apply directly to u^+. However, $u = (u^-, u^+)$ does satisfy a Lagrangian boundary condition in the product manifold $W \times W$. Since L_C is a subset of $C \times C$ it is embedded as a subset of $\psi(\{0\} \times C \times \{0\} \times C) \subset U \times U$ in $W \times W$. Then "splitting" $W^- \times W^+$ along $H_C \times H_C$ by splitting both factors W^- and W^+ along H_C using family of almost complex structures $(-J^\tau_{\mathcal{S}} \times J^\tau_{\mathcal{S}})$, gives rise to a sequence $(P_\tau)_{\tau \geq 0}$ of pearly trajectories. This sequence has uniformly bounded energy by construction. To prove the theorem it remains to show that there exists a subsequence of this sequence which converges to a punctured pearly trajectory pP with the properties (*pP1*) and (*pP2*) stated in the assertion of Theorem 5.1.

In a nutshell, the sequence of pearly trajectories $(P_\tau)_{\tau\geq 0}$ converges to a punctured pearly trajectory pP because the splitting is happening "far away" from L_C. This allows to view each non-trivial holomormpic map u_{i_n} for $i_n \in \{1,\ldots,K_n\}$ with $K_n \in \mathbb{Z}_{\geq 1}$ contributing to the sequence of pearly trajectories as either a single $(-J^n_S \times J^n_S)$-holomorphic map, satisfying Lagrangian boundary condition in the compact parts of $\tilde{W}_C$ and as a pair (u^-, u^+) of a $(-J^n_S)$- and a J^n_S-holomorphic map in the non-compact part of $\tilde{W}_C$. The existence of a punctured pearly trajectory with the properties (pP1) and (pP2) then basically follows from applying Gromov's compactness Theorem in the compact parts (see for example [Fra08]) and by applying the compactness results from [Bou+03] in the non-compact parts. As a result, the limit object pP contains a holomorphic curve with domain S' a nodal Riemann surface. The properties (S1)-(S3) of the holomorphic curve $\tilde{u}$ follow from the fibre bundle structure of L_C by a straightforward argument, which I give at the very end of the proof of Theorem 1.8. A more detailed outline of the proof is given in Section 5.1 of Chapter 5 where I present the proof of the theorem.

Most of the effort of proving Theorem 1.8 lies in adapting the setup of symplectic field theory to the present setting. A priori performing the k-dimensional analogue of a neck-stretch around a codimension k-coisotropic could lead to different results than neck stretching around the associated stable hypersurface H_C. It turns out that the two approaches yield the same result (see Remark 5.9).Thus the machinery developed here allows to use neck stretching techniques for stable coisotropics C via neck stretching around the stable hypersurface H_C

Given these two results, the final step of proving Theorem 1.6 is the following argument: Theorem 1.8 provides, by projection to the first and second factor of the target $-\tilde{W}_C \times \tilde{W}_C$, a pair $(\tilde{u}^-, \tilde{u}^+)$ of a punctured anti-holomorphic and a punctured holomorphic disc which have well defined projections to C and B. All punctures of $(\tilde{u}^-, \tilde{u}^+)$ are either removable by $(S3)$ or approach a pair of generalised Reeb orbits contained in a pair of leaves of the characteristic foliation $\mathcal{F} \times \mathcal{F}$. Since C is fibred, the leaves of the characteristic foliation coincide the fibres of the fibre bundle, and thus generalised Reeb orbits project to points in the symplectic quotient B. Thus after projection to the compact space $B \times B$, the pair $(\pi_B \circ \tilde{u}^-, \pi_B \circ \tilde{u}^+)$ defines a pair of an honest (i.e. without punctures) antiholomorphic u_B^- and an honest holomorphic disc u_B^+, by the removal of singularities theorem. By the definition of L_C as a fibre product over B (recall Equation 1.2 above) and the Lagrangian boundary condition on $\tilde{u}$, the discs u_B^- and u_B^+ agree along their boundaries in ΔB. Thus one may glue them to an (i, J_B)-holomorphic sphere $v : \mathbb{C} \cup \{\infty\} \to B$, which passes trough a given point $b \in B$.

1.4 Previous and related research on coisotropic submanifolds

Coisotropics encompass classes of submanifolds which have been studied extensively in symplectic topology: Lagrangians, hypersurfaces and of course symplectic manifolds themselves. Floer's proof of the Arnold conjecture by developing an intersection theory for Lagrangian submanifolds, see for example [Hof+95], inspired an abundance of research in symplectic topology. Likewise, Viterbo's [Vit87] and Hofer's [Hof93] proofs of the Weinstein conjecture have inspired plenty of research on the dynamics of contact and stable hypersurfaces. Consequently, many questions about coisotropics which have been addressed in the past have their origins either in questions about symplectic manifolds, Lagrangians or hypersurfaces. Put differently, coisotropic submanifolds provide a general framework for addressing many interesting questions in symplectic topology. Consider for example the following:

Question 2.

Given a symplectomorphism ψ of a symplectic manifold W and a coisotropic submanifold C, do there exist leaf-wise fixed points of ψ on C?

Recall the definition of a leaf-wise fixed point. A point x in a coisotropic submanifold C is a *leaf-wise fixed point* of a (Hamiltonian) symplectomorphism ψ of W if $\psi(x)$ lies in the leaf F_x through x, i.e. $\psi(x) \in F_x$. In the case where C is the entire symplectic manifold W, a leaf-wise fixed point is a fixed point of ψ. Thus in the case $C = W$, Question 2 above is about fixed points of symplectomorphisms and thus related to the Arnold conjecture. If C is a Lagrangian L a leaf-wise fixed point is an intersection point of L and $\psi(L)$ and in this case Question 2 is about the self-intersection properties of L. A more detailed exposition of these correspondences is given in Lemma 3.9 and Remark 3.10.

Already in 1978 Moser proved the following result in [Mos78]: given an embedding i of a compact coisotropic submanifold C into a simply connected, exact symplectic manifold W. If the composition of a differentiable, exact symplectic mapping ψ with i is sufficiently C^1 close to i, then $\psi \circ i$ has at least two leaf-wise fixed points on C. These existence results were then extended by Banyaga in [Ban80] to non-simply connected symplectic manifolds even before Gromov's "founding" paper [Gro85] and the advent of modern symplectic geometry.

Dragnev, Ziltener, Kang and Gürel independently proved the existence of leaf-wise fixed points in more general settings using Floer theoretic methods in [Dra08], [Zil10], [Zil14], [Kan13] [Gür10] respectively. The main assumption on the symplectomorphism ψ is that the Hofer norm of ψ does not exceed a symplectic capacity associated to C. Ziltener explains in footnote 2 of [Zil14] how the Hofer norm can be compared to the C^1-norm. The assumptions on C and W vary. For example Ziltener assumes in [Zil10] that W is geometrically bounded and that C is closed and fibred. In [Kan13], Kang assumes W to be convex at infinity and C to be closed and of restricted contact type. To illustrate the kind of results that were proved, I state Dragnev's result from [Dra08]: A symplectomorphisms ψ of $\mathbb{R}^n$ with its standard symplectic structure, has a leaf-wise fixed point on a compact, contact coisotropic (see Definition 2.14), provided the Hofer norm of ψ is smaller than the Floer-Hofer capacity of C.

Recently, Ziltener proved in [Zil14], that, if the inclusion of a closed coisotropic

C into (W, ω) is not necessarily contact, then a Hamiltonian symplectomorphism, which is close to the inclusion in an appropriate sense in the C^0-norm, has at least one leaf-wise fixed point it . Ginzburg and Gürel show in [GG15] that it is not sufficient to assume closeness to the identity in the Hofer norm if one drops the contact condition on C. It seems that either closeness in an appropriate C^0 norm or the contact condition on C are indispensable assumptions for the existence of leaf-wise fixed points. Albers and Frauenfelder have also studied Question 2 in the context of stable and contact hypersurfaces using Rabinowitz Floer theory. See for example [AF12].

A simpler, yet closely related question, originating in the rigidity results for Lagrangian intersections obtained by Chekanov in [Che98], is the following:

Question 3.

Is the displacement energy of a coisotropic submanifold strictly positive?

The displacement energy of a submanifold of a symplectic manifold is, roughly speaking, the infimum over the Hofer norms of all Hamiltonian symplectomorphism displacing the submanifold. For a precise definition see for example page 3 of [Ker08]. Ginzburg in his paper [Gin07], which also provides an overview on the theory of coisotropic intersections, Ziltener [SZ12] and Kerman in [Ker08] obtained affirmative answers to Question 3. Again their assumptions vary. For example Ginzburg assumes that W is either symplectically aspherical and closed or wide and geometrically bounded, and that C is closed and stable. The most general result in this direction is Theorem 1.6 of Usher's paper [Ush11], which implies in particular (Corollary 1.7 of [Ush11]) that any closed stable coisotropic submanifold of a Stein manifold has positive displacement energy.

Questions 2 above has its origin in studying symplectic manifolds, or through Floer's work, in studying Lagrangians. Question 3 originates in the study of Lagrangian submanifolds. A question originating in the interest on the dynamics of hypersurfaces, which can in fact be seen as a generalisation of the Weinstein conjecture to higher codimension $k > 1$ is the following:

Question 4.

Do there exist loops, which are non-contractible in the leaves of the characteristic foliation of a coisotropic (and which bound positive symplectic energy)?

Recall that Bolle, in his 1997 paper [Bol98], introduced the contact and stability condition on coisotropic submanifolds. Using symplectic capacities, he gave a positive answer to Question 4 for contact coisotropics in $\mathbb{R}^{2n}$. The Floer theoretic methods developed in [Gin07], [Kan13], [Ush11], and used to answer Questions 3 and 2 are also applicable to adress Question 4. The most general result for stable coisotropics follows from Theorem 7.5 of [Ush11]: if a closed, stable coisotropic submanifold C of a closed symplectic (or Stein) manifold W is displaceable, then there exists a loop in a leaf F which is a non-contractible in F, bounds positive symplectic area and is contractible in W. Notice the converse implication of this result: If there are no non-contractible loops tangent to the foliation $\mathcal{F}$ of C, bounding positive symplectic area and contractible in W, then C is non-displaceable.

Another interesting direction of research on the rigidity properties of coisotropic submanifolds has been introduced by Humilière, Leclercq and Seyfaddini. In [HLS15] they prove that previously observed C^0-rigidity results for Lagrangians and hypersurfaces are manifestations of the C^0-ridigity of coisotropic submanifolds: the image of a coisotropic submanifold C under a symplectic homeomorphisms θ is a coisotropic $\theta(C)$, given the image of C under θ is smooth. If this is the case, also the image of the characteristic foliation $\mathcal{F}$ of C is smooth under θ.

The phenomena described above can be seen a generalisation of the rigidity of Lagrangian intersections and the fact that a Lagrangian which is displaceable bounds a non-trivial holomorphic disc. In the codimension n case the fact that Lagrangians have non-zero displacement energy was proved by Chekanov in [Che98]. A displaceable Lagrangian bounds a non-trivial holomorphic disc by Lemma 4.23. For a displaceable coisotropic, the leaves of the characteristic foliation, play the role of the Lagrangian. The rigidity of Lagrangian intersections can be interpreted as one of the underlying reasons why Floer could prove the Arnold conjecture by looking at the self intersection properties of Lagrangians. That coisotropics exhibit similar rigidity properties gives hope that an appropriate "coisotropic Arnold conjecture" could be formulated. In fact, Ziltener formulates a coisotropic Arnol'd-Givental conjecture and proves a version of it for fibred coisotropic submanifolds in [Zil10]. Moreover this hope is supported by the Work of Oh, Ginzburg and Ziltener who define coisotropic Maslov indices in [Oh03], [Gin11] and [Zil09]. Ginzburg in [Bat13] and Batoréo [Bat13] show that the coisotropic Maslov index satisfies similar rigidity properties as the Lagrangian Maslov index.

Interestingly this phenomenon can also be seen as a generalisation of the non-triviality of the displacement energy of stable hypersurfaces. In the hypersurface case, the non-triviality of the displacement energy follows from the non-degeneracy of the Hofer norm as observed by Ginzburg in [Gin07] (page 2). The observation that there exists a non-trivial loop bounding positive symplectic energy in the characteristic foliation of a displaceable coisotropic has interesting implications. In the case where C is a contact hypersurface, displaceability of C thus implies the Weinstein conjecture for C. This gives hope that an appropriate "coisotropic Weinstein conjecture" could be formulated.

The generality of coisotropics now allows to link the two conjectures! Hence, one could try to prove a coisotropic Arnold conjecture using methods which were applied to prove the Weinstein conjecture, and one could try to prove the coisotropic Weinstein conjecture using Floer-theoretic methods which were originally developed to prove the Arnold conjecture.

Ultimately one could therefore be tempted to formulate the following (very speculative!) conjecture which would subsume the Arnold, the Arnold-Givental and the Weinstein conjecture in some cases.

Conjecture (very speculative).

Given a Hamiltonian symplectomorphism ψ and a closed and stable coisotropic submanifold C of a symplectic manifold (W, ω).

(A-W) *Either ψ has a number of leaf-wise fixed points which is bounded below by a number depending only on the topology of C or, if not, there exists a non-trivial loop in the characteristic foliation $\mathcal{F}$ of C which bounds non-trivial symplectic energy.*

Notice that the statment about the existence of leaf-wise fixed points is the Arnold conjecture if $C = W$ and the Arnold-Givental conjecture if $C = L$, by taking the sum of the Betti numbers as a lower bounds. Also notice that the statement about the existence of a non-trivial loop, coincides with the Weinstein conjecture if the $W = \mathbb{R}^{2n}$ and the codimension of C is one. In the codimension n case, this statement follows from the Arnold-Liouville theorem, since stable codimension n-coisotropic are necessarily Lagrangian tori. In the codimension 0 case the leaves are the points of W, and thus the question about the existence of non-contractible loops does not make sense.

Besides providing a general framework to investigating these conjectures, coisotropics are also conjectured to play a role in homological mirror symmetry. More precisely, in [KO03], Kapustin and Orlov postulated the integration of objects associated to coisotropics into the Fukaya category as a necessary condition to establish homological mirror symmetry in the context of certain Hyperkähler four-manifolds. They indicate that for these manifolds the K-theory of the Fukaya category is smaller (in an appropriate sense) than the K-theory of the derived category of the mirror and therefore the Fukaya category must be enlarged in some way. They suggest using D-branes associated to coisotropics. Following this paper, in an attempt to understand the space of endomorphisms of coisotropic submanifolds, viewed as objects of a, yet to define, enlarged Fukaya category, the following question has been investigated:

Question 5.

What is the structure of the space of deformations of coisotropics?

In fact, the first result which can be seen as addressing this question, was obtained by Gotay in [Got82] as early as 1982 and thus around thirty years before [KO03]. Gotay proved that all coisotropic embeddings of pre-symplectic manifolds into symplectic manifolds are equivalent up to symplectomorphism.

Recall that the space of infinitesimal deformations of a Lagrangian L modulo Hamiltonian equivalence is diffeomorphic to a neighbourhood of 0 in $H^1(L, \mathbb{R})$. In sharp contrast, Oh and Park explain in [OP05] that for general coisotropics the space of deformations is "non-commutative and fully non-linear", has the structure of an L_∞ algebra and is, in general, obstructed. Ruan demonstrates in [Rua05] that if the coisotropic is fibred, its space of deformations modulo Hamiltonian equivalence is unobstructed and a smooth finite dimensional manifold, which is in accordance with the fact that it is possible to assign the Lagrangian graph L_C to C in this case.

Beyond this conjectured role coisotropic submanifolds play in mirror symmetry, coisotropics occur in related fields of (Quantum-)Physics: Dirac in [Dir67] refers to coisotropics as the configuration space of "the general Hamiltonian theory" of quantum mechanics. To go into more detail about the more recent physics publications on coisotropic submanifolds of Poisson manifolds is beyond the scope of this introduction.

To conclude this introductory chapter, I give a brief outline of the thesis below.

1.5 Outline of the thesis

In Chapter 2, I introduce coisotropic submanifolds in detail . I give the definitions of fibred and stable coisotropic submanifolds and derive some first consequences of these assumptions. I illustrate some important phenomena, which arise for coisotropic submanifolds by examples and expand on the examples given in the introduction. The thesis is then structured according to the goal of accomplishing steps (I), (II) and (III) stated at the beginning of Section 1.3 subsequently which serve to prove Theorem 1.6.

I assign the Lagrangian L_C and the hypersurface H_C to C in Chapter 3. In Section 3.1 I introduce the *Lagrangian graph* L_C of a fibred Coisotropic C in detail. In particular, I explain how the intersection theory of C is related to the intersection theory of L_C. In Section 3.2 I construct the stable hypersurface H_C for a given stable coisotropic C. Moreover I show that the generalised Reeb dynamics on C correspond to to the Reeb dynamics on H_C. Thus, in Chapter 3, I assign to a stable, fibred coisotropic a Lagrangian submanifold L_C and a hypersurface H_C which both capture some relevant parts of the geometry of C. The advantage of L_C and H_C is that they belong to classes of submanifolds, for which more mathematical machinery is available.

Chapter 4 is devoted to proving theorem 1.7. This is done by first adapting the pearl complex machinery developed by Biran and Cornea, in order to make use of the fibre bundle structure on C and L_C and then applying the machinery, i.e. deriving results about C by utilising the algebraic structures at hand.

Chapter 5 is dedicated to the proof of Theorem 1.8. Most of the effort in proving the theorem lies in adapting ideas from symplectic field theory to the present setting. The application of the tools I develop is then a straightforward adaptation of the bubbling-off analysis carried out in proof of the compactness theorem in [Bou+03] to the present setting. I also briefly outline how the machinery developed could be used to formulate and prove a coisotropic SFT compactness theorem.

Finally, in Chapter 6, I explain the small final step of the proof of Theorem 1.6 in detail.

Chapter 2

Introduction to coisotropic submanifolds

Standing assumptions and conventions: I work in the category of smooth manifolds unless stated otherwise. The main object of study of this thesis are coisotropic submanifolds C of symplectic manifolds (W, ω). I will often abbreviate the term "coisotropic submanifold(s) of the symplectic manifold (W, ω)" to just "coisotropic(s)", mimicking the common practice of referring to a "Lagrangian" instead of referring to a "Lagrangian submanifoldf of (W, ω)". I will assume throughout that the symplectic manifold (W, ω) is real $2n$-dimensional, and that coisotropics C have codimension $k \in \{0, \ldots, n\}$ and are thus of dimension $2n - k$. Moreover assume that all coisotropics are embedded, connected and closed (compact and without boundary), unless stated otherwise.

In this section, I introduce coisotropic submanifolds of symplectic manifolds in detail. First, in Section 2.1, I give an overview of the general theory of coisotropic submanifolds and provide some elementary examples. Then, I recall the definition of *fibred* coisotropic submanifolds and explain how one forms the *symplectic quotient* of a coisotropic in Section 2.2. Following this, in Section 2.3, I investigate the notion of *stability* of a coisotropic submanifold. I establish the existence of standard model for a neighbourhood of a stable coisotropic in Proposition 2.18. In Section 2.4, I explore the stability requirement with respect to the dynamics on the coisotropic. It turns out that stable coisotropics fit into the context of Hamiltonian group actions and can be seen as *"locally Hamiltonian group actions"*. Finally, in Proposition 2.22, I state and prove the coisotropic version of the Arnold-Liouville theorem.

2.1 Coisotropic submanifolds

Recall that a *symplectic structure* on a smooth manifold W is a closed, non-degenerate 2-form $\omega \in \Omega^2(W)$. A diffeomorphism

$$\psi : (W, \omega) \to (W', \omega'),$$

which preserves this structure, i.e. $\psi^*\omega' = \omega$, is called a *symplectomorphism*. Given a symplectic manifold (W, ω), the map

$$\begin{aligned} \iota : TW &\to T^*W \\ X &\mapsto \iota(X)\omega = \omega(X, \cdot) \end{aligned} \tag{2.1}$$

defines a canonical isomorphism of the tangent bundle TW and the cotangent bundle T^*W of a symplectic manifold (W, ω). It identifies the sections of these bundles, namely vector fields $\Gamma(W, TW)$ with 1-forms $\Omega^1(W)$. Every function $H : (W, \omega) \to \mathbb{R}$ defines a *Hamiltonian vector field* by

$$\iota(X_H)\omega = dH.$$

Let N be a submanifold and $i_N : N \to W$ the natural inclusion. The tangent bundle TN is a subbundle of the pullback bundle $i_N^* TW$. The *symplectic complement* TN^ω of TN in $i_N^* TW$ is given by

$$TN^\omega := \{v \in i_N^* TW \mid \omega(w, v) = 0 \text{ for all } w \in TN\}, \tag{2.2}$$

and is also a subbundle of $i_N^* TW$. Unlike for the orthogonal complement in Riemannian geometry, it is not necessarily true that $i_N^* TW$ splits as $TN \oplus TN^\omega$. Interesting and natural classes of submanifolds of symplectic manifolds are defined by looking at the relation of TN and TN^ω. A submanifold N of a symplectic manifold (W, ω) is called:

- *coisotropic* if $TN^\omega \subseteq TN$.
- *Lagrangian* if $TN^\omega = TN$.
- *isotropic* if $TN^\omega \supseteq TN$.
- *symplectic* if $TN^\omega \cap TN = \{0\}$.

The following Lemma illustrates the foliation theory of the submanifolds listed above.

Lemma 2.1.
Let N be a submanifold of (W, ω) such that the bundle $TN \cap TN^\omega$ is of constant dimension along N. Then N is foliated by leaves F tangent to $TN \cap TN^\omega$.

Proof. This proof is exactly as the proof of Lemma 5.33 in [MS17]. By the Frobenius theorem, a foliation of N tangent to $TN \cap TN^\omega$ exists if and only if this distribution is closed under the Lie bracket $[\cdot, \cdot]$. Let $q \in N$ and X and Y be vector fields in a neighbourhood of N with values in $TN \cap TN^\omega$. Let Z be any vector field on TN defined in a neighourhood of N. Since $(TN^\omega \cap TN) \subseteq TN$ and N is a submanifold, it follows that $[X, Y] \in TN$. It remains to show that $[X, Y] \in TN^\omega$. By Cartan's identity:

$$\begin{aligned}0 &= d\omega(X,Y,Z)\\ &= \mathcal{L}_X(\omega(Y,Z)) + \mathcal{L}_Z(\omega(X,Y)) + \mathcal{L}_Y(\omega(Z,X))\\ &\quad + \omega([X,Y],Z) + \omega([X,Z],Y) + \omega([Y,Z],X)\\ &= \omega([X,Y],Z)\end{aligned}$$

The three terms in the second line vanish, since the functions $\omega(Y,Z)(q), \omega(X,Y)(q)$ and $\omega(Z,X)(q)$ all vanish identically along N by definition of the ω-complement. The two last terms in the third line vanish since $[X,Z]$ and $[Y,Z]$ are contained in TN. $\square$

Observe that the proof of Lemma 2.1 above uses both the closedness of ω and the isomorphism ι induced by ω. In particular every submanifold N which is coisotropic, isotropic, Lagrangian or symplectic is foliated. For symplectic submanifolds, the foliation consists of 0-dimensional leaves, so each leaf is just a point in the submanifold. Isotropic submanifolds are foliated by just one k-dimensional leaf, namely themselves since here $TN \cap TN^{\omega} = TN$. The same holds for Lagrangians, with the addition that these submanifolds are the maximal isotropic (or minimal coisotropic) submanifolds, and $k = n$.

Coisotropics are the most interesting submanifolds in view of foliation theory: Lemma 2.1 implies that every coisotropic submanifold C is foliated by k dimensional isotropic leaves F, tangent to TC^{ω}. Recall from the introduction that this foliation $\mathcal{F}$ of C is called the characteristic foliation of C. Before embarking on further on the studies of coisotropics I give some elementary examples of coisotropics below. The reader is also invited to revisit Examples 1.1, 1.2, 1.3 and 1.4 from the introduction.

Example 2.2 (Hypersurfaces)**.**

Every hypersurface H in a symplectic manifold (W,ω) is coisotropic. H is foliated by one dimensional leaves.

Example 2.3 (Lagrangians).

Every Lagrangian is a coisotropic, foliated by one n-dimensional leaf, namely itself.

Example 2.4 (Poisson-commuting Hamiltonians).

Assume there exist k Poisson-commuting Hamiltonians $H_1, \ldots, H_k$ on a symplectic manifold i.e. $\omega(X_i, X_j) = 0$. If 0 is a common regular value of all Hamiltonians H_i, then the intersection of level sets,

$$H_1^{-1}(0) \cap \cdots \cap H_k^{-1}(0),$$

is a codimension-k coisotropic.

Example 2.5 (Linear coisotropics).

Consider $\mathbb{R}^{2n}$ with coordinates $(q_1, \ldots q_n, p_1, \ldots, p_n)$ and its standard symplectic structure

$$\omega_0 = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n.$$

For $1 \leq k < l \leq n$ and $k + l \geq n$, define the linear subspace C of $\mathbb{R}^{2n}$ by

$$C_{\text{lin}} = (q_1, \ldots, q_l, 0, \ldots, 0, p_1, \ldots, p_k, 0, \ldots, 0, p_{l+1}, \ldots, p_n).$$

Then, C is a $k + l$ dimensional coisotropic submanifold. More concretely consider the following subspace of $\mathbb{R}^6$:

$$C_{\text{lin}} = (q_1, q_2, 0, p_1, 0, p_3).$$

Then, $TC_{\text{lin}}^{\omega} = (q_2, p_3)$, $TC_{\text{lin}}/TC_{\text{lin}}^{\omega} = (q_1, p_1) \cong \mathbb{R}^2$ and $\mathbb{R}^6 \setminus TC_{\text{lin}} = (q_3, p_2)$.

Returning to the general theory, denote by TW/TC the normal bundle of C defined as the quotient of the bundles i_C^*TW and TC and likewise for TC/TC^ω. Consider the isomorphism ι introduced in Equation (2.12) and denote by ξ^0 the annihalator of a sub-bundle ξ of TW. Along C the isomorphism ι has the following properties:

$$
\begin{aligned}
\iota(TC/TC^\omega) &= (TC^\omega \oplus TW/TC))^0 = (TC/TC^\omega)^* \\
\iota(TC^\omega \oplus TW/TC) &= (TC/TC^\omega)^0 \\
\iota(TC^\omega) &= (TW/TC)^* \\
\iota(TW/TC) &= (TC^\omega)^*.
\end{aligned}
\tag{2.3}
$$

This induces in particular the following splittings of the tangent bundle i_C^*TW along C.

$$
\begin{aligned}
i_C^*TW &\cong TC/TC^\omega \oplus TC^\omega \oplus TW/TC \\
&\cong TC/TC^\omega \oplus TC^\omega \oplus (TC^\omega)^* \\
&\cong TC/TC^\omega \oplus (TW/TC)^* \oplus TW/TC
\end{aligned}
\tag{2.4}
$$

These splittings depend on a choice of complement of TC^ω in TC, since TC/TC^ω is not naturally a sub-bundle of TC and a choice of complement of TC in TW. Such a choice can be made by choosing an identification of TW with TW^*, for example by choosing an ω-compatible almost complex structure J. Recall that an almost complex structure is an endomorphism of TW which squares to $-id$. J is called ω-compatible if $\omega(Jv, v) > 0$ for all $v \in TW$ and $\omega(Jv, Jw) = \omega(v, w)$ for all $v, w \in TW$.

Gotay explains in [Got82] how the splitting 2.4 can be used to show that all embeddings of neighbourhoods of coisotropic submanifolds are symplectomorphic. These splittings of the tangent bundle along C will become important later.

In the case where C is a Lagrangian L, the space TL/TL^ω is a point and the main result from [Got82] recovers the fact that every Lagrangian L has a neighbourhood symplectomorphic to its cotangent bundle T^*L. For a hypersurface H, TH^ω and TW/TH are one dimensional, and the space TH/TH^ω is $2n-2$ dimensional. A famous examples of this is the Hopf fibration $S^1 \to S^3 \to S^2$ (Example 1.1.)

In view of the splittings above one is tempted to consider the quotient space $C/\mathcal{F}$. However this space is not necessarily a manifold as quotienting out by the leaves of $\mathcal{F}$ which are tangent to TC^ω may yield non-Hausdorff spaces. A simple, yet very instructive, illustration of this property of foliations (not necessarily foliations of coisotropics) is the following:

Example 2.6 (Torus foliations).
Consider the two-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with the topology induced from $\mathbb{R}^2$. Let $\lambda \in \mathbb{R} \setminus \{0\}$. The torus admits a vector field

$$X(x,y) = (\frac{\partial}{\partial x}, \lambda \frac{\partial}{\partial x}).$$

Given a point $(x_0, y_0) \in \mathbb{T}^2$, the integral curves

$$\begin{aligned} f_\lambda : \mathbb{R} &\longrightarrow \mathbb{T}^2 \\ x &\mapsto (x_0 + x, \lambda x + y_0). \end{aligned}$$

of X foliate the torus. If λ is a rational number $\lambda = \frac{p}{q}$ for p, q coprime in $\mathbb{Z}$ with $q \neq 0$, then each leaf $F_{\frac{p}{q}}$ through a given point (x_0, y_0) is compact. If λ is an irrational number $\lambda \in \mathbb{R} \setminus \mathbb{Q}$, then each leaf of the foliation $\mathcal{F}_\lambda$ of $\mathbb{T}^2$ is everywhere dense in $\mathbb{T}^2$.

In Example 2.6 above, assume that λ is irrational. Since each leaf F_λ is dense in $\mathbb{T}^2$, the quotient $T^2/\mathcal{F}_\lambda$ is not a Hausdorff space. The topology of F_λ as a leaf of the foliation $\mathcal{F}_\lambda$ does not agree with the topology of F_λ as a submanifold of $\mathbb{T}^2$. If λ is rational, the quotient $\mathbb{T}^2/\mathcal{F}_{\frac{p}{q}}$ is diffeomorphic to a circle. In particular, an arbitrarily small perturbation of the angle defining the foliation drastically changes the properties of the quotient $\mathbb{T}^2/\mathcal{F}_\lambda$.

2.2 Fibredness of coisotropic submanifolds

In this section I introduce the notion of a *fibred coisotropic*. This condition ensures that characteristic foliation of C does not change drastically under small perturbations within B. Moreover I will show that if C is fibred, then the quotient $C/\mathcal{F}$ is a smooth symplectic manifold.

Definition 2.7 (Fibredness of coisotropic submanifolds)**.**
A coisotropic C is called *fibred* if the isotropic leaves F of the characteristic foliation $\mathcal{F}$ of C, which are connected by definition, are closed submanifolds of C and the holonomy of each leaf, as defined in Section 2.1, of [MM03] is trivial.

An important consequence of this requirement is the following Lemma:

Lemma 2.8.
Let C be a fibred coisotropic. Then C is the total space of a smooth fibre bundle

$$F \to C \xrightarrow{\pi} B. \tag{2.5}$$

The base B is called the symplectic quotient of C and carries a natural symplectic structure $\omega_B = \pi_* i_C^* \omega$.

Proof. The holonomy of all leaves is trivial by assumption. Thus it follows from Theorem 2.15 of [MM03] that the quotient $C/\mathcal{F}$ carries a canonical, smooth, second countable, Hausdorff manifold structure. The projection π is induced by the quotient map, which identifies points on the same leaf.

To show that ω_B is well-defined and non-degenerate I argue as follows: assume that $x, y \in C$ lie in the same leaf, say F_x. Choose a finite collection of foliation charts $U = U_1 \cup \cdots \cup U_n$ such that $x, y \in U$. Choose a curve

$$\gamma : [0,1] \longrightarrow U$$

such that $\gamma(0) = x, \gamma(1) = y$ and $\dot{\gamma}(t) \in T_{\gamma(t)} C^\omega$ for all t in $[0,1]$. Define a vector field X on U with values in TC^ω such that

$$X(\gamma(t)) = \frac{d}{dt}\gamma(t) \in T_x C^\omega.$$

Then the Lie derivative $\mathcal{L}_X$ of $i_C^*\omega$ in the direction of X is well defined in U. Calculate at $t_0 \in [0,1]$

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=t_0} \gamma(t)^* i_C^*\omega &= \mathcal{L}_{X(\gamma(t))}(i_C^*\omega(\gamma(t))) \\ &= d(\iota(X(\gamma(t_0))) i_C^*\omega(\gamma(t_0))) \\ &= 0. \end{aligned}$$

Hence $i_C^*\omega(y) = i_C^*\omega(x)$ and ω_B is well defined. By definition

$$\ker(i_C^*\omega)(x) = \ker d\pi(x) = T_x C^\omega.$$

Consequently ω_B is non degenerate on B and varies smoothly with $b \in B$. By the closedness of ω, one has $d(i_C^*\omega) = 0$. Now an elementary computation in a foliation chart shows that this implies that ω_B is closed on B . $\square$

Remark 2.9.

It is also possible to assume the existence of local slices and impose the Hausdorff condition on the quotient in order to prove Lemma 2.8. This is the approach taken in [MS17] Section

5.4. For a detailed proof of Lemma 2.8 under these assumptions see Proposition 5.4.5 of the former reference.

The assumption that the holonomy of each leaf is trivial is necessary even if all leaves are closed submanifolds of C. Otherwise the quotient $C/\mathcal{F}$ can be an orbifold. Many thanks to Dominic Joyce for pointing this mistake out to me! I illustrate this in the example below:

Example 2.10 (Foliation of the Möebius strip).
View the Möbius strip M as the rectangle $[0,1]\times[-1,1]$ with $\{0\}\times[-1,1]$ identified with $\{1\}\times[-1,1]$ via the map $\phi(y) = -y$, for $y \in \{0\}\times[-1,1]$. If y is non-zero, the leaf $F_y := [0,1]\times\{y\}$ has trivial holonomy. Thus there exists a neighbourhood of y such that quotient, $M/\mathcal{F}_y$, where $\mathcal{F}_y$ is the foliation by parallel leaves, in this neighbourhood, is diffeomorphic to an open interval.

The leaf $F_0 = [0,1]\times\{0\}$ has $\mathbb{Z}_2$-holonomy and the quotient $M/\mathcal{F}_0$ can be identified with the orbifold $[0,1]/\mathbb{Z}_2$.

Note that the leaves F of the foliation $\mathcal{F}$ are now the fibres of a fibre bundle and hence nearby leaves are diffeomorphic. Thus the foliation does not change drasitcally under small perturbations in the symplectic quotient B. Fibredness is quite a restrictive assumption. Nonetheless all the interesting Examples 1.1, 1.2, 1.3 and 1.4 from the introduction are fibred coisotropics.

With this property of the characteristic foliation established, in the next section I consider a condition which ensures that the characteristic foliation remains unchanged under small pertubations in the normal directions of C.

2.3 Stability of coisotropic submanifolds

In this section I explore the notion of *stability* for coisotropics. It is a straightforward generalisation of the concept of a stable hypersurface to higher codimension $1 \leq k \leq n$. If $k = 1$ the notion of a stable codimension 1 coisotropic and a stable hypersurface coincide. The stability condition was introduced by Bolle in [Bol98]

Definition 2.11 (Stability (Bolle)).
A codimension k, coisotropic submanifold C of a symplectic manifold (W, ω) is *stable* if there exist k one-forms $\alpha_1, \ldots \alpha_k$ defined on C, which satisfy:

(S1') $\ker i_C^*\omega \subset \ker d\alpha_i$ for all $1 \leq i \leq k$,

(S2') For all x in C, $\alpha_1 \wedge \cdots \wedge \alpha_k \wedge (\omega)^{n-k}(x) \neq 0$.

Remark 2.12.
Condition (S2') in Definition 2.11 above is equivalent to the linear independence of the α_i on $\ker i_C^*\omega = TC^\omega$. Also notice that by applying the isomorphism ι from Equation to the one-forms $\alpha_1, \ldots, \alpha_k$ implies that C has trivial normal bundle.

I would like to advocate an alternative, but equivalent definition which I think better illustrates the fact that stability is a condition on how the coisotropic is embedded into the surrounding manifold. In particular one immediately sees from Definition 2.13 below that stability implies that C has trivial normal bundle. I will prove that the two definitions are equivalent in Lemma 2.16 below. Denote by i_F the inclusion of a leaf F into C.

Definition 2.13 (Stability of coisotropic submanifolds).
A codimension k, coisotropic submanifold C of a symplectic manifold (W, ω) is *stabilizable* if there exist vector fields $Y_1, \ldots Y_k$ on a neighbourhood U of C such that

their pullbacks to $i_C^* TW$ satisfy

(S1) $i_F^* \mathcal{L}_{Y_i}\omega = i_F^*(d(\iota(Y_i)\omega) = 0$.

(S2) $Y_1, \ldots, Y_k$ are linearly independent on $i_C^*(TW/TC)$ and transverse to C.

I call the vector fields $Y_1, \ldots, Y_k$ *stabilising vector fields*. I call $\iota(Y_1)\omega, \ldots, \iota(Y_k)\omega$ *stabilising one forms*, the $(k+1)$-tuple $\mathcal{S} = (\omega, Y_1, \ldots, Y_k)$ a *stable structure* on C and a pair $(C, \mathcal{S})$ a *stable* coisotropic.

This terminology is inspired by [CV15], which deals with stable hypersurfaces. The notions put forward here coincide with the corresponding definitions in [CV15] in the codimension one case. From a dynamical systems point of view Condition (S1) means precisely that the characteristic foliation $\mathcal{F}$ is stable under small perturbations of the coisotropic in the normal directions.

Definition 2.14 (Contact coisotropic)**.**
A *contact structure* on a coisotropic C is a stable structure $\Lambda = (\omega, Y_1, \ldots, Y_k)$ such that

(C1) $\mathcal{L}_{Y_i}\omega = d(\iota(Y_i)\omega) = i_C^*\omega$ for all $1 \leq i \leq k$.

(C2) $Y_1, \ldots, Y_k$ are linearly independent on $i_C^*(TW/TC)$ and transverse to C.

The pair (C, Λ) is called a *contact* coisotropic submanifold.

A codimension one contact coisotropic is thus a contact hypersurface. In this case the vector field Y is usually called *Liouville vector field*.

Remark 2.15.
The product $(C \times C', \mathcal{S} \times \mathcal{S}') \subseteq (W \times W', \omega \times \omega')$ of two stable coisotropics $(C, \mathcal{S}) \subseteq (W, \omega)$ and $(C', \mathcal{S}') \subseteq (W', \omega')$ is again a stable coisotropic. This does not

necessarily hold for contact coisotropic submanifolds. Bolle shows in Remark 3 of [Bol98] *that a contact coisotropic* (C, Λ) *of codimension* k *has to satisfy*

$$\dim(H^1(C)) \geq k - 1.$$

Notice that this provides a large class of examples of submanifolds which are stable-but-not-contact coisotropics. For example, consider the cartesian product of spheres $S^{2m-1} \times S^{2n-1} \subset \mathbb{R}^{2m} \times \mathbb{R}^{2n}$ *with the standard symplectic structure* $\omega_0 \times \omega_0$*. Then* $S^{2m-1} \times S^{2n-1}$ *cannot be contact if* $m, n > 1$*. However* $S^{2m-1} \times S^{2n-1}$ *is a stable codimension* 2 *coisotropic. More generally, the product of any two contact hypersurfaces, which have trivial fundamental groups, is a stable-but-not-contact coisotropic of codimension* 2*.*

Lemma 2.16.
Defintion 2.11 and Definition 2.13 are equivalent.

Proof. Condition $(S2')$ in Definition 2.11 is equivalent to the existence of a trivialisation of $(TC^w)^*$ given by the α_i. Choosing $\alpha_1, \dots, \alpha_k$ corresponds to choosing a trivialisation

$$\tau' : (TC^{\omega})^* \to \mathbb{R}^{k^*} \times C$$

in the same way as choosing $Y_1, \dots, Y_k$ linearly independent and transverse to C corresponds to choosing a trivialisation

$$\tau : TW/TC \to \mathbb{R}^k \times C.$$

Recall from (2.3) that $\iota(TW/TC) = (TC^{\omega})^*$. In particular given $Y_1, \dots, Y_k$ we may choose $\alpha_1, \dots, \alpha_k$ such that $\iota(Y_j)\omega = \alpha_j$ and vice versa. Thus Conditions $(S2')$ and $(S2)$ are equivalent.

By definition, $\ker(i_C^*\omega) = TC^{\omega} = TF$. As described above, one may always arrange $d(\iota(Y_i)\omega) = d\alpha_i$. Thus condition $(S1')$ is equivalent to condition $(S1)$. $\square$

From now on, I will work with Definition 2.13.

Definition 2.17 (Generalised Reeb vector fields)**.**
Given a stable coisotropic C, because $Y_1, \dots, Y_k$ are defined at every point of C, there exist k vector fields $X_1, \dots, X_k$ on C with values in $\in TC^\omega$ which are ω-dual to $Y_1, \dots, Y_k$, that is:

$$\omega(Y_i, X_j) = \alpha_i(X_j) = \delta_{ij}.$$

The vector fields $X_1, \dots, X_k$ are called *generalised Reeb vector fields*. Denote by $\phi_j :$ $\mathbb{R} \times C \to C$ the flow of X_j defined by the equation

$$\frac{d}{dt}\phi_j^t(x) = X_j(\phi_j^t(x)) \quad \phi_j^0(x) = x,$$

where $x \in C$ and $t \in \mathbb{R}$. For $k = 1$ this definition coincides with the usual definition of the Reeb vector field on a hypersurface.

The most important consequence of the stability requirement is the following neighbourhood theorem due to and originally proved by Bolle in [Bol98]. I present a proof it of using Definition 2.13. Denote by $B^k_{\epsilon_0}$ the standard ball of radius $\epsilon_0 > 0$ in $\mathbb{R}^k$.

Proposition 2.18 (Bolle neighbourhood theorem)**.**
Assume $(C, \mathcal{S})$ is stable. Then there exists a neighbourhood U of C, an $\epsilon_0 > 0$ and a diffeomorphism $\psi : B^k_{\epsilon_0} \times C \to U$ which satisfies:

$$\psi^*\omega = \omega_s := \psi^* i_C^*\omega + \sum_{i=1}^k d(p_i\alpha_i). \tag{2.6}$$

where $\alpha_i = \psi^*(\iota(Y_i)\omega)$ and $p_1, \dots, p_k$ are the coordinates on $B^k_{\epsilon_0}$.
Moreover, throughout $B^k_{\epsilon_0}$ the foliations $\mathcal{F}_p$ of $C_p := \{p\} \times C$ are conjugate to the foliation $\mathcal{F}$ of $C = \{0\} \times C$ via a family of diffeomorphisms depending smoothly on p. I will refer

to such a neighbourhood U as a Bolle neighbourhood.

Proof. By condition (S2) in Definition 2.13 choose a smooth trivialisation

$$\tau : TW/TC \to \mathbb{R}^k \times C,$$

given by vector fields $Y_1', \dots Y_k'$ of the normal bundle of C. By the inverse function theorem there exists an $\epsilon' > 0$ and a smooth map $\psi : B^k_{\epsilon'} \times C \to U$ which satisfies $\psi \circ i_C = id$ and

$$\psi^* Y_j' = \partial p_j,$$

where I denote by ∂p_j the canonical vector field of unit length associated to the coordinate p_j using the identification of $\mathbb{R}^k$ with its dual space provided by the standard inner product on $\mathbb{R}^k$. Choose k Reeb vector fields $X_1, \dots, X_k$ as in Definition 2.17. On $T(B_{\epsilon'} \times C)$ view the X_j as $(0, X_j)$. Now perform a symplectic version of Gram-Schmidt to construct vector fields $Y_1, \dots Y_k$ on C such that $\omega(Y_i, Y_j) = 0$. Given $Y_1', \dots, Y_k'$ set

$$\begin{aligned} Y_1 &= Y_1' \\ &\vdots \\ Y_m &= Y_m' - \sum_{n=1}^{m-1} \omega(Y_m', Y_n) \cdot X_n \quad \text{for} \quad 2 \le m \le k. \end{aligned}$$

Then $\omega(Y_i, Y_j) = 0$ for all $1 \le i, j \le k$. First notice that $\omega(Y_i, X_j) = \omega(Y_i', X_j)$ since $\omega(X_i, X_j) = 0$. Next, notice that the one-forms $dH_j := \psi_* dp_j$ are exact for each j and satisfy $dH_j(Y_i) = \omega(Y_i, X_j) = \delta_{ij}$ on TW/TC. By choosing appropriate constants, assume that the H_j are defined on U, satisfy $H_j(\psi(p_j, c)) = p_j$ and give

$$C = \bigcap_{i=1}^{k} H_j^{-1}(0).$$

Define k one-forms on TC^{ω} by setting $\bar{\alpha}_i = \iota(Y_i)\omega$. Then

$$\omega(Y_i, X_j) = dH_j(Y_i) = \bar{\alpha}_i(X_j) = \delta_{ij}. \tag{2.7}$$

Define a symplectic form $\bar{\omega}_s$ on $i_C^* TW$ at each point x in C by

$$\bar{\omega}_s(x) = i_C^*\omega(x) + \sum_{i=1}^{k} d(H_i(x) \cdot \iota(Y_i)\omega(x))$$

Then $\bar{\omega}_s$ is closed and agrees with $i_C^*\omega$ along C by construction. Thus apply Lemma 3.14 from [MS17] (the Weinstein extension theorem) to extend ψ to a symplectomorphism from a possibly smaller ball $B^k_{\epsilon_0} \times C \to U$. By construction ω_s has the required form:

$$\begin{aligned}
\omega_s := \psi^*\bar{\omega}_s &= \psi^* i_C^*\omega + \sum_{j=1}^{k} \psi^* d(H_j(x) \cdot \iota(Y_j)\omega) \\
&= \psi^* i_C^*\omega + \sum_{j=1}^{k} d(\psi^{-1}(H_j(x)) \cdot \psi^*\iota(Y_j)\omega) \\
&= \psi^* i_C^*\omega + \sum_{j=1}^{k} d(p_j \psi^*\iota(Y_j)\omega) \\
&= \psi^* i_C^*\omega + \sum_{j=1}^{k} d(p_j \alpha_j)
\end{aligned}$$

Observe that condition $(S1)$ has not been used so far. The one forms α_i on $T(B^k_{\epsilon_o} \times C)$ are of the form $(0, \alpha_i)$. Recall that (S1) implies that $d\alpha_i$ vanishes on TC^{ω} and thus in particular on the X_j. Calculate

$$\begin{aligned}
\mathcal{L}_{\partial p_j}\omega_s &= d\left(\iota(\partial p_j)\left(\psi^* i_C^*\omega + \sum_{i=1}^{k} d(p_i\alpha_i)\right)\right) \\
&= d\left(\iota(\partial p_j)\left(\sum_{i=1}^{k} dp_i\alpha_i + \sum_{i=1}^{k} p_i d\alpha_i\right)\right) \\
&= d\left(\sum_{i=1}^{k} dp_i(\partial p_j)\alpha_i(\cdot)\right) \\
&= d\alpha_j = d(\iota(\psi^* Y_j)\omega)
\end{aligned} \tag{2.8}$$

Thus it follows from Calculation 2.8 above that ω_s gets scaled by $\sum_{j=1}^{k} p_j d\alpha_j$ as flowing outwards from the origin $\{0\} \times C$ towards the boundary of $B^k_{\epsilon_0} \times C$ via the flow of $(\frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_k})$. Condition (S1) in Definition 2.13 precisely means that

$$i_F^* \left(\sum_{j=1}^{k} d(\iota(\psi^* Y_j)\omega) \right) = 0$$

so that the characteristic foliation $\mathcal{F}$ of C remains "unchanged" throughout $B^k_{\epsilon_0}$. □

The following proposition summarises the relations between Definition 2.11, Definition 2.13 and Proposition 2.18. It is the "stable coisotropics" version of Lemma 2.1 in [CV15].

Proposition 2.19.
Let C be a coisotropic. Then the following are equivalent

(i) C is stable according to Definition 2.11.

(ii) C is stabilizable according to Definition 2.13.

(iii) There exists a Bolle neighbourhood of C.

Proof. The equivalence of (i) and (ii) was proved in Lemma 2.16. That (ii) implies (iii) is the content of the proof of Proposition 2.18. It remains to show that the existence of a Bolle neighbourhood implies that C is stable according to Definition 2.13.

Given a Bolle neighbourhood $B^k_\epsilon \times C$, set $Y_i = \partial p_j$ on $T_{p,c}(B^k_\epsilon \times C) = \mathbb{R}^k \times C$. These vector fields are linearly independent on $\mathbb{R}^k$ and transverse to C by construction. The assertion that $\mathcal{F}_p$ are all conjugate to $\mathcal{F}$ via a family of diffeomorphisms depending smoothly on p implies that

$$i_F^* \mathcal{L}_{\partial p_j} \omega_s = i_F^*(d(\iota(\partial p_j)(\omega_s)) = 0$$

so that condition (S1) in Definition 2.13 is also satisfied for ∂p_j. $\square$

2.4 Dynamics on coisotropics submanifolds and Hamiltonian group actions

In this section I begin to study the k-dimensional dynamics on stable coisotropics. In particular I establish that all compact leaves are tori. I then explain how one can interpret stable coisotropics as "locally Hamiltonian group actions". This section has its roots in chapter 5 of [MS17] where Hamiltonian group actions are treated in detail.

Assume throughout this subsection that C is stable. One question that arises immediately in codimension $k > 1$ is whether the flows of the Reeb vector fields commute. The following Lemma answers this question in the affirmative.

Lemma 2.20.
On a stable codimension-k coisotropic the flows $\phi_1, \dots, \phi_k$ of the Reeb vector fields $X_1, \dots, X_k$ (see Definition 2.17) commute and preserve the symplectic form ω_s

Proof. Recall from the proof of Proposition 2.18 that there exist k Hamiltonians $H_1, \dots, H_k$ on the Bolle neighbourhood of C. These satisfy:

$$-dH_i(\cdot) = \omega(X_i, \cdot).$$

Thus $X_1, \dots, X_k$ are Hamiltonian vector fields on U. Denote by $\{\cdot, \cdot\}$ the Poisson-bracket. Observe that

$$-dH_i(X_j) = -\omega(X_i, X_j) = -\{H_i, H_j\} = 0,$$

because the vector fields X_i and X_j have values in TC^ω. Therefore the Lie brackets

$[X_i, X_j]$ vanish for all $1 \leq i, j \leq k$ and the flows ϕ_i and ϕ_j commute. Moreover

$$\begin{aligned}
\mathcal{L}_{X_i}\omega_s &= d(\iota(X_i)i_C^*\omega + \iota(X_i)\sum_{j=1}^{k} d(p_j\alpha_j)) \\
&= d\left(\sum_{j=1}^{k}(dp_j(\cdot)\alpha_j(X_i) - dp_j(X_i)\alpha_j(\cdot) + p_j d\alpha_j(X_i,\cdot))\right) \\
&= ddp_i = 0
\end{aligned}$$

and thus for each fixed $t \in \mathbb{R}$ and each i, the flow for time t, ϕ_i^t, is a Hamiltonian symplectomorphism of the Bolle neighbourhood. $\square$

I would like to remark that this does not automatically follow from the integrability of TC^ω but is a consequence of the stability condition. While the integrability of TC^ω implies $[X_i, X_j] \in TC^\omega$, stability forces $[X_i, X_j] = 0$ for all generalised Reeb vector fields X_i, X_j.

Definition 2.21 (Generalised Reed flow)**.**
Let $q = (q_1, \ldots, q_k) \in \mathbb{R}^k$ and $x \in C$. The *generalised Reeb flow* on a stable coisotropic C is

$$\begin{aligned}
&\mathbb{R}^k \times C \to C \\
&(q, x) \mapsto \Phi^q(x) := (\phi_k^{q_k} \circ \cdots \circ \phi_1^{q_1})(x).
\end{aligned} \tag{2.9}$$

Thus for each fixed q this is a well defined symplectomorphism by Lemma 2.20 above.

I define below a Hamiltonian group action on the Bolle neighbourhood U. This is a "locally Hamiltonian group action" in the sense that it is not defined on $W \setminus U$ but only on U. More precisely the generalised Reeb flow as defined in 2.21 above, can be interpreted as an action of the non-compact Lie group $\mathbb{R}^k$ on the Bolle neighbourhood $U \cong B_{\epsilon_0}^k \times C$ as follows: define a group action Φ on the Bolle neighbourhood by

$$\begin{aligned}\Phi : \mathbb{R}^k \times B^k_\epsilon \times C \longrightarrow \quad & B^k_\epsilon \times C \\ (q,p,x) \mapsto \quad & (p, \Phi^q(x)).\end{aligned} \tag{2.10}$$

Denote by $e_1, \ldots, e_k$ the standard basis of $\mathbb{R}^k$, viewed as the (trivial) Lie algebra of the Lie group $\mathbb{R}^k$. Denote by $e_1^*, \ldots, e_k^*$ the standard basis of $(\mathbb{R}^k)^*$ viewed as the dual of this (trivial) Lie algebra. Denote the canonical pairing of $(\mathbb{R}^k)^*$ and $\mathbb{R}^k$ by $\langle\langle\cdot,\cdot\rangle\rangle$ and by $\langle\cdot,\cdot\rangle$ the standard inner product on $\mathbb{R}^k$. I claim that the moment map of this action is given by:

$$\begin{aligned}\mu_C : B^k_\epsilon \times C &\longrightarrow (\mathbb{R}^k)^* \\ (p,x) &\longrightarrow (p_1 e_1^*, \ldots, p_k e_k^*).\end{aligned} \tag{2.11}$$

Consider a vector q in the Lie algebra of $\mathbb{R}^k$:

$$\begin{aligned}H_q(p,x) &= \langle\langle \mu(p,x), q\rangle\rangle \\ &= \langle\langle (p_1 e_1^*, \ldots, p_k e_k^*), (q_1 e_1, \ldots, q_k e_k\rangle\rangle \\ &= \langle p, q\rangle.\end{aligned} \tag{2.12}$$

This implies

$$-dH_q(p,x) = \sum_{i=1}^k q_i dp_i.$$

Recall that

$$\begin{aligned}-dH_i(\cdot) = \omega_s(q_i X_i, \cdot) &= \sum_{j=1}^k (dp_j(\cdot)\alpha_j(q_i X_i) - dp_j(q_i X_i)\alpha_j(\cdot) + p_j d\alpha_j(q_i X_i, \cdot)) \\ &= q_i dp_i.\end{aligned}$$

Thus μ_C is indeed the moment map of this $\mathbb{R}^k$ action.

By definition, the stabiliser of $(p, x) \in B^k_\epsilon \times C$ under the $\mathbb{R}^k$ action Φ is given by

$$\text{stab}_{\mathbb{R}^k}(p, x) = \{q \in \mathbb{R}^k \quad | \quad \Phi^q(x) = x\}.$$

It is a discrete subgroup of $\mathbb{R}^k$ and thus, by a standard result (see for example Section 49 of [Arn89]

$$\text{stab}_{\mathbb{R}^k}(x) = \Lambda^l \cong \mathbb{Z}^l \subset \mathbb{R}^k \tag{2.13}$$

for a lattice Λ^l isomorphic to $\mathbb{Z}^l$ for $k \geq l$.

Proposition 2.22 (Arnold-Liouville).
Let C be a stable, fibred coisotropic of codimension k. Then each fibre F of the fibre bundle (2.5) is diffeomorphic to a torus $\mathbb{T}^k$ so that C is the total space of a smooth fibre bundle

$$\mathbb{T}^k \to C \xrightarrow{\pi} B \tag{2.14}$$

over a symplectic base (B, ω_B).

Proof. Since C is fibred we may apply Lemma 2.8. It remains to show that the fibres are diffeomorphic to tori of dimension k. Let $x \in C$. The coisotropic is stable, thus work in a Bolle neighbourhood and consider C as the zero level set of the group action (2.10). The group $\mathbb{R}^k$ acts transitively on F_x the leaf trough x. By equation (2.13) choose an isomorphism from the stabiliser subgroup $\text{stab}_{\mathbb{R}^k}(x)$ to $\mathbb{Z}^l$ for $k \geq l$. With respect to the group action (2.10) the leaf F_x is a homogenous space. Thus there exists a k' such that $k = k' + l$ and a diffeomorphism

$$\Phi : \mathbb{R}^{k'} \times \mathbb{R}^l/\mathbb{Z}^l \longrightarrow F_x.$$

By assumption F_x is compact. Therefore $k' = 0$ and F_x is diffeomorphic to a torus of dimension $k = l$. $\square$

Corollary 2.23.

If C is stable and not necessarily fibred, each leaf F of the characteristic foliation is diffeomorphic to $\mathbb{R}^{k'} \times \mathbb{T}^l$ for $k = k' + l$. In particular each closed leaf is diffeomorphic to a torus $\mathbb{T}^l$.

Remark 2.24.

Lemma 2.22 is an adaptation of the so-called Arnold-Liouville theorem to the present setting. The original result, which is proved in [Arn89], is the special case where $k = n$ and where the action (2.10) is globally defined on the symplectic manifold W. Such an action is called a completely integrable system and was the starting point of what is called KAM theory. See again [Arn89].

Below I quickly revisit Example 1.2 from the introduction. Since the fibres F of the coisotropic are the orbits of the $U(k)$ action, the coisotropic cannot be stable. I explain below how a stable coisotropic arises in this context. Generally speaking, similar constructions work for all (compact) Lie-groups, which contain an appropriate (maximal) torus.

Example 2.25 (The partial flag variety)**.**

Consider for $k \leq n$ the space $hom(\mathbb{C}^k, \mathbb{C}^n)$. Identify this space with the space of n by k complex matrices $\mathbb{C}^{n\times k}$ and equip it with the Hermitian inner product $\mathrm{tr}(A^*B)$, where A^* denotes the conjugate transpose of the matrix $A \in \mathbb{C}^{n\times k}$. Then

$$\omega_{\mathrm{tr}}(A, B) := \mathrm{Im}(\mathrm{tr}(A^*B))$$

is a symplectic form on $\mathbb{C}^{n\times k}$. It is a standard fact (see for example Exercise 5.43 of [MS17]) that the action of $U(k)$ on $\mathbb{C}^{n\times k}$ by right multiplication is Hamiltonian with moment map

$$\mu(A) = \frac{1}{2i} A^* A.$$

The level set $\mu^{-1}(\frac{1}{2i}\mathbb{1})$ is a coisotropic submanifold of $(\mathbb{C}^{n\times k}, \omega_{\mathrm{tr}})$.

Think of $\mathbb{C}^{n\times k}$ as a product $\mathbb{C}^n\times\cdots\times\mathbb{C}^n$ (k-times). Choosing the level set $\mu^{-1}(\frac{1}{2i}\mathbb{1})$ corresponds to restricting to k tuples of vectors $V = (v_1, \dots, v_k)$ in $\mathbb{C}^n$, such that $\langle v_i, v_j\rangle = \delta_{ij}$ for the Hermitian inner product $\langle\cdot,\cdot\rangle$ on $\mathbb{C}^n$. This means that V, viewed as a matrix, is a unitary k-frame.

Under the $U(k)$ action however the isotropic leaves in $\mu^{-1}(\frac{1}{2i}\mathbb{1})$ are the orbits of the $U(k)$ action and thus, unless k is equal to 1, not diffeomorphic to $\mathbb{T}^k$. Therefore $\mu^{-1}(\frac{1}{2i}\mathbb{1})$ under this action cannot be stable by Lemma 2.22. Hence consider the diagonal action of the maximal torus

$$T = U(1)\times\cdots\times U(1)$$

in $\mathrm{U}(k)$. Under this action the level set $\mu^{-1}(\frac{1}{2i}\mathbb{1})$ is a stable coisotropic C. Geometrically the action by elements of T is given by subsequently rotating each of the k vectors in $\mathbb{C}^n$ around a Hopf fibre while leaving the vectors previously rotated untouched. This is different to the $U(k)$ action where each vector in the k by k matrix associated to an element of $U(k)$ acts on each vector in $\mathbb{C}^n\times\cdots\times\mathbb{C}^n$. It follows that the symplectic quotient of B of C is diffeomorphic to the partial flag variety

$$P(k, n, \mathbb{C}) \cong U(n)/(U(1)\times\cdots\times U(1)\times U(n-k)),$$

where the diffeomorphism comes from viewing the space of unitary k-frames as the homogenous space $U(n)$ under the same $U(k)$ action.

Chapter 3

Constructions with coisotropics submanifolds

Coisotropics encompass two extensively studied classes of submanifolds of symplectic manifolds. Every Lagrangian is a coisotropic and so is every hypersurface. In this chapter, I explain that it is also possible to assign Lagrangians and hypersurfaces to certain coisotropics. More precisely, I assign a Lagrangian L_C to a given fibred coisotropic $F \to C \to B$, and construct a stable hypersurface H_C from a given stable coisotropic $(C, \mathcal{S})$.The goal of this chapter is to introduce the *Lagrangian graph*

L_C and *the stable hypersurface* H_C and to explain how these submanifolds capture aspects of the geometry and topology of the coisotropic C.

Concretely, L_C inherits a fibre bundle structure from C. The proof of Theorem 1.7 in Chapter 4 builds on adapting and applying methods from Lagrangian Floer theory to L_C. The Reeb dynamics on the stable hypersurface H_C are in equivalent to the generalised Reeb dynamics of the coisotropic C in an appropriate sense. The proof of Theorem 1.8 in Chapter 5 relies on adapting and applying techniques from symplectic field theory to H_C.

I introduce the Lagrangian graph L_C of C in Section 3.1 and explain how L_C inherits its fibre bundle structure from a fibred C. I define the notions of monotonicity of C and of the minimal Maslov number of C by defining them as notions for L_C (see Definitions 3.4 and 3.5) in Section 3.1.1. I then compute the minimal Maslov number of L_C in a simple case (see Example 3.6). In Section 3.1.2 I explore the relation of displaceability of C and L_C and explain how leaf-wise fixed points of C correspond to the self-intersection theory of the Lagrangian L_C.

I have already derived some elementary facts about the k-dimensional dynamics of stable coisotropics in Section 2.4. Recall in particular that stable coisotropics can be seen as level sets of moment maps of a Hamiltonian group action on the Bolle neighbourhood. Before turning to the construction of H_C , I study a subset of the one dimensional dynamics on C which I call the *generalised Reeb dynamics* on C in Section 3.2.1. This subset of the dynamics was first studied by Bolle in [Bol98]. The generalised Reeb dynamics play an important role in the proof of Theorem 1.8 and Theorem 1.6. I construct H_C as a level set of a moment map of an $\mathbb{R}$-action which has the generalised Reeb dynamics as orbits and prove that H_C is stable in Section 3.2.2. I then explain how the Reeb dynamics of the hypersurface H_C are related to the generalised Reeb dynamics on the stable coisotropic C in Section 3.2.3.

3.1 The Lagrangian graph of a fibred coisotropic submanifold

Given a symplectic manifold (W, ω) one can consider its *twisted product* $(W \times W, -\omega \times \omega)$. For the sake of brevitiy I set

$$W^{-} \times W^{+} = (W \times W, -\omega \times \omega).$$

Throughout I denote by p_{-} the projection onto the first factor $W^{-} = (W, -\omega)$ and by p_{+} the projection onto the second factor $W^{+}(W, \omega)$. Assume throughout this section that C is fibred, so that Lemma 2.8 applies. I continue to denote the projection onto the symplectic quotient B of C by π_B.

Definition 3.1 (Lagrangian graph of a fibred coisotropic submanifold)**.**
The *Lagrangian graph* of C, is defined as the fibre product $C \times_B C$ of the diagram:

$$\begin{array}{ccc} L_C & \xrightarrow{p_+} & C \\ \downarrow{\scriptstyle p_-} & & \downarrow{\scriptstyle \pi_B} \\ C & \xrightarrow{\pi_B} & B. \end{array}$$

As a set, L_C is given by:

$$\begin{aligned} L_C = C \times_B C &= \{(x, y) \in C \times C \mid \pi_B(x) = \pi_B(y)\} \\ &= \{(x, y) \in C \times C \mid y \in F_x\}. \end{aligned} \tag{3.1}$$

Note that this is a special case of a Lagrangian correspondence which were introduced by Weinstein as canonical relations in [Wei77].

Lemma 3.2.

If C is fibred, L_C is a Lagrangian submanifold of $(W \times W, -\omega \times w)$. Moreover L_C is the total space of the smooth fibre bundle

$$F \times F \to L_C \to \Delta B, \tag{3.2}$$

where ΔB denotes the diagonal in $B \times B$.

Proof. Note that $L_C \subseteq C \times C$. For $v, w \in T_{(x,y)} L_C$ write

$$\begin{aligned} v &= (v_x, v_y) \in T_x C \times T_y C \\ w &= (w_x, w_y) \in T_x C \times T_y C. \end{aligned}$$

Let

$$\gamma^v(t) = (\gamma^v_x(t), \gamma^v_y(t)) \text{ and } \gamma^w(t) = (\gamma^w_x(t), \gamma^w_y(t))$$

be curves in L_C such that

$$\begin{aligned} \gamma^v(0) = (x, y) \text{ and } \left.\frac{d}{dt}\right|_{t=0} \gamma^v(t) = (v_x, v_y), \\ \gamma^w(0) = (x, y) \text{ and } \left.\frac{d}{dt}\right|_{t=0} \gamma^w(t) = (w_x, w_y). \end{aligned}$$

Thus by differentiating Equation (3.1) defining L_C along these curves one obtains

$$\begin{aligned} d\pi_B(x) v_x &= d\pi_B(y) v_y \\ d\pi_B(x) w_x &= d\pi_B(y) w_y \end{aligned}$$

Since L_C is a subset of $C \times C$, and C is fibred the kernel of the restriction of ω to C agrees with the kernel of the linearised projection $d\pi_B$:

$$\ker i_C^* \omega(x) = T_x C^\omega = T_x F = \ker d\pi(x).$$

Calculate

$$
\begin{aligned}
(-\omega \times \omega)(v, w) &= -\omega(v_x, w_x) + \omega(v_y, w_y) \\
&= -\omega_B(v_x, w_x) + \omega_B(v_y, w_y) \\
&= -\omega_B(v_x, w_x) + \omega_B(v_x, w_x) \\
&= 0
\end{aligned}
$$

Therefore $i_{L_C}^*(-\omega \times \omega) = 0$ and L_C is Lagrangian.

To see how L_C inherits a fibre bundle structure from C, consider $F \times F$ and the maps

$$
\begin{aligned}
i_- &= i_F \circ p_- : F \times F \to C \\
i_+ &= i_F \circ p_+ : F \times F \to C.
\end{aligned}
$$

By the universal property of the fibre product, there exists a map $i_{F\times F} : F \times F \to L_C$, such that the diagram below commutes. Notice that both, rows and columns, are exact.

$$
\begin{tikzcd}
F \times F \arrow[r, "p_+"] \arrow[d, "p_-"] \arrow[dr, dashed, "i_{F\times F}"] & F \arrow[r, "id"] \arrow[d, "i_F"] & F \arrow[d, "i_F"] \\
F \arrow[r, "i_F"] \arrow[d, "id"] & L_C \arrow[r, "p_+"] \arrow[d, "p_-"] & C \arrow[d, "\pi_B"] \\
F \arrow[r, "i_F"] & C \arrow[r, "\pi_B"] & B
\end{tikzcd}
$$

That L_C is the total space of the fibre bundle (3.2) now follows from equation (3.1) above. $\square$

Lemma 3.2 shows that one may associate to every fibred coisotropic C a Lagrangian L_C which inherits a fibre bundle structure from C. By the universal property of the product, this assignment is unique. Notice also that the embedding of L_C into $W^- \times W^+$ is uniquely determined by the embedding of C into (W, ω).

3.1.1 Montonicity and the minimal Maslov number of coisotropic submanifolds

I now recall two important definitions for Lagrangian submanifolds L of a symplectic manifold (W, ω). Given a disc

$$u : (D, \partial D) \to (W, L),$$

we denote by $E_\omega(u)$ the symplectic energy and by $\mu(u)$ the Maslov index of u. Both maps descend to homomorphisms on $H_2^D(M, L) \subset H_2(M, L)$, the image of the Hurewicz homomorphism $h : \pi_2(M, L) \to H_2(M, L)$.

Definition 3.3 (Monotone Lagrangian)**.**
A Lagrangian L in a symplectic manifold (W, ω) is monotone if there exists a positive real number $\eta > 0$ such that

$$E_\omega(A) = \eta \cdot \mu(A) \quad \text{for all } A \in H_2^D(M, L).$$

Denote by

$$N_L = \min_{A \in H_2^D(M,L)} \mu(A) > 0 \tag{3.3}$$

the *minimal Maslov number* of a monotone Lagrangian L.

Definition 3.4 (Monotone coisotropic)**.**
A fibred coisotropic C of a symplectic manifold (W, ω) is *monotone* if L_C is a monotone Lagrangian submanifold of the twisted product $W^- \times W^+$.

Definition 3.5 (Minimal Maslov number of a coisotropic)**.**

The *minimal Maslov number* N_C of a fibred, monotone coisotropic C is the minimal Maslov number N_{L_C} of the associated Lagrangian graph L_C.

To gain some intuition about these definitions consider the following simple ,yet illuminating example below. This example can be generalised in various directions:

Example 3.6 (Minimal Maslov number of the generalised Hopf fibration)**.**
Consider $\mathbb{R}^{2n} = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ and its standard symplectic structure ω_0. Then the standard almost complex structure J_0 given by

$$J_0 = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

is ω_0-compatible, i.e. for $v, w \in \mathbb{R}^{2n}$

$$\omega_0(v, w) = \langle J_0 v, w \rangle.$$

The standard unit sphere S^{2n-1} is a stable coisotropic with respect to ω_0. Recall that

$$T_v S^{2n-1} = v^{\perp}$$

the orthogonal complement of v. The isotropic distribution at v is given by:

$$\left(T_v S^{2n-1}\right)^{\omega_0} = \left\{ w \in \mathbb{R}^{2n} | \langle J_0 v', w \rangle = 0 \ \forall v' \in v^{\perp} \right\} = \text{span}\{J_0 v\}$$

At each point $v \in S^{2n-1}$ define the 1-form α on $T_v S^{2n-1}$ by

$$\alpha_v(w) \coloneqq \iota_v \omega_0(w) = \langle J_0 v, w \rangle$$

Then

$$\alpha_v(J_0 v) = \langle J_0 v, J_0 v \rangle = 1$$

and

$$d\alpha(v,w) = d(\iota_v \omega_0(w)) = \langle v, J_0 w\rangle = \omega_0(v,w)$$

Thus S^{2n-1} is a contact coisotropic and particular stable. View S^{2n-1} as the total space of the generalised Hopf fibration

$$S^1 \to S^{2n-1} \to \mathbb{CP}^{n-1}.$$

The Lagrangian $L_{S^{2n-1}}$ is thus the total space of the fibre bundle

$$S^1 \times S^1 \to L_{S^{2n-1}} \to \Delta\mathbb{C}P^{n-1}. \tag{3.4}$$

As a set it is given by

$$L_{S^{2n-1}} = \{(v,w) \in S^{2n-1} \times S^{2n-1} | \pi_{\mathbb{CP}^{n-1}}(v) = \pi_{\mathbb{CP}^{n-1}}(w)\}.$$

Denote by $S^1_{\mp}$ the image of the projection $p_{\mp}(S^1 \times S^1)$ to each factor of $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$. To compute the minimal Maslov number of $L_{S^{2n-1}}$ consider the long exact sequence of the fibre bundle (3.4) .

$$\begin{tikzcd}
\pi_2(\Delta\mathbb{C}P^{n-1}) \arrow[r, "\Delta i^*"] \arrow[d, "\cong"] & \pi_1(S^1 \times S^1) \arrow[r, "i_*"] \arrow[d, "\cong"] & \pi_1(L_{S^{2n-1}}) \arrow[r, "\pi_*"] \arrow[d, "\cong"] & \pi_1(\Delta\mathbb{C}P^{n-1}) \arrow[d, "\cong"] \\
\pi_2(\mathbb{C}P^{n-1}) \arrow[r, "\delta_*"] & \pi_1(S^1_-) \times \pi_1(S^1_+) \arrow[r, "i_*"] & \pi_1(L_{S^{2n-1}}) \arrow[r] & 0
\end{tikzcd}$$

From the long exact sequence of the generalised Hopf fibration

$$S^1 \to S^{2n-1} \to \mathbb{C}P^{n-1},$$

it follows that $\pi_2(\mathbb{C}P^{n-1}) \cong \pi_1(S^1) \cong \mathbb{Z}$, where the generator of $\mathbb{Z}$ corresponds to the loop generating the Hopf fibre. One can identify $\Delta\mathbb{C}P^{n-1}$ with $\mathbb{CP}^{n-1}$ either via p_- or via p_+. If one identifies the diagonal $\Delta\mathbb{C}P^{n-1}$ with $p_-(\mathbb{CP}^{n-1} \times \mathbb{CP}^{n-1})$, the image of δ_* is $\pi_1(S^1_1)$ or the $(1,0)$-loop in T^2. Since i_* is surjective it follows in this case that

$$\pi_1(S^1_+) \cong \pi_1(L_{S^{2n-1}}) \cong \mathbb{Z},$$

where the generator of $\mathbb{Z}$ is the $(0,1)$ loop in $\pi_1(T^2)$. Thus the generator of $\pi_1(L_{S^{2n-1}})$ corresponds to the loop around the Hopf fibre in the second factor. In the case where one identifies the diagonal $\Delta\mathbb{C}P^{n-1}$ with $p_+(\mathbb{CP}^{n-1} \times \mathbb{CP}^{n-1})$ the generator of $\pi_1(L_{S^{2n-1}})$ corresponds to $(1,0)$ loop around the Hopf fibre in the first factor. Notice also that the generator $\pi_2(\Delta\mathbb{C}P^{n-1})$ corresponds under the map Δi^* to the $(1,1)$ loop around both Hopf fibres in T^2which is not a minimal loop.

Next, examine long exact sequence of relative homotopy groups

$$\cdots \longrightarrow \pi_i(L_{S^{2n-1}}) \longrightarrow \pi_i(\mathbb{R}^{2n} \times \mathbb{R}^{2n}) \longrightarrow \pi_i(\mathbb{R}^{2n} \times \mathbb{R}^{2n}, L_{S^{2n-1}}) \longrightarrow \dots$$

Since $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ is contractible it follows that $\pi_{i+1}(\mathbb{R}^{2n} \times \mathbb{R}^{2n}, L_{S^{2n-1}}) \cong \pi_i(L_{S^{2n-1}})$ and in particular $\pi_2(\mathbb{R}^{2n} \times \mathbb{R}^{2n}, L_{S^{2n-1}}) \cong \pi_1(L_{S^{2n-1}}) \cong \mathbb{Z}$ is generated by the loop around either the Hopf fibre in the first or the second factor.

I now compute the Maslov number of $L_{S^{2n-1}}$. In complex coordinates

$$L_{S^{2n-1}} = \{(z_1, \dots, z_n, w_1, \dots w_n) \in S^{2n-1} \times S^{2n-1} \mid w_i = e^{i\theta} z_i \text{ for } \theta \in [0, 2\pi]\}$$

Consider the loop

$$\begin{aligned} \gamma : S^1 &\to L_{S^{2n-1}} \\ \theta &\mapsto (e^{i\theta} z_1, \dots, e^{i\theta} z_n, z_1, \dots, z_n). \end{aligned}$$

with base point $\gamma_0 = (e^{i\theta}, 0, \dots, 0, 1, 0, \dots, 0)$, which generates $\pi_1(L_{S^{2n-1}})$. A loop Λ_γ of unitary frames of the tangent spaces $T_{\gamma(\theta)} L_{S^{2n-1}}$ along this loop is given by

$$\Lambda_\gamma = (ie^{i\theta} v_1, e^{i\theta} v_2, \dots, e^{i\theta} v_n, iv_1, v_2 \dots, v_n),$$

for a basis $\{v_1, \dots, v_n\}$ of $T_{\gamma_0} L_{S^{2n-1}}$. Thus the Maslov index of this loops is $2n$ since it rotates each of the first n coordinates once around the origin. This implies that the minimal Maslov number satisfies

$$N_{L_{S^{2n-1}}} = 2n.$$

Moreover, an elementary computation shows that $L_{S^{2n-1}}$ is monotone.

3.1.2 Displaceability and leaf-wise fixed points of C and L_C

It is a well known fact that a diffeomorphism $\psi : W \to W$ is a symplectomorphism if and only if the graph of ψ, given as a set by

$$\text{graph}(\psi) = \{(q, \psi(q)) \in W^- \times W^+ \mid q \in W\}$$

is a Lagrangian submanifold of $W^- \times W^+$ (see Proposition 3.27 in [MS17]).

Definition 3.7 (Displaceability)**.**
A submanifold $N \subset W$ is *Hamiltonian displaceable* if there exists a Hamiltonian symplectomorphism $\psi : W \to W$ such that $\phi(N) \cap N = \emptyset$.

Definition 3.8 (Leaf-wise fixed point)**.**
Let $\psi : W \to W$ be a symplectomorphism and C a coisotropic. A point $x \in C$ is a *leaf-wise fixed point* if $\psi(x)$ lies in the leaf F_x through x.

Lemma 3.9.
Given a fibred coisotropic C and a symplectomorphism ψ there is a one to one correspondence between the set of leafwise fixed points

$$Fix(\psi, \mathcal{F}) = \{x \in C \mid \psi(x) \in F_x\}$$

and the intersection of the graph$(\psi(L_C))$ with L_C.

Proof.

$$\begin{aligned}
\text{graph}(\psi) \cap L_C &= \{(x, \psi(x)) \in W \times W \mid x \in W\} \cap \{(x, y) \in C \times C \mid \pi_B(x) = \pi_B(y)\} \\
&= \{(x, \psi(x)) \in C \times C \mid \pi_B(x) = \pi_B(\psi(x))\} \\
&= \{(x, \psi(x)) \in C \times C \mid \psi(x) \in F_x\}
\end{aligned}$$

$\cong \text{Fix}(\psi, \mathcal{F})$.

□

Remark 3.10.

If the coisotropic $C \subset W$ is fibred, the notion of leafwise fixed points is a generalisation of two well known notions: If C is the entire symplectic manifold W, leafwise fixed points are fixed points of the symplectomorphism ψ. If the coisotropic is Lagrangian i.e. $C = L$, the leafwise fixed points are intersections $\psi(L) \cap L$.

$$\begin{aligned}
k = 0: &\quad \{\psi(q) = q\} \overset{1:1}{\longleftrightarrow} \{\mathit{graph}(\psi) \cap \Delta W\} \\
1 < k < n: &\quad Fix(\psi, \mathcal{F}) \overset{1:1}{\longleftrightarrow} \{\mathit{graph}(\psi) \cap L_C\} \\
k = n: &\quad \{x \in L \mid \psi(x) \in L\} \overset{1:1}{\longleftrightarrow} \{\psi(L) \cap L\}
\end{aligned}$$

Lemma 3.11.

If C is displaceable, so is L_C.

Proof. This follows immediately from the fact that $L_C \subset C \times C$, since $\phi(C) \cap C = \emptyset$ implies $\phi(C) \times \phi(C) \cap C \times C = \emptyset$. □

Remark 3.12.

A natural question to ask is whether displaceability of L_C implies displaceability of C or if not, where exaclty the differences of these notions lie. This would be interesting to investigate in the future.

3.2 The stable hypersurface H_C and generalised Reeb dynamics on C

Assume throughout this section that C is stable. As pointed out at the beginning of Section 1.4 the Reeb dynamics on stable and contact hypersurfaces have been studied extensively. In particular, the Weinstein conjecture has inspired important developments in symplectic geometry and has been proved in some cases. See [Pas12] for a survey and the references therein. In higher codimension $k > 1$, several new questions about the dynamics on C arise. If a leaf of a stable coistropic $(C, \mathcal{S})$ is closed, it is a k-dimensional torus by Corollary 2.23. However as one sees already in Example 2.6 or, in a different context, by considering the Reeb foliation of S^3, nearby leaves of foliated manifolds are not necessarily diffeomorphic. For a stable-but-not-fibred coisotropic this implies that the symplectic quotient is not necessarily Hausdorff. One possible starting point to study the dynamics of leaves is to consider the one dimensional sub dynamics of the leaves. An obvious question is whether the Weinstein conjecture holds for (stable) coisotropics (see also the Conjecture 1.4 in the Introduction):

Conjecture (Weinstein conjecture for stable Coisotropics)**.**

Do there exist (maybe under appropriate additional assumptions) non-contractible loops within the leaves of stable coisotropics?

First of all notice that a closed characteristic on a hypersurface within a leaf of the characteristic foliation is necessarily non contractible. For a loop within a k-dimensional leaf, this is not necessarily true. One therefore has to distinguish between contractible and non-contractible loops within a leaf. Bolle proved the existence of a non-contractible loop on contact coisotropics in $\mathbb{R}^{2n}$ in [Bol98]. As pointed out already at the beginning of this chapter, I will describe in this section how one can translate questions about the generalised Reeb dynamics on the leaves of a stable coisotropic to questions about the Reeb dynamics of the stable hypersurface H_C.

3.2.1 Generalised Reeb dynamics on stable coisotropics

Throughout this section I will identify S^1 with $\mathbb{R}/\mathbb{Z}$, so that a loop $\gamma : S^1 \to W$ has the basepoint $\gamma(0) = \gamma(1)$.

Definition 3.13 (Action vector)**.**
Let C be stable and let $x \in C$. Let $\gamma : S^1 \to F_x$ be a loop on F_x. The *action vector* $\mathcal{A}_k$ of γ is the vector

$$\mathcal{A}_k(\gamma) = (A_1, \dots A_k),$$

where

$$A_i = \int_{S^1} \gamma^* \alpha_i.$$

Lemma 3.14.
Let x in C. A loop $\gamma : S^1 \to F_x$ such that $\gamma(0) = x$ is non-contractible in F_x if and only if $\mathcal{A}_k(\gamma)$ is non-trivial.

Proof. By the stability assumption on C and Stokes' theorem, the action vector depends only on the homotopy class of γ in F_x. Thus contractible loops q have trivial action vector $\mathcal{A}_k(q) = 0$.

Now assume that $\gamma : S^1 \to C$ is non-contractible. By Corollary 2.23, F_x is diffeomorphic to $\mathbb{R}^{k'} \times \mathbb{T}^l$ for $k = k' + l$. If $l = 0$, all loops in F_x are contractible, thus assume $l \geq 1$. Since γ is non-contractible in F_x its homotopy class $[\gamma]$ in $\mathbb{T}^l$ is non-trivial, thus there exists an $1 \leq i \leq l$ such that $A_i(\gamma) \neq 0)$. Thus $\mathcal{A}_k(\gamma) = 0$ if and only if γ is contractible in F. $\square$

Remark 3.15.
A loop γ in a leaf $F_{\gamma(0)}$ satisfies the equation:

$$\dot{\gamma}(t) = \sum_{i=1}^{k} q_i(t) X_i(\gamma(t)).$$

As we have seen above the action vector depends only on its homotopy class. Since $\mathbb{R}$ is contractible every loop γ as above is homotopic to a loop $\beta : S^1 \to F_{\gamma(0)}$ which is a solution to

$$\dot{\beta}(t) = \sum_{i=1}^{k} q_i X_i(\beta(t)). \tag{3.5}$$

Bolle proved the Weinstein conjecture for contact coisotropics in $\mathbb{R}^{2n}$ by showing that there exists a loop of positive action which satisfies Equation 3.5 by considering symplectic capacities. In tribute to him, this equation is usually referred to as *Bolle's equation*.

I now present a point of view on the one dimensional dynamics on C which links them to the action of the generalised Reeb flow and the associated moment map considered in Section 2.4. Observe that contractible loops correspond to trivial solutions of Bolle's equation 3.2.1. Hence, the loop γ is non-contractible if and only if, the vector $q = (q_1, \ldots, q_k)$ is non-trivial. Given any non-trivial vector q of the euclidean vector space $\mathbb{R}^k$, set $\hat{q} = \frac{q}{|q|}$ and $\hat{q}_i = \frac{q_i}{|q|}$ where $|\cdot|$ is the standard euclidean norm. For non-contractible loops one can rewrite Bolle's equation as:

$$\dot{\beta}(t) = \sum_{i=1}^{k} T\hat{q}_i X_i(\beta(t)),$$

for $T = |q| \in \mathbb{R}_{>0}$. I call such a loop β a solution to Bolle's equation of period T.

Recall that the generalised Reeb flow is given by the $\mathbb{R}^k$-action from Definition 2.21. Consider the span of q over $\mathbb{R}$

$$\langle q \rangle = \{t \cdot \hat{q} \mid t \in \mathbb{R}\}$$

as a one parameter subgroup of the group $\mathbb{R}^k$. Denote by $\langle q \rangle^{\perp}$ the span of the orthogonal complement of q with respect to the standard inner product on $\mathbb{R}^k$. In particular, q induces a decomposition of $\mathbb{R}^k$ into the direct sum

$$\mathbb{R}^k = \langle q \rangle \oplus \langle q \rangle^{\perp}. \tag{3.6}$$

of vector spaces and of normal subgroups. Therefore there is a well defined action of these subgroups on C.

Definition 3.16 (Generalised Reeb trajectories and orbits).
For a stable coisotropic C and a point $x \in C$ a *generalised Reeb trajectory* (γ, q) *through* x is an orbit of a subgroup $\langle q \rangle$ of the action described in Definition 2.21 which passes through x. A generalised Reeb trajectory is non-trivial if and only if the vector$q$ is. A non-trivial generalised Reeb trajectory through a point $x \in C$ is closed if there exists a $T \in \mathbb{R}_{>0}$ such that $\Phi^{T\hat{q}}(x) = x$.

A *generalised Reeb orbit through* x is a closed, non-trivial generalised Reeb trajectory through x. I denote generalised Reeb orbits through a point x as triples $(\gamma, \hat{q}, T)$ consisting of a loop $\gamma : S^1 \to F_{\gamma(0)}$which satisfies $\gamma(0) = \gamma(1) = x$ and Bolle's equation 3.5 for a vector $\hat{q}$ in the unit sphere S^{k-1} and a positive real number T, which is the period of the orbit.

The following Lemma follows immediately from this definition by :

Lemma 3.17.

The set of non-contractible loops γ, with basepoint $\gamma(0) = x$, which are solutions to Bolle's equation 3.5 for the vector q and have period $T = |q|$, is in one to one correspondence with the set of generalised Reeb orbits through x of period $T = |q|$.

Remark 3.18.

The set of nontrivial, generalised Reeb trajectories through a given point $x \in C$ is nothing but the set of orbits of subgroups of $\mathbb{R}^k$ of the form $\langle p \rangle$ and thus the set of orbits is isomorphic to the space of lines through the origin in $\mathbb{R}^k$, and thus to either S^{k-1} or the real Grassmanian $G(1,k)$, depending one the whether one wants to consider the loops associated to q and $-q$ as equivalent or not. In the case $k = 1$ the set of generalised Reeb trajectories (orbits) is the set of Reeb trajectories (orbits) on the hypersurface. Here the quotient map from S^0 to $\mathbb{RP}^0$ corresponds to choosing an orientation on $\mathbb{R}$.

Assume the period of a generalised Reeb orbit through x is T. Then the stabiliser of x under the action of $\langle q \rangle$, is the discrete subgroup

$$\mathrm{stab}_q(x) = \{T'\hat{q} \in \langle q \rangle,\ T' \in \mathbb{R} \mid \Phi^{T'\hat{q}}(x) = x\} = \{kT \cdot \hat{q} \in \langle q \rangle \mid k \in \mathbb{Z}\}.$$

and thus isomorphic to a copy of $\mathbb{Z} \subset \langle q \rangle$ by sending T to 1.

Example 3.19 (Generalised Reeb orbits on $\mathbb{T}^2$)**.**

As an example consider a Lagrangian torus $\mathbb{T}^2 \subset (\mathbb{R}^4, \omega_0)$. Then the generalised Reeb orbits are integral curves of rational slope $\frac{p}{q}$ as in Example 2.6. The period is $\frac{q}{p}$.

Example 3.20 (Generalised Reeb orbits on stable codimension 2 coisotropic sub-

manifolds).
Consider a stable coisotropic of codimension 2. Assume there exists a closed leaf $F \cong \mathbb{T}^2$ of the characteristic foliation $\mathcal{F}$. Then the generalised Reeb orbits of C in the leaf F are again the integral curves of rational slope $\frac{p}{q}$ as in Example 2.6.

By Lemma 3.17 above non-trivial solutions to Bolle's equation 3.5 of period T are in one to one correspondence with nontrivial generalised Reeb orbits of period T as in Definition 3.16. I will explain below how one can study the one dimensional dynamics on C by viewing them as the Reeb dynamics of a stable hypersurface, which I now construct. This hypersurface is the hypersurface H_C alluded to in the introduction and at the beginning of this chapter.

3.2.2 Construction of the stable hypersurface H_C

Recall that a stable coisotropic is contained in a Bolle neighbourhood $U \cong B^k_{\epsilon_0} \times C$. The moral of being stable is that the dynamics of the foliation $\mathcal{F}$ of C are the conjugate throughout U. Put differently, at a given point within the Bolle neighbourhood, one is unable to specify one's position within U if the only information one has is the dynamics on the foliation. With this in mind it is not entirely surprising that $k-1$ dimensional spheres S^{k-1}_ϵ for $\epsilon < \epsilon_0$ in $B^k_{\epsilon_0}$ give rise to a stable hypersurfaces, with Reeb dynamics which are in one to one correspondence with the generalised Reeb dynamics on C. Thus, given a stable coisotropic C for each $\epsilon < \epsilon_0$ define:

$$H_{C,\epsilon} = S^{k-1}_\epsilon \times C. \tag{3.7}$$

Note that in case where C is a Lagrangian L, H_L is symplectomorphic to the unit cotangent bundle U^*L and in particular a contact hypersurface of the cotangent bundle T^*L. If C is a hypersurface H, then H_H consists of two copies of H.

Proposition 3.21.

Consider a stable coisotropic C in its Bolle neighbourhood $B^k_{\epsilon_0} \times C$. Denote a vector in $B^k_{\epsilon_0}$ by $p = (p_1, \ldots, p_k)$. Then, for every fixed $0 < \epsilon < \epsilon_0$, the hypersurface $H_{C,\epsilon}$, is the level set $\mu_S^{-1}(\frac{\epsilon^2}{2})$ of the moment map

$$\mu_S(p, x) = \frac{1}{2}\sum_{i=1}^{k} p_i^2$$

associated to the $\mathbb{R}$-action

$$\begin{aligned} \mathbb{R} \times (B^k_{\epsilon_0} \setminus \{0\}) \times C &\to (B^k_{\epsilon_0} \setminus \{0\}) \times C \\ (t, p, x) &\mapsto (p, \Phi^{tp}(x)). \end{aligned} \tag{3.8}$$

Proof. First of all observe $\mu_S(p, x) = \frac{1}{2}\sum_{i=1}^{k} p_i^2 = \frac{\epsilon^2}{2}$ implies $|p|^2 = \epsilon^2$. Thus

$$\mu_S^{-1}(\frac{\epsilon^2}{2}) = S_\epsilon^{k-1} \times C.$$

At an element t_0 in the trivial Lie Algebra $\mathbb{R}$ of the trivial Lie group $\mathbb{R}$, the time dependent Hamiltonian H_{t_0} is given by.

$$\begin{aligned} H_{t_0}(p, x) &= \langle \mu(p, x), t_0 \partial_t \rangle \\ &= \langle \langle (\frac{1}{2}\sum_{i=1}^{k} p_i^2 dt, t_0 \partial_t \rangle \rangle \\ &= \langle \frac{1}{2}\sum_{i=1}^{k} p_i^2, t_0 \rangle = \frac{1}{2}\sum_{i=1}^{k} p_i^2 \cdot t_0 \end{aligned}$$

This implies

$$\begin{aligned} dH_{t_0}(p, c) &= t_0 d\left(\frac{1}{2}\sum_{i=1}^{k} p_i^2\right) \\ &= t_0 \sum_{i=1}^{k} p_i dp_i = \sum_{i=1}^{k} \omega_s(t_0 p_i X_i, \cdot). \end{aligned} \tag{3.9}$$

Since $\sum_{j=1}^{k} t_0 p_j X_j$ generates the flow of $\Phi^{t_0 p}$ it follows that μ_S is indeed the moment map of the $\mathbb{R}$-action 3.8 above. $\square$

Lemma 3.22.

For each x in C there is a one-to-one correspondence of the sets of non-trivial, generalised Reeb trajectories $\mathcal{G}$ on C through x and the set of orbits of the action defined by Equation 3.8 *which pass through x.*

Proof. Fix a point x on C. The set of non-trivial generalised Reeb trajectories through x is the set of orbits of subgroups $\langle q \rangle \subset \mathbb{R}^k$ of the action described in Definition 2.21 which pass through x. As described in Remark 3.18 there is an S^{k-1} worth of these orbits.

An orbit through the point $(p, x) \in H_\epsilon$ of the action defined in Equation 3.8 is given as a set by

$$\{(p, \Phi^{tp}(x)) \in S^{k-1}_\epsilon \times C \mid t \in \mathbb{R}\},$$

and thus is a pair consisting of a vector $p \in S^{k-1}_\epsilon$ and an orbit of the subgroup $\langle p \rangle \subset \mathbb{R}^k$ under the action 3.8, whose C component coincides with the C component of the action described in Definition 2.21 for a fixed vector p. Since for each x, there is an S^{k-1} worth of vectors p to define the subgroup $\langle p \rangle$, the two sets are isomorphic by sending $p \mapsto \frac{p}{|p|}$. $\square$

Level sets of moment maps are not neccessarily stable. See Example 1.2. I show below that the level sets of μ_S are both stable and separating. Moreover, the Reeb dynamics on $H_{C,\epsilon,}$ are independent of the choice of ϵ up to reparametrisation. Preempting this fact, from now on I assume that an appropriate ϵ has been chosen and abuse notation by setting

$$H_C = H_{C,\epsilon}$$

whenever the radius of S^{k-1}_ϵ is either clear from context or irrelevant.

Proposition 3.23.

Given a stable coisotropic $(C, \mathcal{S})$, consider H_C . Denote the stabilising one forms on C by $\alpha_1, \ldots, \alpha_k$. Denote the stabilising vector fields by $Y_1, \ldots, Y_k$. Denote by p a vector in S_ϵ^{k-1} Then the following holds

(i) *H_C is a stable and separating hypersurface in $(B_{\epsilon_0}^k \times C, \omega_s)$.*

(ii) *The Reeb vector field at a point (p, x) on H_C is given by*

$$X(p,x) = \left(0, \sum_{j=1}^{k} \hat{p}_j X_j(x)\right). \tag{3.10}$$

(iii) *The stabilising one form α for H_C is given by*

$$\alpha(p,x) = \sum_{i=1}^{k} \hat{p}_i \alpha_i(x).$$

(iv) *The stable vector field Y at $(p, x) \in S_\epsilon^{k-1} \times C$ is given by the radial vector field $\partial_p \in \Gamma(S_\epsilon^{k-1}, TS_\epsilon^{k-1})$ which satisfies at $\partial_p = \hat{p} \in \langle p \rangle$ at each point $p \in S_\epsilon^{k-1}$. Thus ∂_p satisfies*

$$\iota(\partial_p)\omega_s = \alpha \text{ and in particular } \omega_s(Y, X) = 1.$$

Proof. Recall that by the Bolle neighbourhood theorem 2.18 there exists an $\epsilon_0 > 0$, a neighbourhood U of C in W and a symplectomorphism $\psi : C \times B_{\epsilon_0}^k \to U$ such that

$$\omega_s = \phi^* \omega = i_C^* \omega + d(p_1 \alpha_1) + \cdots + d(p_k \alpha_k) \tag{3.11}$$

where the p_i denote the coordinates on the k-dimensional ball of radius ϵ_0 in $\mathbb{R}^k$. Recall from Section 2.3 that a 1-form α on a hypersurface H is stabilising if α is nonzero on $\ker(i_H^* \omega)$ and $ker(i_H^* \omega) \subset \ker d\alpha$. Denote by $X_1, \ldots, X_k$ the Reeb vector fields associated to the stable 1-forms $\alpha_1, \ldots, \alpha_k$ on C. The tangent space of H_C splits as follows:

$$T_{(p,c)}H_C = T_p S_\epsilon^{k-1} \times T_c C = p^\perp \times T_c C. \tag{3.12}$$

Here, $p^\perp$ denotes the orthogonal complement of the vector p with respect to the standard inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^k$. To prove assertion (ii), I first show that X, defined in Euqation 3.10, lies in the one dimensional kernel of the restriction of ω_s to H_C. Let $Z = (q, Z_C) \in T_{(p,x)}H_C = p^\perp \times T_x C$ and consider:

$$\begin{aligned}
\omega_s(X, Z) &= i_C^*\omega(X, Z) + \sum_{i=1}^{k} d(p_i\alpha_i)(X, Z) \\
&= i_C^*\omega\left(\sum_{j=1}^{k} \hat{p}_j X_j, Z\right) + \sum_{i=1}^{k} p_i d\alpha_i \left(\sum_{j=1}^{k} \hat{p}_j X_j, Z\right) \\
&+ \sum_{i=1}^{k} dp_i \wedge \alpha_i \left(\sum_{j=1}^{k} \hat{p}_j X_j, Z\right) \\
&= 0 + 0 + \sum_{i=1}^{k} dp_i \wedge \alpha_i \left(\sum_{j=1}^{k} \hat{p}_j X_j, Z\right) \\
&= \sum_{i=1}^{k} dp_i \left(\sum_{j=1}^{k} \hat{p}_j X_j\right) \cdot \alpha_i(Z_C) - dp_i(q) \cdot \alpha_i \left(\sum_{j=1}^{k} \hat{p}_j X_j\right) \\
&= \sum_{i,j=1}^{k} (0 \cdot Z_c - q_i \cdot \hat{p}_j \cdot \delta_{ij}) \\
&= -\langle q, \hat{p} \rangle = 0 \\
&= 0
\end{aligned}$$

The two terms in the second line of the calculation vanish since $X_j \in \ker(i_C^*\omega) \subset \ker d\alpha_i$ for all $1 \leq i, j \leq k$ by the stability assumption on C. The last equality follows from the definition of the tangent space of H_C. Thus X is contained in the one dimensional kernel of the restriction of ω_s to H_C. To prove (ii) it remains to show that $\alpha(X) = 1$. Compute:

$$\alpha(X)(p,x) = \sum_{i=1}^{k} \hat{p}_i \alpha_i \left(\sum_{j=1}^{k} \hat{p}_j X_j \right) = \sum_{i,j=1}^{k} \hat{p}_i \hat{p}_j \delta_{ij} = \sum_{i=1}^{k} \hat{p}_i^2 = 1. \tag{3.13}$$

Thus α does not vanish on the one dimensional kernel of ω on H_C and is normalised correctly. Observe that

$$d\alpha = \sum_{i=1}^{k} dp_i \wedge \alpha_i + \sum_{i=1}^{k} p_i d\alpha_i.$$

But we have just seen above that $d\alpha$ vanishes along the one dimensional kernel of $i^*_{H_\epsilon}\omega_s$. Thus α is a stablizing one form for H_C and (iii) is proved.

To prove (iv), calculate:

$$\begin{aligned}
\mathcal{L}_{\partial_p} \omega_s &= d(\iota(\partial_p)\omega_s) \\
&= d\left(i_C^* \omega(\hat{p}, \cdot) + \sum_{i=1}^{k} dp_i \wedge \alpha_i(\hat{p}, \cdot) + \sum_{i=1}^{k} p_i d\alpha_i(\hat{p}, \cdot) \right) \\
&= d\left(\sum_{i=1}^{k} \hat{p}_i \alpha_i \right) \\
&= d\alpha
\end{aligned}$$

Thus $i_F^* \mathcal{L}_{\partial_p} \omega_s = i_F^* d\alpha = 0$, which proves assertion (iv).

By Proposition 2.19 $(H_C, \omega_s, \partial_p)$ is a stable hypersurface. To see that H_C is separating consider the open and disjoint sets $U_C := \psi(C \times \mathring{B}^k_\epsilon)$ and $U_R := W \setminus \psi(\bar{B}^k_\epsilon)$. Assertion (i) follows. $\square$

3.2.3 Relation of generalised Reeb dynamics on C and Reeb dynamics on H_C

I now prove a key Lemma for the proof of the main results of this thesis.

Lemma 3.24.

Let C be a stable coisotropic submanifold of codimension k. Fix $\epsilon < \epsilon_0$. There is a one to one correspondence of the set $\mathcal{G}$ of non-trivial generalised Reeb trajectories on C and the set $\mathcal{R}$ of non-trivial Reeb trajectories on H_C:

$$\mathcal{G} \overset{1:1}{\longleftrightarrow} R. \tag{3.14}$$

In particular for every generalised Reeb orbit $(\gamma, \hat{p}, T)$ on C there exists a unique Reeb orbit $(p, \gamma(tT))$ on H_C.

Proof. By choosing $t_0 = \frac{1}{\epsilon}$ in Equation 3.9 it follows that the Hamiltonian vector field $X_{H_{\frac{1}{\epsilon}}}$ associated to the moment map μ_S agrees with the Reeb vector field X given by Equation 3.10 on H_C. The assertion now follows from Lemma 3.22. □

I summarise the relation of the generalised Reeb dynamics on C and the Reeb dynamics on H_C in the following proposition:

Proposition 3.25.

Given a point $x \in C$, consider a loop $\gamma : S^1 \to F_x$. The following are equivalent:

(*i*) *γ has a non-trivial action vector $\mathcal{A}_k(\gamma)$.*

(*ii*) *γ is non-contractible in F_x.*

(*iii*) *γ is homotopic to a non-trivial solution β of Bolle's equation for some $q \in \mathbb{R}^k$.*

(*iv*) *$(\beta, \hat{q}, T)$ is a generalised Reeb orbit which is homotopic to γ.*

(*v*) *$(\beta, \hat{q}, T)$ is a closed, non-trivial orbit of the action 3.8 and β is homotopic to γ.*

(*vi*) *For β homotopic to γ, there exists a unique, closed, non-trivial Reeb orbit $\tilde{\beta}$ on H_C.*

Proof. The assertion that (i) is equivalent to (ii) follows from Lemma 3.14. That (ii) is equivalent to (iii) from Remark 3.15. Statement (iii) is equivalent to statement (iv) by Lemma 3.17. Lemma 3.22 shows that (iv) is equivalent to (v) and finally Lemma 3.24 proves (v) is equivalent to (vi). □

When studying the Reeb dynamics on stable hypersurface one usually assumes that the dynamics associated to the stable one form α are either of *Morse type* or of *Morse-Bott type*. These conditions ensure that the moduli space of closed Reeb orbits is either discrete up to reparametrisation (Morse type) or has a manifold structure (Morse-Bott type). By Proposition 3.24 the generalised Reeb flow on a coisotropic C coincides with the Reeb flow on the associated stable hypersurface H_C. I extend the notion of Morse-Bott type in a straightforward way to stable coisotropics in Definition 3.26 below. Unless one makes very stringent assumptions, the generalised Reeb flow on C is of Morse-Bott type. The key result of this subsection is Proposition 3.27: The generalised Reeb flow on C is of Morse-Bott type if and only if the Reeb flow on H_C is of Morse-Bott type.

Definition 3.26 (Morse-Bottness of stable coisotropic submanifolds)**.**
A closed, nontrivial, generalised Reeb orbit $(\gamma, \hat{q}, T)$ is of *Morse-Bott type* if the set $\mathcal{G}_T(\hat{q})$ of generalised Reeb orbits of period T in direction $\hat{q} \in S^{k-1}$ is a smooth submanifold of C such that

(i) At each point x in $\mathcal{G}_T(\hat{q})$ the tangent space $T_x\mathcal{G}_T(\hat{q})$ satisfies

$$T_x\mathcal{G}_T(\hat{q}) = \ker(d\Phi^{T\hat{q}} - id)(x).$$

(ii) The rank of $i^*_{\mathcal{G}_T(\hat{q})}\omega$ is constant on each connected component of $\mathcal{G}_T(\hat{q})$.

A stable coisotropic $(C, \mathcal{S})$ is of *Morse-Bott type* if all generalised Reeb orbits $(\gamma, \hat{q}, T)$ are of Morse-Bott type.

This definition coincides with the definition given in [Bou+03] in the case where C is a hypersurface, since, up to sign, there is only one direction $\hat{q} \in \mathbb{R}$ and the generalised Reeb flow on a hypersurface is the Reeb flow on the hypersurface. Recall that every generalised Reeb orbit defines a loop γ contained in a torus $\mathbb{T}^l$ for $l \leq k$. This torus is invariant under the action described in Definition 2.21. Thus there is a $\mathbb{T}^{l-1}$ family of orbits γ, given by translations of γ in $\mathbb{T}^l$. Thus, if nonempty, the set $\mathcal{G}_T(\hat{q})$ contains an l-dimensional torus $\mathbb{T}^l_\gamma$ for each orbit $(\gamma, \hat{q}, T)$. Therefore, unless $l = 1$ for all such tori $\mathbb{T}^l_\gamma$, the space $\mathcal{G}_T(\hat{q})$ cannot be one dimensional and thus C cannot be of Morse type. If C is a hypersurface, necessarily $l \leq k = 1$ and thus the Reeb flow on a hypersurface has a chance to be generically of "Morse type". See again [Bou+03] for reference.

Proposition 3.27.
A stable coisotropic $(C, \mathcal{S})$ is of Morse Bott type if and only if the Reeb flow on H_C is of Morse-Bott type.

Proof. Examine the Reeb flow on H_C. It coincides with the orbits of the action 3.8. For each $t \in \mathbb{R}$ it is a symplectomorphism of the Bolle neighbourhood with restriction to H_C given by

$$\begin{aligned} \Phi^t : S^{k-1}_\epsilon \times C &\to S^{k-1}_\epsilon \times C \\ (\hat{p}, x) &\mapsto (\hat{p}, \Phi^{t\hat{p}}(x)). \end{aligned}$$

Recall the action $\mathbb{R}^k$-action from Definition 2.21 on C. Given any vector $\hat{p}$ one can view this action as the composition of the actions Φ^p of the 1-dimensional subgroup $\langle p \rangle$ and the action $\Phi^{p^\perp}$ of the $(k-1)$ dimensional subgroup $\langle p \rangle^\perp$:

$$\begin{aligned} \mathbb{R}^k \times C &\to C \\ \langle p \rangle \oplus \langle p \rangle^\perp \times C &\to C \\ (p, q) &\mapsto \Phi^{t\hat{p}} \circ \Phi^q(x) \end{aligned}$$

Examine the linearisation of the Reeb flow on H_C:

$$d\Phi^t(p,x) = \begin{pmatrix} id & 0 \\ \frac{d}{d\hat{p}}\Phi^t(p,x) & \frac{d}{dx}\Phi^t(p,x) \end{pmatrix}. \tag{3.15}$$

Observe that the differential of Φ^t with respect to $\hat{p}$ in direction of a vector $q \in \langle p \rangle^{\perp} = T_p S^{k-1}$ is given by the infinitesimal action of $p^{\perp}$ in direction q:

$$\frac{d}{d\hat{p}}\Phi^t(p,x)\hat{q} = \Phi^{tq}(p,x).$$

Thus $\frac{d}{d\hat{p}}\Phi^t(p,x)$ corresponds to the inclusion of the Lie algebra of $\langle p \rangle^{\perp}$ into $T_x C$. I will denote this inclusion by $i(\mathfrak{q})$. Consider

$$\begin{aligned} \ker\left(d\Phi^t(p,x) - id(p,x)\right) &= \ker\left(\begin{pmatrix} id & 0 \\ \Phi^{t\hat{q}} & \frac{d}{dx}\Phi^t(p,x) \end{pmatrix} - \begin{pmatrix} id & 0 \\ 0 & id \end{pmatrix}\right) \\ &= \ker\left(\begin{pmatrix} 0 & 0 \\ i(\mathfrak{q}) & \frac{d}{dx}\Phi^t(p,x) - id \end{pmatrix}\right) \\ &= \ker(\frac{d}{dx}\Phi^t(p,x) - id) \end{aligned} \tag{3.16}$$

By Proposition 3.25 for each generalised Reeb orbit $(\gamma, \hat{q}, T)$ on C there exists a unique Reeb orbit $(\tilde{\gamma}, p, T)$ on H_C. By Equation 3.16 above, the tangent space $T_x\mathcal{G}_T(\hat{p})$ is isomorphic to the tangent space $T_{(\hat{p},x)}\mathcal{R}_T$. The proposition follows. $\square$

Lemma 3.28.

If $(C, \mathcal{S})$ is a stable, fibred, codimension k coisotropic, then $(C, \mathcal{S})$ is of Morse-Bott type.

Proof. By Lemma 2.8, C is the total space of the fibre bundle $\mathbb{T}^k \to C \to B$. Thus for every generalised Reeb orbit $(\gamma, \hat{q}, T)$, the set $T_x\mathcal{G}_T(\hat{q})$ contains the fibre $F_{\gamma(0)} \cong \mathbb{T}^k_{\gamma}$. In a local chart around $\gamma(0)$, the generalised Reeb flow is given by $(b, f) \mapsto (b, \Phi^{t\hat{q}}(b, f))$ and thus leaves the base directions invariant. It follows that

$\frac{d}{dx}\Phi^{Tp}(x) = id(x)$ and thus that $T_x\mathcal{G}_T(\hat{q}) = T_xC$. $\square$

Combining the two previous results one obtains immediately:

Corollary 3.29.

For a stable, fibred coisotropic submanifold C, the Reeb flow on H_C is of Morse-Bott type.

Chapter 4

Existence of pearly trajectories

The goal of this chapter is to prove Theorem 1.7 which I state again below as Theorem 4.1.

Theorem 4.1.
Let C be a fibred, stable coisotropic submanifold of a symplectic manifold (W, ω). Assume that the Lagrangian graph L_C in the product $W^- \times W^+$ is monotone and has minimal Maslov number N_{L_C} at least three. Let b be any point in the symplectic quotient B of C.

If L_C is displaceable, then there exist:

(*M*) *An almost fibred Morse function f on L_C such that the unique global minimum x of f on L_C is contained in $f_B^{-1}(0)$ and projects to $(b, b) \in \Delta B$ the diagonal in*

$B^- \times B^+$.

(*E*) *A constant* $E_0 > 0$*, such that for all* ω*-compatible almost complex structures* J *on* W*, there exists at least one pearly trajectory* P *of energy at most* E_0 *and with the following property:*

(*P*) *The pearly trajectory* P *connects a critical point* y *of* f *contained in* $f_B^{-1}([1, \infty))$ *to the minimum* x *of* f*.*

The Lagrangian graph L_C was introduced in Section 3.1. The notions of monotonicity and the minimal Maslov number of the Lagrangian L_C were introduced and Section 3.1.1 as the as notions for the coisotropic C. I recall these in Definition 4.9 and in Equation 4.11 respectively and define the energy of a pearly trajectory in Definition 4.16 below. An *almost fibred* Morse function is a Morse function on a fibre bundle which takes this structure into account, see Section 4.2. A *pearly trajectory* is, roughly speaking, a configuration of holomorphic discs which lives in moduli spaces which are used to define the the algebraic structures on the *pearl complex*. The cohomology of the pearl complex is a model of the self-Floer cohomology of a Lagrangian. I explain this construction in Section 4.3 where I also recall the definition of a pearly trajectory (see Definition 4.12 and 4.14).

4.1 Outline of Chapter 4

Given a fibred coisotropic C as in Theorem 4.1, assign the Lagrangian graph L_C to C as described in Section 3.1. Recall from Lemma 3.2 that L_C inherits the fibre bundle structure

$$\mathbb{T}^{2k} \to L_C \to \Delta B$$

from the fibred coisotropic C.

By assigning L_C to C the apparatus of Lagrangian Floer theory becomes available to study C. Lagrangian Floer theory can be regarded as a quantum deformation of the classical Morse theory of a Lagrangian. The Lagrangian quantum homology theory defined by Biran and Cornea in [BC07], makes this idea explicit: the vector space underlying the *pearl complex* is still generated by the critical points of a Morse function on the Lagrangian. The differential and the product structure on the pearl complex can be decomposed into a Morse (the classical) part and a Floer (the quantum) part. The quantum part of the differential counts configurations of pseudoholomorphic discs, which are arranged along Morse flow lines like pearls along a string (see Definition 4.12). These are called the *pearly differential trajectories*. The quantum product counts configurations of pseudoholomorphic discs arranged like pearls on the letter 'Y' (see Defintion 4.14). These are called *pearly product trajectories*. I call the collection of pearly product trajectories and pearly differential trajectories, *pearly trajectories*. See [BC09] for an overview of the theory developed by Biran and Cornea.

In oder to prove Theorem 4.1, I adapt the construction of the pearl complex to make it incorporate the fibre bundle structure of L_C. To achieve this I construct in Subsection 4.2.2 a natural class of Morse functions f and almost gradient vector fields Z defined on L_C in the following way: define a Morse function f_B on B and lift it to a Morse function f on L_C by using perturbations of a small Morse function f_F on the typical fibre F. By allowing *almost gradient vector fields* one can ensure that Morse flow-lines of f project to Morse flow lines of f_B. I call such pairs *almost fibred pairs* (see Definition 4.6). With these choices, the critical points x of f are filtered according to the Morse index $|\pi_B(x)|$ of their projection to the symplectic quotient B. Assume for simplicity $f_B(\pi_B(x)) = i$ for all critical points x such $|\pi_B(x)| = i$, i.e. f_B is self indexing. Then L_C can be partitioned into super- and sublevel sets of a fixed value of f_B

$$L_C = \{x \in L_C | f_B(\pi_B(x)) < 1\} \cup \{x \in L_C | f_B(\pi_B(x)) \geq 1\}$$

I learned about the construction of almost fibred Morse functions from Alex Oancea's thesis [OAN03], where this filtration is used to define the Leray-Serre spectral sequence in a Morse theoretic setting.

Next, I quickly recall the definition of the pearl complex and the algebraic structures defined on it in Section 4.3. I then define the *almost fibred pearl complex* as the pearl complex associated to an almost fibred Morse complex in Section 4.3 (see Defintion 4.18). I then explain how the almost fibred pearl complex incorporates some of the fibre bundle structure of L_C in Notation 4.19, Lemma 4.20 and Definition 4.22. This concludes the adaptation of the pearl complex to the fibre bundle structure on L_C.

To prove the existence of a pearly trajectory P with the Property (P) from the assertion of Theorem 4.1 one now uses the algebraic structures on the pearl complex. More precisely one proceeds as follows: choose an almost fibred pair (f, Z) such that the unique minimum x projects to b. The pearl complex is generated as a vector space by Morse critical points, thus the existence of a pearly trajectory ending in the minimum x follows almost immediately from the displaceability of L_C. I prove this in Lemma 4.23 below. To prove Theorem 4.1 one needs to exclude the possibility that all pearly trajectories emanate from critical points y in the fibre above the minimum and are entirely contained in the fibre over the minimum.

Observe that the fibre over the minimum is a $2k$-dimensional torus, $\mathbb{T}^{2k}_x$, by Proposition 2.22. Thus if the Floer part of the differential on the pearl complex decreases the Morse degree of a critical point by at least $2k+2$, there cannot exist any pearly trajectory ending in the minimum x and emanating from a critical point y in the fibre $\mathbb{T}^{2k}_x$ for degree reasons. Thus if one makes this high minimal Maslov assumption Theorem 4.1 is not that hard to prove.

However, the assumption of the Theorem, $N_{L_C} \geq 3$, is independent of the codimension of the coisotropic. One achieves this improvement by the following observation: If there exists a pearly trajectory emanating from a critical point which is not contained in the fibre over the minimum, the theorem follows. If not, all pearly trajectories ending in the minimum emanate from critical points y which are contained in the fibre $\mathbb{T}^{2k}_x$ over the minimum x. Every critical point y of Morse index at least one in the cochain complex of the torus $\mathbb{T}^{2k}_x$ can be generated as sums of Morse cup products of finite linear combinations of critical points $x_1, \dots, x_K$ which are all of degree one for some $K \in \mathbb{Z}_{\geq 0}$. One then considers the quantum deformation of this Morse cup product. This quantum product of $x_1, \dots x_K$ results in a collection y' of critical points in the fibre $\mathbb{T}^{2k}_x$ above the minimum and a collection of critical points y'' which all satisfy $|\pi_B(y'')| \geq 1$. The quantum deformation of the Morse cup product satisfies a Leibnitz rule with respect to the full quantum differential on the almost fibred pearl complex. The critical points $x_1, \dots x_K$ which were used to generate y have Morse index one, and thus, by applying the Leibnitz rule, the assumption $N_L \geq 3$ is now sufficient to eliminate contributions to the Floer differential coming from the collection of points y' above the minimum. By a priori choosing a perfect Morse function on the torus fibre one can then show that the remaining terms $y'' \in f_B^{-1}([1, \dim L])$ in the quantum product of $x_1, \dots x_K$ give rise to a pearly product trajectory with the property (P) from the assertion of Theorem 4.1.

The chapter is structured as follows. In Section 4.2 I explain the construction of an *almost fibred Morse complex* associated to a of a fibre bundle via *almost fibred* pairs of Morse functions and *almost gradient vector fields*. In Section 4.3 I quickly review the construction of the *pearl complex* and explain its adaptation to *almost fibred Morse complexes* resulting in the construction of an *almost fibred pearl complex*. With this in place I carry out the proof of Theorem 4.1 in Section 4.4.

4.2 The Morse complex of an almost fibred Morse function

4.2.1 The Morse complex

To achieve transversality of the moduli spaces involved in the construction of the Morse cohomology ring on needs to allow for certain perturbation data. I will work with a single Morse function and allow for varying almost gradient vector fields. This approach has two main advantages for the proof of Theorem 4.1. The algebraic structures on the pearl complex are defined as counts of elements in moduli spaces associated to pearly trajectories (see Definition 4.13 and 4.15 below or Section 3 of

[BC07] for details). The regularity of these moduli spaces relies on the perturbation data for the Morse complex. If one uses a single Morse function and allows for varying almost gradient vector fields, the critical points of the Morse function f, which generate the pearl complexes associated to sets of perturbation data, remain unchanged under these perturbations. This makes the main argument in the proof of Theorem 4.1 easier to phrase and prove. The second advantage is that one avoids having to to deal with derivatives of cut-off functions in the construction of an almost fibred Morse complex using almost fibred Morse pairs, which also simplifies this argument.

Many thanks to Paul Biran for pointing out the advantages of using almost gradient vector fields in this context to me, an insight that was presented to him by Octav Cornea. See [BK13].

Definition 4.2 (Almost gradient vector field)**.**
Given a Morse function f on a manifold L, a vector field Z on L is *almost gradient* for f if

1. $\mathcal{L}_Z(f) = \iota(Z)df > 0$ throughout the complement of the set of critical points of f.

2. For every critical point x of f there exists a Riemannian metric ρ and a neighbourhood U_x such that $Z = +\nabla_\rho f$ throughout U_x.

I denote a Morse function f and an almost gradient vector field Z for f by (f, Z) and call this an *almost gradient pair*.

Without loss of generality, assume from now on that all the almost gradient pairs (f, Z) used in the constructions satisfy the Morse-Smale condition.

Denote by ϕ_Z^t the flow of Z. For critical point x of f define the *forward (or positive or* *stable*) *manifold*:

$$W^{\leq}(x) = \{q \in L| \lim_{t\to+\infty} \phi^t_Z(q) = x\} \tag{4.1}$$

and the *backward* (*or negative or unstable*) *manifold* of the positive gradient flow as

$$W^{\geq}(x) = \{q \in L| \lim_{t\to-\infty} \phi^t_Z(q) = x\}.$$

The reason for using "$\leq$" and "$\geq$" in this notation will become clear in Remark 4.7. Moreover this notation behaves intuitively when converting from positive almost gradient flows to negative almost gradient flows or passing from (Morse) cohomology to homology:

$$\begin{aligned} W^{\leq}(x) &= W^{\geq}_{-}(x) \\ W^{\geq}_{-}(x) &= W^{\leq}(x), \end{aligned} \tag{4.2}$$

where $W^{\leq}_{-}(x)$ and $W^{\geq}_{-}(x)$ denote the forward and backward manifolds of x with respect to the negative almost gradient flow respectively. The Morse index $|x|$ of a critical point x satisfies

$$|x| := \mathrm{ind}_{\mathrm{Morse}}(f, Z, L; x) = \dim W^{\leq}(x),$$

where $\dim W^{\leq}(x)$ denotes the dimension of $W^{\leq}(x)$ as a manifold.

For the purposes of this thesis it will be sufficient to work with $\mathbb{Z}_2$ coefficients. Therefore set

$$\mathbb{K} = \mathbb{Z}_2.$$

Define

$$C^p_{f,Z}(L) = \bigoplus_{\substack{x\in \mathrm{Crit} f \\ |x|=p}} \mathbb{K}\langle x\rangle \tag{4.3}$$

as the free $\mathbb{K}$-module generated by the finite set of critical points $\mathrm{Crit}_p(f)$ of critical points (f, Z) on L of Morse index p. Since $\mathbb{K}$ is a field, this is nothing but a finite dimensional vector space with basis $\text{Crit}_p(f)$ over $\mathbb{Z}_2$. Viewing C^p as a module will be relevant for the definition of the pearl complex.

In abuse of notation, I drop L from the notation and abbreviate $C^p_{f,Z}(L)$ to $C^p_{f,Z}$. I now recall the definitions of the algebraic structures on the Morse cochain complex.

Definition 4.3 (Morse differential)**.**
For every $p \in \mathbb{Z}_{\geq 0}$. The *Morse differential* d_M counts almost gradient flow lines between critical points of index difference one:

$$
\begin{aligned}
&d_M : C^p_{f,Z} \to C^{p+1}_{f,Z} \\
&d_M(y) = \sum_{\substack{x \in \text{Crit} f \\ |x|=|y|+1=p}} m(y,x)\langle x\rangle,
\end{aligned}
\tag{4.4}
$$

where

$$
m(y,x) := \#_{\mathbb{K}}\{W^{\geq}(y) \cap W^{\leq}(x)\}.
$$

is the count in $\mathbb{K}$ of points in the intersection of the backward manifold of y with the forward manifold of x.

Definition 4.4 (Morse product)**.**
For every pair $(p, q) \in \{0, \ldots \dim(L)\} \times \{0, \ldots \dim(L)\}$. The *Morse cup (or star) product* $\star_0$ is a binary operation:

$$
\star_0 : C^p_{f,Z} \otimes C^q_{f,Z'} \to C^{p+q}_{f,Z''},
$$

where one needs to work with at least two different almost gradient vector fields to ensure transversality of the relevant moduli spaces. It is the Morse theoretic interpretation of the cup product. One may as well work with three different almost gradient vector fields as in [BC09] or with three different Morse functions and their gradients as in [Fuk93]. The notation $\star_0$, which I use here originates in denoting the quantum product on the pearl complex by "$\star$". See Equation 4.17 below.

Let $|x_1| = p$ and $|x_2| = q$. Then the $\star_0$-product is defined as:

$$x_1 \star_0 x_2 := x_1 \cup x_2 = \sum_{\substack{z \in \mathrm{Crit} f \\ |z| = p+q}} n(x_1, x_2, z) \langle z \rangle, \tag{4.5}$$

where

$$n(x_1, x_2, z) := \#_{\mathbb{K}} \{ W^{\geq}(x_1) \cap W^{\geq}(x_2) \cap W^{\leq}(z) \}.$$

is the count in $\mathbb{K}$ of points in the triple intersection of the backward manifold of x and y respectively with the forward manifold of z.

Assume from now on that L is compact and connected. Choose a Morse function with an unique minimum x_{min}. It is almost immediate from the definition of the product that x_{min} represents the identity with respect to the product $\star_0$.

Definition 4.5 (Morse complex and Morse cohomology)**.**
Denote by

$$C^*_{f,Z} = C^0_{f,Z} \xrightarrow{d_M} C^1_{f,Z} \cdots \xrightarrow{d_M} C^{\dim L}_{f,Z}$$

the Morse cochain complex of an almost gradient pair (f, Z) on L. The Morse cohomology ring $HM^*(L)$ is the cohomology of the cochain complex $(C^*_{f,Z}, d_M)$ with the $\star_0$-product.

4.2.2 Almost fibred Morse functions

Assume that the manifold L has a fibre bundle structure $F \to L \to B$ with closed base B and closed fibre F. Note that this is different from the assumption of L to be fibred as a coisotropic. In the application which leads to the proof of Theorem 4.1 L will be L_C.

I now construct an almost gradient pair (f, Z) on L for which the Morse differential preserves the filtration underlying the Leray-Serre spectral sequence. For more details on this construction see [OAN03]. Recall that I assume that choices of almost gradient pairs satisfy the Morse-Smale condition.

Definition 4.6 (Almost fibred pair)**.**
An *almost fibred* pair (f, Z) is the result of the following construction:

Choose an almost gradient pair (f_B, Z_B) on B. The Morse function $f_B \circ \pi_B$ is Morse-Bott on L. Label the critical points of f_B on B by b_i for $i = 0, \ldots, M$. Choose mutually disjoint neighbourhoods U_i of b_i containing smaller neighbourhoods V_i such that the closure $\bar{V}_i$ of V_i is a proper subset of U_i. Next choose smooth cut-off functions θ_i which are identically 1 near b_i and identically 0 on $U_i \setminus \bar{V}_i$. Choose an almost fibred pair (f_F, Z_F) on the typical fibre F. Without loss of generality assume that f_F is self-indexing. Denote the local trivialisations of the fibre bundle by

$$\Psi_i : \pi_B^{-1}(U_i) :\to U_i \times F$$

and by $\pi_F : U_i \times F \to F$ the obvious projection. Denote a point on L by q and extend Z_F to a vector field Z_i on $\pi_B^{-1}(U_i)$ by:

$$Z_i(q) = Z_F \circ \pi_F \circ \Psi_i(q).$$

We denote the zeros of Z_i by c_{ij} for $j = 0, \ldots, N$. Now define the *almost fibred pair* (f, Z) on L by

$$f = f_B + f_F,$$
$$Z(q) = Z_B \circ \pi_B(q) + \epsilon \sum_{i=0}^{N} \theta_i Z_i(q),$$

for a choice of ϵ small enough to guarantee that no new zeros are introduced.

Set

$$x_{ij} = (b_i, c_{ij}) \quad \text{for all pairs} \quad (i,j) \in \{0, \ldots, M\} \times \{0, \ldots N\}. \tag{4.6}$$

Then x_{ij} are critical points of (f, Z) on L_C. It follows from Propositions 3.3.3 and 3.3.4 in [OAN03] that (f, Z) construced as above is Morse-Smale on L if the almost gradient pairs (f_B, Z_B) in the base and (f_F, Z_F) in the fibre satisfy the Morse-Smale condition. Moreover notice that by construction

$$d\pi_B \circ Z(x) = Z_B(x). \tag{4.7}$$

This implies that trajectories of the flow of Z project to trajectories of the flow of Z_B.

Without loss of generality choose $f_B(\pi_B(x)) = i$ for all such $|\pi_B(x)| = i$ (i.e. f_B is self indexing). Define for every $s \geq 0$ the following sets

$$\begin{aligned} S^s &= \{z \in L_C | f_B(\pi_B(z)) = s\} \\ S^{<s} &= \{z \in L_C | f_B(\pi_B(z)) < s\} \\ S^{\geq s} &= \{z \in L_C | f_B(\pi_B(z)) \geq s\}. \end{aligned}$$

One can then for example partition L into super- and sub-level sets of f_B as follows:

$$\begin{aligned} L &= S^{<s} \cup S^{\geq s} \\ &= \{z \in L_C | f_B(\pi_B(x)) < s\} \cup \{z \in L_C | f_B(\pi_B(x)) \geq s\} \\ &= f_B^{-1}([0, s)) \cup f_B^{-1}([s, dim(L)]). \end{aligned} \tag{4.8}$$

Remark 4.7.

Observe the following consequences of Equation 4.7 and the partition from Equation 4.8: If *a critical point x of f on L lies in S^i, then its backward manifold $W^{\geq}(x)$, is contained in $S^{\geq i}$ and its forward manifold $W^{\leq}(x)$, is contained in $S^{\leq i}$ i.e.*

$$W^{\leq}(x) \subseteq S^{\leq i} \tag{4.9}$$

$$W^{\geq}(x) \subseteq S^{\geq i}. \tag{4.10}$$

Definition 4.8 (Almost fibred Morse complex)**.**
Assume that L has a fibre bundle structure $F \to L \to B$. The Morse cochain complex as describe in Defintion 4.5 of an an almost fibred pair (f, Z) as constructed in Defintion 4.6 is called an *almost fibred Morse cochain complex*.

4.3 The pearl complex of an almost fibred Morse function

I now briefly recall the construction of the pearl complex. Consider a closed, connected Lagrangian submanifold L of a symplectic manifold (W, ω). As pointed out above, I follow the ideas of Biran and Cornea presented in [BC09]. My conventions differ slightly from theirs, since I want to formulate the results purely in terms of cohomology, for a "how to convert between conventions" see Equation 4.2.

Denote by $(D, \partial D)$ the closed unit disc in $\mathbb{C}$. Given a map

$$u : (D, \partial D) \to (W, L),$$

denote by $E_\omega(u)$ the symplectic energy of u and by $\mu(u)$ the Maslov index of u. Both maps descend to homomorphisms on $H_2^D(W, L) \subset H_2(W, L)$, the image of the Hurewicz homomorphism $h : \pi_2(W, L) \to H_2(W, L)$.

Definition 4.9 (Monotone Lagrangian)**.**

A Lagrangian L is monotone if there exists a positive real number $\eta > 0$

$$E_\omega(A) = \eta \cdot \mu(A) \quad \forall A \in H_2^D(W, L).$$

Definition 4.10 (Minimal Maslov number of a montone Lagrangian)**.**
Denote by

$$N_L = \min_{A \in H_2^D(M,L)} \mu(A) > 0 \tag{4.11}$$

the *minimal Maslov number* of a monotone Lagrangian L.

Assume from now on that $N_L \geq 2$. Denote by

$$\Lambda = \mathbb{K}[T, T^{-1}]$$

the ring of Laurent polynomials in the formal variable T. Set the degree of T to "$+N_L$". Consider the Morse cochain complex $C^*_{f,Z}$ of L introduced in Defintion 4.5. The module underlying the pearl complex is given by

$$C^* = C^*_{f,Z} \otimes_{\mathbb{K}} \Lambda. \tag{4.12}$$

Here C^* is defined as the tensor product over $\mathbb{K}$ of the $\mathbb{K}$-cochain module $C^*_{f,Z}$ with the ring Λ viewed as a module over $\mathbb{K}$. This is done by including $\mathbb{K}$ as $\mathbb{K} \cdot T^0$ into Λ. For each fixed $p \in \mathbb{Z}$ this means:

$$C^p = \bigoplus_{k \in \mathbb{Z}} C^{p-kN_L}_{f,Z} \otimes_{\mathbb{K}} T^k, \tag{4.13}$$

where T^k denotes monomials of degree $k \in \mathbb{Z}$ in the formal variable T. Since $\mathbb{K}$ is a field C^* is noting but the tensor product of the finitely dimensional $\mathbb{Z}_2$-algebra $C^*_{f,Z}$ with the $\mathbb{Z}_2$-algebra $\mathbb{Z}_2[T, T^{-1}]$.

The grading of C^* is defined to be the sum of the gradings on C_f^* and on Λ. By denoting this differential graded algebra simply by C^*, I am again abusing notation for the sake of brevity and readability.

Notice that for each fixed $p \in \mathbb{Z}$, the sum in Equation 4.13 is finite, since

$$\begin{aligned} C_{f,Z}^{p-k_{\min}N_L-1} &= 0 \quad \text{for } k_{\min} = \frac{p}{N_L} \ , \\ C_{f,Z}^{p-k_{\max}N_L+1} &= 0 \quad \text{for } k_{\max} = \frac{p - \dim L}{N_L} . \end{aligned}$$

If one fixes $k \in \mathbb{Z}$ one obtains $C_{f,Z}^{*-kN_L} \otimes T^k$ and thus a copy of the Morse cochain complex $C_{f,Z}^*$.

Example 4.11 (Pearl complex in four dimensions with $N_L = 2$)**.**
Assume L is 4 dimensional and $N_L = 2$. I denote below some non-zero parts of the complex C^*.

$$\begin{aligned} C^2 &= C_f^4 \otimes T^{-1} \oplus C_f^2 \otimes T^0 \oplus C_f^0 \otimes T^1 \\ C^1 &= C_f^3 \otimes T^{-1} \oplus C_f^1 \otimes T^0 \\ C^0 &= C_f^4 \otimes T^{-2} \oplus C_f^2 \otimes T^{-1} \oplus C_f^0 \otimes T^0 \\ C^{-1} &= C_f^3 \otimes T^{-2} \oplus C_f^1 \otimes T^{-1} \\ C^{-2} &= C_f^4 \otimes T^{-3} \oplus C_f^2 \otimes T^{-2} \oplus C_f^0 \otimes T^{-1} \end{aligned}$$

Similarly to the construction of the algebraic structures for the Morse differential graded algebra a generic choice of perturbation data is necessary to guarantee transversality of moduli spaces used to define the algebraic structures on the (almost fibred) pearl complex. Recall that an almost complex structure on a symplectic manifold (W, ω) is an endomorphism J of the tangent bundle such that $J^2 = -id$. To define the pearl complex, choose a generic ω-compatible almost complex structure J on W. I continue to work with a single Morse function f and allow varying almost gradient vector fields Z, which all satisfy the Morse-Smale condition as auxiliary data. The reference for this approach is [BK13]. I remark that for the definition of the quantum product it is sufficient to work with a fixed Morse function f and two different gradient like vector fields $Z \neq Z'$ at the respective entry flow lines of the core disc. This is explained in the proof of Lemma 5.2.2 in [BC07]. For the purposes of this thesis one may as well work with three distinct almost gradient vector fields.

I now recall the definition of moduli spaces used to define the algebraic structures on the pearl complex. The differential d of the pearl complex is a quantisation of the Morse differential. Its classical part agrees with the Morse differential d_M, its quantum (or Floer) part d_F counts the pearly configurations described below:

Definition 4.12 (Pearly differential trajectory).
Given $y, x \in L$ and $0 \neq A \in H_2^D(M, L)$. Let $l \geq 1 \in \mathbb{Z}$. Consider a sequence $(u_1, \ldots, u_l)$. A *pearly differential trajectory* of length l from y to x is the following configuration:

(PD1) For each $i \in \{1, \ldots, l\}$ the J-holomorphic disc $u_i : (D, \partial D) \to (W, L)$ is non-constant.

(PD2) $[u_1] + \cdots + [u_l] = A$.

(PD3) There exists a $t^- \in [-\infty, 0)$ such that $\phi_Z^{t^-}(u_1(-1)) = y$.

(PD4) For every $1 \leq i \leq l-1$ there exists a $t^i \in (0, \infty)$ such that $\phi_Z^{t^i}(u_i(1)) = u_{i+1}(-1)$.

(PD5) There exists a $t^+ \in (0, +\infty]$ such that $\phi_Z^{t^+}(u_l(1)) = x$.

Denote by

$$\mathcal{P}_{prl} := \mathcal{P}_{prl}(y, x; A; f, Z, J)$$

the moduli space of all possible configurations of all possible lengths $l \geq 1$ described in Definition 4.12. If $A = 0$ define the space

$$\mathcal{P}_{prl}(y, x; 0; f, Z, J)$$

to be the space of unparametrized flowlines of the flow ϕ_Z^t of the almost gradient vector field Z from y to x.

If $A \neq 0$ and y and x are critical points of f, then conditions (PD 3) and (PD 5) become $u_1(-1) \in W^{\geq}(y)$ and $u_l(1) \in W^{\leq}(x)$ respectively. In this case the moduli space $\mathcal{P}_{prl}(y, x; A; f, Z, J)$ can be used to define the *pearly differential* in the following way:

Definition 4.13 (Differential of the pearl complex)**.**

$$\begin{aligned} &d : C^* \to C^{*+1} \\ &d(y) := \sum_{x,A} \#_{\mathbb{K}}(\mathcal{P}_{prl}(y, x; A; f, Z, J))\langle x\rangle \otimes T^{\frac{\mu(A)}{N_L}} \end{aligned} \tag{4.14}$$

where the sum runs over all combinations of x and A such that the moduli space $\mathcal{P}_{prl}(y, x; A; f, Z, J)$ is zero dimensional. See Section 5.1 of [BC07] for more details.

The terms in the pearly differential $d : C^* \to C^{*+1}$ can be grouped as:

$$\begin{aligned} &d = \partial_0 \otimes T^0 + d_F \quad \text{where } \partial_0 = d_M \quad \text{and} \\ &d_F = \partial_1 \otimes \cdot T^1 + \cdots + \partial_m \otimes \cdot T^m + \ldots + \partial_{\frac{(\dim L+1)}{N_L}} \otimes \cdot T^{\frac{(\dim L+1)}{N_L}} \cdot \end{aligned} \tag{4.15}$$

Here

$$\partial_m : C^*_{f,Z} \to C^{*-mN_L+1}_{f,Z} \tag{4.16}$$

is the m-th quantum correction term of d.

To define the $\star$ product on the pearl complex we consider the following configurations:

Definition 4.14 (Pearly product trajectory)**.**
Given $x_1, x_2, y \in \mathrm{Crit} f$ and $0 \neq A \in H_2^D(M, L)$. Consider a tuple $(\tilde{u}, \tilde{u}', \tilde{u}'', v)$. A *pearly product trajectory* from x_1 and x_2 to y is the following configuration:

(PP1) $v : (D, \partial D) \to (W, L)$ is a J-holomorphic disc, which is allowed to be constant.

(PP2) Set $z_1 = v(e^{\frac{-2\pi i}{3}})$, $z_2 = v(e^{\frac{2\pi i}{3}})$ and $z_3 = v(1)$. Let $B_1, B_2, B_3 \in H_2^D(W, L)$.

$$\tilde{u} \in \mathcal{P}_{prl}(x_1, z_1; B_1; f, Z_1, J)$$
$$\tilde{u}' \in \mathcal{P}_{prl}(x_2, z_2; B_2; f, Z_2, J)$$
$$\tilde{u}'' \in \mathcal{P}_{prl}(z_3, y; B_3; f, Z_3, J)$$

(PP3) $B_1 + B_2 + B_3 + [v] = A$.

Again for generic choices of auxiliary data the moduli space

$$\mathcal{P}_{prod} := \mathcal{P}_{prod}(x_1, x_2, y; A; f, Z_1, Z_2, Z_3, J)$$

of all configurations described in Definition 4.14 above can be used to define the $\star$ product:

Definition 4.15 (Product on the pearl complex)**.**
The $\star$-product on C is the binary operation

$$\begin{aligned} \star : & C^i \otimes C^j \longrightarrow C^{i+j} \\ & x_1 \star x_2 := \sum_{y,A} \#_{\mathbb{K}}(\mathcal{P}_{prod}(x_1, x_2, y; A; f, Z_1, Z_2, Z_3, J))\langle y \rangle \otimes T^{\frac{\mu(A)}{N_L}}, \end{aligned} \tag{4.17}$$

where the sum runs over all y and A such that $\mathcal{P}_{prod}$ is zero dimensional.

Like the pearly differential the $\star$-product is a quantisation of the Morse cup product. For $n \geq 0$ its quantisation is given by:

$$\begin{aligned} \star &= \star_0 \otimes T^0 + \star_F \\ \star_F &= \star_1 \otimes \cdot T^1 + \cdots + \star_n \otimes \cdot T^n + \cdots + \star_{\frac{2\dim(L)}{N_L}} \otimes \cdot T^{\frac{2\dim(L)}{N_L}}. \end{aligned} \tag{4.18}$$

Here the n-th quantum correction term is given by:

$$\star_n : C^i_{f,Z} \otimes C^j_{f,Z'} \longrightarrow C^{i+j-nN_L}_{f,Z''} \tag{4.19}$$

The zeroth term $\star_0$ of the $\star$-product coincides with the Morse cup product.

A key fact which we will use is that $\star$ satisfies a Leibnitz rule with respect to d:

$$d(x \star y) = dx \star y + x \star dy. \tag{4.20}$$

This is proved in Proposition 5.2.1 of [BC07]. Care has to be taken since this identity does in general not hold for individual terms if m and n are not both zero:

$$\partial_m(x \star_n y) \neq \partial_m x \star_n y + x \star_n \partial_m x.$$

Definition 4.16 (Energy of a pearly trajectory)**.**
Given a pearly differential or a pearly product trajectory P of homology class A (see condition (PD2) of Definition 4.12 or condition (PP3) of 4.14 respectively), the energy of P is defined as the symplectic energy

$$E_w(u) = \int_D u^*\omega$$

of a disc $u : (D, \partial) \to (W, L)$ such that $[u] = A \in H^2_D(W, L)$. Equivalently,

$$E_w(u) = \sum_{i=1}^{l} \int_D u_i^* \omega$$

for discs $u_i : (D, \partial) \to (W, L)$ such that $[u_1] + \cdots + [u_l] = [A]$.

Definition 4.17 (Pearl complex and quantum cohomology)**.**
The *pearl complex* of (f, Z) on L is the cochain complex (C^*, d), where $C^* = C^*_{f,Z} \otimes_{\mathbb{K}} \Lambda$ and d is as in Defintion 4.13. The $\star$-product give C^* the structure of a (generally) non-commutative, non-associative algebra. The *Lagrangian Quantum cohomology* $QH^*(L)$ is the cohomology of (C^*, d).

Notice also that I have dropped the almost complex structure J from the notation although the algebraic structures defined on C^* depend on it. The Lagrangian Quantum cohomology is independent of the choices of (f, Z) and J, by assertion (i) of Theorem 2.1.1 in [BC07]. By assertion (v) of the same Theorem the Lagrangian Quantum cohomology $QH^*(L)$ is isomorphic to the self-Floer cohomology $HF^*(L, L)$, via the PSS map.

Definition 4.18 (Almost fibred pearl complex)**.**
Assume L has a fibre bundle structure $F \to L \to B$. The *almost fibred pearl complex* C^* of L is the pearl complex $C^*_{f,Z} \otimes_{\mathbb{K}} \Lambda, d$ of an almost fibred pair (f, Z) on L.

Recall that $\Lambda = \mathbb{K}[T, T^{-1}]$. A cochain c of an almost fibred pearl complex C^* is a Λ-linear combination $c = \sum_k \lambda_k x_k$, where the x_k are Morse cochains of $C^*_{f,Z}$ and λ_k are Laurent polynomials in the formal variable T. Notice that c is not necessarily a pure tensor. Since the proof of Theorem 4.1 relies on the partition of L into super- and sub-level sets of f_B as described in Equation 4.8 introduce the following notations:

Notation 4.19.

For a cochain

$$c = \sum_{k=1}^{K} \lambda_k x_k \in C^* \tag{4.21}$$

- Write $c \in S^{\leq i}$ if all critical points x_k contributing nontrivially to the cochain c are contained in $S^{\leq i}$ i.e. $\{x_1, \ldots, x_K\} \subset S^{\leq i}$.

- Write $c \in S^{\geq i}$ if all critical points x_k contributing nontrivially to the cochain c are contained in $S^{\geq i}$ i.e. $\{x_1, \ldots, x_K\} \subset S^{\geq i}$.

- Write $c \in S^{i}$ if all critical points x_k contributing nontrivially to the cochain c are contained in S^{i} i.e. $\{x_1, \ldots, x_K\} \subset S^{i}$.

The following Lemma will be important in the proof of Proposition 4.24.

Lemma 4.20.

Assume that the pearl complex C^* is almost fibred. Let c and $c_1, \ldots, c_k$ be cochains contained in S^i. Then

(D) $d(c) = e + e'$, *where e is a cochain in $S^{\leq i}$ and z' is a cochain in $S^{\geq i+1}$.*

(S) $c_1 \star \cdots \star c_k = e + e'$, *where e is a cochain in $S^{\leq i}$ and e' is a cochain in $S^{\geq i+1}$.*

Proof. This immediately follows from the fact that $S^{\leq i} \cup S^{\geq i+1}$ is a partition of the set of critical points of the almost fibrered Morse function f on L (see Equation 4.8). $\square$

Remark 4.21.

There exists a decreasing filtration on the almost fibred Morse complex $C^*_{f,Z}$*. For each* i *and* $p \in \mathbb{Z}_{\geq 0}$ *the* i*-th filtration of* $C^p_{f,Z}$ *is given by*

$$\mathcal{F}^i(C^p_{f,Z}) = \{x \in C^p_{f,Z} \; | x \in S^{\geq i}\}. \tag{4.22}$$

This filtration is preserved by both the differential d_M *and the* $\star_0$*-product by the same reasoning as in the proof of Lemma 4.20 above. It can be used to define the Leray-Serre spectral sequence. Notice however that the Floer part of the differential* d_F *and the Floer part of the product* $\star_F$ *do not preserve this filtration. One can define a filtration on an almost fibred pearl complex, which takes the fibre bundle structure into account and is preserved by the Floer differential and product. The idea is to filter by distance to a fibre. Given a cochain* $c \in S^i$*,* $\partial_M(c)$ *is contained in* $S^{\geq i-mN_L+1} \cap S^{\leq i+1}$ *and similarly for the product. Despite being interesting, I do not pursue this idea in this thesis, since the application, computing* QH^* *via a spectral sequence, is irrelevant for the proof of Theorem 1.6. Very recently, Schultz defined a spectral sequence in a similar context in [Sch17]. It would be very interesting to further investigate the relation of these spectral sequences and to compute* $QH^*(L_C, L_C)$ *via a spectral sequence.*

Given the Lemma above it makes sense to make the following definition

Definition 4.22 (fibrewise generation)**.**

Assume that the pearl complex C^* is almost fibred.

- Given a cochain $c \in S^i$, say that e is *fibrewise generated* by a set of critical points $G \subset S^i$ via the $\star_0$-product if there exist critical points $x^1_1, \dots x^1_{L_1}, \dots, x^K_1, \dots, x^K_{L_K} \in C^*_{f,Z}$ such that

$$\sum_{k=1}^{K} \lambda^k_1 x^k_1 \star_0 \cdots \star_0 \lambda^k_{L_k} x^k_{L_k} = c + c'$$

for $k, l, K, L_1, \ldots L_K, \geq 1 \in \mathbb{Z}$, $\lambda_l^k \in \Lambda$ and a cochain $c' \in S^{\geq i+1}$.

This means that there exists Laurent polynomials $\lambda^1_{1,\ldots,l_k}, \ldots, \lambda^K_{1,\ldots,l_K} \in \Lambda$ and critical points $x^1_1, \ldots, x^K_{l_K} \in C^*_{f,Z}$ such that

$$\sum_{k=1}^{K} \lambda^k_{1,\ldots,l_k}(x^k_1 \star_0 \cdots \star_0 x^k_{l_k}) = c + c'.$$

This concludes the adaptation of the pearl complex to the present situation. In the following section I apply this algebraic machinery to prove Theorem 4.1.

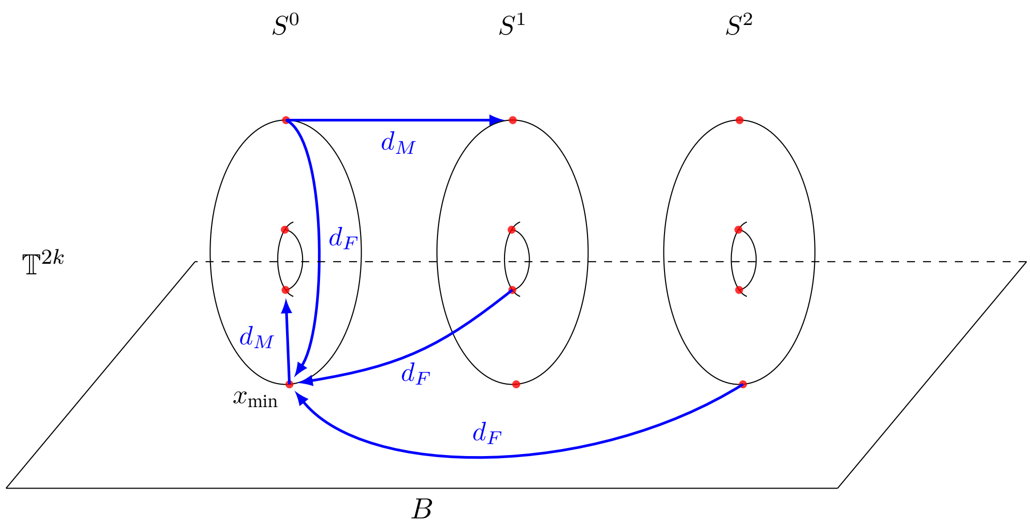


Figure 4.1: A picture of parts of the almost fibred pearl complex of L_C for a fibred, stable coisotropic C

4.4 Proof of Theorem 4.1

To prove the theorem I first prove Lemma 4.23 below, then Proposition 4.24 and finally the theorem by applying Lemma 4.23 and Proposition 4.24.

Assume from now on that L is a closed, connected, monotone Lagrangian submanifold of a symplectic manifold (W, ω), equipped with a generic ω-compatible almost complex structure J. From now on also assume that L is the total space of a fibre bundle $F \to L \to B$. Recall the decomposition of L into sub- and superlevel sets of f_B descibed in Equation 4.8. In particular the unique minimum x_{min} of a Morse function gives rise to a cochain

$$x_{\text{min}} = x_{\text{min}} \otimes T^0 \tag{4.23}$$

in the almost fibred pearl complex which is contained in S^0. It will be convenient in the proof to introduce the following projection:

$$pr_0 : C^* \to C^0_{f,Z} \otimes T^0. \tag{4.24}$$

The Lemma below also holds in the case where L is not fibred.

Lemma 4.23.
Let L be a closed, connected, monotone Lagrangian submanifold. Denote by x_{min} the unique minimum of f on L. Then $QH^*(L) = 0$ if and only if there exists a cochain $c \in C^*$ such that $d_F(c) = x_{\text{min}}$.

Proof. The cochain x_{min} is a Floer cocycle. To see this note that $d_M(x_{min}) = 0$. By Equation 4.15, higher differentials $\partial_{\geq 1}$ lower the Morse index of x_{min} by at least $N_L - 1 \geq 1$ by the assumption that the minimal Maslov number is at least two. But x_{min} has minimal Morse index among all critical points so $d(x_{\text{min}}) = 0$.

Assume that $QH^*(L) = 0$. Thus every cocycle of C^* is a coboundary. Thus there exists a primitive c in C^* such that $d(c) = x_{\text{min}}$. Since there are no non-trivial Morse flow lines ending in the minimum, it follows that c satisfies $d_F(c) = x_{\text{min}}$.

Conversely, the cohomology class $[x_{\text{min}}]$ of the cochain x_{min} is the identity in the Quantum cohomology ring $QH^*(L)$. If x_{min} is a coboundary, this implies that $[x_{\text{min}}] = 1 = 0$ in the quantum cohomology ring $QH^*(L)$. Thus $QH^*(L) = 0$. □

Proposition 4.24.

Assume that L is a compact, connected Lagrangian which is the total space of a fibre bundle $F \to L \to B$. Assume that L is monotone with $N_L \geq 2$. Assume also that f_B is self-indexing and that f_F is perfect, where f_B and f_F are the Morse functions from the construction oofthe almost fibred pair (f, Z), which is used to define the almost fibred pearl complex C^*.

If L is displaceable and all cochains c such that

$$pr_0(d(c)) = \kappa x_{\min} \quad \textit{for} \quad \kappa \neq 0 \in \mathbb{K}$$

satisfy:

(F1) *The cochain c is contained in S^0.*

(F2) *The cochain c is fibrewise generated by a set G of critical points as in Definition 4.22 such that all critical points $x \in \{x_1^1, \ldots, x_{L_1}^1, \ldots, x_1^K, \ldots, x_{L_K}^K\}$ are of the same Morse index g and satisfy:*

$$g = |x_l^k| < N_L - 1 \quad , \textit{for all} \quad k, l.$$

Then there exists a pearly product trajectory P containing a critical point y in $S^{\geq 1}$ and ending in the minimum $x_{\min}$ of f.

Proof. By assumption L is displaceable, so Lemma 4.23 and assertion (D) of Lemma 4.20 imply that there exists a cochain $c \in C^*$ such that

$$d(c) = x_{\min} + e,$$

where e is a cochain in $S^{\geq 1}$. Since $pr_0(d(c)) = x_{\min}$, it follows from Assumption (F1) of the proposition that c is contained in S^0. In particular, by the definition of the differential d, there exist critical points $y \in S^0$ such that

$$pr_0(d(\lambda_y y)) = \kappa x_{\min},$$

for $\lambda_y \in \Lambda, \kappa \neq 0 \in \mathbb{K}$. Among all cochains $c \in S^0$ which satisfy $pr_0(d(c)) = \kappa' x_{\min}$ for some $\kappa' \neq 0 \in \mathbb{K}$, choose a critical point $y_{\min}$ of minimal Morse index amongst these i.e.

$$pr_0(d(\lambda_{\min} y_{\min})) = \kappa_1 x_{\min}.$$

for$\lambda_{\min} \in \Lambda$ and $\kappa_1 \neq 0 \in \mathbb{K}$.

By assumption (F2) and Definition 4.22 there exist critical points $x_1^1, \ldots, x_{L_1}^1, \ldots, x_1^K, \ldots x_{L_K}^K \in G \subset S^0$, which are all of the same Morse index g such that

$$\sum_{k=1}^{K} \lambda_{1,\ldots,l_k}^k (x_1^k \star_0 \cdots \star_0 x_{l_k}^k) = \lambda_{\min} y_{\min} + c'$$

where $\lambda_{1,\ldots,l_k}^k$ are Laurent polynomials and c' is a cochain in $S^{\geq 1}$. Consider the quantum deformation of the $\star_0$-product. Since the $\star$-product is not commutative, the product will depend on the chosen order of the critical points in G. By assertion (S) of Lemma 4.20 one may write

$$\sum_{k=1}^{K} c_1^k \star \cdots \star c_{l_k}^k = \sum_{k=1}^{K} c_1^k \star_0 \cdots \star_0 c_{l_k}^k + \sum_{k=1}^{K} c_1^k \star_F \cdots \star_F c_{l_k}^k \tag{4.25}$$

$$= (\lambda_{\min} y_{\min} + c') + (c'' + c'''), \tag{4.26}$$

where $c' \in S^{\geq 1}$ as above and $c'' \in S^0$ and $c''' \in S^{\geq 1}$. Note that the Morse index of every critical point contributing nontrivially to $c'' \in S^0$ is strictly smaller than the Morse index of $y_{\min}$ by Equation 4.19 and therefore c'' cannot sum to zero with $\lambda_{\min} y_{\min}$.

Next apply the differential d to this equation:

$$d(\sum_{k=1}^{K} c_1^k \star \cdots \star c_{l_k}^k) = d(\lambda_{\min} y_{\min} + c' + c'' + c''') \tag{4.27}$$

$$= d(\lambda_{\min} y_{\min}) + d(c'') + d(c' + c'''). \tag{4.28}$$

I claim that $d(\lambda_{\min} y_{\min})$ is the only non-trivial contribution to $C^0_{f,Z} \otimes T^0$. To see this consider

$$\begin{aligned} pr_0\left(\sum_{k=1}^{K} d(c_1^k \star \cdots \star c_{l_k}^k)\right) =& pr_0(d(\lambda_{\min} y_{\min}) + d(c'') + d(c' + c''')) \\ &= \kappa_1 x_{\min} + pr_0(d(c'')) + pr_0(d(c' + c''')) \\ &= \kappa_1 x_{\min}. \end{aligned}$$

The last equality follows from the fact that the cochains c' and c''' are contained in $S^{\geq 1}$ and therefore cannot contribute non-trivially to $C^0_{f,Z} \otimes T^0$ by assumption of the proposition. The Morse index of every critical point contributing nontrivally to $c'' \in S^0$ is strictly smaller than the Morse index of $y_{\min}$ by Equation 4.19 and therefore cannot contribute non-trivally to $C^0_{f,Z} \otimes T^0$ by the minimality of $y_{\min}$.

Next apply the Leibnitz rule:

$$\sum_{k=1}^{K} d(c_1^k \star \cdots \star c_{l_k}^k) = \sum_{k=1}^{K} \sum_{j=1}^{l_k} c_1^k \star \cdots \star d(c_j^k) \star \cdots \star c_{l_k}^k.$$

By assumption (F2) of this proposition the Morse indices of all critical points x in G satisfy $g = |x| < N_L - 1$. Thus all quantum correction terms of the differential d satisfy $|\partial_m x| < 0$ for $m \geq 1$ by Equation 4.16. Hence

$$\sum_{k=1}^{K} d(c_1^k \star \cdots \star c_{l_k}^k) = \sum_{k=1}^{K} \sum_{j=1}^{l_k} c_1^k \star \cdots \star d_M(c_j^k) \star \cdots \star c_{l_k}^k, \tag{4.29}$$

and

$$pr_0\left(\sum_{k=1}^{K} d(c_1^k \star \cdots \star c_{l_k}^k)\right) = \sum_{k=1}^{K}\sum_{j=1}^{l_k} c_1^k \star \cdots \star d_M(c_j^k) \star \cdots \star c_{l_k}^k$$
$$= \kappa_2 x_{\min}$$

for some $\kappa_2 \neq 0$. Thus there exists $1 \leq k_0 \leq K$ and $1 \leq j_0 \leq l_{k_0}$ such that

$$pr_0\left(c_1^{k_0} \star \cdots \star d_M(c_{j_0}^{k_0}) \star \cdots \star c_{l_k}^{k_0}\right) = \kappa_3 x_{\min}$$

for some $\kappa_3 \neq 0$. Hence $d_M(c_{j_0}^{k_0}) \neq 0 \in C^*$. By the assumption that f_F is perfect, this implies that $d_M(c_{j_0}^{k_0})$ is contained in $S^{\geq 1}$. Set

$$c_{\min} = c_{j_0}^{k_0}$$
$$a = c_1^{k_0} \star \cdots \star c_{j_0-1}^{k_0}$$
$$b = c_{j_0+1}^{k_0} \star \cdots \star c_{l_{k_o}}^{k_0}$$

With this notation

$$pr_0\left(c_1^{k_0} \star \cdots \star d_M(c_{j_0}^{k_0}) \star \cdots \star c_{l_k}^{k_0}\right) = pr_0\left((a \star d_M(c_{\min})) \star b\right)$$
$$= \kappa_3 x_{\min}.$$

While the cochains a, b are not necessarily contained in $S^{\geq 1}$ the cochain $d_M(c_{\min})$ is contained in $S^{\geq 1}$. This implies that there exists a pair of pearly product trajectories consisting of

- A pearly product trajectory emanating from a critical point $a_i \in S^0$ contributing to the cochain a and a critical point $y_1 \in S^{\geq 1}$ contributing to the cochain $d_M(c_{\min})$ ending in a (not necessarily critical) point a_i' on L.

- A pearly trajectory emanating from a_i' and a critical point b_i contributing to the cochain b which ends in $x_{\min}$, since the backward manifold $W^{\leq}(x_{\min})$ consists only of $x_{\min}$ itself.

This pearly trajectory P thus connects a critical point $y = y_1$ contained in $S^{\geq 1}$ to the minimum x_{min}. The proposition follows. $\square$

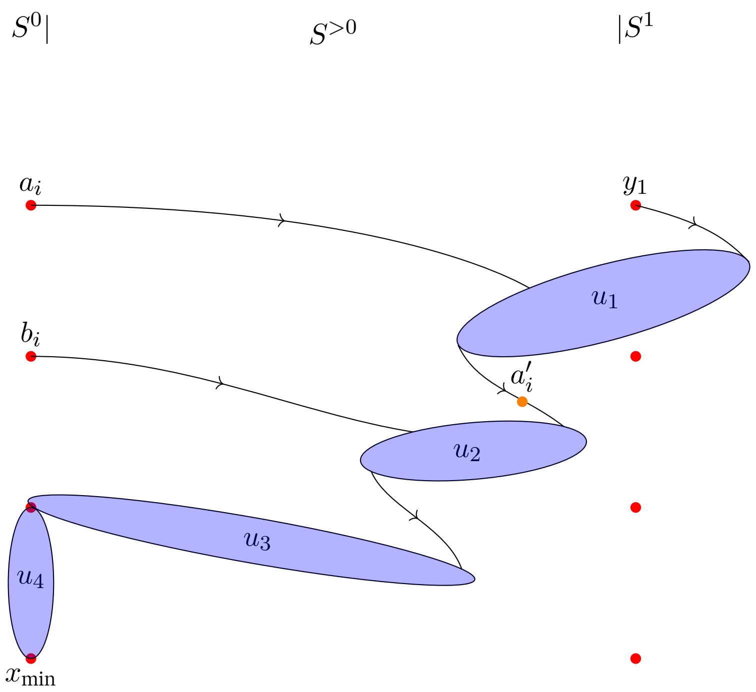


Figure 4.2: A pearly trajectory of the form $(a \star \partial_0 c_{\text{min}}) \star b$ as in the proof of Proposition 4.24

I now prove Theorem 4.1.

Proof of Theorem 4.1. Recall the definition 3.1 of the Lagrangian graph L_C from Section 3.1. By Lemma 3.2 L_C is a Lagrangian submanifold of $W^- \times W^+$ and inherits a fibre bundle structure

$$\mathbb{T}^{2k} \longrightarrow L_C \xrightarrow{\pi_{\Delta B}} \Delta_B$$

from C. Here $\pi_{\Delta B}$ denotes the projection to ΔB.

Choose an almost fibred pair (f, Z) on L_C as described in Definition 4.6. Without loss of generality assume that f_B is self indexing. Choose the fibre component $(f_{\mathbb{T}^{2k}}, Z_{\mathbb{T}^{2k}})$ used to define (f, Z) to be the standard, perfect Morse function on the torus. Moreover since L_C is closed and connected assume f has a unique global minimum x_{min}. Given $b \in B$ one may choose f such that $\pi_{\Delta_B}(x_{\text{min}}) = (b, b)$. The decomposition of L_C described in equation (4.8) gives

$$\begin{aligned} L_C &= S^{<1} \cup S^{\geq 1} \\ &= \{x \in L_C | f_B(\pi_{\Delta_B}(x)) < 1\} \cup \{x \in L_C | f_B(\pi_{\Delta_B}(x)) \geq 1\}. \end{aligned}$$

Notice that all critical points of f in $S^{<1}$ are contained in S^0. Choose a sequence of generic $(J_n)_{n \in \mathbb{N}}$ approaching an arbitrary, but fixed almost complex structure J. Use (f, Z) and J to define the pearl complex $C^*(L_C)$. By assumption of the theorem, L_C is displaceable. Consequently $QH^*(L_C)$ vanishes. By Lemma 4.23 there exists a cochain $c \in C^*$ such that $d(c) = x_{\text{min}}$. Thus there exists at least one pearly trajectory containing a positive, finite number, say K, of non-trivial J-holomorphic discs $u_1, \ldots u_K$. Recall the observation that $W^{\leq}(x_{\text{min}}) = x_{\text{min}}$. Thus the boundary of the K-th, non-trivial J-holomorphic disc contributing to the pearly trajectory passes through x_{min}.

If there exists a cochain $c' \in S^{\geq 1}$ such that $pr_0(d(c')) = \kappa x_{\text{min}}$ for $\kappa \neq 0 \in \mathbb{K}$, this implies that there exists a pearly product trajectory which emanates from a critical point $y' \in S^{\geq 1}$ and ends in x_{min}. By definition of a pearly differential trajectory there are is a positive number, say K' of non-trivial J-holomorphic disc $u_1, \ldots, u_K{}'$ contributing to this pearly trajectory. The minimum x_{min} is contained in $u_{K'}(\partial D)$.

The energy of the pearly differential trajectory is bounded above by $\eta(\dim(L) + 1)N_L$. This follows from the differential degree formula (4.15) and the monotonicity of L_C. The Theorem follows in this case.

If there does not exit a cochain $c' \in S^{\geq 1}$ such that $pr_0(d(c')) = \kappa x_{\text{min}}$ for $\kappa \neq 0 \in \mathbb{K}$, one is in the situation of Proposition 4.24.

I now verify the remaining conditions of Propsition 4.24. The fibre over the minimum is a $2k$ dimensional torus. Recall that I use the standard, perfect Morse function and positive gradient flow to define the almost fibred Morse complex and $\star_0$-product on the fibre. Recall also that every closed form of Morse degree at least one on the torus is generated by sums and products of degree one forms on the torus. Thus every cochain $c \in S^0 \cong \mathbb{T}^{2k}_{x_{\min}}$ such that all critical points contributing non-trivially to c have Morse index at least 1 is fibrewise generated by a set of critical points G as in Definition 4.22. All critical points in G can be chosen to have Morse index $g = 1$. The last condition one needs to check is:

$$g = 1 = |x^k_l| < N_L - 1 \quad \text{for all} \quad k, l.$$

This holds because in the statement of the theorem one assumed $N_L \geq 3$.

Thus all assumptions of Proposition 4.24 are verified. The proposition now implies the existence of a pearly trajectory P with the desired properties. The energy of this pearly product trajectories is bounded from above by $2\eta \dim(L) N_L$ by formula (4.18).

This implies the theorem in this case and thus the proof of Theorem 4.1 is complete.

$\square$

Remark 4.25.
The proof of Theorem 4.1 becomes significantly easier if one assumes $N_L \geq 2k + 2$:

Consider again the decomposition of L_C into $S^{<1}$ and $S^{\geq 1}$. If all holomorphic discs ending in the minimum were entirely contained in $S^{<1}$, there would have to be a critical point of Morse index at least $N_L - 1$ in the torus fibre above the minimum. However, the chain complex of the $2k$-dimensional torus fibre above the minimum is concentrated in degrees 0 to $2k$ thus $N_L \geq 2k + 2$ implies that no such point exists. Thus by the vanishing of Quantum cohomology there must exist a pearly differential trajectory with the desired properties.

Example 4.26.
Consider the product $C = S^{2n-1} \times S^{2m-1} \subset \mathbb{R}^{2n} \times \mathbb{R}^{2m}$ equipped with the standard almost complex structure J_0 and the standard symplectic structure ω_0. I have shown that the minimal Maslov number of $L_{S^{2n-1}}$ is $2n$ in Example 3.6. Thus the

minimal Maslov number $N_{L_{S^{2n-1}\times S^{2m-1}}}$ is equal to the least common multiple of $2m$ and $2n$. Setting $m = n = 2$ one sees that

$$N_{L_{S^3\times S^3}} = lcm(4,4) = 4$$

Thus the assumption $N_{L_{S^3\times S^3}} \geq 3$ is verified but one cannot apply the easier proof from Remark 4.25 since $k = 2$ implies $2k + 2 = 6 > 4$.

Chapter 5

Compactness for pearly trajectories

Theorem 4.1 establishes the existence of a pearly trajectory P emanating from a critical point $y \in f_B^{-1}([1,\infty))$ and ending in the minimum $x \in f_B^{-1}(0)$ of an almost fibred Morse function f on L_C. The main ideas of the proof of Theorem 4.1 were to associate a Lagrangian L_C to C and to then adapt and use techniques from Lagrangian Floer theory. However, Theorem 4.1 asserts nothing about the holomorphic discs contributing to the pearly trajectory P. For example, the interior of these holomorphic discs is not necessarily contained in L_C or even in a neighbourhood of L_C. In order to obtain more information about the holomorphic discs contributing to P, I adapt and apply techniques from symplectic field theory. More precisely, the goal of this chapter is to prove Theorem 1.8 from the Introduction, which I state again below as Theorem 5.1.

Theorem 5.1.
Let C be a fibred, stable coisotropic submanifold of a symplectic manifold (W,ω). Assume that the Lagrangian graph L_C in the product $(W \times W, -\omega \times \omega)$ is monotone and has minimal Maslov number N_{L_C} at least three. Let b be any point in the symplectic quotient B of C.

If L_C is displaceable, then there exist:

(M) *An almost fibred Morse function f on L_C such that the unique global minimum x of f on L_C is contained in $f_B^{-1}(0)$ and projects to $(b,b) \in \Delta B$.*

(E) *A constant $E_0 > 0$, such that for all ω_B-compatible almost complex structures J_B on B, there exists at least one punctured pearly trajectory pP of energy at most E_0 and with the following properties:*

(pP1) *The punctured pearly trajectory pP connects a critical point y of f contained in $f_B^{-1}([1,\infty)$ to the minimum x of f.*

(pP2) *The punctured pearly trajectory pP contains at least one punctured, non-trivial* *holomorphic curve*

$$\tilde{u} : (S, \partial S, j) \to (\tilde{W}_C \times \tilde{W}_C, L_C, -\tilde{J}_C \times \tilde{J}_C)$$

with the following properties:

(S1) *The intersection* $\tilde{u}(\partial S) \cap f_B^{-1}(0)$ *and the intersection* $\tilde{u}(\partial S) \cap f_B^{-1}((0, \infty))$ *is non-empty.*

(S2) *If* $\tilde{u}$ *is unbounded near an interior puncture, then* $\tilde{u}$ *is asymptotic to a cylinder over a generalised Reeb orbit on* C *when approaching the puncture.*

(S3) *All other boundary and interior punctures of* $\tilde{u}$ *are removable.*

Here $(S, \partial S)$ *is a nodal, stable, connected Riemann surface of genus zero with nonempty boundary.*

A *punctured pearly trajectory* is a pearly trajectory in which the domains of the contributing holomorphic discs are allowed to degenerate to *nodal*, connected, *stable*, genus zero Riemann surfaces with nonempty boundary (see Definitions 5.10, 5.11 and 5.12) . The manifold $\tilde{W}_C$ is the *symplectic cobordism* (see Definition 5.4) obtained as the *symplectic completion* of the Bolle neighbourhood of C and diffeomorphic to $\mathbb{R}^k \times C$. The almost complex structure $\tilde{J}_C$ on $\tilde{W}_C$ is the limit of a sequence $(J_{\mathcal{S}}^\tau)_{\tau \geq 0}$ of almost complex structures which is used in a *neck-stretching procedure*. These almost complex structures $J_{\mathcal{S}}^\tau$ are *adjusted* to the stable coisotropic $(C, \mathcal{S})$ and the neck stretching procedure (see Definition 5.5, Section 5.3 and Section 5.5.3).

5.1 Outline of Chapter 5

Roughly speaking, the proof of Theorem 5.1 is a translation of the ideas of the proof of compactness in symplectic field theory from [Bou+03] to the present setting.

Recall the construction of the stable hypersurface in Section 3.2.2. The most important feature of this construction is that there is a one to one correspondence of the set of generalised Reeb trajectories $\mathcal{G}$ on C and the set of Reeb trajectories $\mathcal{R}$ on H_C (see Proposition 3.25). Moreover one may view H_C as the boundary of a Bolle neighbourhood $U \cong B^k_{\epsilon_0} \times C$ where the symplectic form ω_s is given explicitly by Equation 2.6. The coisotropic submanifold C gets embedded into U as $\{0\} \times C$. Thus one can interpret H_C as a stable hypersurface which separates W into *symplectic cobordisms* (see Definition 5.4).

It is a common technique in symplectic and contact topology to "stretch the neck" around a stable hypersurface H in order to obtain information about holomorphic curves in the manifold W (see for example [EGH00], [Bou+03], [CM05] and the references therein). "The neck" refers to a neighbourhood diffeomorphic to $(-\epsilon, \epsilon) \times H$, which gets "stretched" to $\mathbb{R} \times H$. Stretching the neck is also called "splitting" as it results in disjoint, non-compact, symplectic cobordisms. In the present case these disjoint components are $\tilde{W}_C \cong \mathbb{R}^k \times C$, the *symplectic completion* of the Bolle neighbourhood U, $\tilde{W}_H \cong \mathbb{R} \times H_C$, called the *symplectization of* H_C and $\tilde{W}_R$, the symplectic completion of $W \setminus U$. As a result of splitting, a J-holomorphic curve $u : S \to W$ with domain a Riemann surface Swhich satisfies certain assumptions, defines (see again [Bou+03]), a punctured $\tilde{J}_{\mathcal{S}}$-holomorphic curve $\tilde{u}_C : S' \to \tilde{W}_C$, with domain S', which is a nodal Riemann surface.

As alluded to above, the almost complex structure $\tilde{J}_{\mathcal{S}}$ is a limit of a sequence of almost complex structures $J^\tau_{\mathcal{S}}$ for $\tau > 0 \in \mathbb{R}$ on the longer and longer necks $(-\tau, \tau) \times H$. This specific family of almost complex structures plays a key role in obtaining more information about the holomorphic curves via splitting the manifold W. I show in Section 5.3 how to construct such a family of almost complex structures $J_{\mathcal{S}}$ which are adjusted to the stable coisotropic $(C, \mathcal{S})$ and the neck stretching procedure. In particular the $(C, \mathcal{S})$-adjusted almost complex structures $J_{\mathcal{S}}$ are translationally invariant in the normal direction of H_C and render projection to B holomorphic. The correspondence of the generalised Reeb trajectories $\mathcal{G}$ on C and the Reeb trajectories $\mathcal{R}$ on H_C, now implies that if the $\mathbb{R}^k$ component of $\tilde{u}_C$ is unbounded near a puncture, then $\tilde{u}_C$ is asymptotic to a cylinder over a generalised

Reeb orbit on C. I explain this in Proposition 5.14 below. The main assumptions on the holomorphic curve u which are needed to ensure this behaviour are finiteness of energy (see Section 5.5.2), and that, if the domain of u has non-empty boundary ∂S, u maps the boundary to a Lagrangian submanifold L of W i.e. $u(\partial S) \subset L$.

To prove Theorem 5.1 one uses this apparatus as follows: Theorem 4.1 implies that there exists a pearly trajectory, which, by definition of a pearly trajectory, contains at least one non-trivial $(-J \times J)$-holomorphic disc

$$u = (u^-, u^+) : (D, \partial D) \longrightarrow (W^- \times W^+, L_C).$$

The component u^- mapping to the first factor of $W^- \times W^+$ satisfies $u^-(\partial D) \subset C$. If the codimension of C is not n, C is not Lagrangian, and thus the results from [Bou+03] do not apply directly to u^- and likewise do not apply directly to u^+. However, $u = (u^-, u^+)$ does satisfy a Lagrangian boundary condition in the product manifold $W^- \times W^+$. Since L_C is a subset of $C \times C$ it is embedded as a subset of of $\psi(\{0\} \times C \times \{0\} \times C)$ in $W \times W$. Hence a product neighbourhood of L_C in $W^- \times W^+$ is given by $U \times U$. Then "splitting" $W^- \times W^+$ along $H_C \times H_C$ by splitting both factors W along H_C using family of almost complex structures $-J_S^\tau \times J_S^\tau$, gives rise to a sequence $(P_n)_{n \geq 0}$ of pearly trajectories. The goal is now to show that there exists a subsequence of this sequence which converges to a punctured pearly trajectory pP with the desired properties.

In a nutshell, the pearly trajectory P from Theorem 4.1 converges to a punctured pearly trajectory pP as described above, because the splitting is happening "far away" from L_C. This allows us to view each non-trivial holomormpic map u_{i_n} for $i \in \{1, \ldots, L_n\}$ contributing to the sequence of pearly trajectory as either a single $(-J \times J)$-holomorphic map, satisfying Lagrangian boundary condition in the compact parts of $\tilde{W}_C$ or as a pair (u^-, u^+) of a $(-J)$- and a J-holomorphic map in the non-compact part of $\tilde{W}_C$. Roughly speaking the existence of a punctured pearly trajectory with the properties (pP1) and (pP2) then follows from applying Gromov's compactness Theorem in the compact parts (see for example [Fra08]) and by applying the compactness results from [Bou+03] in the non-compact part. As a result of stretching the neck the domains of the pearly trajectories degenerate to connected, noded, stable Riemann surfaces S' with non-empty boundary . The limit object pP containes a holomorphic curve with domain S', which contains a disc component. The properties (S1)-(S3) of the holomorphic curve $\tilde{u}$ follow from the fibre bundle structure of L_C by a straightforward argument, which I give at the very end of the proof of Theorem 5.1

A detailed outline of the individual steps of the proof is given at the beginning of Section 5.6, where I present the proof of the theorem.

I have structured this chapter as follows: Sections 5.2 - 5.5 are dedicated to the setup of the machinery for the proof. In Section 5.2 I recall the notion of symplectic cobordisms and explain how W can be separated along H_C into three symplectic cobordisms. In section 5.3 I construct the class of $(C, \mathcal{S})$-adjusted almost complex structures described above. I describe the neck stretching procedure in Section 5.4. I then recall the relevant notions for Riemann surfaces with boundary and holomorphic curves in Section 5.5 in order to introduce punctured pearly trajectories and define a notion of energy for these objects. The last section of the chapter contains a very rough outline of how to use the machinery developed in this chapter to define the analogues of holomorphic buildings for stable coisotropics. I call these *holomorphic chessboards*.

Most of the effort of proving Theorem 5.1 lies in adapting the setup of symplectic field theory to the present setting. The actual proof is a simple adaptation of the arguments and ideas in [Bou+03].

Remark 5.2.
The standard approach when considering discs with boundary on a Lagrangian is to neck-stretch around the unit cotangent bundle U^*L, which is a contact hypersurface. As a result of neck stretching around U^*L one obtains that the holomorphic discs with boundary on L_C converge to holomorphic buildings in a split manifold and are asymptotic at their non

removable punctures to cylinders over Reeb orbits of U^*L. In the present situation this approach does not lead to the desired outcomes. Recall that the goal is to produce holomorphic spheres in B. In order to produce a holomorphic building which has a disc component with boundary on L_C, which projects holomorphically to $B \times B$ and has only removable punctures one needs that the projection to both factors of $B \times B$ is holomorphic and that the Reeb orbits of U^*L_C are contained in the fibres F. The Reeb orbits of U^*L_C are however not necessarily contained in the fibres F of the characteristic foliation. To see this, recall that L_C is a fibre product over ΔB. Thus the normal directions of L_C in its cotangent disc bundle D^*L involve directions in a (chosen) orthogonal complement of the diagonal ΔB in $B \times B$ with respect to a chosen Riemannian metric $g_{B\times B}$ on $B \times B$. Thus, after the stretching the neck in these directions, the projections to each factor are not necessarily holomorphic and the rest of the argument would not work. Moreover there is, to my knowledge, no obvious family of almost complex structures on $B \times B$ which is translation invariant in these "off-diagonal" directions and leads to the asymptotic behaviour of holomorphic curves which is desirable in order to prove Theorem 1.6.

5.2 Symplectic cobordisms

To explain how C and H_C fit into the symplectic cobordism setting, I would like to expand on how H_C is embedded into the Bolle neighbourhood U of C. Recall that by Proposition 2.18 there exists an $\epsilon_0 > 0$ and a symplecmorphism $\psi_C : B^k_{\epsilon_0} \times C \to U$ of C such that $\psi_C^*\omega = \omega_s$. By Proposition 3.23 $H_{C,\epsilon}$ is a stable hypersurface for every $\epsilon < \epsilon_0$. Applying the Bolle neighbourhood theorem to $H_{C,\epsilon} \subset B^k_{\epsilon_0} \times C$ one concludes that there exists an $\epsilon' < \min\{\epsilon, \epsilon_0 - \epsilon\}$, a neighbourhood $U_H \subset B^k_{\epsilon_0} \times C$ of $H_{C,\epsilon}$, and a symplectomorphism

$$\phi_H : (\epsilon - \epsilon', \epsilon + \epsilon') \times H_{\epsilon,C} \longrightarrow U_H$$

such that the symplectic form ω_s pulls back under ϕ_H to

$$\omega_H := \phi_H^* \omega_s = i_H^* \omega_s + d(r\alpha)$$

On the other hand, a neighbourhood U_s of $H_{C,\epsilon}$ in the symplectic manifold U is given by a family

$$U_s = \bigcup_{r \in (\epsilon_-, \epsilon_+)} H_{C,r} = \bigcup_{r \in (\epsilon_-, \epsilon_+)} S_r^{k-1} \times C,$$

where $0 < \epsilon_- < \epsilon < \epsilon_+ < \epsilon_0$. The Lemma below shows that these two neighbourhoods are symplectomorphic.

Lemma 5.3.
Let C be a stable coisotropic and $H_{C,\epsilon}$ the associated stable hypersurface. Then there exists symplectomorphism $\psi_H : ((\epsilon - \epsilon', \epsilon + \epsilon') \times H_{\epsilon,C}, \omega_H) \to (U_s, \omega_s)$, for (U_s, ω_s) as above. The map ψ_H is given by the restriction to $(\epsilon - \epsilon', \epsilon + \epsilon') \times C$ of:

$$\begin{aligned} \psi :& (0,\infty) \times S_\epsilon^{k-1} \times C \longrightarrow \mathbb{R}^k \setminus \{0\} \times C \\ & (r, p, x) \qquad\qquad \mapsto \quad \Big(\frac{rp_1}{\epsilon}, \dots, \frac{rp_k}{\epsilon}, x\Big) \end{aligned} \tag{5.1}$$

Proof. Write

$$\psi(r,p,x) = (\psi_1, \dots, \psi_k, id)(r,p,x),$$

where

$$\psi_i(r,p,x) = \frac{rp_i}{\epsilon}$$

The symplectic form on $\mathbb{R}^k \setminus \{0\} \times C$ is given by ω_s. The symplectic form on $(0,\infty) \times S_\epsilon^{k-1} \times C$ is given by

$$\omega_H = i^*_{H_{C,\epsilon}} \omega_s + d(r\alpha),$$

where

$$\alpha = (\hat{p}_1 \alpha_1 + \dots \hat{p}_k \alpha_k) \quad \text{and} \quad \hat{p} = (\hat{p}_1, \dots, \hat{p}_k) \in S_1^{k-1}.$$

To see that ψ is a symplectomorphism compute

$$\begin{aligned}
\psi_i^* \omega_s &= d(\psi_i(r,p,x)\alpha_i) \\
&= d\left(\frac{rp_i\alpha_i}{\epsilon}.\right)
\end{aligned}$$

Thus

$$\begin{aligned}
\psi^* \omega_s &= (\psi_1, \ldots, \psi_k, id)^* \omega_s \\
&= i_C^* \omega_s + \sum_{i=1}^{k} d(r(\frac{p_i}{\epsilon})\alpha_i) \\
&= i_C^* \omega_s + i_{H_{C,\epsilon}}^* d(\frac{p_i}{\epsilon}\alpha_i) + d(r(\hat{p}_i\alpha_i)) \\
&= \omega_H.
\end{aligned}$$

□

Note that given ϵ_0 and ϵ one may choose ϵ_-, ϵ_+ such that $\epsilon - \epsilon' = \epsilon_-$ and $\epsilon + \epsilon' = \epsilon_+$. Assume from now on that such a choice has been made. Thus one may identify the neighbourhood U_s of $H_{C,\epsilon}$ in the Bolle neighbourhood of C via $\psi_H$with the neighbourhood U_H of $H_{C,\epsilon}$ which is symplectomorphic to the standard model $(\epsilon - \epsilon', \epsilon + \epsilon') \times H_{\epsilon,C}$ of the neighbourhood of a stable hypersurface. Pictorially speaking, ψ converts a neighbourhood consisting of concentric spheres S_r^{k-1} into a cylinder of spheres S_ϵ^{k-1} of constant radius ϵ. This compatibility of neighbourhoods is relevant for the construction of almost complex structures which are adapted to the stable structure $\mathcal{S}$ on C and the neck-stretching procedure on which the proof of Theorem 5.1 relies.

Definition 5.4 (Symplectic cobordism).
A *symplectic cobordism* is a compact, symplectic manifold (W, ω) with stable boundary $\partial W = V = V^+ \sqcup V^-$, where one or both components of the boundary are allowed to be empty. For simplicity, I also assume that W is connected.

A connected component of V belongs to V_+ if it has a collar neighbourhood symplectomorphic to

$$(5.2) \qquad (-\epsilon + R, R] \times V_+, d(r\alpha) + i^*_{V_+}\omega)$$

and to V_- if it has a collar neighbourhood symplectomorphic to

$$(5.3) \qquad [R, R+\epsilon) \times V_-, d(r\alpha) + i^*_{V_-}\omega)$$

for a some $R \in \mathbb{R}_{\geq 0}$. Since all boundary components are stable it is posible to extend the symplectic form $d(r\alpha) + i^*_{V_+}\omega$ from collar neighbourhoods of V_+ to bicollar neighbourhoods $(-\epsilon + R, R+\epsilon) \times V_+)$ and likewise for V_-. Then identifying $[R, R+\epsilon)$ with $\mathbb{R}_{\geq 0}$ and $(-\epsilon + R, R]$ with $\mathbb{R}_{\leq 0}$ one obtains a symplectic cobordism $\tilde{W}$, which is diffeomorphic to W and has a positive end $V_+ \times \mathbb{R}_{\geq 0}$ and negative end $V_- \times \mathbb{R}_{\leq 0}$ attached.

$$\tilde{W} := \mathbb{R}_{\leq 0} \times V_- \cup_{V_-} W \cup_{V_+} \mathbb{R}_{\geq 0} V_+.$$

In the case where either V_+ or V_- are the empty set such that ∂W consists of a single component $\tilde{W} \cong W \cup_V \mathbb{R}_{\geq 0} \times V$ is called the *symplectic completion of* W. In the case where $W = I \times V$, for an an interval $I \in \mathbb{R}$, the manifold $\tilde{W} \cong \mathbb{R} \times V$ is called the *symplectization* of V.

Given a stable coisotropic submanifold C, the hypersurface H_C is separating by Proposition 3.23. One may thus write the surrounding symplectic manifold W as a union of three symplectic cobordisms. To obtain this decomposition first cut W open along the boundary of the neighbourhood U_s as above, i.e. form $W \setminus U_s$. Since this neighbourhood is symplectomorphic to U_H by Lemma 5.3 above one may write:

$$(5.4) \qquad W = W_C \bigcup_{H_{C,\epsilon_-} = \{\epsilon_-\} \times H_{C,\epsilon}} W_H \bigcup_{H_{C,\epsilon} \times \{\epsilon_+\} = H_{C,\epsilon_+}} W_R,$$

where

$$
\begin{aligned}
W_C &\cong (\bar{B}^k_{\epsilon_-} \times C, H_{C,\epsilon_-}, \omega_s) \\
W_H &\cong ((\epsilon - \epsilon', \epsilon + \epsilon') \times H_C, \omega_H) \qquad (5.5) \\
W_R &\cong (W \setminus \mathring{B}^k_{\epsilon_+}, H_{C,\epsilon_+}, \omega)
\end{aligned}
$$

5.3 Almost complex structures adjusted to stable coisotropic submanifolds

In this section I explain how one can equip the symplectic cobordisms introduced in Equation 5.5 above with almost complex structures which make projection to B holomorphic and are natural with respect to a given stable structure $\mathcal{S}$ on C.

Let C be a coisotropic submanifold of (W, ω). Recall that ω induces an isomorphism $\iota : TW \to T^*W$ which gives a splitting

$$
i_C^* TW \cong TC/TC^\omega \oplus (TC^\omega \oplus TW/TC)
$$

of the bundle i_C^*TW into symplectic vector bundles $\xi_C \cong TC/TC^\omega$ and $\xi_C^\omega \cong TC^\omega \oplus TW/TC$ over C. This splitting depends on a choice of complement of TC^ω in TC and a choice of complement of TC in TW. Such a choice can be made by choosing a complex structure J_1 on the bundle TC/TC^ω which is compatible with the induced symplectic form on the quotient bundle TC/TC^ω and a choice of a complex structure J_2 on the bundle $TC^\omega \oplus TW/TC$ which is compatible with the induced symplectic structure on on this bundle. Note that a choice of an almost complex structure J on W does not necessarily induces complex structure of the type $J_1 \oplus J_2$ as above.

From now on assume that $(C, \mathcal{S})$ is a stable coisotropic submanifold of (W, ω). In now construct a natural class of almost complex structures which are adjusted to C and to the stable structure $\mathcal{S}$. Recall from Section 2.3 that the stabilising vector fields $Y_1, \dots, Y_k$ define stabilising one-forms $\alpha_1, \dots, \alpha_k$. Define corresponding generalised Reeb vector fields $X_1, \dots, X_k$ by:

$$\alpha_i(X_j) = dH_j(Y_i) = \omega(Y_i, X_j) = \delta_{ij}.$$

Given a stable structure $\mathcal{S}$, the sub-bundle

$$\xi_C := \cap_{i=1}^k \ker \alpha_i$$

of TC is isomorphic to TC/TC^w and a complement of TC^ω in TC. The splitting

$$TC = \cap_{i=1}^k \ker \alpha_i \oplus TC^\omega \tag{5.6}$$

depends only on $\mathcal{S}$ and $X_1, \dots X_k$. Likewise the splitting

$$i_C^* TW = \xi_c \oplus \xi_C^\omega = (\cap_{i=1}^k \ker \alpha_i) \oplus (X_1 \oplus \cdots \oplus X_k \oplus Y_1 \oplus \cdots \oplus Y_k) \tag{5.7}$$

depends only on $\mathcal{S}$ and $X_1, \dots X_k$. Denote the symplectic forms arising as the restrictions of ω to ξ_C by ω_{ξ_C} and the restriction of ω to ξ_C^ω by $\omega_{X,Y}$.

Definition 5.5 ($(C, \mathcal{S})$-adjusted almost complex structures)**.**
Let $(C, \mathcal{S})$ be a stable coisotropic in a symplectic manifold (W, ω). A $(C, \mathcal{S})$-adjusted almost complex structure J_C on a Bolle neighbourhood U of C in (W, ω) is constructed as follows:

choose a pair $(J_{\xi_C}, J_{\xi^\omega})$, where J_{ξ_C} is any ω_{ξ_C}-compatible complex structure on the bundle $\xi_C = \cap_{j=1}^k \ker \alpha_j$. Define $J_{X,Y}$ on ${\xi_C}^\omega$ by

$$\begin{aligned} J_{X,Y} X_i &= Y_i \\ J_{X,Y} Y_i &= -X_i. \end{aligned} \tag{5.8}$$

Then J_{ξ_C} and J_{C^ω} fit together to define an complex structure J_C on $i_C^* TW$, which by construction preserves the splitting (5.7).

Recall from Proposition 2.18 that there exists a symplectomorphism $\psi_C : B^k_{\epsilon_0} \times C \to U$. Given $q \in U$ choose (p, x) such that $\phi(p, x) = q$ and define

$$J_C(q) \coloneqq J_C(p, x) \coloneqq J_C(x).$$

At a point $(p, x) \in B^k_{\epsilon_0} \times C$ one may write J_C as a matrix

$$J_C(p,x) = \begin{pmatrix} J_{X,Y}(x) & 0 \\ 0 & J_{\xi_C}(x) \end{pmatrix} \tag{5.9}$$

where $J_{X,Y}(x) : \xi^w \to \xi^w$ denotes the $2k \times 2k$ matrix satisfying Equation 5.8 and $J_{\xi_C}(x)$ denotes a $2(n-k) \times 2(n-k)$ matrix representing the complex structure J_{ξ_C} on $T_x\xi_C$.

When $(C, \mathcal{S})$ is clear from the context, I will refer to a $(C, \mathcal{S})$-adjusted almost complex structure J_C by an *adjusted* J_C.

Remark 5.6.
If $(C, \mathcal{S})$ is also fibred, an adjusted J_C can be constructed by first choosing an ω_B-compatible almost complex structure J_B on the symplectic quotient B of C and then defining J_C by

$$J_C(p,x) = \begin{pmatrix} J_{X,Y}(x) & 0 \\ 0 & \pi_B^* J_B(x) \end{pmatrix} \tag{5.10}$$

on all of U.

The symplectic cobordism W_C inherits an adjusted $(C, \mathcal{S})$-compatible almost complex structure J_C from C by definition. To equip W_H with an almost complex structure, first pull back J_C via ψ_H to $H_{C,\epsilon}$. More precisely set

$$J_{H_C}(p, x) = \psi_H^* J_C(p, x). \tag{5.11}$$

Let $(p,x) \in S^{k-1}_\epsilon \times C$. Denote by ∂_r the vector field spanning the tangent space of $(\epsilon-\epsilon', \epsilon+\epsilon')$, by X_H the Reeb vector field of $H_{C,\epsilon}$. Denote by ∂_p the vector field which is the unit $\hat{p}$ vector in the direction of p at $p \in S^{k-1}_\epsilon$ and by ∂_{q_i} a vector field which is the unit vector $\hat{q}_i$ in the direction q_i of the tangent space to the sphere $p^\perp$ for $1 \leq i \leq k$. A simple calculation in coordinates shows that

$$\begin{aligned}
J_{H_C}(p,x)\partial_r &= J_C(p,x)d\psi(p,x)\partial_r = J_C(p,x)\partial_p = \hat{p}X_H \\
J_{H_C}(p,x)X_H &= J_C(p,x)d\psi(p,x)X_H = J_C(p,x)X_H = -\partial_p \\
J_{H_C}(p,x)\partial_{q_i} &= J_C(p,x)d\psi(p,x)\partial_{q_i} = J_C(p,x)\partial_{q_i} = \hat{q}_iX_i \\
J_{H_C}(p,x)\hat{q}_iX_i &= J_C(p,x)d\psi(p,x)\hat{q}_iX_i = J_C(p,x)\hat{q}_idX_i = -\partial_{q_i}.
\end{aligned}$$

Hence J_{H_C} is $(H_C, (\omega_H, \partial_p))$-adjusted and moreover preserves the splitting

$$i^*_{H_C}TW = \xi_C \oplus \xi_H/\xi_C \oplus \xi_H^\omega.$$

At a point $(p,x) \in H_{C,\epsilon}$ one may write J_{H_C} as a matrix

(5.12)

$$J_{H_C}(p,x) = \begin{pmatrix} J_{X,\partial_p}(p,x) & 0 & 0 \\ 0 & J_{S,R}(p,x) & 0 \\ 0 & 0 & J_{\xi_C}(x) \end{pmatrix} = \begin{pmatrix} J_{X,\partial_p}(p,x) & 0 \\ 0 & J_{\xi_H(p,x)} \end{pmatrix}.$$

Here $J_{\xi_C}(x)$ denotes a $2(n-k)$ by $2(n-k)$ matrix representing the ω_{ξ_C}-compatible complex structure on the bundle ξ_C as before. $J_{S,R}(p,x)$ denotes a $2(k-1)$ by $2(k-1)$ matrix representing the almost complex structure pairing directions in TS^{k-1} and $TC^\omega \setminus TH^\omega$. $J_{X,\partial_p}(p,x)$ denotes a 2 by 2 matrix pairing the Reeb vector field X_H of H_C with the normal direction ∂_p of H_C. Finally $J_{\xi_H}(p,x)$ denotes a $2(n-1)$ by $2(n-1)$ matrix representing the $\omega_{H_{\xi_H}}$-compatible almost complex structure on ξ_H, This is, by construction, the matrix

$$J_{\xi_{H_C}}(p,x) = \begin{pmatrix} J_{S,R}(p,x) & 0 \\ 0 & J_{\xi_C(x)} \end{pmatrix}. \tag{5.13}$$

Extend this almost complex structure on $H_{C,\epsilon}$ to an almost complex structure on $W_H = (\epsilon - \epsilon', \epsilon + \epsilon') \times S_\epsilon^{k-1} \times C$ by setting

$$J_H(r, p, x) = J_{H_C}(p, x)$$

for all r in $(\epsilon - \epsilon', \epsilon + \epsilon')$. By construction J_C and J_H fit together smoothly to give an almost complex structure on $W_C \cup W_H$ by using the restriction to the boundaries of the map ψ from Lemma 5.3 and possibly a perturbation as described in Section 3.1 of [Bou+03]. Extend this almost complex structure to an ω-compatible almost complex structure J_R on W_R in the same way to obtain an ω-compatible almost complex structure $J_\mathcal{S}$ on all of W.

I will call such an almost complex structure constructed as above a $(C, \mathcal{S})$ -adjusted alomst complex structure on W and denote it by $J_\mathcal{S}$. If it is clear from the context I will just call $J_\mathcal{S}$ *adjusted*. Slightly abusing notation I will denote the restriction of $J_\mathcal{S}$ to W_C by J_C and likeswise for W_H and W_R.

5.4 Stretching the neck

In this section I briefly review the neck stretching or splitting construction from symplectic field theory (see Section 3.4 of [Bou+03] or Section 2.7 of [CM05]), that will be performed to obtain more information about the pearly trajectories provided by Theorem 4.1.

Recall the separation of W into the three symplectic cobordims W_C, W_H and W_R defined in Equation 5.5 above. Equip W with a $(C, \mathcal{S})$-adjusted almost complex structure $J_\mathcal{S}$ as describe in Section 5.3 above.

For $\tau > 0$, the intervals $(\epsilon - \epsilon', \epsilon + \epsilon')$ and $(\epsilon - \tau, \epsilon + \tau)$ are diffeomorphic, for example via the linear diffeomorphism $\phi_\tau : (\epsilon - \tau, \epsilon + \tau) \to (\epsilon - \epsilon', \epsilon + \epsilon')$ given by

$$\phi_\tau(t) = \frac{t\epsilon'}{\tau} + \epsilon - \frac{\epsilon\epsilon'}{\tau}. \tag{5.14}$$

If one lets τ grow to infinity, the "neck"

$$W_H^\tau := (\epsilon - \tau, \epsilon + \tau) \times H_C \tag{5.15}$$

will expand to the symplectization $\mathbb{R} \times H_C$. If one now considers $(W_H^\tau, \phi_\tau^* J_H)$ one obtains

$$\phi_\tau^* J_H = \begin{pmatrix} 0 & -\frac{\epsilon'}{\tau} & 0 \\ \frac{\tau}{\epsilon'} & 0 & 0 \\ 0 & 0 & J_{\xi_H} \end{pmatrix}.$$

Letting τ grow to infinity, applying this almost complex structure "blows up" the $\mathbb{R}$-direction and degenerates the Reeb direction to zero. Any orientation preserving sequence of diffeomorphisms $\{\phi'_\tau\}_{\tau>0}$ of the intervals $(\epsilon-\tau, \epsilon+\tau)$ and $(\epsilon-\epsilon', \epsilon+\epsilon')$ mapping the ends to the ends has to exhibit this behaviour. To avoid this degeneration and to be able to extract information about the asymptotic behaviour of holomorphic curves in $\tilde{W}$ set:

$$(W_H^\tau, J_H^\tau) := (W_H^\tau, J_H). \tag{5.16}$$

This neighbourhood of H_C is diffeomorphic to W_H but carries a translationally invariant almost complex structure which does not degenerate as τ grows to infinity.

To fit this into the symplectic cobordism setting introduced above, set:

$$W^\tau := W_C \bigcup_{H_{C,\epsilon_-} = \{\epsilon-\tau\}\times H_{C,\epsilon}} W_H^\tau \bigcup_{H_{C,\epsilon_+} = \{\epsilon+\tau\}\times H_{C,\epsilon}} W_R \tag{5.17}$$

This manifold is diffeomorphic to W. Define an almost complex structure on W_τ by

$$J_S^\tau = \begin{cases} J_R & \text{on } W_R \\ J_H & \text{on } (\epsilon - \tau, \epsilon + \tau) \times H_C \\ J_C & \text{on } W_C \end{cases} \tag{5.18}$$

Again letting τ grow to infinity, one can write the resulting "split" manifold $\tilde{W}$ as a the result of attaching cylindrical ends to the three symplectic cobordisms W_C, W_H and W_R and thus as the non compact symplectic cobordisms

$$\begin{aligned}\tilde{W}_C &= W_C \bigcup_{\{\epsilon_-\}\times H_C} \mathbb{R}_+ \times H_C. \\ \tilde{W}_H &= \mathbb{R} \times H_C. \\ \tilde{W}_R &= W_R \bigcup_{\{\epsilon_+\}\times H_C} \mathbb{R}_- \times H_C.\end{aligned} \tag{5.19}$$

Set

$$\tilde{W} = \tilde{W}_C \cup \tilde{W}_H \cup \tilde{W}_R. \tag{5.20}$$

The almost complex structures $J_{\mathcal{S}}^{\tau}$ converge pointwise in an appropriate sense to a $(C, \mathcal{S})$-adjusted almost complex structure $\tilde{J}_{\mathcal{S}}$ on $\tilde{W}$. See again Section 3.4 of [Bou+03] for details. Denote the restriction of $\tilde{J}_{\mathcal{S}}$ to $\tilde{W}_H$ by $\tilde{J}_H$ the restriction to $\tilde{W}_C$ by $\tilde{J}_C$ and by $\tilde{J}_R$ for $\tilde{W}_R$.

Remark 5.7.
Notice that by the stability assumption on C, one has

$$\mathcal{L}_X \omega_s = \mathcal{L}_X \alpha = 0.$$

The symplectic form ω_H is compatible with J_H as constructed. In the language of [Bou+03], $J_{\mathcal{S}}^{\tau}$ and $\tilde{J}$ are symmetric, cylindrical almost complex structures adjusted to ω.

The symplectic forms ω_s and ω_H, will "blow up" as τ goes to infinity. This problem is overcome by adapting the notion of symplectic energy to a notion which takes the rescaling into account. I give the relevant definitions in Section 5.5.2.With the proof of Theorem 5.1 in mind I summarise the relevant data from this section in the definition below:

Definition 5.8 (neck stretching data).
Neck stretching data $\mathcal{N} := \mathcal{N}(W, \omega, J, C)$ consist of the following:

(N1) A sequence of symplectic manifolds $(W^n \times W^n, -\omega_n \times \omega_n)$ indexed by an increasing sequence of non-negative integers $n \in \mathbb{Z}_{\geq 0}$ diverging to $+\infty$. The manifolds W^n and its parts are defined by equation (5.17)

(N2) A sequence of $(C, \mathcal{S})$-adjusted almost complex structures $J^n_{\mathcal{S}}$ on W^n as constructed in Section 5.3. These define a sequence of almost complex manifolds $(W^n \times W^n, -J^n_{\mathcal{S}} \times J^n_{\mathcal{S}})$. Notice that this product can be separated into nine parts:

$$\begin{aligned} &W_C \times W_C, W^n_H \times W_C, W_C \times W^n_H, W_R \times W_C, W^n_H \times W^n_H, \\ &W_C \times W_R, W_R \times W^n_H, W^n_H \times W_R, W_R \times W_R \end{aligned} \tag{5.21}$$

(N3) The limit object $\tilde{W} \times \tilde{W}$ which splits up into nine parts:

$$\begin{aligned} &\tilde{W}_C \times \tilde{W}_C, \tilde{W}_H \times \tilde{W}_C, \tilde{W}_C \times \tilde{W}_H, \tilde{W}_R \times \tilde{W}_C, \tilde{W}_H \times \tilde{W}_H, \\ &\tilde{W}_C \times \tilde{W}_R, \tilde{W}_R \times \tilde{W}_H, \tilde{W}_H \times \tilde{W}_R, \tilde{W}_R \times W_R, \end{aligned} \tag{5.22}$$

where each factor is as defined in (5.19) and (5.20). The products are equipped with the respective almost complex structure , $(-\tilde{J}_C \times \tilde{J}_C), (-\tilde{J}_C \times \tilde{J}_H), (-\tilde{J}_H \times \tilde{J}_C), (-\tilde{J}_H \times \tilde{J}_H), (-\tilde{J}_R \times \tilde{J}_H), (-\tilde{J}_H \times \tilde{J}_R), (-\tilde{J}_R \times \tilde{J}_R)$ and $(-\tilde{J}_S \times \tilde{J}_S)$ respectively.

Remark 5.9.
It is also possible to define a "k-dimensional neck stretch", by cutting out $B^k_r \times C$ from W and then letting r grow to infinity. The resulting symplectic cobordisms will be diffeomorphic to the three completed symplectic cobordisms $\tilde{W}_C, \tilde{W}_H$ and $\tilde{W}_R$ above and can also be equipped with $(C, \mathcal{S})$-adjusted almost complex structures. In this sense one may view

$\tilde{W}_C$ as the k-dimensional symplectization of the coisotropic C. In this setting a notion of k-dimensional Hofer energy can be developed, which is similar to the notion of energy put forward in Section 5.5.2. Thus the k-dimensional analogue of a neck-stretch around a codimension k-coisotropic does not seem to lead to different results than neck stretching around the associated stable hypersurface H_C. Since irrelevant for the proof of Theorem 5.1, I do not investigate the relationship of these two neck-stretching operations here.

5.5 Holomorphic curves

5.5.1 Punctured pearly trajectories

I follow the notations and conventions used in [Abb14] and [Bou+03]. Many thanks to Chris Wendl for explaining the "doubling operation" to me. See appendix B of [Wen05].

Definition 5.10 (Riemann surface data).
Riemann surface data

$$S = (S, \partial S, j, \mathring{M} \cup M_\partial, \mathring{Z} \cup Z_\partial, \mathring{D} \cup D_\partial) = (S, \partial S, j, M, Z, D) \tag{5.23}$$

consist of

(RS 1) A Riemann surface S consisting of collection of disjoint connected Riemann surfaces $S_1, \ldots, S_k$ with possibly nonempty boundaries ∂S_i.

(RS 2) An (almost) complex structure j on TS.

(RS 3) The finite set of interior marked points $\mathring{M} \subset \mathring{S}$, and the finite set of boundary marked points $M_\partial \subset \partial S$. Set $M = \mathring{M} \cup M_\partial$.

(RS 4) The finite sets $\mathring{Z} \subset \mathring{S}$ and $Z_\partial \subset \partial S$ of interior and boundary punctures . Set $Z = \mathring{Z} \cup Z_\partial$.

(RS 5) The finite set $\mathring{D} \subset \mathring{S}$ of pairs $\{d, d'\}$ of interior marked points and the finite set $D_\partial \subset \partial S$ of pairs $\{b, b'\}$ of boundary marked points. Set $D = \mathring{D} \cup D_\partial$. These points will be identified to form a nodal (or singular) Riemann surface. So I will call them *nodal pairs.*

To S we can associate a nodal or singular surface S^{sing} by identifying nodal pairs:

$$S^{sing} = S/\{z_j \sim z_j' \quad \text{for each pair} \quad \{z_j, z_j'\} \in D\}.$$

Say that S is connected if S^{sing} is connected.

For each Riemann surface $(S, \partial S, j)$ with non-empty boundary $\partial S \neq \emptyset$ there exists a conjugate Riemann surface $S^c = (S, \partial S, -j)$ which can be glued to S along ∂S to form a surface

$$(S^d, j^d) = (S \cup_{\partial S} S^c, j \cup -j)$$

without boundary, a natural almost complex structure j^d, and a natural anti-holomorphic involution $\sigma : S^d \to S^d$ whose fixed point set is ∂S. If S has empty boundary define the doubled Riemann surface data S^d by

$$(S^d, M^d, Z^d, D^d) = (S, M, Z, D).$$

If S has at least one boundary component, define the doubled Riemann surface data S^d to be

$$(S^d, M^d, Z^d, D^d) = (S^d, j^d, \mathring{M} \cup \mathring{M}^c \cup M_\partial, \mathring{Z} \cup \mathring{Z}^c \cup Z_\partial, \mathring{D} \cup \mathring{D}^c \cup D_\partial),$$

where $\mathring{M}^c = \sigma(\mathring{M}), \mathring{Z}^c = \sigma(\mathring{Z}), \mathring{D}^c = \sigma(\mathring{D})$. The sets $M_\partial, Z_\partial, D_\partial$ are fixed by σ. Set $\dot{S}^d = S^d \setminus (M^d \cup Z^d \cup D^d)$.

Connected Riemann surface data S with boundary are *stable* if:

$$\chi(\dot{S}^d_j) = 2 - 2g_j - |(M^d \cup Z^d \cup D^d) \cap S_j| < 0$$

holds for each S_j. Here g_j is the genus of S_j^d and $|\cdot|$ denotes the cardinalities of the sets of M, Z and D. Say that Riemann surface data S are stable if every connected component of S^d is stable. For example a disc $D = (D, \partial D, \{m_{\partial D}^1, m_{\partial D}^2\} \in \partial D, m_3 \in \mathring{D})$ with one interior and two boundary marked points satisfies

$$2 - 2 \cdot 0 - |\{m_{\partial D}^1, m_{\partial D}^2, m_3, m_3^d\}| = -2 < 0,$$

and is thus stable.

If S is connected, its *arithmetic genus* ag is defined by:

$$ag(S) = |D| - C + \sum_{i=1}^{C} g_j + 1$$

Here g_j is the genus of a connected component S_j of S and C is the number of connected components of S. The signature sig of S is given by

$$sig(S) = (ag(S), |\mathring{M}|, |M_\partial|, |\mathring{Z}|, |Z_\partial|),$$

Thus the signature of S^d is given by

$$sig(S^d) = (ag(S^d), |M^d|, |Z^d|)$$

For more details on Riemann surfaces with boundary the reader is referred to Section 1.3.3 of [Abb14] and Appendix B of [Wen05] and the references therein. In particular one can prove a version of the Deligne-Mumford compactness theorem for Riemann surfaces with boundary. The main idea of the proof is to double the surface with boundary as described above and then follow the strategy of proof for the case without boundary.

Since Theorem 4.1 establishes the existence of a pearly trajectory with certain properties one will have to deal with the possible degenerations of pearly trajectories in the neck stretching process. In order to absorb bubbling phenomena into alterations of the domains of the pearly trajectories one thus allows for slightly more general pearly trajectories. I will describe this bubbling-off procedure in more detail in Section 5.6.3.Throughout I will use the shorthand notation $u : (S, \partial S) \to (T, L)$ for a holomorphic curve

$$u(S, \partial S, j, M, Z, D) \to (T, L, J)$$

mapping to a symplectic manifold $T = (W, \omega)$ equipped with an ω-compatible almost complex structure J and respecting the boundary condition $u(\partial S) \subset L$ and defined away from $M \cup Z \cup D$. The relevant definitions for pearly trajectories are below. For simplicity I will restrict to the case where one can split the manifold $\tilde{W}$ into three symplectic cobordisms $\tilde{W}_C, \tilde{W}_H$ and $\tilde{W}_R$ defined in Equation 5.5. The case in which a hypersurface is non-sperating is similar and for example dealt with in [Abb14].

Definition 5.11 (Punctured pearly differential trajectory).
A *punctured pearly differential trajectory* is a pearly differential trajectory as in Definition 4.12, where condition (PD1) is replaced by

(pPD1) At least one of the J-holomorphic curves $u_i : S \to (T, L)$ for $i \in \{1, \dots, l\}$has at least one non-constant disc component $(D, \partial D)$. Here S are Riemann surface data as in Definition 5.10 with the additional conditions that S is connected and that $g = 0$. Moreover there is a set $E_d = \{z_{\text{in}}, z_{\text{out}}\} \subset \partial S$ of entry and exit points of S which is disjoint from the sets of marked points, double points and punctures i.e.:

$$(M_\partial \cup Z_\partial \cup D_\partial) \cap E = \emptyset.$$

Condition (PD2) is replaced by

(pPD2) The energy of the punctured pearly differential trajectory defined below in Equation 5.37 is bounded above by a constant $E_0 > 0$.

Conditions (PD3)-(PD5) are replaced by

(pPD3) There exists a $t^{-} \in [-\infty, 0)$ such that $\phi_Z^{t^-}(u_1(z_{\text{in}})) = y$.

(pPD3) For every $1 \leq i \leq l-1$ there exists a $t^i \in (0, \infty)$ such that $\phi_Z^{t^i}(u_i(z_{\text{out}})) = u_{i+1}(z_{\text{in}})$.

(pPD3) There exists a $t^{+} \in (0, +\infty]$ such that $\phi_Z^{t^+}(u_l(z_{\text{out}})) = x$.

Denote by

$$p\mathcal{P}_{\text{diff}} \coloneqq p\mathcal{P}_{\text{diff}}(y, x; A; f, Z, J)$$

the moduli space of all possible configurations of all possible lengths $l \geq 1$ described in definition 5.11

Definition 5.12 (Punctured pearly product trajectory).
A *punctured pearly product trajectory* is a pearly product trajectory as in Definition 4.14 where conditions (PP1)-(PP3) are replaced by:

(pPP1) $v : (S_p, \partial S_p) \to (T, L)$ is a J-holomorphic curve, which is allowed to be constant. Here S are Riemann surface data as in definition 5.10 with the additional conditions that S_p is connected and that $g = 0$. Moreover there is a set

$$E_p = \{z^p_{\text{in},1}, z^p_{\text{in},2}, z^p_{\text{out}}\}$$

of entry and exit points of S_p, the central, product component of the pearly trajectory which is disjoint from the sets of marked points, double points and punctures of S_p i.e.:

$$(M^p_\partial \cup Z^p_\partial \cup D^p_\partial) \cap E_p = \emptyset.$$

(pPP2) Set $z_1 = v(z_{\text{in}}^1)$, $z_2 = v(z_{\text{in}}^2)$ and $z_3 = v(z_{\text{out}}^p)$.

$$u \in p\mathcal{P}_{\text{diff}}(x_1, z_1; B_1; f, Z_1, J)$$
$$u' \in p\mathcal{P}_{\text{diff}}(x_2, z_2; B_2; f, Z_2, J)$$
$$u'' \in p\mathcal{P}_{\text{diff}}(z_3, y; B_3; f, Z_3, J)$$

Require again that the set of entry and exit points $E \cup E_p$ of discs contributing to the pearly trajectories is disjoint from the sets of marked points, double points and punctures:

$$(M_\partial \cup Z_\partial \cup D_\partial) \cap (E \cup E_p) = \emptyset.$$

(pPP3) The energy E as in definition below 5.37 of the punctured pearly product trajectory is bounded above by a constant $E_0 > 0$.

In abuse of notation denote a punctured pearly product or a punctured pearly differential trajectory as defined above by pP. I will call both kinds of punctured pearly trajectories just punctured pearly trajectories.

5.5.2 Energy

Variations of *Hofer's* energy as defined in [Hof93] are used throughout the literature. For example [Bou+03], [CM05] use slightly different conventions. The definitions in these references are equivalent in the sense that a uniform bound on one implies a uniform bound on the other and vice versa. (see Lemma 4.1(b) in [CM05] and Lemma 9.2 in [Bou+03] respectively)

I now adapt the notion of energy put forward in [Bou+03] to the present setting. One needs to adapt these notions since in the proof of Theorem 5.1 I will be dealing with pearly trajectories converging to punctured pearly trajectories rather than with holomorphic curves converging to holomorphic buildings. Moreover notice that the target of the pearly trajectories is a product of two symplectic manifolds. I give the relevant definitions below. Recall that H_C is separating, thus

$$W^\tau = W_C \cup W_H^\tau \cup W_R. \tag{5.24}$$

where $W_H^\tau = (\epsilon - \tau, \epsilon + \tau) \times H_C$. Set

$$W^{\tau,-} \times W^{\tau,+} = (W^\tau \times W^\tau, -\omega \times \omega, -J_S^\tau \times J_S^\tau), \tag{5.25}$$

and likewise for the products of the different parts of the symplectic cobordism. Recall that $W^\tau \times W^\tau$ has nine different parts listed in Equation 5.21, which converge to the parts listed in Equation 5.22. For each $\tau > 0$ a pearly trajectory P_τ in $W^{\tau,-} \times W^{\tau,+}$ consists of a finite collection $u_1, \ldots, u_{l_\tau}$ of holomorphic maps $u_i : (S^\tau, \partial S) \to (W^{\tau,-} \times W^{\tau,+}, L_C)$. First define the ω-energy of P_τ in the respective parts of the product of the symplectic cobordisms W_C, W_H^τ and W_R:

$$\begin{aligned}
E_\omega^{C,C}(P_\tau) &= \sum_{i=1}^{l_\tau} \int_{{u_i}^{-1}(W_C^- \times W_C^+)} u_i^*(-\omega_s \times \omega_s) \\
E_\omega^{C,H}(P\tau) &= \sum_{i=1}^{l_\tau} \int_{{u_i}^{-1}(W_C^- \times W_H^{\tau,+})} (u_i^-, \pi_H \circ u_i^+)^*(-\omega_s \times \omega_H) \\
E_\omega^{H,C}(P_\tau) &= \sum_{i=1}^{l_\tau} \int_{{u_i}^{-1}(W_H^{\tau,-} \times W_C^+)} (\pi_H \circ u_i^-, u_i^+)^*(-\omega_H \times \omega_s) \\
E_\omega^{C,R}(P_\tau) &= \sum_{i=1}^{l_\tau} \int_{{u_i}^{-1}(W_C^- \times W_R^+)} u_i^*(-\omega_s \times \omega) \\
E_\omega^{R,C}(P_\tau) &= \sum_{i=1}^{l_\tau} \int_{{u_i}^{-1}(W_R^- \times W_C^+)} u_i^*(-\omega \times \omega_s) \\
E_\omega^{H,H}(P_\tau) &= \sum_{i=1}^{l_\tau} \int_{{u_i}^{-1}(W_H^{\tau,-} \times W_H^{\tau,+})} ((\pi_H \times \pi_H) \circ u_i)^*(-\omega_H \times \omega_H)
\end{aligned} \tag{5.26}$$

$$E_\omega^{H,R}(P_\tau) = \sum_{i=1}^{l_\tau} \int_{u_i{}^{-1}(W_H^{\tau,-} \times W_R^+)} (\pi_H \circ u_i^-, u_i^+)^*(-\omega_H \times \omega)$$

$$E_\omega^{R,H}(P_\tau) = \sum_{i=1}^{l_\tau} \int_{u_i{}^{-1}(W_R^- \times W_H^{\tau,+})} (u_i^-, \pi_H \circ u_i^+)^*(-\omega \times \omega_H)$$

$$E_\omega^{R,R}(P_\tau) = \sum_{i=1}^{l_\tau} \int_{u_i{}^{-1}(W_R^- \times W_R^+)} u_i^*(-\omega \times \omega)$$

The total ω-energy of P_τ, $E_\omega(P_\tau)$, is then defined as the sum of the ω-energies in different parts of the product of the symplectic cobordisms, i.e.

(5.27)

$$\begin{aligned} E_\omega(P_\tau) = {} & E_\omega^{C,C}(P_\tau) + E_\omega^{C,H}(P_\tau) + E_\omega^{H,C}(P_\tau) + E_\omega^{C,R}(P_\tau) + E_\omega^{R,C}(P_\tau) \\ & + E_\omega^{H,H}(P_\tau) + E_\omega^{H,R}(P_\tau) + E_\omega^{R,H}(P_\tau) + E_\omega^{R,R}(P_\tau) \end{aligned}$$

Define the α-energy in the respective parts of the product by

(5.28)

$$E_\alpha^{C,H}(P_\tau) = \sup_\phi \sum_{i=1}^{l_\tau} \int_{u_i^{-1}(W_C \times W_H^{\tau,+})} (\phi \circ \pi_\mathbb{R} \circ u_i^+)(u_i^+)^*(dr \wedge \alpha)$$

$$E_\alpha^{H,C}(P_\tau) = \sup_\phi \sum_{i=1}^{l_\tau} \int_{u_i^{-1}(W_H^{\tau,-} \times W_C)} (\phi \circ \pi_\mathbb{R} \circ u_i^-)(u_i^-)^*(-dr \wedge \alpha)$$

$$E_\alpha^{R,H}(P_\tau) = \sup_\phi \sum_{i=1}^{l_\tau} \int_{u_i^{-1}(W_R \times W_H^{\tau,+})} (\phi \circ \pi_\mathbb{R} \circ u_i^+)(u_i^+)^*(dr \wedge \alpha)$$

$$E_\alpha^{H,R}(P_\tau) = \sup_\phi \sum_{i=1}^{l_\tau} \int_{u_i^{-1}(W_H^{\tau,-} \times W_R)} (\phi \circ \pi_\mathbb{R} \circ u_i^-)(u_i^-)^*(-dr \wedge \alpha)$$

$$E_\alpha^{H,H}(P_\tau) = \sup_{(\phi^-,\phi^+)} \sum_{i=1}^{l_\tau} \int_{u_i^{-1}(W_H^{\tau,-} \times W_H^{\tau,+})} ((\phi^-, \phi^+) \circ \pi_\mathbb{R} \times \pi_\mathbb{R} \circ u_i) u_i^*(-dr \wedge \alpha \times dr \wedge \alpha)$$

Here $\pi_\mathbb{R} : (\epsilon - \tau, \epsilon + \tau) \times H_C \to (\epsilon - \tau, \epsilon + \tau)$ denotes the obvious projection and the supremum is taken over either functions $\phi : (\epsilon - \tau, \epsilon + \tau) \to \mathbb{R}_+$ with

$\int_{(\epsilon-\tau,\epsilon+\tau)} \phi(r)dr = 1$, or functions $\phi^{\mp} : (\epsilon-\tau, \epsilon+\tau) \to \mathbb{R}_+$

$$\int\limits_{(\epsilon-\tau,\epsilon+\tau)} (\phi^{\mp})(r)(dr = 1.$$

Note that one needs not to define the α-energy in $W_C \times W_C, W_C \times W_R, W_R \times W_C$ and $W_R \times W_R$ since no stretching is taking place in these parts of the product. The total α-energy of P^τ, $E_\alpha(P^\tau)$, is defined by

$$E_\alpha(P^\tau) = E_\alpha^{C,H}(P^\tau) + E_\alpha^{H,C}(P^\tau) + E_\alpha^{R,H}(P^\tau) + E_\alpha^{H,R}(P^\tau) + E_\alpha^{H,H}(P^\tau)$$

Finally, the total energy of P_τ, $E(P_\tau)$, is defined by

$$E(P_\tau) = E_\omega(P_\tau) + E_\alpha(P_\tau). \tag{5.29}$$

As a result of stretching the neck, W splits into the three symplectic cobordisms described in (5.19). Thus $W \times W$ splits into nine pieces. For a punctured pearly trajectory pP as in Definition 5.11 or 5.12 again first define the ω-energy of pP, $E_\omega(pP)$, in the different parts of the product of the symplectic cocbordisms $\tilde{W}_C$, $\tilde{W}_H$ and $\tilde{W}_R$:

$$\begin{aligned} E_\omega^{C,C}(pP) = &\sum_{i=1}^{l} \int\limits_{u_i{}^{-1}(W_C^- \times W_C^+)} u_i^*(-\omega_s \times w_s) \\ &+ \sum_{i=1}^{l} \int\limits_{u_i{}^{-1}(W_C^- \times \mathbb{R}_+ \times H_C^+)} (u_i^-, \pi_H \circ u_i^+)^*(-\omega_s \times \omega_H) \\ &+ \sum_{i=1}^{l} \int\limits_{u_i{}^{-1}(\mathbb{R}_+ \times H_C^- \times W_C^+)} (\pi_H \circ u_i^-, u_i^+)^*(-\omega_H \times \omega_s) \\ &+ \sum_{i=1}^{l} \int\limits_{u_i{}^{-1}(\mathbb{R}_+ \times H_C^- \times \mathbb{R}_+ \times H_C^+)} ((\pi_H \times \pi_H) \circ u_i)^*(-\omega_H \times \omega_H) \end{aligned} \tag{5.30}$$

$$
\begin{aligned}
E_\omega^{C,H}(pP) &= \sum_{i=1}^{l} \int_{{u_i}^{-1}(W_C^- \times \tilde{W}_H^+)} (u_i^-, \pi_H \circ u_i^+)^*(-\omega_s \times \omega_H) \\
&+ \sum_{i=1}^{l} \int_{{u_i}^{-1}(\mathbb{R}_+ \times H_C^- \times \tilde{W}_H^+)} (\pi_H \circ u_i^-, \pi_H \circ u_i^+)^*(-\omega_H \times \omega_H) \\
E_\omega^{H,C}(pP) &= \sum_{i=1}^{l} \int_{{u_i}^{-1}(\tilde{W}_H^- \times W_C^+)} (\pi_H \circ u_i^-, u_i^+)^*(-\omega_H \times \omega_s) \\
&+ \sum_{i=1}^{l} \int_{{u_i}^{-1}(\tilde{W}_H^- \times \mathbb{R}_+ \times H_C^+)} (\pi_H \circ u_i^-, \pi_H \circ u_i^+)^*(-\omega_H \times \omega_H) \\
E_\omega^{C,R}(pP) &= \sum_{i=1}^{l} \int_{{u_i}^{-1}(W_C^- \times W_R^+)} u_i^*(-\omega_s \times \omega) \\
&+ \sum_{i=1}^{l} \int_{{u_i}^{-1}(W_C^- \times \mathbb{R}_- \times H_C^+)} (u_i^-, \pi_H \circ u_i^+)^*(-\omega_s \times \omega_H) \\
&+ \sum_{i=1}^{l} \int_{{u_i}^{-1}(\mathbb{R}_+ \times H_C^- \times W_R^+)} (\pi_H \circ u_i^-, u_i^+)^*(-\omega_H \times \omega) \\
&+ \sum_{i=1}^{l} \int_{{u_i}^{-1}(\mathbb{R}_+ \times H_C^- \times \mathbb{R}_- \times H_C^+)} ((\pi_H \times \pi_H) \circ u_i)^*(-\omega_H \times \omega_H) \\
E_\omega^{R,C}(pP) &= \sum_{i=1}^{l} \int_{{u_i}^{-1}(W_R^- \times W_C^+)} u_i^*(-\omega \times \omega_s) \\
&+ \sum_{i=1}^{l} \int_{{u_i}^{-1}(W_R^- \times \mathbb{R}_+ \times H_C^+)} (u_i^-, \pi_H \circ u_i^+)^*(-\omega \times \omega_H) \\
&+ \sum_{i=1}^{l} \int_{{u_i}^{-1}(\mathbb{R}_- \times H_C^- \times W_C^+)} (\pi_H \circ u_i^-, u_i^+)^*(-\omega_H \times \omega_s) \\
&+ \sum_{i=1}^{l} \int_{{u_i}^{-1}(\mathbb{R}_- \times H_C^- \times \mathbb{R}_+ \times H_C^+)} ((\pi_H \times \pi_H) \circ u_i)^*(-\omega_H \times \omega_H)
\end{aligned}
\tag{5.31}
$$

$$
\begin{aligned}
E_\omega^{H,H}(pP) &= \sum_{i=1}^{l} \int_{{u_i}^{-1}(\tilde{W}_H^- \times \tilde{W}_H^+)} ((\pi_H \times \pi_H) \circ u_i)^*(-\omega_H \times \omega_H) \\
E_\omega^{R,H}(pP) &= \sum_{i=1}^{l} \int_{{u_i}^{-1}(W_R^- \times \tilde{W}_H^+)} (u_i^-, \pi_H \circ u_i^+)^*(-\omega \times \omega_H) \\
&\quad + \sum_{i=1}^{l} \int_{{u_i}^{-1}(\mathbb{R}_- \times H_C^- \times \tilde{W}_H^+)} (\pi_H \circ u_i^-, \pi_H \circ u_i^+)^*(-\omega \times \omega_H) \\
E_\omega^{H,R}(pP) &= \sum_{i=1}^{l} \int_{{u_i}^{-1}(\tilde{W}_H^- \times W_R^+)} (\pi_H \circ u_i^-, u_i^+)^*(-\omega_H \times \omega) \\
&\quad + \sum_{i=1}^{l} \int_{{u_i}^{-1}(\tilde{W}_H^- \times \mathbb{R}_- \times H_C^+)} (\pi_H \circ u_i^-, \pi_H \circ u_i^+)^*(-\omega_H \times \omega_H) \\
E_\omega^{R,R}(pP) &= \sum_{i=1}^{l} \int_{{u_i}^{-1}(W_R^- \times W_R^+)} u_i^*(-\omega \times w) \\
&\quad + \sum_{i=1}^{l} \int_{{u_i}^{-1}(W_R^- \times \mathbb{R}_- \times H_C^+)} (u_i^-, \pi_H \circ u_i^+)^*(-\omega \times \omega_H) \\
&\quad + \sum_{i=1}^{l} \int_{{u_i}^{-1}(\mathbb{R}_- \times H_C^- \times W_R^+)} (\pi_H \circ u_i^-, u_i^+)^*(-\omega_H \times \omega) \\
&\quad + \sum_{i=1}^{l} \int_{{u_i}^{-1}(\mathbb{R}_- \times H_C^- \times \mathbb{R}_- \times H_C^+)} ((\pi_H \times \pi_H) \circ u_i)^*(-\omega_H \times \omega_H)
\end{aligned}
\tag{5.32}
$$

The ω energy is then defined as the sum over the ω energies in the respective symplectic cobordisms.

(5.33)

$$
\begin{aligned}
E_\omega(pP) = {} & E_\omega^{C,C}(pP) + E_\omega^{C,H}(pP) + E_\omega^{H,C}(pP) + E_\omega^{C,R}(pP) + E_\omega^{R,C}(pP) \\
& + E_\omega^{H,H}(pP) + E_\omega^{H,R}(pP) + E_\omega^{R,H}(pP) + E_\omega^{R,R}(pP)
\end{aligned}
$$

Similarily, the α-energy of pP, $E_\alpha(pP)$ is first defined in the different parts of the product:

(5.34)

$$\begin{aligned}
E_\alpha^{C,C}(pP) &= \sup_{\phi_+}\sum_{i=1}^{l}\int_{u_i^{-1}(W_C^-\times\mathbb{R}_+\times H_C^+)}(\phi_+\circ\pi_\mathbb{R}\circ u_i^+)(u_i^+)^*(dr\wedge\alpha)\\
&+\sup_{\phi_+}\sum_{i=1}^{l}\int_{u_i^{-1}(\mathbb{R}_+\times H_C^-\times W_C^+)}(\phi_+\circ\pi_\mathbb{R}\circ u_i^-)(u_i^-)^*(dr\wedge\alpha)\\
&+\sup_{(\phi_+,\phi'_+)}\sum_{i=1}^{l}\int_{u_i^{-1}(\mathbb{R}_+\times H_C^-\times\mathbb{R}_+\times H_C^+)}((\phi^-,\phi'^+)\circ\pi_\mathbb{R}\times\pi_\mathbb{R}\circ u_i)u_i^*(-dr\wedge\alpha\times dr\wedge\alpha)\\
E_\alpha^{C,H}(pP) &= \sup_{\phi}\sum_{i=1}^{l}\int_{u_i^{-1}(W_C\times\tilde W_H^+)}(\phi\circ\pi_\mathbb{R}\circ u_i^+)(u_i^+)^*(dr\wedge\alpha)\\
&+\sup_{(\phi_+,\phi)}\sum_{i=1}^{l}\int_{u_i^{-1}(\mathbb{R}_+\times H_C^-\times\tilde W_H^+)}((\phi_+,\phi)\circ\pi_\mathbb{R}\times\pi_\mathbb{R}\circ u_i)u_i^*(-dr\wedge\alpha\times dr\wedge\alpha)\\
E_\alpha^{H,C}(pP) &= \sup_{\phi}\sum_{i=1}^{l}\int_{u_i^{-1}(\tilde W_H^-\times W_C)}(\phi\circ\pi_\mathbb{R}\circ u_i^-)(u_i^-)^*(-dr\wedge\alpha)\\
&+\sup_{(\phi,\phi_+)}\sum_{i=1}^{l}\int_{u_i^{-1}(\tilde W_H^-\times\mathbb{R}_+\times H_C^+)}((\phi,\phi_+)\circ\pi_\mathbb{R}\times\pi_\mathbb{R}\circ u_i)u_i^*(-dr\wedge\alpha\times dr\wedge\alpha)\\
E_\alpha^{R,H}(pP) &= \sup_{\phi}\sum_{i=1}^{l}\int_{u_i^{-1}(W_R\times W_H^{\tau,+})}(\phi\circ\pi_\mathbb{R}\circ u_i^+)(u_i^+)^*(dr\wedge\alpha)\\
&+\sup_{(\phi_-,\phi)}\sum_{i=1}^{l}\int_{u_i^{-1}(\mathbb{R}_-\times H_C^-\times\tilde W_H^+)}((\phi_-,\phi)\circ\pi_\mathbb{R}\times\pi_\mathbb{R}\circ u_i)u_i^*(-dr\wedge\alpha\times dr\wedge\alpha)\\
E_\alpha^{H,R}(pP) &= \sup_{\phi}\sum_{i=1}^{l}\int_{u_i^{-1}(\tilde W_H^-\times W_R)}(\phi\circ\pi_\mathbb{R}\circ u_i^-)(u_i^-)^*(-dr\wedge\alpha)\\
&+\sup_{(\phi,\phi_-)}\sum_{i=1}^{l}\int_{u_i^{-1}(\tilde W_H^-\times\mathbb{R}_-\times H_C^+)}((\phi,\phi_-)\circ\pi_\mathbb{R}\times\pi_\mathbb{R}\circ u_i)u_i^*(-dr\wedge\alpha\times dr\wedge\alpha)\\
E_\alpha^{H,H}(pP) &= \sup_{(\phi^-,\phi^+)}\sum_{i=1}^{l}\int_{u_i^{-1}(\tilde W_H^-\times\tilde W_H^+)}((\phi^-,\phi^+)\circ\pi_\mathbb{R}\times\pi_\mathbb{R}\circ u_i)u_i^*(-dr\wedge\alpha\times dr\wedge\alpha)
\end{aligned}$$

(5.35)

$$\begin{aligned}
E_\alpha^{R,R}(pP) &= \sup_{\phi_-}\sum_{i=1}^{l}\int_{u_i^{-1}(W_R^-\times\mathbb{R}_-\times H_C^+)}(\phi\circ\pi_{\mathbb{R}}\circ u_i^+)(u_i^+)^*(dr\wedge\alpha)\\
&+\sup_{\phi_-}\sum_{i=1}^{l}\int_{u_i^{-1}(\mathbb{R}_-\times H_C^-\times W_R^+)}(\phi\circ\pi_{\mathbb{R}}\circ u_i^+)(u_i^+)^*(dr\wedge\alpha)\\
&+\sup_{(\phi_-,\phi'_-)}\sum_{i=1}^{l}\int_{u_i^{-1}(\mathbb{R}_-\times H_C^-\times\mathbb{R}_-\times H_C^+)}((\phi_-,\phi'_-)\circ\pi_{\mathbb{R}}\times\pi_{\mathbb{R}}\circ u_i)u_i^*(-dr\wedge\alpha\times dr\wedge\alpha)\\
E_\alpha^{C,R}(pP) &= \sup_{\phi_-}\sum_{i=1}^{l}\int_{u_i^{-1}(W_C^-\times\mathbb{R}_-\times H_C^+)}(\phi_-\circ\pi_{\mathbb{R}}\circ u_i^+)(u_i^+)^*(dr\wedge\alpha)\\
&+\sup_{\phi_+}\sum_{i=1}^{l}\int_{u_i^{-1}(\mathbb{R}_+\times H_C^-\times W_R^+)}(\phi_+\circ\pi_{\mathbb{R}}\circ u_i^-)(u_i^-)^*(-dr\wedge\alpha)\\
&+\sup_{(\phi_+,\phi_-)}\sum_{i=1}^{l}\int_{u_i^{-1}(\mathbb{R}_+\times H_C^-\times\mathbb{R}_-\times H_C^+)}((\phi_+,\phi_-)\circ\pi_{\mathbb{R}}\times\pi_{\mathbb{R}}\circ u_i)u_i^*(-dr\wedge\alpha\times dr\wedge\alpha)\\
E_\alpha^{R,C}(pP) &= \sup_{\phi_+}\sum_{i=1}^{l}\int_{u_i^{-1}(W_R^-\times\mathbb{R}_+\times H_C^+)}(\phi_+\circ\pi_{\mathbb{R}}\circ u_i^-)(u_i^-)^*(dr\wedge\alpha)\\
&+\sup_{\phi_-}\sum_{i=1}^{l}\int_{u_i^{-1}(\mathbb{R}_-\times H_C^-\times W_C^+)}(\phi_-\circ\pi_{\mathbb{R}}\circ u_i^-)(u_i^-)^*(-dr\wedge\alpha)\\
&+\sup_{(\phi_-,\phi_+)}\sum_{i=1}^{l}\int_{u_i^{-1}(\mathbb{R}_-\times H_C^-\times\mathbb{R}_+\times H_C^+)}((\phi_-,\phi^+)\circ\pi_{\mathbb{R}}\times\pi_{\mathbb{R}}\circ u_i)u_i^*(-dr\wedge\alpha\times dr\wedge\alpha)
\end{aligned}$$

The total α-energy is then defined as the sum over the different parts, i.e.

$$\begin{aligned}
E_\alpha(pP) &= E_\alpha^{C,C}(pP)+E_\alpha^{C,H}(pP)+E_\alpha^{H,C}(pP)+E_\alpha^{R,H}(pP)+E_\alpha^{H,R}(pP)\\
&+E_\alpha^{H,H}(pP)+E_\alpha^{C,R}(pP)E_\alpha^{R,C}(pP)+E_\alpha^{R,R}(pP)
\end{aligned}$$

The suprema for are taken over the sets of all C^∞-functions $\phi,\phi^{\mp},\phi'^{\mp}:\mathbb{R}\to\mathbb{R}_+$

and $\phi_\mp : \mathbb{R}_\mp \to \mathbb{R}_+$, such that

$$\int_{\mathbb{R}} \phi(r)dr = \int_0^{\mp\infty} \phi_\mp^\mp(r)dr = \int_{\mathbb{R}} \phi^\mp(r)dr = \int_{\mathbb{R}} \phi'^\mp(r)dr = 1. \tag{5.36}$$

Notice that this definition of energy differs slightly from the definition given in [Bou+03] where the maximum is taken instead of summing over the different parts of the α-energy. Again these two choices are equivalent in the sense that a uniform bound on the maximum of the α-energies implies a uniform bound on the sum of the α-energies and vice versa. Finally the total energy of a punctured pearly trajectory pP is given by:

$$E(pP) = E_\omega(pP) + E_\alpha(pP). \tag{5.37}$$

It follows from Lemma 9.1 in [Bou+03] that the energy of a sequence of pearly trajectories $(P)_{n\in N}$ in (W^n, J^n) which converges to a punctured pearly trajectory pP in $(\tilde{W}, \tilde{J})$ satisfies

$$\lim_{n\to\infty} E_\omega((P)_{n\in\mathbb{N}}) = E_\omega(pP). \tag{5.38}$$

By Lemma 9.2 of the same reference there exists a constant $C > 0$ depending only on W, J, C,and $\mathcal{S}$such that for every $\tau > 0$ every pearly trajectory P satisfies

$$E(P) \leq CE_\omega(P) \tag{5.39}$$

Remark 5.13.
I have modelled the definition of energy put forward in this section on the definition of energy put forward in Section 9.2 of [Bou+03]. Thus, given a holomorphic curve u : $(S, j) \to (T, J)$ with domain a Riemann surface S and target $T = W, W^\tau$ or $\tilde{W}^- \times$ as above, the energy of u is defined exactly in the same manner as I have defined the energy of pearly and punctured pearly trajectories above. I will therefore, abusing notation, also

refer to the energy of a (single) holomorphic curve by $E(u)$.

5.5.3 Holomorphic projections and asymptotics

For the proof of Theorem 5.1 and later the proof of Theorem 1.6 the following consequences of using $\mathcal{S}$-adapted almost complex structures of the form of J_C and J_H are relevant:

For the obvious projections in the symplectic cobordisms W_C and W_H and W_H^τ defined in Equation 5.5 and their completions $\tilde{W}_C, \tilde{W}_H$ defined in Equation 5.19 above, set

$$\pi_N \circ u = u_N \quad \text{for the manifolds} \quad N = H_C, C, B.$$

Moreover denote the projection $TH_C \to \xi_H$ along the Reeb direction of H_C by pr_{ξ_H}.

A map $u : S \to T$ with domain a Riemann surface (S, j) and target

$$(T, J) \in \{(W_C, J_C), (\tilde{W}_C, \tilde{J}_C), (W_H, J_H), (W_H^\tau, J^\tau), (\tilde{W}_H, \tilde{J}_H)\}$$

is defined by a $(k+1)$-tuple of maps:

$$u = (a_1, \ldots, a_k, u_C) : (S, j) \to (T, J),$$

where, if necessary, the change of coordinates, is provided by the symplectomorphism ψ of Lemma 5.3.

The map u is (j, J) holomorphic if it satisfies the $(k+1)$ equations:

$$\begin{aligned} J_B \circ du_B &= du_B \circ j \\ {u_C}^* \alpha_i &= da_i \circ j \quad \text{for } i = 1, \ldots, k. \end{aligned} \tag{5.40}$$

A map $u : S \to T'$ with domain a Riemann surface (S, j) and target

$$(T', J') \in \{(W_H, J_H), (\tilde{W}_H, \tilde{J}_H)\}$$

is also defined as a pair of maps

$$\tilde{u} = (a, u_H) : (S, j) \to (T', J').$$

The map u' is (j, J') holomorphic if it satisfies the equations

$$J_{\xi_H} \circ pr_{\xi_H} \circ du' = pr_{\xi_H} \circ du \circ j. \qquad (5.41)$$
$$u_H{}^* \alpha = da \circ j.$$

Thus for a (j, J')-holomorphic map u as above, there are two ways in which its holomorphicity can be expressed.

Moreover notice that

$$\begin{aligned}
u_C &= \pi_C \circ u. && (5.42)\\
&= \pi_C \circ \pi_H \circ u.\\
u_B &= \pi_B \circ u_C && (5.43)\\
&= \pi_B \circ \pi_C \circ u. && (5.44)\\
&= \pi_B \circ \pi_C \circ \pi_H \circ u.
\end{aligned}$$

An important ingredient of any compactness proof is the asymptotic behaviour of holomorphic curves near punctures in the domain. This is described Theorem 9.6 [Wen16], which is the generalisation of Proposition 5.8 to the stable case. A more detailed description is given in Theorem 9.8 of [Wen16], which generalises Proposition 5.6, 5.7 in [Bou+03] to the stable case. I present the main implications of Theorem 9.8 in [Wen16] to the present situation below:

Denote by $\dot{D} = D \backslash \{0\}$ the punctured unit disc and define two biholomorphic maps

$\varphi_\pm$ by

$$\varphi_+ : Z_+ := [0,\infty) \times S^1 \to \dot{D} : (s,t) \to e^{-2\pi(s+it)}$$
$$\varphi_- : Z_- := (-\infty,0] \times S^1 \to \dot{D} : (s,t) \to e^{2\pi(s+it)}.$$

Proposition 5.14.

Let C be a fibred, stable coisotropic submanifold of a symplectic manifold (W,ω). Assume $\tilde{W}_C$ and $\tilde{W}_H$ are equipped with $\mathcal{S}$-adapted almost complex structures. If one of the following conditions hold

(i) $\tilde{u} : \dot{D} \to \tilde{W}_C$ is a $\tilde{J}_C$-holomorphic curve of finite energy $E(\tilde{u}) < \infty$.

(ii) $\tilde{u} : \dot{D} \to \tilde{W}_H$ is a $\tilde{J}_H$-holomorphic curve of finite energy $E(\tilde{u}) < \infty$.

Then either the singularity at $0 \in D$ is removable or $\tilde{u}$ is a proper map. In the latter case the map

$$u(a_1,\dots,a_k,u_C) := \tilde{u}\circ\varphi_\pm \quad \textit{for } (s,t) \in Z_\pm \textit{ near infinity}$$

satisfies

$$a_i(s,\cdot) - s(T\hat{p}_i) \to c \quad in \quad C^\infty(S^1) \quad as\ s \to \pm\infty \quad for\ all\ 1 \le i \le k \tag{5.45}$$
$$\tilde{u}_C(s,\cdot) \to \gamma(\cdot) \qquad in\ C^\infty(S^1,C) \quad as\ s\to\pm\infty \tag{5.46}$$

for a constant $c \in \mathbb{R}$ and where the triple $(\gamma,\hat{p},T)$ is a generalised Reeb orbit, i.e. $\hat{p} \in S_1^{k-1}, T \in \mathbb{R}_{>0}$ and $\gamma : S^1 \to C$ is a solution to Bolle's equation $\dot{\gamma}(t) = \sum_{i=1}^k T\hat{p}_i X_i(\gamma(t))$ of period T. The energy $E(\tilde{u})$ is bounded below by $|T|$. Moreover the J_B-holomorphic map $\tilde{u}_B$ mapping to the symplectic reduction B of C approaches a point $b \in B$.

Proof. By assumption C is stable and fibred, thus by 3.23 H_C is stable and the Reeb flow on H_C is of Morse-Bott type by Corollary 3.29. By Theorem 9.8 of [Wen16] the holomorphic curve $(a, u_H) := \tilde{u} \circ \varphi_\pm$ satisfies

$$a(s,\cdot) - s(T\hat{p}) \to c \quad \text{in } C^\infty(S^1) \quad \text{as } s \to \pm\infty \tag{5.47}$$

$$\tilde{u}_H(s,\cdot) \to (\hat{p}, \gamma(T\cdot)) \in S^{k-1} \times C \qquad \text{in } C^\infty(S^1, H_C) \quad \text{as } s \to \pm\infty. \tag{5.48}$$

for a Reeb orbit (p,γ) of period T on H_C. Recall that by Proposition 3.25 there is a one to one correspondence of Reeb trajectories on H_C and generalised Reeb trajectories on C. In particular, given (p,γ,T) as a Reeb orbit of H_C there exists a unique generalised Reeb orbit $(\gamma,\hat{p},T)$ on C. The result now follows by applying the change of coordinates ψ from Lemma 5.3. By Equations 5.40 and 5.41 $\tilde{u}_C$ is holomorphic. Since C is fibred the leaf $\mathbb{T}^k_{\gamma(0)}$ is the kernel of $\pi_{B_C}(\gamma(o)$. Now Equation 5.42 implies that $\tilde{u}_B$ approaches a point $b \in B$. □

5.6 Proof of Theorem 5.1

Before embarking on the proof of Theorem 5.1 I give an outline of the structure of the proof below.

5.6.1 Outline of the proof

Strategy of the proof

Given a coisotropic C satisfying the assumptions of Theorem 5.1 choose neck stretching data $\mathcal{N}$ as in Definition 5.8. Apply Theorem 4.1 to each manifold $W^n \times W^n$ to obtain a sequence of pearly trajectories $(P_n)_{n\in\mathbb{Z}_{\geq 0}}$ with the properties listed in the assertion of Theorem 4.1. The strategy of the proof is to subsequently extract subsequences of $(P_n)_{n\in\mathbb{Z}_{\geq 0}}$ which eventually converge in an appropriate sense to a punctured pearly trajectory which has the properties from the assertion of Theorem 5.1.

By elliptic bootstrapping and the Arzela-Ascoli theorem , the only obstruction to the existence of a converging subsequence is the lack of a uniform bound on the gradient of the sequence of pearly trajectories $(P_n)_{n\in\mathbb{Z}_{\geq 0}}$ (see Section 2.2.3 of [Abb14] for details). To establish a uniform gradient bound one repeatedly carries out a bubbling off analysis, which "absorbs" gradient blow ups in the targets as alterations of the domains by a local, conformal rescaling procedure. I describe this procedure in more detail in Section 5.6.3.

As a result of the analysis the local gradient blow up no longer occurs and the domain has a new part which serves as the domain of the "bubble". Each such bubble carries a positive amount of energy as defined in Section 5.5.2. Thus if one shows that the energy of the sequence of pearly trajectories is bounded and diminishes in this process, this algorithmic bubbling-off process terminates after finitely many repetitions and a uniform gradient bound exists. By Arzela-Ascoli and elliptic bootstrapping this implies the existence of a converging subsequence.

In this proof I follow the exposition given in [Abb14], which relies on the ideas presented in [Bou+03].

The proof of the theorem is structured into four main parts, which I list and explain briefly below

Section 5.6.2: Preliminaries

I show first how pearly trajectories fit into the framework of stable Riemann surfaces. I then explain how one can decompose domain and target of the pearly trajectories into different parts which can be analysed separately. The domain consists of a thin and a thick part (see Equation 5.49). The target consists of the products of symplectic cobordisms listed in Equation 5.21. A key point is that one may view the holomorphic curves contributing to the pearly trajectories as either a single holomorphic curve satisfying a Lagrangian boundary condition or as a pair of holomorphic curves depending on which part of the target one considers. Moreover I show that there exists a uniform bound on the energy of the sequence of pearly trajectories provided by Theorem 4.1.

Section 5.6.3: The bubbling Lemma

I describe how gradient blow-ups of the pearly trajectories in the target can be "absorbed" by alterations of the domain: A conformal rescaling of a neighbourhood of the blow up in the domain makes it possible to bound the gradient in that neighbourhood. A case by case analysis, depending on the local data of the Riemann surfaces in the neighbourhood, shows that one can "absorb" the blow up by adding one or two sphere or one or two disc components to the domains.

Section 5.6.4: Algorithmic removal of obstructions to compactness

It takes three main steps to establish uniform gradient bounds for the sequence of pearly trajectories:

Step 1 Gradient bounds away from punctures:

One proceeds algorithmically in a case by case analysis to establish uniform gradient bounds away from finitely many points in the domain. If there is a sequence of points along which the gradient of the pearly trajectory is unbounded one alters the domains according to the procedure described in Section 5.6.3 on the bubbling Lemma. Each sphere or a disc bubble has positive energy, thus this bubbling-off process terminates after finitely many steps. One can treat each of the parts of the target manifold, listed in Equation 5.21, separately. In each part one has to make the necessary case distinctions. In $W_C \times W_C$ one has to distinguish between a gradient blow up occurring along a sequence of points converging to the boundary of the domain and a gradient blow up along a sequence of points remaining in the interior. In the other eight parts , one has to analyse all possible cases that lead to a gradient blow -up of P_n. There are essentially two of these: The gradient of P^n blows up if either the gradient of projection to the first factor, P_n^- blows up while the gradient of P_n^+, the projection to the second factor, remains bounded, or both the gradients of ${P_n}^-$ and ${P_n}^+$ blow up, possibly at different speeds.

Step 2 Convergence in the thick part:

One establishes convergence in the thick part of the Riemann surface by using the uniform gradient bounds obtained above and the estimate for the injectivity radius on the thick part. This part is not different from the standard literature. I include it for the sake of completeness

Step 3 Convergence in the thin part:

I establish convergence in the thin part of the Riemann surface. I use the description of the thin part from the preliminaries and a rescaling metric to obtain uniform gradient bounds. Thus one has established convergence on the entire Riemann surface, and thereby shown the existence of limit punctured pearly trajectory pP. This part is also not much different from the standard procedure described in Section 10.2.3 of [Bou+03]. Again I include it for the sake of completeness.

Section 5.6.5: Properties of the limit punctured pearly trajectory

Finally I prove that the limit object pP satisfies the properties listed in the assertion of Theorem 5.1.

5.6.2 Preliminaries

Given a coisotropic C satisfying the assumptions of Theorem 5.1, choose neck stretching data $\mathcal{N}$ as in Definition 5.8. Apply Theorem 4.1 to each manifold $W^n \times W^n$ to obtain a sequence of pearly trajectories $(P_n)_{n\in\mathbb{Z}_{\geq 0}}$ with the properties listed in the assertion of Theorem 4.1.

Since the domain of P_n is $(D, \partial D)$ add the set $M_\partial = \{m_1, m_2, m_3\}$ of three boundary marked points to $(D, \partial D)$ (one could also add two marked points in the interior). Associate the (now stable) Riemann surface data $S_n = (D, \partial D, i, M_\partial)$ to the domains of the pearly trajectories as described in Section 5.5.1 and denote the pearly trajectories by pP^n. Notice that the sets of D_n and Z_n of nodal pairs and punctures are empty. The uniformisation theorem (Theorem 1.14 of [Abb14]) now guarantees the existence of a unique complete hyperbolic metric h_n on S_n^d with constant curvature -1. Denote by $\rho_n(z)$ the injectivity radius of h_n. Decompose S_n^d into a thick and a thin part given by

$$
\begin{aligned}
&\mathrm{Thick}_\epsilon(S_n) = \{z \in (S_n \setminus M) | \rho_n(z) \geq \epsilon\}. \\
&\mathrm{Thin}_\epsilon(S_n) \;= \{z \in (S_n \setminus M) | \rho_n(z) < \epsilon\}.
\end{aligned}
\tag{5.49}
$$

One may choose ϵ universally in such a way that every thin component in the sequence S_n^d is conformally equivalent to either a finite cylinder $[-R, R] \times S^1$ or to $[0, \infty) \times S^1$. Another fact from hyperbolic geometry we will use is Bers' theorem stated at the beginning of Section 1.3 of [Abb14]. It asserts the existence of a pair of pants decomposition of each S_n^d where the length of the boundaries of each pair of pants is bounded above. By having added the marked points to the domains, one may now view the sequence of pearly trajectories $P_n : S_n \to W^n$ as a sequence of punctured pearly trajectories pP_n.

In the proof one subsequently extracts subsequences (of subsequences) of punctured pearly trajectories, such that a subsequence of $(pP^n)_{n\geq 0}$ eventually converges to a finite energy punctured pearly trajectory pP which satisfies the properties (pP1) and (pP2) stated in Theorem 5.1 above. In abuse of notation I will denote all subsequence of $(pP^n)_{n\geq 0}$ still by $(pP^n)_{n\geq 0}$.

For all $n \geq 0$, I continue to denote the projection onto the first factor of the cartesian products $W^{n,-} \times W^{n,+}$ and $\tilde{W}^- \times \tilde{W}^+$ by p_-, and the projection onto the second factor by p_+. It can be helpful to keep the diagrams below in mind.

$$
\begin{tikzcd}
W_C^- \arrow[d, "\pi_C"] & W_C^- \times W_C^+ \arrow[l, "p_-"'] \arrow[r, "p_+"] & W_C^+ \arrow[d, "\pi_C"] \\
\{0\} \times C & L_C \arrow[l, "i_-"'] \arrow[r, "i_+"] \arrow[u, "i_{L_C}"] & \{0\} \times C
\end{tikzcd}
$$

$$
\begin{tikzcd}
W_H^{\tau,-} \arrow[d, "\pi_H"] & W_H^{\tau,-} \times W_H^{\tau,+} \arrow[l, "p_-"'] \arrow[r, "p_+"] \arrow[dl, dashed] \arrow[d, dashed] \arrow[dr, dashed] & W_H^{\tau,+} \arrow[d, "\pi_H"] \\
H_C \arrow[r, "\pi_C"] & C & H_C \arrow[l, "\pi_C"']
\end{tikzcd}
$$

The following seemingly trivial observation is important for the proof: By the definition of the cartesian product and by our choice of product almost complex structures $-J^n_{\mathcal{S}} \times J^n_{\mathcal{S}}$ and $-\tilde{J}_{\mathcal{S}} \times \tilde{J}_{\mathcal{S}}$ the sequence of punctured pearly trajectories pP_n can be interpreted in two ways:

1. As a sequence of finite collections $(u_1, \dots, u_{k_n})_{n \geq 0}$ of $(j^n, -J^n_{\mathcal{S}} \times J^n_{\mathcal{S}})$-holomorphic maps
(5.50)
$$u_{n,i} : (S_n, \partial S_n, j^n) \to (W^{n,-} \times W^{n-+}, L_C, -J^n_{\mathcal{S}} \times J^n_{\mathcal{S}}). \quad \text{for } i = 1, \dots, k_n$$

   To simplify the notation I will continue to denote such a sequence of finite collections of holomorphic maps contributing to the pearly trajectory by pP_n.

2. As a sequence of finite collections $(u_1, \dots, u_{k_n})_{n \in \mathbb{Z}_{\geq 0}}$ of pairs $(u^-_{n,i}, u^+_{n,i})$ consisting of:

$$\begin{aligned} &\text{finite collections of } (j_n, -J^n_{\mathcal{S}})\text{-holomorphic maps:} \\ &u^-_{n,i} : (S_n, \partial S_n, j_n) \to (W^{n,-}, C^-, -J^n_{\mathcal{S}}) \quad \text{for } i = 1, \dots, k_n, \\ &\text{and finite collections of } (j_n, J^n_{\mathcal{S}})\text{-holomorphic maps:} \\ &u^+_{n,i} : (S_n, j_n) \to (W^{n,+}, C^+, J_{\mathcal{S}^n}) \quad \text{for } i = 1, \dots, k_n. \end{aligned} \tag{5.51}$$

   Again simplifying notation I will denote these projections of sequences of finite collections by pP^-_n and pP^+_n respectively.

The energy of a sequence of finite pearly trajectories was defined in Section 5.5.2. By the definition of the algebraic structures on the pearl complex, see Equations 4.15 and 4.18 in Chapter 4, the Maslov index $\mu(P_n)$ of each element of the sequence is bounded above by either 2 dim L_C, or dim L_C+1 depending on whether it is a pearly product or a pearly differential trajectory ending in the minimum. By the monotonicity assumption the energy of a pearly trajectory P_n, as defined in section 5.5.2 is positively proportional to the Maslov index. Thus the constant E_0 from assertion (E) of Theorem 4.1 can be chosen to serve as a uniform bound on the energy of the sequence of pearly trajectories.

5.6.3 The bubbling Lemma

A key step in proving the existence of a convergent subsequence is to understand how a concentration of energy leading to a gradient blow up of a sequence of pearly trajectories in the target can be absorbed by an alteration of the domains of the sequence. This analysis is not limited to the sequences of pearly trajectories, but valid for all stable Riemann surfaces and standard. Despite the name of this Section, I do not state the results of this section as a Lemma, since the formulation is very cumbersome. For a precise statement see for example Proposition 4.3 in [Bou+03] or Section 3.2 in [Abb14]. Instead I include a detailed description of the phenomena for the sake of clarity of the exposition and in an attempt to increase the readability of the manuscript. I suggest consulting the picutres, which illustrate the phenonema in each of the possible cases, before reading the description of the respective case.

Given a sequence of stable Riemann surface data with S_n with fixed signature, then S_n converges to a stable, nodal Riemann surface S by the Deligne-Mumford compactness theorem (see Section 1.3 of [Abb14] and the references therin). Now assume that there exists a sequence of points $z_n \in S_n$ such that the gradient blows up, i.e.

$$R_n := ||dP_n(z_n)|| \to \infty.$$

By this I mean that there exists a sequence of holomorphic maps, denoted in abuse of notation by u_n contributing to the pearly trajectory such that $||du_n(z_n)|| \to \infty$. I now describe how one can bound derivatives in a sequence of neighbourhoods in the sequence of Riemann surfaces S_n by conformal rescaling .

First assume that the sequence z_n stays away from the boundary of the Riemann surface. Then there exists a sequence of holomorphic coordinate charts $\psi_n : B_{\epsilon_n R_n}(0) \to U_n$, where U_n is a neighbourhood of z_n. By Lemma 3.8 in [Abb14] one has for all $z \in U_n$

$$d_n(z_n, z) \leq \frac{C_2 \rho_n(z_n) |\psi_n^{-1}(z)|_{\mathbb{C}}}{R_n} \leq C_2 \rho_n(z_n) \epsilon_n. \tag{5.52}$$

Choose $\epsilon_n \to 0$, while $\epsilon_n R_n \to \infty$. By (5.49), the injectivity radius ρ_n is strictly less than a fixed ϵ if U_n is contained in the thin part of the Riemann surface for all n large enough or greater than or equal to ϵ if U_n is contained in the thick part of S_n for all n large enough. In essence, the idea is now to consider the boundary $\psi_n(\partial B_{\epsilon_n R_n}) = \partial U_n$ as a degenerating boundary component in the sequence of Riemann surfaces, which is the situation considered in the proof of the Deligne-Mumford compactness theorem as given in Theorem 1.91 of [Abb14]. The key idea here is to associate nodal pairs to degenerating boundary components and vice versa. Notice that by this one alters the pair of pants decomposition of the Riemann surface and therefore needs to make sure that the Riemann surface one creates by adding marked points remains stable. Depending on the position of the marked points and nodal pairs in the pair of pants decomposition of the sequence S_n relative to U_n, there are three cases to consider:

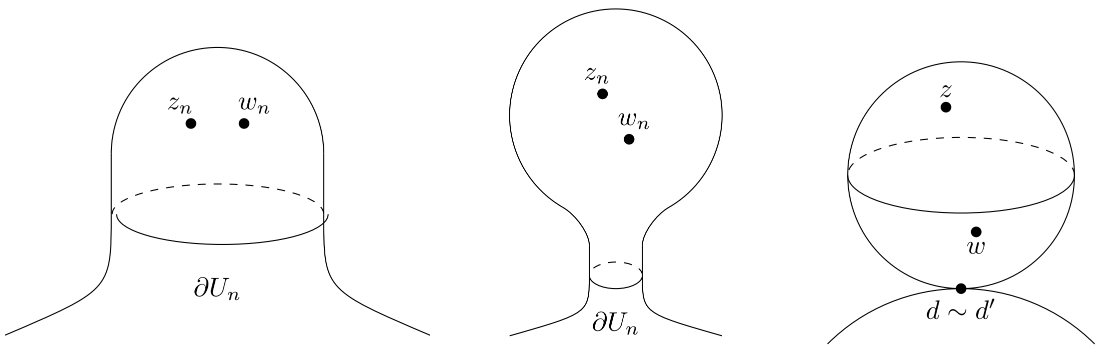
Bubbling, Case 1: U_n contains neither a marked nor a nodal point for all n large enough.

Bubbling, Case 2: U_n contains a marked w_n point from $M_n \cup D_n$ for all n large enough.

Bubbling, Case 3: U_n contains a nodal pair $\{w_n, w_n'\}$ for all n large enough.

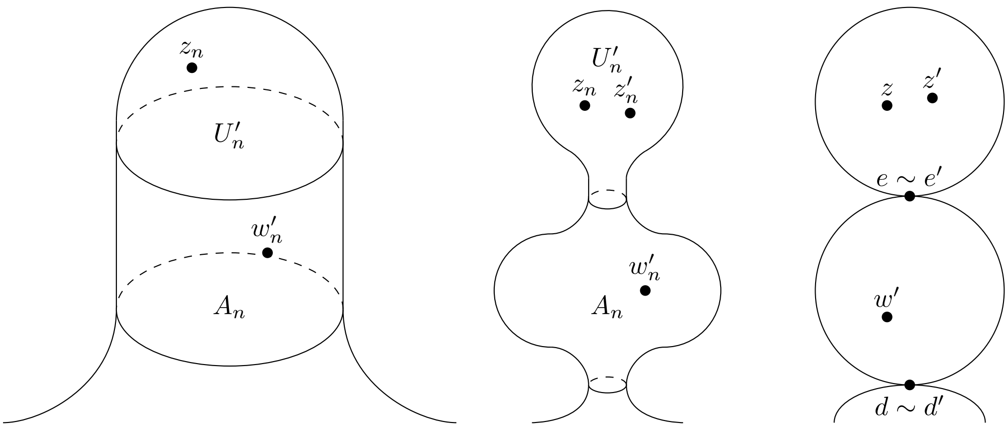
These are all the cases one has to consider since double and marked points are isolated.

Bubbling, Case 1:



One needs to add two marked points in the interior of U_n order to stabilise U_n, which one views as a disc. A natural choice for one of the two marked points is z_n. Denote the other marked point by w_n. Removing these two marked points and ∂U_n will make U_n into a pair of pants. This is the situation described in the proof of the Deligne-Mumford compactness theorem as presented in Section 1.3.2 of [Abb14]. Remove ∂U_n from S_n and replace it with two boundary components (one in the pair of pants obtained from U_n, one in $S_n \setminus U_n$). Treat these two boundary components as a pair of geodesics of the hyperbolic metric h_n degenerating to nodal points $\{d, d'\}$. Thus S_n with the two added marked points z_n and w_n converges to a stable nodal Riemann surface $S' = S \cup_{\{d \sim d'\}} S^2$ obtained by attaching a sphere with the marked points $z, w \in S^2$ corresponding to the limits of z_n and w_n at $\{d, d'\}$ to S.

Bubbling, Case 2:

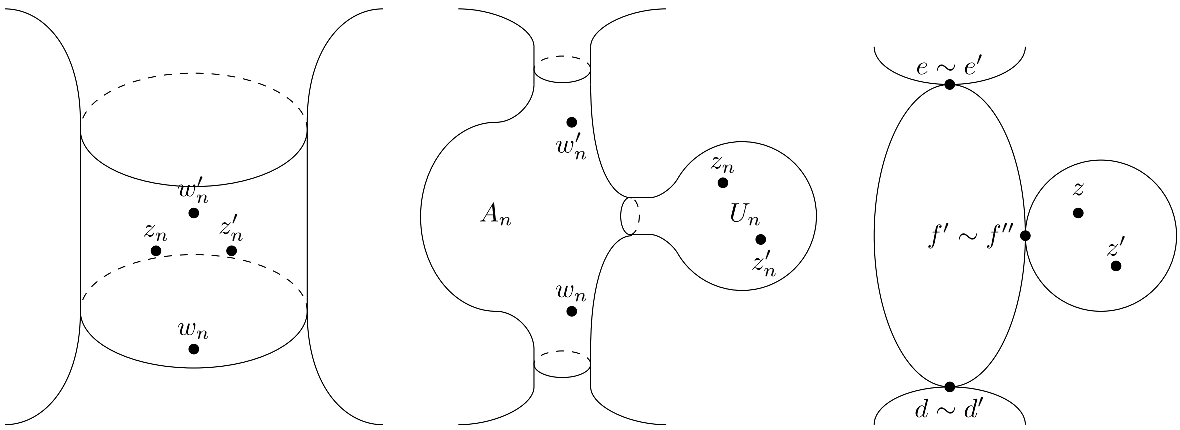


There are two subcases to consider. If the marked point w_n contained in U_n does not correspond to a boundary component that degenerates to a point as $n \to \infty$, one adds z_n and w_n as marked points to U_n and is back in the situation considered in Bubbling, Case 1.

Otherwise the marked point w_n contained in U_n corresponds to a boundary component of a pair of pants composition that collapses faster to a point than ∂U_n. Then there exists a sequence of annuli A_n which separate w_n from z_n in U_n. By this I mean that one of the boundary components of A_n coincides with ∂U_n and that z_n is contained in the interior of $U_n \setminus A_n$. The inner boundary component of A_n corresponds to the marked point w_n.

One now adds a marked point z_n' to $U_n' = U_n \setminus A_n$ and another marked point w_n' to A_n to stablise both A_n and U_n'. By assumption, the boundary of $S_n \setminus U_n$ and the outer boundary of the annulus A_n collapse to a pairs of nodal points $\{d, d'\}$ as $n \to \infty$. Moreover the inner boundary of A_n and the boundary of U_n' collapses to a pair of nodal points $\{e, e'\}$ as as $n \to \infty$. So, repeating the procedure describe in case 1, the sequence S_n with the marked points z_n, z_n' and w_n', converges to a stable nodal Riemann surface S'' obtained from S by attaching one sphere containing w_n' along the nodal pair $\{d, d'\}$ corresponding to the boundary of $S_n \setminus U_n$ and ∂U_n and a sphere containing z and z', the limits of z_n and z_n' attached along a pair of nodal points $\{e, e'\}$ corresponding to the inner boundary component of A_n and the boundary of U_n'.

Bubbling, Case 3:



Represent the pair of nodal points $\{w_n, w_n'\}$ as a pair of degenerating boundary components. If there exists an annulus in A_n that separates one of the points in $\{w_n, w_n'\}$ and z_n from the other point in $\{w_n, w_n'\}$, the situation is as the one considered in Bubbling, Case 2.

Otherwise U_n is contained in an annulus A_n whose degenerating boundary components are represented by the pair $\{w_n, w_n'\}$ of nodal points. Add a marked point z_n' to U_n to stabilise it. Now both boundary components of the annuli A_n and the boundary component of U_n collapse to pairs of nodal points $\{d, d'\}, \{e, e'\}$

and $\{f', f''\}$. Repeating the analysis from the previous cases, we see that S_n with marked points z_n, z_n', w_n, w_n' converges to a nodal Riemann surface S'''. This surface is obtained from S by attaching in a sphereS' along the pairs of nodal points $\{d, d'\}$ and $\{e, e'\}$. The sphere S' has another sphere S'' attached to it along the nodal pair $\{f, f'\}$. The pair of nodal points $\{d, d'\}$ corresponds to one boundary component of A_n and the corresponding boundary component in $S_n \setminus A_n$. The pair of nodal points $\{e, e'\}$ corresponds to the other boundary component of A_n and its corresponding boundary component in $S_n \setminus A_n$. The pair $\{f', f''\}$ corresponds to the boundary ∂U_n in U_n and the corresponding boundary component in $A_n \setminus U_n$. The sphere S' is stable since it contains $\{d', e', f'\}$. The sphere S'' has z and z' the limits of z_n and z_n' on it and is thus also stable since it is attached along $\{f', f''\}$.

If the sequence of points converges to the boundary of the Riemann surfaces, one uses rescaling coordinate charts described in the following Lemma, which is the boundary version of Lemma 3.8 from [Abb14]. I state it for the sake of completeness:

Lemma 5.15 (boundary version of Lemma 3.8 in [Abb14]).
There are holomorphic charts $\psi_n : B_1^+ \to V_n \subset (\dot{S}_n, j_n)$ with $\psi_n(B_R^+ \cap \mathbb{R}) \subset \partial \dot{S}_n$ and $\psi_n(0) = z_n$ for $z_n \in \partial \dot{S}_n$ and positive constants C_3, C_4 such that for all $z \in D^+$ and all large n

$$C_3 \rho_n(z_n) \leq ||d\psi_n(z)|| \leq C_4 \rho_n(z_n), \tag{5.53}$$

where $B_R^+ = \{z \in \mathbb{C} | \quad ||z|| < R, Im(z) \geq 0\}$ and V_n is a neighbourhood of z_n.

The boundary of V_n degenerates to a point as n grows to infinity. Double the Riemann surface as described in Section 5.5.1, so that V_n becomes a neighbourhood without boundary like U_n with additional data remembering that U_n sits at the boundary of the original Riemann surface. Then carry out exactly the same bubbling off analysis as for interior points keeping track of the additional boundary data.

Having carried out the analysis described above, one can go back to the original Riemann surface with boundary. It turns out that one is attaching one or two discs along boundary nodal pairs instead of one or two spheres along interior nodal pairs. Thus if a gradient blow up occurs along the boundary, one can absorb this by attaching one or two disc components to the original sequence of Riemann surfaces. For further details see section 3.2 of [Abb14].

Summing up, if a gradient blow up occurs, one adds a set of marked points to the original sequence S_n of Riemann surfaces, forming a new sequence of stable Riemann surfaces S'_n. This new sequence converges to a nodal Riemann surface S' which differs from the limit S of the original sequence S_n by one or two sphere or disc components. These spheres or discs serve as the domains of the sphere or disc bubble which now contributes to the new sequence of pearly trajectories $pP_{n'}$.

5.6.4 Algorithmic removal of obstructions to compactness

Step 1: Gradient bounds

In this section I explain how to obtain gradient bounds for the sequence of pearly trajectories $(pP_n)_{n\geq 0}$ away from finitely many points in the domain. I will use the bubbling-off procedure described in the preceding section to jump back and forth between the sequences of domains and the sequences of images of $(pP_n)_{n\geq 0}$ in an algorithmic procedure. More precisely I prove the following proposition in this section:

Proposition 5.16 (Prop 3.7 in [Abb14]).
There exists an integer $K \geq 0$ and a constant $C > 0$ which depend only on E_0 and points

$$Y_n = \{y_n^{(1)}, {y'}_n^{(1)}, \dots, y_n^{(K)}, {y'}_n^{(K)}\} \subset S_n \setminus (M_n) \tag{5.54}$$

such that

$$||dP_n(z)|| \leq \frac{C}{\rho_n(z)} \forall z \in \dot{S}_n := S_n \setminus (M_n \cup Y_n). \tag{5.55}$$

Here ρ_n denotes the injectivity radius with respect to the Poincaré metric $h^{\dot{S}_n \setminus Y_n}$ on $(\dot{S}_n \setminus Y_n, j_n)$. The gradient is computed with respect to $h^{\dot{S}_n \setminus Y_n}$ and the metric induced by the respective compatible choices of ω and J_S in the corresponding parts of the target manifold and for the corresponding holomorphic maps contributing to pP_n.

Proof. The sets of double points and punctures are empty, i.e.$D_n \cup Z_n = \emptyset$. In Section 5.6.2 marked points were added to stabilise the Riemann surfaces S_n underlying the pearly trajectories. By the Deligne-Mumford compactness result for Riemann surfaces with boundary one may assume that, after passing to a subsequence, S_n converges to a noded surface with boundary S. Note that S may have nonempty sets D and Z. Recall that by elliptic bootstrapping, the only obstruction to applying the Arzela-Ascoli theorem can come from the lack of a C^1 bound of pP_n. Thus, if the gradient of pP_n is uniformly bounded on $S_n \setminus M_n$, Proposition 5.16 follows.

Otherwise there exists a sequence $z_n \in S_n \setminus M_n$ such that

$$\lim_{n \to \infty} \rho(z_n) ||dP_n(z_n)|| = +\infty. \tag{5.56}$$

Recall that this mean that there exists a sequence of holomorphic maps contributing to the pearly trajectory such that the gradient blows up. One may treat each such sequence individually, one after the other since there are only finitely many holomorphic maps contributing to pP_n and finitely many possibilities of configurations of holomorphic discs due to the uniform energy bound. By the notation (5.56) I want to indicate that one performs the relevant steps whenever necessary. There are two main cases to consider:

Step1, Case 1: $pP_n(z_n)$ is contained in a compact subset of $\tilde{W}_C \times \tilde{W}_C$ or of $\tilde{W}_R \times \tilde{W}_R$ for all n large enough.

Step1, Case 2: $pP_n(z_n)$ is contained in any other of the remaining seven parts of $\tilde{W} \times \tilde{W}$ listed in Equation 5.22 for all n large enough.

In all cases, the procedure is similar: first choose an appropriate rescaling (either Lemma 3.8 of [Abb14] or Lemma 5.15 above), which bounds the gradient on a neighbourhood of a blow-up and extract a subsequence of holomorphic curves, with domain the rescaled neighbourhood, converging to a non constant holomorphic curve (the bubble) . Second, use the appropriate quantisation of energy theorem (Proposition 4.1.4 in [MS12], Proposition 2.59 in [Abb14] or Proposition 5.14) to show that the limit holomorphic curve has positiv energy bounded away from zero. Finally add a set of marked points representing the domain of the limit holomorphic curve to the domain of the original Riemann surfaces according to the procedure described in Section 5.6.3. If there are still sequences along which the gradient blows up, repeat this series of steps. In order to avoid notation like $\mathring{\tilde{y}}_n^{5'}$ I will abuse notation and always denote the marked points one adds by y_n and y_n'. Likewise I will always denote the set containing y_n, y_n' by Y_n. This process terminates, since the energy of the sequence of pearly trajectories is finite. Each time one runs this "algorithm" consumes a positive amount of energy. I now describe this procedure in more detail:

Step1, Case 1:

In this case view pP_n as in equation (5.50), since one wants to use the Lagrangian boundary condition for the analysis. Recall that the Lagrangian L_C is defined only as a submanifold of the product $\tilde{W}_C \times \tilde{W}_C$ and not in a single factor.

There are two subcases to analyse:

Step 1, Case 1.1: The sequence z_n stays away from ∂S_n for all n large enough.

Step1, Case 1.2: The sequence z_n converges to the boundary ∂S_n for all n large enough.

Remark 5.17.

Notice that one needs to make another case distinction in Case 1.2. If the sequence z_n con-

verges to the boundary at slower speed than the rescaling parameter R_n, a spherical bubble forms, thus one is back in Case1.1. This is explained in detail for example in [Fra08].

Step1, Case 1.1: First notice that this is always the case given the sequence stays in a compact subset of $\tilde{W}_R \times \tilde{W}_R$ for all n large enough. Define

$$Q_n = P_n \circ \psi_n : B_{\epsilon_n R_n} \to \tilde{W}_C \times \tilde{W}_C \tag{5.57}$$

By the standard bubbling off analysis this sequence converges to a non constant holomorphic sphere $Q_\infty : \mathbb{C} \cup \{\infty\} \to \tilde{W}_C \tilde{W}_C$, This sphere bubble has positive energy by Proposition 4.1.4 in [MS12]. Add the set Y_n consisting of the marked points y_n and y'_n to the sequences S_n according to the procedure described in Section 5.6.3. Then $S'_n = (S_n, M_n \cup Y_n)$, converges to a stable nodal Riemann surface S', which differs from S by one or two spherical components, depending on the local Riemann surface data as described in Section 5.6.3 above.

Step1, Case 1.2: Use Lemma 5.15 to define a sequence of holomorphic curves

$$Q_n = P_n \circ \psi_n : B^+_{\epsilon_n R_n} \to U_n. \tag{5.58}$$

The standard bubbling off analysis shows that the limit map is a non-constant holomorphic disc with boundary on L_C, because the puncture is always removable in the present case, since the image of Q_n is contained in the compact part of $\tilde{W}_C$ and L_C is compact by assumption. This disc has positive energy by Proposition 4.1.4 in [MS12]. Again, add the set Y_n consisting of the marked points y_n and y'_n to the sequences S_n as describe in Section 5.6.3. Then $S'_n = (S_n, M_n \cup Y_n)$, converges to a stable nodal Riemann surface S', which differs from S by one or two disc components as described in Section 5.6.3 above. This concludes the analysis for Step1, Case 1.

Step1, Case 2:

In this case view pP_n as in equation (5.51). This is possible since in this case there cannot be a sequence z_n along which the gradient blows up and which converges to the boundary of ∂S_n and thus as sequence $pP_n(z_n)$ converging to L_C. To see this, recall that L_C is a subset of $\psi_C(\{0\}\times C)\times\phi_C(\{0\}\times C)$, where ψ_C is the symplectomorphism provided by the Bolle neighbourhood theorem, Proposition 2.18 and thus increasingly far away as n grows to infinity. Note that

$$||dP_n(z_n)|| = ||dP_n^-(z_n)|| + ||dP_n^+(z_n)||. \tag{5.59}$$

Thus there are again two sub-cases to analyse:

Step 1, Case 2.1: $dP_n^{\mp}$ is bounded and $dP_n^{\pm}$ is unbounded.

Step 1, Case 2.2: Both dP_n^- and dP_n^+ are unbounded.

First analyse **Step 1, Case 2.1:**

Without loss of generality assume

$$||dP_n^-(z_n)|| \to \infty \quad , ||dP_n^+(z_n)|| \leq C \tag{5.60}$$

Choose holomorphic rescaling charts

$$\psi_n : B_{\epsilon_n R_n}(0) \to U_n, \tag{5.61}$$

where $R_n = ||dP_n^-(z_n)||$ and define:

$$(Q_n^-, Q_n^+) = (P_n^- \circ \psi_n, P_n^+ \circ \psi_n). \tag{5.62}$$

Then it follows from the usual bubbling off analysis in each separate factor that Q_n^- converges to a finite energy holomorphic plane Q_∞^-, while Q_n^+ converges to a constant map on U_n. If the $\mathbb{R}$-component of P_∞^- is unbounded, then by Proposition 5.14, P_∞^- is asymptotic to a cylinder over a generalised Reeb orbit and the energy of P_∞^- is bounded below by the period of the generalised Reeb orbit it converges to. Otherwise apply the removal of singularities theorem (Theorem 2.68 in [Abb14]) to obtain a non-constant holomorphic sphere, which has positive energy again by Proposition 4.1.4 in [MS12]. Add a set Y_n consisting of the marked points y_n and y_n' to the sequence S_n. Then $S_n' = (S_n, M_n \cup Z_n \cup Y_n)$, converges to a stable nodal Riemann surface S', which differs from S by one or two spherical components as described in Section 5.6.3 above.

Step1, Case 2.2

One cannot, without loss of generality, assume that there is any relation between $||dP_n^-(z_n)||$ and $||dP_n^+(z_n)||$.

Choose holomorphic rescaling charts

$$\psi_n : B_{\epsilon_n R_n^-}(0) \to U_n, \tag{5.63}$$

where $R_n^- = ||dP_n^-(z_n)||$ and define

$$(Q_n^-, Q_n^+) = (P_n^- \circ \psi_n, P_n^+ \circ \psi_n). \tag{5.64}$$

Then it follows from the usual bubbling off analysis in each factor that Q_n^- converges to a finite energy holomorphic plane Q_∞^-. If the $\mathbb{R}$-component of P_∞^- is unbounded, then by Proposition 5.14 P_∞^- is asymptotic to a cylinder over a generalised Reeb orbit and the energy of P_∞^- is bounded below by the period of the generalised Reeb orbit it converges to by Proposition 5.14. Otherwise apply the removal of singularities theorem and obtain a non-constant holomorphic sphere, which has again has positive energy by Proposition 4.1.4 in [MS12]. Again add a set Y_n consisting of the marked points y_n and y_n' to the sequence S_n. Then $S_n' = (S_n, M_n \cup Z_n \cup Y_n)$, converges to a stable nodal Riemann surface S', which differs from S by one or two spherical components as described in section 5.6.3 above.

Now there are two cases to consider

Step 1, Case 2.2.1: Q_n^+ also converges to a finite energy holomorphic plane or to a constant map.

Step 1, Case 2.2.2: Q_n^+ does not converge.

Step 1, Case 2.2.1:

If Q_n^+ converges to a constant map one is back in Case 2.1. Thus assume Q_n^+ is non-constant. It follows from the usual bubbling off analysis in each factor that both Q_n^- and Q_n^+ converge to finite energy holomorphic planes $Q_\infty^\mp$. Notice that in the present case, by the choice of R_n^-, $0 < ||dQ_n^+|| \leq ||dQ_n^-|| \leq C$. If If both $\mathbb{R}$-components of $Q_\infty^\mp$ are unbounded, the energy of $Q_\infty^\mp$ is bounded below by the smaller period of the pair of generalised Reeb orbits to which $Q_\infty^\mp$ converge by Proposition 5.14. If both $Q_\infty^\mp$ are contained in some compact subset of the symplectisations one may apply removal of singularities and obtain two non-constant holomorphic spheres of positive energy. Notice that all possible combinations may occur, for example Q_∞^- could asymptote to a generalised Reeb orbit while Q_∞^+ has a removable singularity or vice versa. In each case, the energy of the pair $Q_\infty^\mp$ is strictly positive and bounded away from zero. Thus add a set Y_n consisting of the marked points y_n and y_n' to the sequences S_n. Then $S_n' = (S_n, M_n \cup Z_n \cup Y_n)$, converges to a stable nodal Riemann surface S', which differs from S by one or two spherical components as described in section 5.6.3 above. These spherical components serve as the domains of both maps $Q_\infty^\mp$.

Step 1, Case 2.2.2:

If Q_n^+ does not converge, one has not formed a bubble for P_n^+ although the gradient of P_n^+ explodes. Thus there could be a sequence z_n' such that $||dP_n^+(z_n')|| \to \infty$ on the sequence of Riemann surfaces S_n'. If along this sequence $||dP_n^-(z_n')||$ is bounded, the analysis of the bubble arising from $||dP_n^+(z_n')|| \to \infty$ is now the same as in Step 1, Case 2.1. If not, repeat the procedure of Step1, Case 2.2 just described. As soon as there do not exist any sequences z_n such that $||dP_n^-(z_n)|| \to \infty$, choose holomorphic rescaling charts

$$\psi_n : B_{\epsilon_n R_n}(0) \to U_n, \tag{5.65}$$

where $R_n = ||dP_n^+(z_n)||$ and define

$$(Q_n^-, Q_n^+) = (P_n^- \circ \psi_n, P_n^+ \circ \psi_n). \tag{5.66}$$

Then carry out the procedure described above with all minuses replaces by pluses and vice versa. Since every sphere or a Reeb cylinder that bubbles off has positive energy, this bubbling-off process terminates, possibly after jumping back and forth between rescalings of $||dP_N^-||$ and $||dP_N^+||$ finitely many times, after a finite number of repetitions. Note that this process terminates independently of the order in which the analysis is carried out. Thus regardless of the choice of rescaling, it is possible to bound the gradient in both factors of the target.

This finishes the proof of Proposition 5.16 and concludes Step 1 in the proof. □

Step 2: Convergence in the thick part

This part of the proof is exaclty as in [Abb14]. I include it for the sake of completeness. By Proposition 5.16 one may assume that

$$||dP_n(z)|| \leq \frac{C}{\rho_n(z)} \forall z \in S_n \setminus (M_n \cup Z_n \cup Y_n).$$

Absorb the set Y_n into $M_n \cup Z_n \cup D_n$ and denote this set by $M_n' \cup Z_n' \cup D_n'$. By Deligne-Mumford compactness:

$$S_n = (S_n, \partial S_n, j_n, M_n', D_n', Z_n',) \xrightarrow{n\to\infty} (S = S, \partial S, j, M, D, Z).$$

Abusing notation I will denote $S_n \setminus M_n' \cup Z_n' \cup Y_n'$ still by $\dot{S}_n$ and likewise for S and $\dot{S}$. I now establish a uniform gradient bound in terms of the injectivity radius ρ on the thick part of the nodal Riemann surface S. Recall that on the thick part one has $\rho(z) \geq \epsilon$ by equation (5.49) for a fixed ϵ. By the definition of convergence in the Deligne-Mumford space there exist maps $\varphi : \dot{S}_n \to \dot{S}$ such that $\varphi_n^* h_n \to h$. Thus assume for sufficiently large n

$$\sup\{|\rho_n(z) - \rho(z)| | z \in \mathrm{Thick}(\dot{S})\} \leq \frac{\epsilon}{4}. \tag{5.67}$$

Thus $\rho_n(z) \geq \frac{3}{4}\epsilon$. Estimate:

$$\begin{aligned} ||dP_n \circ \varphi_n(z)|| &\leq |\frac{C}{\rho_n(z)} - \frac{C}{\rho(z)}| + \frac{C}{\rho(z)} \\ &\leq C\frac{|\rho_n(z) - \rho(z)|}{\rho_n(z)\rho(z)} + \frac{C}{\rho(z)} \\ &\leq \frac{\frac{\epsilon}{4}}{\frac{3}{4}\epsilon\rho(z)} + \frac{C}{\rho(z)} \\ &\leq \frac{4}{3}\frac{C}{\rho(z)}. \end{aligned}$$

Hence, for every $\epsilon > 0$ one obtains a uniform gradient bound on Thick_ϵ. By elliptic bootstrapping and repeated application of Arzela Ascoli, extract a subsequence of punctured pearly trajectories pP_n which converges in $C^\infty_{\mathrm{loc}}\left(\cup_\epsilon \mathrm{Thick}_\epsilon(\dot{S})\right)$. Denote this limit by pP'. This establishes Step 2 of the proof.

Step 3: Convergence in the thin part

This part of the proof is also exactly as in [Abb14] or in [Bou+03]. I include a summary of the necessary analysis for the sake of completeness. Denote by $C_1, \dots C_k$ the connected components of the Riemann surface $S \setminus D$ obtained in Step 2. There are two kinds of nodal pairs, interior nodal pairs $\{d, d'\} \subset \mathring{D}$ and boundary nodal pairs $\{b, b'\} \subset D_\partial$. First of all notice that if pP' is bounded near a node, pP' extends continuously over this boundary node by the removal of singularities theorem. Since L_C is compact this holds for all pairs in D_∂. If pP' is unbounded near a node apply Proposition 5.14, to conclude that pP' is asymptotic to a pair of generalised Reeb orbit as it approaches the node.

The goal is now to establish a uniform gradient bound on all components of the thin part of the Riemann surface in order to extract a subsequence of pP_n which converges on all parts of the underlying Riemann surface. By the preliminary choices, each ϵ-thin component $T_{n,\epsilon}$ of S_n which degenerates to a pair of nodal points is conformally equivalent to $[-R_{n,\epsilon}, R_{n,\epsilon}] \times S^1$ or to $[0, R^+_{n,\epsilon}) \times S^1$. If D_∂ is nonempty, double $[0, R^+_{n,\epsilon}) \times S^1$. Thus one has to consider thin components of the form $[-R_{n,\epsilon}, R_{n,\epsilon}] \times S^1$ only. By using the flat metric on $[-R_{n,\epsilon}, R_{n,\epsilon}] \times S^1$ the holomorphic parameterisations

$$\phi_{n,\epsilon} : A_{n,\epsilon} := [-R_{n,\epsilon}, R_{n,\epsilon}] \times S^1 \to T_{n,\epsilon}.$$

satisfy

$$||\phi_{n,\epsilon}(z)|| \leq C' \rho_n(\phi_{n,\epsilon}(z).$$

Use the estimate from Proposition 5.16 to obtain

$$\begin{aligned} ||dP_n \circ \varphi_n(z) \circ \phi_{n,\epsilon}(z)|| &\leq ||dP_n \circ \varphi_n(z)|| ||\phi_{n,\epsilon}(z)|| \\ &\leq \frac{C}{\rho_n(\phi_{n,\epsilon}(z))} \cdot C' \rho_n(\phi_{n,\epsilon}(z)) \\ &\leq C''. \end{aligned}$$

Thus again by Arzela Ascoli one may extract a subsequence of pP_n which converges also on the thin parts of S to a punctured pearly trajectory pP which is asymptotic to a pair of generalised Reeb orbits or has one or two removable singularities at each nodal pair.

The asymptotic limits on the thin components are a priori not equal to the asymptotic limits on the thick part. By carrying out yet another bubbling off analysis on the thin part and by possibly adding components to the domains S_n as described in section 5.6.3, one can arrange that the limits within the thin part and on thin and thick parts match up. In essence the origin of the bubbling lies in differences of the action vectors of the generalised Reeb orbits on the different components C_i and C_j adjacent to the puncture. Since this more detailed analysis is not needed for the proof of Theorem 1.6 I refer the reader to [Abb14] or [Bou+03] for details. This concludes Step 3 in the proof.

So far I have shown that there exists a subsequence of pP_n which converges on S to a punctured pearly trajectory pP. It remains to show that the assertions of Theorem 5.1 hold and in particular that pP has the desired properties.

5.6.5 Properties

By assertion (M) of Theorem 4.1 there exists a Morse function f_n on L_C in each manifold $W_n \times W_n$ considered in the spitting process. Since one is not changing L_C in the neck stretching procedure one may choose $f_n = f$ to be identical for all n. Thus assertion (M) of the theorem follows.

The energy of pP is finite by Equation (5.38) and (5.39). Moreover since $(C, \mathcal{S})$ is also fibred by assumption of the theorem, the adjusted almost complex structure $J_{\mathcal{S}}$ may be constructed by first choosing any ω_B almost complex structure on the symplectic reduction B of C, as explained in Remark 5.6. Thus assertion (E) of the theorem follows if pP satisfies properties (pP1) and (pP2). Recall that each pearly trajectory in the original sequence $(P_n)_{n\geq 0}$ connects a a critical point y_n of f contained in $f_B^{-1}([1,\infty))$ to the minimum x of f contained in $f_B^{-1}(0)$. This is a closed condition, therefore the limit pP has the same property which means nothing but that it has property (pP1).

I now show that pP also satisfies assertions (S1)-(S3) and thus has property (pP2). Recall that the genus of the Riemann surfaces S_n underlying the sequence $(P_n)_{n\geq 0}$ is zero and that all S_n are connected by the definition of a punctured pearly trajectory. Thus the Riemann surface S underlying pP is also connected and has (arithmetic) genus zero. Consider only the component pP_C of pP contained in $\tilde{W}_C \times \tilde{W}_C$. Recall that the backward manifold of the minimum $W^{\leq}(x)$ consists only of x alone by Equation 4.1 and that a punctured pearly trajectory has at least one non-trivial J-holomorphic discs component. Say there are $l \geq$ non-trivial components

$$\tilde{u}_1, \dots \tilde{u}_l : (S, \partial S) \to (\tilde{W}_C \times \tilde{W}_C, L_C)$$

of genus zero contributing to pP_C (see Definitions 5.11 and 5.12). The exit point of the l-th non-trivial component contributing to the punctured pearly trajectory has to be contained in $W^{\leq}(x) = x$ by the definition of a punctured pearly trajectory. Thus pP_C contains a non-trivial holomorphic curve u such that $x \in \tilde{u}_l(\partial S)$.

Recall the notation S^0 and $S^{>0}$ for level- and super-level-sets of an almost fibred pair (f, Z) introduced above Equation 4.8. To prove that pP_C contains a holomorphic curve $\tilde{u}$ which satisfies $\tilde{u}(\partial_S) \cap S^0 \neq \emptyset$ and $\tilde{u}(\partial_S) \cap S^{>0}$, I argue as follows.

If the entry point p of u_l is contained in $S^{>0}$, then the claim follows. If not, the entry point must be contained in S^0. The exit point q of the $(l-1)$-th holomorphic curve u_{l-1} flows to the entry point p of the l-th holomorphic curve under the positive gradient flow of an almost fibred Morse function by the definition of a punctured pearly trajectory. Recall that a trajectory of this flow cannot enter $S^{>0}$ and then return S^0 by construction of an almost fibred Morse function. This implies that if p is contained in S^0, then so is q. Thus if all l holomorphic curves contributing to the pearly product trajectory were contained in S^0 it would follow that the entry point r of the first holomorphic curve u_1, which is contained in the forward manifold $W^{\geq}(y)$, is also contained in S^0. However y is contained in $S^{\geq 1}$ by (pP1). Since these sets are disjoint there exists l_0 such that $1 \leq l_0 \leq l$ and such that $u_{l_0}(\partial S)$ intersects both S^0 and $S^{>0}$. Thus $\tilde{u}_{l_0}$ has property (S1).

I have shown in Steps 1-3 of the proof of Theorem 5.1 that each boundary puncture or boundary nodal pair is removable since it is contained in the compact Lagrangian L_C. If pP_C is unbounded near a node or puncture it is asymptotic to a pair of cylinders over a generalised Reeb orbits on C by Proposition 5.14. In particular, $\tilde{u}_{l_0}$ has properties (S2) and (S3). Setting $\tilde{u} = \tilde{u}_{l_0}$ completes the proof of Theorem 5.1.

5.7 Holomorphic chessboards

I conclude this chapter by briefly outlining how the machinery developed in this chapter can be used to define *holomorphic chessboards*. This is the analogue of a holomorphic builidings, as defined in [Bou+03], for stable coisotropics.

To define a holomorphic chessboard, instead of considering sequences of pearly trajectories and punctured pearly trajectories one considers holomorphic curves u with domain S a general Riemann surface as in Definition 5.10. For the purposes of this exposition I will stick to the splitting scenario where the target T consists of the symplectic manifold W equipped with a $(C, \mathcal{S})$ adjusted almost complex structure $J_{\mathcal{S}}$ and split along H_C as in Equation 5.5. It is also possible to develop similar notions for the symplectic completion $\tilde{W}_C$ of a stable coisotropic. In the splitting case a (k,l)-*holomorphic chessboard*, which I denote by $U(k,l)$ consists of the following data

- A holomorphic curve $\tilde{u}(1,1) = \tilde{u}_C : S_C \to \tilde{W}_C \times \tilde{W}_C$, with domain a Riemann surface S_C. The curve $\tilde{u}_C$ maps the boundary ∂S to L_C if $\partial S_C \neq \emptyset$. Moreover $\tilde{u}(1,1)$ is asymptotic at its non-removable punctures to generalised Reeb orbits and such that the asymptotics match the asymptotics of the adjacent fields of the chessboard, i.e. of$\tilde{u}(1,2)$, $\tilde{u}(2,2)$ and $\tilde{u}(2,1)$ described below.
- Holomorphic curves

  $$\tilde{u}(i,j) : S_{i,j} \to \tilde{W}_H^i \times \tilde{W}_H^J$$

  For $1 \leq i \leq k$ and $1 \leq j \leq l$ excluding the pairs $(i,j) = (k,l)$ and $(i,j) = (1,1)$. Each map $\tilde{u}_{i,j}$ is asymptotic at its non-removable punctures to generalised Reeb orbits which match the asymptotics of all adjacent fields of the chessboard i.e. of $\tilde{u}(i-1,j-1), \tilde{u}(i,j-1), \tilde{u}(i+1,j-1), \tilde{u}(i+1,j)$ and $\tilde{u}(i+1,j+1), \tilde{u}(i,j+1), \tilde{u}(i-1,j+1), \tilde{u}(i-1,j)$.
- A holomorphic curve $\tilde{u}(k,l) = u_R : S_R \to \tilde{W}_R \times \tilde{W}_R$. Which is asymptotic at its non-removable punctures to generalised Reeb orbits and matches the asymptotics of adjacent fields of the chessboard.

The analysis carried out in Section 5.6 goes through without any major changes. For the matching of asymptotics one carries out Step 3 “Convergence in the thin part” of the proof in detail. The assumption that C is either fibred, or that $(C, \mathcal{S})$ is of

Morse-Bott type, which both result in Morse-Bottnes of the Reeb flow on H_C is important to guarantee uniqueness in Proposition 5.14. Given this assumption, in order to derive the holomorphic chessboard structure one proceeds exactly as in 10.2.4 [Bou+03], where the level structure of a holomorphic building is derived. The only difference being that one uses the order " $\leq''$ and the equivalence relation " $\sim''$ from [Bou+03] in both factors of the cartesian product $\tilde{W} \times \tilde{W}$.

To formulate a "stable coisotropic SFT compactness theorem" the notions of convergence have to be adapted accordingly. The notion of energy developed for pearly trajectories in Section 5.5.2 carries over in a straightforward way.

Since not relevant for proving the main result of this thesis I do not pursue this direction here. Given an interesting application, it would be very interesting to develop this theory in the future. Below is a picture of holomorphic disc with boundary on L_C and a possible limit holomorphic chessboard $U(3,3)$.

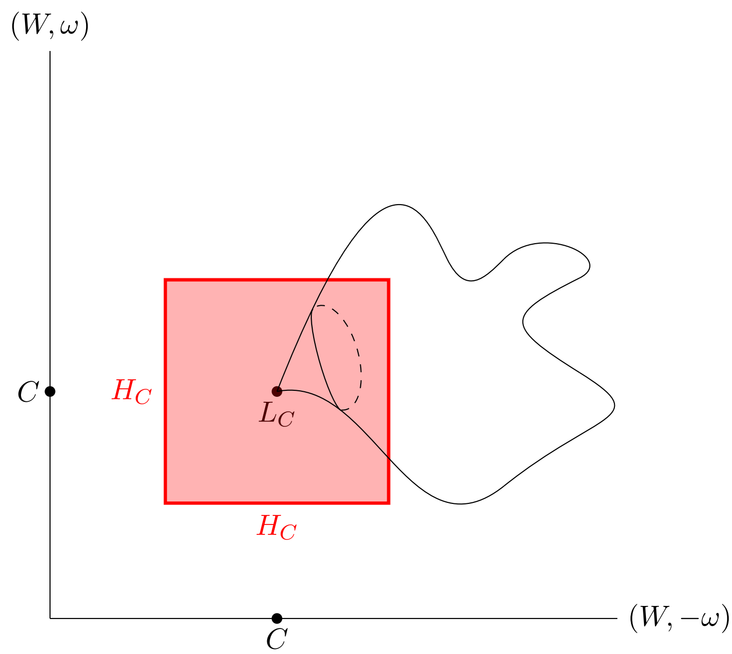


Figure 5.1: A picture disc of (looking like a genie) with boundary on L_C escaping the product Bolle neighbourhood(red)

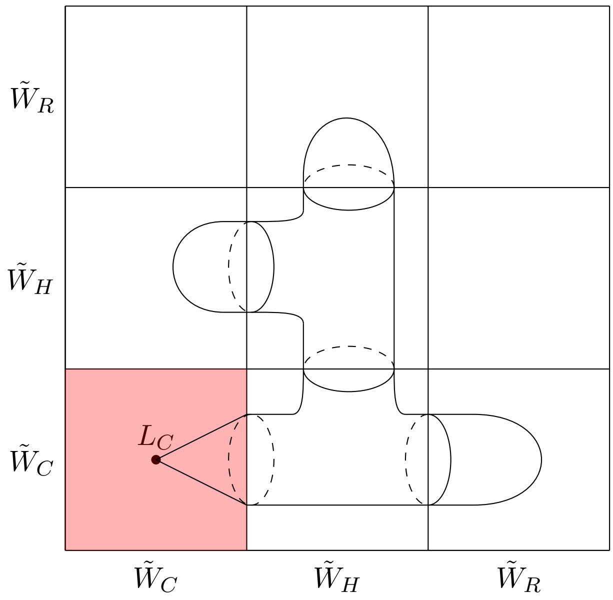


Figure 5.2: A $(3,3)$ holomorphic chessboard which is a possible limit of the genie disc above.

Chapter 6

Geometric uniruling of the symplectic quotient

In this chapter I prove the main result of this thesis, Theorem 1.6, which I state again below.

Theorem 6.1.
Let C be a closed, codimension k, coisotropic submanifold of a symplectic manifold (W, ω). If C is fibred and stable, C is the total space of a torus fibre bundle

$$\mathbb{T}^k \to C \to B$$

over its symplectic quotient (B, ω_B). Assume that C is monotone and has minimal Maslov number at least three. If C is displaceable, then the symplectic quotient (B, ω_B) has the following property:

Given any point $b \in B$, for every ω_B- compatible almost complex structure J_B on B, there exists a non-constant J_B-holomorphic sphere

$$v : (\mathbb{C} \cup \{\infty\}, i) \to (B, J_B)$$

passing through b.

Proof. By Theorem 5.1 there exists a punctured pearly trajectory pP which contains at least one punctured $(-\tilde{J}_C \times \tilde{J}_C)$ holomorphic curve

$$\tilde{u} : (S, \partial S) \longrightarrow (\tilde{W}_C \times \tilde{W}_C, L_C, -\tilde{J}_C \times \tilde{J}_C),$$

where S is a connected Riemann surface of genus zero with non-empty boundary and satisfying the following properties:

(S1) The intersections $\tilde{u}(\partial S) \cap f_B^{-1}(0)$ and the intersection $\tilde{u}(\partial S) \cap f_B^{-1}((0, \infty))$ is non-empty.

(S2) If $\tilde{u}$ is unbounded near a puncture, then $\tilde{u}$ is asymptotic to a pair of cylinders over generalised Reeb orbits on $C \times C$ when approaching the puncture.

(S3) All other boundary and interior punctures of $\tilde{u}$ are removable.

By the choice of a $(C, \mathcal{S})$-adjusted almost complex structure $\tilde{J}_C$ on $\tilde{W}_C$ (see Sections 5.3 and 5.4), projection to B is holomorphic. Recall that L_C inherits a fibre bundle structure from C:

$$\mathbb{T}^{2k} \to L_C \to \Delta B.$$

Thus one has the following holomorphic projections:

$$\begin{tikzcd}
 & \tilde{W}_C \times \tilde{W}_C \arrow[d, "\pi_C \times \pi_C"] \\
L_C \arrow[ur, hook, "i_{L_C}"] \arrow[d, "\pi_B \times \pi_B"] & C \times C \arrow[d, "\pi_B \times \pi_B"] \\
\Delta_B \arrow[r, hook, "i_{\Delta B}"] & \times B \times B
\end{tikzcd}$$

Here i_{L_C} the inclusion of L_C into $\psi(\{0\} \times C \times \{0\} \times C) \subset \tilde{W}_C \times \tilde{W}_C$. I continue to denote the projection to the $\mp$ factors of the cartesian product by $p_{\mp}$. Moreover

- Denote by W_C^- the image of the projection $p_-((\tilde{W}_C \times \tilde{W}_C, -\tilde{J}_C \times \tilde{J}_C))$ and by W_C^+ the image of the projection $p_+((\tilde{W}_C \times \tilde{W}_C, -\tilde{J}_C \times \tilde{J}_C)))$

- Denote by $C^\mp$ the coisotropic submanifold $\{0\} \times C$ contained in the factor $\tilde{W}_C^\mp$.
- Denote by $B^\mp = (B^\mp, J_B)$ the symplectic reduction of $C^\mp$.

Recall that L_C contains the diagonal $\Delta C \subset C \times C$. By this and with the notations above, the following projections are defined $\tilde{W}_C \times \tilde{W}_C$:

$$\begin{tikzcd}
\tilde{W}_C^- \arrow[d, "\pi_C"] & \tilde{W}_C^- \times \tilde{W}_C^+ \arrow[l, "p_-"'] \arrow[r, "p_+"] & \tilde{W}_C^+ \arrow[d, "\pi_C"] \\
C^- \arrow[d, "\pi_B"] & L_C \arrow[u, hook, "i_{L_C}"] \arrow[l, "p_-"'] \arrow[r, "p_+"] \arrow[d, "\pi_B \times \pi_B"] & C^+ \arrow[d, "\pi_B"] \\
B^- & \Delta B \arrow[l, "p_-"'] \arrow[r, "p_+"] & B^+
\end{tikzcd}$$

As described in Section 5.6.2 one may also view $\tilde{u}$ as a pair

$$\begin{aligned} \tilde{u}^- &: (S, \partial S) \to (\tilde{W}_C^-, p_-(L_C)) \\ \tilde{u}^+ &: (S, \partial S) \to (\tilde{W}_C^+, p_+(L_C)) \end{aligned} \tag{6.1}$$

of a punctured $(j, -\tilde{J}_C)$-holomorphic curve $\tilde{u}^-$ and a punctured $(j, \tilde{J}_C)$-holomorphic curve $\tilde{u}^+$.

Define by u_B the punctured $(-J_B \times J_B)$-holomorphic curve

$$u_B = (\pi_B \times \pi_B) \circ (\pi_C \times \pi_C) \circ \tilde{u} : (S, \partial S) \longrightarrow (B \times B, \Delta B). \tag{6.2}$$

I claim that all punctures of u_B are removable. To see this assume first that $\tilde{u}$ is unbounded near a puncture. By property (S2) $\tilde{u}_B$ is asymptotic to cylinders over generalised Reeb orbits in both factors . After projection to $C \times C$ these are entirely contained in the fibres $\mathbb{T}^k \times \mathbb{T}^k$ of the fibration $\mathbb{T}^k \times \mathbb{T}^k \to C \times C \to B \times B$. Thus near each puncture z_i one has:

$$\lim_{z \to z_i} (u_B^-(z), u_B^+(z)) = (b_i^-, b_i^+)$$

for a pair of points $(b_i^-, b_i^+) \in B \times B$. By assumption B is compact, so apply the removal of singularities theorem to extend $u_B^{\mp}$ holomorphically over (b_i^-, b_i^+). Since all other punctures are removable by property (S3) of $\tilde{u}$ it follows that u_B is a holomorphic curve without punctures. Thus u_B defines a pair of holomorphic curves without punctures which I will continue to denote by $u_B^{\mp}$. The maps

$$u_B^{\mp} := \pi_B \circ \pi_C \circ \tilde{u}^{\mp} : (S, \partial S, j) \to (B^{\mp}, \pi_{B^{\mp}}(p_i(L_C)))$$

define a pair of an anti-holomorphic and a holomorphic curve with respect to J_B. By the definition of L_C

$$L_C = \{(x, y) \in C \times C | \pi_B(x) = \pi_B(y)\},$$

or again the fact the L_C fibres over ΔB the maps $u_B^{\mp}$ agree along their boundary.

Recall that $(S, \partial S)$ is a connected Riemann surface with non-empty boundary of genus zero. In fact, S is a collection of punctured discs and spheres which are identified along nodal pairs, which arose from the original sequence of discs in the neck stretching procedure. Since, by Property (S1) of Theorem 5.1, $u(\partial S)$ intersects $f_B^{-1}((0, \infty))$ there exists a disc component $(D, \partial D)$ of S which also hast this property. Thus the pair $u_B^{\mp}$ gives rise to at least one pair of J_B-holomorphic discs $u^{\mp} : (D, \partial D) \to (B^{\mp}, \pi_{B^{\mp}}(p_i(L_C)))$ which are non-trivial in $B^{\mp}$ by choosing such a disc component of the domain.

To establish the existence of a non-trivial J_B-holomorphic sphere, perform the doubling operation for Riemann surfaces described in Section 5.10 explicitly to glue $u^{\mp}$ along their common boundary: Denote by c complex conjugation $z \mapsto \bar{z}$. Given u^- and u^+ define:

$$v(z) = \begin{cases} u^+(z) & , \text{if } |z| \leq 1 \\ u^- \circ c\left(\frac{1}{z}\right) & , \text{if } |z| \geq 1 \end{cases}$$

First notice that if $|z| = 1$ we have $\frac{1}{\bar{z}} = z$ so that v is well defined. Since u^- is (i, J_B)-

anti-holomorphic, $u^- \circ c$ is (i, J_B)-holomorphic. Since $z \mapsto \frac{1}{z}$ is holomorphic on $\mathbb{C} \cup \{\infty\}$ and u^+ is (i, J_B) holomorphic, the map v is an (i, J_B)-holomorphic sphere in B. This sphere is non-constant by construction and contains a given point b in B by property (pP1). By Definition 1.5 this means precisely that B is geometrically uniruled.

This completes the proof of Theorem 1.6. $\square$

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