Abstract—We introduce a new approach to Constant Envelope Precoding (CEP) based on an interference-driven optimization region for generic phase-shift-keying (PSK) modulations in the multi-user (MU) multiple-input-multiple-output (MIMO) downlink channel. Whilst conventional precoding approaches in the literature aim to minimize the multi-user-interference (MUI) with a total sum-power constraint at the transmitter side, in the proposed scheme we consider MUI as a source of additional energy to increase the signal-to-interference-and-noise-ratio (SINR) at the receiver. In our studies, we focus on two different CEP approaches: a first technique, where the power at each antenna is fixed to a specific value, and a two-step approach, where we first relax the power constraints to be lower than a defined parameter and then enforce per-antenna power constraints. The algorithms are studied in terms of computational costs, with a detailed comparison between the proposed approach and the classical interference suppression schemes from the literature. Moreover, we derive the analytical definition of a robust optimization region to counteract the effects of channel state estimation errors. The presented schemes are evaluated in terms of achievable symbol error rate (SER) in a perfect and imperfect channel state information (CSI) scenario for different modulation orders. Our results show that the proposed techniques further extend the benefits of classical CEP by judiciously relaxing the optimization region.

Index Terms—Constant-Envelope, Multiuser, Massive MIMO

I. INTRODUCTION

MASSIVE Multi-Input-Multi-Output (M-MIMO) communication systems have experienced an increasing growth of interest from the scientific community, because of the significant benefits they provide in terms of spectral efficiency when compared to classical MIMO approaches [1], [2]. The pioneering work from [1] proved that a base station (BS) equipped with high dimensional antenna arrays can achieve high throughput values by exploiting the innate high degrees of freedom offered by a large number of antennas at the transmitter. At the same time, M-MIMO systems are known to require lower values of radiated energy, thanks to the higher beamforming gains provided by large antenna arrays. Moreover, it has been proven that simple linear precoding techniques, such as matched filtering and linear precoding [3], [4], are asymptotically optimal [5] for massive systems, because of the favorable propagation effects that rise for infinitely large arrays. In addition, recent works [6], [7] have shown that transmit mutual coupling at the base station can be exploited with the aim to further increase the dimensions of antenna arrays in fixed physical spaces.

When considering linear precoding techniques, it is common in the literature [8], [9] to apply sum-power constraints at the transmitter side, where the average or instantaneous total transmitted power is constrained to a specific value. While a sum-power constraint approach is easier to model and study, it is important to consider that, in a realistic scenario, each antenna of the base station is typically connected to its own power amplifier (PA), which has to meet specific power constraints. This is particularly relevant in M-MIMO, because the benefits of using a large number of antennas at the transmitter side are followed by heavy burdens in terms of hardware costs and power consumption, which strongly affect its feasibility for future communication systems. In fact, the use of very large arrays (VLA) leads to an equally large number of radio-frequency (RF) chains, where the role of PAs is particularly critical, as inefficient PAs are accountable for $\sim 40 - 50\%$ of the total power consumption [10].

Toward this end, the employment of non-linear RF components in conjunction with low peak-to-average-power-ratio (PAPR) precoding techniques [11] can positively affect the power efficiency of M-MIMO [12]–[14]. More specifically, [12] presents a transmission scheme for orthogonal frequency-division multiplexing (OFDM) modulations based on low PAPR precoding, while [13], [14] propose a constant envelope precoding technique where the transmitted signal amplitude corresponding to each antenna is constant and independent from the channel realization, i.e., leading to a unitary PAPR and therefore facilitating low cost PAs. In [13] the precoding technique is designed by minimizing the error norm function of the received signal for a single user scenario, while in [14] the transmitted symbols vector is designed for multiuser MIMO with the aim to reduce the interference caused by other users. CEP was further analyzed in [15], where the precoding design for frequency-selective MIMO channels is presented. Still, the performances of CEP with interference reduction are strongly affected by the number of iterations used and by the array size at the transmitter side [14]. More recently, the authors in [16] further improved the performances of interference reduction CEP by employing cross-entropy optimization instead of gradient descent based algorithms.

While the above approaches focus on interference minimization, previous works on linear precoding [17]–[19] showed that interference minimization does not necessarily lead to the best performances in a communication system. In fact, since interference is data dependent, the transmitter is able to predict the MUI at the receiver and can use this knowl-
edge to influence it and benefit from it. Early works in [20], [21] focused on reducing the the negative effects of interference while preserving its positive components, defined according to the correlation between the substreams of a MIMO PSK-modulated transmission. Further results in [22] instead, showed that the transmitted signal can be precoded in order to rotate the destructive component of interference into constructive or beneficial interference. Therefore, future research is focusing onto identifying new optimization metrics that exploit CSI and data knowledge at the transmitter side to maximize the SINR of each user by capitalizing on the power contained within multi-user interference. More specifically, recent works [23]–[25] on PSK modulated signals have introduced different metrics that prove how the known interference can be effectively used as a source of green signal power for downlink transmissions with high-order PSK modulations.

In this paper, we present two novel CEP techniques which exploit concepts of constructive interference for PSK-modulated signals. In the proposed techniques, we relax the conditions over interference, allowing the transmitter to use the interfering signal as a green source of power to increase the signal to interference and noise ratio at the receiver side. It is important to highlight that the proposed schemes are particularly suitable for high-interference and low-SNR scenarios, where low order modulations such as BPSK and 4-PSK are often preferred to ensure reliable communications [26]. Nevertheless, constructive interference concepts could also be applied over the outer constellation points of Quadrature Amplitude Modulation (QAM) signals or to the whole constellation by means of adaptive decision thresholds [19]. Here we list the contributions of the paper:

- We analytically describe and define a new optimization region for constant envelope precoding, based on the concepts of constructive interference.
- We introduce two different CEP approaches, when both equality and inequality power constraints are considered.
- We study the computational costs of the proposed techniques in comparison with the classical CEP approach in the literature.
- We introduce a CSI-robust precoding scheme based on a relaxation of the interference optimization region.
- We evaluate the performances of the proposed schemes for different PSK modulation orders and in scenarios where the transmitter holds perfect and imperfect CSI.

The rest of the paper is organized as follows: Section II introduces the system model used throughout this work and describes the classical CEP approach from the literature. Section III describes the proposed interference-based optimization region, while Section IV is dedicated to a thorough description of the proposed techniques. In Section V the computational complexity of the proposed scheme is analyzed and compared with the previous approach, based on interference mitigation. In Section VI a robust optimization region is analytically derived as a function of the CSI error upperbound and Section VII shows the performance achieved by the proposed techniques in different scenarios. Finally, in Section VIII the main contributions of the paper are summarized.

**Notation:** Upper case boldfaced letters are used for matrices (i.e., $X$), lower case boldfaced letters denote vectors (i.e., $x$), subindices in vectors are used to identify rows of a matrix (i.e., $x_m$ is the $m$-th row of $X$), $tr[\cdot]$ represents the trace of the argument and superscripts $(\cdot)^H$ and $(\cdot)^*$ stand for Hermitian transpose and complex conjugate, respectively. Operators $\Re(\cdot)$ and $\Im(\cdot)$ respectively represent the imaginary and real part of the argument.

## II. System Model

Consider a downlink multi-user scenario where the BS employs an $N$ dimensional antenna array to communicate with a population of $M$ single-antenna users. The received signal $y$ is a $CM \times 1$ vector that collects the $M$ user received signals $y_m$, and is analytically defined as:

$$y = Hx + w,$$

where $H$ is the $CM \times N$ channel matrix, $x$ represents the $CN \times 1$ vector of transmitted symbols and $w$ is the $CM \times 1$ zero mean additive white Gaussian noise vector, i.e., $w \sim CN(0, \sigma^2)$. Complex channel gains $h_{m,n}$ in M-MIMO are modeled to include both the complex small scale fading $g_{m,n}$ between the $n$-th antenna and the $m$-th user and the real large scale fading coefficient $\beta_m$ experienced by the $m$-th user [1], leading to the following analytical definition

$$h_{m,n} = g_{m,n} \sqrt{\beta_m}.$$  

In our studies, we consider a single cell scenario where channel gains are modeled by independent Rayleigh fading [27], i.e., small scale fading $g_{m,n}$ are zero mean i.i.d. Gaussian variables and large scale coefficients $\beta_m = 1, \forall m \in \{1, ..., M\}$.

Given the total transmitted power by the antenna array $P_t$, we can define the $n$-th transmitted symbol from the $n$-th antenna of the BS as [14]

$$x_n = \sqrt{P_n}e^{j\theta_n},$$

where $P_n$ is the power transmitted from the $n$-th antenna, so that $\sum_{n=1}^{N} P_n = P_t$, and $\theta_n$ represents the precoding phase of the CEP signal. We can then similarly define the received signal at the $m$-th user as

$$y_m = \sum_{n=1}^{N} h_{m,n} \sqrt{P_n}e^{j\theta_n} + w_m.$$  

For simplicity and to ease the notation, throughout the paper we assume unitary transmitted power $P_t = 1$ and equally distributed power among the $N$ antennas at the BS, i.e., $P_n = 1/N, \forall n \in \{1, ..., N\}$, hence leading to

$$y_m = \sum_{n=1}^{N} \frac{1}{\sqrt{N}} h_{m,n} e^{j\theta_n} + w_m.$$  

The first term of the received signal $y_m$ can be rearranged in order to explicitly discriminate between the desired signal and the interference. Analytically we have

$$y_m = u_m + t_m + w_m,$$
where \( u_m = d_m e^{j\phi_m} \) is the PSK desired symbol for the \( m \)-th user, with magnitude \( d_m \) and phase \( \phi_m \), and \( t_m \) represents the interfering signal for the \( m \)-th user

\[
t_m = \left( \sum_{n=1}^{N} \frac{1}{\sqrt{N}} h_{m,n} e^{j\theta_n} - d_m e^{j\phi_m} \right) \tag{7}
\]

Accordingly, we can identify the total MUI energy as

\[
E_{MUI} = \sum_{m=1}^{M} \left\| \left( \sum_{n=1}^{N} \frac{1}{\sqrt{N}} h_{m,n} e^{j\theta_n} - d_m e^{j\phi_m} \right) \right\|^2 \tag{8}
\]

First approaches to CEP were based on the minimization of the MUI energy [14]. In order to minimize (8), the base station proceeds in identifying the \( N \) dimensional transmit phase angle vector \( \theta = [\theta_1, \ldots, \theta_N] \) that leads to the lowest MUI energy. Accordingly, the constant envelope precoding algorithm can be formulated as follows [14], [16]

\[
\mathcal{P}_1: \text{ minimize } \theta \quad \text{subject to } |\theta_n| \leq \pi, \forall n \in \{1, \ldots, N\}, \tag{9}
\]

which represents a non-convex nonlinear least squares (NLS) problem, affected by local minima. The optimization problem (9) was first solved in [14] with a gradient descent (GD) based approach, and further improved in [16] with a direct application of cross-entropy method [28].

III. CONSTRUCTIVE INTERFERENCE OPTIMIZATION REGION

When considering PSK-modulated signals, interference can be differentiated between constructive and destructive according to simple geometrical concepts [20]. In fact, the interference signal \( t_m \) can be considered beneficial for system performances when it leads the noise free received symbol \( \tilde{r}_m = y_m - u_m \) further away from the decision thresholds of the desired constellation symbol \( u_m \). A visual representation of the distinction between constructive and destructive interference is presented in Fig.1a, where the desired symbol \( u_m \) is considered to be the \((1/\sqrt{2} + j1/\sqrt{2})\) point of the 8-PSK constellation. Here the superscripts \( \{\cdot\}^c \) and \( \{\cdot\}^d \) are used to differentiate between two different cases, where the received symbol falls in the constructive region (i.e., the green shaded area) or in destructive region (i.e., the red dot-pattern area), respectively. As per above, we can see that when the received symbol falls in the destructive region it resides closer to the decision thresholds, represented by the bold lines, when compared to the desired symbol. On the other hand, when \( \tilde{r}_m \) lays in the constructive region, its distance from the decision thresholds is greater than the one which characterizes \( u_m \). In [23], for the case when the received signal fully aligns with the desired symbol, constructive interference conditions are analytically expressed for the \( m \)-th received signal as follows:

\[
\Re \left( \tilde{r}_m \cdot e^{-j\phi_m} \right) = \Re \left( \sum_{n=1}^{N} \frac{1}{\sqrt{N}} h_{m,n} e^{j\theta_n} e^{-j\phi_m} \right) \geq \eta \sqrt{N_0} \tag{10}
\]

where \( \eta \leq R^+ \) is a direct proportionality coefficient used to set a threshold for constructive interference over the real part of \( t_m \) and determines the resulting signal-to-noise ratio (SNR). Note that the conditions in (10) and (11) are imposed over the phase-shifted received signal \( \tilde{r}_m \cdot e^{-j\phi_m} \), according to the phase of the symbol of interest for the \( m \)-th user \( \phi_m \). The \( \phi_m \) phase-shift is a fundamental operation, as it allows to isolate the received amplitude and phase shift over the desired symbol \( u_m \) caused by the interference \( t_m \). It is important to stress that these conditions are valid for any PSK modulation order.

The condition in (11) can be further relaxed, as the phase of the interfering signal \( t_m \) does not need to be strictly aligned with the phase of the desired symbol \( u_m \). In fact, the interfering signal \( t_m \) is to be considered constructive and beneficial for the transmission as long as it is contained in the constructive area of the constellation, as in the 8-PSK example of Fig.1a.

From basic geometry properties and from the conditions (10) and (11), we can analytically define the constructive interference region for the \( m \)-th user as

\[
\Re \left( \tilde{r}_m \cdot e^{-j\phi_m} \right) \leq \left( \Re \left( \tilde{r}_m \cdot e^{-j\phi_m} \right) - \eta \sqrt{N_0} \right) \tan \Phi, \tag{12}
\]

where \( \Phi \) is the central angle of the constructive interference sectors, which depends on the constellation order \( L \) and can be readily computed as \( \Phi = \pm \pi/L \).

The constructive interference constraint definition in (12) allows the identification of a new precoding optimization region that exploits the interfering signal power, instead of reducing it. In fact, as shown in Fig.1a, the constructive interference regions can be defined as sectors with infinite radii whose central angle depends on the constellation order. This assumption allows to relax the classical optimization metrics based on interference minimization, as the constructive interference region is only constrained by the proximity to the decision thresholds and extends infinitely in the directions away from them. Optimization region constraints are visually represented in Fig.1a by the dashed lines.

The optimization region described in (10) is applied to the total received signal, hence it does not differentiate between desired signal \( u_m \) and interfering signal \( t_m \). However, a direct application over the interfering signal can be readily extended, as shown in Lemma 1.

**Lemma 1.** Interference signal \( t_m \) is considered constructive when

\[
|\Re \left( \tilde{r}_m \cdot e^{-j\phi_m} \right)| \leq \Re \left( t_m \cdot e^{-j\phi_m} \right) \tan \Phi. \tag{13}
\]

**Proof.** The condition follows directly from the derivation of equation (12) by substituting the received signal \( \tilde{r}_m \) with \( t_m \) and by removing the SNR condition over the thresholds (i.e., \( \eta \sqrt{N_0} \)). \( \square \)

The condition (13) is visually described in Fig.1b for the 8-PSK case, where \( \tilde{r}_m = t_m \cdot e^{-j\phi_m} \) represents the rotated interfering signal for the \( m \)-th user and \( \tilde{r}_m^c \) and \( \tilde{r}_m^d \) respectively identify the real and imaginary part of \( t_m \). As previously
stated, $\bar{t}_m^R$ and $\bar{t}_m^I$ essentially represent the shift from $u_m$ suffered by the received symbol by means of interference. It follows that $\bar{t}_m^R$ can be seen as a measure of the amplification of the received constellation point along the axis of $u_m$ thanks to constructive interference, while $\bar{t}_m^I$ represents a linear measure of the angle shift from the original constellation point, i.e., the deviation from the axis of the constellation point with phase $\phi_m$. It is important to stress that $\bar{t}_m^R$ and $\bar{t}_m^I$ can grow infinitely, as long they respect the condition in (12). The reader is referred to [21]–[23] for more details on the definition of the constructive interference region.

IV. CONSTANT ENVELOPE PRECODING WITH CONSTRUCTIVE INTERFERENCE OPTIMIZATION

Existing studies in M-MIMO systems mostly consider precoding techniques with sum-power constraints at the transmitter side. However, this is not a realistic assumption, since each transmitting antenna is typically characterized by its own amplifier and is hence affected by specific power constraints. Moreover, the use of precoding techniques where the power at each antenna is fixed also allows the employment of power-efficient amplifiers, hence reducing the total operational power consumption of the system.

Toward this end, we introduce two different CEP approaches, both based on constructive interference exploitation concepts: one with CEP equality constraints, i.e., $|x_n| = p, \forall n \in \{1, \ldots, N\}$, and a two-stage approach where the constraints are initially relaxed to inequality conditions, i.e., $|x_n| \leq p, \forall n \in \{1, \ldots, N\}$, to be successively reapplied by means of normalization in order to perform CEP.

Following the concepts of constructive interference in (13), it is possible to define a new optimization metric that maximizes the interference power, while imposing constraints over the phase of $t_m$. Thanks to simple analytical operations, we can rearrange (13) as

$$\Re \left( t_m e^{-j\phi_m} \right) \tan \Phi - \Im \left( t_m e^{-j\phi_m} \right) \geq 0.$$  \hspace{1cm} (14)

The difference on the left side of the inequation can be used as an indicator of how constructive or destructive the interfering signal $t_m$ is. More specifically, it implicitly describes both the power, with the real part, and the phase, with the imaginary part, of the interfering signal $t_m$. In fact, if (14) is negative, the interfering signal lies in the constructive region of interference, while if (14) is positive it implies that the interfering signal is destructive. In addition, since the real part of (13) represents the power of the interfering signal, we can infer that higher and positive values of (14) lead to stronger forms of constructive interference. Accordingly, we define the optimization problem $\mathcal{P}_2$ as follows:

$$\mathcal{P}_2 : \max_{\theta} \min_{m} \left\{ \Re \left( t_m e^{-j\phi_m} \right) \tan \Phi - \Im \left( t_m e^{-j\phi_m} \right) \right\} \hspace{1cm} (15)$$

subject to $|\theta_n| \leq \pi, \forall n \in \{1, \ldots, N\}$, where $m \in \{1, \ldots, M\}$ and the operator $\min \{\cdot\}$ represents the minimum value of the argument among each of the $M$ values. In $\mathcal{P}_2$ we maximize the minimum value of the constructive interference metric. With this approach, when the minimum value of the metric is positive, we can automatically infer that the constructive interference condition is verified and maximized for all the $M$ users. In cases where the solution to $\mathcal{P}_2$ leads to negative values of the minimum, instead, it implies that the precoding phases minimize the destructive interference as its least constructive component is maximized, as visually described for the 8-PSK case in Fig.1b. The formulation in $\mathcal{P}_2$ is clearly non-convex, however it can be efficiently solved via the cross-entropy method (CEM).

A. A CEM Application for Constructive Interference Optimization

The cross-entropy method can be described as an adaptive algorithm that aims to the identification of rare events by means of variance reduction. The algorithm is characterized
by an iterative approach [28], where each iteration presents two main steps:

- Generation of random samples based on a specific distribution \( f(\theta, u) \).
- Update distribution parameters \( u \in \mathbb{R} \), according to the computed values of a chosen cost function, in order to improve the random samples generation in the following iterations.

The use of cross-entropy method to perform combinatorial optimization can be described as follows. Consider the maximization problem described in \( P_2 \), we can define the global optimum \( \gamma^* \) as

\[
\gamma^* = \min_m \{ R(\tilde{f}_m) \tan \Phi - |3(\tilde{f}_m)| \}
\]

where \( \tilde{f}_m \) represents the \( m \)-th element of the normalized interfering signal, analytically expressed as

\[
\tilde{f}_m = \left( \sum_{n=1}^{N} \frac{1}{\sqrt{N}} h_{m,n} e^{j \theta_n} - d_n e^{j \phi_m} \right) e^{-j \phi_m},
\]

with \( \theta_n \) being the \( n \)-th element of the optimal solution \( \theta^* \) to the optimization problem. The application of CEM to optimization problems is based on the association of the maximization problem with the probability estimation of a rare event. Given a performance threshold \( \gamma \), we can evaluate the probability of the rare event \( m \{ R(\tilde{f}_m) \tan \Phi - |3(\tilde{f}_m)| \} \geq \gamma \) as

\[
\mathcal{L}(\gamma) = \mathbb{P}_u \left\{ \min_m \{ R(\tilde{f}_m) \tan \Phi - |3(\tilde{f}_m)| \} \geq \gamma \right\}
\]

\[
= \mathbb{E}_u \left\{ \int \mathcal{I} \left\{ \min_m \{ R(\tilde{f}_m) \tan \Phi - |3(\tilde{f}_m)| \} \geq \gamma \right\} \right\} f(\theta, u) d\theta
\]

where the operator \( \mathbb{P}_u (\cdot) \) evaluates the probability of the event in argument, the operator \( \mathbb{E}_u (\cdot) \) represents the expectation of the argument with respect to the distribution \( f(\theta, u) \) and \( \mathcal{I} \{ \cdot \} \) is boolean indicator function that returns 1 or 0 values when its argument true or false, respectively. The estimation of \( \mathcal{L}(\gamma) \) can be performed through Monte Carlo simulations, by drawing a set of \( K \) random states \( \Theta_1, ..., \Theta_K \) from \( f(\theta, u) \) and by computing

\[
\hat{\mathcal{L}}(\gamma) = \frac{1}{K} \sum_{k=1}^{K} \mathcal{I} \left\{ \min_m \{ R(\tilde{f}_m) \tan \Phi - |3(\tilde{f}_m)| \} \geq \gamma \right\},
\]

where \( \tilde{f}_m \) is the \( m \)-th element of the interfering signal for the \( k \)-th state \( \Theta_k = [\theta_1^k, ..., \theta_n^k, ..., \theta_N^k] \)

\[
\tilde{f}_m = \left( \sum_{n=1}^{N} \frac{1}{\sqrt{N}} h_{m,n} e^{j \theta_n^k} - d_n e^{j \phi_m} \right) e^{-j \phi_m}.
\]

A direct application of (19) becomes rapidly prohibitive when the probability of the event is very small, i.e., on the order of \( \sim 10^{-3} \). This can be addressed by means of importance sampling, where we estimate a different probability density function \( g(\Theta) \) that more frequently generates such rare events. Under importance sampling, the estimation problem becomes

\[
\hat{\mathcal{L}}(\gamma) = \frac{1}{K} \sum_{k=1}^{K} \mathcal{I} \left\{ \min_m \{ R(\tilde{f}_m) \tan \Phi - |3(\tilde{f}_m)| \} \geq \gamma \right\} \frac{g(\Theta_k, u)}{g(\Theta_k)},
\]

where \( g(\Theta_k, u) \) represents the importance sampling distribution and \( \frac{g(\Theta_k, u)}{g(\Theta_k)} \) is defined as the likelihood ratio (LR) estimator.

The importance sampling function is commonly chosen as a probability density function from the same family of \( f(\theta, u) \), as

\[
g(\Theta) = f(\theta, v),
\]

where \( v \in \mathbb{R} \) is the tilting parameters vector and is obtained by computing the function with the minimum Kullback-Leibler distance from the ideal solution \( g^*(\Theta) = \frac{I(S(\Theta) \geq \gamma)f(\theta, u)}{L(\gamma)} \), where \( S(\Theta) \) is a real valued function of the optimization parameter \( \theta \). The Kullback-Leibler distance or cross-entropy between two densities \( s(x) \) and \( t(x) \) is analytically defined as

\[
D(s, t) = \int s(x) \ln s(x) dx - \int s(x) \ln t(x) dx
\]

and its minimization can be achieved through the maximization of the second term in the equation. The tilting parameters \( v \) deriving from the minimization of the Kullback-Leibler distance between \( g^*(\Theta) \) and \( f(\theta, u) \) can be obtained as

\[
v^* = \arg \max_v \int \mathcal{I} \{ S(\Theta) \geq \gamma \} f(\theta, u) \ln f(\Theta, v) d\Theta,
\]

which, for the proposed optimization problem, is equivalent to the maximization [28]:

\[
v^* = \arg \max_v \mathbb{E}_u \left\{ \int \mathcal{I} \left\{ \min_m \{ R(\tilde{f}_m) \tan \Phi - |3(\tilde{f}_m)| \} \geq \gamma \right\} \ln f(\Theta, v) \right\}.
\]

A solution to (25) can be numerically estimated as

\[
\hat{\gamma}^* = \frac{1}{K} \sum_{k=1}^{K} \mathcal{I} \left\{ \min_m \{ R(\tilde{f}_m) \tan \Phi - |3(\tilde{f}_m)| \} \geq \gamma \right\} \ln f(\Theta_k, v).
\]

In our study we consider \( f(\theta, v) \) to be a Gaussian distribution, i.e. \( f(\theta, v) = f(\theta, [\mu, \sigma]) \), which allows to analytically estimate (25) as

\[
\hat{\mu} = \frac{\sum_{k=1}^{K} \mathcal{I} \left\{ \min_m \{ R(\tilde{f}_m) \tan \Phi - |3(\tilde{f}_m)| \} \geq \gamma \right\} \mathbb{E}_u f(\Theta_k, v)}{\sum_{k=1}^{K} \mathcal{I} \left\{ \min_m \{ R(\tilde{f}_m) \tan \Phi - |3(\tilde{f}_m)| \} \geq \gamma \right\}},
\]

\[
\hat{\sigma} = \sqrt{\frac{\sum_{k=1}^{K} \mathcal{I} \left\{ \min_m \{ R(\tilde{f}_m) \tan \Phi - |3(\tilde{f}_m)| \} \geq \gamma \right\} (\Theta_k - \hat{\mu})^2}{\sum_{k=1}^{K} \mathcal{I} \left\{ \min_m \{ R(\tilde{f}_m) \tan \Phi - |3(\tilde{f}_m)| \} \geq \gamma \right\}}},
\]

\[1\) This assumption is not uncommon for continuous optimization problems [16] and leads to efficient solutions.
where \( \hat{\mu} \) and \( \hat{\sigma} \) respectively represent mean and standard deviation of the importance sampling distribution, i.e., \( \tilde{v}^* = [\hat{\mu}, \hat{\sigma}] \). As previously mentioned, CEM is based on an iterative approach and requires the tilting parameters to be updated at each iteration. However, a direct update from (26) is often undesirable, as it might rapidly converge to suboptimal solutions [28]. The occurrence of these events can be reduced by using smooth updating procedures, as follows
\[
\mu^{(n)} = \alpha \hat{\mu}^{(l)} + (1 - \alpha) \mu^{(l-1)} \tag{29}
\]
\[
\sigma^{(l)} = \alpha \hat{\sigma}^{(l)} + (1 - \alpha) \sigma^{(l-1)}, \tag{30}
\]
where the superscript \( (\cdot)^{(l)} \) represents the \( l \)-th iteration of the value in argument.

An analytical description of the constructive interference optimization precoding based on cross-entropy optimization (CEO-CIO) technique is presented in Algorithm 1. Here, \( T \) represents the number of iterations, \( K \) identifies the random sample size and \( \rho \) is direct proportionality coefficient used to compute the intermediate threshold \( \gamma^{(l)} \).

The application of the proposed algorithm leads to received symbols \( \tilde{x} \) which prevalently reside in the constructive interference region. To illustrate this effect, Fig. 2 shows the received constellation of CEP precoded signals for the example of 8-PSK constellation in the noise free transmission over 100 different channel realizations, in a scenario where the BS is equipped with \( N = 100 \) antennas and communicates with \( M = 20 \) single-antenna users.

**B. Two-Step Convex CEP**

In addition to the previous approach, we propose an additional technique for constant envelope transmissions where the power constraints are initially relaxed into inequality, allowing to use standard convex optimization techniques, and subsequently enforced to equality via normalization at a later stage (i.e., by dividing the antenna outputs that do not respect power constraints by their absolute value). In order to relax the conditions in \( P_2 \), we reformulate the optimization problem in its equivalent form where the cost function is dependent on the transmitted signal \( x \):

\[
P_3 : \text{maximize } x \quad \text{subject to } \quad \|x_n\| \leq 1/\sqrt{N}, \forall n \in \{1,\ldots, N\}.
\]

Similarly to the optimization in \( P_2 \), the above problem is non-convex, because of the equality constraint over a convex set. In order to tackle this, we can convexify the problem by imposing relaxed conditions to the transmitted signal \( x_n \in \mathbb{C}, \forall n \in \{1,\ldots, N\} \) and its absolute value \( |x_n| \leq 1/\sqrt{N}, \forall n \in \{1,\ldots, N\} \). Thanks to this, we can reformulate the optimization problem \( P_3 \) into its relaxation \( P'_3 \) as

\[
P'_3 : \text{maximize } x \quad \text{subject to } \quad |x'_n| \leq 1/\sqrt{N}, \forall n \in \{1,\ldots, N\}.
\]

**Algorithm 1 CEO-CIO Precoding**

**Input:** \( H, u, T, L, K \)

**Output:** \( x \)

1. Initialize \( \mu^{(0)} \) and \( \sigma^{(0)} \)
2. for \( l = 1 \rightarrow T \)
   1. \( \Theta^{(l)} = [\theta_1^{(l)}, \ldots, \theta_K^{(l)}] \) where the columns \( \theta_k^{(l)} \sim \mathcal{N}(\mu^{(l-1,)}, (\sigma^{3(l-1)})^2) \)
   2. for \( k = 1 \rightarrow K \)
      1. \( x_k^{(l)} = \frac{1}{\sqrt{N}} e^{j\theta_k^{(l)}} \)
      2. \( t_k^{(l)} = H \cdot x_k^{(l)} - u \)
      3. \( C_k = \min_m \left\{ \Re (\tilde{t}_m e^{-j\phi}) \tan \Phi - |3 (\tilde{t}_m e^{-j\phi})| \right\} \)
3. Sort \( C_1 \geq C_2 \geq \ldots \geq C_K \)
4. \( \gamma^{(l)} = C_{(\rho K)} \)
5. \( \hat{\mu}^{(l)} \) and \( \hat{\sigma}^{(l)} \) from (27) and (28)
6. \( \mu^{(l)} \) and \( \mu^{(l)} \) from (29) and (30)

Return \( x = x^{(T)} \)
transmission for all the antennas at the BS we need to force the equality constrained before transmission. More specifically, in the second and final stage of the algorithm, we can proceed by normalizing the elements where when $|x_n'| \neq 1/\sqrt{N}, \forall n \in \{1, ..., N\}$ as follows

$$x_n = \begin{cases} x_n' / \left(\sqrt{N} |x_n'|\right) & \forall n \text{ where } |x_n'| \neq 1/\sqrt{N} \\ x_n' & \forall n \text{ where } |x_n'| = 1/\sqrt{N}. \end{cases}$$ \hspace{1cm} (33)

The precoding scheme, which we refer to as Convex Constructive Interference Optimization (CVX-CIO), is analytically described in Algorithm 2.

Algorithm 2 CVX-CIO Precoding

Input: $H$, $u$

Output: $x$

Derive $x'$ from $P_3'$ via standard convex optimization methods

Return $x = [x_1, ..., x_N]^T = \left[\frac{x_1'}{\sqrt{N}}, ..., \frac{x_N'}{\sqrt{N}}\right]^T$

V. COMPUTATIONAL COMPLEXITY

In this section, we compute and analyze the complexity of the proposed CEO-CIO in comparison with the CEO approach to interference reduction (CEO-IR) precoding from [16] in terms of floating-point operations (flops), following the operational costs listed in the literature [29]. More specifically, we consider addition, subtraction and multiplication between two floating-point numbers as a flop. Since both approaches are characterized by the same number of iterations $T$, we focus our analysis on the computational burdens of the two different cost functions.

For our study, we consider a simple time-division duplexing (TDD) scenario [30] where coherence time $T_{\text{cohe}}$ indicates the maximum number of data symbols that can be transmitted within a channel realization, i.e., when the elements of the channel matrix $H$ can be considered constants. The TDD assumption is not uncommon in M-MIMO literature, as it allows to exploit the reciprocity of the channel, enabling the CSI acquisition for downlink via uplink pilots. This property is fundamental in M-MIMO systems, as the time required by CSI acquisition $T_{\text{CSI}}$ becomes proportional to the number of users $M$ instead of the number of antennas $N$. In our analysis, we consider a simple TDD case where $T_{\text{CSI}} = \mu M$, with $\mu \geq 1$ being the number of pilot slots.

Finally, we consider a symmetrical transmission case where the time for data transmission $T_{\text{data}} = T_{\text{cohe}} - T_{\text{CSI}}$ is divided between downlink and uplink transmissions according to a parameter $0 \leq \epsilon_{DL} \leq 1$. The parameter $\epsilon_{DL}$ explicitly represents the portion of $T_{\text{data}}$ devoted to downlink symbol transmission. Analytically, we have

$$T_{DL} = \epsilon_{DL} \left(T_{\text{cohe}} - T_{\text{CSI}}\right) = \epsilon_{DL} \left(T_{\text{cohe}} - \mu M\right).$$ \hspace{1cm} (34)

A. CEO-CIO Costs

As previously mentioned, main costs of the proposed CEO-CIO algorithm reside in the need to compute the cost function for each of the randomly generated samples. We can synthesize the computation of the cost function in the following main operations:

- Computation of the received vector in a noise free scenario $\tilde{r} = Hx$,
- Identification of the interfering signal vector $t = \tilde{r} - u$,
- Projection of the interfering signal $\tilde{t} = t \circ u^*$,
- Identification of $\min \{|\Re(\tilde{t})\tan \Phi - |\Im(\tilde{t})|\}$,

where $\circ$ represents the Hadamard product.

From the literature [29], we know the costs of each of the aforementioned operations: the multiplication between a $M \times N$ matrix and an $N \times 1$ vector requires $M(2N-1)$ flops, while the computation of the interfering signal and its rotation can be performed with $M$ flops each, since they can be achieved by $M$ subtractions and multiplications, respectively. Finally, we can compute the costs of the identification of the minimum as a search through an $M$-sized vector, hence leading to $M$ flops. It follows that the proposed approach is characterized by a total flop count of $M(2N-1) + 4M$ flops, which includes the cost of the separation between the real and imaginary part of the rotated interfering signal. Computational costs for the derivation and transmission of a CEO-CIO signal are listed in Table I.

B. CEO-IR Costs

The application of the conventional CEO-IR follows a similar pattern to CEO-CIO, due to the fact that they both require the computation of the interfering signal for all the randomly generated samples. More specifically, the computational costs of CEO-IR can be highlighted in the following operations:

- Computation of the received vector in a noise free scenario $\tilde{r} = Hx$,
- Identification of the interfering signal vector $t = \tilde{r} - u$,
- Computation of the interference energy $\sum_{m=1}^{M} |m|^2$.

Following a similar approach to the previous section, we identify the multiplication costs in $M(2N-1)$ flops and the computation of the interfering signal as $M$ flops. Since the interfering energy can be computed as the inner product of two $M$-sized vectors, i.e., by a cost of $2M - 1$ flops, the total cost of the CEO-IR algorithm is $M(2N-1) + 3M - 1$ flops.

As we can see, the computational costs of the proposed technique CEO-CIO are comparable to the ones of the CEO-IR approach from the literature, as the flop count difference is almost negligible. The total costs of the application of the precoding techniques in a coherence time are listed in Table I, which includes the effects deriving by both the number of iterations $T$ and the sample size $K$.

VI. CSI-ROBUST CONSTANT ENVELOPE PRECODING

In the previous sections we assumed the transmitter to possess a perfect knowledge over the channel, allowing the definition of the constructive and destructive regions of interference in absence of uncertainty. When the CSI acquisition is imperfect, however, the received signal region extends according to the CSI error. We consider the BS to be aware of
TABLE I: Computational Costs in flops.

<table>
<thead>
<tr>
<th>CEO-CIO</th>
<th>T · K · M(2N − 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>r = Hx</td>
<td>T · K · M</td>
</tr>
<tr>
<td>t = r − u</td>
<td>T · K · M</td>
</tr>
<tr>
<td>t = t ∘ u*</td>
<td>T · K · M</td>
</tr>
<tr>
<td>min {ℜ(t) tan Φ −</td>
<td>3( t )</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>TDL · T · K [M(2N − 1) + 4M]</strong></td>
</tr>
</tbody>
</table>

CEO-IR

| r = Hx  | T · K · M(2N − 1) |
| t = r − u | T · K · M       |
| JH t | T · K · 2M − 1 |
| **Total** | **TDL · T · K [M(2N − 1) + 3M − 1]** |

an estimated channel matrix, defined analytically as follows [23]

\[ \hat{H} = H + S, \]

where the error matrix S represents the CSI uncertainty at the BS, statistically independent from H, and characterized as a constrained spherical error, i.e., each element \( s_{m,n} : \left\{ |s_{m,n}|^2 \leq \delta_{m,n}^2 \right\} \) [23]. Following [23], we consider a scenario where the base station is aware of the error bounds \( \delta_{m,n} \) but has no knowledge over the error matrix S. Differently from classical robust precoding approaches from the literature [23], [31], [32], where the transmitted power is increased in order to overcome the effects of CSI estimation errors, we propose a worst-case approach where the optimization region is redefined according to the CSI uncertainty, while preserving CEP constraints. The estimated interfering signal for the m-th user, in case of imperfect CSI, can be defined as follows

\[
\hat{t}_m = \left\{ \sum_{n=1}^{N} \frac{1}{\sqrt{N}} \hat{h}_{m,n} e^{j\theta_n} - d_m e^{j\phi_m} \right\} 
= \left\{ \sum_{n=1}^{N} \frac{1}{\sqrt{N}} \left( h_{m,n} + s_{m,n} \right) e^{j\theta_n} - d_m e^{j\phi_m} \right\} 
= \left\{ \sum_{n=1}^{N} \frac{1}{\sqrt{N}} h_{m,n} e^{j\theta_n} - d_m e^{j\phi_m} \right\} + \sum_{n=1}^{N} s_{m,n} e^{j\theta_n},
\]

where \( s_{m,n} \) represents the n-th element of the m-th row of the CSI uncertainty matrix S. As we can see in the last step of (36), the estimated interfering signal \( \hat{t}_m \) is characterized by two different components: the actual interfering signal \( t_m \), i.e., when considering perfect CSI, and the uncertainty error signal \( z_m = \sum_{n=1}^{N} s_{m,n} e^{j\theta_n} \). It follows that the estimated interfering signal can be defined as the sum of the two terms

\[
\hat{t}_m = t_m + z_m.
\]

In (13), the interfering signal is rotated according to the desired symbol, with the aim to have a region definition that is independent from the specific phase of the symbol of interest \( u_m \). In a similar manner, we can define the rotated interfering signal for the m-th user in presence of CSI errors \( \hat{t}_m \) as

\[
\tilde{t}_m = \hat{t}_m e^{-j\phi_m} = \hat{t}_m + \bar{z}_m.
\]

The second term in (38) can be described as the shift from the ideal interfering signal \( \tilde{t}_m \) caused by the CSI errors and can be represented as a circular constrained region of uncertainty, as visually presented in Fig. 3a. Accordingly, we can identify the worst-case scenario in the event where the actual interfering signal \( \tilde{t}_m \) is within the constructive interference region, but the uncertainty error signal \( \bar{z}_m \) moves the estimated \( \hat{t}_m \) away from it, as shown in Fig. 3b. Since we assume the CSI errors to be constrained within a spherical region, it is possible to analytically derive amplitude and phase of the worst-case scenario uncertainty error signal \( \bar{z}_m \).

**Theorem 1.** The amplitude of \( \bar{z}_m \) is characterized by the following analytical upperbound

\[
|\bar{z}_m| \leq \frac{\sum_{n=1}^{N} \delta_{m,n}}{\sqrt{N}}.
\]

**Proof.** Following the definition of \( \bar{z}_m \) we have

\[
|\bar{z}_m| = \left| \sum_{n=1}^{N} \frac{1}{\sqrt{N}} s_{m,n} e^{j\theta_n} e^{-j\phi_m} \right| = \left| \sum_{n=1}^{N} \frac{1}{\sqrt{N}} |s_{m,n}| e^{j\{\phi(s_{m,n}) + \theta_n - \phi_m\}} \right|,
\]

where \( s_{m,n} \) has been represented in order to show amplitude and phase and the operator \( \phi \{ \cdot \} \) identifies the phase extraction of the argument. The absolute value of \( z_m \) is evaluated as the absolute value of the sum of complex values. According to the triangle inequality (i.e., given two complex numbers \( a, b \in \mathbb{C} \) they satisfy the property \( |a + b| \leq |a| + |b| \)) we have

\[
\left| \sum_{n=1}^{N} \frac{1}{\sqrt{N}} |s_{m,n}| e^{j\{\phi(s_{m,n}) + \theta_n - \phi_m\}} \right| \leq \sum_{n=1}^{N} \frac{1}{\sqrt{N}} |s_{m,n}|.
\]

Given the assumption of a spherical constrained error during CSI estimation we have

\[
\sum_{n=1}^{N} \frac{1}{\sqrt{N}} |s_{m,n}| \leq \sum_{n=1}^{N} \frac{1}{\sqrt{N}} \delta_{m,n}.
\]

which ends the proof.
Finally, the worst-case scenario phase of $\bar{\theta}_m$ can be readily identified as the phase that is orthogonal to the constructive interference threshold identified by $\Phi$.

The knowledge of the worst-case effects of CSI errors at the transmitter can be used to relax of the optimization region, in order to include the events that would be affected by the uncertainty error signal. Thanks to this relaxation, we can achieve a CSI errors robust precoding, without the need to increase the transmitted power.

More specifically, according to simple geometrical analysis, the phase threshold $\Phi$ is relaxed as

$$
\Phi_R(\delta_m) = \Phi_L + \arctan\left(\frac{\sum_{n=1}^{N} \delta_{m,n}}{E\{|t_m|\} \sqrt{N}}\right),
$$

(43)

where $\Phi_L = \pi/L$ identifies the threshold angle for the $L$ order PSK modulation used in transmission. Accordingly, we can define a new optimization problem, specifically designed for the imperfect CSI case.

$$
P_4: \quad \text{maximize} \quad \min_m \{\Re(\bar{\theta}_m + \pi) \tan \Phi_R(\delta_m) - |\Im(\bar{\theta}_m)|\}
$$

subject to $|\bar{\theta}_n| \leq \pi, \forall n \in \{1, \ldots, N\}$,

(44)

Without loss of generality, in our studies we consider a case where $\delta_{m,n} = \delta, \forall n \in \{1, \ldots, M\}, \forall n \in \{1, \ldots, N\}$ and $E\{|t_m|\}$ is unitary, which leads to a simplified definition of the robust relaxation

$$
\Phi_R(\delta) = \begin{cases} 
\Phi_L + \arctan\left(\delta \sqrt{N}\right) & \text{if } \arctan\left(\delta \sqrt{N}\right) \leq \pi/L \\
\Phi_{L-1} - \epsilon & \text{otherwise},
\end{cases}
$$

(45)

where $\epsilon$ is an arbitrarily small positive quantity, which imposes an upperbound to the growth of $\Phi_R$ for high values of $\delta$, and $L-1$ identifies the modulation order which is immediately lower than the one used during data transmission. The defined upperbound is particularly important, given the fact that very high values of $\delta$ could cause ambiguity with lower modulation orders, i.e., when their values lead the robust region $\Phi_R(\delta)$ to coincide with or exceed $\Phi_{L-1}$.

VII. RESULTS

This section shows the performances of the proposed precoding techniques through Monte Carlo simulations over 50000 channel realizations. We consider the downlink transmission described in the previous sections, where the BS employs $N = 64$ antennas to communicate with a population of $M = 12$ mobile users. Since the proposed technique can be applied independently from the modulation order, results are presented for both 4-PSK and 8-PSK. Legends are characterized by the following notation: CEO-CIO identifies constructive interference driven precoding based on CEM. CEO-IR is used to represent interference minimization CEO precoding and finally, CVX-CIO represents the two-step convex CEP approach to constructive interference optimization. Both CEO techniques are applied while considering the same parameter settings: $T = 1000, \rho = 0.05$ and $\alpha = 0.08$ [16]. In addition to CEO-IR, we compare the proposed techniques with a CEP approach to linear zero-forcing(ZF) precoding [14], ZF-P in the legends, which can be analytically defined as

$$
x_{ZF-P} = \frac{e^{j\mu[G_{ZF}u]}}{\sqrt{N}},
$$

(46)

where $G_{ZF} = H^H (HH^H)^{-1}$ is the ZF precoding matrix.

Figures 4 and 5 present the SER as a function of the transmitted SNR for 4-PSK and 8-PSK modulation respectively when considering a BS with $N = 64$ and $M = 12$ users. As we can see from Fig.4 and Fig.5, the proposed approaches strongly outperform the classical CEO-IR and ZF-P. This is due to the fact that CEO-CIO wisely exploits the interference signal $t_m, \forall m \in \{1, \ldots, M\}$ to increase the received signal power, while CEO-IR aims to a direct minimization of the interference energy. Regarding the ZF-P approach, we can see that a direct normalization of the precoded signal leads to a significant decrease in performances, due to its sub-optimal
In our simulations we assume the desired symbols to have unitary energy constellation, i.e., $d_m = d = 1, \forall m \in \{1, ..., M\}$. While this assumption is not uncommon in CEP literature [13]–[16], the constellation energy can be increased to improve CEP-IR performances. This represents one of the key drawbacks of the CEP-IR approach, as its performances are strongly dependent on the constellation energy $E = d^2$.

A. Constellation Energy

In fact, since the expected value of the MUI is a function of both topology (i.e., number of antennas at the BS and number of users) and modulation used in transmission [14], it is not possible to know a priori the optimal constellation amplitude $d^*$. More specifically, the identification of the optimal energy would require to dynamically estimate the SER at the transmitter side as a function of the constellation energy $E$, hence increasing the computational complexity of the system. Otherwise, the search for a sub-optimal constellation energy for CEO-IR could be performed at the transmitter side via an additional topology-dependent optimization problem [14]. The optimization problem that identifies the optimal constellation amplitude $d^*$ is defined as follows [14]

$$
\begin{align*}
\text{maximize} & \quad d \\
\text{subject to} & \quad E\left\{ \sum_{m=1}^{M} \left( \sum_{n=1}^{N} \frac{h_{m,n}}{\sqrt{N}} e^{j\theta_n} - d_m e^{j\theta_m} \right)^2 \right\} \leq \gamma \\
& \quad d_m = d, \forall m \in \{1, ..., M\}
\end{align*}
$$

where $\gamma \geq 0 \in \mathbb{R}^+$ is a chosen threshold parameter to the MUI energy. The optimization problem aims to identify the maximum constellation energy that preserves the expected MUI energy within a desired threshold.

It is important to stress that for classic CEO-IR, the choice of the constellation energy is critical. These considerations are visually presented in Fig.8 and Fig.9, for the $M = 6, N = 32$ and $M = 12, N = 64$ scenario respectively. Both figures consider the perfect-CSI case, while similar results can be seen for the imperfect-CSI case. In fact, the aforementioned figures show that the performances of CEO-IR worsen as we incautiously increase the constellation energy $d$, with this effect being particularly visible for higher modulation orders such as 8-PSK. This is due to the MUI-based metric used for CEO-IR, which aims only to minimize the energy of the interference signal, without having any control over its phase. Moreover, we can see that the optimal $d^*$ changes when considering...
identify the optimal transmitted constellation energy, hence of the interference at the user side. In other words, the desired symbol energy as they allow a constrained portion of the energy spectrum. This is supported by the fact that the proposed metric are able to outperform the classical CEO-IR for most of different scenarios and different modulations, supporting how it is not possible to identify $d^*$ before transmitting. On the other hand, the performances of the proposed techniques are not affected by the desired symbol energy, as they aim to maximize the constructive effects of interference over the received signal. Therefore, a critical benefit of the proposed scheme is that the additional optimization of $E$ can be avoided, along with the significant associated computational costs. In fact, as shown in Fig.8 and Fig.9 the proposed techniques are able to outperform the classical CEO-IR for most of the energy spectrum. This is supported by the fact that the performances of the proposed metric are independent from the desired symbol energy as they allow a constrained portion of the interference at the user side. In other words, the proposed metric adaptively increases the received constellation in function of the current CSI, without the need to additionally identify the optimal transmitted constellation energy, hence showing a very positive complexity-performance trade-off.

VIII. CONCLUSIONS

This paper proposes a CEP scheme where multi-user interference is effectively exploited to increase the performances of systems with constant envelope constraints at the base station. The proposed techniques show that a relaxation of the optimization region in function of the constructive interference can be beneficial to achieve reliable communications. The computational burdens of the proposed techniques has been analyzed in terms of flops, and compared with the approaches from the literature, showing negligible differences. In addition, a precoding approach robust to bounded CSI errors that does not require to increase the transmitted power has been analytically derived for scenarios that involve imperfect CSI. Finally, performances have been shown in terms of symbol error rate for different modulation orders, proving the benefits introduced by the proposed scheme when compared to classical CEP approaches.

Fig. 8: Symbol Error Rate as a function of the constellation energy $E = d_m^2 = d^2, \forall m \in \{1, ..., M\}$ when $M = 6$ and $N = 32$.

Fig. 9: Symbol Error Rate as a function of the constellation energy $E = d_m^2 = d^2, \forall m \in \{1, ..., M\}$ when $M = 12$ and $N = 64$.

ACKNOWLEDGMENT

This work was supported by the Royal Academy of Engineering, UK and the Engineering and Physical Sciences Research Council (EPSRC) project EP/M014150/1.

REFERENCES


