Lattice point problems in the hyperbolic plane

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I, Dimitrios Chatzakos, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the work.

Abstract

In this thesis we investigate two different lattice point problems in the hyperbolic plane, the classical hyperbolic lattice point problem and the hyperbolic lattice point problem in conjugacy classes. In order to study these problems we use tools from the harmonic analysis on the hyperbolic plane $\mathbb{H}$.

In Chapter 1, we give a short introduction to Euclidean lattice counting problems. We also give a description of the two hyperbolic lattice counting problems we are interested in and we summarize some of our results.

In Chapter 2 we give a introduction to the spectral theory of $GL(2)$-automorphic forms that we will use and we study Ω-results for the error term of the classical hyperbolic lattice point problem. This type of results were first investigated by Phillips and Rudnick. We use their ideas and specific properties of the Selberg/Harish-Chandra transform to prove Ω-results for the normalized error term and $\Omega_{\pm}$-results for the error term pointwise.

In Chapter 3 we study the error term of the lattice point problem in conjugacy classes. This problem is related to counting geodesic segments from a fixed point to a fixed geodesic of the Riemann surface $\Gamma\backslash\mathbb{H}$. Using the large sieve inequalities of Chamizo for $\Gamma\backslash\mathbb{H}$, we prove upper bounds for the second moments of the error term which are conjecturally optimal. We also discuss upper bounds on average for the error terms of both problems on closed geodesics.

In Chapter 4 we prove mean value and Ω-results for the lattice point problem in conjugacy classes. This work extends the results of Phillips-Rudnick to this problem. We prove that, after normalization, the error term of the conjugacy class problem has mean value in the radial parameter. We also deduce various Ω-results for the average of the error term on closed geodesics and pointwise.

Finally, in Chapter 5, we briefly review some arithmetic applications of the classical problem in counting solutions of quadratic forms and correlation sums of arithmetic functions, and we prove arithmetic applications of the conjugacy class problem. The main results here are about counting solutions of indefinite quadratic forms in four variables under restrictions.

In Chapter 6 we give a brief summary of these results in two tables.

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Contents

Chapter 1

Introduction

The spectral theory of automorphic forms is the area of number theory that studies the properties of automorphic forms using the tools of harmonic analysis. In this thesis we study applications of the spectral theory of the Laplace operator for the Riemann surfaces $\Gamma\backslash\mathbb{H}$ on lattice counting problems on the hyperbolic plane $\mathbb{H}$.

1.1 Euclidean lattice point problems

Lattice point counting problems arise naturally in various areas of number theory. Historically, these kind of problems first appeared in the Euclidean case, in the work of Gauss of Dirichlet. The most famous Euclidean lattice point problems are the Gauss' circle problem (1834) and the Dirichlet's divisor problem (1849). These two problems are closely related [42]. We consider a Euclidean circle D with center at the origin $(0,0)$, and radius $x^{1/2}$. Call $N(x)$ the number of integer points in the interior of this circle, i.e.

$$N(x) = \#\left\{w = (a,b) \in \mathbb{Z}^2 : \|w\| \leq x^{1/2}\right\}. \tag{1.1}$$

Using an elementary geometric packing method, Gauss first proved (1834) that, as $x \to \infty$,

$$N(x) = \sum_{m \leq x} r(m) = \pi x + O(x^{1/2}),$$

where $r(m) = \#\{(a,b) \in \mathbb{Z}^2 : a^2 + b^2 = m\}$, the number of representations of m as sum of two squares. The key point in Gauss' proof is bounding the error term $E(x) = N(x) - \pi x$ by the area of a boundary strip $\left\{x^{1/2} - 1/\sqrt{2} < |z| < x^{1/2} + 1/\sqrt{2}\right\}$. The Gauss circle problem asks to estimate the order of growth of the error term $E(x)$ as $x \to \infty$.

The first improvement of Gauss' bound was obtained by Voronoi (1903), Sierpinski (1906), Landau (1913) and van der Corput (1923). Using tools of harmonic analysis on $\mathbb{R}^2$ (Poisson summation formula), they obtained the bound

$$E(x) = O(x^{1/3}).$$

This upper bound has been improved several times by many authors. The optimal existing bound

$$E(x) = O\left(x^{131/416}(\log x)^{18637/8320}\right)$$

is due to Huxley [40]. We refer to [1], [42] for a detailed history of these results.

The error term $E(x)$ has a Fourier expansion (called Hardy's identity) involving special functions (Bessel functions). This formula can be simplified to the form

$$E(x) = \frac{x^{1/4}}{\pi} \sum_{n=1}^{\infty} \frac{r(n)}{n^{3/4}} \sin\left(2\pi\sqrt{nx} - \frac{\pi}{4}\right) + O(1). \tag{1.2}$$

In 1914 Hardy conjectured that one should expect $E(x) = O_\epsilon(x^{1/4+\epsilon})$ for every $\epsilon > 0$. He supported this conjecture by proving Ω-results for the error term. More specifically, in [28], [29] he proved that

$$E(x) = \Omega_-(x^{1/4}\log^{1/4} x), \quad E(x) = \Omega_+(x^{1/4}).$$

These results were subsequently improved by many authors. The current record is due to Soundararajan [67], who proved that

$$E(x) = \Omega\left(x^{1/4}(\log x)^{1/4}(\log\log x)^{(3/4)(2^{1/3}-1)}(\log\log\log x)^{-5/8}\right).$$

We refer again to [1] for a history of these Ω-results.

Although the optimal growth of $E(x)$ is still not known, one can prove average results supporting Hardy's conjecture. Hardy in 1917 [30] proved that, averaging over the radius x, one gets

$$\int_1^T |E(x)|\, dx = O(T^{5/4+\epsilon}).$$

Further, Cramér in 1922 [16] proved that

$$\int_1^T |E(x)|^2\, dx = cT^{3/2} + O(T^{5/4+\epsilon}),$$

where c is the explicit constant

$$c = \frac{1}{3\pi^2} \sum_{n=1}^{\infty} \frac{r^2(n)}{n^{3/2}}.$$

Notice that in Cramér's result the asymptotic gets rid of the extra factor x^ϵ in the main term. Instead of averaging over the radii, Kendall averaged over the centers. If $E(x, \alpha, \beta)$ denotes the error term of the circle problem with radius x and center $(\alpha, \beta) \in [0,1]^2$, then Kendall [45] proved that

$$\int_0^1 \int_0^1 |E(x, \alpha, \beta)|^2 d\alpha d\beta = O(x^{1/2}),$$

and

$$\int_0^1 \int_0^1 |E(x, \alpha, \beta)|^2 d\alpha d\beta = \Omega(x^{1/2}).$$

Here also the upper bound gets rid ot the extra factor x^ϵ. Both the results of Cramér and Kendall support the conjectured optimal bound for $E(x)$.

1.2 The general lattice point problem

Let X be a topological space and Γ a group that acts discontinuously on X. Let $D = \{D_i\}$ be a family of compact subsets $D_i \subset X$ and z a point in X. The lattice point problem is to estimate the number of points of the orbit $\Gamma z = \{\gamma z : \gamma \in \Gamma\}$ which meet D_i. In this setting, Gauss' circle problem is the special case $X = \mathbb{R}^2$, $\Gamma = \mathbb{Z}^2$ and D is the family of circles with center at the origin $(0,0)$ and radius $x \to \infty$.

Quite often, the space X is the homogeneous space G/K of a Lie group G, where K is a maximal compact subgroup of G and Γ is a lattice in G. We may consider D be a family of well-shaped compact sets, for instance, circles, ellipsoids or more general well-rounded sets (see [18], [20]). We are interested in the case $G = \text{SL}_2(\mathbb{R})$, $K = \text{SO}_2(\mathbb{R})$, and we identify the symmetric space G/K with the hyperbolic plane:

$$\mathbb{H} = \text{SL}_2(\mathbb{R})/\text{SO}_2(\mathbb{R}).$$

In this case, a lattice Γ is a discrete (Fuchsian) subgroup of $\text{PSL}_2(\mathbb{R})$ of the first kind. For such a group, the volume of the surface $\Gamma\backslash\mathbb{H}$ is always finite.

1.3 Summary of our results

We will study two different kinds of lattice counting problems in $\mathbb{H}$. The first problem we are interested in is the (classical) hyperbolic lattice counting problem. This is the hyperbolic analogue of Gauss circle problem. Assume $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ is a Fuchsian group of the first kind and z, w are two (perhaps different) fixed points in $\mathbb{H}$. The problem asks to estimate the counting function

$$N(X; z, w) = \#\left\{\gamma \in \Gamma : 2\cosh\rho(z, \gamma w) \leq X\right\},$$

as $X \to \infty$, where $\rho(z, w)$ is the hyperbolic distance of the points z and w. Due to the geometry of the plane $\mathbb{H}$ (surface of negative curvature) the elementary packing method of Gauss fails to provide an asymptotic estimate for the behaviour of $N(X; z, w)$ (see Chapter 2 for details). Using properties of special functions, Delsarte [17] first found that $N(X; z, w)$ has the asymptotic behaviour

$$N(X; z, w) \sim \frac{\pi}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} X.$$

Selberg [65] et al. used the spectral theory of automorphic forms to prove the stronger result

$$N(X; z, w) = M(X; z, w) + E(X; z, w)$$

where $M(X; z, w)$ is a finite sum of main terms and the error term satisfies the bound

$$E(X; z, w) = O(X^{2/3}).$$

The main term $M(X; z, w)$ depends on the small eigenvalues $\lambda_j < 1/4$ of the hyperbolic Laplacian $-\Delta$ of the surface $\Gamma\backslash\mathbb{H}$. When Γ is arithmetic, the $1/4$-conjecture of Selberg implies there are no secondary summands in $M(X; z, w)$.

The upper bound $O(X^{2/3})$ for the error term is not expected to be optimal; however, it has not been improved for any group Γ and any fixed points z, w. The main contribution to the error term comes from the 'large' eigenvalues $\lambda_j > 1/4$. Understanding the distribution of these eigenvalues and the behaviour of the corresponding eigenfunctions $u_j(z)$ (Maaß forms) is one of the major problems of the analytic theory of $GL(2)$-automorphic forms. Numerical investigations by Phillips and Rudnick [60] suggest we should expect

$$E(X; z, w) = O_\epsilon(X^{1/2+\epsilon})$$

for every $\epsilon > 0$. Any progress towards this conjecture is difficult, as it amounts to detecting subtle cancelation between the Selberg/Harish-Chandra transform and the Maaß forms $u_j(z)$.

Chamizo, in his PhD thesis [5], [6], [7] proved large sieve inequalities for the Riemann surfaces $\Gamma\backslash\mathbb{H}$ and he used them to deduce upper bounds for the second moments of the error term $E(X; z, w)$. He proved the following two theorems.

Theorem 1.3.1 (Chamizo [5], [7])**.** *Let Γ be a cocompact or cofinite Fuchsian group and z, w two points in $\mathbb{H}$. Then*

$$\frac{1}{X}\int_X^{2X} |E(x; z, w)|^2 dx \ll X \log^2 X, \tag{1.3}$$

where the constant implied in '$\ll$' depends on Γ, z and w.

Theorem 1.3.2 (Chamizo [5], [7])**.** *Let Γ be a cocompact Fuchsian group and z, w two points in $\mathbb{H}$. Then, for $n = 1, 2$*

$$\int_{\Gamma\backslash\mathbb{H}} |E(X; z, w)|^{2n} d\mu(z) \ll X^n \log^{2n} X, \tag{1.4}$$

where the constant implied in '$\ll$' depends on Γ.

The extra $\log^2 X$ can be improved to $\log X$, see Cherubini [13, Th. 12, p. 2].

We extract the contribution of $\lambda_j = 1/4$ from $E(X; z, w)$ and we denote the new error by $e(X; z, w)$. Phillips and Rudnick [60] investigated the behaviour of $e(X; z, z)$ and they deduced mean value limit in the radial parameter $r \sim \log X$ and Ω-results for the error. In contrast to their results, in Chapter 2 we investigate mean value results in the X-parameter and for different points z and w. Let $E_{\mathfrak{a}}(z, s)$ denote the nonholomorphic Eisenstein series associated with the cusp $\mathfrak{a}$. A null-vector is an Eisenstein series such that $E_{\mathfrak{a}}(z, 1/2) \neq 0$ (see [58, p. 64–66] for an explanation of this definition). We prove the following results.

Theorem 1.3.3. *Let Γ be a cocompact or cofinite Fuchsian group and let z be a fixed point. If Γ has an eigenvalue $\lambda_j > 1/4$ with $u_j(z) \neq 0$ and $\lambda_1 > 2.7823...$ then there exists a fixed $\delta = \delta_{\Gamma,z} > 0$ such that for every point $w \in B(z, \delta)$ the limit*

$$\lim_{X\to\infty} \frac{1}{X}\int_2^X \frac{e(x; z, w)}{x^{1/2}} dx$$

does not exist.

Theorem 1.3.4. *Let Γ be a cofinite but not cocompact Fuchsian group, z be a fixed point and assume that Γ has at least one null-vector. If Γ has sufficiently many cusp forms at z in the sense that the series*

$$\sum_{|t_j|\leq T}\frac{|u_j(z)|^2}{t_j^{3/2}}$$

diverges, then there exists a fixed $\delta=\delta_{\Gamma,z}>0$ such that for every point $w\in B(z,\delta)$ we have:

$$e(X;z,w)=\Omega_{\pm}\left(X^{1/2}\right).$$

For instance, $\mathrm{SL}_2(\mathbb{Z})$ satisfies the conditions of Theorem 1.3.3 and every $\Gamma(N)$ with $N=5$ or ≥ 7 satisfies the conditions of Theorem 1.3.4.

The second problem we are interested in is the hyperbolic lattice point problem in conjugacy classes. Let $\mathcal{H}$ denote a hyperbolic conjugacy class of Γ and μ the length of the $\mathcal{H}$-invariant closed geodesic ℓ. The main problem here is to study the asymptotic behaviour of the quantity

$$N(\mathcal{H},X;z)=\#\left\{\gamma\in\mathcal{H}:\sinh\left(\frac{\rho(z,\gamma z)}{2}\right)\leq\sinh\left(\frac{\mu}{2}\right)X\right\}$$

as $X\to\infty$. This problem is related with counting distances of points in the orbit of z from a closed geodesic of $\Gamma\backslash\mathbb{H}$. In Chapter 3 we use spectral theory to study the problem. Here again we write

$$N(\mathcal{H},X;z)=M(\mathcal{H},X;z)+E(\mathcal{H},X;z),$$

where the main term $M(\mathcal{H},X;z)$ is a finite sum over the small eigenvalues of the hyperbolic Laplacian. The conjugacy class problem has been studied by Huber [36], [38] and Good [26] using spectral theory of automorphic forms, while recently it was investigated by Parkkonen and Paulin [54] with ergodic methods. Using estimates for the Huber transform, the local Weyl's law for Maaß forms and estimates for their period integrals, we give a new and simpler proof of the following result.

Theorem 1.3.5. *Let Γ be a cofinite or cocompact Fuchsian group. Then*

$$E(\mathcal{H},X;z)=O(X^{2/3}).$$

As in the classical problem, the bound $O(X^{2/3})$ has not been improved for any group Γ, class $\mathcal{H}$ and point z. In Chapter 3 we apply Chamizo's large sieve to study the second moments of the error term of the conjugacy class problem.

Theorem 1.3.6. *Let Γ be a cocompact or cofinite Fuchsian group, and $\mathcal{H}$ a hyperbolic conjugacy class of Γ. Then*

$$\frac{1}{X}\int_X^{2X} |E(\mathcal{H},x;z)|^2 dx \ll X\log^2 X,$$

where the constant implied in '$\ll$' depends on $\Gamma,\mathcal{H}$ and z.

Theorem 1.3.7. *Let Γ be a cocompact Fuchsian group, and $\mathcal{H}$ a hyperbolic conjugacy class of Γ. Then, for $n=1,2$*

$$\int_{\Gamma\backslash\mathbb{H}} |E(\mathcal{H},X;z)|^{2n} d\mu(z) \ll X^n \log^{2n} X,$$

where the constant implied in '$\ll$' depends on Γ and $\mathcal{H}$.

We extract the contribution of $\lambda_j = 1/4$ from $E(\mathcal{H},X;z)$ and we denote the new error by $e(\mathcal{H},X;z)$. In Chapter 4 we study mean value and Ω-results for the conjugacy class problem. We prove that the error term $e(\mathcal{H},X;z)$ has mean value in the radial parameter.

Theorem 1.3.8. *Let Γ be a cocompact or cofinite Fuchsian group and for $x\geq 1$ let r be defined as $r=\log\left(x+\sqrt{x^2-1}\right)$. Then*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{e\left(\mathcal{H},x;z\right)}{x^{1/2}}dr = \frac{|\Gamma(3/4)|^2}{\pi^{3/2}}\sum_{\mathfrak{a}} \hat{E}_{\mathfrak{a}}(1/2)E_{\mathfrak{a}}(z,1/2), \tag{1.5}$$

where $\hat{E}_{\mathfrak{a}}(1/2)$ is the period integral of the Eisenstein series $E_{\mathfrak{a}}(z,1/2)$ across a specific segment of ℓ and the sum is understood to be 0 if Γ is cocompact.

Further, we use an averaging argument combined with mean results for the periods $\hat{u}_j$ of Maaß forms to prove the following bound.

Theorem 1.3.9. *Let Γ be cocompact or cofinite with sufficiently small Eisenstein periods in the sense that*

$$\int_{-T}^{T} |\hat{E}_{\mathfrak{a}}(1/2+it)|^2 dt \ll \frac{T}{(\log T)^{1+\delta}}$$

for a fixed $\delta > 0$. Then:

$$\int_{\ell} e(\mathcal{H}, X; z) d\mu(z) = \Omega\left(X^{1/2} \log\log\log X\right).$$

We also study pointwise Ω-results for the error $e(\mathcal{H}, X; z)$, using a discrete average. These results depend on the nonvanishing of the periods $\hat{u}_j$ and $\hat{E}_{\mathfrak{a}}(1/2)$.

For specific arithmetic groups Γ one can deduce arithmetic applications of the classical problem in counting solutions of definite quadratic forms in four variables, as well as in studying correlation sums of the arithmetic function $r(n)$. For instance, for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and $z = i$ one can get the estimate (see [41, ch.12]):

$$\#\left\{(a, b, c, d) \in \mathbb{Z}^4 : ad - bc = 1, a^2 + b^2 + c^2 + d^2 \leq X\right\} = 6X + O(X^{2/3}).$$

In Chapter 5 we work with the modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ to deduce arithmetic applications of the conjugacy class problem in counting solutions of indefinite quadratic forms in four variables under restrictions. Let $Q(x, y) = ax^2 + cy^2$ be a quadratic form with integer coefficients such that $d = -4ac > 0$ is not a square and let M be the generator matrix of the group of automorphs $\mathrm{Aut}(Q) \subset \mathrm{SL}_2(\mathbb{Z})$. Let also ε_d be given by:

$$\varepsilon_d = \frac{t_0 + \sqrt{d}u_0}{2}$$

where $t_0, u_0 > 0$ is the fundamental solution of Pell's equation $x^2 - dy^2 = 4$. We prove the following proposition.

Proposition 1.3.10. *Let $P(X)$ be the number of 4-tuples $(\alpha, \beta, \gamma, \delta) \in \mathbb{Z}^4$ such that $\alpha\delta - \beta\gamma = 1$ and*

$$\left|\alpha^2 - \frac{a}{c}\beta^2 + \frac{c}{a}\gamma^2 - \delta^2\right| \leq X,$$

under the equivalence: $(\alpha, \beta, \gamma, \delta) \sim (\alpha', \beta', \gamma', \delta')$ if and only if there exists an integer n such that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = M^n \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}.$$

Then

$$P(X) = \frac{6 \log \varepsilon_d}{\pi} X + O(X^{2/3}).$$

In the case of arithmetic groups there exists a special family of operators that commute with the hyperbolic Laplacian, the *Hecke operators*. In the last section of Chapter 5 we apply the theory of Hecke operators to generalize 1.3.10 for the case $\alpha\delta - \beta\gamma = n$. We also prove similar asymptotic results for a more general family of indefinite quadratic forms.

Chapter 2

The classical hyperbolic lattice point problem

2.1 Spectral theory of automorphic forms for $\mathbf{SL}_2(\mathbb{R})$

We begin with a short introduction to the spectral theory of automorphic forms. The standard reference for this theory is [41]. We work on the hyperbolic plane $\mathbb{H}$, which can be viewed as the complex upper half-plane

$$\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\},$$

endowed with the hyperbolic metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2},$$

i.e., for a path $\gamma = \{z(t) = x(t) + iy(t), t \in [0, 1]\}$ the hyperbolic length is given by

$$h(\gamma) = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \frac{dt}{y(t)}.$$

The distance $\rho(z, w)$ is now defined as $\rho(z, w) = \min h(\gamma)$, where the minimum is considered over all the paths γ from z to w. The distance function can be expressed as $\cosh \rho(z, w) = 1 + 2u(z, w)$, where $u(z, w)$ is the fundamental point pair invariant function

$$u(z, w) = \frac{|z - w|^2}{4\Im(z)\Im(w)}. \tag{2.1}$$

The hyperbolic metric induces a measure $d\mu$ on $\mathbb{H}$ given by

$$d\mu = \frac{dxdy}{y^2}.$$

The space $(\mathbb{H}, ds, d\mu)$ is a Riemannian manifold of constant negative curvature $K = -1$. The group $\mathcal{M}$ of Möbius transformations acts on $\mathbb{H}$, where for $g \in \mathcal{M}$ the action is defined by:

$$g \cdot z = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. The group $\mathcal{M}$ is the group of isometries of $\mathbb{H}$ that preserve orientation. If I denotes the identity matrix of the group

$$\mathrm{SL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : (a,b,c,d) \in \mathbb{R}^4, \quad ad - bc = 1 \right\},$$

then the elements I and $-I$ induce the same Möbius transformation, hence the group $\mathcal{M}$ is isomorphic to the group $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm I\}$. Poincaré proved that a subgroup Γ' of $\mathrm{SL}_2(\mathbb{R})$ is discrete in the norm topology if and only if the projection Γ of Γ' in $\mathrm{PSL}_2(\mathbb{R}) \simeq \mathcal{M}$ acts discontinuously on $\mathbb{H}$ (the orbit of a point does not have accumulation points in $\mathbb{H}$). Such a group is called a Fuchsian group.

Among the family of cofinite Fuchsian groups, there are specific groups of 'arithmetic' nature. Among them, the most important are the modular group

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : (a,b,c,d) \in \mathbb{Z}^4, \quad ad - bc = 1 \right\},$$

and the congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ defined by

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\} \subset \mathrm{SL}_2(\mathbb{Z}),$$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : N|c \right\} \subset \mathrm{SL}_2(\mathbb{Z})$$

for every $N \geq 1$. Moreover, for any of the groups defined above the quotient space $\Gamma\backslash\mathbb{H}$ has finite hyperbolic area. These arithmetic groups are examples of cofinite Fuchsian groups, i.e. a Fuchsian group Γ is called cofinite if the surface $\Gamma\backslash\mathbb{H}$ satisfies

$$\mathrm{vol}(\Gamma\backslash\mathbb{H}) < \infty.$$

Further, if $\Gamma\backslash\mathbb{H}$ is compact then Γ is called cocompact. The quotient space $\Gamma\backslash\mathbb{H}$ has a structure of a Riemann surface. The distance function $\rho(z,w)$ in $\mathbb{H}$ induces a distance $\rho_\Gamma(z,w)$ in $\Gamma\backslash\mathbb{H}$ given by

$$\rho_\Gamma(z,w) = \inf_{\gamma\in\Gamma} \rho(\gamma z, w),$$

where each point in $\Gamma\backslash\mathbb{H}$ is identified with one of its representatives in $\mathbb{H}$.

An element γ of Γ is classified as elliptic, parabolic or hyperbolic if $|\text{tr}(\gamma)| < 2$, $|\text{tr}(\gamma)| = 2$ or $|\text{tr}(\gamma)| > 2$. A point $z \in \overline{\mathbb{H}}$ is called a cusp for Γ if it is the fixed point of a parabolic $\gamma \in \Gamma$. Two points $z, w \in \overline{\mathbb{H}}$ are said to be Γ-equivalent if $w \in \Gamma z$. We have the following definition.

Definition 2.1.1. *A set $F \subset \mathbb{H}$ is called a fundamental domain for Γ if:*
(a) F is a domain in $\mathbb{H}$,
(b) any two distinct points in F are not Γ-equivalent,
(c) any orbit of Γ contains a point in $\overline{F}$.

A fundamental domain for Γ is a model for the quotient space $\Gamma\backslash\mathbb{H}$. The cusps of Γ are on the boundary of $\mathbb{H}$ and there exist only finitely many Γ-inequivalent cusps. They indicate whether a fundamental domain for Γ touches the topological boundary of $\mathbb{H}$. A cofinite group Γ is cocompact if and only if $\Gamma\backslash\mathbb{H}$ does not have cusps, i.e. if and only if Γ does not contain parabolic elements.

For every cusp $\mathfrak{a}$ of Γ there exists a matrix $\sigma_\mathfrak{a} \in \text{PSL}_2(\mathbb{R})$ such that

$$\sigma_\mathfrak{a}\infty = \mathfrak{a}, \quad \sigma_\mathfrak{a}^{-1}\Gamma_\mathfrak{a}\sigma_\mathfrak{a} = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} /\{\pm I\} : n \in \mathbb{Z} \right\},$$

where $\Gamma_\mathfrak{a}$ is ths stabilizer of $\mathfrak{a}$ in Γ. We are interested in functions defined on the Riemann surface $\Gamma\backslash\mathbb{H}$.

Definition 2.1.2. *A function $f : \mathbb{H} \to \mathbb{C}$ is said to be automorphic with respect to Γ if*

$$f(\gamma z) = f(z)$$

for every $\gamma \in \Gamma$.

Clearly, such a function defines a function on the surface $f : \Gamma\backslash\mathbb{H} \to \mathbb{C}$. We denote the space of Γ-automorphic functions by $\mathcal{A}(\Gamma\backslash\mathbb{H})$.

The group $\text{PSL}_2(\mathbb{R})$ leaves invariant the Laplace-Beltrami operator in $\mathbb{H}$, defined by

$$\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

The Laplace operator Δ generates the algebra of $\mathrm{SL}_2(\mathbb{R})$-invariant differential operators on $\mathbb{H}$, i.e. the hyperbolic plane $\mathbb{H}$ is a rank one symmetric space.

Definition 2.1.3 (Maaß [52]). *A smooth automorphic function $f \in \mathcal{A}(\Gamma\backslash\mathbb{H})$ that is an eigenfunction of $-\Delta$*

$$(\Delta + \lambda) f = 0, \quad \lambda = s(1-s),$$

is called an automorphic form.

We denote by $\mathcal{A}_s(\Gamma\backslash\mathbb{H})$ the space of eigenfunctions of $-\Delta$ on $\Gamma\backslash\mathbb{H}$ with eigenvalue $\lambda = s(1-s)$. We also write $s = 1/2 + it$, hence $\lambda = 1/4 + t^2$. Notice that $t \in \mathbb{R}$ if and only if $\Re(s) = 1/2$, i.e. if and only if $\lambda \geq 1/4$. The eigenvalues $\lambda < 1/4$ are called *small* (or *exceptional*) eigenvalues of Γ. If Γ has cusps, the eigenfunctions of Δ that are smooth functions in $\Gamma\backslash\mathbb{H}$ and have exponential decay at every cusp $\mathfrak{a}$ of Γ are called *(Maaß) cusp forms*. We define the inner product

$$\langle f, g \rangle = \int_{\Gamma\backslash\mathbb{H}} f(z)\overline{g(z)} d\mu(z).$$

Maaß cusp forms belong in the space

$$L^2(\Gamma\backslash\mathbb{H}) = \left\{ f \in \mathcal{A}(\Gamma\backslash\mathbb{H}) : \|f\|_2^2 = \langle f, f \rangle < \infty \right\}.$$

Further, for every cusp $\mathfrak{a}$ and $s \in \mathbb{C}$ with $\Re(s) > 1$ one defines the Eisenstein series

$$E_{\mathfrak{a}}(z,s) = \sum_{\gamma \in \Gamma_{\mathfrak{a}}\backslash\Gamma} \left(\Im(\sigma_{\mathfrak{a}}^{-1}\gamma z) \right)^s.$$

The series $E_{\mathfrak{a}}(z,s)$ converges for $\Re(s) > 1$ and belongs in $\mathcal{A}_s(\Gamma\backslash\mathbb{H})$. A very important fact for understanding the spectral theory in $\mathbb{H}$ is that the Eisenstein series $E_{\mathfrak{a}}(z,s)$ has a meromorphic continuation in $\mathbb{C}$ as a function of s and the only poles in $\Re(s) > 1/2$ are simple and real, see [41], [64]. The residues are eigenfunctions of Δ in $L^2(\Gamma\backslash\mathbb{H})$. The meromorphically continued Eisenstein series are orthogonal to cusp forms.

We can now explain the spectral decomposition of $L^2(\Gamma\backslash\mathbb{H})$. The Laplace operator acts on all smooth $f \in \mathcal{A}(\Gamma\backslash\mathbb{H})$. For the purpose of the spectral resolution of Δ on $L^2(\Gamma\backslash\mathbb{H})$, let $D(\Gamma\backslash\mathbb{H})$ be the space of functions $f \in \mathcal{A}(\Gamma\backslash\mathbb{H})$ such that f and Δf are smooth and bounded. Then, $D(\Gamma\backslash\mathbb{H})$ is dense in $L^2(\Gamma\backslash\mathbb{H})$. Further, $-\Delta$ is symmetric and non-negative, hence by Friedrichs Extension Theorem it has a unique self-adjoint extension to $L^2(\Gamma\backslash\mathbb{H})$.

If Γ is cocompact, the operator $-\Delta$ has only discrete spectrum $\{\lambda_j\}_{j=0}^{\infty}$ such that $\lambda_0 = 0$ corresponds to the constant eigenfunction and $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_j \leq \ldots$ with $\lambda_j \to \infty$ as $j \to \infty$. We denote by $\{u_j(z)\}_{j=0}^{\infty}$ a set of Maaß forms, i.e. an orthogonal system of eigenfunctions of the Laplacian:

$$(\Delta + \lambda_j)\, u_j(z) = 0.$$

We will also assume that u_j are L^2-normalized, i.e. $\|u_j\|_2 = 1$.

If Γ is cofinite but not cocompact, the operator $-\Delta$ has also continuous spectrum covering the interval $[1/4, \infty)$ with multiplicity the number of cusps. For a point $\lambda = 1/4 + t^2 \geq 1/4$ in the continuous spectrum the corresponding eigenfunction is the Eisenstein series $E_{\mathfrak{a}}(z, 1/2 + it)$:

$$\left(\Delta + \left(1/4 + t^2\right)\right) E_{\mathfrak{a}}\,(z, 1/2 + it) = 0.$$

In that case, an orthonormal basis $\{u_j(z)\}$ of eigenfunctions for the (possibly infinite) discrete spectrum $\{\lambda_j\}$ consists of the Maaß cusp forms and the residues of the Eisenstein series.

Around 1956 Selberg proved the analytic continuation of the Eisenstein series $E_{\mathfrak{a}}\,(z, s)$ for $\Re(s) < 1$. Using this result, he proved the following main theorem. Independently, around the same time Roelcke (and later Huber) arrived at a weak version of the same result (for the cocompact case).

Theorem 2.1.4 (Spectral theorem in $L^2(\Gamma\backslash\mathbb{H})$ (Selberg [66], Huber [37], Roelcke [62]))**.** *Every function $f \in L^2(\Gamma\backslash\mathbb{H})$ has a spectral expansion*

$$f(z) = \sum_{\lambda_j \geq 0} \langle f, u_j \rangle u_j(z) + \frac{1}{4\pi} \sum_{\mathfrak{a}} \int_{-\infty}^{\infty} \langle f, E_{\mathfrak{a}}(\cdot, 1/2 + it) \rangle E_{\mathfrak{a}}(z, 1/2 + it) dt, \quad (2.2)$$

which converges in the norm topology. Here, the second sum is over all the (finitely many) inequivalent cusps $\mathfrak{a}$ of Γ, if Γ contains parabolic elements. If $f(z)$ belongs to the domain $D(\Gamma\backslash\mathbb{H})$ of functions $f \in \mathcal{A}(\Gamma\backslash\mathbb{H})$ such that f and Δf are smooth and bounded, then the expansion (2.2) converges pointwise absolutely and uniformly on compact sets.

We are particularly interested in the spectral theorem for automorphic kernels, the so-called *pre-trace formula* (see [41, Chapters 1, 7]). Let L be an integral operator in $\mathbb{H}$ defined by

$$(Lf)(z) = \int_{\mathbb{H}} k(z,w)f(w)d\mu(w), \tag{2.3}$$

where $k : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$ is a function called the *kernel* of L, and the functions k, f are such that the integral converges absolutely. The operator L is $\mathrm{SL}_2(\mathbb{R})$-invariant if and only if $k(gz, gw) = k(z,w)$ for all $g \in \mathrm{SL}_2(\mathbb{R})$, i.e. k is a *point-pair invariant* kernel. It follows that k depends only on the distance $\rho(z,w)$. We will write $k(u)$ for the function

$$k(z,w) = k(u(z,w)),$$

for $u(z,w)$ given by formula (2.1).

Assume that $k(u)$ is smooth enough. The invariant integral operators commute with the Laplace operator. Hence, an eigenfunction of Δ in $\mathbb{H}$ is also an eigenfunction for all invariant integral operators. If

$$(\Delta + \lambda) f(z) = 0,$$

for $\lambda = 1/4 + t^2$, then

$$\int_{\mathbb{H}} k(u(z,w))f(w)d\mu(w) = h(t)f(z), \tag{2.4}$$

where $h(t)$ is the *Selberg/Harish-Chandra transform* of the kernel $k(u)$. For given $k(u)$ the transform $h(t)$ is the Fourier transform of the Abel transform of k; more precisely, it can be computed in three steps by the formulas:

$$\begin{aligned} q(v) &= \int_v^{+\infty} \frac{k(u)}{\sqrt{u-v}} du, \\ g(r) &= 2q\left(\left(\sinh\frac{r}{2}\right)^2\right), \\ h(t) &= \int_{-\infty}^{+\infty} e^{irt} g(r) dr. \end{aligned} \tag{2.5}$$

The Selberg/Harish-Chandra transform is an even function of t. The required smoothness of $k(u)$ can be expressed in terms of $h(t)$: it is holomorphic in the strip $|\Im t| \leq 1/2 + \epsilon$ for an $\epsilon > 0$ and satisfies the bound (see [41, Chapter 1, eq. (1.63)])

$$h(t) \ll (|t| + 1)^{-2-\epsilon}. \tag{2.6}$$

Under these assumptions we can invert the process and compute $k(u)$ for a given $h(t)$:

$$\begin{aligned} g(r) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{irt}h(t)dt, \\ q(v) &= \frac{1}{2}g\left(2\log\left(\sqrt{v+1}+\sqrt{v}\right)\right), \\ k(u) &= -\frac{1}{\pi}\int_{u}^{+\infty}\frac{1}{\sqrt{v-u}}dq(u). \end{aligned} \tag{2.7}$$

If we restrict the domain of the operator L to Γ-automorphic functions f, then clearly we get

$$(Lf)(z) = \int_{\Gamma\backslash\mathbb{H}} K(z,w)f(w)d\mu(w), \tag{2.8}$$

where $K(z,w)$ is an *automorphic kernel* given by

$$K(z,w) = \sum_{\gamma\in\Gamma} k(u(\gamma z, w)). \tag{2.9}$$

We view $K(z,w)$ as a function of z. Then, we compute the Fourier coefficients of K, which are given by (see [41, p. 104, eq. (7.16)]):

$$\begin{aligned} \langle K(\cdot,w), u_j\rangle &= h(t_j)\overline{u_j(w)}, \\ \langle K(\cdot,w), E_{\mathfrak{a}}(\cdot,1/2+it)\rangle &= h(t)\overline{E_{\mathfrak{a}}(w,1/2+it)}. \end{aligned}$$

The spectral theorem implies that the automorphic kernel K has the following spectral expansion ([41, Theorem 7.4]).

Theorem 2.1.5 (Pre-trace formula)**.** *Assume the pair $k(u)$ and $h(t)$ is related by equations (2.5) and $h(t)$ satisfies (2.6). Then the automorphic kernel given by eq. (2.9) has the spectral expansion*

$$\begin{aligned} K(z,w) &= \sum_j h(t_j)u_j(z)\overline{u_j(w)} \\ &+ \frac{1}{4\pi}\sum_{\mathfrak{a}}\int_{-\infty}^{\infty} h(t)E_{\mathfrak{a}}(z,1/2+it)\overline{E_{\mathfrak{a}}(w,1/2+it)}dt, \end{aligned} \tag{2.10}$$

which converges absolutely and uniformly on compact sets.

An automorphic kernel $K(z,w)$ that is absolutely integrable on the diagonal $z=w$ is said to be *of trace class*. For those kernels one can go further and deduce the *Selberg* *trace formula*, which relates the spectrum of the Laplacian with the *length spectrum* of $\Gamma\backslash\mathbb{H}$ (see [41, Chapter 10]). An immediate application of the Selberg trace formula is Weyl's law which, roughly speaking, counts the size of the spectrum up to height T. For instance, in the simple case that Γ is cocompact Weyl's law states that, as $T \to \infty$ we have the asymptotic formula

$$\#\{j : |t_j| \leq T\} \sim \frac{\mathrm{vol}(\Gamma\backslash\mathbb{H})}{4\pi} T^2. \tag{2.11}$$

There are specific arithmetic groups for which we know a stronger form of Weyl's law. Selberg (see [41, p. 159, eq. (11.5)]) has proved that for congruence groups we have

$$\#\{j : |t_j| \leq T\} = \frac{\mathrm{vol}(\Gamma\backslash\mathbb{H})}{4\pi} T^2 + O(T \log T). \tag{2.12}$$

Quite often the following local version of Weyl's law is needed for working with lattice counting problems.

Theorem 2.1.6 (Local Weyl's law, [60]). *For every z, as $T \to \infty$,*

$$\sum_{|t_j|<T} |u_j(z)|^2 + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-T}^{T} |E_{\mathfrak{a}}(z, 1/2 + it)|^2 dt \sim cT^2, \tag{2.13}$$

where $c = c(z)$ depends only on the number of elements of Γ fixing z.

When z remains in a bounded region of $\mathbb{H}$ (more specifically in a compact set), the constant $c(z)$ is uniformly bounded, depending only on Γ. For z not remaining in a compact set, instead of the asymptotic (2.13) we will often refer to the following Bessel's inequality (see [41, p. 101, Proposition 7.2]), where the secondary term depends on the height function of z:

$$\sum_{|t_j|<T} |u_j(z)|^2 + \sum_{\mathfrak{a}} \int_{-T}^{T} |E_{\mathfrak{a}}(z, 1/2 + it)|^2 dt \ll T^2 + T y_\Gamma(z), \tag{2.14}$$

where the height function $y_\Gamma(z)$ is defined as

$$y_\Gamma(z) = \max_{\mathfrak{a}} \max_{\gamma \in \Gamma} \{\Im(\sigma_{\mathfrak{a}}^{-1} \gamma z)\}. \tag{2.15}$$

In Chapters 3 and 4 we will see that similar estimates hold for period integrals of automorphic forms.

2.2 Overview of old and new results

For Γ a cocompact or cofinite Fuchsian group, we are interested in the problem of estimating, as $r \to \infty$, the quantity

$$N_r(z, w) = \#\{\gamma \in \Gamma : \rho(z, \gamma w) \leq r\}.$$

The study of the asymptotic behaviour of $N_r(z, w)$ is traditionally called the (classical) hyperbolic lattice point problem. This problem has a really rich history [7, 14, 17, 37, 55, 59, 60, 65, 72] and various generalizations [26, 27, 49].

In the Euclidean circle problem, Gauss' argument works because the area of a large Euclidean disc (providing the main term of $N(x)$ in (1.1)) dominates the length of the Euclidean circle, which bounds the error term $E(x)$. The isoperimetric inequality for a Riemannian surface of constant curvature takes the form

$$4\pi A - KA^2 \leq L^2,$$

where A is the area of a domain D, L is the length of the bounday of D and K is the curvature of the surface. In the Euclidean plane we have $K = 0$ and Gauss' argument applies. However, in $\mathbb{H}$ the isoperimetric inequality gives $A \leq L$. Indeed, the area of a hyperbolic disc of radius r is $4\pi \sinh^2(\frac{r}{2}) \sim \pi e^r$ as $r \to \infty$ and the length of the circumference is $2\pi \sinh r \sim \pi e^r$ as $r \to \infty$. Hence, the area of the disc and the length of the boundary have the same order of growth. This explains the reason one cannot estimate the error term for $N_r(z, w)$ based on an elementary geometric argument.

For $u(z, w)$ the point pair-invariant function given by eq. (2.1) we get $\cosh \rho(z, w) = 2u(z, w) + 1$ hence, after the change of variable $X = 2 \cosh r$, the problem is equivalent to studying the quantity

$$N(X; z, w) = \#\{\gamma \in \Gamma : 4u(z, \gamma w) + 2 \leq X\} \tag{2.16}$$

as $X \to \infty$. Set

$$M(X; z, w) = \sum_{1/2 < s_j \leq 1} \sqrt{\pi} \frac{\Gamma(s_j - 1/2)}{\Gamma(s_j + 1)} u_j(z) \overline{u_j(w)} X^{s_j}. \tag{2.17}$$

We have the following theorem.

Theorem 2.2.1 (Selberg [65], Günther [27], Good [26]). *Let z, w be two fixed points in $\mathbb{H}$ and Γ be a cocompact or cofinite Fuchsian group. Then, as $X \to \infty$, we have*

$$N(X; z, w) = M(X; z, w) + E(X; z, w),$$

where the error term satisfies the bound

$$E(X; z, w) = O(X^{2/3}).$$

The $O(X^{2/3})$-bound should be regarded as the analogue of the $O(X^{1/3})$-bound in the Euclidean case; however, it has not been improved for any group Γ or any pair of points z, w. Selberg [65] was the first who proved the bound $O(X^{2/3})$, but he didn't publish it. For Γ cofinite, Patterson [55, 56] obtained the bound $O(X^{3/4})$. Earlier, Fricker [22] had already deduced the analogue of Theorem 2.2.1 for the 3-dimensional hyperbolic space $\mathbb{H}^3$. This result finally first appeared in Günther [27] for all rank one symmetric spaces. Good [26], in his book 'Local analysis of Selberg trace formula' used a different approach to give a new proof of the $O(X^{2/3})$-bound. He proved a general sum formula that covers many cases of decompositions of the group $G = \mathrm{SL}_2(\mathbb{R})$. One of these cases corresponds to the classical lattice problem, which in his notation is the ${}_\zeta G_\zeta$ case (see [26, p. 20, eq. (3.12)]). We will discuss Good's approach further in Chapter 3, in relationship with the lattice counting problem in conjugacy classes.

We briefly describe the idea of the proof of Theorem 2.2.1, which is quite close to the ideas used by Voronoi in the Euclidean problem, see [41, Chapter 12]. Assume Γ is cocompact. Let $k(u)$ be the characteristic function

$$k(u) = k_X(u) = \chi_{[0,(X-2)/4]}(u). \tag{2.18}$$

One can easily see that

$$N(X; z, w) = \sum_{\gamma \in \Gamma} k(u(\gamma z, w)). \tag{2.19}$$

The naive approach in the proof of Theorem 2.2.1 is to apply the pre-trace formula for this kernel $k(u)$. However, $k(u)$ is not smooth enough. If we let $h(t) = h_X(t)$ be the Selberg/Harish-Chandra transform of the kernel $k(u)$, then for t and X big enough the Selberg/Harish-Chandra transform behaves like

$$h_X(t) \sim |t|^{-3/2} X^{1/2+it}. \tag{2.20}$$

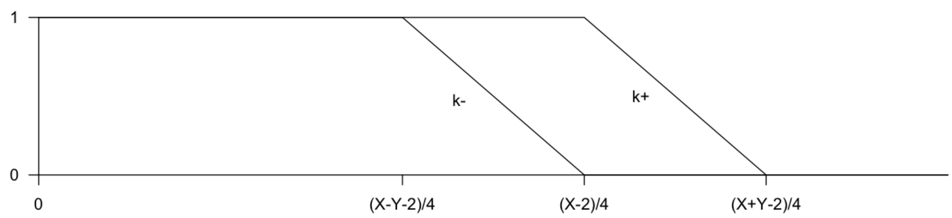

Figure 2.1: The kernels $k_+(u)$ and $k_-(u)$.

In this case the pre-trace formula implies a formula of the form

$$E(X;z,w) = \sum_{t_j \in \mathbb{R}} h_X(t_j) u_j(z) \overline{u_j(w)} + o(X^{1/2}). \tag{2.21}$$

In view of (2.20) and local Weyl's law (Theorem 2.1.6), the series in the expansion (2.21) diverges, and hence this choice of kernel fails to give an upper bound for $E(X;z,w)$. In order to have good estimates about the Selberg/Harish-Chandra transform one has to work with smoothed versions of the kernel $k(u)$. We define the kernels $k_\pm(u)$ by

$$k_+(u) = \begin{cases} 1, & \text{for} \quad u \leq \frac{X-2}{4}, \\ \dfrac{-4u}{Y} + \dfrac{X+Y-2}{Y}, & \text{for} \quad \frac{X-2}{4} \leq u \leq \frac{X+Y-2}{4}, \\ 0, & \text{for} \quad \frac{X+Y-2}{4} \leq u, \end{cases} \tag{2.22}$$

$$k_-(u) = \begin{cases} 1, & \text{for} \quad u \leq \frac{X-Y-2}{4}, \\ \dfrac{-4u}{Y} + \dfrac{X-2}{Y}, & \text{for} \quad \frac{X-Y-2}{4} \leq u \leq \frac{X-2}{4}, \\ 0, & \text{for} \quad \frac{X-2}{4} \leq u. \end{cases} \tag{2.23}$$

We obtain the upper bound

$$E(X;z,w) \ll \sum_{t_j \neq 0} h_\pm(t_j) u_j(z) \overline{u_j(w)} + O(X^{1/2}\log X + Y), \tag{2.24}$$

where the Selberg/Harish-Chandra transform $h_\pm(t)$ of $k_\pm(u)$ for $t \neq 0$ satisfies the bound (see [41, p. 173, eq. (12.9)])

$$h_\pm(t) \ll |t|^{-5/2} \{\min\{|t|, X/Y\}\} X^{1/2}. \tag{2.25}$$

The upper bound (2.24) implies $E(X;z,w) = O(Y + XY^{-1/2} + X^{1/2}\log X)$ and the choice $Y = X^{2/3}$ implies the bound of Theorem 2.2.1. For Γ cofinite but not cocompact we must take into account the contribution of the continuous spectrum, which does not affect the general argument of the proof.

The classical problem asks to study the optimal growth of the error term $E(X; z, w)$ as $X \to \infty$. Numerical investigations by Phillips and Rudnick [60] indicate that the bound of Theorem 2.2.1 is far from being optimal; in fact one should expect square root cancellation for the error term, i.e. we have the conjecture

$$E(X; z, w) = O_\epsilon(X^{1/2+\epsilon}) \tag{2.26}$$

for every $\epsilon > 0$. In the Euclidean circle problem, we know the eigenvalues $r(n)$ and the eigenfunctions appearing in the spectral expansion of the error term explicitly. This allows to improve the bound $O(X^{1/3})$. However, in the hyperbolic case only few things are known for the eigenfunctions $u_j(z)$ and the eigenvalues λ_j in general. For this reason any improvement of the bound $O(X^{2/3})$ towards the conjecture (2.26) is a much more difficult problem.

Phillips and Rudnick supported conjecture (2.26) with mean value results for the error term. To explain their result we need to be more specific about the expansion (2.21). Ignoring convergence issues, if we apply the pre-trace formula for the kernel $k(u)$ we conclude that the error term in the cocompact case has a spectral expansion

$$E(X; z, w) = \sum_{t_j \in \mathbb{R}} h(t_j) u_j(z) \overline{u_j(w)} + O\left(X^{-1} + \sum_{1/2 < s_j \leq 1} \frac{X^{1-s_j}}{2s_j - 1} \right), \tag{2.27}$$

see [7, p. 320, Lemma (2.4)]. Since the s_j's are discrete, there exists a constant $\sigma = \sigma_\Gamma \in (0, 1/2]$, depending only on Γ, such that $s_j - 1/2 \geq \sigma$ for all small eigenvalues. We conclude the above O-term is bounded as

$$X^{-1} + \sum_{1/2 < s_j \leq 1} \frac{X^{1-s_j}}{2s_j - 1} = O(X^{1/2-\sigma}). \tag{2.28}$$

Consider the contribution of the terms coming from the eigenvalue $\lambda_j = 1/4$ (corresponding to $t_j = 0$). From [60, p. 86, Lemma 2.2]) we have $h_X(0) = O(X^{1/2} \log X)$, hence they contribute

$$h_X(0) \sum_{t_j = 0} u_j(z) \overline{u_j(w)} = O(X^{1/2} \log X).$$

We subtract this quantity from $E(X; z, w)$ and we define the error term $e(X; z, w)$ to be the difference

$$e(X;z,w)=E(X;z,w)-h_X(0)\sum_{t_j=0}u_j(z)\overline{u_j(w)}. \tag{2.29}$$

Thus, the conjectural bound (2.26) is equivalent with the bound

$$e(X;z,w)=O_\epsilon(X^{1/2+\epsilon}) \tag{2.30}$$

for every $\epsilon>0$. Patterson proved that the bound (2.30) is true on average.

Theorem 2.2.2 (Patterson, [55], [56]). *Let Γ be a cocompact or cofinite Fuchsian group. Then the error term $e(X;z,w)$ satisfies the average bound*

$$\frac{1}{X}\int_2^X e(x;z,w)dx=O(X^{1/2}).$$

Notice that the upper bound does not have the extra X^ϵ-factor. Motivated by their numerical results for the Fermat groups $\Phi(N)\subset\Gamma(2)$, Phillips and Rudnick [60] proved the following mean value result for the error term in the radial parameter $r=\cosh^{-1}(X/2)$ and for $z=w$.

Theorem 2.2.3 (Phillips-Rudnick, [60]). *(a) If Γ is cocompact, then*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\frac{e(2\cosh r;z,z)}{e^{r/2}}dr=0. \tag{2.31}$$

(b) If Γ is cofinite, then

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\frac{e(2\cosh r;z,z)}{e^{r/2}}dr=\sum_{\mathfrak{a}}\left|E_{\mathfrak{a}}(z,1/2)\right|^2. \tag{2.32}$$

In case (b) of Theorem 2.2.3, the limit of the Phillips-Rudnick normalized average error term is positive only when $E_{\mathfrak{a}}(z,1/2)\neq 0$ for at least one cusp $\mathfrak{a}$. Such an Eisenstein series is called a null-vector for Γ.

We recall the definition of the Ω-notation. For $g(x)$ a positive function we say that a function $f(x)$ is $\Omega(g(x))$ as $x\to\infty$ if and only if $f(x)\neq o(g(x))$, i.e.

$$\limsup\frac{|f(x)|}{g(x)}>0.$$

Phillips and Rudnick proved the following Ω-results for the error term $e(X;z,z)$.

Theorem 2.2.4 (Phillips-Rudnick, [60]). *(a) If Γ is cocompact or a subgroup of finite index in PSL$_2(\mathbb{Z})$, then for all $\delta > 0$*

$$e(X;z,z) = \Omega_\delta \left(X^{1/2} (\log\log X)^{1/4-\delta} \right).$$

(b) If Γ is cofinite but not cocompact, and either at least one eigenvalue $\lambda_j > 1/4$ or there is a null-vector, then

$$e(X;z,z) = \Omega \left(X^{1/2} \right).$$

(c) In any other cofinite case for all $\delta > 0$ we have

$$e(X;z,z) = \Omega \left(X^{1/2-\delta} \right).$$

We distinguish further between the two cases of Ω-results: for $g(x)$ a positive function of x, we write $f(x) = \Omega_+(g(x))$ if $\limsup f(x)/g(x) > 0$ and $f(x) = \Omega_-(g(x))$ if $\liminf f(x)/g(X) < 0$. Instead of the normalization of Theorem 2.2.3, we are interested in studying the more natural normalization

$$m(X;z,w) = \frac{1}{X} \int_2^X \frac{e(x;z,w)}{x^{1/2}} dx \tag{2.33}$$

as $X \to \infty$. Theorem 2.2.2 and integration by parts imply that $m(X;z,w) = O(1)$.

For Theorem 2.2.4, after choosing $z = w$, Phillips and Rudnick work with the average of the function

$$\sum_{0 \neq t_j \in \mathbb{R}} \frac{|u_j(z)|^2}{t_j^{3/2}} e^{it_j \log X}, \tag{2.34}$$

which is an almost periodic function in the variable $s = \log X$. However, this is not an almost periodic function in the variable X. In contrast with Theorem 2.2.3, we deal with $m(X;z,w)$ and for $z \neq w$. We prove that, under specific conditions, $m(X;z,w)$ does not have a limit as $X \to \infty$.

Theorem 2.2.5. *Let Γ be a cocompact or cofinite Fuchsian group and z a fixed point. Then there exists a fixed $\delta = \delta_{\Gamma,z} > 0$ such that for every point $w \in B(z,\delta)$ we have:*

(a) if Γ is cocompact then, as $X \to \infty$,

$$m(X; z, w) = \Omega_-(1).$$

Moreover, there exists an explicit constant $C > 1/4$ such that if $\lambda_1 > C$, the limit of $m(X; z, w)$ as $X \to \infty$ does not exist for $w \in B(z, \delta)$.
(b) if Γ is cofinite and has at least one eigenvalue $\lambda_j > 1/4$ with $u_j(z) \neq 0$ then, as $X \to \infty$,

$$m(X; z, w) - \sum_{\mathfrak{a}} |E_{\mathfrak{a}}(z, 1/2)|^2 = \Omega_-(1).$$

Further, if C is as in part (a) and $\lambda_1 > C$, the limit of $m(X; z, w)$ as $X \to \infty$ does not exist for $w \in B(z, \delta)$.

In section 2.3 we will see that the exact value of the constant C is approximately 2.7823.... We know specific arithmetic groups that satisfy the bound, for instance $\mathrm{PSL}_2(\mathbb{Z}), \Gamma(2)$ and many more (see [32, Appendix C], [41, chapter 11], [3, p. 10], [70, p. 34-39] and the LMFDB database [48] for explicit numerical results).

The following proposition is an immediate corollary of our detailed analysis in the proof of Theorem 2.2.5.

Proposition 2.2.6. *Assume Γ is cofinite but not cocompact. If Γ either does not have eigenvalues with $\lambda_j > 1/4$ or $u_j(z) = 0$ for every such λ_j then for every $\epsilon_1 > 0$ there exists a $\delta = \delta_{\epsilon_1} > 0$ such that for every point $w \in B(z, \delta)$ we have:*

$$m(X; z, w) = \sum_{\mathfrak{a}} |E_{\mathfrak{a}}(z, 1/2)|^2 + O(\epsilon_1),$$

as $X \to \infty$. In this case, we conclude that $e(X; z, z)$ has finite mean value in the $X-$parameter:

$$\lim_{X \to \infty} m(X; z, z) = \sum_{\mathfrak{a}} |E_{\mathfrak{a}}(z, 1/2)|^2 .$$

In particular, if Γ does not have null vectors, the error $e(X; z, z)$ has zero mean value in the $X-$parameter:

$$\lim_{X \to \infty} m(X; z, z) = 0.$$

Theorem 2.2.5 also implies that even if $\lambda_1 < C$, in many cases $m(X; z, w)$ does not have mean value zero. If Γ is either (i) cocompact, or (ii) cofinite, has some eigenvalue $\lambda_j > 1/4$ with $u_j(z) \neq 0$ and does not have null-vectors, then for every

$w \in B(z, \delta)$

$$m(X; z, w) \not\to 0,$$

as $X \to \infty$. Thus, in these cases Theorem 2.2.5 implies as an immediate corollary pointwise Ω-results for the error $e(X; z, w)$ with $w \in B(z, \delta)$.

Corollary 2.2.7. *With the notation of Theorem 2.2.5 we have:*
(a) if Γ is either i) cocompact, or ii) as in case (b) of Theorem 2.2.5 and does not have null-vectors, then

$$e(X; z, w) = \Omega_{-}\left(X^{1/2}\right)$$

for every $w \in B(z, \delta)$.
(b) if $\lambda_1 > C$, then

$$e(X; z, w) = \Omega\left(X^{1/2}\right)$$

for every $w \in B(z, \delta)$.

Corollary 2.2.7 does not cover all cases of cofinite Fuchsian groups. However, using a more careful analysis of $e(X; z, w)$, there are some more cases of cofinite groups for which we can deduce refined Ω-results. For this purpose, we have the following definition, which is related to local Weyl's law (see Theorem 2.1.6).

Definition 2.2.8. *Let Γ be a cofinite Fuchsian group. We say that Γ has sufficiently many cusp forms at the point z if the series*

$$\sum_{t_j > 0} \frac{|u_j(z)|^2}{t_j^{3/2}}$$

diverges.

Local Weyl's law stipulates an asymptotic for the partial sums up to height T of the series appearing in Definition 2.2.8 of order $\gg T^{1/2}$. We prove the following result.

Theorem 2.2.9. *Let Γ be a cofinite but not cocompact Fuchsian group and $z \in \mathbb{H}$ fixed. Then, there exists a fixed $\delta = \delta_{\Gamma,z} > 0$ such that for every point $w \in B(z, \delta)$ we have:*
(a) if Γ has sufficiently many cusp forms at the point z, then

$$e(X; z, w) = \Omega_{-}\left(X^{1/2}\right).$$

(b) if Γ *has null vectors, then*

$$e(X; z, w) = \Omega_+ \left(X^{1/2}\right).$$

Hence, we conclude that:

Corollary 2.2.10. *If* Γ *is cofinite but not cocompact, has null vectors and sufficiently many cusp forms at the point* z*, then*

$$e(X; z, w) = \Omega_\pm \left(X^{1/2}\right).$$

The proofs of Theorems 2.2.5 and 2.2.9 use a detailed analysis of the signs of the Fourier coefficients appearing in the spectral expansions of $e(X; z, w)$ and $m(X; z, w)$; for this reason they depend crucially on specific 'fixed sign' properties of the Γ-function. We summarize these properties in section 2.3 (Lemma 2.3.1). We prove Theorem 2.2.5 in section 2.4 and Theorem 2.2.9 in section 2.5. Our analysis of both the discrete and the continuous spectrum in section 2.5 can be used to give a second proof of case (a) of Corollary 2.2.7.

Remark 2.2.11. In Theorem 2.2.4, Phillips and Rudnick actually prove that

$$e(X; z, z) = \Omega_- \left(X^{1/2} (\log\log X)^{1/4-\delta}\right)$$

if the partial sums of the series in Definition 2.2.8 grows like $\gg T^{1/2}$ (see [60, p. 99, Theorem 3.2]). For Γ a subgroup of finite index in $\mathrm{PSL}_2(\mathbb{Z})$, the statement follows from the work of Young [73] and Huang-Xu [35]. Further, in case (b) of Theorem 2.2.4 the sign of their $\Omega(X^{1/2})$-result depends on the group, whereas in case (c), the sign of the $\Omega(X^{1/2-\delta})$-result cannot be determined by their method. If Γ has sufficiently many cusp forms and null vectors, their method implies both a Ω_+ and a Ω_--result (not always of the same order of growth) for $e(X; z, z)$. Our analysis allows us to prove a $\Omega_\pm$-result for $w \in B(z, \delta)$, which is one of the main reasons we choose to emphasize the sign of our Ω-results.

Remark 2.2.12. There are known groups that satisfy the conditions of Corollary 2.2.10. If Γ is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of finite index, then it follows from [73, 35] that in many cases Γ has sufficiently many cusp forms at z for z remaining in a compact subset of the surface. Further, every $\Gamma(N)$ with $N = 5$ or ≥ 7 has null vectors. Among the subgroups $\Gamma_0(N)$ there are also groups having null vectors, for instance $\Gamma_0(25)$. For further discussion on the null vectors of these arithmetic groups see [57, p. 80-81], [39, p. 151-153].

For $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and certain other groups it is conjectured that the real Satake parameters t_j are linearly independent over $\mathbb{Q}$. Such a conjecture would allow to apply Kronecker's Theorem [31, p. 510, Theorem 444] and find a sequence of $R_m \to \infty$ such that the exponentials $\{e^{it_j R_m}\}_{j=1}^n$ approach the point -1 simultaneously (see Lemma 2.4.1 in section 2.4). Using the fixed sign properties of the Γ-function from section 2.3, this would allow to substitute Ω_- and Ω_+ in Theorems 2.2.5 and 2.2.9 with $\Omega_\pm$. Recent numerical investigations by Steeples [69, Chapter 11] for $\mathrm{SL}_2(\mathbb{Z})$ and specific (same or different) points z, w suggest we should expect $e(X; z, w)/X^{1/2}$ remaining unbounded in both sides.

Remark 2.2.13. Cramér [15] studied the normalized error term of the Chebyshev's prime counting function $\psi(x)$. He proved that

$$\frac{\psi(e^x) - e^x}{e^{x/2}}$$

has mean square average [15, p. 148, eq. (1)], whereas

$$\frac{\psi(u) - u}{u^a}$$

does not have mean square average for $a < 1$ [15, p. 148, eq. (2)]. For the hyperbolic lattice point problem Theorems 2.2.3 and 2.2.5 show a similar phenomenon for the error term $e(X; z, w)$.

2.3 Some useful lemmas

One of the key ingredients in the proofs of our results is the following lemma.

Lemma 2.3.1. *For every $t \in \mathbb{R}$, we have:*

(a)

$$\Re\left(\frac{\Gamma(it)}{\Gamma(3/2 + it)}\right) < 0, \tag{2.35}$$

(b)

$$\Re\left(\frac{\Gamma(it)}{\Gamma(3/2 + it)(1 + it)}\right) < 0. \tag{2.36}$$

The proof of Lemma 2.3.1 uses an elementary result about real functions.

Lemma 2.3.2. *Let* $f : (-\infty, 0) \to \mathbb{R}$ *be a continuous and strictly increasing real-valued function such that* $f(x)\sin(x)$ *is integrable in* $(-\infty, 0)$*. Then*

$$\int_{-\infty}^{0} f(x)\sin(x)dx < 0.$$

Proof. (of Lemma 2.3.2) Since $f(x)\sin(x)$ is integrable in $(-\infty, 0)$, we split the integral as

$$\begin{aligned}\int_{-\infty}^{0} f(x)\sin(x)dx &= \sum_{n=0}^{\infty}\int_{-2(n+1)\pi}^{-2n\pi} f(x)\sin(x)dx \\ &= \sum_{n=0}^{\infty}\int_{-2n\pi-\pi}^{-2n\pi} (f(x) - f(x-\pi))\sin(x)dx.\end{aligned}$$

Since f is strictly increasing and $\sin(x)$ is negative in the interval $(-2n\pi - \pi, -2n\pi)$, the statement follows. □

Proof. (of Lemma 2.3.1) (a) Since $\Gamma(\overline{z}) = \overline{\Gamma(z)}$, it suffices to prove the lemma for $t > 0$. Using [23, p. 909, eq. (8.384.1)] we get

$$\frac{\Gamma(it)}{\Gamma(3/2 + it)} = \frac{2}{\sqrt{\pi}} B(it, 3/2),$$

where $B(x, y)$ is the Beta function. Using the formula

$$B(x + 1, y) = B(x, y)\frac{x}{x + y}$$

we get

$$B\left(it, \frac{3}{2}\right) = B\left(it + 1, \frac{3}{2}\right) - \frac{3i}{2t} B\left(it + 1, \frac{3}{2}\right),$$

hence

$$\Re\left(B\left(it, \frac{3}{2}\right)\right) = \Re\left(B\left(it + 1, \frac{3}{2}\right)\right) + \frac{3}{2t}\Im\left(B\left(it + 1, \frac{3}{2}\right)\right). \tag{2.37}$$

By the definition of the Beta function [23, p. 908, eq. (8.380.1)] we have

$$\begin{aligned}B(it + 1, 3/2) &= \int_0^1 s^{it}(1 - s)^{1/2}ds \\ &= \int_0^1 \cos(t\log s)(1 - s)^{1/2}ds + i\int_0^1 \sin(t\log s)(1 - s)^{1/2}ds.\end{aligned}$$

Thus, using equation (2.37), we see that inequality (2.35) is equivalent with

$$\frac{2t}{3}\int_0^1 \cos(t\log s)(1-s)^{1/2}ds + \int_0^1 \sin(t\log s)(1-s)^{1/2}ds < 0.$$

Setting $s = e^u$ and using integration by parts we have

$$\frac{2t}{3}\int_0^1 \cos(t\log s)(1-s)^{1/2}ds = -\frac{2}{3}\int_{-\infty}^0 \sin(tu)((1-e^u)^{1/2}e^u)'du.$$

We conclude that it suffices to prove

$$\int_{-\infty}^0 \sin(tu)\left((1-e^u)^{1/2}e^u - \frac{2}{3}((1-e^u)^{1/2}e^u)'\right)du < 0.$$

In order to apply Lemma 2.3.2, we need to notice that the function

$$f_1(u) = (1-e^u)^{1/2}e^u - \frac{2}{3}((1-e^u)^{1/2}e^u)'$$

is strictly increasing. Thus, setting $x = tu$ and applying Lemma 2.3.2 for $f(x) = f_1(x/t)$, part (a) follows.

(b) It follows the same way. Again, it suffices to prove it for $t > 0$. We have

$$\frac{\Gamma(it)}{\Gamma(3/2+it)(1+it)} = -\frac{i+t}{\sqrt{\pi}t(1+t^2)}B(1+it,1/2).$$

Using the definition of the Beta function we calculate

$$B(it+1,1/2) = \int_0^1 \cos(t\log s)(1-s)^{-1/2}ds + i\int_0^1 \sin(t\log s)(1-s)^{-1/2}ds.$$

Hence it follows that (2.36) is equivalent with

$$-t\int_0^1 \cos(t\log s)(1-s)^{-1/2}ds + \int_0^1 \sin(t\log s)(1-s)^{-1/2}ds < 0.$$

Setting $s = e^u$, using integration by parts and applying Lemma 2.3.2 for the strictly increasing function

$$f_2(u) = (1-e^u)^{-1/2}e^u + ((1-e^u)^{-1/2}e^u)'$$

we finish the proof working as in part (a). □

Remark 2.3.3. There exists a positive constant $c > 0$ such that for $|t| > c$ we have

$$\Re\left(\frac{\Gamma(it)}{\Gamma(3/2+it)}\frac{it}{1+it}\right) < 0. \tag{2.38}$$

This can be deduced easily from Stirling's formula. Inequality (2.38) is equivalent with

$$\int_0^1 \cos(t\log s)(1-s)^{-1/2}ds + t\int_0^1 \sin(t\log s)(1-s)^{-1/2}ds < 0.$$

Using Lemma 2.3.1 and working as in (a) we can estimate $c \approx 2.30277....$ Using Mathematica to investigate (2.38), we find the optimal value of c to be approximately $\approx 1.59135....$ For this c, we can choose C of Theorem 2.2.5 by $C = 1/4 + c^2 \approx 2.7823...$ (see section 2.4).

Remark 2.3.4. With the same method as in the proof of 2.3.1 we see that for fixed $\beta > 0$ and for every $t \in \mathbb{R}$,

$$\Re\left(\frac{\Gamma(it)}{\Gamma(3/2+it)(\beta+it)}\right) < 0.$$

Using this for $\beta = 3/2 - a$ with $a \in [0, 1/2]$ and following the steps of our proof we get

$$\frac{1}{X}\int_2^X \frac{e(x;z,w)}{x^a}dx = \Omega(X^{1/2-a}).$$

The case $a = 0$ implies that

$$\frac{1}{X}\int_2^X e(x;z,w)dx = \Omega(X^{1/2}),$$

which is a lower bound for the averaged error term of Patterson (Theorem 2.2.2).

2.4 Ω-results for the average $m(X;z,w)$

2.4.1 The cocompact case

The quantity $N(X;z,w)$ can be interpreted as

$$N(X;z,w) = K(z,w) = \sum_{\gamma\in\Gamma} k(u(z,\gamma w)), \tag{2.39}$$

for $k(u) = \chi_{[0,(X-2)/4]}(u)$. Since k is not smooth, we cannot apply the pre-trace formula to the kernel $K(z,w)$. Instead, we work with the smooth average $m(X;z,w)$. We use [7, p. 321, eq. (2.7)] to write the Selberg Harish-Chandra transform $h(t) = h_X(t)$ of $k(u)$ as

$$h(t) = 2\pi(\sinh r)P^{-1}_{-1/2+it}(\cosh r), \tag{2.40}$$

where $r = \cosh^{-1}(X/2)$ and $P^{\mu}_{\nu}(z)$ is the associated Legendre function of the first kind. Using the formula [23, p. 971, eq. (8.776.1)], for $t \in \mathbb{R}$ we get

$$h(t) = 2\sqrt{\pi}\Re\left(\frac{\Gamma(it)}{\Gamma\left(\frac{3}{2}+it\right)}X^{it}\right)\left(X^{1/2} + O(X^{-3/2})\right). \tag{2.41}$$

We first deal with $m(X; z, w)$ for the cocompact case.

Proof. For z fixed, consider a sequence of points $\{w_n\}_{n=1}^{\infty}$ such that $w_n \to z$. Then, for every j we get

$$\overline{u_j(w_n)} \to \overline{u_j(z)},$$

as $n \to \infty$ (where we do not know uniformity in the limit). For $X = e^R$ we define

$$\tilde{m}(R; z, w) := m(X; z, w).$$

Using Theorem 2.1.5, Theorem 2.2.1, equations (2.27), (2.28), (2.29), (2.33) and estimates about $h(t)$ ([7, p. 320, Lemma 2.4.(b)]) we get

$$m(X; z, w_n) = \sum_{t_j>0} u_j(z)\overline{u_j(w_n)}\left(\frac{1}{X}\int_2^X \frac{h_x(t_j)}{x^{1/2}}dx\right) + O(X^{-\sigma}). \tag{2.42}$$

We use (2.41) to obtain

$$\tilde{m}(R; z, w_n) = 2\sqrt{\pi}\sum_{t_j>0} u_j(z)\overline{u_j(w_n)}\Re\left(\frac{\Gamma(it_j)}{\Gamma\left(\frac{3}{2}+it_j\right)}F(R,t_j)\right) + O(e^{-\sigma R}), \tag{2.43}$$

where

$$F(R,t_j) = e^{-R}\int_2^{e^R} x^{it_j}\left(1 + O\left(x^{-2}\right)\right)dx = \frac{e^{it_jR}}{1+it_j} + O\left(\frac{e^{-R}}{1+|t_j|}\right). \tag{2.44}$$

Using Stirling's formula [23, p. 895, eq. (8.328.1)] and Theorem 2.1.6 we see that

$$\tilde{m}(R; z, w_n) = 2\sqrt{\pi}\sum_{t_j>0} u_j(z)\overline{u_j(w_n)}\Re\left(\frac{\Gamma(it_j)}{\Gamma\left(\frac{3}{2}+it_j\right)(1+it_j)}e^{it_jR}\right) + O(e^{-\sigma R}).$$

For $A > 1$, we split the sum in the intervals $[0, A)$ and $[A, +\infty)$. Stirling's formula, Theorem 2.1.6 and the Cauchy-Schwarz inequality imply the bound

$$\sum_{t_j \geq A} u_j(z)\overline{u_j(w_n)}\Re\left(\frac{\Gamma(it_j)}{\Gamma\left(\frac{3}{2}+it_j\right)(1+it_j)}e^{it_jR}\right) = O(A^{-1/2}).$$

Let $\epsilon_1 > 0$. Since for every j we have $\overline{u_j(w_n)} \to \overline{u_j(z)}$, we can find an integer $n_0 = n_0(\epsilon_1, A)$ such that

$$\overline{u_j(w_n)} = \overline{u_j(z)} + O(\epsilon_1)$$

for every $n \geq n_0$ and for every j such that $0 < t_j < A$. Thus, using Theorem 2.1.6 and Cauchy-Schwarz inequality, for $n \geq n_0(\epsilon_1, A)$ we get

$$\begin{aligned} \tilde{m}(R; z, w_n) &= 2\sqrt{\pi}\sum_{0<t_j<A}|u_j(z)|^2\Re\left(\frac{\Gamma(it_j)}{\Gamma\left(\frac{3}{2}+it_j\right)(1+it_j)}e^{it_jR}\right) \\ &\quad +O\left(A^{-1/2}+\epsilon_1+e^{-\sigma R}\right). \end{aligned} \quad (2.45)$$

The sum for $t_j < A$ can be handled by applying Dirichlet's principle (see [60, p. 96, Lemma 3.3]).

Lemma 2.4.1 (Dirichlet's box principle)**.** *Let $r_1, r_2, ..., r_n$ be n distinct real numbers and $M > 0$, $T > 1$. Then, there is an R satisfying $M \leq R \leq MT^n$, such that*

$$|e^{ir_jR} - 1| < \frac{1}{T}$$

for all $j = 1, ..., n$.

We apply Dirichlet's principle to the sequence e^{it_jR}. For $M > 0$ and $T > 1$ sufficiently large we find an R such that

$$\begin{aligned} \tilde{m}(R; z, w_n) &= 2\sqrt{\pi}\sum_{0<t_j<A}|u_j(z)|^2\Re\left(\frac{\Gamma(it_j)}{\Gamma\left(\frac{3}{2}+it_j\right)(1+it_j)}\right) \\ &\quad +O\left(T^{-1}+A^{-1/2}+\epsilon_1+e^{-\sigma R}\right). \end{aligned}$$

We apply local Weyl's law (Theorem 2.1.6) and Lemma 2.3.1 to the sum. Local Weyl's law implies that as $A \to \infty$ the sum remains bounded and, for Γ cocompact, there exist infinitely many j's such that $u_j(z) \neq 0$. Lemma 2.3.1 implies that all the nonzero terms are negative. Hence, there exists an A_0 such that for every $A \geq A_0$:

$$\left| \sum_{0<t_j<A} |u_j(z)|^2 \Re \left(\frac{\Gamma(it_j)}{\Gamma\left(\frac{3}{2}+it_j\right)(1+it_j)} \right) \right| \gg 1. \tag{2.46}$$

Choosing T sufficiently large, A fixed and sufficiently large and ϵ_1 fixed and sufficiently small, we can choose n_0 fixed such that $m(X;z,w_n) = \Omega_-(1)$ for every $n \geq n_0$. Hence, $m(X;z,w) = \Omega_-(1)$ for w in a fixed δ-neighbourhood of z. This proves the first statement of part (a) of Theorem 2.2.5.

To prove that if $\lambda_1 > C$ the limit does not exist, we consider the sum in equation (2.45) which we denote by

$$S_{z,A}(R) = 2\sqrt{\pi} \sum_{0<t_j<A} |u_j(z)|^2 \Re \left(\frac{\Gamma(it_j)}{\Gamma\left(\frac{3}{2}+it_j\right)(1+it_j)} e^{it_jR} \right).$$

Here A is chosen finite and sufficiently large. We first prove that $S_{z,A}(R)$ attains at least two different values. We differentiate $S_{z,A}(R)$. We compute

$$\frac{\partial S_{z,A}}{\partial R}(R) = 2\sqrt{\pi} \sum_{0<t_j<A} |u_j(z)|^2 \Re \left(\frac{\Gamma(it_j)}{\Gamma\left(\frac{3}{2}+it_j\right)} \frac{it_j}{(1+it_j)} e^{it_jR} \right).$$

Applying again Dirichlet's principle, we find a sufficiently large T_0 and an R_0, depending on T_0, such that

$$\frac{\partial S_{z,A}}{\partial R}(R_0) = 2\sqrt{\pi} \sum_{0<t_j<A} |u_j(z)|^2 \Re \left(\frac{\Gamma(it_j)}{\Gamma\left(\frac{3}{2}+it_j\right)} \frac{it_j}{(1+it_j)} \right) + O(A^{1/2}T_0^{-1}).$$

Assume $t_1 > c$, with c as in Remark 2.3.3 (hence $\lambda_1 > 1/4 + c^2 = C$). We conclude that $\frac{\partial S_{z,A}}{\partial R}(R_0) \neq 0$, hence $S_{z,A}(R)$ is not constant. In particular, it admits at least two different values B_1, B_2. Assume we express B_ν as

$$B_\nu = 2\sqrt{\pi} \sum_{0<t_j<A} |u_j(z)|^2 \Re \left(\frac{\Gamma(it_j)}{\Gamma\left(\frac{3}{2}+it_j\right)(1+it_j)} e^{it_jR_\nu} \right)$$

for $\nu = 1, 2$. We estimate

$$\begin{aligned} |\tilde{m}(R;z,w_n) - B_\nu| \ll & \sum_{0<t_j<A} |u_j(z)|^2 \left| \frac{\Gamma(it_j)}{\Gamma\left(\frac{3}{2}+it_j\right)(1+it_j)} \left(e^{it_j(R-R_\nu)} - 1\right) \right| \\ & + O\left(A^{-1/2} + \epsilon_1 + e^{-\sigma R}\right). \end{aligned}$$

Applying Dirichlet's principle for $\nu = 1, 2$ we find sequences $T_{\mu,\nu} \to \infty$ and sequences $R_{\mu,\nu} \to \infty$ as $\mu \to \infty$ such that for all $t_j \in (0, A)$:

$$e^{it_j(R_{\mu,\nu}-R_\nu)} = 1 + O(T_{\mu,\nu}^{-1}).$$

Hence, we conclude that

$$|\tilde{m}(R_{\mu,\nu}; z, w_n) - B_\nu| = O\left(T_{\mu,\nu}^{-1} + A^{-1/2} + \epsilon_1 + e^{-\sigma R}\right).$$

Since $R_{\mu,\nu} \to \infty$, we conclude that $m(X, z, w)$ approaches both values B_1, B_2 infinitely many times as close as we want as $X \to \infty$. Since $B_1 \neq B_2$, we conclude that the $m(X, z, w)$ does not have a limit as $X \to \infty$. □

We notice that in order to prove the lower bound (4.44), it is enough to assume that there exists at least one $\lambda_j > 1/4$ such that $u_j(z) \neq 0$. In any such case, the contribution of the discrete spectrum in $m(X; z, w_n)$ is $\Omega_-(1)$ for $n \geq n_0$. The same argument holds for the last statement of Theorem 2.2.5.

We also notice that in order to prove that $m(X, z, w)$ does not have a limit as $X \to \infty$ we only need the first real t_j satisfying $t_j > c \approx 1.59135...$. Thus, our result also holds if Γ does not have eigenvalues in the interval $[1/4, C]$.

2.4.2 The cofinite case

We now consider the case that Γ is cofinite but not cocompact (part (b) of Theorem 2.2.5). We have to deal with the contribution of the continuous spectrum in $m(X; z, w_n)$, which is spanned by the Eisenstein series $E_{\mathfrak{a}}(z, 1/2 + it)$ (see section 2.1).

Proof. (Part (b) of Theorem 2.2.5) Working as in the proof of the cocompact case we get

$$\begin{aligned} m(X; z, w_n) &= \sum_{t_j > 0} u_j(z)\overline{u_j(w_n)} \left(\frac{1}{X}\int_2^X \frac{h_x(t_j)}{x^{1/2}} dx\right) + O(X^{-\sigma}) \\ &\quad + \sum_{\mathfrak{a}} \frac{1}{4\pi}\int_{-\infty}^{\infty} E_{\mathfrak{a}}(z, 1/2 + it)\overline{E_{\mathfrak{a}}(w_n, 1/2 + it)} \left(\frac{1}{X}\int_2^X \frac{h_x(t)}{x^{1/2}} dx\right) dt, \end{aligned}$$

where the second sum is over the cusps $\mathfrak{a}$ of Γ. Hence, the contribution of the continuous spectrum for the cusp $\mathfrak{a}$ in $m(X; z, w_n)$ is equal to

$$\frac{1}{4\pi}\int_{-\infty}^{\infty} E_{\mathfrak{a}}(z, 1/2+it)\overline{E_{\mathfrak{a}}(w_n, 1/2+it)}\left(\frac{1}{X}\int_2^X \frac{h_x(t)}{x^{1/2}}dx\right)dt. \tag{2.47}$$

In order to control the above contribution we will need the following lemma, which is the analogue of Lemma 2.4 in [60] for our average error term. Define

$$A(X) = \int_{-\infty}^{\infty}\frac{1}{X}\int_2^X \frac{h_x(t)}{x^{1/2}}dxdt.$$

Lemma 2.4.2. *As $X \to \infty$ we have*

$$\lim_{X\to\infty} A(X) = 4\pi.$$

Proof. The Selberg/Harish-Chandra transform $h(t)$ of $\chi_{[0,(\cosh r-1)/2]}$ can be written in the form

$$h(t) = 2\sqrt{2}\int_{-\infty}^{\infty} e^{itu}(\cosh r - \cosh u)^{1/2}\chi_{[-r,r]}(u)du,$$

see [60, p. 84, 85, eq. (2.9), (2.10)]. Hence

$$A(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{itu}\Phi_X(u)dudt,$$

where $\Phi_X(u)$ is given by

$$\Phi_X(u) = \frac{4}{X}\int_{|u|}^{\cosh^{-1}(X/2)} \sinh r\sqrt{1-\frac{\cosh u}{\cosh r}}dr.$$

Using the Fourier inversion formula and easy estimates we get

$$A(X) = 2\pi\Phi_X(0) = \frac{8\pi}{X}\left(\frac{X}{2}-1\right) + O\left(\frac{\log X}{X}\right) \to 4\pi.$$

□

Let $\phi_{\mathfrak{a},n}(t), \phi_{\mathfrak{a}}(t)$ be defined as:

$$\begin{aligned} \phi_{\mathfrak{a},n}(t) &= E_{\mathfrak{a}}(z, 1/2+it)\overline{E_{\mathfrak{a}}(w_n, 1/2+it)} - |E_{\mathfrak{a}}(z, 1/2)|^2, \\ \phi_{\mathfrak{a}}(t) &= |E_{\mathfrak{a}}(z, 1/2+it)|^2 - |E_{\mathfrak{a}}(z, 1/2)|^2. \end{aligned} \tag{2.48}$$

We conclude that the contribution of the cusp $\mathfrak{a}$, given by (2.47), can be written in the form

$$\frac{1}{4\pi}|E_{\mathfrak{a}}(z,1/2)|^2 A(X)+\frac{1}{4\pi}\int_{-\infty}^{\infty}\phi_{\mathfrak{a},n}(t)\left(\frac{1}{X}\int_2^X\frac{h_x(t)}{x^{1/2}}dx\right)dt. \tag{2.49}$$

Using equation (2.41), the second summand of (2.49) takes the form

$$\begin{aligned}&\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\phi_{\mathfrak{a},n}(t)\Re\left(\frac{\Gamma(it)}{\Gamma(3/2+it)}\frac{1}{1+it}X^{it}\right)dt\\&+O\left(\frac{1}{X}\int_{-\infty}^{\infty}\phi_{\mathfrak{a},n}(t)\Re\left(\frac{\Gamma(it)}{\Gamma(3/2+it)}\frac{2^{it}}{1+it}\right)dt\right)\\&+O\left(\frac{1}{X^2}\int_{-\infty}^{\infty}\phi_{\mathfrak{a},n}(t)\Re\left(\frac{\Gamma(it)}{\Gamma(3/2+it)}\frac{1}{1-it}X^{it}\right)dt\right)\end{aligned} \tag{2.50}$$

For any $A>1$ we split the first integral as:

$$\int_{-A}^{A}\phi_{\mathfrak{a},n}(t)\Re\left(\frac{\Gamma(it)}{\Gamma(3/2+it)}\frac{1}{1+it}X^{it}\right)dt+\int_{|t|>A}\phi_{\mathfrak{a},n}(t)\Re\left(\frac{\Gamma(it)}{\Gamma(3/2+it)}\frac{1}{1+it}X^{it}\right)dt. \tag{2.51}$$

Since $w_n\to z$, Cauchy-Schwarz inequality, Theorem 2.1.6 and Stirling's asymptotics imply that the integral for $|t|>A$ is bounded independently of n as:

$$\int_{|t|>A}\phi_{\mathfrak{a},n}(t)\Re\left(\frac{\Gamma(it)}{\Gamma(3/2+it)}\frac{1}{1+it}X^{it}\right)dt=O(A^{-1/2}). \tag{2.52}$$

In $[-A,A]$, we approximate $\phi_{\mathfrak{a},n}(t)$: for every $\epsilon_1>0$ there exists a $n_0=n_0(\epsilon_1,A)$ such that for every $n\geq n_0$:

$$\phi_{\mathfrak{a},n}(t)=\phi_{\mathfrak{a}}(t)+O(\epsilon_1)$$

for every $t\in[-A,A]$. Thus, we get

$$\begin{aligned}\int_{-A}^{A}\phi_{\mathfrak{a},n}(t)\Re\left(\frac{\Gamma(it)}{\Gamma(3/2+it)}\frac{1}{1+it}X^{it}\right)dt &= \int_{-\infty}^{\infty}\phi_{\mathfrak{a}}(t)\Re\left(\frac{\Gamma(it)}{\Gamma(3/2+it)}\frac{1}{1+it}X^{it}\right)dt\\&- \int_{|t|>A}\phi_{\mathfrak{a}}(t)\Re\left(\frac{\Gamma(it)}{\Gamma(3/2+it)}\frac{1}{1+it}X^{it}\right)dt\\&+ O\left(\epsilon_1\int_{-A}^{A}\Re\left(\frac{\Gamma(it)}{\Gamma(3/2+it)}\frac{1}{1+it}X^{it}\right)dt\right).\end{aligned}$$

Using the bound $\phi_{\mathfrak{a}}(t)=O(t)$ for small t and Theorem 2.1.6 for $t\to\infty$ we get that the function

$$\theta_{\mathfrak{a}}(t)=\phi_{\mathfrak{a}}(t)\frac{\Gamma(it)}{\Gamma(3/2+it)}\frac{1}{1+it}$$

is in $L^1(\mathbb{R})$. Applying the Riemann–Lebesgue Lemma to the first term we conclude that it converges to 0 as $X \to \infty$. As for (2.52) we see that the second term is bounded by $O(A^{-1/2})$. The function

$$g_{\mathfrak{a}}(t) = \Re\left(\frac{\Gamma(it)}{\Gamma(3/2+it)}\frac{1}{1+it}X^{it}\right)$$

has no pole as $t \to 0$:

$$\lim_{t\to 0} g_{\mathfrak{a}}(t) = \lim_{t\to 0}\frac{1}{2it}\left(\frac{e^{itR}}{\Gamma(3/2+it)(1+it)} - \frac{e^{-itR}}{\Gamma(3/2-it)(1-it)}\right) < \infty,$$

hence $g_{\mathfrak{a}}(t)$ is in $L^1(\mathbb{R})$ uniformly in X. We conclude that the third term is $O(\epsilon_1)$. For the O-terms in (2.50) we use trivial estimates instead of the Riemann–Lebesgue Lemma. We conclude that for every $\epsilon_1 > 0, A > 1$ there exists a $n_0 = n_0(\epsilon_1, A)$ such that for every $n \geq n_0$:

$$\frac{1}{4\pi}\int_{-\infty}^{\infty}\phi_{\mathfrak{a},n}(t)\left(\frac{1}{X}\int_2^X \frac{h(t)}{x^{1/2}}dx\right)dt = O(\epsilon_1 + A^{-1/2}) + o(1).$$

Thus, choosing $\epsilon_1 = A^{-1/2}$ we conclude that for every $\epsilon_1 > 0$ there exists a $n_0 = n_0(\epsilon_1)$ such that for every $n \geq n_0$ the contribution of the continuous spectrum to $m(X; z, w_n)$ is equal to

$$\sum_{\mathfrak{a}} |E_{\mathfrak{a}}(z, 1/2)|^2 + O(\epsilon_1) + o(1). \tag{2.53}$$

Case (b) of Theorem 2.2.5 follows for ϵ_1 sufficiently small and fixed. □

Remark 2.4.3. Phillips and Rudnick in [60] generalize Theorem 2.2.3 and case (a) of Theorem 2.2.4 in the case of the n-dimensional hyperbolic space $\mathbb{H}^n$ [60, p. 106]. Let us write $e_n(X; z, z)$ for the analogue of the error term $e(X; z, z)$ in the n-dimensional space. They prove that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{e_n(2\cosh r; z, z)}{e^{(n-1)r/2}}dr = \sum_{\mathfrak{a}}\left|E_{\mathfrak{a}}\left(z, \frac{n-1}{2}\right)\right|^2, \tag{2.54}$$

where the above sum is understood to be 0 when Γ is cocompact. Further, if Γ is cocompact or a congruence subgroup, then for all $\delta > 0$

$$e_n(X; z, z) = \Omega\left(X^{\frac{n-1}{2}}(\log\log X)^{\frac{n-1}{2n}-\delta}\right).$$

However, for $n \geq 4$ they construct arithmetic lattices with error $e_n(X; z, z) = \Omega(X^{n-2})$. This lower bound for the error term is not due to the existence of exceptional eigenvalues; instead, it follows from the number of ways we can write a number as sum of n squares. This same feature causes the error in the Euclidean circle problem to be $\Omega(x^{n-2})$ for $n \geq 4$ (with x the radius of the Euclidean circle).

In the n-th dimensional hyperbolic space the Selberg/Harish-Chandra transform of the kernel $k(u) = \chi_{[0,(X-2)/4]}$, with $X = 2\cosh r$, is given by the formula

$$\begin{aligned} h_{X,n}(t) &= c_n 2^{\frac{n-1}{2}} \int_{-r}^{r} (\cosh r - \cosh u)^{\frac{n-1}{2}} e^{itu} du \\ &= c_n 2^{\frac{n+1}{2}} \int_{0}^{r} (\cosh r - \cosh u)^{\frac{n-1}{2}} \cos(tu) du, \end{aligned}$$

for a specific constant c_n (see [60, p. 102]). Using [23, eq. (8.715.1)] we can write $h_{X,n}(t)$ in the form

$$h_{X,n}(t) = \sqrt{\pi} c_n 2^{n/2} \Gamma\left(\frac{n+1}{2}\right) (\sinh r)^{n/2} P^{-n/2}_{-1/2+it}(\cosh r),$$

where $P^{-n/2}_{-1/2+it}(\cosh r)$ is the associated Legendre function. To prove the analogues of Theorem 2.2.5 for

$$\frac{1}{X} \int_{2}^{X} \frac{e_n(x; z, w)}{x^{(n-1)/2}} dx$$

and Theorem 2.2.9 for $e_n(X; z, w)$ we would need that the following expressions

$$\Re\left(\frac{\Gamma(it)}{\Gamma\left(\frac{n+1}{2} + it\right)(1 + it)}\right), \quad \Re\left(\frac{\Gamma(it)}{\Gamma\left(\frac{n+1}{2} + it\right)}\right)$$

have fixed sign for all $t \in \mathbb{R}$ and

$$\Re\left(\frac{\Gamma(it)it}{\Gamma\left(\frac{n+1}{2} + it\right)(1 + it)}\right)$$

has fixed sign for t sufficiently large. The first real part does not have fixed sign for $n \geq 4$, and the second real part does not have fixed sign for $n \geq 6$. However, all the above real parts have fixed signs for t sufficiently large: for fixed dimension n we can find a constant C_n such that if $\lambda_1 > C_n$ then the same technique applies.

2.5 Ω-results for the error term $e(X; z, w)$

We now come to the proof of Theorem 2.2.9. Assume that Γ is cofinite but not co-compact. In order to deal with the error term $e(X; z, w)$ we mollify it as in Phillips and Rudnick [60]. Let ψ be a smooth, even, non-negative function that is compactly supported in $[-1, 1]$ and such that

$$\int_{-\infty}^{+\infty} \psi(x)e^{-itx}dx = \hat{\psi}(t) \geq 0 \tag{2.55}$$

and $\int_{-\infty}^{\infty} \psi(x)dx = 1$. For every $\epsilon > 0$ we also define the family of functions $\psi_\epsilon(x) = \epsilon^{-1}\psi(x/\epsilon)$. We trivially have $0 \leq \hat{\psi}_\epsilon(x) \leq 1$ and $\hat{\psi}_\epsilon(0) = 1$. We study the contribution of the discrete spectrum first.

2.5.1 The contribution of the discrete spectrum

For z fixed we pick again a sequence $\{w_n\}_{n=1}^{\infty}$ converging to z. For every $n \geq 1$ we define

$$\tilde{e}_n(R, z) = \frac{e(e^R; z, w_n)}{e^{R/2}}, \tag{2.56}$$

and we consider the convolution

$$\tilde{e}_{n,\epsilon}(R, z) = (\psi_\epsilon * \tilde{e}_n(\cdot, z))(R) := \int_{-\infty}^{\infty} \psi_\epsilon(R - Y)\tilde{e}_n(Y, z)dY.$$

In order to prove a lower bound for $\tilde{e}_n(R, z)$ it suffices to prove a lower bound for $\tilde{e}_{n,\epsilon}(R, z)$. Since we smooth the error by taking convolution with ψ_ϵ, we are working again with the characteristic kernel $k(u)$.

Using the pre-trace formula (Theorem 2.1.5), the expression (2.41) and the bound

$$\hat{\psi}_\epsilon(t_j) = O_k\left((\epsilon|t_j|)^{-k}\right) \tag{2.57}$$

for every $k \in \mathbb{N}$, we conclude that the contribution of the discrete spectrum in $\tilde{e}_{n,\epsilon}(R, z)$ is equal to:

$$2\sqrt{\pi}\sum_{t_j>0} u_j(z)\overline{u_j(w_n)}\Re\left(\frac{\Gamma(it_j)}{\Gamma\left(\frac{3}{2} + it_j\right)}e^{it_jR}\right)\hat{\psi}_\epsilon(t_j) + O\left(e^{-\sigma R}\right). \tag{2.58}$$

We bound the tail of the sum for $t_j > A$, using the bound (2.57), Theorem 2.1.6 and

Stirling's formula. We conclude that (2.58) takes the form

$$2\sqrt{\pi} \sum_{0<t_j<A} u_j(z)\overline{u_j(w_n)}\Re\left(\frac{\Gamma(it_j)}{\Gamma\left(\frac{3}{2}+it_j\right)}e^{it_jR}\right)\hat{\psi}_\epsilon(t_j) + O_k\left(A^{1/2-k}\epsilon^{-k} + e^{-\sigma R}\right). \tag{2.59}$$

Let $\epsilon_1 > 0$. We find again an integer $n_0 = n_0(\epsilon_1, A)$ such that

$$\overline{u_j(w_n)} = \overline{u_j(z)} + O(\epsilon_1)$$

for every $n \geq n_0$ and for every j such that $0 < t_j < A$. Since $\hat{\psi}_\epsilon(x)$ is bounded, Cauchy-Schwarz inequality, weak Weyl's law, i.e. $\{j : |t_j| \leq T\} \ll T^2$, and Theorem 2.1.6 yield that the quantity in (2.59) is

$$\sum_{0<t_j<A} 2\sqrt{\pi}|u_j(z)|^2\Re\left(\frac{\Gamma(it_j)}{\Gamma\left(\frac{3}{2}+it_j\right)}e^{it_jR}\right)\hat{\psi}_\epsilon(t_j) + O\left(A^{1/2-k}\epsilon^{-k} + \epsilon_1 A + e^{-\sigma R}\right).$$

Applying Dirichlet's principle for the exponentials e^{it_jR}, for any $T > 1$ sufficiently large we find an R such that $e^{it_jR} = 1 + O(T^{-1})$, thus concluding that the contribution of the discrete spectrum to $\tilde{e}_{n,\epsilon}(R, z)$ takes the form

$$\sum_{0<t_j<A} 2\sqrt{\pi}|u_j(z)|^2\Re\left(\frac{\Gamma(it_j)}{\Gamma\left(\frac{3}{2}+it_j\right)}\right)\hat{\psi}_\epsilon(t_j) + O\left(T^{-1}A^{1/2} + A^{1/2-k}\epsilon^{-k} + \epsilon_1 A + e^{-\sigma R}\right).$$

By part (a) of Lemma 2.3.1, the coefficients in the sum are all negative. We balance the error term by taking $\epsilon^{-1} = A^{1-3/(2k+2)}$, $\epsilon_1 = A^{-1}\epsilon$. Then the error term is $O(A^{1/2}T^{-1} + \epsilon + e^{-R\sigma R})$. For the function ψ there exists one $\tau \in (0, 1)$ such that $\hat{\psi}(x) \geq 1/2$ whenever $|x| \leq \tau$. Using this, local Weyl's law and the fact that $\hat{\psi}_\epsilon(t_j) = \hat{\psi}(\epsilon t_j)$, we bound the modulus of the above main term from below by

$$\sum_{0<t_j<A} |u_j(z)|^2\hat{\psi}_\epsilon(t_j)\left|\Re\left(\frac{\Gamma(it_j)}{\Gamma\left(\frac{3}{2}+it_j\right)}\right)\right| \gg \sum_{0<t_j<\tau/\epsilon} |u_j(z)|^2\left|\Re\left(\frac{\Gamma(it_j)}{\Gamma\left(\frac{3}{2}+it_j\right)}\right)\right|.$$

If Γ has sufficiently many cusp forms at the point z, we obtain the bound

$$\sum_{0<t_j<\tau/\epsilon} |u_j(z)|^2\left|\Re\left(\frac{\Gamma(it_j)}{\Gamma\left(\frac{3}{2}+it_j\right)}\right)\right| \gg f(\epsilon^{-1})$$

for some function f with $f(\epsilon^{-1}) \to \infty$ as $\epsilon \to 0$. Since the error is $O(A^{1/2}T^{-1} + \epsilon + e^{-R\sigma})$, there exists a fixed, sufficiently small $\epsilon_0 > 0$ such that, for T and R sufficiently large, $f(\epsilon_0^{-1})$ dominates the error. Therefore there exists a fixed integer $n_0 = n_0(\epsilon_0)$

such that for every $n \geq n_0$ the contribution of the discrete spectrum in $\tilde{e}_{n,\epsilon_0}(R,z)$ is $\Omega_-(1)$, i.e. for every $n \geq n_0$ the contribution in $e(X;z,w_n)$ is $\Omega_-(X^{1/2})$.

2.5.2 The contribution of the continuous spectrum

We come to the study of the contribution of the continuous spectrum in $\tilde{e}_{n,\epsilon}(R,z)$. Using the pre-trace formula we deduce that this is given by the expansion

$$\sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^{\infty} E_{\mathfrak{a}}(z,1/2+it)\overline{E_{\mathfrak{a}}(w_n,1/2+it)} \left(\int_{-\infty}^{\infty} \psi_\epsilon(R-Y) \frac{h_{e^Y}(t)}{e^{Y/2}} dY \right) dt. \tag{2.60}$$

Let also $\phi_{\mathfrak{a},n}(t)$ be as in equation (2.48). For $h(t) = h_{e^Y}(t)$ the contribution of the cusp $\mathfrak{a}$ in (2.60) takes the form

$$\begin{aligned} &\frac{1}{4\pi}|E_{\mathfrak{a}}(z,1/2)|^2 \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \psi_\epsilon(R-Y) \frac{h(t)}{e^{Y/2}} dY \right) dt \\ &+\frac{1}{4\pi} \int_{-\infty}^{\infty} \phi_{\mathfrak{a},n}(t) \left(\int_{-\infty}^{\infty} \psi_\epsilon(R-Y) \frac{h(t)}{e^{Y/2}} dY \right) dt. \end{aligned} \tag{2.61}$$

Using [60, p. 98, eq. (3.30)] we get

$$\frac{1}{4\pi}|E_{\mathfrak{a}}(z,1/2)|^2 \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \psi_\epsilon(R-Y) \frac{h(t)}{e^{Y/2}} dY \right) dt = |E_{\mathfrak{a}}(z,1/2)|^2 + O(e^{-R}).$$

Using that $\psi(x)$ has support in $[-1,1]$ and eq. (2.41) we see that the second summand of (2.61) takes the form

$$\begin{aligned} &\frac{1}{2\sqrt{\pi}} \Re \left(\int_{-\infty}^{\infty} \phi_{\mathfrak{a},n}(t) \frac{\Gamma(it)}{\Gamma(3/2+it)} e^{itR} \hat{\psi}_\epsilon(t) dt \right) \\ &+O\left(e^{-2R} \int_{-\infty}^{\infty} \phi_{\mathfrak{a},n}(t) \Re \left(\frac{\Gamma(it)}{\Gamma(3/2+it)(-2+it)} \right) dt \right). \end{aligned} \tag{2.62}$$

For the first term of (2.62), we imitate the method in subsection 2.4.2: we split the integral for $t \in [-A,A]$ and $|t| > A$. For $|t| > A$ we apply the local Weyl's law and the bound (2.57) to get

$$\Re \left(\int_{|t|>A} \phi_{\mathfrak{a},n}(t) \frac{\Gamma(it)}{\Gamma(3/2+it)} e^{itR} \hat{\psi}_\epsilon(t) dt \right) = O(\epsilon^{-1} A^{-1/2}),$$

independently of n. We approximate $\phi_{\mathfrak{a},n}(t)$ uniformly by $\phi_{\mathfrak{a}}(t)$: $\phi_{\mathfrak{a},n}(t) = \phi_{\mathfrak{a}}(t) + O(\epsilon_1)$ for every $n \geq n_0 = n_0(\epsilon_1)$ and for every $t \in [-A,A]$. The function $\phi_{\mathfrak{a}}(t)$ satisfies the bounds $\phi_{\mathfrak{a}}(t) = O(t)$ for small t. Using $\hat{\psi}_\epsilon(0) = 1$, $\hat{\psi}_\epsilon(t) = O((\epsilon|t|)^{-2}))$

and local Weyl's law we deduce that, for any fixed $\epsilon > 0$, the function

$$\epsilon^2 \phi_{\mathfrak{a}}(t) \frac{\Gamma(it)}{\Gamma(3/2+it)} \hat{\psi}_\epsilon(t)$$

is in $L^1(\mathbb{R})$ independently of ϵ. Applying the Riemann-Lebesgue Lemma and local Weyl's law we deduce that

$$\frac{1}{2\sqrt{\pi}} \Re \left(\int_{-A}^{A} \phi_{\mathfrak{a}}(t) \frac{\Gamma(it)}{\Gamma(3/2+it)} e^{itR} \hat{\psi}_\epsilon(t) dt \right) = \epsilon^{-2} G(R) + O(\epsilon^{-1} A^{-1/2}),$$

with $G(R) = o(1)$, independently of ϵ. Since the function

$$\Re \left(\frac{\Gamma(it)}{\Gamma(3/2+it)} e^{itR} \right) = \Re \left(\frac{\Gamma(it)}{\Gamma(3/2+it)} \right) \cos(tR) - \Im \left(\frac{\Gamma(it)}{\Gamma(3/2+it)} \right) \sin(tR)$$

is bounded as $t \to 0$ we conclude it is in $L^1(\mathbb{R})$, hence using Stirling's formula we deduce

$$\frac{1}{2\sqrt{\pi}} \Re \left(\epsilon_1 \int_{-A}^{A} \frac{\Gamma(it)}{\Gamma(3/2+it)} e^{itR} \hat{\psi}_\epsilon(t) dt \right) = O(\epsilon_1).$$

Working similarly we get

$$e^{-2R} \int_{-\infty}^{\infty} \phi_{\mathfrak{a},n}(t) \Re \left(\frac{\Gamma(it)}{\Gamma(3/2+it)(-2+it)} \right) dt = O(e^{-2R})$$

uniformly, i.e. independently of n. Balancing $\epsilon^{-2} = A^{1/2}$ and $\epsilon_1 = \epsilon$, we conclude that the contribution of the continuous spectrum in $\tilde{e}_{n,\epsilon}(R, z)$ can be finally written in the form

$$\sum_{\mathfrak{a}} |E_{\mathfrak{a}}(z, 1/2)|^2 + \epsilon^{-2} G(R) + O(\epsilon + e^{-R}). \tag{2.63}$$

2.5.3 Proof of part (a) of Theorem 2.2.9

Since Γ has sufficiently many cusp forms at the point z, the contribution of the discrete spectrum in $\tilde{e}_{n,\epsilon}(R, z)$ is of the form $b(\epsilon) + O(A^{1/2} T^{-1} + \epsilon + e^{-\sigma R})$ with $b(\epsilon) = \Omega_-(f(\epsilon^{-1}))$. Fix an R and pick a fixed ϵ_0 such that $f(\epsilon_0^{-1})$ dominates the sum $\sum_{\mathfrak{a}} |E_{\mathfrak{a}}(z, 1/2)|^2$ and the $O(\epsilon_0)$-terms (from the discrete spectrum and from eq. (2.63)). Since ϵ_0 is fixed, $\epsilon_0^{-2} G(R) \to 0$ as $R, T \to \infty$. Thus, for every $n \geq n_0(\epsilon_0)$:

$$\tilde{e}_{n,\epsilon_0}(R, z) = \Omega_-(1),$$

which implies $e(X;z,w_n) = \Omega_-(X^{1/2})$ for every $n \geq n_0$.

2.5.4 Proof of part (b) of Theorem 2.2.9

Assume that Γ has null-vectors. We have to prove that $e(X;z,w) = \Omega_+(X^{1/2})$ for w in a small neighboorhood $B(z,\delta)$ of z. By Theorem 2.2.3 of Phillips and Rudnick in [60] we have $e(X;z,z) = \Omega_+(X^{1/2})$. Hence, in order to prove part (b), it suffices to prove the following proposition.

Proposition 2.5.1. *If Γ has null-vectors, then there exists a $\delta = \delta_{\Gamma,z} > 0$ such that for every $w \in B(z,\delta)$*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{e(e^r;z,w)}{e^{r/2}}dr > \frac{1}{2}\sum_{\mathfrak{a}} |E_{\mathfrak{a}}(z,1/2)|^2 > 0. \tag{2.64}$$

Since the proof is a routine using the ideas used in the proof of Theorem 2.2.3 in [60] and in section 2.4, we only sketch the basic steps.

Proof. For any $w_n \to z$, the contribution of the Maaß forms in

$$\frac{1}{T}\int_0^T \frac{e(e^r;z,w_n)}{e^{r/2}}dr$$

is estimated using expression (2.41), the local Weyl's law and the Cauchy-Schwarz inequality to be equal to

$$2\sqrt{\pi}\sum_{t_j>0} u_j(z)\overline{u_j(w_n)}\Re\left(\frac{\Gamma(it_j)}{\Gamma\left(\frac{3}{2}+it_j\right)}\frac{1}{T}\int_0^T e^{it_jR}dr\right)$$
$$+O\left(T^{-1} + \sum_{1/2<s_j\leq 1}\frac{1}{T}\int_0^T \frac{e^{r(1/2-s_j)}}{2s_j-1}dr\right).$$

Using local Weyl's law, this is bounded by

$$T^{-1} + T^{-1}\sum_{t_j>0}\frac{u_j(z)\overline{u_j(w_n)}}{|t_j|^{5/2}} = O\left(T^{-1}\right).$$

The contribution of the Eisenstein series is equal to

$$\sum_{\mathfrak{a}}\frac{1}{4\pi}\int_{-\infty}^{\infty} E_{\mathfrak{a}}(z,1/2+it)\overline{E_{\mathfrak{a}}(w_n,1/2+it)}\left(\frac{1}{T}\int_0^T\frac{h(t)}{e^{r/2}}dr\right)dt.$$

Let $\phi_{\mathfrak{a},n}(t)$ be the functions defined in eq. (2.48). Using [60, p. 87, Lemma. 2.4] and working as in subsection 2.4.2 for $\phi_{\mathfrak{a},n}(t)$, for every $\epsilon_1 > 0$ there a n_0 such that for every $n \geq n_0$ the contribution of the continuous spectrum is

$$\sum_{\mathfrak{a}} |E_{\mathfrak{a}}(z, 1/2)|^2 + O(\epsilon_1).$$

Thus, for ϵ_1 sufficiently small and fixed the proposition follows. □

Chapter 3

Lattice point counting in conjugacy classes: average results

3.1 Description of the problem and results

In this chapter we turn our attention to the study of the lattice counting problem in conjugacy classes. Let $\mathcal{H} \subset \Gamma$ be a hyperbolic conjugacy class of Γ. Write $\mathcal{H}$ as $\mathcal{H} = \mathcal{P}^\nu$ where $\mathcal{P}$ is a primitive conjugacy class, i.e. $\mathcal{H} = \{ag^\nu a^{-1}, a \in \Gamma\}$, where g is a primitive hyperbolic element of Γ. For $\gamma \in \Gamma$ define

$$\mu(\gamma) = \inf_{z \in \mathbb{H}} \rho(z, \gamma z).$$

Notice that $\mu(\gamma)$ is constant in conjugacy classes, hence we can define $\mu := \mu(\mathcal{H}) = \mu(g^\nu)$. Thus μ is the length of the closed geodesic corresponding to the hyperbolic class $\mathcal{H}$. Let also z be a fixed point in $\mathbb{H}$, and define the quantity

$$N_z(t) = \#\{\gamma \in \mathcal{H} : \rho(z, \gamma z) \leq t\}.$$

For Γ cocompact Huber [36] was the first one who posed and studied the problem of estimating the asymptotic behaviour of $N_z(t)$. He proved that, as $t \to \infty$, the asymptotic behaviour of $N_z(t)$ is

$$N_z(t) \sim \frac{2}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} \frac{\mu}{\nu} X, \tag{3.1}$$

where

$$X = \frac{\sinh(t/2)}{\sinh(\mu/2)}. \tag{3.2}$$

There is a nice geometric interpretation of this problem, explained in [36] and [38]. Let ℓ be the invariant closed geodesic of g (and of $\mathcal{H}$). Then, $N_z(t)$ counts the number of $\gamma \in \Gamma/\langle g\rangle$ such that $\rho(\gamma z, \ell) \leq t$. This is the number of geodesic segments on $\Gamma\backslash\mathbb{H}$ from z perpendicular to ℓ of length less than or equal to t. After conjugation, one can assume that ℓ lies on $\{yi, y > 0\}$. Huber's interpretation shows that $N_z(t)$ actually counts γ in $\Gamma/\langle g\rangle$ such that $\cos v \geq X^{-1}$, where v is the angle defined by the ray from 0 to γz and the geodesic $\{yi, y > 0\}$.

For Γ cocompact or cofinite, Good [26] proved a general sum formula that covers many cases of decompositions of the group $G = \mathrm{SL}_2(\mathbb{R})$. One of these cases corresponds to Huber's hyperbolic lattice point problem in conjugacy classes. In Good's notation the hyperbolic lattice point problem in conjugacy classes corresponds to the ${}_\eta G_\zeta$ case, see Chapter 2 and [26, p. 20, Eq. (3.12)]. His method is based on defining certain Poincaré series $P_\xi(z, s, m)$ [26, p. 73, Eq. (7.1)] as sums over cosets of a hyperbolic subgroup of Γ of his basic eigenfunctions $V_\xi(z, s, \lambda)$ [26, p. 28, Eq.(4.8)]. These Poincaré series generalise the Eisenstein series and the resolvent kernel. He then expands a modification of $P_\xi(z, s, m)$ into automorphic eigenfunctions, and computes the Fourier expansion around ζ. This involves generalizations of Kloosterman sums, leading to a local trace formula [26, p. 98, Theorem 1]. The end result is Good's general formula [26, Theorem 4, p. 116]. After matching notation for $m = n = 0$ this formula implies

$$N_z(t) = \frac{2}{\mathrm{vol}(\Gamma\backslash\mathbb{H})}\frac{\mu}{\nu}X + 2\lambda_{\mathcal{H}}\lambda_z \sum_{\frac{1}{2}<s_j<1} a_j(\mathcal{H}, z)X^{s_j} + E_z(t),$$

where

$$E_z(t) = O(X^{2/3}),$$

$\lambda_{\mathcal{H}}$, λ_z are specific constants and $a_j(\mathcal{H}, z)$ are functions depending on $s_j, u_j, \mathcal{H}, z$ and special functions.

Later Huber [38] proved that, for Γ cocompact we have

$$\left|N_z(t) - \frac{2}{\mathrm{vol}(\Gamma\backslash\mathbb{H})}\frac{\mu}{\nu}X\right| \leq c\left(\frac{2^{5/4}}{\pi}X^{\tau} + 1.8X^{3/4} + 6.75X^{1/2}\right),$$

where $\tau = s_1 > 3/4$ if $\lambda_1 < 3/16$, $\tau = 3/4$ else and

$$c = \frac{\mu}{\nu}\max\{2, \sinh^{-2}(d/2)\},$$

where d is the injective radius of Γ, defined as $d = 1/2\min\{\mu(\gamma), \gamma \in \Gamma, \gamma \neq 1\}$.

Although Huber's result is worse than the bound of Good, his proof is much more enlighting than Good's method. His proof uses a spectral expansion of an automorphic function $A(f)$ (see eq. (3.7), (3.8)) and explicit estimates. We study further Huber's method in the section 3.2.

As the natural parametrization is given by (3.2), we denote

$$N(\mathcal{H}, X; z) = N_z(t), \tag{3.3}$$

and work with $N(\mathcal{H}, X; z)$ for the rest of this thesis. Clearly

$$N(\mathcal{H}, X; z) = \#\left\{\gamma \in \mathcal{H} : \frac{\sinh(\rho(z, \gamma z)/2)}{\sinh(\mu/2)} \leq X\right\}.$$

We also define the main term

$$M(\mathcal{H}, X; z) = \sum_{1/2 < s_j \leq 1} A(s_j)\hat{u}_j u_j(z) X^{s_j}, \tag{3.4}$$

where

$$A(s) = 2^{s-1}\left(e^{i\frac{\pi}{2}(s-1)} + e^{-i\frac{\pi}{2}(s-1)}\right)\frac{\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(s-\frac{1}{2}\right)}{\pi\Gamma(s+1)} \tag{3.5}$$

and

$$\hat{u}_j = \int_\sigma \overline{u}_j \, ds \tag{3.6}$$

is the period integral of $\overline{u}_j$ across a segment σ of the invariant geodesic of $\mathcal{H}$ with length $\int_\sigma ds = \mu/\nu$ (see Lemma 3.2.1). The sum in (3.4) is over the small eigenvalues of the hyperbolic Laplacian of $\Gamma\backslash\mathbb{H}$. We denote by $E(\mathcal{H}, X; z)$ the error term

$$E(\mathcal{H}, X; z) = N(\mathcal{H}, X; z) - M(\mathcal{H}, X; z).$$

In section 3.2 we refine the machinery of Huber in [38]. We compute his special functions $\xi_\lambda(v)$ (see [38, Eq.(10), (11)]) in terms of the Legendre functions $P^0_{s-1}(i\tan v)$. This allows to show the oscillatory behaviour of the Huber transform $d(f^\pm, t)$, see Proposition 3.2.4. In sections 3.3 and 3.4 we give a new proof of the following theorem.

Theorem 3.1.1 (Good, [26]). *Let Γ be a cocompact or cofinite Fuchsian group, and $\mathcal{H}$ a hyperbolic conjugacy class of Γ. Then*

$$E(\mathcal{H}, X; z) = O(X^{2/3}).$$

We combine the results and techniques of section 3.2 with the large sieve inequalities obtained by Chamizo in [6] to prove average results for the error term $E(\mathcal{H}, X; z)$, similar to those in [7] for the error term of the classical hyperbolic lattice point problem. In section 3.6 we prove the following main theorems.

Theorem 3.1.2. *Let Γ be a cocompact or cofinite Fuchsian group, and $\mathcal{H}$ a hyperbolic conjugacy class of Γ. Then*

$$\frac{1}{X}\int_{X}^{2X} |E(\mathcal{H}, x; z)|^2 dx \ll X \log^2 X,$$

where the constant implied in '$\ll$' depends on $\Gamma, \mathcal{H}$ and z.

Theorem 3.1.3. *Let Γ be a cocompact Fuchsian group, and $\mathcal{H}$ a hyperbolic conjugacy class of Γ. Then, for $n = 1, 2$*

$$\int_{\Gamma\backslash\mathbb{H}} |E(\mathcal{H}, X; z)|^{2n} d\mu(z) \ll X^n \log^{2n} X,$$

where the constant implied in '$\ll$' depends on Γ and $\mathcal{H}$.

Before we give the proofs of our results, we briefly summarize some remarks about lattice counting in conjugacy classes.

Remark 3.1.4. Our use of piecewise linear functions $f^{\pm}$ to define the smooth automorphic functions $A(f^{\pm})(z)$ in section 3.2 is much simpler than the construction and spectral expansion of Poincaré series in Good [26]. Moreover, the oscillatory behaviour that is crucial in the application of the large sieve seems difficult to identify in the local trace formula in [26]. Even matching Good's expansion [26, Theorem 4, p. 116] with $M(\mathcal{H}, X; z)$ seems to be a complicated task needing extensive calculations. Only the leading term of $M(\mathcal{H}, X; z)$ is easy to match.

Remark 3.1.5. Eskin and McMullen used ergodic methods to study the asymptotics of various counting problems on Lie groups, one of which is the conjugacy class problem [20, III.2, p. 187]. Duke, Rudnick and Sarnak [18] give another proof of the main term [18, Example 1.5, p. 147]. For the classical hyperbolic lattice point problem one deals with the locally symmetric space $\Gamma\backslash\mathrm{SL}_2(\mathbb{R})/SO(2)$. Our case involves the space $\Gamma\backslash\mathrm{SL}_2(\mathbb{R})/A$ which is not even Hausdorff, where A is the group of diagonal matrices.

Remark 3.1.6. Hill and Parnovski [34] studied the asymptotic behaviour of the variance of the hyperbolic lattice point counting function for the classical hyperbolic lattice point problem. When Γ has no small eigenvalues, in Theorem 3.1.3 we provide an upper bound for the variance of the hyperbolic lattice point function in our situation.

Remark 3.1.7. For $\mathcal{M}$ an n-dimensional compact manifold of constant negative curvature, Herrmann [33] studied the problem of counting the number of geodesic arcs on $\mathcal{M}$ from a point z on the manifold to a Jordan measurable subset of a totally geodesic submanifold.

Recently Parkkonen and Paulin [54] studied the hyperbolic lattice point problem in conjugacy classes for n-dimensional negatively curved manifolds using the geodesic flow. In special cases i.e. for compact manifolds or arithmetic group of isometries, they obtain bounds for the error term. It would be interesting to prove error bounds analogous to Theorem 3.1.1, and average results analogous to Theorems 3.1.2, 3.1.3. For dimension $n = 3$, this has been recently studied by Laaksonen [46, 47].

Remark 3.1.8. An interesting application of the hyperbolic lattice point problem in conjugacy classes and its geometric interpretation concerns degenerating Riemann surfaces and the appearance of Eisenstein series, see [24].

3.2 Certain automorphic functions and their spectral expansion

3.2.1 Plan of proof and comparison with the classical problem.

We have already seen that if $K(z,w)$ is the automorphic kernel defined as

$$K(z,w) = \sum_{\gamma\in\Gamma} k(u(\gamma z, w)),$$

then for $k(u)$ being the characteristic function of the interval $[0,(X-2)/4]$ we have $K(z,w) = N(X;z,w)$, and the asymptotics of $N(X;z,w)$ can be studied using the pre-trace formula for approximations of the kernel $k(u)$.

When we restrict the summation to the conjugacy class $\mathcal{H} \subset \Gamma$, we do not get an automorphic kernel $g(u)$ in the place of $k(u)$. Therefore, Selberg theory does not apply in this case. Huber, however, defined an automorphic function $A(f)$ that plays the role of $K(z,w)$ for a suitable test function f when $z = w$, see (3.7). The spectral expansion of $A(f)$ provides the asymptotic behaviour of $N(\mathcal{H}, X; z)$.

Assume that Γ is cocompact. Let $C_0^*[1,\infty)$ be the space of real functions of compact support that are bounded in $[1,\infty)$ and have at most finitely many discontinuities. For an f in $C_0^*[1,\infty)$, define the Γ-automorphic function

$$A(f)(z) = \sum_{\gamma\in\mathcal{H}} f\left(\frac{\cosh\rho(z,\gamma z)-1}{\cosh\mu(\gamma)-1}\right). \tag{3.7}$$

Since Γ is cocompact and f has compact support, the sum in (3.7) is finite. Then $A(f)$ has an L^2-expansion:

$$A(f) = \sum_j c(f,t_j)u_j(z), \tag{3.8}$$

where

$$c(f,t_j) = \int_{\Gamma\backslash\mathbb{H}} A(f)(z)u_j(z)\,d\mu(z)$$

is the j-th Fourier coefficient of $A(f)$. We have the following lemma of Huber:

Lemma 3.2.1 (Huber, [38])**.** *We have*

$$c(f,t_j) = 2\hat{u}_j d(f,t_j),$$

where $\hat{u}_j$ is the period integral

$$\hat{u}_j = \int_\sigma \overline{u}_j ds \tag{3.9}$$

across a segment σ of the invariant geodesic ℓ of g with length $\int_\sigma ds = \mu/\nu$,

$$d(f,t) = \int_0^{\frac{\pi}{2}} f\left(\frac{1}{\cos^2 v}\right)\frac{\xi_\lambda(v)}{\cos^2 v}dv, \tag{3.10}$$

with $\lambda = 1/4+t^2$, and ξ_λ is the solution of the differential equation

$$\xi_\lambda''(v) + \frac{\lambda}{\cos^2 v}\xi_\lambda(v) = 0, \quad v\in\Big(-\frac{\pi}{2},\frac{\pi}{2}\Big), \tag{3.11}$$

with $\xi_\lambda(0)=1$, $\xi_\lambda'(0)=0$.

The coefficient $d(f,t)$, which we call the Huber transform of f, now plays the role of the Selberg–Harish-Chandra transform. For our choice of test functions f we can use properties of special functions to estimate the Fourier coefficients $d(f,t_j)$, see Proposition 3.2.4. The next table summarizes the analogies between the two problems.

Classical problem	For conjugacy classes
z, w	$z, \mathcal{H}$
$k(u)$	$f\left(\frac{\cosh\rho-1}{\cosh\mu-1}\right)$
$K(z,w)$	$A(f)(z)$
$h(t)$	$d(f,t)$
$u_j(w)$	$\hat{u}_j$

In order to bound $E(\mathcal{H}, X; z)$, we will need the following bound for the period integrals of $\hat{u}_j$'s defined in Lemma 3.2.1:

Lemma 3.2.2 (Huber). *For the sequence of the period integrals $\{\hat{u}_j\}_{j=0}^{\infty}$, the following estimate holds:*

$$\sum_{t_j \leq T} |\hat{u}_j|^2 \ll T.$$

A proof of this upper bound is given in Huber [38, eq. (63), p. 24]. The exact estimate was first proved by Good [26, Theorem 2, p. 108] (see also [53]) and in bigger generality by Tsuzuki [71, Theorem 1, p. 2]. In Lemma 3.4.3 we give a proof analogous of this bound for the periods of Eisenstein series, which is analogous to the proof of Huber. We will discuss the asymptotic result of Good and Tsuzuki in Theorem 4.1.3, which combines both the contributions of the discrete and the continuous spectrum. We refer to [53, p. 3-4] for a detailed history of these results.

3.2.2 Special functions and test functions

For the proof of Theorem 3.1.1 it is crucial to identify the special function $\xi_\lambda(v)$ and its relevant properties. Using [21, p. 185, eq. (87)], [19, p. 111, eq. (10), (12)] and [23, p. 1009, eq. (9.132.2)] we see that the general solution of equation (3.11) can be written in the form

$$\xi_\lambda(v) = a(s)F\left(s, 1-s, 1; \frac{1-i\tan(v)}{2}\right) + b(s)F\left(s, 1-s, 1; \frac{1+i\tan(v)}{2}\right),$$

where $F(a,b,c;z)$ is the Gauss hypergeometric function. The initial conditions of Lemma 3.2.1 imply that

$$a(s) = b(s) = \left(2 \cdot F(s, 1-s, 1; 1/2)\right)^{-1}.$$

Using [23, p. 959, eq. (8.702)] and [19, p. 104, eq. (50)] we can write $\xi_\lambda(v)$ as

$$\xi_\lambda(v) = (2\sqrt{\pi})^{-1}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(1-\frac{s}{2}\right)\left(P^0_{s-1}\left(i\tan v\right) + P^0_{s-1}\left(-i\tan v\right)\right),$$

where, as in Chapter 2, $P^\mu_\nu(z)$ is the associated Legendre function of the first kind. Using the change of variable $x = \tan(v)$, we get

$$\begin{aligned} d(f,t) &= (2\sqrt{\pi})^{-1}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(1-\frac{s}{2}\right) \\ &\quad \cdot \int_0^\infty f(x^2+1)\left(P^0_{s-1}(ix) + P^0_{s-1}(-ix)\right)dx. \end{aligned} \tag{3.12}$$

Huber's interpretation shows that we are counting $\gamma \in \mathcal{H}$ such that $(\cos v)^{-1} \leq X$, i.e. $x^2+1 \leq X^2$. Hence, choosing

$$f(x^2+1) = \begin{cases} 1, & \text{for} \quad x \leq \sqrt{X^2-1}, \\ 0, & \text{for} \quad x > \sqrt{X^2-1}, \end{cases} \tag{3.13}$$

we get

$$A(f)(z) = N(\mathcal{H}, X; z). \tag{3.14}$$

Let us set

$$U = \sqrt{X^2-1}. \tag{3.15}$$

Motivated by [4, p. 269] we define the following test functions for $x > 0$ and $0 < U/2 < T < U < V < 2U$:

$$f^+(x^2+1) = \begin{cases} 1, & \text{for} \quad x \leq U, \\ \dfrac{V-x}{V-U}, & \text{for} \quad U \leq x \leq V, \\ 0, & \text{for} \quad V \leq x, \end{cases} \tag{3.16}$$

$$f^-(x^2+1) = \begin{cases} 1, & \text{for} \quad x \leq T, \\ \dfrac{U-x}{U-T}, & \text{for} \quad T \leq x \leq U, \\ 0, & \text{for} \quad U \leq x. \end{cases} \tag{3.17}$$

Denote $Y = V - U$. Notice that

$$f(x^2+1) = \begin{cases} 1, & \text{for} \quad x \leq U, \\ 0, & \text{for} \quad U < x, \end{cases}$$

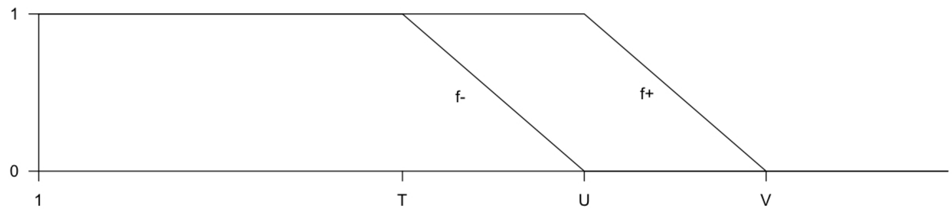

Figure 3.1: The functions f^+ and f^-.

hence $f^- \leq f \leq f^+$. This gives

$$A(f^-)(z) \leq N(\mathcal{H}, X; z) \leq A(f^+)(z).$$

Since $U = X + O(X^{-1})$ as $U, X \to \infty$, we can translate estimates involving X to ones with U and vice versa. We compute $d(f^+, t)$ and $d(f^-, t)$. The analysis for $d(f^-, t)$ is similar to the one for $d(f^+, t)$ with U and T instead of V and U. Therefore, we will discuss in details estimates only $d(f^+, t)$.

For an $A > 0$, we define $I(A)$ and $J(A)$ by

$$I(A) = \int_0^A \left(P^0_{s-1}(ix) + P^0_{s-1}(-ix)\right)(A - x)dx,$$

$$J(A) = (A^2 + 1)\left(P^{-2}_{s-1}(iA) + P^{-2}_{s-1}(-iA)\right).$$

Then, it is easy to see that

$$d(f^+, t) = (2\sqrt{\pi})^{-1}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(1 - \frac{s}{2}\right) \cdot \frac{I(V) - I(U)}{V - U}.$$

Lemma 3.2.3. *The functions $I(A)$ and $J(A)$ satisfy the relation*

$$I(A) = J(A) - 2P^{-2}_{s-1}(0).$$

Proof. Using integration by parts, the formula [23, p. 968, eq. 8.752.3], and the fact that the function $(z^2 - 1)^{1/2} P^{-1}_{s-1}(z)$ is single-valued in the disk with center $(1, 0)$ and radius 2, we get

$$I(A) = -i \int_0^A (-x^2 - 1)^{1/2} \left(P^{-1}_{s-1}(ix) - P^{-1}_{s-1}(-ix)\right) dx.$$

Using again twice [23, p. 968, eq. 8.752.3] for $m = 1, 2$ we get

$$I(A) = (x^2+1)\Big(P_{s-1}^{-2}(ix) + P_{s-1}^{-2}(-ix)\Big)\Big|_0^A .$$

The result is immediate. □

3.2.3 Estimates for the Huber transform

For the Huber transform of f^+, Lemma 3.2.3 implies that

$$d(f^+,t) = (2\sqrt{\pi})^{-1}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(1-\frac{s}{2}\right)\cdot\frac{J(V)-J(U)}{V-U}. \tag{3.18}$$

Relation [23, p. 971, eq. 8.776.1] implies

$$\begin{aligned} d(f^+,t) &= B(s)\cdot\frac{(V^2+1)V^{s-1}-(U^2+1)U^{s-1}}{V-U}\cdot\left(1+O(U^{-2})\right) \\ &+ D(s)\cdot\frac{(V^2+1)V^{-s}-(U^2+1)U^{-s}}{V-U}\cdot\left(1+O(U^{-2})\right), \end{aligned} \tag{3.19}$$

where

$$B(s) = 2^{s-2}\left(e^{i\frac{\pi}{2}(s-1)} + e^{-i\frac{\pi}{2}(s-1)}\right)\frac{\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(s-\frac{1}{2}\right)}{\pi\Gamma(s+2)}, \tag{3.20}$$

$$D(s) = \left(e^{i\frac{\pi}{2}(-s)} + e^{-i\frac{\pi}{2}(-s)}\right)\frac{\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(\frac{1}{2}-s\right)}{\pi\Gamma(3-s)2^{s+1}}. \tag{3.21}$$

For the asymptotic behaviour of $d(f^+,t)$ we have the following proposition.

Proposition 3.2.4. *(a) For any $s = 1/2 + it$ we have*

$$\begin{aligned} d(f^+,t) &= B\left(\frac{1}{2}+it\right)\left(\frac{3}{2}+it\right)X^{1/2+it} + D\left(\frac{1}{2}+it\right)\left(\frac{3}{2}-it\right)X^{1/2-it} \\ &+ O\left(B\left(\frac{1}{2}+it\right)|t|^2X^{-1/2+it}Y + D\left(\frac{1}{2}+it\right)|t|^2X^{-1/2-it}Y\right) \end{aligned}$$

(b) Let $t\in\mathbb{R}$ (i.e $\Re(s) = 1/2$) and $t \neq 0$. Then, $d(f^+,t)$ can be written in the form

$$d(f^+,t) = a(t,Y/X)X^{1/2+it} + b(t,Y/X)X^{1/2-it},$$

where the coefficients $a(t,Y/X)$ *and* $b(t,Y/X)$ *satisfy the bound*

$$a(t,Y/X), b(t,Y/X) = O\left(|t|^{-2}\min\{|t|, XY^{-1}\}\right).$$

Hence

$$d(f^+,t) = O\left(|t|^{-2}\min\{|t|, XY^{-1}\}X^{1/2}\right).$$

(c) Let $t \notin \mathbb{R}$*, i.e* $s \in (1/2,1]$*. Then*

$$\begin{aligned} d(f^+,t) &= B(s)(s+1)X^s + D(s)(2-s)X^{1-s} \\ &\quad +O(\Gamma(s-1/2)Y + \Gamma(1/2-s)X^{1/2}). \end{aligned}$$

(d) For $t=0$ *we get*

$$d(f^+,0) = O(X^{1/2}\log X).$$

Proof. (a) First, apply the mean value theorem to the function $f(x) = x^{s+1} + x^{s-1}$ to get

$$\frac{(V^2+1)V^{s-1} - (U^2+1)U^{s-1}}{V-U} = (s+1)X^s + O(s(s+1)X^{s-1}Y + (s-2)X^{-1}).$$

Applying it again to the function $g(x) = x^{2-s} + x^{-s}$, we have

$$\frac{(V^2+1)V^{-s} - (U^2+1)U^{-s}}{V-U} = (2-s)X^{1-s} + O((2-s)(1-s)X^{-s}Y + (-s)X^{-3/2}).$$

Plugging $s = 1/2 + it$ in (3.19) and using that $O(U^{-2}) = O(X^{-2})$ and the above estimates, we get the result.

(b) First, consider the function $f(x)$ as above. We know from part (a) that the terms containing $X^{1/2+it}$ come from the terms containing $f(x)$. The mean value theorem implies

$$\frac{(V^2+1)V^{s-1} - (U^2+1)U^{s-1}}{V-U} \ll |t| \cdot X^{1/2},$$

whereas, trivial estimates imply

$$\frac{(V^2+1)V^{s-1} - (U^2+1)U^{s-1}}{V-U} \ll X^{3/2}Y^{-1}.$$

Hence, if we set

$$a(t,Y/X) = \left(1+O(U^{-2})\right)B(s)\cdot\frac{(V^2+1)V^{s-1} - (U^2+1)U^{s-1}}{V-U}X^{-(1/2+it)},$$

and use the Stirling's formula for the Γ-function, we get the bound

$$a(t, Y/X) = O\left(|t|^{-2}\min\{|t|, XY^{-1}\}\right).$$

Doing the same for $g(x)$ as above and the coefficient $b(t, Y/X)$ defined as

$$b(t, Y/X) = \left(1 + O(U^{-2})\right) D(s) \cdot \frac{(V^2+1)V^{-s} - (U^2+1)U^{-s}}{V-U} X^{-(1/2-it)},$$

we get (b).

(c) It follows from (a). We estimate three of the Γ-factors in $B(s)$, $D(s)$ (eq. (3.20), (3.21)) and keep the factors $\Gamma(s-1/2)$ and $\Gamma(1/2-s)$ accordingly.

(d) Putting $t = 0$ in (3.18) we get

$$d(f^+, 0) = (2\sqrt{\pi})^{-1}\Gamma^2(3/4)\frac{H(V) - H(U)}{V-U}$$

where $H(z) = (z^2+1)\left(P^{-2}_{-1/2}(iz) + P^{-2}_{-1/2}(-iz)\right)$. Thus, applying once again the mean value theorem, there exists a $\xi \in [U, V]$ such that

$$d(f^+, 0) = (2\sqrt{\pi})^{-1}\Gamma^2(3/4)H'(\xi).$$

For $H'(z)$ we have

$$H'(z) = 2z\left(P^{-2}_{-1/2}(iz) + P^{-2}_{-1/2}(-iz)\right) + (z^2+1)\frac{d}{dz}\left(P^{-2}_{-1/2}(iz) + P^{-2}_{-1/2}(-iz)\right).$$

Formula [23, p. 964, eq. (8.731.1)] implies

$$H'(z) = \frac{3z}{2}\left(P^{-2}_{-1/2}(iz) - P^{-2}_{-1/2}(-iz)\right) - \frac{5i}{2}\left(P^{-2}_{1/2}(iz) - P^{-2}_{1/2}(-iz)\right). \tag{3.22}$$

Consider the first bracket. Using formula [23, p. 961, eq. (8.713.2)] we get

$$P^{-2}_{-1/2}(i\xi) - P^{-2}_{-1/2}(-i\xi) \ll (\xi^2+1)\int_0^\infty \left(\cosh^2 t + \xi^2\right)^{-5/4} dt$$

which is bounded by

$$\xi^{-1/2}\int_0^\infty \left(\left(\frac{\cosh t}{\xi}\right)^2 + 1\right)^{-5/4} dt. \tag{3.23}$$

Setting $x = \cosh t/\xi$ we split (3.23) as

$$\begin{aligned}\int_0^\infty \left(\left(\frac{\cosh t}{\xi}\right)^2 + 1\right)^{-5/4} dt &= \int_{1/\xi}^1 \left(x^2+1\right)^{-5/4} \frac{\xi}{(\xi^2x^2-1)^{1/2}}dx \\ &+ \int_1^\infty \left(x^2+1\right)^{-5/4} \frac{\xi}{(\xi^2x^2-1)^{1/2}}dx.\end{aligned}$$

Since $U, V \to \infty$, we can assume that $\xi \geq 2$. We see that

$$\int_1^\infty \left(x^2+1\right)^{-5/4} \frac{\xi}{(\xi^2x^2-1)^{1/2}}dx \ll \int_1^\infty \left(x^2+1\right)^{-5/4} dx = O(1)$$

and, after setting $u = x\xi$,

$$\begin{aligned}\int_{1/\xi}^1 \left(x^2+1\right)^{-5/4} \frac{\xi}{(\xi^2x^2-1)^{1/2}}dx &= \int_1^\xi \left(\frac{\xi^2}{u^2+\xi^2}\right)^{5/4} \frac{\xi}{(u^2-1)^{1/2}} \frac{du}{\xi} \\ &\leq \int_1^\xi \frac{1}{\sqrt{u^2-1}}du \ll \log \xi.\end{aligned}$$

Combining these estimates we get

$$P^{-2}_{-1/2}(i\xi) + P^{-2}_{-1/2}(-i\xi) \ll \xi^{-1/2}\log\xi.$$

For the second bracket, using once again [23, p. 961, eq. (8.713.2)], we get

$$\begin{aligned}P^{-2}_{1/2}(i\xi) - P^{-2}_{1/2}(-i\xi) &\ll (\xi^2+1)\int_0^\infty \cosh t \left(\cosh^2 t + \xi^2\right)^{-5/4} dt \\ &\ll \xi^{1/2}\int_0^\infty \frac{\cosh t}{\xi}\left(\left(\frac{\cosh t}{\xi}\right)^2 + 1\right)^{-5/4} dt.\end{aligned}$$

As above, set $x = \cosh t/\xi$ and split the integral into two integrals:

$$\int_{1/\xi}^1 \left(x^2+1\right)^{-5/4} \frac{\xi x}{(\xi^2x^2-1)^{1/2}}dx + \int_1^\infty \left(x^2+1\right)^{-5/4} \frac{\xi x}{(\xi^2x^2-1)^{1/2}}dx.$$

As above, assuming $\xi \geq 2$, the second integral is easily seen to converge, whereas the first one, setting $u = x\xi$ is again bounded by $\int_1^\xi (u^2-1)^{-1/2}du$. Finally, combining all the above estimates, we get

$$d(f^+, 0) \ll H'(\xi) \ll \xi^{1/2}\log\xi \ll V^{1/2}\log V,$$

which implies the desired bound, since $X \sim U$ and $V < 2U$. □

3.3 The cocompact case

We can now give the proof of Theorem 3.1.1 when Γ is cocompact. The proof follows the ideas sketched in subsection 2.2 for the classical problem, where we now use Proposition 3.2.4 instead of estimates for the Selberg/Harish-Chandra transform and Lemma 3.2.2 instead of local Weyl's law (Theorem 2.1.6).

Theorem 3.3.1. *Let Γ be a cocompact Fuchsian group and $\mathcal{H}$ a hyperbolic conjugacy class of Γ. Then the error $E(\mathcal{H}, X; z)$ satisfies the upper bound:*

$$E(\mathcal{H}, X; z) = O(X^{2/3}).$$

Proof. We begin with the spectral expansion of $A(f^+)$:

$$A(f^+)(z) = \sum_j c(f^+, t_j) u_j(z) = \sum_j 2d(f^+, t_j)\hat{u}_j u_j(z).$$

Using Proposition 3.2.4, we write it in the form

$$\begin{aligned} A(f^+)(z) = & \sum_{1/2<s_j\leq 1} 2B(s_j)(s_j+1)\hat{u}_j u_j(z) X^{s_j} + \sum_{1/2<s_j\leq 1} 2D(s_j)(2-s_j)X^{1-s_j} \\ & + O\left(\sum_{1/2<s_j\leq 1} \Gamma(s_j - 1/2)\hat{u}_j u_j(z) Y + \sum_{1/2<s_j\leq 1} \Gamma(1/2 - s_j)\hat{u}_j u_j(z) X^{1/2} \right) \\ & + \sum_{0\neq t_j \in \mathbb{R}} 2d(f^+, t_j)\hat{u}_j u_j(z) + O(X^{1/2}\log X). \end{aligned}$$

As in subsection 2.2, for s_j corresponding to a small eigenvalue, $s_j - 1/2$ is bounded away from zero. As the number of small eigenvalues is finite, we get

$$\sum_{1/2<s_j\leq 1} \Gamma(s_j - 1/2)\hat{u}_j u_j(z) Y + \sum_{1/2<s_j\leq 1} \Gamma(1/2 - s_j)\hat{u}_j u_j(z) X^{1/2} = O(Y + X^{1/2}).$$

By the same argument,

$$\sum_{1/2<s_j\leq 1} 2D(s_j)(2-s_j)X^{1-s_j} = O(X^{1/2}).$$

Let $A(s)$ be the function defined in equation (3.5). Then

$$A(s) = 2B(s)(s+1).$$

After setting

$$G(f^+,z) = \sum_{0\neq t_j\in\mathbb{R}} 2d(f^+,t_j)\hat{u}_j u_j(z),$$

using (d) of Proposition 3.2.4 we rewrite the spectral expansion of $A(f^+)(z)$ as

$$A(f^+)(z) = \sum_{1/2<s_j\leq 1} A(s_j)\hat{u}_j u_j(z) X^{s_j} + G(f^+,z) + O(Y + X^{1/2}\log X). \quad (3.24)$$

Using again Proposition 3.2.4 and the discreteness of the spectrum, we get

$$G(f^+,z) \ll \sum_{t_j>1} 2d(f^+,t_j)\hat{u}_j u_j(z) + O(X^{1/2}). \quad (3.25)$$

After using dyadic decomposition, we get the bound

$$G(f^+,z) \ll \sum_{n=0}^{\infty} \sup_{2^n<t_j\leq 2^{n+1}} d(f^+,t_j) \left(\sum_{2^n\leq t_j<2^{n+1}} \hat{u}_j u_j(z) \right).$$

Using estimate (2.14), Proposition 3.2.4 and Lemma 3.2.2, we get

$$\begin{aligned} G(f^+,z) &\ll \sum_{n=0}^{\infty} 2^{-2n} \min\left\{2^n, XY^{-1}\right\} X^{1/2} \left(\sum_{t_j<2^{n+1}} |\hat{u}_j|^2 \right)^{1/2} \left(\sum_{t_j<2^{n+1}} |u_j(z)|^2 \right)^{1/2} \\ &\quad + X^{1/2} \\ &\ll X^{1/2} \left(\sum_{n=0}^{\infty} 2^{-n/2} \min\left\{2^n, XY^{-1}\right\} \right) + X^{1/2}. \end{aligned}$$

We split the sum according to $n < \log_2(X/Y)$ and $n > \log_2(X/Y)$. We get

$$\begin{aligned} G(f^+,z) &\ll X^{1/2} \left(\sum_{n<\log_2(X/Y)} 2^{-n/2} \min\left\{2^n, XY^{-1}\right\} \right) \\ &\quad + X^{1/2} \left(\sum_{n\geq\log_2(X/Y)} 2^{-n/2} \min\left\{2^n, XY^{-1}\right\} \right) + X^{1/2}, \end{aligned}$$

which is bounded by

$$\begin{aligned} X^{1/2} \sum_{n<\log_2(X/Y)} 2^{n/2} \quad &+ \quad X^{3/2}Y^{-1} \sum_{n\geq\log_2(X/Y)} 2^{-n/2} + X^{1/2} \\ &\ll \quad XY^{-1/2} + X^{1/2}. \end{aligned}$$

By (3.24) we finally get

$$A(f^+)(z) = \sum_{1/2<s_j\leq 1} A(s_j)\hat{u}_j u_j(z) X^{s_j} + O(XY^{-1/2} + Y + X^{1/2}\log X). \quad (3.26)$$

We work similarly for $A(f^-)$ and we get

$$A(f^-)(z) \leq N(\mathcal{H}, X; z) \leq A(f^+)(z).$$

We conclude that

$$E(\mathcal{H}, X; z) = O(XY^{-1/2} + Y + X^{1/2}\log X).$$

The optimal error arises for $Y = XY^{-1/2}$, i.e. $Y = X^{2/3}$. The proof is complete. □

For $\lambda_0 = 0$, i.e. $s_0 = 1$, the contribution to $M(\mathcal{H}, X; z)$ is $2\hat{u}_0 u_0(z)X$. We have

$$u_0(z) = \frac{1}{\sqrt{\mathrm{vol}(\Gamma\backslash\mathbb{H})}}, \quad \hat{u}_0 = \frac{1}{\sqrt{\mathrm{vol}(\Gamma\backslash\mathbb{H})}}\frac{\mu}{\nu},$$

hence we get Huber's main term (3.1).

3.4 The cofinite case

Now, let Γ be a cofinite but not cocompact Fuchsian group, and define $A(f)$ as in eq. (3.7). The first obstacle we face is to examine whether $A(f)$ is in $L^2(\Gamma\backslash\mathbb{H})$. To see this, suppose that f is compactly supported in $[1, K]$ with $K > 0$ fixed, and consider the counting function

$$\tilde{N}(z, \delta) = \#\{\gamma \in \mathcal{H} : u(\gamma z, z) \leq \delta\}.$$

An element γ contributes to the summation in $A(f)$ exactly when $\gamma \in \tilde{N}(z, \delta)$, with $\delta = K(\cosh(\mu) - 1)$. To prove that $A(f)$ is in $L^2(\Gamma\backslash\mathbb{H})$, it suffices to prove that $\tilde{N}(z, \delta)$ in uniformly bounded (i.e. independently of z).

Lemma 3.4.1. *The $\tilde{N}(z, \delta)$ in uniformly bounded, hence $A(f) \in L^2(\Gamma\backslash\mathbb{H})$.*

Proof. For simplicity, assume that Γ has only one cusp at $\mathfrak{a}$. Conjugating, we can assume that $\mathfrak{a} = \infty$. Then, for $Y > 0$, consider the set

$$A(Y) = \{\gamma \in \Gamma_\infty \backslash \Gamma : \Im(\gamma z) > Y\}.$$

Lemma [41, Lemma 2.10, p. 50] shows that

$$\#A(Y) < 1 + \frac{10}{c_\infty Y},$$

where c_∞ is a constant depending only on the cusp ∞. That means there exists a Y_0 such that for every $Y > Y_0$,

$$\#A(Y) \leq 1,$$

i.e. for Y large enough $A(Y)$ contains at most one class $\Gamma_\infty \gamma$. Since the fundamental domain contains points of deformation ≤ 1, we have $\Im(z) \geq \Im(\gamma z) > Y$. Since the trivial class leaves the $\Im(\gamma z) = \Im(z)$, we have $\gamma \in \Gamma_\infty$. Hence, if $\gamma \in \mathcal{H}$, then, for all z, $\Im(\gamma z) < Y_0$, where Y_0 depends only on the cusp ∞.

Now, let γ be an element of $\mathcal{H}$ such that $u(\gamma z, z) \leq \delta$. Using the formula (2.1) we obtain

$$\Im(z) - \Im(\gamma z) \leq |z - \gamma z| \leq \sqrt{4\delta \Im(z) \Im(\gamma z)} \leq \sqrt{4\delta Y_0 \Im(z)},$$

hence

$$\Im(z) \leq \sqrt{4\delta Y_0 \Im(z)} + Y_0.$$

This inequality implies an upper bound $\Im(z) \leq M$, where M depends only on Y_0 and δ. We also get

$$\Re(\gamma z) - \Re(z) \leq |z - \gamma z| \leq \sqrt{4\delta \Im(z) \Im(\gamma z)} \leq \sqrt{4\delta Y_0 M},$$

and since, for Γ cofinite, we have a uniform bound $|\Re(z)| < M$, we get also get a uniform bound for $\Re(\gamma z)$. That means, for all $\gamma \in \mathcal{H}$ satisfying $u(\gamma z, z) \leq \delta$, γz lies in a compact set, which does not depend of z but only on δ. This proves $\tilde{N}(z, \delta)$ is uniformly bounded, and thus $A(f) \in L^2(\Gamma \backslash \mathbb{H})$. □

Lemma 3.4.1 allows us to write a spectral expansion for $A(f)$. For Γ cofinite but not cocompact the continuous spectrum of $-\Delta$ contributes to the spectral expansion of

$A(f)$. We have

$$A(f) = \sum_j c(f, t_j) u_j(z) + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^{\infty} c_{\mathfrak{a}}(f, t) E_{\mathfrak{a}}(z, 1/2 + it) dt. \tag{3.27}$$

The rest of the proof for the cofinite case is the same as for the cocompact case: the estimates of $d(f, t)$ in Proposition 3.2.4 do not depend on the compactness of the group Γ, but only on the spectral parameter t. To complete the proof of Theorem 3.1.1, we need the analogues of Lemmas 3.2.1 and 3.2.2 for the Eisenstein series.

Examining the proof of Lemma 3.2.1 in [38], we notice that, along the same lines, we can prove the following version for Eisenstein series.

Lemma 3.4.2. *We have*

$$c_{\mathfrak{a}}(f, t) = 2\hat{E}_{\mathfrak{a}}(1/2 + it) d(f, t),$$

where $\hat{E}_{\mathfrak{a}}(1/2 + it)$ is the integral

$$\hat{E}_{\mathfrak{a}}(1/2 + it) = \int_{\sigma} E_{\mathfrak{a}}(z, 1/2 - it) ds$$

across a segment σ of the invariant geodesic of γ with length $\int_{\sigma} ds = \mu/\nu$, $d(f, t)$ given by eq. (3.10) and ξ_{λ} satisfying eq. (3.11) with the same initial conditions.

The analogue of Lemma 3.2.2 for Eisenstein series is the following lemma.

Lemma 3.4.3. *We have the bound*

$$\int_{-T}^{T} |\hat{E}_{\mathfrak{a}}(1/2 + it)|^2 dt \ll T.$$

Proof. For $T > 0$, define the angle $v_T \in (0, \frac{\pi}{2})$ by the relation

$$\tan(v_T) = \frac{\sqrt{2}}{T},$$

and the function f as

$$f(u) = \begin{cases} 1, & \text{for} \quad 1 \le u \le \cos^{-2}(v_T), \\ 0, & \text{for} \quad \cos^{-2}(v_T) < u. \end{cases}$$

Thus, for

$$X = \sqrt{1 + \frac{2}{T^2}},$$

we have $A(f)(z) = N(\mathcal{H}, X; z)$. By Lemma 3.4.1, we get that $A(f)$ is in $L^2(\Gamma\backslash\mathbb{H})$. Moreover, Lemma 3.4.1 shows that

$$M_X := \sup_{z\in\mathbb{H}} N(\mathcal{H}, X; z) < \infty.$$

Since we are interested about the estimate as $T \to \infty$, X remains bounded and hence M_X can be chosen uniformly bounded by some M. Then, we have the trivial bound

$$\int_{\Gamma\backslash\mathbb{H}} (A(f)(z))^2 \, d\mu(z) \leq M \int_{\Gamma\backslash\mathbb{H}} A(f)(z) d\mu(z),$$

and, by [38, p. 24, eq. (60)], we get the bound

$$\int_{\Gamma\backslash\mathbb{H}} A(f)(z) d\mu(z) \ll T^{-1}.$$

By Lemma 3.4.2, for $\lambda = 1/4 + t^2$ we get

$$c_{\mathfrak{a}}(f, t) = 2\hat{E}_{\mathfrak{a}}(1/2 + it) \int_0^{v_T} \frac{\xi_\lambda(v)}{\cos^2(v)} dv.$$

On the other hand, from the Parseval's identity we get

$$\begin{aligned} \int_F (A(f)(z))^2 \, d\mu(z) &= \sum_j |c(f, t_j)|^2 + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^{+\infty} |c_{\mathfrak{a}}(f, t)|^2 dt \\ &\geq \sum_{\mathfrak{a}} \frac{1}{\pi} \int_{-T}^{T} |\hat{E}_{\mathfrak{a}}(1/2 + it)|^2 \left(\int_0^{v_T} \frac{\xi_\lambda(v)}{\cos^2(v)} dv \right)^2 dt. \end{aligned}$$

We use [38, Appendix, eq. (5), p. 39], as in the proof of Lemma 3.2.2 in [38, p. 24] to get

$$\int_0^{v_T} \frac{\xi_\lambda(v)}{\cos^2(v)} dv \gg T^{-1},$$

hence

$$\int_{-T}^{T} |\hat{E}_{\mathfrak{a}}(1/2 + it)|^2 \ll T.$$

□

We can now finish the proof of Theorem 3.1.1 as follows.

Theorem 3.4.4. *Let* Γ *be a cofinite Fuchsian group, and* $\mathcal{H}$ *a hyperbolic conjugacy class of* Γ. *Then*

$$E(\mathcal{H}, X; z) = O(X^{2/3}).$$

Proof. The contribution of the Maaß cusp forms can be handled exactly as in the co-compact case. For the contribution of Eisenstein series in the spectral expansion of $A(f^+)(z)$ we use Proposition 3.2.4, Lemmas 3.4.2 and 3.4.3 exactly the same way as in the proof of Theorem 3.3.1 for $G(f^+, z)$. We obtain the estimate

$$\sum_{\mathfrak{a}} \int_{-\infty}^{\infty} d(f,t)\hat{E}_{\mathfrak{a}}(1/2+it)E_{\mathfrak{a}}(z,1/2+it)dt = O(XY^{-1/2} + X^{1/2}).$$

Choosing $Y = X^{2/3}$ we complete the proof of the theorem. □

3.5 The large sieve for Riemann surfaces

To explain motivation, it is useful to state the large sieve inequality for the Euclidean case, which is a non-trivial estimate for exponential sums (see [2, Theorem 4], [6, p. 303]).

Theorem 3.5.1. *Let* $x_1, x_2, ..., x_R \in \mathbb{R}/\mathbb{Z}$, *such that* $\min_{n\in\mathbb{Z}} |x_i - x_j - n| > \delta > 0$ *for* $i \neq j$. *Then for* $a_1, ..., a_n \in \mathbb{C}$ *we have:*

$$\sum_{m=1}^{R} \left| \sum_{n\leq N} a_n e^{2\pi i n x_m} \right|^2 \ll (N + \delta^{-1}) \sum_{n\leq N} |a_n|^2.$$

Let a_j be a sequence of complex numbers and, for each cusp $\mathfrak{a}$, let $a_{\mathfrak{a}}(t)$ be a continuous function of t. Chamizo [6] proved the following large-sieve inequalities for the Riemann surfaces $\Gamma\backslash\mathbb{H}$; the first is a large sieve inequality for averaging over the radii and the second for averaging over points in $\Gamma\backslash\mathbb{H}$.

Theorem 3.5.2 (Chamizo, [6]). *Given* $z \in \Gamma\backslash\mathbb{H}$, $T, X > 1$ *and* $x_1, x_2, ...x_R \in [X, 2X]$, *if* $|x_k - x_\ell| > \delta > 0$ *for* $k \neq \ell$, *then*

$$\sum_{m=1}^{R} \left| \sum_{|t_j|\leq T} a_j x_m^{it_j} u_j(z) + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-T}^{T} a_{\mathfrak{a}}(t) x_m^{it} E_{\mathfrak{a}}(z, 1/2+it)dt \right|^2$$

$$\ll (T^2 + XT\delta^{-1})\|a\|_*^2,$$

where

$$\|a\|_* = \left(\sum_{|t_j| \le T} |a_j|^2 + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-T}^{T} |a_{\mathfrak{a}}(t)|^2 dt \right)^{1/2},$$

and the '$\ll$' constant depends on Γ and $y_\Gamma(z)$.

Theorem 3.5.3 (Chamizo, [6])**.** *Given $T > 1$ and $z_1, z_2, ... z_R \in \Gamma\backslash\mathbb{H}$, if $\rho(z_k, z_\ell) > \delta > 0$ for $k \neq \ell$, then*

$$\sum_{m=1}^{R} \left| \sum_{|t_j| \le T} a_j u_j(z_m) + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-T}^{T} a_{\mathfrak{a}}(t) E_{\mathfrak{a}}(z_m, 1/2 + it) dt \right|^2$$

$$\ll (T^2 + \delta^{-2}) \|a\|_*^2,$$

where $\|a\|_*$ is defined as above and and the '$\ll$' constant depends on Γ and $\max y_\Gamma(z_m)$.

3.6 Statements of the averaging results

We will apply the large sieve results for the Riemann surfaces $\Gamma\backslash\mathbb{H}$ (Theorems 3.5.2, 3.5.3) to obtain averaging results for $E(\mathcal{H}, X; z)$. We first use Theorem 3.5.2 to prove the following result for the radial averaging of $E(\mathcal{H}, X; z)$.

Proposition 3.6.1. *Let $X > 2$ and $X_1, X_2, ..., X_R \in [X, 2X]$, satisfying the condition $|X_i - X_j| > \delta$ for some $\delta > 0$, when $i \neq j$. Then we have*

$$\sum_{m=1}^{R} |E(\mathcal{H}, X_m; z)|^2 \ll R^{1/3} X^{4/3} \log X + \delta^{-1} X^2 \log^2 X,$$

where the '$\ll$' constant depends on Γ, $\mathcal{H}$ and z.

We conclude the following upper bound for the second moment of the error term.

Theorem 3.6.2. *If $R\delta \gg X$ and $R > X^{1/2}$, then*

$$\frac{1}{R} \sum_{m=1}^{R} |E(\mathcal{H}, X_m; z)|^2 \ll X \log^2 X. \tag{3.28}$$

Letting R go to infinity, we get

$$\frac{1}{X} \int_X^{2X} |E(\mathcal{H}, x; z)|^2 dx \ll X \log^2 X. \tag{3.29}$$

For the spatial average, we use Theorem 3.5.3 to prove the following analogue of Proposition 3.6.1.

Proposition 3.6.3. *Let $X > 2$ and $z_1, z_2, ..., z_R$ be points in $\Gamma\backslash\mathbb{H}$ away from the cusps, satisfying the condition $\rho(z_i, z_j) > \delta$ for some $\delta > 0$, when $i \neq j$. Then, we have*

$$\sum_{m=1}^{R} |E(\mathcal{H}, X; z_m)|^2 \ll \delta^{-2} X + R^{1/3} X^{4/3} \log^2 X, \tag{3.30}$$

and

$$\sum_{m=1}^{R} |E(\mathcal{H}, X; z_m)|^4 \ll \delta^{-2} X^2 \log^4 X + R^{1/3} X^{8/3} \log^3 X, \tag{3.31}$$

where the '$\ll$' constants depend on Γ, $\mathcal{H}$ and $\max y_\Gamma(z_m)$.

In particular, for the spatial average we conclude upper bounds for the second and the fourth moment of the error term.

Theorem 3.6.4. *If $R\delta^2 \gg 1$ and $R > X^{1/2}$, then, for $n = 1, 2$*

$$\frac{1}{R} \sum_{m=1}^{R} |E(\mathcal{H}, X; z_m)|^{2n} \ll X^n \log^{2n} X.$$

Letting R go to infinity, if Γ is cocompact, we get

$$\int_{\Gamma\backslash\mathbb{H}} |E(\mathcal{H}, X; z)|^{2n} d\mu(z) \ll X^n \log^{2n} X.$$

Remark 3.6.5. It is possible that, using the methods of Cherubini [13], one can substitute the extra $\log^2 X$ factor by $\log X$.

Before giving the proof of the above results, we need to fix the following notation. For a function $f \in C_0^*[1, \infty)$ denote by $G(f, z)$ the difference

$$G(f, z) = A(f)(z) - \sum_{1/2 \le s_j \le 1} 2d(f, t_j) \hat{u}_j u_j(z).$$

In the proofs of Theorems 3.3.1 in section 3.3 and 3.4.4 in section 3.4 we proved that for Γ cocompact or cofinite we have

$$e_{f^+}(\mathcal{H}, X; z) := G(f^+, z) = O(XY^{-1/2} + X^{1/2}),$$

$$e_{f^-}(\mathcal{H}, X; z) := G(f^-, z) = O(XY^{-1/2} + X^{1/2}),$$

$$e_{f^-}(\mathcal{H}, X; z) < E(\mathcal{H}, X; z) + O(Y + X^{1/2}\log X) < e_{f^+}(\mathcal{H}, X; z).$$

3.7 Proofs of the averaging results

We begin with the proof of Proposition 3.6.1.

Proof. (of Proposition 3.6.1) We choose Y such that $X^{1/2}\log X \ll Y \ll X$. We get

$$e_{f^-}(\mathcal{H}, X; z) < E(\mathcal{H}, X; z) + O(Y) < e_{f^+}(\mathcal{H}, X; z).$$

We choose f to be f^+ or f^- as in (3.16) and (3.17) with $X = X_m$ and U given by (3.15). We have

$$\sum_{m=1}^{R} |E(\mathcal{H}, X_m; z)|^2 \ll \sum_{m=1}^{R} |e_f(\mathcal{H}, X_m; z)|^2 + RY^2.$$

The estimates below are true for $f = f^+$ or f^-. We write

$$\begin{aligned} S(X, z, T) &= 2\sum_{T<|t_j|\le 2T} d(f, t_j)\hat{u}_j u_j(z) \\ &\quad + \frac{1}{\pi}\sum_{\mathfrak{a}}\left(\int_T^{2T} + \int_{-2T}^{-T}\right) d(f,t)\hat{E}_{\mathfrak{a}}(1/2+it)E_{\mathfrak{a}}(z, 1/2+it)dt. \end{aligned}$$

We break the set of t_j, t's in the following sets

$$\begin{aligned} A_1 &= \{t_j : 0 < |t_j| \le 1\}. \\ B_1 &= \{t : 0 < |t| \le 1\}, \\ A_2 &= \{t_j : 1 < |t_j| \le X^2Y^{-2}\}, \\ B_2 &= \{t : 1 < |t| \le X^2Y^{-2}\}, \\ A_3 &= \{t_j : |t_j| > X^2Y^{-2}\}, \\ B_3 &= \{t : |t| > X^2Y^{-2}\}. \end{aligned}$$

Using the notation

$$S_i(z) := 2\sum_{t_j\in A_i} d(f, t_j)\hat{u}_j u_j(z) + \frac{1}{\pi}\sum_{\mathfrak{a}}\int_{B_i} d(f,t)\hat{E}_{\mathfrak{a}}(1/2+it)E_{\mathfrak{a}}(z, 1/2+it)dt,$$

$e_f(\mathcal{H},X;z)$ can be written as

$$e_f(\mathcal{H},X;z) = S_1(z)+S_2(z)+S_3(z).$$

We first estimate $S_3(z)$. Using the estimates for $t_j \in \mathbb{R}$ in Proposition 3.2.4 we get the bound

$$\begin{aligned}\sum_{t_j\in A_3} 2d(f,t_j)\hat{u}_j u_j(z) &\ll \sum_{|t_j|>X^2Y^{-2}} |t_j|^{-2}\min\{t_j, X/Y\}X^{1/2}\hat{u}_j u_j(z)\\ &\ll \sum_{t_j>X^2Y^{-2}} t_j^{-2}X^{3/2}Y^{-1}\hat{u}_j u_j(z).\end{aligned}$$

Using dyadic decomposition, this is trivially bounded by

$$\begin{aligned}&\ll X^{3/2}Y^{-1}\sum_{n=0}^{\infty}\left(\sum_{2^nX^2Y^{-2}<t_j\le 2^{n+1}X^2Y^{-2}} t_j^{-2}\hat{u}_j u_j(z)\right)\\ &\ll X^{3/2}Y^{-1}\sum_{n=0}^{\infty}2^{-2n}X^{-4}Y^4\left(\sum_{2^nX^2Y^{-2}<t_j\le 2^{n+1}X^2Y^{-2}} \hat{u}_j u_j(z)\right).\end{aligned}$$

Using Cauchy-Schwarz, bound (2.14) and Lemma 3.2.2 we get the bound

$$\begin{aligned}&\ll X^{-5/2}Y^3\sum_{n=0}^{\infty}2^{-2n}\left(\sum_{t_j\le 2^{n+1}X^2Y^{-2}}|\hat{u}_j|^2\right)^{1/2}\left(\sum_{t_j\le 2^{n+1}X^2Y^{-2}}|u_j(z)|^2\right)^{1/2}\\ &\ll X^{-5/2}Y^3\sum_{n=0}^{\infty}2^{-2n}(2^{n/2}XY^{-1})(2^nX^2Y^{-2}) \ll X^{1/2} \ll Y.\end{aligned}$$

Similarly we deal with the case of the Eisenstein series over B_3. We conclude that $S_3(z) = O(Y)$. The case of the sum $S_1(z)$ is easier. We have

$$\sum_{t_j\in A_1} d(f,t_j)\hat{u}_j u_j(z) \ll X^{1/2}\sum_{|t_j|<1} t_j^{-2}\min\{t_j, X/Y\}\hat{u}_j u_j(z) \ll X^{1/2} \ll Y,$$

since there exist finitely many eigenvalues with spectral parameter $|t_j|\le 1$. Similarly, we prove the $O(Y)$ bound for the Eisenstein series' contribution over B_1. We conclude that $S_1(z)=O(Y)$. Combining all the above we get

$$\begin{aligned}e_f(\mathcal{H},X;z) &= 2\sum_{t_j\in A_2} d(f,t_j)\hat{u}_j u_j(z)\\ &\quad+\frac{1}{\pi}\sum_{\mathfrak{a}}\int_{A_2} d(f,t)\hat{E}_{\mathfrak{a}}(1/2+it)E_{\mathfrak{a}}(z,1/2+it)dt + O(Y).\end{aligned}$$

Adding for $T = 2^k$, $k = 0, 1, \ldots, [\log_2(X^2Y^{-2})]$, we get the bound

$$e_f(\mathcal{H}, X; z) \ll \sum_{1 \le T < X^2Y^{-2}} S(X, z, T) + O(Y),$$

and adding for $X_1, ...X_R$ we get

$$\sum_{m=1}^{R} |e_f(\mathcal{H}, X_m; z)|^2 \ll \sum_{m=1}^{R} \left| \sum_{1 \le T < X^2Y^{-2}} S(X_m, z, T) \right|^2 + RY^2. \tag{3.32}$$

Cauchy-Schwarz inequality yields

$$\left| \sum_{1 \le T < X^2Y^{-2}} S(X_m, z, T) \right|^2 \ll \log X \sum_{1 \le T < X^2Y^{-2}} |S(X_m, z, T)|^2 \tag{3.33}$$

which, combinded with the bound (3.32) gives

$$\sum_{m=1}^{R} |e_f(\mathcal{H}, X_m; z)|^2 \ll \log X \sum_{1 \le T < X^2Y^{-2}} \left(\sum_{m=1}^{R} |S(X_m, z, T)|^2 \right) + RY^2. \tag{3.34}$$

Using the estimates of Proposition 3.2.4 we can now write

$$d(f, t) = X^{1/2}(a(t, Y/X)X^{it} + b(t, Y/X)X^{-it})$$

where $a(t, Y/X)$ and $b(t, Y/X)$ are functions satisfying

$$a(t, Y/X), b(t, Y/X) \ll |t|^{-2} \min\left\{|t|, XY^{-1}\right\}.$$

We apply Theorem 3.5.2, which implies that, for $a_j x_m^{it_j} = d(f, t_j)\hat{u}_j$ and $a(t)x_m^{it} = d(f, t)\hat{E}_{\mathfrak{a}}(1/2 + it)$

$$\sum_{m=1}^{R} \left| \sum_{T < |t_j| \le 2T} d(f, t_j)\hat{u}_j u_j(z) + \frac{1}{\pi} \sum_{\mathfrak{a}} \int_{T}^{2T} d(f, t)\hat{E}_{\mathfrak{a}}(1/2 + it)E_{\mathfrak{a}}(z, 1/2 + it)dt \right|^2$$

is bounded by

$$(T^2 + XT\delta^{-1})\|a\|_*^2,$$

i.e.

$$\sum_{m=1}^{R} |S(X_m, z, T)|^2 \ll (T^2 + XT\delta^{-1})\|a\|_*^2,$$

where

$$\begin{aligned}\|a\|_*^2 \ll & \sum_{T<|t_j|\leq 2T} \left| |t_j|^{-2} \min\{|t_j|, XY^{-1}\} X^{1/2} \hat{u}_j \right|^2 \\ & + \frac{1}{\pi} \sum_{\mathfrak{a}} \int_T^{2T} \left| |t|^{-2} \min\{|t|, XY^{-1}\} X^{1/2} \hat{E}_{\mathfrak{a}}(1/2+it) \right|^2 .\end{aligned}$$

The last expression can be bounded by

$$XT^{-4} \min\{T^2, X^2Y^{-2}\} \left(\sum_{T\leq|t_j|\leq 2T} |\hat{u_j}|^2 + \sum_{\mathfrak{a}} \int_T^{2T} |\hat{E}_{\mathfrak{a}}(1/2+it)|^2 dt \right)$$

and, using Lemmas 3.2.2 and 3.4.3, we obtain

$$\|a\|_*^2 \ll XT^{-3} \min\{T^2, X^2Y^{-2}\}. \tag{3.35}$$

Thus, we conclude

$$\sum_{m=1}^{R} |S(X_m, z, T)|^2 \ll (T^2 + XT\delta^{-1})(XT^{-3} \min\{T^2, X^2Y^{-2}\}), \tag{3.36}$$

hence

$$\begin{aligned}\sum_{m=1}^{R} |E(\mathcal{H}, X_m; z)|^2 & \ll \log X \sum_{1\leq T<X^2Y^{-2}} \left(\sum_{m=1}^{R} |S(X_m, z, T)|^2 \right) + RY^2 \\ & \ll \log X \sum_{1\leq T<X^2Y^{-2}} (T^2 + XT\delta^{-1})(XT^{-3} \min\{T^2, X^2Y^{-2}\}) + RY^2.\end{aligned}$$

We get the bound

$$\begin{aligned}\sum_{m=1}^{R} |E(\mathcal{H}, X_m; z)|^2 \ll & X \log X \left(\sum_{1\leq T<XY^{-1}} T \right) + X^2\delta^{-1} \log X \left(\sum_{1\leq T<XY^{-1}} 1 \right) \\ & + X^3Y^{-2} \log X \left(\sum_{XY^{-1}\leq T<X^2Y^{-2}} T^{-1} \right) \\ & + X^4\delta^{-1}Y^{-2} \log X \left(\sum_{XY^{-1}\leq T<X^2Y^{-2}} T^{-2} \right) + RY^2.\end{aligned}$$

Trivial bounds for each term seperately yield the bound

$$\sum_{m=1}^{R} |E(\mathcal{H}, X_m; z)|^2 \ll X^2 Y^{-1} \log X + \delta^{-1} X^2 \log^2 X + RY^2.$$

The optimal choice for Y is $Y = R^{-1/3} X^{2/3}$ which implies the bound

$$\sum_{m=1}^{R} |E(\mathcal{H}, X_m; z)|^2 \ll R^{1/3} X^{4/3} \log X + \delta^{-1} X^2 \log^2 X, \tag{3.37}$$

which completes the proof of Proposition 3.6.1. □

Proof. (of Theorem 3.6.2) Choosing $\delta^{-1} \ll RX^{-1}$ and $R > X^{1/2}$ in the bound (3.37) we get

$$\begin{aligned}\sum_{m=1}^{R} |E(\mathcal{H}, X_m; z)|^2 &\ll R^{1/3} X^{4/3} \log X + RX \log^2 X \\ &\ll RX \log^2 X\end{aligned}$$

and we conclude the bound (3.28). For the bound (3.29), we take the points X_i equally spaced in the interval $[X, 2X]$ with $\delta = R^{-1}X$. As $R \to \infty$,

$$\sum_{m=1}^{R} |E(\mathcal{H}, X_m; z)|^2 \frac{X}{R} \to \int_X^{2X} |E(\mathcal{H}, x; z)|^2 dx,$$

hence

$$\frac{1}{X} \int_X^{2X} |E(\mathcal{H}, x; z)|^2 dx \ll X \log^2 X. \tag{3.38}$$

□

We now proceed to the proof of the radial average.

Proof. (of Proposition 3.6.3) For a sequence $\{a_k\}$, Cauchy-Schwarz inequality implies

$$\begin{aligned}\left(\sum_{k=0}^{n} a_k\right)^2 &\ll \sum_{k=0}^{n} (n+1-k)^2 a_k^2, \\ \left(\sum_{k=0}^{n} a_k\right)^2 &\ll \sum_{k=0}^{n} (k+1)^2 a_k^2.\end{aligned}$$

The first inequality for $a_k = S(X, z_m, 2^k)$ implies the bound

$$\left| \sum_{1 \le T < X^2Y^{-2}} S(X, z_m, T) \right|^2 \ll \sum_{1 \le T < X^2Y^{-2}} \left| \log T^{-1}X^2Y^{-2} + 1 \right|^2 |S(X, z_m, T)|^2, \tag{3.39}$$

whereas the second gives

$$\left| \sum_{1 \le T < X^2Y^{-2}} S(X, z_m, T) \right|^2 \ll \sum_{1 \le T < X^2Y^{-2}} (\log T + 1)^2 |S(X, z_m, T)|^2. \tag{3.40}$$

The bounds (3.39) and (3.40) give

$$\left| \sum_{1 \le T < X^2Y^{-2}} S(X, z_m, T) \right|^2 \ll \sum_{1 \le T < X^2Y^{-2}} |c_T|^2 |S(X, z_m, T)|^2, \tag{3.41}$$

where $c_T = \min\{\log T^{-1}X^2Y^{-2} + 1, \log T + 1\}$. Using Theorem 3.5.3, bound (3.35) and summing over $T = 2^k$, $k = 0, 1, \dots, [\log_2(X^2Y^{-2})]$, we get

$$\begin{aligned} \sum_{m=1}^{R} |E(\mathcal{H}, X; z_m)|^2 &\ll \sum_{1 \le T < X^2Y^{-2}} |c_T|^2 (T^2 + \delta^{-2})(XT^{-1} \min\{1, X^2T^{-2}Y^{-2}\}) + RY^2 \\ &\ll \delta^{-2}X + X^2Y^{-1}\log^2 X + RY^2, \end{aligned}$$

where the last bound yields as in the proof of Proposition 3.6.1. The bound (3.30) is obtained for $Y = X^{2/3}R^{-1/3}$. For the fourth moment, we use Hölder's inequality to prove

$$\sum_{m=1}^{R} |E_f(\mathcal{H}, X; z_m)|^4 \ll \log^3 X \sum_{1 \le T < X^2Y^{-2}} \left(\sum_{m=1}^{R} |S(X, z_m, T)|^4 \right) + RY^4, \tag{3.42}$$

We can now finish the proof assuming the following large sieve inequality

$$\sum_{m=1}^{R} \left| \sum_{|t_j| \le T} a_j u_j(z_m) + \frac{1}{4\pi} \sum_{\mathfrak{a}} \int_{-T}^{T} a_{\mathfrak{a}}(t) E_{\mathfrak{a}}(z_m, 1/2 + it) dt \right|^4 \ll (T^4 + T^2\delta^{-2}) \|a\|_*^4. \tag{3.43}$$

For the proof see [5]. We can now derive the second part of the proposition applying (3.43) to the $a_j = d(f, t_j)\hat{u}_j$ and $a(t) = d(f, t)\hat{E}_{\mathfrak{a}}(1/2 + it)$. □

Proof. (of Theorem 3.6.4) Consider first the $n = 1$ case. Choosing $\delta^{-2} \ll R$ and

$R > X^{1/2}$ in the bound

$$\sum_{m=1}^{R} |E(\mathcal{H}, X; z_m)|^2 \ll \delta^{-2} X + R^{1/3} X^{4/3} \log^2 X,$$

we get

$$\frac{1}{R} \sum_{m=1}^{R} |E(\mathcal{H}, X; z_m)|^2 \ll X + R^{-2/3} X^{4/3} \log^2 X \ll X \log^2 X,$$

and the first part follows. For the integral estimate we notice that, since Γ is cocompact and $\delta^{-2} \ll R$, the points become well distributed on the surface hence as $R \to \infty$

$$\frac{1}{R} \sum_{m=1}^{R} |E(\mathcal{H}, X; z_m)|^2 \to \int_{\Gamma\backslash\mathbb{H}} |E(\mathcal{H}, X; z)|^2 d\mu(z).$$

Hence

$$\int_{\Gamma\backslash\mathbb{H}} |E(\mathcal{H}, X; z)|^2 d\mu(z) \ll X \log^2 X.$$

The $n = 2$ case follows in exactly the same way. □

For the error term $E(X; z, w)$ of the classical hyperbolic lattice point problem we saw (equation (2.26) that the optimal bound is conjectured to be $E(X; z, w) = O_\epsilon(X^{1/2+\epsilon})$ for every $\epsilon > 0$. Theorems 3.6.2 and 3.6.4 lead us to formulate the analogous conjecture.

Conjecture 3.7.1. For Γ cocompact or cofinite and $\mathcal{H}$ a hyperbolic conjugacy class of Γ, the error term $E(\mathcal{H}, X; z)$ satisfies the bound

$$E(\mathcal{H}, X; z) = O_\epsilon(X^{1/2+\epsilon})$$

for every $\epsilon > 0$.

3.8 Bounds for higher moments of the error term

It is a natural question to consider an upper bound for every k-moment of the error term. We prove the following results.

Proposition 3.8.1. *Let $k \geq 3$. For the k-moment of the error term we have:*

$$\left(\frac{1}{X}\int_X^{2X} |E(\mathcal{H},x;z)|^k dx\right)^{1/k} \ll X^{2/3-1/(3k)} \log^{2/k} X, \tag{3.44}$$

where the '$\ll$' constant does not depend on k.

Proposition 3.8.2. *If Γ is cocompact then*

$$\left(\int_{\Gamma\backslash\mathbb{H}} |E(\mathcal{H},X;z)|^3 d\mu(z)\right)^{1/3} \ll X^{1/2}\log X,$$

and for $k \geq 5$ we have

$$\left(\int_{\Gamma\backslash\mathbb{H}} |E(\mathcal{H},X;z)|^k d\mu(z)\right)^{1/k} \ll X^{2/3-2/(3k)} \log^{4/k} X, \tag{3.45}$$

where the '$\ll$' constant does not depend on k.

Proof. (of Proposition 3.8.1) For $k \geq 3$ we get

$$\frac{1}{X}\int_X^{2X} |E(\mathcal{H},x;z)|^k dx \ll \sup_{x\in[X,2X]} |E(\mathcal{H},x;z)|^{k-2} \frac{1}{X}\int_X^{2X} |E(\mathcal{H},x;z)|^2 dx.$$

Since $|E(\mathcal{H},X;z)| \leq CX^{2/3}$ for some constant C that depends only on Γ, $\mathcal{H}$ and z we get

$$\begin{aligned}\sup_{x\in[X,2X]} |E(\mathcal{H},x;z)|^{k-2} \frac{1}{X}\int_X^{2X} |E(\mathcal{H},x;z)|^2 dx &\ll (2C)^k X^{(2k-4)/3} X \log^2 X \\ &\ll (2C)^k X^{2k/3-1/3} \log^2 X.\end{aligned}$$

□

Proof. (of Proposition 3.8.2) The case $k = 3$ is trivial using Cauchy-Schwarz inequality. We notice that for z in a compact set the constant C depends only on Γ and $\mathcal{H}$ (see [44, p. 2]). For Γ cocompact and $k \geq 5$ we get

$$\begin{aligned}\int_{\Gamma\backslash\mathbb{H}} |E(\mathcal{H},X;z)|^k d\mu(z) &\ll \sup_{z\in\Gamma\backslash\mathbb{H}} |E(\mathcal{H},X;z)|^{k-4} \int_{\Gamma\backslash\mathbb{H}} |E(\mathcal{H},X;z)|^4 d\mu(z) \\ &\ll C^k X^{(2k-8)/3} X^2 \log^4 X \\ &\ll C^k X^{2k/3-2/3} \log^4 X.\end{aligned}$$

□

Remark 3.8.3. One can apply the large sieve inequalities to deduce upper bounds for the sums

$$\sum_{m=1}^{R} |E(\mathcal{H}, X_m; z)|^{2k} \tag{3.46}$$

with $X_1, X_2, ..., X_R \in [X, 2X]$ and

$$\sum_{m=1}^{R} |E(\mathcal{H}, X; z_m)|^{2k} \tag{3.47}$$

with z_m well distributed on the surface $\Gamma\backslash\mathbb{H}$ and apply them to obtain upper bounds for the $2k$-moment of the error term.

For the classical problem, Chamizo has derived large sieve inequalities for the $2k$-moment [5, Corollary 2.1.1, Corollary 2.2.1]. However, in his inequalities the implied constant depends on Γ, $\max y_\Gamma(z_m)$ and k, where the dependence on k is not explicitly investigated. For this reason, Chamizo in [5] does not state any result for the $2k$-moment as $k \to \infty$.

It is plausible that one can use Chamizo's method to deduce an upper bound for the moment of the form

$$\left(\frac{1}{X}\int_X^{2X} |E(\mathcal{H}, x; z)|^{2k} dx\right)^{1/2k} \ll X^{2/3-1/3(3k-1)} \log X. \tag{3.48}$$

where the '$\ll$' constant does not depend on k (and, for Γ cocompact, a similar upper bound for the spatial average as well). Notice that in this case the large sieve method implies a worse upper bound than the trivial estimates in the proofs of Propositions (3.8.1) and (3.8.2).

For Γ cocompact, Cherubini [13, Theorem 3.33] used the theory of almost periodic functions to deduce an upper bound similar to (3.44) for the error term in the radial average. We refer also to [46, Section 6.3.4] for a discussion on the limitations of the large sieve method for large moments and dimension $n \geq 3$.

These results show that for every finite moment we have a power saving to the pointwise bound $O(X^{2/3})$, and further, as $k \to \infty$, for both the radial and the spatial average, the k-moment of the error terms has an upper bound of the form $O(X^{2/3-f(k)} \log X)$ with $f(k) \to 0$ as $k \to \infty$. We notice that if one could prove an upper bound $O_\epsilon(X^{a+\epsilon-f(k)})$ for the k-moment with $f(k) \to 0$, $a < 2/3$ and for every $\epsilon > 0$ then this would imply $E(\mathcal{H}, X; z) = O(X^a)$. This would be a pointwise improvement towards the the Conjecture 3.7.1.

3.9 Upper bounds on geodesics

As applications of Lemmas 3.2.2 and 3.4.3, we give upper bounds on average for both the errors $E(X; z, w)$ and $E(\mathcal{H}, X; z)$.

Improving the pointwise upper bound $O(X^{2/3})$ amounts to detecting subtle cancelation between the eigenvalues λ_j and the eigenfunctions u_j of the hyperbolic Laplacian. Petridis and Risager [59] captured this cancellation using QUE-information on average. We describe their result.

Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. The Quantum Unique Ergodicity conjecture (QUE), which is now a theorem due to the work of Lindenstrauss [50] and Soundararajan [68], states that for every function f which is smooth and compactly supported on $\Gamma\backslash\mathbb{H}$ (or similar arithmetic surfaces) we have

$$\int_{\Gamma\backslash\mathbb{H}} f(z)|u_j(z)|^2 d\mu(z) \to \frac{1}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} \int_{\Gamma\backslash\mathbb{H}} f(z) d\mu(z)$$

as $j \to \infty$. Sarnak has conjectured the precise rate of convergence in QUE:

$$\int_{\Gamma\backslash\mathbb{H}} f(z)|u_j(z)|^2 d\mu(z) - \frac{1}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} \int_{\Gamma\backslash\mathbb{H}} f(z) d\mu(z) = O_\epsilon(t_j^{-1/2+\epsilon}). \tag{3.49}$$

Luo and Sarnak [51] proved that (3.49) holds on average, i.e.:

$$\sum_{|t_j| \le T} \left| \int_{\Gamma\backslash\mathbb{H}} f(z)|u_j(z)|^2 d\mu(z) - \int_{\Gamma\backslash\mathbb{H}} f(z) d\mu(z) \right|^2 = O_{f,\epsilon}(T^{1+\epsilon}). \tag{3.50}$$

Using (3.50) and estimates for exponential sums over eigenvalues for $\mathrm{PSL}_2(\mathbb{Z})$, Petridis and Risager proved the following local average result when we allow the center to vary in a small region and $z = w$, which goes half the way towards the $O(X^{1/2+\epsilon})$-conjecture for the error $E(X; z, z)$.

Theorem 3.9.1 (Petridis-Risager [59]). *Let* $\Gamma = PSL_2(\mathbb{Z})$ *and* f *a positive, compactly supported function on* $\Gamma\backslash\mathbb{H}$*. Then*

$$\int_{\Gamma\backslash\mathbb{H}} f(z) E(X; z, z) d\mu(z) = O_{f,\epsilon}(X^{7/12+\epsilon}) \tag{3.51}$$

for all $\epsilon > 0$.

In another direction we describe below how we can get bounds for the error term of the order $O_\epsilon(X^{1/2+\epsilon})$ for the error terms of both hyperbolic lattice problems, when we average on closed geodesics. The key ingredient is the average behaviour for the sums of period integrals (Lemmas 3.2.2 and 3.4.3), which is slower than the growth in the local Weyl's law 2.1.6.

3.9.1 An upper bound for the classical problem

Theorem 3.9.2. *Let* Γ *be a cocompact or cofinite group and* ℓ_0 *a closed geodesic of* $\Gamma\backslash\mathbb{H}$. *Then*

$$\int_{\ell_0} E(X;z,w)ds(w) = O_{\ell_0}(X^{1/2}\log X).$$

Notice that we integrate in one of the two variables only, in contrast with Theorem 3.9.1. The proof of this result follows the steps of the proof for the classical pointwise bound $O(X^{2/3})$, sketched in section 2.2. The standard idea here is to approximate the kernel $k(u) = \chi_{[0,(x-2)/4]}$ by appropriate step functions $k_\pm(u)$ and use the observation

$$\sum_{|t_j|\leq T} \frac{u_j(z)\hat{u}_j}{t_j^{3/2}} \ll \log T. \tag{3.52}$$

Proof. Assume first that Γ is cocompact. Let $k(u)$ be the characteristic of the interval $[0,(X-2)/4]$ and the functions $k_-(u) \leq k(u) \leq k_+(u)$ be as in eq. (2.22), (2.23). We denote their Selberg/Harish-Chandra transform by $h_\pm(t)$. From eq. (2.24) we have

$$E(X;z,w) \ll \sum_{t_j\neq 0} h_\pm(t_j)u_j(z)\overline{u_j(w)} + O(X^{1/2}\log X + (X^{1/2}+Y)), \tag{3.53}$$

where the term $O(X^{1/2}\log X)$ comes from the eigenvalue $t_j = 0$ and the term $O(X^{1/2}+Y)$ comes from the sum

$$O\left((Y+X^{1/2})\sum_{1/2<s_j\leq 1} u_j(z)\overline{u_j(w)}\right).$$

Since this sum is finite we obtain

$$\int_{\ell_0} E(X;z,w)ds(w) \ll \sum_{t_j>0} h_\pm(t_j)u_j(z)\hat{u}_j + O_\ell(X^{1/2}\log X + Y).$$

We use the estimate (2.25) to get

$$\sum_{t_j>0} h_\pm(t_j)u_j(z)\hat{u}_j \ll X^{1/2} \sum_{t_j>0} |t_j|^{-5/2} \min\{|t_j|, XY^{-1}\} u_j(z)\hat{u}_j.$$

Using Cauchy-Schwarz inequality, local Weyl's law for the Maaß forms $u_j(z)$, Lemma 3.2.2 for the periods $\hat{u}_j$ and observation (3.52) this is bounded by

$$\begin{aligned} X^{1/2} \sum_{t_j \le X/Y} |t_j|^{-3/2} u_j(z)\hat{u}_j \quad &+ \quad X^{3/2}Y^{-1} \sum_{t_j > X/Y} |t_j|^{-5/2} u_j(z)\hat{u}_j \\ &\ll \quad X^{1/2}\log(X/Y) + X^{1/2}. \end{aligned}$$

We conclude

$$\int_{\ell_0} E(X; z, w)ds(w) \ll X^{1/2}\log(X/Y) + Y + X^{1/2}\log X,$$

and the statement follows for $Y \ll X^{1/2}$. For the cofinite case, the result follows from the bound

$$\sum_{\mathfrak{a}} \int_{-\infty}^{\infty} h_\pm(t)\hat{E}_{\mathfrak{a}}(1/2+it)E_{\mathfrak{a}}(z, 1/2+it)dt = O(X\log(X/Y) + X^{1/2})$$

for the Eisenstein series and their period integrals. □

3.9.2 An upper bound for the lattice counting in conjugacy classes

For the error term $E(\mathcal{H}, X; z)$ of the conjugacy class problem we deduce the following upper bound.

Theorem 3.9.3. *Let ℓ_0 be a closed geodesic on $\Gamma\backslash\mathbb{H}$ (perhaps different from the invariant closed geodesic ℓ of $\mathcal{H}$). Then*

$$\int_{\ell_0} E(\mathcal{H}, X; z)ds(z) = O(X^{1/2}\log X).$$

Proof. For Γ cocompact, we let $f = f_X$ be the test function defined in (3.13) and the functions $f_- \le f \le f_+$ be as in eq. (3.16), (3.17). Using (3.24) and (3.25) we get

$$\begin{aligned} E(\mathcal{H}, X; z) \ll \sum_{t_j > 1} d(f^{\pm}, t_j)\hat{u}_j u_j(z) \quad &+ \quad O(X^{1/2}\log X) \\ &+ \quad O\left((Y + X^{1/2}) \sum_{1/2 < s_j \le 1} u_j(z)\hat{u_j} \right), \end{aligned}$$

hence, using Proposition 3.2.4 we deduce

$$\begin{aligned} \int_{\ell_0} E(\mathcal{H}, X; z) ds(z) \quad &\ll \quad \sum_{t_j > 1} d(f^{\pm}, t_j)\hat{u}_j \int_{\ell_0} u_j(z) ds(z) + O(Y + X^{1/2}\log X) \\ &\ll \quad X^{1/2} \sum_{t_j > 1} |t_j|^{-2} \min\{|t_j|, XY^{-1}\}\hat{u}_j \int_{\ell_0} u_j(z) ds(z) \\ &\quad\; + O(Y + X^{1/2}\log X). \end{aligned}$$

We apply Cauchy-Schwarz inequality and Lemma 3.2.2 for both sequences of period integrals $\hat{u}_j$ and $\int_{\ell_0} u_j(z)ds(z)$ and we get

$$\int_{\ell_0} E(\mathcal{H}, X; z) ds(z) \ll X^{1/2}\log(X/Y) + Y + X^{1/2}.$$

The statement follows for $Y = X^{1/2}$. For the cofinite case, we finish the proof using Lemma 3.4.3 to obtain:

$$\sum_{\mathfrak{a}} \int_{-\infty}^{\infty} d(f^{\pm}, t)\hat{E}_{\mathfrak{a}}(1/2 + it) \int_{\ell_0} E_{\mathfrak{a}}(z, 1/2 + it) ds(z) dt = O(X\log(X/Y) + X^{1/2}).$$

□

Remark 3.9.4. Picking $z_1, ..., z_R$ in Theorem 3.6.3 which are equidistributed on the geodesic ℓ_0, choosing $\delta^{-1} = R/\mu(\ell_0)$ and allowing $R \to \infty$ we get

$$\begin{aligned} \int_{\ell_0} |E(\mathcal{H}, X; z)|^2 ds(z) \quad &\ll \quad X^{6/5}\log^2 X, \\ \int_{\ell_0} |E(\mathcal{H}, X; z)|^4 ds(z) \quad &\ll \quad X^{12/5}\log^4 X. \end{aligned}$$

Thus, working as in Proposition 3.8.2 we conclude

$$\left(\int_{\ell_0} |E(\mathcal{H}, X; z)|^k ds(z) \right)^{1/k} \ll X^{2/3 - 4/(15k)} \log^{4/k} X.$$

for every $k \geq 5$. The same bounds hold for the classical problem if we pick the points on a geodesic and apply [7, p. 317, Proposition 2.2].

Chapter 4

Lattice point counting in conjugacy classes: mean value results and Ω-results

4.1 Statements of results

In this chapter we study mean value results and Ω-results for the hyperbolic lattice problem in conjugacy classes. Our results extend the work of Phillips and Rudnick to the conjugacy class problem.

As in the classical problem, we subtract the contribution of the eigenvalue $\lambda_j = 1/4$ from the error term $E(\mathcal{H}, X; z)$ and we denote the difference by $e(\mathcal{H}, X; z)$:

$$e(\mathcal{H}, X; z) = E(\mathcal{H}, X; z) - d(f_X, 0) \sum_{t_j=0} \hat{u}_j u_j(z), \tag{4.1}$$

where f_X is the test function defined in (3.13). In Proposition 4.2.1 we prove that

$$d(f_X, 0) \sum_{t_j=0} \hat{u}_j u_j(z) = O(X^{1/2} \log X),$$

hence Conjecture 3.7.1 is equivalent with the bound

$$e(\mathcal{H}, X; z) = O_\epsilon(X^{1/2+\epsilon}). \tag{4.2}$$

The first result we prove is the analogue of Patterson's result (Theorem 2.2.2) for the conjugacy class problem.

Proposition 4.1.1. *Let* Γ *be a cocompact or cofinite Fuchsian group. Then the error term* $e(\mathcal{H}, X; z)$ *satisfies the average bound*

$$\frac{1}{X}\int_1^X e(\mathcal{H}, x; z)dx = O(X^{1/2}).$$

The second result we prove in this chapter is the analogue of Theorem 2.2.3 for the conjugacy class problem. From the change of variables (3.2) it follows that Theorem 4.1.2 is indeed a mean value result in the radial parameter $t \asymp r$ (where the parameter t counts the distance between the closed geodesic of $\mathcal{H}$ and the orbit of z).

Theorem 4.1.2. *Let* Γ *be a cocompact or cofinite Fuchsian group and for* $x \geq 1$ *let* r *be defined as* $r = \log(x + \sqrt{x^2 - 1})$.
(a) If Γ *is cocompact, then*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{e\left(\mathcal{H}, x; z\right)}{x^{1/2}}dr = 0. \tag{4.3}$$

(b) If Γ *is cofinite, then*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{e\left(\mathcal{H}, x; z\right)}{x^{1/2}}dr = \frac{|\Gamma(3/4)|^2}{\pi^{3/2}} \sum_{\mathfrak{a}} \hat{E}_{\mathfrak{a}}(1/2)E_{\mathfrak{a}}(z, 1/2). \tag{4.4}$$

In particular we see that the radial parameter t satisfies asymptotically $t \sim 2r + \mu - 2\log 2$.

We give the proof of Proposition 4.1.1 in section 4.3 and the proof of Theorem 4.1.2 in section 4.4.

For the conjugacy class problem, proving pointwise Ω-results is a more subtle problem comparing to the classical one, due to the appearance of the period integrals in the spectral expansion of $e(\mathcal{H}, X; z)$. In the proof of Theorem 2.2.4, Phillips and Rudnick choose $z = w$ so that the series expansion of the error term $e(X; z, w)$ contains the expressions $|u_j(z)|^2$ which are nonnegative. In this setting, the natural choice is to average over the $\mathcal{H}$-invariant geodesic ℓ.

The following result of Good and Tsuzuki describes the exact asymptotic behaviour for the period integrals (see Lemmas 3.2.2 and 3.4.3).

Theorem 4.1.3 (Good [26], Tsuzuki [71]). *The period integrals $\hat{u}_j$ of Maaß forms and $\hat{E}_{\mathfrak{a}}(1/2+it)$ of Eisenstein series satisfy the asymptotic*

$$\sum_{|t_j|<T} |\hat{u}_j|^2 + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-T}^{T} |\hat{E}_{\mathfrak{a}}(1/2+it)|^2 dt \sim \frac{\mu(\ell)}{\pi} \cdot T,$$

where $\mu(\ell)$ denotes the length of the invariant closed geodesic ℓ.

We refer to [53, p. 3-4] for a detailed history of this result. In order to state our next results we need the following definition which is related to Theorem 4.1.3.

Definition 4.1.4. *Let $\mathcal{H}$ be a fixed hyperbolic conjugacy class of a cofinite group Γ. We say that the group Γ:*
(a) has sufficiently many cusp forms in the sense of period integrals associated to $\mathcal{H}$ if

$$\sum_{|t_j|<T} |\hat{u}_j|^2 \gg T.$$

(b) has sufficiently small Eisenstein periods associated to $\mathcal{H}$ if for all cusps $\mathfrak{a}$:

$$\int_{-T}^{T} |\hat{E}_{\mathfrak{a}}(1/2+it)|^2 dt \ll \frac{T}{(\log T)^{1+\delta}}$$

for a fixed $\delta > 0$.

We write $\int_{\mathcal{H}} ds$ to indicate that we average over a segment of the invariant geodesic ℓ of length μ/ν. In section 4.5 we prove the following theorem, which is an average Ω-result on the closed geodesic of $\mathcal{H}$.

Theorem 4.1.5. *(a) If Γ is either (i) cocompact or (ii) cofinite but not cocompact and has sufficiently small Eisenstein periods associated to $\mathcal{H}$ according to Definition 4.1.4, then*

$$\int_{\mathcal{H}} e(\mathcal{H}, X; z) ds(z) = \Omega_+(X^{1/2} \log\log\log X).$$

(b) If Γ is cofinite but not cocompact and either (i) $\hat{u}_j \neq 0$ for at least one $\lambda_j > 1/4$ or (ii) $\hat{E}_{\mathfrak{a}}(1/2) \neq 0$ for a cusp $\mathfrak{a}$ then

$$\int_{\mathcal{H}} e(\mathcal{H}, X; z) ds(z) = \Omega_+(X^{1/2}).$$

Notice that the asymptotic behaviour for the sums of period integrals in Theorem 4.1.3 is $c \cdot T$, where in local Weyl's law (Theorem 2.1.6) we get an asymptotic $c \cdot T^2$. If Γ is cocompact or cofinite but it has sufficiently many cusp forms in the sense of period integrals associated to $\mathcal{H}$ then

$$\sum_{0<t_j<T} \frac{|\hat{u}_j|^2}{t_j} \gg \log T. \tag{4.5}$$

In case (a) of Theorem 4.1.5 the triple logarithm should be compared with the extra factor $(\log\log X)^{1/4-\delta}$ in case (a) of Theorem 2.2.4. The first is a consequence of the asymptotic behaviour of period integrals in Theorem 4.1.3, and the second is a consequence of the local Weyl's law.

In subsection 4.5.3 we will see that the modular group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ has sufficiently small Eisenstein periods associated to a fixed conjugacy class $\mathcal{H} \subset \Gamma$. This follows from a subconvexity bound on the critical line for an Epstein zeta function associated to $\mathcal{H}$.

Further, for part (a) of Theorem 4.1.5, in Remark 4.5.1 we will prove that a $\Omega_+(X^{1/2}\log\log\log X)$-result follows assuming only the weaker condition that Γ has sufficiently many cusp forms, if we have the extra condition of a polynomial upper bound

$$\frac{d}{dt}\hat{E}_{\mathfrak{a}}(1/2+it) \ll |t|^N$$

for the derivatives of the Eisenstein periods.

To prove pointwise Ω-results for $e(\mathcal{H},X;z)$ we would like to have a fixed pair $(z,\mathcal{H})$ with $e(\mathcal{H},X;z)$ large, i.e. a pair $(z,\mathcal{H})$ with a uniform 'fixed sign' property of all $\hat{u}_j u_j(z)$. That would allow us to prove a pointwise Ω-result of the form

$$\limsup_{X} \frac{e(\mathcal{H},X;z)}{X^{1/2}} = \infty.$$

However, Maaß forms have complicated behaviour on the surface $\Gamma\backslash\mathbb{H}$; for instance, the nodal domains have very complicated shapes. For this reason we have not been able to determine any such specific pair $(z,\mathcal{H})$ with the desired fixed sign property. To overcome this problem we notice that the period integral is the limit of Riemann sums. Starting with a fixed conjugacy class $\mathcal{H}$, a discrete average allows us to prove the existence of at least one point $z = z_{\mathcal{H}}$ for which the error $e(\mathcal{H},X;z_{\mathcal{H}})$ cannot be small.

We first prove the following proposition for discrete averages.

Proposition 4.1.6. *Let $\mathcal{H}$ be a fixed hyperbolic class in Γ. If Γ is either (i) cocompact or (ii) if Γ is as in part (b) of Theorem 4.1.5, then there exist an integer $K = K_{\mathcal{H}}$ depending only on $\mathcal{H}$ and $z_1, z_2, ..., z_K$ points on ℓ such that:*

$$\frac{1}{K}\sum_{m=1}^{K} e(\mathcal{H}, X; z_m) = \Omega_+(X^{1/2}).$$

We finally deduce the following pointwise Ω-results for the error term $e(\mathcal{H}, X; z)$.

Theorem 4.1.7. *Let Γ be a Fuchsian group, $\mathcal{H}$ a hyperbolic conjugacy class of Γ and ℓ the invariant closed geodesic of $\mathcal{H}$.*
(a) If Γ is as in Proposition 4.1.6 then there exist at least one point $z_{\mathcal{H}} \in \ell$ such that:

$$e(\mathcal{H}, X; z_{\mathcal{H}}) = \Omega_+(X^{1/2}).$$

(b) If Γ is either (i) cocompact or (ii) cofinite but not cocompact, $\hat{u}_j \neq 0$ for at least one $\lambda_j > 1/4$ and $\hat{E}_{\mathfrak{a}}(1/2) = 0$ for all cusps $\mathfrak{a}$, then there exists at least one point $z_{\mathcal{H}} \in \ell$ such that:

$$e(\mathcal{H}, X; z_{\mathcal{H}}) = \Omega_-(X^{1/2}).$$

(c) If the sum $\sum_{\mathfrak{a}} \hat{E}_{\mathfrak{a}}(1/2) E_{\mathfrak{a}}(z, 1/2)$ does not vanish then

$$e(\mathcal{H}, X; z) = \Omega(X^{1/2}).$$

Part (a) of Theorem 4.1.7 is an immediate corollary of Proposition 4.1.6, and part (c) is a corollary of Theorem 4.1.2.

The proof of part (b) is more subtle and uses the ideas from Chapter 2. In comparison with our results in Chapter 2 we investigate the asymptotic behaviour of the average error term

$$M_{\mathcal{H},z}(X) = \frac{1}{X}\int_1^X \frac{e(\mathcal{H}, x; z)}{x^{1/2}} dx.$$

It is immediate to prove that $M_{\mathcal{H},z}(X) = O(1)$. In order to prove a Ω_--result for $e(\mathcal{H}, X; z)$ we are lead to study a discrete average of $M_{\mathcal{H},z}(X)$ on the geodesic. It follows that if Γ is as in part (b), then there exist an integer K and $z_1, z_2, ..., z_K$ points in ℓ such that, as $X \to \infty$, the quantity

$$\frac{1}{K}\sum_{m=1}^{K} M_{\mathcal{H},z_m}(X)$$

is $\Omega_{-}(1)$, and the statement of part (b) follows. We give the proof of part (b) in the subsection 4.6.3. Notice that, as in Chapter 2, in order to prove both signs for our $\Omega_{\pm}$ -results we make use of Ω-results on average. In comparison with Chapter 2, we notice here that we could also investigate the limit of $\int_{\mathcal{H}} M_{\mathcal{H},z}(X)ds(z)$ as $X \to \infty$. In contrast with Theorem 2.2.5, there is no influence of the first real Satake parameter in the limit (and therefore, of the first eigenvalue λ_1) for the conjugacy class problem in dimension 2 (see our comments after Lemma 4.2.2).

Remark 4.1.7. One of the main ingredients in the proofs of Theorems 4.1.5, 4.1.7 and Proposition 4.1.6 is the asymptotic behaviour of the period integrals of the eigenfunctions (Theorem 4.1.3). For the proofs of Theorem 4.1.5, Proposition 4.1.6 and Theorem 4.1.7 we will also need two extra 'fixed-sign' properties of the Γ-function stated in Lemmas 4.2.2 and 4.6.1. We emphasize that the differences of the sign in these Lemmas causes the different sign of our Ω-results.

Remark 4.1.8. Our results imply that in order to prove a pointwise result $e(\mathcal{H}, X; z_{\mathcal{H}}) = \Omega(X^{1/2})$ for one point $z_{\mathcal{H}}$ in the general cofinite case, we must only assume the nonvanishing of one period $\hat{u}_j$. In this case, the sign of our Ω-result can be determined by the vanishing or not of the Eisenstein period integrals. If all Eisenstein periods vanish then there exists at least two points $z, w \in \ell$ such that:

$$e(\mathcal{H}, X; z) = \Omega_{+}(X^{1/2}), \quad e(\mathcal{H}, X; w) = \Omega_{-}(X^{1/2}).$$

If all $\hat{u}_j$ vanish, then the existence of a $\hat{E}_{\mathfrak{a}}(1/2) \neq 0$ implies a $\Omega_{+}(X^{1/2})$-result. These Eisentein periods are of particular arithmetic interest; in fact $\hat{E}_{\mathfrak{a}}(1/2)$ is the constant term of the hyperbolic Fourier expansion of $E_{\mathfrak{a}}(z, s)$ (see [25, section 3.2]). In the arithmetic case, these periods are associated to special values of Epstein zeta functions (see subsection 4.5.2). We notice that, in principle, it is easier to check the nonvanishing of one period $\hat{E}_{\mathfrak{a}}(1/2)$ than the nonvanishing of the sum $\sum_{\mathfrak{a}} \hat{E}_{\mathfrak{a}}(1/2)E_{\mathfrak{a}}(z, 1/2)$.

Remark 4.1.9. In order to generalize Theorems 4.1.2, 4.1.5 and 4.1.6 in dimensions $n \geq 3$, we need an explicit expression for the Huber transform $d_n(f, t)$ in the n-th dimension. In dimension $n = 3$, $d_3(f, t)$ has been studied explicitly in [46].

4.2 Some auxiliary results

We will need two auxiliary results. The first one is about explicit estimates for the Huber transform $d(f,t)$, where $f = f_X = \chi_{[1,X^2]}$ is the characteristic function of the interval $[1, X^2]$ defined in (3.13). This is the analogue of Proposition 3.2.4 for the characteristic function. As in Chapter 3 we fix the notation $U = \sqrt{X^2-1}$.

Proposition 4.2.1. *Let $s = 1/2 + it$. The Huber transform of f_X has the following asymptotic behaviour.*
(a) If $s \in (1/2, 1]$ then

$$2d(f_X,t) = A(s)X^s + O\left(\Gamma(s-1/2)X^{s-2} + \Gamma(1/2-s)X^{1-s}\right),$$

where $A(s)$ is the Γ-product defined in (3.5).
(b) For $t \in \mathbb{R} - \{0\}$

$$\begin{aligned} 2d(f_X,t) &= \frac{\sqrt{2}}{\pi}\left|\Gamma\left(\frac{3}{4}+\frac{it}{2}\right)\right|^2 \Re\left(\frac{\Gamma(it)}{\Gamma(3/2+it)}\left(e^{-\frac{i\pi}{4}-\frac{\pi t}{2}} + e^{\frac{i\pi}{4}+\frac{\pi t}{2}}\right)(X+U)^{it}\right)X^{1/2} \\ &\quad + O\left((1+|t|)^{-2}X^{-3/2}(X+U)^{it}\right). \end{aligned}$$

(c) For $t = 0$ we have

$$d(f_X, 0) = O(X^{1/2}\log X).$$

Before giving the proof of Proposition 4.2.1 we fix also the notation

$$\begin{aligned} a(t) &= e^{-\frac{i\pi}{4}-\frac{\pi t}{2}} + e^{\frac{i\pi}{4}+\frac{\pi t}{2}}, \\ D(t) &= \frac{\sqrt{2}}{\pi}\left|\Gamma\left(\frac{3}{4}+\frac{it}{2}\right)\right|^2, \\ G(t) &= \frac{D(t)a(t)}{\Gamma(3/2+it)}. \end{aligned} \tag{4.6}$$

Stirling's formula implies that, as $|t| \to \infty$,

$$D(t) \sim 4e^{-\frac{\pi|t|}{2}}(1+|t|)^{1/2}, \quad \frac{|\Gamma(it)|}{|\Gamma(3/2+it)|} \sim (1+|t|)^{-3/2}. \tag{4.7}$$

Therefore, we have the asymptotic

$$|G(t)\Gamma(it)| \asymp (1+|t|)^{-1}. \tag{4.8}$$

Proof. (of Proposition 4.2.1) (a) Using the integral representation (3.12) for $d(f_X, t)$ we get

$$d(f_X,t) = (2\sqrt{\pi})^{-1}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(1-\frac{s}{2}\right)\int_0^U \left(P^0_{s-1}(iv)+P^0_{s-1}(-iv)\right)dv. \quad (4.9)$$

Using [23, p. 968, eq. (8.752.3)], this takes the form

$$d(f_X,t) = (2\sqrt{\pi})^{-1}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(1-\frac{s}{2}\right)X\left(P^{-1}_{s-1}(iU)-P^{-1}_{s-1}(-iU)\right). \quad (4.10)$$

Using formula [23, p. 971, eq. (8.776)], the statement follows.
(b) We use [23, p. 971, eq. (8.774)], so that equation (4.10) gives

$$2d(f_X,t) = \Re\left(G(t)\Gamma(it)(X+U)^{it}F\left(-\frac{1}{2},\frac{3}{2};1+it;\frac{X-U}{2X}\right)\right)X^{1/2}, \quad (4.11)$$

where $F(a,b;c;z)$ denotes the Gauss' hypergeometric function. As $X\to\infty$, the definition of the hypergeometric function [23, p. 1005, eq. (9.100)] implies

$$F\left(-\frac{1}{2},\frac{3}{2};1+it;\frac{X-U}{2X}\right) = 1+O\left((1+|t|)^{-1}X^{-2}\right). \quad (4.12)$$

We finish the proof of (b) using the asymptotics (4.7).
(c) It follows after elementary calculations, using eq. (4.10), formula [23, p. 961, eq. (8.713.2)] and estimates. □

The following lemma is one of the key ingredients in the proofs of our results.

Lemma 4.2.2. *For every $t\in\mathbb{R}-\{0\}$, we have:*

$$\Re\left(\frac{\Gamma(it)}{\Gamma(3/2+it)}a(t)\right) > 0.$$

Lemma 4.2.2 can be proved elementarily, as Lemma 2.3.1. Let

$$H(u) = (1-e^u)^{1/2}e^u), \quad h_1(t) = e^{\frac{\pi t}{2}}+e^{-\frac{\pi t}{2}}, \quad h_2(t) = e^{\frac{\pi t}{2}}-e^{-\frac{\pi t}{2}}.$$

and $g(u)$ be the function

$$g(u) = -2th_1(t)H'(u)+3th_1(t)H(u)+3h_2(t)H'(u)+2t^2h_2(t)H(u).$$

Using properties of the Beta function $B(x,y)$ and integration by parts, we deduce that

Lemma 4.2.2 is equivalent to

$$\int_{-\infty}^{0} \sin(tu)g(u)du < 0.$$

Lemma 4.2.2 can also be verified using Mathematica. In view of our comments after Theorem 4.1.7 we get that, for all $t \neq 0$, the real parts

$$\Re\left(\frac{\Gamma(it)}{\Gamma(3/2+it)(1+it)}a(t)\right), \quad \Re\left(\frac{\Gamma(it)}{\Gamma(3/2+it)}\frac{it}{1+it}a(t)\right) \tag{4.13}$$

have also fixed signs. Compare (4.13) with Lemma 2.3.1 and Remark 2.3.3.

4.3 Proof of Proposition 4.1.1

We start with the proof of the upper bound for the normalized error

$$\frac{1}{X}\int_{1}^{X} e(\mathcal{H},x;z)dx = O(X^{1/2}). \tag{4.14}$$

Proof. Ignoring for a while convergence issues, applying the spectral expansion (3.8) to $A(f_X)$ and using (3.14), (3.4), (4.1) and Proposition 4.2.1, for Γ cocompact we derive

$$\begin{aligned} e(\mathcal{H},X;z) &= \sum_{t_j>0} 2d(f_X,t_j)\hat{u}_j u_j(z) \qquad (4.15)\\ &+O\left(\sum_{1/2<s_j\leq 1}\Gamma(s_j-1/2)X^{s_j-2}+\Gamma(1/2-s_j)X^{1-s_j}\right). \end{aligned}$$

Since the s_j's are discrete we find a constant $\sigma=\sigma_\Gamma\in(0,1/2]$ such that $s_j-1/2\geq\sigma$ for all $s_j\in(1/2,1]$. This implies that the above O-term is $O(X^{1/2-\sigma})$, see also bound (2.28). However, the series in (4.15) does not converge, hence, we cannot apply the expansion (3.8) for the automorphization of the characteristic function f_X. For this reason, we apply the spectral expansion to the smoothed error (4.14) and we conclude that the contribution of the discrete spectrum in (4.14) is given by

$$\sum_{t_j>0}\frac{1}{X}\int_{1}^{X}\Re\left(G(t)\Gamma(it)\left(x+\sqrt{x^2-1}\right)^{it}\right)x^{1/2}dx\hat{u}_j u_j(z)+O(X^{1/2-\sigma}). \tag{4.16}$$

The asymptotic (4.8) and Theorem 4.1.3 imply that the above expansion is bounded by

$$\sum_{t_j>0} X^{1/2}|t_j|^{-2}\hat{u}_j u_j(z)+O(X^{1/2-\sigma})\ll X^{1/2}. \tag{4.17}$$

To finish the proof for the cofinite case, we notice that the contribution of the continuous spectrum

$$\sum_{\mathfrak{a}} \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{E}_{\mathfrak{a}}(1/2+it) E_{\mathfrak{a}}(z, 1/2+it) \left(\frac{1}{X} \int_{1}^{X} \frac{d(f_x, t)}{x^{1/2}} dx \right) dt$$

is bounded by

$$\ll X^{1/2} \sum_{\mathfrak{a}} \int_{1}^{\infty} |t|^{-2} \hat{E}_{\mathfrak{a}}(1/2+it) E_{\mathfrak{a}}(z, 1/2+it) dt.$$

We complete the proof applying Theorem 4.1.3. □

4.4 Radial mean values

4.4.1 The cocompact case

In this section we give the proof of Theorem 4.1.2. We first prove that the error term $e(\mathcal{H}, X; z)$ has zero mean value for Γ cocompact.

Proof. As we cannot apply the spectral theorem for $L^2(\Gamma\backslash\mathbb{H})$ 2.1.4 directly to $A(f_X)$, we will work with the average

$$M_{\mathcal{H}}(T) = \frac{1}{T} \int_{0}^{T} \frac{e(\mathcal{H}, x; z)}{x^{1/2}} dr.$$

By part (b) of Lemma 4.2.1 we get

$$\begin{aligned} M_{\mathcal{H}}(T) &= \sum_{t_j>0} \Re \left(G(t_j)\Gamma(it_j) \frac{1}{T} \int_{0}^{T} e^{it_j r} dr \right) \hat{u}_j u_j(z) \\ &+ O\left(\sum_{t_j>0} |t_j|^{-2} \hat{u}_j u_j(z) \frac{1}{T} \int_{0}^{T} e^{(-2+it_j)r} dr + \frac{1}{T} \int_{0}^{T} e^{-r\sigma} dr \right). \end{aligned}$$

Using Theorems 2.1.6, 4.1.3 and estimate (4.8) we bound the main term and the O-terms of $M_{\mathcal{H}}(T)$ as

$$M_{\mathcal{H}}(T) \ll \frac{1}{T} \sum_{t_j>0} |t_j|^{-2} \hat{u}_j u_j(z) + O\left(\frac{1}{T} \sum_{t_j>0} |t_j|^{-3} \hat{u}_j u_j(z) + \frac{1}{T} \int_{0}^{T} e^{-r\sigma} dr \right) = O(T^{-1}),$$

and the statement follows. □

4.4.2 The cofinite case

To prove case (b) of Theorem 4.1.2 we consider the contribution of the continuous spectrum in $M_{\mathcal{H}}(T)$, which given by

$$\sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \hat{E}_{\mathfrak{a}}(1/2+it) E_{\mathfrak{a}}(z,1/2+it) \left(\frac{1}{T} \int_0^T \frac{2d(f_x,t)}{x^{1/2}} dr \right) dt. \quad (4.18)$$

To deal with this expansion, we need the following lemma for the Huber transform.

Lemma 4.4.1. *As $T \to \infty$ we have*

$$\lim_{T\to\infty} \int_{-\infty}^{\infty} \frac{1}{T} \int_0^T \frac{2d(f_x,t)}{x^{1/2}} drdt = \frac{4}{\sqrt{\pi}} |\Gamma(3/4)|^2.$$

Proof. Using expression (4.11) we write

$$\int_{-\infty}^{\infty} \frac{1}{T} \int_0^T \frac{2d(f_x,t)}{x^{1/2}} drdt \;=\; \Re\left(\int_{-\infty}^{\infty} \frac{1}{T} \int_0^T G(t)\Gamma(it) e^{irt} F\left(-\frac{1}{2},\frac{3}{2};1+it;\frac{1}{e^{2r}+1}\right) drdt \right).$$

Let $\varepsilon > 0$ be a fixed small number and $M > 0$ be a fixed large number. We consider the path integral

$$\int_\gamma G(z)\Gamma(iz) \frac{1}{T} \int_0^T e^{irz} F\left(-\frac{1}{2},\frac{3}{2};1+iz;\frac{1}{e^{2r}+1}\right) drdz,$$

where γ is the contour $\gamma = \bigcup_{i=1}^6 C_i$ with

$$\begin{aligned} C_1 &= [\epsilon, M], \\ C_2 &= \{M+iv, v \in [0,1/2]\}, \\ C_3 &= [-M+i/2, M+i/2], \\ C_4 &= \{-M+iv, v\in[0,1/2]\}, \\ C_5 &= [-M,-\epsilon], \\ C_6 &= \{\varepsilon \cdot e^{i\theta}, \theta \in [0,\pi]\}, \end{aligned}$$

traversed counterclockwise. We write $G(z)$ as

$$G(z) = \frac{\sqrt{2}}{\pi} \frac{\Gamma\left(\frac{3}{4}+\frac{iz}{2}\right)\Gamma\left(\frac{3}{4}-\frac{iz}{2}\right)}{\Gamma(3/2+iz)} \left(e^{-\frac{i\pi}{4}-\frac{\pi z}{2}} + e^{\frac{i\pi}{4}+\frac{\pi z}{2}}\right),$$

hence we see that the integrand is holomorphic inside the contour. The simple pole at $z = 0$ is coming from $\Gamma(iz)$. We note that $\text{Res}_{z=0}\Gamma(iz) = -i$. Applying Stirling's formula and the asymptotics of the hypergeometric function (4.12) we deduce

$$\begin{aligned}\int_{C_2+C_4} G(z)\Gamma(iz)\frac{1}{T}\int_0^T e^{irz}F\left(-\frac{1}{2},\frac{3}{2};1+iz;\frac{1}{e^{2r}+1}\right)drdz &= O\left(M^{-2}T^{-1}\right),\\ \int_{C_3} G(z)\Gamma(iz)\frac{1}{T}\int_0^T e^{irz}F\left(-\frac{1}{2},\frac{3}{2};1+iz;\frac{1}{e^{2r}+1}\right)drdz &= O\left(T^{-1}\right).\end{aligned}$$

Further, as $\varepsilon \to 0$ we see that the term

$$\int_{C_6} G(z)\Gamma(iz)\frac{1}{T}\int_0^T e^{irz}F\left(-\frac{1}{2},\frac{3}{2};1+iz;\frac{1}{e^{2r}+1}\right)drdz$$

converges to

$$-i\pi G(0)\frac{1}{T}\int_0^T F\left(-\frac{1}{2},\frac{3}{2};1;\frac{1}{e^{2r}+1}\right)dr\text{Res}_{z=0}\Gamma(iz) = -\pi G(0)(1+O(T^{-1})).$$

From Cauchy's Theorem we conclude

$$\begin{aligned}\int_{-M}^{M} G(t)\Gamma(it)\frac{1}{T}\int_0^T e^{irt}F\left(-\frac{1}{2},\frac{3}{2};1+it;\frac{1}{e^{2r}+1}\right)drdt &= \pi G(0)(1+O(T^{-1}))\\ &\quad +O(M^{-2}T^{-1}+T^{-1}).\end{aligned}$$

As $M \to \infty$ we get

$$\int_{-\infty}^{\infty}\frac{1}{T}\int_0^T \frac{2d(f_x,t)}{x^{1/2}}drdt = 2\frac{\Gamma(3/4)^2}{\Gamma(3/2)} + O(T^{-1}).$$

We use $\Gamma(3/2) = \sqrt{\pi}/2$ and the statement follows. □

We let $\phi_{\mathcal{H},\mathfrak{a}}(t)$ denote the function

$$\phi_{\mathcal{H},\mathfrak{a}}(t) = \hat{E}_{\mathfrak{a}}(1/2+it)E_{\mathfrak{a}}(z,1/2+it) - \hat{E}_{\mathfrak{a}}(1/2)E_{\mathfrak{a}}(z,1/2).$$

Thus, the contribution of the cusp $\mathfrak{a}$ in eq. (4.18) can be written in the form

$$\begin{aligned}&\frac{1}{4\pi}\hat{E}_{\mathfrak{a}}(1/2)E_{\mathfrak{a}}(z,1/2)\int_{-\infty}^{\infty}\left(\frac{1}{T}\int_0^T \frac{2d(f_x,t)}{x^{1/2}}dr\right)dt \\ &+\frac{1}{4\pi}\int_{-\infty}^{\infty}\phi_{\mathcal{H},\mathfrak{a}}(t)\left(\frac{1}{T}\int_0^T \frac{2d(f_x,t)}{x^{1/2}}dr\right)dt.\end{aligned} \tag{4.19}$$

Using Proposition 4.2.1 we calculate the second term of eq. (4.19) is splitted in

$$\frac{1}{2\sqrt{2}\pi^2}\int_{-\infty}^{\infty}\phi_{\mathcal{H},\mathfrak{a}}(t)G(t)\Gamma(it)\frac{e^{itT}-1}{itT}dt+O\left(\frac{1}{T}\int_{-\infty}^{\infty}\phi_{\mathcal{H},\mathfrak{a}}(t)\frac{G(t)\Gamma(it)}{(1+|t|)(2+|t|)}dt\right).$$

Since $\phi_{\mathcal{H},\mathfrak{a}}(0)=0$, applying Theorems 2.1.6 and 4.1.3 we conclude the bound

$$\int_{-\infty}^{\infty}\phi_{\mathcal{H},\mathfrak{a}}(t)\left(\frac{1}{T}\int_0^T\frac{2d(f_x,t)}{x^{1/2}}dr\right)dt=O(T^{-1}).$$

Hence, as $T\to\infty$ the contribution of the continuous spectrum converges to

$$\pi^{-3/2}|\Gamma(3/4)|^2\sum_{\mathfrak{a}}\hat{E}_{\mathfrak{a}}(1/2)E_{\mathfrak{a}}(z,1/2).$$

This completes the proof of part (b) of Theorem 4.1.2.

4.5 Ω-results for the average error term on geodesics

In this section we give the proof of Theorem 4.1.5. For this reason, we mollify the average of the error term on the geodesic ℓ. Let $\psi\geq 0$ be a smooth even function as in section 2.5: ψ is compactly supported in $[-1,1]$, such that $\hat{\psi}\geq 0$ and $\int_{-\infty}^{\infty}\psi(x)dx=1$. For every $\epsilon>0$ we consider the family of functions $\psi_\epsilon(x)=\epsilon^{-1}\psi(x/\epsilon)$. We have $0\leq\hat{\psi}_\epsilon(x)\leq 1$ and $\hat{\psi}_\epsilon(0)=1$. As before, we study separately the contributions of the discrete and the continuous spectrum.

4.5.1 The contribution of the discrete spectrum

Let us denote by $e(\mathcal{H},R)$ the average of the normalized error term on the geodesic, evaluated at the parameter $R=\log(X+U)=\log(X+\sqrt{X^2-1})$, i.e.

$$e(\mathcal{H},R)=:\int_{\mathcal{H}}\frac{e(\mathcal{H},X;z)}{X^{1/2}}ds(z),$$

and we consider the convolution

$$\left(e(\mathcal{H},\cdot)*\psi_\epsilon\right)(R)=:\int_{-\infty}^{+\infty}\psi_\epsilon(R-Y)e(\mathcal{H},Y)dY.$$

In order to prove an Ω-result for the average $\int_{\mathcal{H}}e(\mathcal{H},X;z)ds$, it suffices to prove an Ω-result for the convolution $(e(\mathcal{H},\cdot)*\psi_\epsilon)(R)$. Using Proposition 4.2.1, Stirling's asymptotic (4.7), Theorem 4.1.3 and the properties of ψ we calculate the contribution of the discrete spectrum in $(e(\mathcal{H},\cdot)*\psi_\epsilon)(R)$ is given by

$$\begin{aligned}&\sum_{t_j>0}|\hat{u}_j|^2\Re\left(G(t_j)\Gamma(it_j)\int_{-\infty}^{+\infty}\psi_\epsilon(Y-R)e^{it_jY}dY\right)\\&+O\left(e^{-2R}\sum_{t_j>0}\frac{|\hat{u}_j|^2}{t_j^2}\int_{-\infty}^{+\infty}\psi_\epsilon(Y-R)e^{it_jY}dY+e^{-\sigma R}\right)\\&=\sum_{t_j>0}|\hat{u}_j|^2\Re\left(G(t_j)\Gamma(it_j)e^{it_jR}\right)\hat{\psi}_\epsilon(t_j)+O\left(e^{-\sigma R}\right).\end{aligned}$$

Let $A>1$. We split the sum of the above main term for $t_j\geq A$ and $t_j<A$. Using the bound

$$\hat{\psi}_\epsilon(t_j)=O_k((\epsilon|t_j|)^{-k}) \tag{4.20}$$

for every $k\geq 1$, for $t_j\geq A$ we get

$$\sum_{t_j\geq A}|\hat{u}_j|^2\Re\left(G(t_j)\Gamma(it_j)e^{it_jR}\right)\hat{\psi}_\epsilon(t_j)=O_k(\epsilon^{-k}A^{-k}).$$

For the partial sum of the series we apply Lemma 2.4.1 to the sequence e^{it_jR} and Theorem 4.1.3. Given T large we find an R sufficiently large such that the contribution of the discrete spectrum in the convoluted error term $(e(\mathcal{H},\cdot)*\psi_\epsilon)(R)$ takes the form

$$\sum_{t_j<A}|\hat{u}_j|^2\Re\left(G(t_j)\Gamma(it_j)\right)\hat{\psi}_\epsilon(t_j)+O_k\left(T^{-1}\log A+\epsilon^{-k}A^{-k}+e^{-\sigma R}\right) \tag{4.21}$$

with $M\ll R\ll MT^n\ll MT^{A^2}$. The balance $A\log A=T$, $\log M\asymp\epsilon^{-1}$, $\epsilon^{-2}=A$ implies $\log\log R\asymp\log(\epsilon^{-1})$ and for $\epsilon\leq 1$:

$$T^{-1}\log A+\epsilon^{-k}A^{-k}+e^{-\sigma R}=O(\epsilon+e^{-\sigma R}).$$

From Lemma 4.2.2 we conclude the sum in (4.21) is positive. On the other hand there exists one $\tau\in(0,1)$ such that $\hat{\psi}(x)\geq 1/2$ for $|x|\leq\tau$. Since $\hat{\psi}_\epsilon(t_j)=\hat{\psi}(\epsilon t_j)$, we get

$$\begin{aligned}\sum_{t_j<A}\Re\left(G(t_j)\Gamma(it_j)\right)\hat{\psi}_\epsilon(t_j)|\hat{u}_j|^2 &\gg \sum_{t_j<\tau/\epsilon}\Re\left(G(t_j)\Gamma(it_j)\right)|\hat{u}_j|^2\\&\gg\sum_{t_j<\tau/\epsilon}t_j^{-1}|\hat{u}_j|^2.\end{aligned}$$

When Γ is cocompact or has sufficiently many cusp forms in the sense of Definition 4.1.4, we have

$$\sum_{t_j<\tau/\epsilon} t_j^{-1}|\hat{u}_j|^2 \gg \log(\epsilon^{-1}) \gg \log\log R.$$

We conclude that the contribution of the discrete spectrum in $e(\mathcal{H}, R)$ is $\Omega_+(\log\log R)$. This implies that if Γ is cocompact or has sufficiently many cusp forms in the sense of Definition 4.1.4, the contribution of the discrete spectrum in $\int_{\mathcal{H}} e(\mathcal{H}, X; z)ds$ is $\Omega_+(X^{1/2}\log\log\log X)$. In particular, this completes the proof of Theorem 4.1.5 for Γ cocompact.

4.5.2 The contribution of the continuous spectrum

The contribution of the continuous spectrum in $(e(\mathcal{H}, \cdot) * \psi_\epsilon)(R)$ is given by the quantity

$$\sum_{\mathfrak{a}} \frac{1}{4\pi}\int_{-\infty}^{\infty} |\hat{E}_{\mathfrak{a}}(1/2+it)|^2 \Re\left(G(t)\Gamma(it)e^{iRt}F\left(-\frac{1}{2},\frac{3}{2};1+it;\frac{1}{e^{2R}+1}\right)\right)\hat{\psi}_\epsilon(t)dt.$$

Let $\chi_{\mathcal{H},\mathfrak{a}}(t)$ denote the function

$$\chi_{\mathcal{H},\mathfrak{a}}(t) = |\hat{E}_{\mathfrak{a}}(1/2+it)|^2 - |\hat{E}_{\mathfrak{a}}(1/2)|^2. \tag{4.22}$$

Thus the contribution of cusp $\mathfrak{a}$ in $(e(\mathcal{H}, \cdot) * \psi_\epsilon)(R)$ splits in

$$\frac{|\hat{E}_{\mathfrak{a}}(1/2)|^2}{4\pi}\int_{-\infty}^{\infty} \Re\left(G(t)\Gamma(it)e^{iRt}F\left(-\frac{1}{2},\frac{3}{2};1+it;\frac{1}{e^{2R}+1}\right)\right)\hat{\psi}_\epsilon(t)dt$$
$$+\frac{1}{4\pi}\int_{-\infty}^{\infty} \chi_{\mathcal{H},\mathfrak{a}}(t)\Re\left(G(t)\Gamma(it)e^{iRt}F\left(-\frac{1}{2},\frac{3}{2};1+it;\frac{1}{e^{2R}+1}\right)\right)\hat{\psi}_\epsilon(t)dt.$$

Let γ be the contour $\gamma = \bigcup_{i=1}^{6} C_i$ defined in the proof of Lemma 4.4.1. The function $\psi_\epsilon(x)$ is compactly supported in the interval $[-\epsilon, \epsilon]$. Hence, appyling the Paley-Wiener Theorem [43, Theorem 7.4] we deduce that the holomorphic Fourier transform of $\psi_\epsilon(x)$:

$$\hat{\psi}_\epsilon(z) = \int_{-\infty}^{\infty} \psi_\epsilon(x)e^{-ixz}dx$$

is an entire function of type ϵ, i.e. $|\hat{\psi}_\epsilon(z)| \ll e^{\epsilon|z|}$, and it is square-integrable over horizontal lines:

$$\int_{-\infty}^{\infty} |\hat{\psi}_\epsilon(v+iu)|^2 dv < \infty.$$

For fixed $\epsilon > 0$ we have

$$\int_{-\infty}^{\infty} |\hat{\psi}_\epsilon(v+iu)|^2 dv = \epsilon^{-1} \int_{-\infty}^{\infty} |\hat{\psi}(v+i\epsilon u)|^2 dv$$

and since $\int_{-\infty}^{\infty} |\hat{\psi}(v+i\epsilon u)|^2 dv$ converges uniformly to $\int_{-\infty}^{\infty} |\hat{\psi}(v)|^2 dv$ as $\epsilon \to 0$ we get

$$\int_{-\infty}^{\infty} |\hat{\psi}_\epsilon(v+i/2)^2 dv \ll \epsilon^{-1}. \tag{4.23}$$

Consider the integral

$$\int_\gamma G(z)\Gamma(iz)e^{iRz} F\left(-\frac{1}{2}, \frac{3}{2}; 1+iz; \frac{1}{e^{2R}+1}\right) \hat{\psi}_\epsilon(z) dz.$$

The integrand is holomorphic inside the contour. Working as in the proof of Lemma 4.4.1 and appyling Cauchy-Schwarz inequality and bound (4.23) for the integral over C_3 we deduce

$$\begin{aligned}\int_{-\infty}^{\infty} G(t)\Gamma(it)e^{iRt} F\left(-\frac{1}{2}, \frac{3}{2}; 1+it; \frac{1}{e^{2R}+1}\right) \hat{\psi}_\epsilon(t) dt &= \pi G(0)\hat{\psi}_\epsilon(0)\left(1+O\left(e^{-2R}\right)\right) \\ &\quad +O(\epsilon^{-1}e^{-R/2}).\end{aligned}$$

To prove part (a) of Theorem 4.1.5, we notice that if

$$\int_{-T}^{T} |\hat{E}_{\mathfrak{a}}(1/2+it)|^2 dt \ll \frac{T}{(\log T)^{1+\delta}},$$

then the function

$$H_1(t) = \chi_{\mathcal{H},\mathfrak{a}}(t) G(t)\Gamma(it) F\left(-\frac{1}{2}, \frac{3}{2}; 1+it; \frac{1}{e^{2R}+1}\right) \hat{\psi}_\epsilon(t)$$

is in $L^1(\mathbb{R})$ independently of ϵ and R. To obtain this we notice that $\chi_{\mathcal{H},\mathfrak{a}}(t)\Gamma(it)$ remains bounded close to $t=0$, we use the trivial bound $\hat{\psi}_\epsilon(t) \leq 1$, Proposition 4.2.1 and we estimate

$$\begin{aligned}\int_{-\infty}^{\infty}|H_1(t)|dt &\ll \int_{-1}^{1}|H_1(t)|dt+\sum_{n=0}^{\infty}2\int_{2^n}^{2^{n+1}}|t|^{-1}|\hat{E}_{\mathfrak{a}}(1/2+it)|^2dt\\ &\ll \int_{-1}^{1}|H_1(t)|dt+\sum_{n=0}^{\infty}2^{-n}\int_{2^n}^{2^{n+1}}|\hat{E}_{\mathfrak{a}}(1/2+it)|^2dt \qquad (4.24)\\ &\ll \int_{-1}^{1}|H_1(t)|dt+\sum_{n=0}^{\infty}\frac{1}{(n+1)^{1+\delta}}=O(1).\end{aligned}$$

Applying the Riemann–Lebesgue Lemma we conclude that

$$\lim_{R\to\infty}\int_{-\infty}^{\infty}H_1(t)e^{iRt}dt=0. \qquad (4.25)$$

Since $\hat{\psi}_\epsilon(0)=1$ and $\pi G(0)=4\pi^{-1/2}|\Gamma(3/4)|^2$, the contribution of the continuous spectrum in $(e(\mathcal{H},\cdot)*\psi_\epsilon)(R)$ takes the form

$$\frac{1}{\pi^{3/2}}|\Gamma(3/4)|^2\sum_{\mathfrak{a}}|\hat{E}_{\mathfrak{a}}(1/2)|^2+O(\epsilon^{-1}e^{-R/2})+o(1). \qquad (4.26)$$

As in the discrete spectrum (see the balance after expansion (4.21)) we choose the balance $\epsilon^{-1}\ll\log R\ll\log\log X$. Hence (4.26) takes the form

$$\frac{1}{\pi^{3/2}}|\Gamma(3/4)|^2\sum_{\mathfrak{a}}|\hat{E}_{\mathfrak{a}}(1/2)|^2+O(X^{-1/2}\log\log X)+o(1). \qquad (4.27)$$

In particular, this completes the proof of part (a) of Theorem 4.1.5.

To prove part (b), we first notice that in this case the contribution from the discrete spectrum is $c(R)+O_k\left(T^{-1}\log A+\epsilon^{-k}A^{-k}+e^{-\sigma R}\right)$, where $c(R)=\Omega_+(1)$ if there exists one $\hat{u}_j\neq 0$ and $c(R)$ vanishes otherwise. The contribution of the continuous spectrum takes the form

$$\begin{aligned}\frac{1}{\pi^{3/2}}|\Gamma(3/4)|^2\sum_{\mathfrak{a}}|\hat{E}_{\mathfrak{a}}(1/2)|^2 &+ O(\epsilon^{-1}e^{-R/2})\\ &+ \epsilon^{-1}\int_{-\infty}^{\infty}H_2(t)e^{iRt}dt \qquad (4.28)\end{aligned}$$

where, using Theorem 4.1.3 and estimate (4.20), we deduce

$$H_2(t)=\epsilon H_1(t)$$

is in $L^1(\mathbb{R})$ independently of ϵ and R. Applying Riemann–Lebesgue Lemma we conclude that $(e(\mathcal{H},\cdot) * \psi_\epsilon)(R)$ is given by

$$\pi^{-3/2}|\Gamma(3/4)|^2 \sum_{\mathfrak{a}} |\hat{E}_{\mathfrak{a}}(1/2)|^2 + O(\epsilon^{-1}e^{-R/2}) + \epsilon^{-1}Q(R),$$

with $Q(R) = o(1)$ as $R \to \infty$. We choose the balance $\epsilon^{-2} = A$. For $\epsilon = \epsilon_0$ sufficiently small and fixed and letting $R, T \to \infty$ the contribution of the continuous spectrum takes the form

$$\pi^{-3/2}|\Gamma(3/4)|^2 \sum_{\mathfrak{a}} |\hat{E}_{\mathfrak{a}}(1/2)|^2 + o(1),$$

which is $\Omega_+(1)$ if and only if $\hat{E}_{\mathfrak{a}}(1/2) \neq 0$ for at least one cusp $\mathfrak{a}$. The statement of part (b) follows.

Remark 4.5.1. For part (a) of Theorem 4.1.5, even if Γ has not sufficiently small Eisenstein periods associated to $\mathcal{H}$ but we have sufficiently many cusp forms, we can derive the $\Omega_+(X^{1/2}\log\log\log X)$ bound if we have a polynomial bound for the derivatives of the Eisenstein series on the critical line

$$\int_{-T}^{T} \left| \frac{d}{dt} E_{\mathfrak{a}}(z, 1/2+it) \right|^2 \ll |T|^M. \tag{4.29}$$

Since $\chi_{\mathcal{H},\mathfrak{a}}(t) = \hat{E}_{\mathfrak{a}}(1/2+it)\hat{E}_{\mathfrak{a}}(1/2-it) - |\hat{E}_{\mathfrak{a}}(1/2)|^2$ is smooth, we conclude that the function $H_1(t)$ is smooth. By bound (4.20), for all $k \geq 2$ we get that $\epsilon^k H_1(t)$ is in $L^1(\mathbb{R})$ independently of ϵ.

Using (4.29) and estimates for the derivatives of the Γ-function and the hypergeometric function we find a sufficiently large N such that $\epsilon^N H_1'(t)$ is in $L^1(\mathbb{R})$ independently of ϵ. A quantitative version of the Riemann–Lebesgue Lemma states that if $f \in C^1(\mathbb{R}) \cap L^1(\mathbb{R})$ and $f' \in L^1(\mathbb{R})$ then

$$\int_{-\infty}^{\infty} f(t)e^{iRt}dt = O(R^{-1})$$

as $R \to \infty$. In this case we conclude

$$\int_{-\infty}^{\infty} H_1(t)e^{iRt}dt = O(\epsilon^{-N}R^{-1}). \tag{4.30}$$

The previous balance $\epsilon^{-1} \ll \log R \ll \epsilon^{-2}$ finishes the proof.

4.5.3 An arithmetic case: the modular group

In this subsection we concentrate to $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. The set of primitive indefinite quadratics forms $Q(x,y) = ax^2 + bxy + cy^2$ in two variables (that means $(a,b,c) = 1$ and $b^2 - 4ac = d > 0$ is not a square) is in one-to-one correspondence with the set of primitive hyperbolic elements of Γ (see [63, p. 232]). Here we describe this correspondence.

The automorphs of Q is the cyclic group $\mathrm{Aut}(Q) \subset \mathrm{SL}_2(\mathbb{Z})$ which fixes Q, under the action

$$\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} = \gamma^t \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \gamma.$$

Let M_Q be a generator of $\mathrm{Aut}(Q)$. In Proposition 5.2.1 we will see that the correspondence $Q \to M_Q$ is bijective between indefinite integral quadratic forms in two variables and primitive hyperbolic elements of the modular group. Denote by $\mathcal{H}_Q$ the conjugacy class of M_Q and by ℓ_Q the M_Q-invariant geodesic. Define

$$r(Q,n) = \#(\{(x,y) \in \mathbb{Z}^2 : Q(x,y) = n\}/\mathrm{Aut}(Q)),$$

and let $\zeta(Q,s)$ denote the Epstein zeta function

$$\zeta(Q,s) = \sum_{n=1}^{\infty} \frac{r(Q,n)}{n^s}, \tag{4.31}$$

which is absolutely convergent in $\Re(s) > 1$. Hecke proved that the Eisenstein period $\hat{E}_{\mathfrak{a}}(s)$ along a normalized segment of ℓ_Q satisfies

$$\hat{E}_{\mathfrak{a}}(s) = \frac{d^{s/2}\Gamma^2(s/2)}{\zeta(2s)\Gamma(s)}\zeta(Q,s). \tag{4.32}$$

(see [71, eq. (9.5)]). The functional equation of the Eisenstein series implies the functional equation of the Epstein zeta function:

$$d^{(1-s)/2}\Gamma^2\left(\frac{1-s}{2}\right)\pi^{s-1}\zeta(Q,1-s) = d^{s/2}\Gamma^2\left(\frac{s}{2}\right)\pi^{-s}\zeta(Q,s).$$

In particular, the functional equation implies the convexity bound on the critical line:

$$\zeta(Q, 1/2 + it) \ll_\epsilon (1 + |t|)^{1/2+\epsilon}, \quad t \in \mathbb{R}. \tag{4.33}$$

Tsuzuki [71, Proposition 84] noticed that any subconvexity bound of the form

$$\zeta(Q, 1/2 + it) \ll_\epsilon (1 + |t|)^{\delta} \tag{4.34}$$

with $\delta < 1/2$ fixed implies the asymptotic

$$\sum_{|t_j|<T} |\hat{u}_j|^2 \sim \frac{\mu(\ell)}{\pi} \cdot T.$$

We remark that in this case we can deduce the stronger statement that Γ has sufficiently small Eisenstein periods; in fact

$$\int_{-T}^{T} |\hat{E}_{\mathfrak{a}}(1/2 + it)|^2 dt \ll T^{1-\alpha}$$

for some fixed $\alpha > 0$. To prove this, we use the bound

$$|\zeta(1 + 2it)|^{-1} \ll (\log |t|)^7$$

as $|t| \to \infty$ and Stirling's formula, which imply

$$\frac{|\Gamma^2(1/4 + it/2)|}{|\Gamma(1/2 + it)|} \ll (1 + |t|)^{-1/2}.$$

Thus

$$\hat{E}_{\mathfrak{a}}(1/2 + it) \ll (1 + |t|)^{-1/2} (\log |t|)^7 \zeta(Q, 1/2 + it) \ll_\epsilon (1 + |t|)^{\delta - 1/2 + \epsilon}$$

for every $\epsilon > 0$. We conclude

$$\int_{-T}^{T} |\hat{E}_{\mathfrak{a}}(1/2 + it)|^2 dt \ll_\epsilon T^{2\delta+\epsilon}$$

for every $\epsilon > 0$, and the statement follows.

As a fact, the subconvexity bound (4.34) holds with $\delta = 1/3 + \epsilon$. Indeed, the Epstein zeta function $\zeta(Q, s)$ is a linear combination of L-functions $L(s, \chi)$, where χ runs through the class group characters of the number field $Q(\sqrt{d})$. If χ is real, then $L(s, \chi)$

factors into two Dirichlet L-functions and the bound follows from the convexity bound for Dirichlet L-functions. If χ is complex then $L(s,\chi)$ is an L-function associated to a Maass form of eigenvalue $1/4$, i.e. an $L(1/2,u_j)$ (the last following by automorphic induction).

In particular, in any case we conclude that if $\mathcal{H}$ is the conjugacy class of the element M_Q, then

$$\int_{\ell_Q} e(\mathcal{H}_Q,X;z)ds(z) = \Omega_+(X^{1/2}\log\log\log X).$$

In the next chapter we have a much more detailed study of the correspondence described above between primitive hyperbolic elements of the modular group and indefinite quadratic forms to obtain explicit arithmetic applications of our results for the conjugacy class problem.

4.6 Pointwise Ω-results for the error term

In this section we prove Proposition 4.1.6 and Theorem 4.1.7, where we consider pointwise Ω-results for the error term $e(\mathcal{H},X;z)$. We first start with the study of the discrete average.

4.6.1 The discrete spectrum

For $K>0$ an integer we pick $z_1,z_2,...,z_K$ points on the invariant closed geodesic ℓ of $\mathcal{H}$. Assume also that z_i are equally spaced and $\rho(z_{i+1},z_i)=\delta$, hence $\delta=\mu(\ell)/K$. For $R=\log(X+\sqrt{X^2-1})$ we define the quantity

$$N_K(\mathcal{H},R) = \frac{1}{K}\sum_{m=1}^{K}\frac{e(\mathcal{H},X;z_m)}{X^{1/2}}$$

and we consider the convolution

$$\left(\psi_\epsilon * N_K(\mathcal{H},\cdot)\right)(R) \;=\; \int_{-\infty}^{\infty}\psi_\epsilon(R-Y)N_K(\mathcal{H},Y)dY.$$

Using Proposition 4.2.1, the properties of ψ_ϵ, Theorem 2.1.6 and Theorem 4.1.3 we conclude

$$\left(\psi_\epsilon * N(\mathcal{H},\cdot)_K\right)(R) = \sum_{t_j>0}\hat{u}_j\left(\frac{1}{K}\sum_{m=1}^{K}u_j(z_m)\right)\Re\left(G(t_j)\Gamma(it_j)e^{it_jR}\right)\hat{\psi}_\epsilon(t_j) + O(e^{-\sigma R}).$$

For $A > 1$, we apply the estimate (4.20) to bound the tail of the series for $t_j > A$. Using Stirling's formula, Theorem 2.1.6, Theorem 4.1.3 and estimate 4.20 for $k \geq 1$ we conclude

$$\sum_{t_j > A} t_j^{-1} \hat{u}_j \left(\frac{1}{K} \sum_{m=1}^{K} u_j(z_m) \right) \hat{\psi}_\epsilon(t_j) = O_k(\epsilon^{-k} A^{1/2-k}).$$

The partial sum of the series can be handled as follows: by the definition of the period integral $\hat{u}_j$ we get

$$\frac{\mu(\ell)}{K} \sum_{m=1}^{K} u_j(z_m) = \sum_{m=1}^{K} u_j(z_m)\delta \to \overline{\hat{u}_j}$$

uniformly, for every $j = 1, ..., n$ (where n is such that $t_n \leq A < t_{n+1}$, hence $n \asymp A^2$). That means for every small $\epsilon_1 > 0$ there exists a $K_0 = K_0(\epsilon_1) \geq 1$ such that

$$\hat{u}_j \left(\frac{1}{K} \sum_{m=1}^{K} u_j(z_m) \right) = \frac{|\hat{u}_j|^2}{\mu(\ell)} + O\left(\epsilon_1 \hat{u}_j\right). \tag{4.35}$$

for every $K \geq K_0$. We get

$$\begin{aligned} \left(\psi_\epsilon * N(\mathcal{H}, \cdot)_K\right)(R) &= \frac{1}{\mu(\ell)} \sum_{t_j \leq A} |\hat{u}_j|^2 \Re\left(G(t_j)\Gamma(it_j)e^{it_jR}\right) \hat{\psi}_\epsilon(t_j) \\ &\quad + O\left(\epsilon_1 \sum_{t_j \leq A} \hat{u}_j \Re\left(G(t_j)\Gamma(it_j)e^{it_jR}\right) \hat{\psi}_\epsilon(t_j)\right) \\ &\quad + O_k(\epsilon^{-k} A^{1/2-k} + e^{-\sigma R}). \end{aligned}$$

We now proceed as in the proof of Theorem 4.1.5. Using Theorem 4.1.3 the O-term is bounded by $O(\epsilon_1 A^{1/2})$. For the main term, apply Dirichlet's principle (Lemma 2.4.1) to the exponentials e^{it_jR}. For every M and T we can find $M \ll R \ll MT^{A^2}$ such that

$$\begin{aligned} \left(\psi_\epsilon * N(\mathcal{H}, \cdot)_K\right)(R) &= \frac{1}{\mu(\ell)} \sum_{t_j \leq A} |\hat{u}_j|^2 \Re\left(G(t_j)\Gamma(it_j)\right) \hat{\psi}_\epsilon(t_j) \\ &\quad + O_k(\epsilon^{-k} A^{1/2-k} + T^{-1} \log A + \epsilon_1 A^{1/2} + e^{-\sigma R}). \end{aligned}$$

The balance $\epsilon^{-1} = A^{1-3/(2k+2)}$, $\epsilon_1 = A^{-1/2}\epsilon$ implies the above O-term is $O(T^{-1} \log A + \epsilon + e^{-\sigma R})$. By Lemma 4.2.2, the coefficients of the above sum are all positive. For the function ψ, pick $\tau \in (0, 1)$ as in the proof of Theorem 4.1.5 (hence

$\hat{\psi}(x) \geq 1/2$ for $|x| \leq \tau$). Using this and the fact that $\hat{\psi}_\epsilon(t_j) = \hat{\psi}(\epsilon t_j)$, in the case that Γ is cocompact or has sufficiently many cusp forms in the sense of Definition 4.1.4, we can bound the above sum from below by

$$\frac{1}{\mu}\sum_{t_j \leq A} |\hat{u}_j|^2 \Re\left(G(t_j)\Gamma(it_j)\right)\hat{\psi}_\epsilon(t_j) \gg \log(\epsilon^{-1}).$$

We deduce that for every $\epsilon > 0$ we can find a sufficiently large $K = K(\epsilon)$ such that

$$\left(\psi_\epsilon * N_K(\mathcal{H},\cdot)\right)(R) = k(\epsilon) + O(\epsilon + e^{-\sigma R}).$$

with $k(\epsilon) = \Omega_+(\log(\epsilon^{-1}))$. If Γ is cocompact, choosing $\epsilon = \epsilon_0$ sufficiently small and $K = K(\epsilon_0)$ sufficiently large, for $R, T \to \infty$ we conclude Proposition 4.1.6.

4.6.2 The continuous spectrum

The contribution of the continuous spectrum in the convolution $\left(\psi_\epsilon * N_K(\mathcal{H},\cdot)\right)(R)$ is given by

$$\begin{aligned}\sum_{\mathfrak{a}} \frac{1}{4\pi}\int_{-\infty}^{\infty} \quad & \hat{E}_{\mathfrak{a}}(1/2+it)\left(\frac{1}{K}\sum_{m=1}^{K} E_{\mathfrak{a}}(z_m, 1/2+it)\right) \\ & \cdot\Re\left(G(t)\Gamma(it)F\left(-\frac{1}{2},\frac{3}{2};1+it;\frac{1}{e^{2R}+1}\right)e^{itR}\right)\hat{\psi}_\epsilon(t)dt.\end{aligned} \tag{4.36}$$

For $A > 0$, we split the integrals for $|t| \leq A$ and $|t| > A$. By Theorem 4.1.3, asymptotic (4.8) and (4.20) for $k \geq 1$ it follows that the contribution of $|t| > A$ in the above integral is $O(\epsilon^{-k}A^{1/2-k})$.

For $|t| \leq A$, for any small $\epsilon_2 > 0$ we approximate the Eisenstein period integral as

$$\frac{1}{K}\sum_{m=1}^{K} E_{\mathfrak{a}}(z_m, 1/2+it) = \hat{E}_{\mathfrak{a}}(1/2-it) + O(\epsilon_2) \tag{4.37}$$

for every $K \geq K_0$ with $K_0 = K_0(\epsilon_2)$ sufficiently large. The contribution of the continuous spectrum (4.36) takes the form

$$\sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{|t|\leq A} |\hat{E}_{\mathfrak{a}}(1/2+it)|^2 \Re\left(G(t)\Gamma(it)e^{iRt}F\left(-\frac{1}{2},\frac{3}{2};1+it;\frac{1}{e^{2R}+1}\right)\right)\hat{\psi}_\epsilon(t)dt$$
$$+O\left(\epsilon_2 \sum_{\mathfrak{a}} \int_{|t|\leq A} \hat{E}_{\mathfrak{a}}(1/2+it)G(t)\Gamma(it)e^{iRt}F\left(-\frac{1}{2},\frac{3}{2};1+it;\frac{1}{e^{2R}+1}\right)\hat{\psi}_\epsilon(t)dt\right)$$
$$+O\left(\epsilon^{-k}A^{1/2-k}\right). \tag{4.38}$$

By subsection 4.5.2 and Theorem 4.1.3, the first summand of (4.38) takes the form

$$\frac{1}{\pi^{3/2}}|\Gamma(3/4)|^2 \sum_{\mathfrak{a}} |\hat{E}_{\mathfrak{a}}(1/2)|^2 + O(\epsilon^{-1}Q_1(R)+\epsilon^{-k}A^{-k}), \tag{4.39}$$

with $Q_1(R)\to 0$ as $R\to\infty$. For the second summand of (4.38), we let $\theta_{\mathcal{H},\mathfrak{a}}(t)$ denote the function

$$\theta_{\mathcal{H},\mathfrak{a}}(t) = \hat{E}_{\mathfrak{a}}(1/2+it) - \hat{E}_{\mathfrak{a}}(1/2) \tag{4.40}$$

and we split the integral as

$$\sum_{\mathfrak{a}} \hat{E}_{\mathfrak{a}}(1/2) \int_{|t|\leq A} G(t)\Gamma(it)e^{iRt}F\left(-\frac{1}{2},\frac{3}{2};1+it;\frac{1}{e^{2R}+1}\right)\hat{\psi}_\epsilon(t)dt \tag{4.41}$$
$$+\sum_{\mathfrak{a}} \int_{|t|\leq A} \theta_{\mathcal{H},\mathfrak{a}}(t)G(t)\Gamma(it)e^{iRt}F\left(-\frac{1}{2},\frac{3}{2};1+it;\frac{1}{e^{2R}+1}\right)\hat{\psi}_\epsilon(t)dt.$$

The first summand of (4.41) can be handled by calculating the contour integral

$$\int_\gamma G(z)\Gamma(iz)e^{iRz}F\left(-\frac{1}{2},\frac{3}{2};1+iz;\frac{1}{e^{2R}+1}\right)\hat{\psi}_\epsilon(z)dz.$$

where γ is the contour $\gamma = \bigcup_{i=1}^{6} C_i$ defined in the proof of Lemma 4.4.1 for $M=A$. We conclude it is bounded by $O(1+\epsilon^{-1}e^{-R/2})$. The second summand of (4.41) is trivially estimated to be $O(\log A)$. Thus, the contribution of the continuous spectrum in $(\psi_\epsilon * N_K(\mathcal{H},\cdot))(R)$ is

$$\frac{1}{\pi^{3/2}}|\Gamma(3/4)|^2 \sum_{\mathfrak{a}} |\hat{E}_{\mathfrak{a}}(1/2)|^2 + O(\epsilon^{-1}Q_1(R)+\epsilon^{-k}A^{-k+1/2}+\epsilon_2+\epsilon_2\epsilon^{-1}e^{-R/2}+\epsilon_2\log A).$$

Choosing $\epsilon_2 = \epsilon^2$ and $\epsilon^{-1} = A^{1-3/(2k+2)}$ as before we conclude the O-term is $O(\epsilon^{-1}Q_1(R)+\epsilon)$. If Γ has at least one $\hat{u}_j \neq 0$ with $\lambda_j > 1/4$ then for fixed and sufficiently small ϵ the contribution of the discrete spectrum in $(\psi_\epsilon * N_K(\mathcal{H},\cdot))(R)$ is $\Omega_+(1)$. If Γ has at least one nonzero Eisenstein period integral then for fixed and sufficiently small ϵ we get that the contribution of the continuous spectrum in $(\psi_\epsilon * N_K(\mathcal{H}, \cdot))(R)$ is also $\Omega_+(1)$. This completes the proof of Proposition 4.1.6.

4.6.3 Proof of Theorem 4.1.7

As we have already mentioned, parts (a) and (c) of Theorem 4.1.7 follow immediately from Proposition 4.1.6 and Theorem 4.1.2. In this subsection we complete the proof of the theorem by proving part (b). For the rest of this subsection we define $Y = X + \sqrt{X^2 - 1}$ and $y = x + \sqrt{x^2 - 1}$. Here we deduce pointwise Ω_--results for the error term by studying a discrete average of the mean

$$\frac{1}{X}\int_1^X \frac{e(\mathcal{H}, x; z)}{x^{1/2}} dx$$

on the geodesic ℓ (actually, we will work with a modification of this error with Y in the place of X, see Proposition 4.6.3). Notice that here x is not the distance. To prove this average result we have to apply a combination of the arguments in Chapter 2 with the arguments in the proofs of Theorems 4.1.5 and 4.1.6, hence we only sketch the basic steps.

We will use the first of the fixed sign properties of Γ-function stated in (4.13).

Lemma 4.6.1. *For every $t \in \mathbb{R} - \{0\}$, we have:*

$$\Re\left(\frac{\Gamma(it)}{\Gamma(3/2 + it)(1 + it)} a(t)\right) < 0. \tag{4.42}$$

This fixed sign property can either be deduced working as in the proof of Lemma 2.3.1 or it can also be verified using Mathematica.

We will also need the following lemma for the Huber transform. Compare this with Lemmas 2.4.2 and 4.4.1.

Lemma 4.6.2. *As $Y \to \infty$ we have*

$$\int_{-\infty}^{\infty} \frac{1}{Y}\int_1^Y \frac{2d(f_x, t)}{x^{1/2}} dy dt \to \frac{4}{\sqrt{\pi}}|\Gamma(3/4)|^2.$$

Proof. Using expressions (4.11) and (4.12) we write

$$\int_{-\infty}^{\infty}\frac{1}{Y}\int_1^Y\frac{2d(f_x,t)}{x^{1/2}}dydt = \Re\left(\int_{-\infty}^{\infty}\frac{1}{Y}\int_1^Y G(t)\Gamma(it)e^{irt}F\left(-\frac{1}{2},\frac{3}{2};1+it;\frac{1}{e^{2r}+1}\right)dydt\right),$$

hence $e^r=y$. We consider the contour integral

$$\int_\gamma G(z)\Gamma(iz)\frac{1}{Y}\int_1^Y e^{irz}F\left(-\frac{1}{2},\frac{3}{2};1+iz;\frac{1}{e^{2r}+1}\right)dydz,$$

where γ is the contour defined in the proof of Lemma 4.4.1. At $z=0$ we have the simple pole of $\Gamma(iz)$ with $\mathrm{Res}_{z=0}\Gamma(iz)=-i$. Applying Stirling's formula and the asymptotics of the hypergeometric function (4.12) we get

$$\int_{C_2+C_4}G(z)\Gamma(iz)\frac{1}{Y}\int_1^Y e^{irz}F\left(-\frac{1}{2},\frac{3}{2};1+iz;\frac{1}{e^{2r}+1}\right)dydz = O\left(M^{-1}\right),$$

$$\int_{C_3}G(z)\Gamma(iz)\frac{1}{Y}\int_1^Y e^{irz}F\left(-\frac{1}{2},\frac{3}{2};1+iz;\frac{1}{e^{2r}+1}\right)dydz = O\left(Y^{-1/2}\right).$$

For $\varepsilon\to 0$ the term

$$\int_{C_6}G(z)\Gamma(iz)\frac{1}{Y}\int_1^Y e^{irz}F\left(-\frac{1}{2},\frac{3}{2};1+iz;\frac{1}{e^{2r}+1}\right)dydz$$

converges to

$$-i\pi G(0)\frac{1}{Y}\int_1^Y F\left(-\frac{1}{2},\frac{3}{2};1;\frac{1}{e^{2r}+1}\right)dy\mathrm{Res}_{z=0}\Gamma(iz)=-\pi G(0)\left(1+O(Y^{-1})\right).$$

Using Cauchy's Theorem and for $M\to\infty$ we conclude

$$\lim_{Y\to\infty}\int_{-\infty}^{\infty}\frac{1}{Y}\int_1^Y\frac{2d(f_x,t)}{x^{1/2}}dydt = \frac{4}{\sqrt{\pi}}|\Gamma(3/4)|^2.$$

□

We can now prove the following result.

Proposition 4.6.3. *Let Γ be either (i) cocompact or (ii) cofinite but not cocompact, $\hat{u}_j\neq 0$ for at least one $\lambda_j>1/4$ and $\hat{E}_{\mathfrak{a}}(1/2)=0$ for all cusps $\mathfrak{a}$. For $Y=X+\sqrt{X^2-1}$ and $y=x+\sqrt{x^2-1}$ let*

$$M_{\mathcal{H},z}(X)=\frac{1}{Y}\int_1^Y\frac{e(\mathcal{H},x;z)}{x^{1/2}}dy.$$

Then there exist an integer K and $z_1, z_2, ..., z_K$ points in ℓ such that, as $X \to \infty$:

$$\frac{1}{K}\sum_{m=1}^{K} M_{\mathcal{H},z_m}(X) = \Omega_{-}(1).$$

Proof. Assume first that Γ is cocompact. We pick $z_1, z_2, ..., z_K$ equally spaced points on the invariant closed geodesic ℓ of $\mathcal{H}$ and $\rho(z_{i+1}, z_i) = \delta$, hence $\delta = \mu(\ell)/K$. Using Proposition 4.2.1, Theorem 2.1.6 and Theorem 4.1.3 we conclude

$$\frac{1}{K}\sum_{m=1}^{K} M_{\mathcal{H},z_m}(X) = \sum_{t_j>0} \hat{u}_j \left(\frac{1}{K}\sum_{m=1}^{K} u_j(z_m)\right) \Re\left(G(t_j)\Gamma(it_j)\frac{1}{Y}\int_1^Y e^{it_j r}dy\right) + O(Y^{-\sigma}),$$

For $A > 1$, we apply the estimate (4.20) and we bound the tail of the series by

$$\sum_{t_j>A} t_j^{-1}\hat{u}_j \left(\frac{1}{K}\sum_{m=1}^{K} u_j(z_m)\right)\frac{Y^{it_j}}{1+it_j} = O(A^{-1/2}).$$

For the partial sum of the series, we approximate the period integral $\hat{u}_j$ uniformly, for every $j = 1, ..., n$ (where $n \asymp A^2$). For any $\epsilon_1 > 0$ we find a $K_0 = K_0(\epsilon_1) \geq 1$ such that for every $K \geq K_0$:

$$\hat{u}_j \left(\frac{1}{K}\sum_{m=1}^{K} u_j(z_m)\right) = \frac{|\hat{u}_j|^2}{\mu(\ell)} + O\left(\epsilon_1 \hat{u}_j\right). \tag{4.43}$$

We get

$$\begin{aligned}\frac{1}{K}\sum_{m=1}^{K} M_{\mathcal{H},z_m}(X) &= \frac{1}{\mu(\ell)}\sum_{t_j\leq A} |\hat{u}_j|^2 \Re\left(G(t_j)\Gamma(it_j)\frac{Y^{it_j}}{1+it_j}\right)\\ &\quad + O(Y^{-1} + \epsilon_1 + A^{-1/2} + Y^{-\sigma}).\end{aligned}$$

For the main term, apply Dirichlet's principle (Lemma 2.4.1) to the exponentials $e^{it_j R} = Y^{it_j}$. For each T we can find $R \ll T^{A^2}$ such that

$$\begin{aligned}\frac{1}{K}\sum_{m=1}^{K} M_{\mathcal{H},z_m}(X) &= \frac{1}{\mu(\ell)}\sum_{t_j\leq A} |\hat{u}_j|^2 \Re\left(\frac{G(t_j)\Gamma(it_j)}{1+it_j}\right)\\ &\quad + O(T^{-1} + \epsilon_1 + A^{-1/2} + Y^{-\sigma}).\end{aligned}$$

By Theorem 4.1.3, as $A \to \infty$ the sum remains bounded and, for Γ cocompact, there exist infinitely many j's such that $\hat{u}_j \neq 0$. By Lemma 4.6.1, all the nonzero terms are negative. Hence, there exists an A_0 such that for every $A \geq A_0$:

$$\left| \sum_{t_j \leq A} |\hat{u}_j|^2 \Re \left(\frac{G(t_j)\Gamma(it_j)}{1+it_j} \right) \right| \gg 1. \tag{4.44}$$

Choosing T, Y and A fixed and sufficiently large and ϵ_1 fixed and sufficiently small, we can choose a $K = K_0$ fixed such that

$$\frac{1}{K} \sum_{m=1}^{K} M_{\mathcal{H}, z_m}(X) \quad = \quad \Omega_-(1).$$

Notice that the lower bound (4.44) holds if and only if there exists at least one nonzero $\hat{u}_j$ with $\lambda_j > 1/4$.

Assume now that Γ is not cocompact. In this case, the contribution of the discrete spectrum in

$$\frac{1}{K} \sum_{m=1}^{K} M_{\mathcal{H}, z_m}(X)$$

is given by

$$\frac{1}{K} \sum_{m=1}^{K} \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{1}{Y} \int_0^Y \frac{2d(f_x, t)}{x^{1/2}} dy \hat{E}_{\mathfrak{a}}(1/2+it) E_{\mathfrak{a}}(z_m, 1/2+it) dt. \tag{4.45}$$

We cut the integral for $|t| \leq A$ and $|t| > A$. In the interval $|t| \leq A$ we approximate the Eisenstein period ϵ_2-close. Applying Lemma 4.6.2 and following a standard calculation, expansion (4.45) takes the form

$$\begin{aligned} \frac{|\Gamma(3/4)|^2}{\pi^{3/2}} \sum_{\mathfrak{a}} |\hat{E}_{\mathfrak{a}}(1/2)|^2 \quad &+ \quad \Re \left(\sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \chi_{\mathcal{H}, \mathfrak{a}}(t) \frac{G(t)\Gamma(it)}{1+it} Y^{it} dt \right) \\ &+ \quad O(A^{-1/2} + \epsilon_2 + Y^{-1}), \end{aligned}$$

with $K = K(\epsilon_2, A)$. Since for Γ all the Eisenstein periods $\hat{E}_{\mathfrak{a}}(1/2)$ vanish, applying Riemann–Lebesgue Lemma for the second term the proposition follows for A, Y sufficiently large and ϵ_2 sufficiently small. □

Proposition 4.6.3 implies that there exists a point $z_{\mathcal{H}} \in \ell$ such that we have the pointwise Ω_--result:

$$\frac{1}{Y} \int_1^Y \frac{e(\mathcal{H}, x; z_{\mathcal{H}})}{x^{1/2}} dy = \Omega_-(1).$$

Part (b) of Theorem 4.1.7 follows.

Remark 4.6.4. We also notice here that for Proposition 4.1.6 and part (a) of Theorem 4.1.7, instead of the part (a) of Definition 4.1.4, we could had only make the weaker assumption that the series

$$\sum_{0<t_j} \frac{|\hat{u}_j|^2}{t_j}$$

diverges.

Chapter 5

Arithmetic applications

In this chapter we will focus on some arithmetic applications of the hyperbolic lattice point problems. In particular, we are interested in applications on counting solutions of (definite or indefinite) quadratic forms, as well as applications to arithmetic functions. For the classical problem, applications of this type have been studied in [55], [41] and [7]. In the last section we extend our results by applying the theory of Hecke operators.

5.1 Review of the arithmetic applications for the classical problem

We first review some arithmetic applications of the classical problem worked out by Iwaniec and Chamizo in [41, Chapter 12], [7]. Assume first that $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ and $z = w = i$. Then, for

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we can easily see that $N(X; i, i)$ counts sums of four squares under the determinant condition:

$$N(X; i, i) = \# \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : \|\gamma\|_2^2 := a^2 + b^2 + c^2 + d^2 \leq X \right\}. \tag{5.1}$$

The modular group does not have small eigenvalues [41, Corollary 11.5], hence, applying Theorem 2.2.1 we deduce the following corollary.

Proposition 5.1.1. *The asymptotic behaviour of the quantity $N(X;i,i)$ defined in eq. (5.1) as $X \to \infty$ is*

$$N(X;i,i) = 6X + O(X^{2/3}). \tag{5.2}$$

We now look for applications for correlation sums of $r(n)$, the function counting the number of ways n can be written as sum of two squares. Let $\Gamma' \subset \mathrm{PSL}_2(\mathbb{Z})$ be the group of the transformations such that $a \equiv d \mod 2$ and $b \equiv c \mod 2$. Then Γ' is conjugate to $\Gamma_0(2)$. In particular, it is of index 3 in the modular group and

$$\Gamma' = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}^{-1} \Gamma_0(2) \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}. \tag{5.3}$$

Using the transformations $a+d = 2k, a-d = 2\ell, b+c = 2m, b-c = 2n$, we are counting k,ℓ,m,n such that $k^2+n^2 = \ell^2+m^2+1$ and $k^2+n^2+\ell^2+m^2 \leq X/2$. We conclude

$$N(4X+2;i,i) = \sum_{m \leq X} r(m)r(m+1). \tag{5.4}$$

Since $\Gamma_0(2)$ has no small eigenvalues [41, Corollary 11.5], Theorem 2.2.1 implies the following asymptotic.

Proposition 5.1.2. *The correlation sum of the arithmetic function $r(n)$ satisfies the asymptotic*

$$\sum_{m \leq X} r(m)r(m+1) = 8X + O(X^{2/3}). \tag{5.5}$$

In comparison with the Gauss estimate (1.1) in the Euclidean circle problem, Proposition 5.1.2 indicates a significant cancellation for the correlation sum of $r(n)$. For the modular group and its subgroups, the theory of Hecke operators allows us to extend these results to transformations acting on the point $z = i$ satisfying $ad - bc = n$.

We first briefly review the basic elements from the theory of the Hecke operators, see [41, section 8.5, chapter 12]. For $n \in \mathbb{N}$, let $T_n : \mathcal{A}(\Gamma\backslash\mathbb{H}) \to \mathcal{A}(\Gamma\backslash\mathbb{H})$ be the n-th

Hecke operator defined by

$$T_n(f)(z) = \frac{1}{\sqrt{n}} \sum_{\tau \in \Gamma \backslash \Gamma_n} f(\tau z),$$

where Γ_n is the set

$$\Gamma_n = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}^{2\times 2} : ad - bc = n \right\}.$$

As the Hecke operators commute with Δ, we choose a joint orthonormal basis u_j. We denote by $\lambda_j(n)$ the eigenvalue of T_n for $u_j(z)$, i.e.

$$T_n u_j(z) = \lambda_j(n) u_j(z),$$

and $\eta_t(n)$ for the Eisenstein series, i.e.

$$T_n E_\infty(z, 1/2 + it) = \eta_t(n) E_\infty(z, 1/2 + it),$$

where

$$\eta_t(n) = \sum_{ad=n} \left(\frac{a}{d}\right)^{it}.$$

For level $N \geq 1$ we have $\mathcal{A}(\Gamma_0(1)\backslash\mathbb{H}) \subset \mathcal{A}(\Gamma_0(N)\backslash\mathbb{H})$. For the rest of this Chapter we assume that $(n, N) = 1$. Every operator T_n acts on $\mathcal{A}(\Gamma_0(N)\backslash\mathbb{H})$.

For $n \geq 1$ we define the counting function

$$P_n(X) = \#\{\gamma \in \Gamma_n : \|\gamma\|_2^2 := a^2 + b^2 + c^2 + d^2 \leq X\}. \tag{5.6}$$

We have the following result.

Proposition 5.1.3. *The asymptotic behaviour of the quantity $P_n(X)$ as $X \to \infty$ is*

$$P_n(X) = \frac{6\sigma(n)}{n} X + O\left(\frac{\sigma(n)}{n^{2/3}} X^{2/3}\right), \tag{5.7}$$

uniformly in n.

Proof. Define the kernel

$$K_n(z,w) = \sum_{\gamma\in\Gamma_n} k(\gamma z, w). \tag{5.8}$$

where $k(\gamma z, w) = k(u(\gamma z, w))$ is the characteristic function of $[0,(X-2)/4]$. For $\gamma \in \Gamma_n$ we have

$$4u(\gamma i, i) + 2 = \frac{a^2+b^2+c^2+d^2}{n} \leq \frac{X}{n} \tag{5.9}$$

hence the counting function $P_n(X)$ counts points in the orbit $\Gamma_n \cdot i$ in a disc of radius X/n. It turns out that we can determine the asymptotic behaviour of $P_n(X)$ from an estimate for the kernel $K_n(z,w)$. For this reason, we apply T_n on both expressions of the kernel $K(z,w) = K_1(z,w)$. Applying T_n to the spectral expansion (2.10) we get

$$\begin{aligned} T_n K(z,w) &= \sum_j \lambda_j(n) h(t_j) u_j(z) \overline{u_j(w)} \\ &+ \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^{\infty} h(t)\eta_t(n) E_\infty(z,1/2+it)\overline{E_\infty(w,1/2+it)}dt. \end{aligned} \tag{5.10}$$

On the geometric side, we have

$$\begin{aligned} T_n K(z,w) &= \frac{1}{\sqrt{n}} \sum_{\tau\in\Gamma\backslash\Gamma_n} \left(\sum_{\gamma\in\Gamma} k(\tau\gamma z, w) \right) \\ &= \frac{1}{\sqrt{n}} \sum_{\gamma'\in\Gamma_n} k(\gamma' z, w) = \frac{1}{\sqrt{n}} K_n(z,w). \end{aligned} \tag{5.11}$$

From eq. (5.10) and (5.11) we conclude

$$\begin{aligned} K_n(z,w) &= \sqrt{n} \sum_j \lambda_j(n) h(t_j) u_j(z) \overline{u_j(w)} \\ &+ \sum_{\mathfrak{a}} \frac{\sqrt{n}}{4\pi} \int_{-\infty}^{\infty} h(t)\eta_t(n) E_\infty(z,1/2+it)\overline{E_\infty(w,1/2+it)}dt. \end{aligned} \tag{5.12}$$

Using the bound

$$|\lambda_j(n)| \leq \lambda_0(n) = \frac{\sigma(n)}{\sqrt{n}}, \tag{5.13}$$

(see [41, eq. (8.35]) and working as in the case of Γ_1 we finish the proof. □

Let Γ' be the group defined in 5.3. For $(n,2) = 1$ the action of T_n on the Γ'-automorphic forms is non-trivial, hence we can repeat the previous argument to conclude the following result.

Proposition 5.1.4. *For n odd and $X \geq n \geq 1$ we have*

$$\sum_{m \leq X} r(m)r(m+n) = \frac{8\sigma(n)}{n} X + O\left(\frac{\sigma(n)}{n^{2/3}} X^{2/3}\right). \tag{5.14}$$

For n even the asymptotic behaviour of this sum is more complicated. We refer to [8] for a study of this problem.

Working with various subgroups of $\mathrm{SL}_2(\mathbb{Z})$ and for various different points (among them, Heegner points on the modular curve) Chamizo [7, Section 3] has obtained various arithmetic results of this type for correlations of arithmetic functions and solutions of (both definite and indefinite) quadratic forms.

The applications we described so far are about definite quadratic forms in four variables and $r(n)$. Working with the modular group, Patterson in [55], obtained applications of the lattice counting problem for a certain indefinite quadratic form in three variables: for $D > 0$ fixed he counted the number of primitive integral solutions of

$$4AC - B^2 = D$$

under the restriction $0 < A + C \leq X$. The number $n_D(X)$ of such solutions satisfies

$$n_D(X) = \frac{h(D)}{\sqrt{D}} X + O(X^{2/3}), \tag{5.15}$$

where $h(D)$ is the class number of D.

5.2 Arithmetic corollaries for the conjugacy class problem

In this section we use Huber's geometric interpretation to study some arithmetic consequences of our results in Chapters 3, 4. More specifically, for $\mathcal{H}$ a hyperbolic class of $\mathrm{PSL}_2(\mathbb{Z})$, we interpret the quantity $N(\mathcal{H}, X; z)$ in terms of the number of solutions of indefinite quadratic forms in four variables with restrictions. We also use the theory of Hecke operators to generalize our results for the case $ad - bc = n$.

Fix a point z in $\mathbb{H}$. Huber's interpretation in [38] shows that $N(\mathcal{H}, X; z)$, perhaps after a conjugation, counts γ in $\Gamma/\langle g\rangle$ such that $\cos v \geq X^{-1}$, where v is the angle defined by the ray from 0 to γz and the geodesic $\{yi, y > 0\}$. Denote by ℓ_1 the geodesic from γz perpendicular to $\{yi, y > 0\}$ and by $\mu(\ell_1)$ its length. Then

$$\rho(\gamma z, \{iy, y > 0\}) = \mu(\ell_1) = \int_{\pi/2-v}^{\pi/2} \csc(t)\, dt = \log\left(\frac{1+\sin v}{\cos v}\right). \tag{5.16}$$

On the other hand, the distance of γz to the imaginary axis is given by

$$\cosh \rho(\gamma z, i|\gamma z|) = \frac{|\gamma z|}{\Im(\gamma z)}. \tag{5.17}$$

For

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

we define

$$f_z(A,B,C,D) = \cosh \rho(\gamma z, i|\gamma z|) = \frac{|\gamma z|}{\Im(\gamma z)} = \frac{|Az+B||Cz+D|}{\Im(z)}. \tag{5.18}$$

Using $|Cz+D| = |C\bar{z}+D|$ and $AD - BC = 1$, f_z can be written in the form

$$f_z(A,B,C,D) = \frac{\left(\left(AC|z|^2 + BD + AD\Re(z) + BC\Re(z)\right)^2 + \Im(z)^2\right)^{1/2}}{\Im(z)}. \tag{5.19}$$

Using (5.16), the condition $\cos v \geq X^{-1}$ can now be written as

$$f_z(A,B,C,D) \leq \cosh\left(\log(X + \sqrt{X^2-1})\right) = X. \tag{5.20}$$

Let $z = \alpha + \beta i$. Inequality (5.20) takes the form

$$\left|(\alpha^2+\beta^2)AC + BD + \alpha AD + \alpha BC\right| \leq \beta\sqrt{X^2-1} = \beta X + O(X^{-1}). \tag{5.21}$$

To get simple results, we take specific choices for z.

5.2.1 Quadratic forms

For the basics of indefinite quadratic forms we refer to [63]. Let $Q(x,y) = ax^2 + bxy + cy^2$ be a primitive indefinite quadratic form in two variables, i.e. $(a,b,c) = 1$ and $b^2 - 4ac = d > 0$ is not a square. We denote Q by $[a,b,c]$. Two forms $[a,b,c]$, $[a',b',c']$

are called equivalent ($[a,b,c] \sim [a',b',c']$) if there is a $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that

$$\begin{pmatrix} a' & b'/2 \\ b'/2 & c' \end{pmatrix} = \gamma^t \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \gamma.$$

The automorphs of Q is the group $\mathrm{Aut}(Q) = \Gamma \subset \mathrm{PSL}_2(\mathbb{Z})$ which fixes Q, under the action above. This group is infinite and cyclic, with generator

$$M_{[a,b,c]} = \begin{pmatrix} \frac{t_0 - bu_0}{2} & -cu_0 \\ au_0 & \frac{t_0+bu_0}{2} \end{pmatrix},$$

where $t_0, u_0 > 0$ is the fundamental solution of Pell's equation $x^2 - dy^2 = 4$. Since $t_0 > 2$, the matrix $M_{[a,b,c]}$ is hyperbolic. We denote by ε_d the quantity

$$\varepsilon_d = \frac{t_0 + \sqrt{d}u_0}{2}.$$

The quadratic form Q is associated with two real quadratic numbers

$$\theta_1 = \frac{-b+\sqrt{d}}{2a}, \quad \theta_2 = \frac{-b-\sqrt{d}}{2a}, \tag{5.22}$$

the roots of the polynomial $a\theta^2 + b\theta + c$.

The main reason we are interested in primitive indefinite quadratic forms in two variables is that, according to the next proposition, they are in one-to-one correspondence with primitive hyperbolic conjugacy classes of the modular group (a proof of this result can be found in [63, p. 232]).

Proposition 5.2.1. *Define the map ϕ by*

$$\phi([a,b,c]) = M_{[a,b,c]}. \tag{5.23}$$

Then:
(a) ϕ commutes with the action of $SL_2(\mathbb{Z})$: $[a,b,c] \sim [a',b',c']$ if and only if $M_{[a,b,c]}$ is conjugate to $M_{[a',b',c']}$.
(b) ϕ is a bijective map of the set of primitive indefinite quadratic forms onto the set of primitive hyperbolic elements of $SL_2(\mathbb{Z})$.

The matrix $M_{[a,b,c]}$ has eigenvalues:

$$\lambda_{1,2} = \frac{t_0 \pm \sqrt{d}u_0}{2} = \varepsilon_d^{\pm 1}. \tag{5.24}$$

Its diagonalization is

$$M_{[a,b,c]} = T \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} T^{-1}, \quad T = \begin{pmatrix} \theta_1 & \theta_2 \\ 1 & 1 \end{pmatrix}. \tag{5.25}$$

Calculations imply that

$$M^n_{[a,b,c]} = \frac{1}{\theta_1 - \theta_2} \begin{pmatrix} \lambda_1^n\theta_1 - \lambda_2^n\theta_2 & (\lambda_2^n - \lambda_1^n)\theta_1\theta_2 \\ \lambda_1^n - \lambda_2^n & \lambda_2^n\theta_1 - \lambda_1^n\theta_2 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}). \tag{5.26}$$

Let us first examine a simple case. Consider the case $b = 0$, i.e. we consider the form $Q(x, y) = ax^2 + cy^2$. Set $\theta = \theta_1 = -\theta_2 = \sqrt{d}/2a$. Thus, $\theta^2 = -c/a > 0$. In this case, $M^n_{[a,b,c]}$ takes the form

$$M^n_{[a,0,c]} = M^n = \frac{1}{2\theta} \begin{pmatrix} (\lambda_1^n + \lambda_2^n)\theta & (\lambda_1^n - \lambda_2^n)\theta^2 \\ \lambda_1^n - \lambda_2^n & (\lambda_1^n + \lambda_2^n)\theta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}). \tag{5.27}$$

We now prove an application of Theorems 3.1.1 and 3.1.2.

Proposition 5.2.2. *Given $Q(x, y) = ax^2 + cy^2$ with $\theta^2 = -c/a > 0$ and $-ac$ not a square, let F_Q denote the indefinite quadratic form*

$$F_Q(\alpha, \beta, \gamma, \delta) = \alpha^2 - \frac{a}{c}\beta^2 + \frac{c}{a}\gamma^2 - \delta^2.$$

Let $P(X)$ be the number of solutions $(\alpha, \beta, \gamma, \delta) \in \mathbb{Z}^4$ such that $\alpha\delta - \beta\gamma = 1$ and

$$|F_Q(\alpha, \beta, \gamma, \delta)| \le X,$$

under the equivalence: $(\alpha, \beta, \gamma, \delta) \sim (\alpha', \beta', \gamma', \delta')$ iff there exists an integer n such that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = M^n \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}.$$

Here M^n is given by (5.27). Then:

(a) $P(X)$ satisfies

$$P(X) = \frac{6 \log \varepsilon_d}{\pi} X + E(X),$$

with ϵ_d given by (5.24) and $E(X) = O(X^{2/3})$.
(b) the error term $E(X)$ satisfies the average bound

$$\frac{1}{X} \int_X^{2X} |E(x)|^2 dx \ll X \log^2 X,$$

where the '$\ll$' constant depends on the quadratic form Q.

Proof. For (a), let ℓ be the invariant geodesic of M. Conjugating Γ with T, we bring ℓ on the imaginary axis. Let also $z = T^{-1}(i)$. In this case we are counting the number of

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z})/\langle M \rangle \simeq T^{-1}\mathrm{PSL}_2(\mathbb{Z})T/G,$$

where G is the cyclic group

$$G = \left\langle \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} \epsilon_d & 0 \\ 0 & \epsilon_d^{-1} \end{pmatrix} \right\rangle,$$

such that asymptotically

$$|AC + BD| \leq X.$$

For

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = T^{-1} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} T \in T^{-1}\mathrm{PSL}_2(\mathbb{Z})T$$

we get

$$2|AC + BD| = |F_Q(\alpha, \beta, \gamma, \delta)|.$$

Thus, $N(\mathcal{H}, X; z)$ counts points of $\mathrm{PSL}_2(\mathbb{Z})$ under the extra equivalence that comes from the quotient with $\langle M \rangle$ such that

$$|F_Q(\alpha, \beta, \gamma, \delta)| \leq 2X,$$

i.e. $P(X) = N(\mathcal{H}, X/2; z)$. By Theorem 3.4.4 and the fact that $\mathrm{PSL}_2(\mathbb{Z})$ has no eigenvalues $\lambda \in (0, 1/4)$, we get

$$P(X) = \hat{u}_0 u_0(z) X + O(X^{2/3}).$$

For $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ we have

$$u_0(z) = \sqrt{\frac{3}{\pi}}, \quad \hat{u}_0 = \sqrt{\frac{3}{\pi}} \frac{\mu}{\nu}.$$

Since M is primitive we have $\nu = 1$ and, for $\mathrm{PSL}_2(\mathbb{Z})$, we know that $\mu = 2 \log \varepsilon_d$, which equals the length of the closed geodesic ℓ (see for example [63, Corollary 1.5]). Part (a) now follows. Part (b) follows immediately as $E(X) = E(\mathcal{H}, X/2; z)$. □

We note that, for $z = i$, by (5.18) we count also solutions of the product of two definite quadratic forms:

$$(A^2 + B^2)(C^2 + D^2) \leq X^2,$$

with specific restrictions, as above (see also [61, Theorem 6.1]).

We have dealt with the simple case that $Q(x, y) = ax^2 + cy^2$ (i.e. $b = 0$), $z = T^{-1}(i)$. Obviously, one would like to generalize Proposition 5.2.2 for the general case $b \neq 0$ and $z \neq T^{-1}(i)$. The general case is not particularly different. The form F_Q takes a more complicated form that comes from the inequality (5.18) and the form of the matrix T.

Proposition 5.2.3. *Let z be a fixed point in $\mathbb{H}$ and $Q(x, y) = ax^2 + bxy + cy^2$ a definite quadratic form. Let $M_{[a,b,c]}$ and θ_1, θ_2, the two quadratic real numbers corresponding to Q, defined in (5.22). Denote by $F_{Q,z}$ the indefinite quadratic form*

$$F_{Q,z}(\alpha, \beta, \gamma, \delta) = |z|^2 AC + BD + \Re(z) AD + \Re(z) BC,$$

where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = T^{-1} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} T$$

and T given by 5.25. Denote with $P_{Q,z}(X)$ the number of solutions $(\alpha, \beta, \gamma, \delta) \in \mathbb{Z}^4$ such that $\alpha\delta - \beta\gamma = 1$ and

$$|F_{Q,z}(\alpha, \beta, \gamma, \delta)| \leq X,$$

under the equivalence $\sim$ such that: $(\alpha,\beta,\gamma,\delta)\sim(\alpha',\beta',\gamma',\delta')$ iff there exists an integer n such that

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = M_{[a,b,c]}^{n}\begin{pmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{pmatrix}.$$

Then:

(a) $P_{Q,z}(X)$ has the asymptotic behaviour

$$P_{Q,z}(X) = \frac{12\log\varepsilon_d}{\pi}X + E_{Q,z}(X),$$

where $E_{Q,z}(X) = O(X^{2/3})$.

(b) $E_{Q,z}(X)$ satisfies the average bound

$$\frac{1}{X}\int_X^{2X} |E_{Q,z}(x)|^2 dx \ll X\log^2 X,$$

where the '$\ll$' constant depends on the quadratic form Q and the point z.

Proof. The proof is similar to the proof of Proposition 5.2.2, where we interpret $P_{Q,z}(X) = N(\mathcal{H}, X, T^{-1}(z))$ for $\mathcal{H}$ the conjugacy class of $M_{[a,b,c]}$. □

For the conjugacy class problem we have deduced applications only for indefinite quadratic forms: the quadratic forms $F_{Q,z}$ of Propositions 5.2.2, 5.2.3. This reflects the fact that we count the distance $\rho(\gamma z, \{iy, y>0\})$ which is related with the $\cos v$ of the angle v (eq. (5.16)).

We come back to the case $b = 0$ and z as in the proof of Proposition 5.2.2. Let $\theta^2 = -c/a = k \in \mathbb{N}$. Using the transformation $\alpha+\delta = 2A$, $\alpha-\delta = 2\Delta$, $k\beta+\gamma = 2B$, $k\beta-\gamma = 2\Gamma$, we notice that we count the integer solutions (A, B, Γ, Δ) such that

$$\left|A\Delta + \frac{B\Gamma}{k}\right| \leq X,$$

under the determinant condition

$$A^2 - \Delta^2 + \frac{B^2-\Gamma^2}{k} = 1,$$

as well as the extra conditions that come from $\sim$: $(A, B, \Gamma, \Delta) \sim (A', B', \Gamma', \Delta')$ iff

$$\begin{pmatrix} A+\Delta & \frac{B+\Gamma}{k}\\ B-\Gamma & A-\Delta\end{pmatrix} = M^n \begin{pmatrix} A'+\Delta' & \frac{B'+\Gamma'}{k}\\ B'-\Gamma' & A'-\Delta'\end{pmatrix} \tag{5.28}$$

for some integer n. For $m \in \mathbb{N}$, define the set

$$A_k(m) = \{(\Gamma, \Delta) \in \mathbb{Z} : m = k\Delta^2 + \Gamma^2\}.$$

The following immediate corollary of Proposition 5.2.2 can be viewed as the analogue of Theorem 12.3 in [41] for the lattice point problem in conjugacy classes.

Proposition 5.2.4. *Fix $k \in \mathbb{N}$, and let $P_k(X)$ denote the number of integer solutions (A, B, Γ, Δ) of*

$$\left| A\Delta + \frac{B\Gamma}{k} \right| \leq X$$

such that $(\Gamma, \Delta) \in A_k(m)$ iff $(A, B) \in A_k(m+k)$ and under the equivalence condition $\sim$ of 5.28. Then:
(a) $P_k(X)$ has the asymptotic behaviour

$$P_k(X) = \frac{6 \log \varepsilon_d}{\pi} X + E_k(X),$$

where $E_k(X) = O(X^{2/3})$.
(b) $E(X)$ satisfies the average bound

$$\frac{1}{X} \int_X^{2X} |E(x)|^2 dx \ll X \log^2 X,$$

where the '$\ll$' constant depends on k.

Since $\mathrm{PSL}_2(\mathbb{Z})$ has $\lambda_1 > 1/4$ and all Eisenstein periods $\hat{E}_{\mathfrak{a}}(1/2) = 0$, the subconvexity bound (4.34) implies that the error terms $E(X)$, $E_{Q,z}(X)$ and $E_k(X)$ satisfy also part (a) of Theorem 4.1.5 and part (b) of Theorem 4.1.7.

Working with the one-sheeted hyperboloid model, Duke, Rudnick and Sarnak [18] interpret the counting function $N(\mathcal{H}, X; z)$ in terms of the arithmetic function $r(n)$. They prove that for d not a square

$$\sum_{|n| \leq X} r(dn^2 + 1) \sim c(d) \cdot X, \tag{5.29}$$

for some constant $c(d) > 0$, whereas for d square the asymptotic of (5.29) as $X \to \infty$ is $\sim c(d) \cdot X \log X$. The case of the non-square d corresponds to the conjugacy class problem.

5.2.2 Hecke operators

Applying Hecke operators as for the classical lattice point counting problem, we can count solutions of $|F(\alpha,\beta,\gamma,\delta)|\leq X$ lying in the hypersurface $\alpha\delta-\beta\gamma=n$, with $n>1$. Notice that counting solutions

$$|F_Q(\alpha,\beta,\gamma,\delta)|\leq X$$

with $\alpha\delta-\beta\gamma=n$ is equivalent to counting solutions

$$|f_{z=i}(A,B,C,D)|\leq nX$$

with $AD-BC=n$, where f_z is defined in (5.19).

We apply T_n on both expressions of $A(f)(z)$. Applying T_n to the spectral expansion (3.27) we get

$$T_nA(f)(z)=\sum_j c(f,t_j)\lambda_j(n)u_j(z)+\frac{1}{4\pi}\int_{-\infty}^{\infty}c_\infty(f,t)\eta_t(n)E_\infty(z,1/2+it)dt. \tag{5.30}$$

On the geometric side, we have

$$T_nA(f)(z)=\frac{1}{\sqrt{n}}\sum_{\tau\in\Gamma\backslash\Gamma_n}\left(\sum_{\gamma\in\mathcal{H}}f\left(\frac{\cosh\rho(\tau^{-1}\gamma\tau z,z)-1}{\cosh\mu(\gamma)-1}\right)\right).$$

If $\mathcal{H}$ is the conjugacy class of the primitive hyperbolic matrix M, we define the set

$$\mathcal{H}_n=\{\gamma^{-1}M\gamma:\gamma\in\Gamma_n\}.$$

The set $\mathcal{H}_n$ is in one-to-one correspondence with the quotient set $\Gamma_n/\langle M\rangle$. Notice also that $\mu(\tau\gamma\tau^{-1})=\mu(\gamma)$. Therefore,

$$T_nA(f)(z)=\frac{1}{\sqrt{n}}\sum_{\gamma\in\mathcal{H}_n}f\left(\frac{\cosh\rho(\gamma z,z)-1}{\cosh\mu(\gamma)-1}\right). \tag{5.31}$$

Using that $|\lambda_j(n)|\leq\lambda_0(n)=\sigma(n)n^{-1/2}$ and the uniform bound $|\eta_t(n)|\ll d(n)\ll\lambda_0(n)$ we conclude the following result.

Proposition 5.2.5. *Denote with $P_{Q,n}(X)$ the number of solutions $(\alpha,\beta,\gamma,\delta)\in\mathbb{Z}$ such* *that $\alpha\delta - \beta\gamma = n$ and*

$$|F_Q(\alpha, \beta, \gamma, \delta)| \leq X,$$

under the equivalence $\sim$ such that: $(\alpha, \beta, \gamma, \delta) \sim (\alpha', \beta', \gamma', \delta')$ iff there exists an integer m such that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = M^m_{[a,b,c]} \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}.$$

Then
(a) $P_{Q,n}(X)$ has the asymptotic behaviour

$$P_{Q,n}(X) = \frac{6 \log \varepsilon_d}{\pi} \frac{\sigma(n)}{n} X + E_n(X),$$

with

$$E_n(X) = O\left(\frac{\sigma(n)}{n^{2/3}} X^{2/3}\right).$$

(b) $E_n(X)$ satisfies the bound

$$\frac{1}{X} \int_X^{2X} |E_n(x)|^2 dx \ll \sigma^2(n) \frac{X}{n} \log^2 \left(\frac{X}{n}\right),$$

where the '$\ll$' constant depends on the quadratic form Q.

An analogous result can be easily proved for the general case covered in Proposition 5.2.3.

Chapter 6

Conclusions

The goal of this PhD thesis was to study two different lattice point problems in the hyperbolic plane and provide evidence towards the conjectures that the error terms of their counting functions satisfy square root cancellation.

The first problem is the classical hyperbolic lattice point problem, which is the hyperbolic analogue of the Gauss circle problem. The second problem is the hyperbolic lattice point problem in the conjugacy classes, which is related to counting distances between the orbit of a point from a closed geodesic. We denoted the error term of the classical problem by $E(X; z, w)$ and the error term of the conjugacy class problem by $E(\mathcal{H}, X; z)$. When we subtract the contribution of the eigenvalue $\lambda_j = 1/4$ from $E(X; z, w)$ and $E(\mathcal{H}, X; z)$ we denote their differences by $e(X; z, w)$ and $e(\mathcal{H}, X; z)$, respectively. Thus, we have the conjecture

$$E(X; z, w) = O_\epsilon(X^{1/2+\epsilon}),$$

which is equivalent with the bound $e(X; z, w) = O_\epsilon(X^{1/2+\epsilon})$. We also make the conjecture

$$E(\mathcal{H}, X; z) = O_\epsilon(X^{1/2+\epsilon}),$$

which is equivalent with the bound $e(\mathcal{H}, X; z) = O_\epsilon(X^{1/2+\epsilon})$.

In the two following tables we summarize some of the previously known results and the most important of our results on these two hyperbolic lattice counting problems.

Pointwise (Selberg [65] et. al.): Γ cofinite: $E(X;z,w)=O(X^{2/3})$
Radial second moment (Chamizo [7]): Γ cofinite: $\int_X^{2X} \lvert E(x;z,w)\rvert^2 dx \ll X^2\log^2 X$
Spatial second and fourth moments ($n=1,2$) (Chamizo [7]): Γ cocompact: $\int_{\Gamma\backslash\mathbb{H}} \lvert E(X;z,w)\rvert^{2n} d\mu(z)=O(X^n\log^{2n}X)$
Geodesic average: Γ cofinite: $\int_{\ell_0} E(X;z,w)ds(z)=O(X^{1/2}\log X)$
Mean value in distance r (Phillips-Rudnick [60]): Γ cofinite: $\frac{1}{T}\int_0^T \frac{e(2\cosh r;z,z)}{e^{r/2}}dr \to \sum_{\mathfrak{a}} \lvert E_{\mathfrak{a}}(z,1/2)\rvert^2$
Mean value in X: $\lambda_1(\Gamma)>2.7823...$: $\frac{1}{X}\int_2^X \frac{e(x;z,w)}{x^{1/2}}dx$ does not converge
Ω-results (first result: in Phillips-Rudnick [60]): Γ cocompact/arithmetic: $e(X,z,z)=\Omega_-\left(X^{1/2}(\log\log X)^{1/4-\delta}\right)$ Γ many cusp forms and null vectors: $e(X;z,w)=\Omega_\pm(X^{1/2})$
Arithmetic applications (Chamizo [7], Iwaniec [41]): Γ arithmetic: Solutions of definite quadratic forms in four variables. $\Gamma_0(2)$: Correlation sums of $r(n)$: $\sum_{n\le X} r(n)r(n+m)$

Figure 6.1: Synopsis of results for the classical hyperbolc lattice point problem

Pointwise (Good [26]):
Γ cofinite: $E(\mathcal{H}, X; z) = O(X^{2/3})$
Radial second moment:
Γ cofinite: $\int_X^{2X} \lvert E(\mathcal{H}, x; z)\rvert^2 dx \ll X^2 \log^2 X$
Spatial second and fourth moments ($n = 1, 2$):
Γ cocompact: $\int_{\Gamma\backslash\mathbb{H}} \lvert E(\mathcal{H}, X; z)\rvert^{2n} d\mu(z) = O(X^n \log^{2n} X)$
Geodesic average:
Γ cofinite: $\int_{\ell_0} E(\mathcal{H}, X; z) ds(z) = O(X^{1/2} \log X)$
Mean value in distance r:
Γ cofinite: $\frac{1}{T}\int_0^T \frac{e(\mathcal{H}, x; z)}{x^{1/2}} dr \to \frac{\lvert\Gamma(3/4)\rvert^2}{\pi^{3/2}} \sum_{\mathfrak{a}} \hat{E}_{\mathfrak{a}}(1/2) E_{\mathfrak{a}}(z, 1/2).$
Mean value in X:
Γ has $\hat{u}_j \neq 0$ and all $\hat{E}_{\mathfrak{a}}(1/2) = 0$: $\frac{1}{X}\int_1^X \frac{e(\mathcal{H}, x; z_{\mathcal{H}})}{x^{1/2}} dx = \Omega_-(1)$
Ω-results:
Γ cocompact/$\mathrm{PSL}_2(\mathbb{Z})$: $\int_{\ell} e(\mathcal{H}, X; z) ds(z) = \Omega_+ \left(X^{1/2}(\log\log\log X)\right)$ Γ has $\hat{u}_j \neq 0$: $e(\mathcal{H}, X; z_{\mathcal{H}}) = \Omega_+(X^{1/2})$ or $\Omega_-(X^{1/2})$, sign depends on $\lvert\hat{E}_{\mathfrak{a}}(1/2)\rvert$
Arithmetic applications:
$\Gamma = \mathrm{PSL}_2(\mathbb{Z})$: Solutions of indefinite quadratic forms in four variables up to equivalence

Figure 6.2: Synopsis of results for the hyperbolc lattice point problem in conjugacy classes

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