

## EXPECTED SIGNATURE OF BROWNIAN MOTION UP TO THE FIRST EXIT TIME FROM A BOUNDED DOMAIN

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The signature of a path provides a top down description of the path in terms of its effects as a control [*Differential Equations Driven by Rough Paths* (2007) Springer]. The signature transforms a path into a group-like element in the tensor algebra and is an essential object in rough path theory. The expected signature of a stochastic process plays a similar role to that played by the characteristic function of a random variable. In [Chevyrev (2013)], it is proved that under certain boundedness conditions, the expected value of a random signature already determines the law of this random signature. It becomes of great interest to be able to compute examples of expected signatures and obtain the upper bounds for the decay rates of expected signatures. For instance, the computation for Brownian motion on  $[0, 1]$  leads to the “cubature on Wiener space” methodology [Lyons and Victoir, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **460** (2004) 169–198]. In this paper we fix a bounded domain  $\Gamma$  in a Euclidean space  $E$  and study the expected signature of a Brownian path starting at  $z \in \Gamma$  and stopped at the first exit time from  $\Gamma$ . We denote this tensor series valued function by  $\Phi_\Gamma(z)$  and focus on the case  $E = \mathbb{R}^d$ . We show that  $\Phi_\Gamma(z)$  satisfies an elliptic PDE system and a boundary condition. The equations determining  $\Phi_\Gamma$  can be recursively solved; by an iterative application of Sobolev estimates we are able, under certain smoothness and boundedness condition of the domain  $\Gamma$ , to prove geometric bounds for the terms in  $\Phi_\Gamma(z)$ . However, there is still a gap and we have not shown that  $\Phi_\Gamma(z)$  determines the law of the signature of this stopped Brownian motion even if  $\Gamma$  is a unit ball.

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Received May 2013; revised June 2014.

<sup>1</sup>Supported by EPSRC (Grant Reference: EP/F029578/1) and ERC Grant (Grant Agreement No. 291244 Esig).

<sup>2</sup>Supported by ERC Grant (Grant Agreement No. 291244 Esig).

*AMS 2000 subject classifications.* 60J60, 60J65, 60J10, 60J35, 47D03, 47D07, 35K05, 35K08, 35K10, 35K51.

*Key words and phrases.* Expected signature, rough path, diffusion, cubature.

<p>This is an electronic reprint of the original article published by the Institute of Mathematical Statistics in <i>The Annals of Probability</i>, 2015, Vol. 43, No. 5, 2729–2762. This reprint differs from the original in pagination and typographic detail.</p>
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**1. Introduction.** An essential notion in the theory of rough paths is the signature of a path, which represents the path information in terms of its effects as a control [12]. In [11], Lyons and Victoir developed a methodology called a *cubature on the Wiener Space* to numerically solve high dimensional SDEs and semi-elliptic PDEs, which can be seen as an alternative to the Monte Carlo method. The Lyons–Victoir method relies on the computation of the expected signature of the Brownian motion up to an arbitrarily fixed time  $T \in \mathbb{R}^+$ . Let  $S(B_{[0,T]})$  denote the signature of the Brownian motion up to time  $T$ . In [3], a closed-form expression for the expected signature of the Brownian motion up to time  $T$ , that is, the expectation of  $S(B_{[0,T]})$ , is given by the following:

$$(1) \quad \mathbb{E}[S(B_{[0,T]})] = \exp\left(\frac{T}{2} \sum_{i=1}^d e_i \otimes e_i\right),$$

where  $B$  is a standard  $d$ -dimensional Brownian motion with canonical basis  $(e_1, e_2, \dots, e_d)$  for  $\mathbb{R}^d$ .

In [4], Fawcett’s formula (1) has been extended to Lévy processes up to fixed time. In this paper, we consider the Brownian case when  $T$  is a certain kind of the stopping time as another extension of Fawcett’s formula (1). More specifically, we consider the first exit time of the Brownian motion  $B$  from  $\Gamma$ , denoted by  $\tau_\Gamma$ , where  $\Gamma$  is a bounded domain in  $E := \mathbb{R}^d$ . Denote the expected signature of the Brownian motion  $B$  starting at  $z \in \Gamma$  up to  $\tau_\Gamma$  by  $\Phi_\Gamma(z) := \mathbb{E}^z[S(B_{[0,\tau_\Gamma]})]$ . We prove that  $\Phi_\Gamma$  satisfies an elliptic PDE taking values in the tensor algebra space, with a boundary condition and an initial condition given as follows:

$$\begin{cases} \Delta(\Phi_\Gamma(z)) = -\left(\sum_{i=1}^d e_i \otimes e_i\right) \otimes \Phi_\Gamma(z) - 2 \sum_{i=1}^d e_i \otimes \frac{\partial \Phi_\Gamma(z)}{\partial z_i}, & \forall z \in \Gamma, \\ \lim_{t \uparrow \tau_\Gamma} \Phi_\Gamma(B_t) = \mathbf{1} \text{ a.s. } P^z, & \forall z \in \Gamma \\ \rho_0(\Phi_\Gamma(z)) = 1, \rho_1(\Phi_\Gamma(z)) = 0, & \forall z \in \bar{\Gamma}, \end{cases}$$

where  $\rho_n$  is defined in Definition 2.10.

Using this PDE, we compute each term of the expected signature recursively. In the case of the domain  $\Gamma$  being a unit disk we extend this result further, by demonstrating that the expected signature has polynomial form with common factor  $(1 - |z|^2)$ . In addition we derive a finite difference equation that characterizes the expected signature of a simple  $d$ -dimensional random walk up to the first exit time from a bounded domain.

In Section 3.5, we study the decay rate of each term in  $\Phi_\Gamma$  and prove that  $\Phi_\Gamma(z)$  is geometrically bounded if  $\Gamma$  is strongly Lipschitz and belongs to the class  $C^m$  where  $m = \lfloor \frac{d}{2} \rfloor + 1$  by using our PDE approach in conjunction with Sobolev’s theorem. This is motivated by a series of studies on whether the

law of a signature can be determined by its expectation. The first relevant result was due to Fawcett [3]. Recently this result has been extended significantly in [2], and a sufficient condition for the law of random signature  $S(X)$  to be uniquely determined by its expected value is that its expected signature is compact-like in the sense of the second version of the paper [2], that is, for every  $\delta > 0$ , there exists a positive integer  $N_\delta$ , such that for every  $n \geq N_\delta$ ,

$$\|\rho_n(\mathbb{E}[S(X)])\| < \delta^n.$$

However, the geometric boundedness of  $\Phi_\Gamma$  does not imply that  $\Phi_\Gamma(z)$  is compact-like. It is still an open question whether  $\Phi_\Gamma(z)$  determines the law of the signature of the Brownian motion up to  $\tau_\Gamma$ .

In [14], a parabolic result of the expected signature of a general time-homogenous Itô diffusion up to fixed time  $T$  is obtained using the similar PDE approach. More specifically, let  $X_t$  be a time-homogeneous  $E$ -valued Itô diffusion satisfying the following SDE:

$$dX_t = \mu(X_t) dt + V(X_t) dW_t,$$

where  $W_t$  is a standard multi-dimensional Brownian motion. Let  $\Phi(T, x)$  denote the expected signature of  $X$  from time 0 to  $T$ . Under some regularity assumptions of  $\mu$ ,  $V$  and  $\Phi$ ,  $\Phi$  satisfies the following PDE:

$$\begin{cases} \left( -\frac{\partial}{\partial T} + A \right) \Phi(T, x) + \sum_{j=1}^d \left( \sum_{j_1=1}^d b_{j_1, j}(x) e_{j_1} \right) \otimes \frac{\partial \Phi(T, x)}{\partial x_j} \\ + \left( \sum_{j=1}^d \mu^j(x) e_j + \frac{1}{2} \sum_{j_1=1}^d \sum_{j_2=1}^d b_{j_1, j_2}(x) e_{j_1} \otimes e_{j_2} \right) \otimes \Phi(T, x) = 0, \\ \Phi(0, x) = \mathbf{1}, \rho_0(\Phi(T, x)) = 1, \end{cases}$$

where  $b(x) = V(x)V(x)^T$ .

**2. Preliminary.** In this paper, we are mostly interested in describing probability measures on paths. Despite our examples being quite specific, they can be understood in a more wider context. For the sake of precision, we start by introducing some notation, making essential definitions and stating the basic results we require. These can also be found in [12]. A reader experienced in rough path theory might prefer to go directly to Section 3.

2.1. *Tensors products.* Throughout the rest of the paper, fix  $E = \mathbb{R}^d$  as the space in which paths will take their values. Then  $E$  has the basis  $\{e_1, \dots, e_d\}$ . Consider the successive tensor powers  $E^{\otimes n}$  of  $E$  (equipped with some tensor norm). If we think of the elements  $e_i$  as letters, then  $E^{\otimes n}$  is

spanned by the words of length  $n$  in the letters  $\{e_1, \dots, e_d\}$ , and can be identified with the space of real homogeneous noncommuting polynomials of degree  $n$  in  $d$  variables. We note that  $E^{\otimes 0} = \mathbb{R}$ . In order for our analysis to work it will be necessary to constrain the norms we use when considering tensor products (the injective and projective norms satisfy our constraints).

DEFINITION 2.1. We say that the tensor powers of  $E$  are endowed with an admissible norm  $|\cdot|$ , if the following conditions hold:

(1) for each  $n \geq 1$ , the symmetric group  $S_n$  acts by isometry on  $E^{\otimes n}$ , that is,

$$|\sigma v| = |v| \quad \forall v \in E^{\otimes n}, \forall \sigma \in S_n;$$

(2) the tensor product has norm 1, that is,  $\forall n, m \geq 1$ ,

$$|v \otimes w| \leq |v||w| \quad \forall v \in E^{\otimes n}, w \in E^{\otimes m}.$$

## 2.2. The algebra of tensor series.

DEFINITION 2.2. A formal  $E$ -tensor series is a sequence of tensors, denoted by  $(a_n \in E^{\otimes n})_{n \in \mathbb{N}}$ , which we write  $\mathbf{a} = (a_0, a_1, \dots)$ . There are two binary operations on  $E$ -tensor series, an addition  $+$  and a product  $\otimes$ , which are defined as follows. Let  $\mathbf{a} = (a_0, a_1, \dots)$  and  $\mathbf{b} = (b_0, b_1, \dots)$  be two  $E$ -tensor series. Then we define

$$(2) \quad \mathbf{a} + \mathbf{b} = (a_0 + b_0, a_1 + b_1, \dots)$$

and

$$(3) \quad \mathbf{a} \otimes \mathbf{b} = (c_0, c_1, \dots),$$

where for each  $n \geq 0$ ,

$$(4) \quad c_n = \sum_{k=0}^n a_k \otimes b_{n-k}.$$

The product  $\mathbf{a} \otimes \mathbf{b}$  is also denoted by  $\mathbf{ab}$ . We use the notation  $\mathbf{1}$  for the series  $(1, 0, \dots)$ , and  $\mathbf{0}$  for the series  $(0, 0, \dots)$ . If  $\lambda \in \mathbb{R}$ , then we define  $\lambda \mathbf{a}$  to be  $(\lambda a_0, \lambda a_1, \dots)$ .

DEFINITION 2.3. The space  $T((E))$  is defined to be the vector space of all formal  $E$ -tensors series.

REMARK 2.1. The space  $T((E))$  with  $+$  and  $\otimes$  and the action of  $\mathbb{R}$  is an associative and unital algebra over  $\mathbb{R}$ . An element of  $\mathbf{a} = (a_0, a_1, \dots)$  of

$T((E))$  is invertible if and only if  $a_0 \neq 0$ . Its inverse is then given by the series

$$(5) \quad \mathbf{a}^{-1} = \frac{1}{a_0} \sum_{n \geq 0} \left( \mathbf{1} - \frac{\mathbf{a}}{a_0} \right)^n,$$

which is well defined because, for each given degree, only finitely many terms of the sum produce nonzero tensors of this degree. In particular, the subset  $\{\mathbf{a} \in T((E)) | a_0 = 1\}$  forms a group.

DEFINITION 2.4. The dilation operator denoted by  $\delta_\varepsilon$  is a mapping from  $\mathbb{R}^+ \times T((E)) \rightarrow T((E))$  defined by

$$\delta_\varepsilon(\mathbf{a}) = (a_0, \varepsilon a_1, \dots, \varepsilon^n a_n, \dots) \quad \forall \varepsilon \in \mathbb{R}^+, \mathbf{a} \in T((E)).$$

DEFINITION 2.5. Let  $n \geq 1$  be an integer. Let  $B_n = \{\mathbf{a} = (a_0, a_1, \dots) | a_0 = \dots = a_n = 0\}$ . The truncated tensor algebra  $T^{(n)}(E)$  of order  $n$  over  $E$  is defined as the quotient algebra

$$(6) \quad T^{(n)}(E) = T((E))/B_n.$$

The canonical homomorphism  $T((E)) \rightarrow T^{(n)}(E)$  is denoted by  $\pi_n$ .

2.3. *Paths.* Paths also have algebraic properties:

DEFINITION 2.6. Let  $X: [r, s] \rightarrow E$  and  $Y: [s, t] \rightarrow E$  be two continuous paths. Their concatenation is the path  $X * Y$  defined by

$$(X * Y)_u = \begin{cases} X_u, & u \in [r, s], \\ X_s + Y_u - Y_s, & u \in [s, t]. \end{cases}$$

REMARK 2.2.  $*$  is an associative operation. Let  $X: [r, s] \rightarrow E, Y: [s, t] \rightarrow E$  and  $Z: [t, v] \rightarrow E$  be three continuous paths. Then

$$(X * Y) * Z = X * (Y * Z).$$

2.4. *The signature of a path.*

DEFINITION 2.7. Let  $J$  denote a compact interval. Let  $X: J \rightarrow E$  be a path of bounded variation or a rough path of finite  $p$ -variation such that the following integration makes sense. The signature of  $X$ , denoted by  $S(X_J)$ , is an element  $(1, X^1, \dots, X^n, \dots)$  of  $T((E))$  defined for each  $n \geq 1$  as follows:

$$X^n = \int \cdots \int_{u_1 < \cdots < u_n, u_1, \dots, u_n \in J} dX_{u_1} \otimes \cdots \otimes dX_{u_n}.$$

The truncated signature of  $X$  of order  $n$  is denoted by  $S^n(X)$ , that is,  $S^n(X) = \pi_n(S(X_J))$ , for every  $n \in \mathbb{Z}$ .

EXAMPLE 2.1. (1) If  $X_t$  is a continuous path with finite  $p$ -variation, where  $1 \leq p < 2$ , then its signature can be defined in the sense of the Young integral [12]. More generally, if  $X_t$  is a  $p$ -rough path ( $p \geq 1$ ), then the integrals will exist as a result of the general theory of rough paths (Theorem 3.7, page 45, [12]).

(2) For a Brownian path, its signature can be defined in the sense of the Itô integral or the Stratonovich integral. There is a simple rewriting rule that allows one to go between them [7]. Almost all Brownian paths, with their Lévy area processes, are  $p$ -rough paths for any  $p > 2$ . With probability one, for all  $(s, t)$  the Stratonovich signature agrees with the canonical rough path signature ([12], page 57).

Chen's identity is a fundamental theorem, which asserts that the signature is a homomorphism between path space and rough path space.

THEOREM 2.1. *Let  $X : [r, s] \rightarrow E$  and  $Y : [s, t] \rightarrow E$  be two continuous paths with finite  $p$ -variation where  $1 \leq p < 2$ . Then*

$$(7) \quad S(X * Y) = S(X) \otimes S(Y).$$

The proof can be found in [12].

2.5. *Real-valued functions on the signatures of paths.* We now introduce a special class of linear forms on  $T((E))$ ; see [12]. Suppose  $(e_1^*, \dots, e_i^*, \dots)$  are elements of  $E^*$ . We can introduce coordinate iterated integrals by setting

$$X_u^{(i)} := \langle e_i^*, X_u \rangle$$

and rewriting  $\langle e_{i_1}^* \otimes \dots \otimes e_{i_n}^*, S(X) \rangle$  as the scalar iterated integral of coordinate projections

$$\int \dots \int_{\substack{u_1 < \dots < u_n \\ u_1, \dots, u_n \in J}} dX_{u_1}^{(i_1)} \otimes \dots \otimes dX_{u_n}^{(i_n)}.$$

In this way, we realize  $n$ th degree coordinate iterated integrals as the restrictions of linear functionals in  $E^{\otimes n}$  to the space of signatures of paths. If  $(e_1, \dots, e_d)$  is a basis for a finite dimensional space  $E$ , and  $(e_1^*, \dots, e_d^*)$  is a basis for the dual space  $E^*$ , we can write

$$\mathbf{X}_J = \sum_{n=0}^{\infty} \sum_{i_1, \dots, i_n \in \{1, \dots, d\}} \left( \int \dots \int_{\substack{u_1 < \dots < u_n \\ u_1, \dots, u_n \in J}} dX_{u_1}^{(i_1)} \otimes \dots \otimes dX_{u_n}^{(i_n)} \right) e_{i_1} \otimes \dots \otimes e_{i_n}.$$

DEFINITION 2.8. We define  $T(E) \subset T((E))$  to be the tensor series  $\mathbf{a} = (a_0, a_1, \dots)$  for which there exists  $N$  depending on  $\mathbf{a}$  so that  $a_i = 0$  for all  $i > N$ . In other words it is the space of polynomials (instead of series) in elements of  $E$ .

REMARK 2.3. There is a natural inclusion

$$T(E^*) \rightarrow T((E))^*.$$

REMARK 2.4. For any  $\mathbf{e}^* \in T(E^*)$ , we denote by  $\mathcal{U}_{\mathbf{e}^*}$  the restriction of the linear map  $\langle \mathbf{e}^*, \cdot \rangle$  to the range of the signature. It is a real-valued function on the set of the signature of paths. It is well known that for any  $\mathbf{e}^*$  and  $\mathbf{f}^*$  in  $T(E^*)$ , the pointwise product  $\mathcal{U}_{\mathbf{e}^*}\mathcal{U}_{\mathbf{f}^*}$  equals  $\mathcal{U}_{\mathbf{e}^*\mathbf{f}^*}$  for the shuffle product  $\mathbf{e}^*\mathbf{f}^* \in T(E^*)$ . In other words any polynomial in coordinate iterated integrals can be written as a linear combination of iterated integrals [12].

REMARK 2.5. The previous remark explains why, if we have the probability measure on the signatures of paths, then the expected signature is a powerful piece of information if it exists. The integral of any polynomial against this measure can be calculated as follows. Use the shuffle product to identify the linear functional on tensors that coincides with the polynomial on the set of signatures. The polynomial and the linear functional have the same integral, and this will be the contraction of the linear functional with the expectation of the measure (the expected signature). So the expected signature determines the integral of the measure against any polynomial. Of course if the measure were compactly supported, then the polynomials are dense in the continuous real-valued functions and the measure would be completely determined by its expectation. This theorem has been extended in [2] recently.

REMARK 2.6. Re-parameterizing a path inside the interval of definition does not change its signature over the maximal interval. Translating a path does not change its signature. We may define an equivalence relation between paths by asking that they have the same signature.

Returning to the finite dimensional example and the notation introduced above, the linear forms  $e_I^*$ , as  $I$  spans the set of finite words in the dual letters  $e_1^*, \dots, e_d^*$ , form a basis for  $T(E^*)$ . For convenience, we fix two useful functions on  $T((E))$ :  $\pi^I$  and  $\rho_n$ .

DEFINITION 2.9.  $\pi^I$  is defined by

$$\begin{aligned} \pi^I : T((E)) &\rightarrow \mathbb{R}; \\ \mathbf{a} &\mapsto e_I^*(\mathbf{a}). \end{aligned}$$

In particular,

$$\pi^I(S(X)) = \mathcal{U}_{e_I^*}(S(X)) = \int \cdots \int_{\substack{u_1 < \cdots < u_n \\ u_1, \dots, u_n \in J}} dX_{u_1}^{(i_1)} \otimes \cdots \otimes dX_{u_n}^{(i_n)},$$

where  $I = (i_1, \dots, i_n)$ . We call  $\pi^I(S(X))$  the coordinate signature of  $X$  indexed by  $I$ .

DEFINITION 2.10.  $\rho_n$  is defined by

$$\begin{aligned} \rho_n : T((E)) &\rightarrow E^{\otimes n}, \\ (a_0, a_1, \dots) &\mapsto a_n, \end{aligned}$$

where  $a_n \in E^{\otimes n}$  and  $n \in \mathbb{N}$ .

2.6. *The homogeneous Carnot–Caratheodory norm on the truncated signature of a path.* In this subsection, we consider continuous paths of finite 1-variation and look at their signatures truncated to order  $n$ . The range of this signature map, a subset of  $T^{(n)}(E)$ , forms a group. We will work with the Carnot–Caratheodory norm (CC norm) on this space. We refer to [6] for a detailed discussion of this norm but note that it is obvious from the definition that the CC norm is invariant under rotation of paths. We start with the definition of the space of continuous paths of finite 1-variation.

DEFINITION 2.11. Let  $E$  be a Euclidean space endowed with Euclidean metric  $\mathbf{d}$  and  $x : [0, T] \rightarrow E$ . For  $0 \leq t \leq \bar{t} \leq T$ , the 1-variation of  $x$  on  $[t, \bar{t}]$  is defined as

$$|x|_{1\text{-var}; [t, \bar{t}]} = \sup_{\mathcal{D}=(t=u_1 < u_2 < \cdots < u_n = \bar{t})} \sum_{i=0}^{n-1} \mathbf{d}(x_{u_i}, x_{u_{i+1}}).$$

If  $|x|_{1\text{-var}; [t, \bar{t}]} < \infty$ , we say that  $x$  is of bounded variation or of finite 1-variation on  $[t, \bar{t}]$ . The space of continuous paths of finite 1-variation on  $[0, T]$  is denoted by  $C^{1\text{-var}}([0, T], E)$ .

DEFINITION 2.12. The set of all signatures, truncated at order  $N$ , of bounded variation paths is denoted by

$$G^N(E) := \{S^N(x)_{0,1} : x \in C^{1\text{-var}}([0, 1], E)\}.$$

The group  $G^N(E)$  admits a number of equivalent metric structures. Two will be important in this paper. The so-called *Carnot–Caratheodory norm* is defined (and is finite) for every  $g \in G^N(E)$  as

$$\|g\| = \inf\{\|\gamma\|_{1\text{-var}} : S^N(\gamma) = g, \gamma \in C^{1\text{-var}}([0, 1], E)\}.$$

A second “norm” can be built directly out of the signature

$$\|g\| := \max_{i=1,\dots,N} |\rho_i(g)|^{1/i} \quad \forall g \in G^N(E).$$

Both “norms” are homogeneous of degree 1 in scaling  $\delta_\varepsilon$ . The group is finite dimensional. Therefore  $\|g\|$  and  $\|g\|$  are equivalent (page 11, [6]).

DEFINITION 2.13. The  $p$ -variation of a geometric rough path  $\gamma$  defined on  $[0, T]$ , and written  $\|S^N(\gamma)\|_{p\text{-var};[0,T]}$ , is defined on paths in  $G^N(E)$  to be

$$\|S^N(\gamma)\|_{p\text{-var};[0,T]} = \sup_{\mathcal{D}=(0=u_1 < u_2 < \dots < u_n=T)} \left( \sum_{\mathcal{D}} \|S^N(\gamma)_{u_i, u_{i+1}}\|^p \right)^{1/p},$$

where  $\|\cdot\|$  is the Carnot–Caratheodory norm.

LEMMA 2.1. *There is a constant  $C$  depending on  $d$  and  $p$  such that for any path  $\gamma$  of bounded variation,<sup>3</sup>*

$$\begin{aligned} \left| \int_{0 < u_1 < \dots < u_N < T} d\gamma_{u_1} \otimes \dots \otimes d\gamma_{u_N} \right|^{1/N} &\leq \|S^N(\gamma)\| \leq C \|S^N(\gamma)\| \\ &\leq C \|S^N(\gamma)\|_{p\text{-var};[0,T]}. \end{aligned}$$

### 2.7. Sobolev space.

DEFINITION 2.14. Let  $u$  be a locally integrable function in  $\Gamma$  and  $\alpha$  be a multi-index. Then a locally integrable function  $r_\alpha u$  such that for every  $g \in C_c^\infty(G)$ ,

$$\int_{\Gamma} g(x) r_\alpha(x) dx = (-1)^{|\alpha|} \int_{\Gamma} D^\alpha g(x) u(x) dx,$$

will be called the weak derivatives of  $u$ , and  $r_\alpha$  is denoted by  $D^\alpha u$ . By convention,  $D^\alpha u = u$  if  $|\alpha| = 0$ .

Throughout the rest of discussion in this subsection,  $\Gamma$  is a subset of  $E := \mathbb{R}^d$ .

DEFINITION 2.15. The Sobolev space is defined to be the set of all  $\mathbb{R}^{\bar{d}}$ -valued functions  $u \in L^p(\Gamma)$  such that for every multi-index  $\alpha$  with  $|\alpha| \leq k$ , the weak partial derivative  $D_\alpha u$  belongs to  $L^p(\Gamma)$ , that is,

$$W^{k,p}(\Gamma) = \{u \in L^p(\Gamma) : D^\alpha u \in L^p(\Gamma) \ \forall |\alpha| \leq k\},$$

where  $k \in \mathbb{N}$ , and  $\Gamma$  is an open set.

<sup>3</sup> $\|\cdot\|$  and  $\|\cdot\|$  are different.

The Sobolev norm is defined as follows:

$$\|u\|_{W^{k,p}(\Gamma)} = \sum_{j=1}^{\bar{d}} \left( \sum_{|\alpha| \leq k} \int_{\Gamma} |D_{\alpha} u^j(x)|^p dx \right)^{1/p}.$$

When  $k = 0$ ,  $\|u\|_{W^{k,p}(\Gamma)}$  is also  $L^p(\Gamma)$ -norm, that is,

$$\|u\|_{W^{k,p}(\Gamma)} = \|u\|_{L^p(\Gamma)}.$$

**DEFINITION 2.16.** A domain  $\Gamma \subset E$  is said to be strongly Lipschitz if and only if  $\Gamma$  is bounded and each point  $x_0$  of  $\partial\Gamma$  is in a neighborhood  $\mathcal{R}$  which is the image under a rotation and translation of axes of a domain  $\{x := (x_1, \dots, x_d) : |x'_d| < R, |x_d| < 2LR\}$  in which  $x_0$  corresponds to the origin,  $\mathcal{R} \cap \partial\Gamma$  corresponds to the locus  $x_d = f(x'_d)$  where  $x'_d = (x_1, \dots, x_{d-1})$ , and  $f$  satisfies the Lipschitz condition with constant  $L$  as well as

$$f(\mathcal{R} \cap \Gamma) = \{x : |x'_d| < R, f(x'_d) < x_d < 2LR\}.$$

Let us introduce one useful theorem, which states how the  $L^\infty$  norm of  $u$  can be controlled in terms of  $L^p$  norm of the weak derivatives of  $u$  up to some degree under certain regularity condition of a domain  $\Gamma$ ; see Theorem 3.5.1 in [13].

**THEOREM 2.2.** *Suppose  $u \in W^{m,p}(\Gamma)$  where  $\Gamma$  is strongly Lipschitz in  $E$  and  $m > d/p$ . Then  $u$  is continuous on  $\Gamma$  and there is a constant  $C(d, m, p, \Gamma)$  such that*

$$|u(x)| \leq C|\Gamma|^{-1/p} \left\{ \sum_{j=0}^{m-1} \frac{R^j}{j!} \|\nabla^j u\|_{L^p(\Gamma)} + (m - v/p)^{-1} \frac{R^m}{(m-1)!} \|\nabla^m u\|_{L^p(\Gamma)} \right\},$$

where  $R := R(\Gamma)$  is the diameter of  $\Gamma$  and  $u$  can be a vector valued function.

**LEMMA 2.2.** *Let  $u : \Gamma \rightarrow \mathbb{R}^{\bar{d}}$  where  $\Gamma$  is a strongly Lipschitz domain in  $E$ . Let  $m = \lfloor d/2 \rfloor + 1$  and  $u \in W^{m,2}(\Gamma)$ . Then  $u$  is continuous on  $\Gamma$  and there is a constant  $C := C(d, \Gamma)$  such that*

$$|u(x)| \leq C|\Gamma|^{-1/2} \max \left\{ \max_{j=0}^{m-1} \left( \frac{R^j}{j!} \right), \frac{R^m}{(m-1)!} \right\} \|u\|_{W^{m,2}(\Gamma)},$$

where  $R$  is the diameter of  $\Gamma$ .

**PROOF.** This is just a special case when  $p = 2$ ,  $m = \lfloor d/2 \rfloor + 1$  in Theorem 2.2. The proof is completed by noticing that

$$\sum_{j=0}^m \|\nabla^j u\|_2^0 \leq m \|u\|_{W^{m,2}(\Gamma)}.$$

□

Then we will introduce an auxiliary result about boundary regularity for solutions of linear elliptic equations [9]. Let  $\mathbb{D}(z, r)$  denote an open  $d$ -dimensional ball centered at  $z$  with radius  $r$ , and for simplicity let  $\mathbb{D}$  denote the  $d$ -dimensional open unit ball centered at the origin. To state it precisely, we introduce the definition of a domain of the class  $C^k$ .

DEFINITION 2.17. An open and bounded set  $\Gamma \subset E$  is of class  $C^k$  ( $k = 0, 1, \dots, \infty$ ) if for any  $x_0 \in \partial\Gamma$  there exist  $r > 0$  and a bijective map  $F: \mathbb{D}(x_0, r) \subset \mathbb{R}^d$  with the following properties:

- (1)  $F(\Gamma \cap \mathbb{D}(x_0, r)) \subset \{(x_1, \dots, x_d) : x_d \geq 0\}$ .
- (2)  $F(\partial\Gamma \cap \mathbb{D}(x_0, r)) \subset \{(x_1, \dots, x_d) : x_d = 0\}$ .
- (3)  $F$  and  $F^{-1}$  are of class  $C^k$ .

REMARK 2.7. This means that  $\partial\Gamma$  is a  $(d-1)$ -dimensional  $C^k$  submanifold of  $\mathbb{R}^d$ .

Let us consider the following class of elliptic differential equations:

$$Mu := \sum_{i,j} \frac{\partial}{\partial x^j} \left( a^{i,j}(x) \frac{\partial}{\partial x^i} u(x) \right),$$

where  $a^{i,j}$  satisfies the ellipticity condition, namely that, there exists some  $\lambda > 0$ , with

$$\sum_{i,j=1}^d a^{ij}(x) \xi_i \xi_j \geq \lambda |\xi|^2 \quad \forall x \in \Gamma, \xi \in \mathbb{R}^d.$$

THEOREM 2.3. Let  $u$  be a weak solution of

$$\begin{aligned} Mu &= f(x) \in \Gamma, \\ u - g &\in H_0^{1,2}(\Gamma). \end{aligned}$$

Suppose that the ellipticity condition holds. Let  $f \in W^{k,2}(\Gamma)$ ,  $g \in W^{k+2,2}(\Gamma)$ . Let  $\Gamma$  be of class  $C^{k+2}$ , and let the coefficients of  $M$  be of class  $C^{k+1}(\bar{\Gamma})$ . Then

$$\|u\|_{W^{k+2,2}(\Gamma)} \leq c(\|f\|_{W^{k,2}(\Gamma)} + \|g\|_{W^{k+2,2}(\Gamma)})$$

with  $c$  depending on  $\lambda, d, \Gamma$  and on the  $C^{k+1}$ -norms for the  $a^{i,j}$ .

PROOF. The proof of Lemma 8.3.3 is given on page 207 in [9].  $\square$

**3. Expected signature of planar Brownian motion up to the first exit time from a bounded domain.** Recall that  $E = \mathbb{R}^d$ , and so it has a canonical basis  $(e_1, \dots, e_d)$ . Let  $(B_t)_{t \geq 0}$  denote a standard  $d$ -dimensional Brownian motion on  $E$  under a probability space  $(\Omega, P^z, \mathcal{F})$  with its canonical filtration  $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$  where  $P^x(B_0 = z) = 1$  and  $z \in E$ .

DEFINITION 3.1. Let  $\Gamma$  be a domain (a connected open subset) in  $E$ . Then  $\tau_\Gamma = \inf\{t \geq 0 : B_t \in \Gamma^c\}$  is the first exit time of Brownian motion from  $\Gamma$ .

The (Stratonovich) signature is defined for almost every Brownian path  $B$ , and for all pairs of times  $(s, t)$ . We are interested in the random signature  $S(B_{[0, \tau_\Gamma]})$  of the Brownian path up to the first exit time from the domain  $\Gamma$ .

DEFINITION 3.2 (Expected signature of Brownian motion). Assume  $\tau_\Gamma < \infty$  a.s. and the componentwise integrability of  $S(B_{[0, \tau_\Gamma]})$ . We denote by  $\Phi_\Gamma(z)$  the expected signature of Brownian motion starting at  $z$  and stopped upon the first exit time  $\tau_\Gamma$  from a domain  $\Gamma$ , that is,

$$(8) \quad \Phi_\Gamma(z) = \mathbb{E}^z[S(B_{[0, \tau_\Gamma]})].$$

REMARK 3.1. For the case of Brownian motion in the upper half plane  $\mathbb{H}$ , the projection  $\pi^{(1)}$  of the signature at the exit time onto the horizontal axis has a Cauchy distribution, and therefore does not have finite expectation. Similarly the projection  $\pi^{(1,1)}$  is positive and has a  $\frac{1}{2}$ -stable distribution and so more obviously does not have finite expectation.

In order to discuss the expected signature of Brownian motion further, we introduce an auxiliary function  $\Psi$  mapping  $E$  to  $T((E))$ .

DEFINITION 3.3. We denote by  $\Psi(z)$  the expected signature of a Brownian motion started at 0, run until it leaves the ball  $\mathbb{D}(0, |z|)$ , and conditioned to exit the ball at the point  $z$ . In formula,

$$\Psi(z) = \mathbb{E}^0[S(B_{[0, \tau_{\mathbb{D}(0, |z|)}]}) | B_{\tau_{\mathbb{D}(0, |z|)}} = z].$$

In particular,  $\Psi(0) = \mathbf{1} = (1, 0, 0, \dots)$ . The function  $\Psi$  plays an important role in the rest of the paper. In part, its importance comes from the strong Markov property. In Lemma 3.6, we easily prove that  $\Psi$  is well defined, and a smooth function of  $z$ . A more serious challenge is to show that the boundedness of the domain  $\Gamma$  can guarantee the existence and twice differentiability of  $\Phi_\Gamma$ . Then we will be in the position to derive a PDE which characterizes  $\Phi_\Gamma$ .

In the case of the disk, which is clearly a bounded domain, the PDE is so explicit that one can use it to identify the solution as a combination of polynomials in an explicit manner. The solution to this PDE contains an enormous amount of information about Brownian motion in the disk, and, for example, it easily gives the moments of the Lévy area for Brownian motion stopped at the first exit time from the disk in a quite explicit form; see [14].

3.1. *The componentwise smoothness of  $\Phi_\Gamma$ .* We start by proving that the signature of Brownian motion upon its first exit time  $\tau_\Gamma$  from a bounded domain  $\Gamma$  has finite expectation, and so  $\Phi_\Gamma$  is well defined. We will then go on to discuss the componentwise smoothness of  $\Phi_\Gamma$ . The expectation can be controlled by using extension of Lepingle's BDG inequality obtained in [5].

DEFINITION 3.4. We say that  $F: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is moderate if:

- (1)  $F$  is continuous and increasing;
- (2)  $F(x) = 0$  if and only if  $x = 0$ ;
- (3) for some (and thus for every  $\alpha > 1$ )

$$\sup_{x>0} \frac{F(\alpha x)}{F(x)} < \infty.$$

THEOREM 3.1. *Let  $M$  be a continuous,  $\mathbb{R}^d$ -valued local martingale starting at 0 and  $S^n(M)$  be the truncated Stratonovich signature of  $M$  up to level  $n$  viewed as a path in  $T^{(n)}(\mathbb{R}^d)$ . Then for any moderate function  $F$ , there exists  $C_i = C_i(n, F, d, |\cdot|)$  for  $i = 1, 2$  such that for all stopping times  $\tau$ ,*

$$\begin{aligned} & \mathbb{E} \left( F \left( \left| \int_{0 < u_1 < u_2 < \dots < u_n < \tau} \circ dM_{u_1} \circ \dots \circ dM_{u_n} \right|^{1/n} \right) \right) \\ & \leq C_1 \mathbb{E}(F(\|S^n(M)\|_{p\text{-var};[0,\tau]})) \\ & \leq C_2 \mathbb{E}(F(|\langle M \rangle_\tau|^{1/2})). \end{aligned}$$

The proof can be found in [5].

COROLLARY 3.1. *Let  $\Gamma$  be a bounded domain. Then for every  $n \in \mathbb{N}^+$ , there exists a constant  $C = C(n)$  such that for every  $z \in \Gamma$ ,*

$$\begin{aligned} & \mathbb{E}^z \left[ \left| \int \dots \int_{0 < u_1 < u_2 < \dots < u_n < \tau_\Gamma} \circ dB_{u_1} \circ \dots \circ dB_{u_n} \right| \right] \\ & \leq C(n) \mathbb{E}^z [\|S^n(B)\|_{p\text{-var};[0,\tau]}^n] \\ & \leq C(n) \sup_{z \in \Gamma} \{\mathbb{E}^z[\tau_\Gamma^{n/2}]\} < +\infty. \end{aligned}$$

PROOF. Applying Theorem 3.1 to the case with  $M_t = B_t$  and  $F(x) = x^n$ , we get

$$\begin{aligned} \mathbb{E}^z \left[ \left| \int \cdots \int_{0 < t_1 < t_2 < \cdots < t_n < \tau_\Gamma} \circ dB_{t_1} \circ \cdots \circ dB_{t_n} \right| \right] &\leq C(n) \mathbb{E}^z [\|S^n(B)\|_{p\text{-var};[0,\tau]}^n] \\ &\leq C(n) \mathbb{E}^z [\tau_\Gamma^{n/2}]. \end{aligned}$$

The only thing we need to show that is  $\sup_{z \in \Gamma} \{\mathbb{E}^z[\tau_\Gamma^{n/2}]\} < +\infty$ . It can be shown that there exists a positive number  $\alpha > 0$  such that

$$\mathbb{E}^0[e^{\alpha\tau_\mathbb{D}}] < +\infty,$$

and due to the Brownian scaling property we have

$$\forall r > 0 \quad \mathbb{E}^0[e^{\alpha\tau_\mathbb{D}}] = \mathbb{E}^0[e^{\alpha/r^2\tau_{\mathbb{D}(0,r)}}].$$

Since  $\Gamma$  is bounded, there exists a positive constant  $R > 0$  such that  $\Gamma \subseteq \mathbb{D}(0, R)$ . Thus  $0 \leq \tau_\Gamma \leq \tau_{\mathbb{D}(0,R)}$ . This implies that

$$\mathbb{E}^z[e^{\alpha/(4R^2)\tau_\Gamma}] \leq \mathbb{E}^z[e^{\alpha/(4R^2)\tau_{\mathbb{D}(z,2R)}}] = \mathbb{E}^0[e^{\alpha/(4R^2)\tau_{\mathbb{D}(0,2R)}}] = \mathbb{E}^0[e^{\alpha\tau_\mathbb{D}}] < \infty.$$

Thus  $\tau_\Gamma$  has finite moments, since for every  $z \in \Gamma$ ,

$$\frac{1}{n!} \left( \frac{\alpha}{4R^2} \right)^n \mathbb{E}^z[\tau_\Gamma^n] \leq \mathbb{E}^z[e^{\alpha/(4R^2)\tau_\Gamma}] \leq \mathbb{E}^0[e^{\alpha\tau_\mathbb{D}}] < \infty.$$

Then it follows that

$$\forall n \in \mathbb{N} \quad \sup_{z \in \Gamma} \{\mathbb{E}^z[\tau_\Gamma^n]\} < \infty$$

and

$$\sup_{z \in \Gamma} \{\mathbb{E}^z[\tau_\Gamma^{n/2}]\} \leq \sqrt{\sup_{z \in \Gamma} \{\mathbb{E}^z[\tau_\Gamma^n]\}} < \infty.$$

Now our proof is complete.  $\square$

In the rest of this subsection, we are going to discuss the smoothness of  $\Phi_\Gamma$  in componentwise sense providing that  $\Psi$  is a smooth function, which is proved later in Lemma 3.6.

**THEOREM 3.2.** *Suppose that  $\Gamma$  is a nonempty bounded domain in  $E$ . Then the following statements are true:*

- (1)  $\Phi_\Gamma$  is a well defined function and moreover  $\Phi_\Gamma \in L^1$ ;
- (2)  $\Phi_\Gamma$  is twice differentiable in componentwise sense, that is, for all index  $I$ ,  $\pi^I \circ \Phi_\Gamma$  is twice differentiable.

PROOF. We start with proving the first statement. By Corollary 3.1, it is easy to see that  $\Phi_\Gamma$  is well defined and uniformly bounded in  $\Gamma$ , since for every index  $I$ , there exists a constant  $C > 0$ , such that

$$\sup_{z \in \Gamma} |\pi^I(\Phi_\Gamma(z))| \leq \sup_{z \in \Gamma} \mathbb{E}^z[|\pi^I(S(B_{[0, \tau_\Gamma]}))|] < C.$$

Obviously  $\Phi_\Gamma$  is a measurable function. Furthermore,  $\pi^I(\Phi_\Gamma)$  has a compact support. Hence for every index  $I$ ,

$$\int_\Gamma |\pi^I(\Phi_\Gamma(z))| dz \leq CA(\Gamma) < \infty,$$

where  $A(\Gamma)$  is the volume of  $\Gamma$ . So  $\Phi_\Gamma \in L^1$ . Thus the proof of the first statement is complete. Now the only thing left to prove is that  $\Phi_\Gamma$  is twice differentiable.

For every  $\varepsilon > 0$ , let  $\Gamma_\varepsilon = \{z \in \Gamma \mid \text{dist}(z, \partial\Gamma) > \varepsilon\}$ . The Markov property of the expected signature of Brownian motion, which is described in detail in Lemma 3.8, ensures that for every  $z \in \Gamma_\varepsilon$ ,

$$\Phi_\Gamma(z) = \frac{1}{d\omega_d r^{d-1}} \int_{\partial\mathbb{D}(0, r)} \Psi(y) \otimes \Phi_\Gamma(z+y) d\sigma(y),$$

where  $r < \frac{\varepsilon}{2}$ ,  $\omega_d$  is the volume of the unit ball, and  $\sigma(y)$  is the  $d-1$ -dimensional surface measure.

Then this implies that

$$(9) \quad \Phi_\Gamma(z) = \int_0^\infty \left( \frac{1}{d\omega_d r^{d-1}} \int_{\partial\mathbb{D}(0, r)} \Psi(y) \otimes \Phi_\Gamma(z+y) d\sigma(y) \right) K_\varepsilon(r) dr,$$

for a smooth distribution  $K_\varepsilon(r)$  with compact support  $[0, \frac{\varepsilon}{2}]$ . Let  $F_\varepsilon$  be a map from  $E$  to  $T((E))$  defined by

$$F_\varepsilon(z) = \Psi(-z) K_\varepsilon(|z|).$$

Rewriting (9) we have

$$\Phi_\Gamma(z) = \int_\Gamma F_\varepsilon(z-y) \otimes \Phi_\Gamma(y) dy = F_\varepsilon * \Phi_\Gamma(z),$$

where  $*$  is the convolution. Since  $\Psi$  is smooth and  $K_\varepsilon$  is a smooth function with compact support,  $F_\varepsilon$  is a smooth function with compact support. It is easy to show that

$$\|F_\varepsilon\|_{L^1} + \|\nabla F_\varepsilon\|_{L^1} + \|\Delta F_\varepsilon\|_{L^1} < +\infty.$$

On the other hand,  $\Phi_\Gamma \in L^1$  as well, and so we have

$$\|F_\varepsilon * \Phi_\Gamma\|_{L^1} + \|\nabla F_\varepsilon * \Phi_\Gamma\|_{L^1} + \|\Delta F_\varepsilon * \Phi_\Gamma\|_{L^1} < +\infty.$$

Thus  $F_\varepsilon * \Phi_\Gamma$  is twice differentiable, since  $F_\varepsilon * \Phi_\Gamma$ ,  $(\nabla F_\varepsilon) * \Phi_\Gamma$  and  $(\Delta F_\varepsilon) * \Phi_\Gamma \in L^1$ . Moreover, because  $\Gamma = \bigcup_{\varepsilon > 0} \Gamma_\varepsilon$ ,  $\Phi_\Gamma$  is smooth on  $\Gamma$ .  $\square$

REMARK 3.2. Actually following this proof we can show that  $\Phi_\Gamma$  is infinitely differentiable in a bounded domain. Alternatively, since we have proved that  $\Phi_\Gamma$  is twice differentiable so that our PDE provides the integral representation for  $\Phi_\Gamma$ , this also implies that  $\Phi_\Gamma$  is infinitely differentiable.

3.2. *Properties of the expected signature of stopped Brownian motion.* Throughout this subsection, suppose that  $\Gamma$  is a bounded domain. In view of Theorem 3.2  $\Phi_\Gamma(z)$  is well defined and twice differentiable. Moreover  $\Phi_\Gamma(z)$  has the following properties.

3.2.1. *Translation invariance.*

LEMMA 3.1 (Translation invariance). *Let  $z + \Gamma = \{z + y | y \in \Gamma\}$ . Then*

$$\begin{aligned}\mathbb{E}^z[S(B_{[0, \tau_{z+\Gamma}])}] &= \mathbb{E}^0[S(B_{[0, \tau_\Gamma])}], \\ \mathbb{E}^z[S(B_{[0, \tau_{z+\Gamma}])} | B_{\tau_{z+\Gamma}} = z + a] &= \mathbb{E}^0[S(B_{[0, \tau_\Gamma])} | B_{\tau_\Gamma} = a],\end{aligned}$$

where  $a \in \partial\Gamma$ .

3.2.2. *Scaling property.*

LEMMA 3.2 (Scaling property). *For  $n \in \mathbb{N}$  and  $\varepsilon \in [0, +\infty)$ , we have that*

$$\begin{aligned}\Phi_{\varepsilon\mathbb{D}}(0) &= \delta_\varepsilon(\Phi_{\mathbb{D}}(0)), & \text{that is, } \rho_n(\Phi_{\varepsilon\mathbb{D}}(0)) &= \varepsilon^n \rho_n(\Phi_{\mathbb{D}}(0)); \\ \Psi(\varepsilon) &= \delta_\varepsilon(\Psi(1)), & \text{that is, } \rho_n(\Psi(\varepsilon)) &= \varepsilon^n \rho_n(\Psi(1)).\end{aligned}$$

3.2.3. *Rotation property.* Let  $O(d)$  denote the rotation group in  $d$  dimensions. Let us introduce the rotation operator  $\bar{\delta}_\theta$  on  $T((E))$ , where  $\theta \in O(d)$ . The rotation operator  $\bar{\delta}_\theta$  is defined as follows: for every  $a = (a_0, a_1, \dots) \in T((E))$ ,

$$\bar{\delta}_\theta(a) = (a_0, \theta a_1, \dots, \theta^{\otimes n} a_n, \dots),$$

where  $\cdot^{\otimes}$  is the Kronecker power, and  $bc$  is the matrix multiplication of  $b$  and  $c$ .

LEMMA 3.3 (Rotation property). *For every fixed  $x \in \partial\mathbb{D}$  and some neighborhood  $U$  contained in  $\partial\mathbb{D}$  around  $x$ , there is a smooth map  $u$  from  $U$  into  $O(d)$  so that for every  $y \in U$ ,  $u(y)e_1 = y$ . Then*

$$\Psi(y) = \bar{\delta}_{u(y)}(\Psi(e_1)).$$

PROOF. Let  $B_t$  be the Brownian motion starting at the origin and stopped for the first exit time of the unit disk on condition that the exit point is  $e_1$ . Then for every  $y \in U$ ,  $\tilde{B}_t = u(y)B_t$  is the Brownian motion starting at the origin and stopped at the first exit time from the unit disk on condition that the exit point is  $y$ . It is obvious that  $\tilde{B}_t$  is just a linear transformation of  $B_t$ . Then immediately it follows that for every  $y \in U$ ,

$$\rho^n(\Psi(y)) = u(y)^{\otimes n} \rho^n(\Psi(e_1))$$

or equivalently

$$\Psi(y) = \bar{\delta}_{u(y)}(\Psi(e_1)). \quad \square$$

### 3.2.4. An application of the strong Markov property.

LEMMA 3.4 (Strong Markov property). *For every  $\varepsilon > 0$  such that  $\mathbb{D}(z, \varepsilon) \subseteq \Gamma$ ,*

$$(10) \quad \begin{aligned} \Phi_\Gamma(z) &= \frac{1}{d\omega_d \varepsilon^{d-1}} \\ &\times \int_{\partial \mathbb{D}(0, \varepsilon)} \mathbb{E}^z[S(B_{[0, \tau_{\mathbb{D}(z, \varepsilon)}]}) | B_{\tau_{\mathbb{D}(z, \varepsilon)}} = z + y] \otimes \Phi_\Gamma(z + y) d\sigma(y), \end{aligned}$$

where  $\partial \mathbb{D}(0, \varepsilon)$  denotes the boundary of  $\mathbb{D}(0, \varepsilon)$ ,  $\omega_d$  is the volume of  $d$ -dimensional unit ball and  $\sigma$  is the  $d - 1$ -dimensional surface measure.

PROOF. By Chen's identity (Theorem 2.1) and the fact that  $S(B_{[0, \tau_{\mathbb{D}(z, \varepsilon)}]})$  is  $\mathcal{F}_{\tau_{\mathbb{D}(z, \varepsilon)}}$ -measurable, we obtain

$$\mathbb{E}^z[S(B_{[0, \tau_\Gamma]}) | \mathcal{F}_{\tau_{\mathbb{D}(z, \varepsilon)}}] = S(B_{[0, \tau_{z+\varepsilon \mathbb{D}}]}) \otimes \mathbb{E}[S(B_{[\tau_{\mathbb{D}(z, \varepsilon)}, \tau_\Gamma]}) | \mathcal{F}_{\tau_{\mathbb{D}(z, \varepsilon)}}],$$

where  $\mathbb{D}(z, \varepsilon) \subseteq \Gamma$  and  $\mathcal{F}$  is defined as before, that is, the filtration generated by the Brownian motion.

As it is known that  $B$  has the strong Markov property,

$$\mathbb{E}^z[S(B_{[\tau_{\mathbb{D}(z, \varepsilon)}, \tau_\Gamma]}) | \mathcal{F}_{\tau_{\mathbb{D}(z, \varepsilon)}}] = \mathbb{E}^z[S(B_{[\tau_{\mathbb{D}(z, \varepsilon)}, \tau_\Gamma]}) | B_{\tau_{\mathbb{D}(z, \varepsilon)}}] = \Phi_\Gamma(B_{\tau_{\mathbb{D}(z, \varepsilon)}}),$$

which implies that

$$(11) \quad \mathbb{E}^z[S(B_{[0, \tau_\Gamma]}) | \mathcal{F}_{\tau_{\mathbb{D}(z, \varepsilon)}}] = S(B_{[0, \tau_{\mathbb{D}(z, \varepsilon)}]}) \otimes \Phi_\Gamma(B_{\tau_{\mathbb{D}(z, \varepsilon)}}).$$

By the tower property, we have

$$(12) \quad \Phi_\Gamma(z) = \mathbb{E}^z[\mathbb{E}^z[S(B_{[0, \tau_\Gamma]}) | \mathcal{F}_{\tau_{\mathbb{D}(z, \varepsilon)}}]].$$

Substituting equation (11) into it, we obtain

$$\begin{aligned}\Phi_\Gamma(z) &= \mathbb{E}^z[S(B_{[0, \tau_{z+\varepsilon\mathbb{D}}]}) \otimes \Phi_\Gamma(B_{\tau_{\mathbb{D}}(z, \varepsilon)})] \\ &= \frac{1}{d\omega_d \varepsilon^{d-1}} \\ &\quad \times \int_{\partial\mathbb{D}(0, \varepsilon)} \mathbb{E}^z[S(B_{[0, \tau_{z+\varepsilon\mathbb{D}}]}) | B_{\tau_{z+\varepsilon\mathbb{D}}} = z + y] \otimes \Phi_\Gamma(z + y) d\sigma(y). \quad \square\end{aligned}$$

3.2.5. *The smoothness of  $\Psi$ .* In this subsection, we focus on the discussion of the componentwise smoothness of  $\Psi$ . We start with the proof of the finiteness of  $\Psi(e_1)$ , and then show that the function  $\Psi$  is smooth. We end with the derivation of the first two gradings of  $\Psi$ , which is important for us to derive the PDE for  $\Phi$  later.

LEMMA 3.5.  *$\Psi(e_1)$  is well defined.*

PROOF. Let  $\tilde{B}_t$  be a  $d$ -dimensional Brownian motion starting at zero in the unit ball. Let  $M_t = \tilde{B}_{\tau_{\mathbb{D}} \wedge t}$  be the stopped process. It is obvious from BDG inequality that the exit time  $\tau_{\mathbb{D}}$  of  $\tilde{B}_t$  from the ball (or any bounded set) has finite moments of all orders.

By the theorem of the equivalence of homogeneous norms on  $G^N(E)$  (Lemma 2.1), we have

$$\left| \int_{0 < u_1 < u_2 < \dots < u_N < \tau_{\mathbb{D}}} \circ dM_{u_1} \circ \dots \circ dM_{u_N} \right| \leq C \|S^N(M)\|_{p\text{-var}; [0, \tau_{\mathbb{D}}]}^N.$$

Thus it holds that

$$\begin{aligned}\mathbb{E}^0 \left[ \left| \int_{0 < u_1 < u_2 < \dots < u_N < \tau} \circ dM_{u_1} \circ \dots \circ dM_{u_N} \right| \middle| M_{\tau_{\mathbb{D}}} = e_1 \right] \\ \leq C \mathbb{E}^0 [\|S^N(M)\|_{p\text{-var}; [0, \tau_{\mathbb{D}}]}^N | M_{\tau_{\mathbb{D}}} = e_1].\end{aligned}$$

By the definition of the Carnot–Caratheodory norm, it is obvious that for every  $\theta \in O(d)$

$$\|S^N(\theta M)\|_{p\text{-var}; [0, \tau]}$$

does not depend on  $\theta$ . Therefore

$$\mathbb{E}^0 [\|S^N(M)\|_{p\text{-var}; [0, \tau_{\mathbb{D}}]}^N | M_{\tau_{\mathbb{D}}} = e_1] = \mathbb{E}^0 [\|S^N(M)\|_{p\text{-var}; [0, \tau_{\mathbb{D}}]}^N],$$

because there is nothing significant in choosing the exit point to be  $e_1$  providing the norm used is invariant under rotation; using extension of Lepingle’s BDG inequality obtained in [5] one can control the  $p$ -variation of a martingale in terms of its bracket process, and so there exists a positive constant

$C > 0$  such that

$$\mathbb{E}^0[\|S^N(M)\|_{p\text{-var};[0,\tau_{\mathbb{D}}]}^N] \leq C\mathbb{E}^0[\tau_{\mathbb{D}}^{N/2}].$$

Finally we have that there exists a constant  $C \geq 0$  such that

$$\begin{aligned} |\rho_N(\Psi(1))| &= \left| \mathbb{E}^0 \left[ \int_{0 < u_1 < u_2 < \dots < u_N < \tau_{\mathbb{D}}} \circ dM_{u_1} \circ \dots \circ dM_{u_N} \middle| M_{\tau_{\mathbb{D}}} = 1 \right] \right| \\ &\leq \mathbb{E}^0 \left[ \left| \int_{0 < u_1 < u_2 < \dots < u_N < \tau_{\mathbb{D}}} \circ dM_{u_1} \circ \dots \circ dM_{u_N} \right| \middle| M_{\tau_{\mathbb{D}}} = 1 \right] \\ &\leq C\mathbb{E}^0[\tau_{\mathbb{D}}^{N/2}] < +\infty. \end{aligned} \quad \square$$

LEMMA 3.6.  $\Psi$  is a componentwise smooth function in  $E \setminus \{0\}$ , that is, for every index  $I$ ,  $\pi^I(\Psi)$  is a smooth function.

PROOF. We recall the definition of  $\Psi(x)$ . It is clear that if  $\alpha > 0$  and  $\theta$  is a rotation, then

$$\begin{aligned} \Psi(\theta(\alpha x)) &= \delta_\alpha(\Psi(\theta x)) \\ &= \delta_\alpha(\bar{\delta}_\theta(\Psi(x))). \end{aligned}$$

In particular the map  $(\alpha, \theta) \mapsto \Psi(\theta(\alpha x))$  is smooth and defined on  $\mathbb{R}^+ \times O(d)$  to  $T((E))$ . Fix  $x$  on the sphere and some small neighborhood  $U$  contained in  $\partial\mathbb{D}$  around  $x$  chosen so that there is a smooth map  $u$  from  $U$  into  $O(d)$  so that for every  $y \in U$ ,  $u(y)(e_1) = y$ . We observe that on  $\mathbb{R}^+ \times U$ ,

$$\Psi(\alpha y) = \delta_\alpha(\bar{\delta}_{u(y)}(\Psi(e_1))).$$

As the right-hand side is the composition of two smooth maps  $\delta_\alpha$  and  $\bar{\delta}_{u(x)}$ , the left-hand side must be smooth on  $\mathbb{R}^+ \times U$ . Since  $x$  is an arbitrary point in the sphere, we have proved the result.  $\square$

LEMMA 3.7.  $\Psi$  is a twice continuously differentiable function in  $E$  in componentwise sense. Moreover,

$$\begin{aligned} \pi_2(\Psi(z)) &= 1 + \sum_{i=1}^d z_i e_i + \sum_{i=1}^d \frac{1}{2} z_i^2 e_i \otimes e_i, \\ \frac{\partial \Psi(0)}{\partial z_i} &= e_i \quad \text{for } i = 1, 2, \dots, d \quad \text{and} \\ \Delta \Psi(0) &= \sum_{i=1}^d e_i \otimes e_i. \end{aligned}$$

PROOF. The term of tensor degree one in  $\Psi$  is the expected increment of the conditioned path, so it is not random and  $\rho_1(\Psi(z)) = z$ . The expectation of the Lévy area of the Brownian motion conditioned on leaving the disk at

any particular point equals to zero due to the symmetry of the Brownian motion. Thus the term  $\rho_2(\Psi(z))$  only contains the symmetric “increment squared” part, thus we have

$$\pi_2(\Psi(z)) = 1 + \sum_{i=1}^d z_i e_i + \sum_{i=1}^d \frac{1}{2} z_i^2 e_i \otimes e_i.$$

This implies that  $\pi_2(\Psi(z))$  is smooth in  $E$ , and moreover it follows that

$$\begin{aligned} \frac{\partial \pi_2(\Psi(0))}{\partial z_i} &= e_i \quad \text{for } i = 1, 2, \dots, d \quad \text{and} \\ \Delta \pi_2(\Psi(0)) &= \sum_{i=1}^d e_i \otimes e_i. \end{aligned}$$

Now let us focus on  $\rho_n \circ \Psi$  for any  $n \geq 3$ . By Lemma 3.6,  $\Psi$  is smooth in  $E \setminus \{0\}$ ; the scaling property of  $\Psi$  can guarantee the smoothness of  $\Psi$  at the origin, which we explain in more detail as follows. By the definition of  $\Psi$ ,  $\Psi(0) = \mathbf{1}$ . By the scaling property of  $\rho_n \circ \Psi$ , it is easy to verify that for every integer  $n \geq 3$ ,

$$\begin{aligned} \frac{\partial \rho_n(\Psi(0))}{\partial z_i} &= \lim_{\alpha \rightarrow 0} \frac{\rho_n(\Psi(\alpha e_i)) - \rho_n(\Psi(0))}{\alpha} = \lim_{\alpha \rightarrow 0} \alpha^{n-1} \rho_n(\Psi(e_i)) = 0; \\ \frac{\partial \rho_n(\Psi(z_i e_i))}{\partial z_i} &= \frac{\partial z_i^n \rho_n(\Psi(e_i))}{\partial z_i} = n z_i^{n-1} \rho_n(\Psi(e_i)) \quad \forall z_i \in (-1, 1); \\ \frac{\partial^2 \rho_n(\Psi(0))}{\partial z_i^2} &= \lim_{\alpha \rightarrow 0} \frac{1}{\alpha} \left( \frac{\partial \rho_n(\Psi(\alpha e_i))}{\partial z_i} - \frac{\partial \rho_n(\Psi(0))}{\partial z_i} \right) \\ &= \lim_{\alpha \rightarrow 0} n \alpha^{n-2} \rho_n(\Psi(e_i)) = 0. \end{aligned}$$

Thus it follows that for every integer  $n \geq 3$ ,

$$\Delta \rho_n(\Psi(0)) = 0.$$

The proof is complete.  $\square$

**3.3. The PDE for the expected signature of multi-dimensional Brownian motion up to the first exit time from a bounded domain.** In this section, our goal is to derive the PDE for the expected signature of Brownian motion up to the first exit time from a bounded domain.

**LEMMA 3.8.** *For every  $z \in \Gamma$  and every  $\varepsilon > 0$  sufficiently small such that  $\mathbb{D}(z, \varepsilon) \subset \Gamma$ , it holds that*

$$(13) \quad \Phi_\Gamma(z) = \frac{1}{d\omega_d \varepsilon^{d-1}} \int_{\partial \mathbb{D}(0, \varepsilon)} \Psi(y) \otimes \Phi_\Gamma(z + y) d\sigma(y),$$

where  $\omega_d$  is the volume of the  $d$ -dimensional unit ball, and  $\sigma$  is the  $(d-1)$ -dimensional surface measure.

PROOF. From the strong Markov property, we have

$$(14) \quad \begin{aligned} \Phi_\Gamma(z) &= \frac{1}{d\omega_d\varepsilon^{d-1}} \\ &\times \int_{\partial\mathbb{D}(0,\varepsilon)} \mathbb{E}^z[S(B_{[0,\tau_{\mathbb{D}(z,\varepsilon)}]})|B_{\tau_{\mathbb{D}(z,\varepsilon)}} = z+y] \otimes \Phi_\Gamma(z+y) d\sigma(y). \end{aligned}$$

By the translation invariance property, substituting

$$\Psi(y) = \mathbb{E}^z[S(B_{[0,\tau_{\mathbb{D}(z,\varepsilon)}]})|B_{\tau_{\mathbb{D}(z,\varepsilon)}} = z+y]$$

into (14), we obtain that

$$\Phi_\Gamma(z) = \frac{1}{d\omega_d\varepsilon^{d-1}} \int_{\partial\mathbb{D}(0,\varepsilon)} \Psi(y) \otimes \Phi_\Gamma(z+y) d\sigma(y),$$

where  $\varepsilon$  is sufficiently small such that  $\mathbb{D}(z,\varepsilon) \subseteq \Gamma$ .  $\square$

LEMMA 3.9. *If  $\Gamma$  is a bounded domain, then*

$$\lim_{t \uparrow \tau_\Gamma} \Phi_\Gamma(B_t) = \mathbf{1} \quad \text{a.s. } P^z, z \in \Gamma.$$

PROOF. Fix  $z \in \Gamma$ , and let  $B_t$  be the Brownian motion starting at  $z$ -defined on  $(\Omega, P^z, (\mathcal{F}_t), \mathcal{F})$ . Let  $N_t := \mathbb{E}^z[S(B_{[0,\tau_\Gamma]})|\mathcal{F}_t \wedge \tau_\Gamma]$ . By Corollary 3.1, the boundedness of  $\Gamma$  ensures that  $S(B_{[0,\tau_\Gamma]})$  is  $L_1$ -integrable, and it implies that  $N_t$  is a uniformly integrable martingale and by the martingale convergence theorem  $\lim_{t \uparrow \infty} N_t$  exists a.s. and in  $L^1$ . Since  $\Gamma$  is bounded,  $\mathbb{E}[\tau_\Gamma] < \infty$ , and thus  $\tau_\Gamma$  is finite a.s. in  $P^z$ . Therefore it follows that

$$\lim_{t \uparrow \infty} N_t = \lim_{t \uparrow \tau_\Gamma} \mathbb{E}^z[S(B_{[0,\tau_\Gamma]})|\mathcal{F}_t].$$

Let  $\{D_k\}$  be an increasing sequence of open sets such that  $D_k \subset \subset \Gamma$  and  $\Gamma = \bigcup_k D_k$ . Let  $\tau_k := \tau_{D_k}$ , and let  $\mathcal{F}_k$  denote the  $\sigma$ -algebra generated by the Brownian motion up to  $\tau_k$ . Since  $\tau_k$  is finite a.s.,  $\lim_{k \uparrow \infty} \tau_k = \tau_\Gamma$ . Let  $\mathcal{F}_\infty$  be the  $\sigma$ -algebra generated by  $\{\mathcal{F}_k\}_{\tau_k \geq 0}$ . Since  $S(B_{[0,\tau_\Gamma]})$  is  $L_1$ -integrable, then  $M_k := N_{\tau_k}$  is a discrete martingale. By the martingale convergence theorem, it holds both a.s. and in  $L^p(P^z)$  for every  $p > 0$  that

$$(15) \quad \lim_{k \rightarrow \infty} M_k = \mathbb{E}^z[S(B_{[0,\tau_\Gamma]})|\mathcal{F}_\infty] = S(B_{[0,\tau_\Gamma]}).$$

Since  $\tau_k \uparrow \tau_\Gamma$  a.s., and  $\lim_{t \uparrow \infty} N_t$  exists, then  $\lim_{t \uparrow \infty} N_t$  must coincide with  $\lim_{k \uparrow \infty} N_{\tau_k} = S(B_{[0,\tau_\Gamma]})$ , that is,

$$(16) \quad \lim_{t \uparrow \tau_\Gamma} \mathbb{E}^z[S(B_{[0,\tau_\Gamma]})|\mathcal{F}_t] = S(B_{[0,\tau_\Gamma]}) \quad \text{a.s.}$$

Recall that  $\Phi_\Gamma(z) := \mathbb{E}^z[S(B_{[0,\tau_\Gamma]})]$ . By the strong Markov property of Brownian motion, the multiplicativity of the signatures and  $L^1$  integrability of the signature of the stopped Brownian motion, we have that

$$(17) \quad \mathbb{E}^z[S(B_{[0,\tau_\Gamma]})|\mathcal{F}_t] = S(B_{[0,t]}) \otimes \Phi_\Gamma(B_t) \quad \text{if } t < \tau_\Gamma.$$

Substituting (17) into (16), we obtain that

$$\lim_{t \uparrow \tau_\Gamma} S(B_{[0,t]}) \otimes \Phi_\Gamma(B_t) = S(B_{[0,\tau_\Gamma]}) \quad \text{a.s. } P^x, x \in \Gamma.$$

Due to the continuity of the signature with respect to time, it is obvious that

$$(18) \quad \lim_{t \uparrow \tau_\Gamma} S(B_{[0,t]}) = S(B_{[0,\tau_\Gamma]}) \quad \text{a.s. } P^x, x \in \Gamma.$$

Since  $S(B_{[0,t]})$  is invertible and it is nothing else but the signature of the Brownian motion running backward from time  $t$  to 0, (18) implies that

$$(19) \quad \lim_{t \uparrow \tau_\Gamma} (S(B_{[0,t]}))^{-1} = (S(B_{[0,\tau_\Gamma]}))^{-1} \quad \text{a.s. } P^x, x \in \Gamma.$$

Combining (18) and (19), we obtain that

$$\begin{aligned} \lim_{t \uparrow \tau_\Gamma} \Phi_\Gamma(B_t) &= \lim_{t \uparrow \tau_\Gamma} (S(B_{[0,t]}))^{-1} \otimes \lim_{t \uparrow \tau_\Gamma} S(B_{[0,t]}) \otimes \Phi_\Gamma(B_t) \\ &= (S(B_{[0,\tau_\Gamma]}))^{-1} \otimes S(B_{[0,\tau_\Gamma]}) = \mathbf{1}. \end{aligned}$$

Now the proof is complete.  $\square$

**THEOREM 3.3** (A PDE for the expected signature of Brownian motion). *Assume  $\Gamma$  is a bounded domain. Then  $\Phi_\Gamma$  satisfies the following PDE:*

$$(20) \quad \Delta \Phi_\Gamma(z) = - \left( \sum_{i=1}^d e_i \otimes e_i \right) \otimes \Phi_\Gamma(z) - 2 \sum_{i=1}^d \left( e_i \otimes \frac{\partial \Phi_\Gamma(z)}{\partial z_i} \right) \quad \text{if } z \in \Gamma,$$

with the boundary condition that for every  $z \in \Gamma$ ,

$$(21) \quad \lim_{t \uparrow \tau_\Gamma} \Phi_\Gamma(B_t) = \mathbf{1} \quad \text{a.s. in } P^z,$$

and the initial condition:

$$(22) \quad \rho_0(\Phi_\Gamma(z)) = 1 \quad \text{if } z \in \bar{\Gamma},$$

$$(23) \quad \rho_1(\Phi_\Gamma(z)) = 0 \quad \text{if } z \in \bar{\Gamma}.$$

**PROOF.** For any fixed  $z \in \Gamma$ , let us consider a function

$$\begin{aligned} \varphi: \bar{\varepsilon}\mathbb{D} &\rightarrow T((E)), \\ y &\mapsto \Psi(y) \otimes \Phi_\Gamma(z + y), \end{aligned}$$

where  $\bar{\varepsilon} = \text{dist}(z, \partial\Gamma)$ .

By Lemma 3.8, for every  $\varepsilon < \bar{\varepsilon}$ ,

$$\Phi_\Gamma(z) = \frac{1}{d\omega_d \varepsilon^{d-1}} \int_{\partial\mathbb{D}(0,\varepsilon)} \Psi(y) \otimes \Phi_\Gamma(z+y) d\sigma(y)$$

for every  $z \in \Gamma$ .

This implies that  $\varphi$  satisfies the mean value property at 0, that is,

$$(24) \quad \varphi(0) = \Psi(0) \otimes \Phi_\Gamma(z) = \Phi_\Gamma(z) = \frac{1}{d\omega_d \varepsilon^{d-1}} \int_{\partial\mathbb{D}(0,\varepsilon)} \varphi(y) d\sigma(y),$$

for every  $\varepsilon \leq \bar{\varepsilon}$ .

Since  $\Gamma$  is a bounded domain, and in Lemma 3.6 we prove that  $\Phi_\Gamma$  is a well defined and twice differentiable function in the componentwise sense, we see that so is  $\varphi$ . By the mean value property and differentiability of  $\varphi$  at point 0 we immediately have that

$$\Delta(\varphi(y))|_{y=0} = 0.$$

By the chain rule, we obtain

$$(25) \quad \begin{aligned} & \Delta(\varphi(y))|_{y=0} \\ &= \Delta(\Psi(0)) \otimes \Phi_\Gamma(z) + 2 \sum_{i=1}^d \frac{\partial(\Psi(0))}{\partial z_i} \otimes \frac{\partial(\Phi(z))}{\partial z_i} + \Psi(0) \otimes \Delta(\Phi_\Gamma(z)) \\ &= 0. \end{aligned}$$

Recalling Lemma 3.7, we have the following equalities:

$$(26) \quad \Psi(0) = \mathbf{1},$$

$$(27) \quad \frac{\partial(\Psi(0))}{\partial z_i} = e_i \quad \text{for } i = 1, 2, \dots, d \quad \text{and}$$

$$(28) \quad \Delta(\Psi(0)) = \sum_{i=1}^d e_i \otimes e_i.$$

Thus after substituting these into (25) and rearranging the equation, we finally obtain

$$(29) \quad \Delta\Phi_\Gamma(z) = - \left( \sum_{i=1}^d e_i \otimes e_i \right) \otimes \Phi_\Gamma(z) - 2 \sum_{i=1}^d \left( e_i \otimes \frac{\partial\Phi_\Gamma(z)}{\partial z_i} \right).$$

The boundary condition is proved in Lemma 3.9. Moreover by the definition of the signature, it is obvious that  $\rho_0(\Phi(z)) = 1$  and

$$\rho_1(\Phi(z)) = \sum_{i=1}^d \mathbb{E}^z \left[ \int_0^{\tau_\Gamma} \circ dB_t^{(i)} \right] e_i = \sum_{i=1}^d \mathbb{E}^z [B_{\tau_\Gamma}^{(i)} - z^{(i)}] e_i = 0,$$

since Brownian motion is a martingale.  $\square$

REMARK 3.3. In the proof of Theorem 3.3, the essential condition for the domain  $\Gamma$  is actually the well definedness and the componentwise differentiability of the function  $\Phi_\Gamma$ , for which the boundedness of the domain is a sufficient, but not a necessary condition.

The following corollary is an equivalent statement of Theorem 3.3, which states how to use the PDE system to solve each term of the expected signature recursively.

COROLLARY 3.2. *Let  $\Gamma$  be a bounded domain. For every  $n \in \mathbb{N}$  and  $n \geq 2$ , the  $n$ th term of  $\Phi_\Gamma$  satisfies the following PDE:*

$$(30) \quad \begin{aligned} & \Delta(\rho_n(\Phi_\Gamma(z))) \\ &= -2 \sum_{i=1}^d e_i \otimes \frac{\partial \rho_{n-1}(\Phi_\Gamma(z))}{\partial z_i} - \left( \sum_{i=1}^d e_i \otimes e_i \right) \otimes \rho_{n-2}(\Phi_\Gamma(z)), \end{aligned}$$

with the boundary condition that for each  $z \in \partial\Gamma$ ,

$$\lim_{t \uparrow \tau_\Gamma} \rho_n(\Phi_\Gamma(B_t)) = \begin{cases} 0, & \text{if } n \geq 1; \\ 1, & \text{if } n = 0. \end{cases}$$

Moreover  $\rho_0(\Phi_\Gamma(z)) = 1, \rho_1(\Phi_\Gamma(z)) = 0, \forall z \in \bar{\Gamma}$ .

REMARK 3.4. From the corollary it is important to notice that if we have computed the  $(n-1)$ th and  $(n-2)$ th term of expected signature, the right-hand side of the PDE of the  $n$ th term is known. There is a stochastic representation to the solution to this generalized Poisson equation problem with the appropriate Dirichlet boundary condition (Lemma 3.9), and as the solutions are all bounded, this implies that from the PDE and the boundary condition we can resolve all the terms of expected signature recursively [15]. The result is summarized in the following theorem.

THEOREM 3.4. *Let  $\Gamma$  be a bounded domain. For each  $n \in \mathbb{N}$ ,*

$$\varphi_n(z) := - \left( \sum_{i=1}^d e_i \otimes e_i \right) \otimes \rho_{n-2}(\Phi_\Gamma(z)) - 2 \left( \sum_{i=1}^d e_i \otimes \frac{\partial \rho_{n-1}(\Phi_\Gamma(z))}{\partial z_i} \right).$$

Suppose for each  $n \in \mathbb{N}$ ,

$$\mathbb{E}^z \left[ \int_0^{\tau_\Gamma} \varphi_n(B_t) dt \right] < \infty \quad \forall z \in \Gamma.$$

Then it follows that

$$\rho_n(\Phi_\Gamma(z)) = \mathbb{E}^z \left[ \int_0^{\tau_\Gamma} \varphi_n(B_t) dt \right].$$

3.4. *A concrete example: Brownian motion in the unit ball.* Recall that  $E = \mathbb{R}^d$ , and  $\mathbb{D}$  denotes the open unit ball in  $E$  centered at the origin. In this subsection, we will discuss the expected signature of  $d$ -dimensional Brownian motion starting at  $z \in \mathbb{D}$  upon the first exit time from  $\mathbb{D}$ , denoted by  $\Phi_{\mathbb{D}}(z)$  as before.

Let us start with the one-dimensional case, where  $\Phi_{\mathbb{D}}$  can be solved explicitly. For  $d = 1$ ,  $\mathbb{D}$  is just the interval  $(-1, 1)$ . After an easy computation, the probability of hitting 1 at the exit time is

$$\mathbb{P}^x(B_{\tau_{(-1,1)}} = 1) = \frac{x+1}{2},$$

and it follows immediately that

$$\begin{aligned} \rho_n(\Phi_{(-1,1)}) &= \frac{(1-x)^n}{n!} \mathbb{P}^x(B_{\tau_{(-1,1)}} = 1) + \frac{(-1-x)^n}{n!} \mathbb{P}^x(B_{\tau_{(-1,1)}} = -1) \\ &= \frac{1}{2n!} (1-x^2) ((1-x)^{n-1} - (-1-x)^{n-1}). \end{aligned}$$

It is easy to verify that  $\Phi_{(-1,1)}(x)$  satisfies the following ODE, which is consistent with Corollary 3.2:

$$\frac{d^2(\rho_n(\Phi_{(-1,1)}(x)))}{dx^2} = -\rho_{n-1}(\Phi_{(-1,1)}(x)) - 2 \frac{d(\rho_{n-2}(\Phi_{(-1,1)}(x)))}{dx} \quad \forall n \geq 2,$$

with the boundary condition

$$\rho_n(\Phi_{(-1,1)}(x)) = 0 \quad \text{if } x = \pm 1 \quad \forall n \geq 2$$

and the initial condition

$$\begin{aligned} \rho_0(\Phi_{(-1,1)}(x)) &= 1 \quad \text{and} \\ \rho_1(\Phi_{(-1,1)}(x)) &= 0 \quad \forall x \in [-1, 1]. \end{aligned}$$

After computing  $\Phi_{(-1,1)}$ , we are going to show that the expected signature of the  $d$ -dimensional Brownian motion upon the first exit time from the unit ball is in polynomial form using Corollary 3.2. To be precise, we introduce the definition of a polynomial form for a function mapping from a domain  $\Gamma \subset E$  to  $E^{\otimes n}$ .

**DEFINITION 3.5.** Let  $f_n$  be a map from  $\Gamma$  to  $E^{\otimes n}$ , where  $\Gamma$  is a domain in  $E$  endowed with the canonical basis. Let  $g$  be a polynomial in  $E$ . We say that  $f_n$  is in polynomial form with a factor  $g$  if for every index  $I$  with length  $n$ ,  $\pi^I \circ f_n$  is a polynomial with a common factor  $g$ . The degree of the polynomial form  $f_n$  is defined as the maximum of degrees of all  $\pi^I \circ f_n$  over all the indexes  $I$  of length  $n$ .

To show that for every  $n \in \mathbb{N}$ ,  $\rho_n \circ \Phi_{\mathbb{D}}$  is in polynomial form, which is summarized in Theorem 3.5, we need the following auxiliary lemma.

LEMMA 3.10. *For any given polynomial of degree  $n$  denoted by  $f: \mathbb{D} \rightarrow \mathbb{R}$ , the solution to the PDE*

$$\begin{cases} \Delta F(z) = f(z), & \text{if } z \in \mathbb{D}; \\ F(z) = 0, & \text{if } |z| = 1 \end{cases}$$

*exists and it is unique. Moreover it is a polynomial of degree  $n + 2$  with a factor of  $(1 - \sum_{i=1}^d z_i^2)$ .*

PROOF. For any fixed  $n \in \mathbb{N}$ , let us denote the space of the polynomials with degree no more than  $n$  by  $P[n]$ . Define the linear operator  $L$  which maps  $P[n]$  to  $P[n]$  as follows:

$$L(f)(z) := \Delta \left( \left( 1 - \sum_{i=1}^d z_i^2 \right) f(z) \right),$$

where  $f \in P[n]$ .

We are going to show that  $\dim(\text{Im}(L)) = \dim(P[n])$ . Let us consider  $\text{Ker}(L) = \{g \in P[n] \mid L(g) = 0\}$ . For every  $g \in \text{Ker}(L)$ , let  $G$  be  $G(z) := (1 - \sum_{i=1}^d z_i^2)g(z)$ . Then  $G \in P[n+2]$  and satisfies the PDE problem

$$\Delta G(z) = 0 \quad \text{if } z \in \mathbb{D},$$

and the boundary condition

$$G(z) = 0 \quad \text{if } |z| = 1.$$

By the strong maximum principle  $G(z) = 0$  is the unique solution to this PDE problem, so we have  $\text{Ker}(L) = \{0\}$  and  $\dim(\text{Ker}(L)) = 0$ . Since  $L$  is a linear operator, by rank-nullity theorem, it is easy to see that  $\text{Im}(L) \subseteq P[n]$ . Then we can claim that  $L$  is a bijection and  $\text{Im}(L) = P[n]$ , which means that for every  $f \in P[n]$ , there exists a unique polynomial  $g \in P[n]$  such that

$$L(g) = f.$$

Equivalently for every  $f \in P[n]$ , there exists a unique polynomial  $F \in P[n+2]$  such that  $F$  is the unique solution to the following PDE:

$$\begin{cases} \Delta F(z) = f(z), & \text{if } z \in \mathbb{D}; \\ F(z) = 0, & \text{if } |z| = 1, \end{cases}$$

and moreover  $F(z)$  has a factor  $(1 - \sum_{i=1}^d z_i^2)$ .  $\square$

The following proof is an alternative way to prove Lemma 3.10 restricted for the two-dimensional case, but it gives us an algorithm to compute  $F$  explicitly.

PROOF OF LEMMA 3.10. We adopt the notation of  $P[n]$  and the linear operator  $L$  used in the above proof. To prove the lemma, it is equivalent to prove that for  $f \in P[n]$  with  $n \geq 0$ , there exists a homogeneous polynomial denoted by  $g_n$  and a polynomial  $R_n$  which is a polynomial of degree strictly less than  $n$  if  $n \geq 1$  and  $R_0 = 0$  such that

$$L(g_n) = f^* + R_n,$$

where  $f^*$  is the leading term of  $f$ .

For  $n = 0$ , each polynomial  $f \in P[n]$  is a constant function, that is,  $f(z) = c \forall z = (z_1, z_2) \in \mathbb{D}$ . Then  $g(z) := -\frac{c}{4}$  satisfies

$$(31) \quad \Delta \left( \left( 1 - \sum_{i=1}^2 z_i^2 \right) g(z) \right) = f(z) = c.$$

It is easy to show that the solution to this PDE is unique. Thus the statement is true for  $n = 0$ .

For  $n \geq 1$ , since  $g_n$  is a homogeneous polynomial of degree  $n$ , we have

$$L(g_n)(z) = -4(n+1)g_n(z) + \left( 1 - \sum_{i=1}^2 z_i^2 \right) \Delta(g_n(z)).$$

The leading term of  $L(g_n)(z)$  is

$$-4(n+1)g_n(z) - \sum_{i=1}^2 z_i^2 \Delta(g_n(z)),$$

which should match  $f^*$ .

Let us write  $g_n$  in the form

$$g_n(z) = \sum_{j=0}^n a_j z_1^j z_2^{n-j}$$

and  $f^*$  in the form

$$f^*(z) = \sum_{j=0}^n b_j z_1^j z_2^{n-j}.$$

Then

$$\begin{aligned} & -4(n+1)g_n(z) - \sum_{i=1}^2 z_i^2 \Delta(g_n(z)) \\ &= \sum_{j=0}^n [-4(n+1) - j(j-1)1_{\{2 \leq j \leq n\}}] \end{aligned}$$

$$\begin{aligned}
& - (n-j)(n-j-1)1_{\{0 \leq j \leq n-2\}}]a_j z_1^j z_2^{n-j} \\
& - \sum_{j=0}^{n-2} (j+2)(j+1)a_{j+2} z_1^j z_2^{n-j} \\
& - \sum_{j=2}^n (n-j+2)(n-j+1)a_{j-2} z_1^j z_2^{n-j}.
\end{aligned}$$

Whether there exists  $g_n$  such that the leading term of  $\Delta((1 - \sum_{i=1}^2 z_i^2)g_n(z))$  is  $f^*(z)$  depends on whether the following linear equation has solution or not:

$$\begin{aligned}
& [-4(n+1) - j(j-1)1_{\{2 \leq j \leq n\}} - (n-j)(n-j-1)1_{\{0 \leq j \leq n-2\}}]a_j \\
& - (j+2)(j+1)a_{j+2}1_{\{0 \leq j \leq n-2\}} - a_{j-2}(n-j+2)(n-j+1)1_{\{2 \leq j \leq n\}} \\
& = b_j
\end{aligned}$$

for  $j = 0, 1, \dots, n$ , or equivalently in the matrix form

$$(32) \quad M_n \vec{a} = \vec{b},$$

where  $\vec{a} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix}$ ,  $\vec{b} = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{pmatrix}$ , and  $M_n$  is a  $(n+1) \times (n+1)$  matrix which is defined as below:

$$\left\{ \begin{array}{l} M_n(j, j) = [-4(n+1) - (n-j)(n-j-1)], \\ \quad \text{if } j < 2; \\ M_n(j, j) = [-4(n+1) - j(j-1) - (n-j)(n-j-1)], \\ \quad \text{if } j \geq 2, j \leq n-2; \\ M_n(j, j) = [-4(n+1) - j(j-1)], \\ \quad \text{if } j > n-2, j \leq n; \\ M_n(j, j+2) = -(j+2)(j+1), \\ \quad \text{if } j \leq n-2; \\ M_n(j, j-2) = -(n-j+2)(n-j+1), \\ \quad \text{if } j \geq 2; \\ M_n(i, j) = 0, \quad \text{otherwise.} \end{array} \right.$$

The final step is to prove that  $M_n$  is invertible. It is easily verified that  $M_n$  is strictly diagonally dominant, since we have

$$|M_n(j, j)| - \sum_{\substack{i=0 \\ i \neq j}}^n |M_n(i, j)| = 4(n+1) > 0.$$

By the Gershgorin's theorem [8], a strictly diagonally dominant matrix is nonsingular. So  $M_n$  is invertible, and there exists a unique solution  $\vec{a}$  to equation (32),

$$\vec{a} = M_n^{-1} \vec{b}$$

that is, there exists a unique  $g_n$  such that the leading term of  $\Delta((1 - \sum_{i=1}^2 z_i^2)g_n(z))$  equals to  $f^*$ . By induction, we conclude that for every polynomial  $f$  of degree  $n$ , there exists a unique polynomial  $g$  of degree  $n$  such that

$$L(g)(z) = f(z). \quad \square$$

**THEOREM 3.5.** *For each  $n \in \mathbb{N}^+$ ,  $\rho_n \circ \Phi_{\mathbb{D}}(z)$  is in polynomial form of degree no more than  $n$  with a factor  $(1 - \sum_{i=1}^d z_i^2)$ .*

**PROOF.** We prove this by induction.

For  $n = 1$ ,  $\rho_1(\Phi_{\mathbb{D}}(z)) = 0$  and trivially has a factor of  $(1 - \sum_{i=1}^d z_i^2)$ .

For  $n = 2$ , by Corollary 3.2 we have that  $\rho_2(\Phi_{\mathbb{D}}(z))$  satisfies the following PDE:

$$(33) \quad \begin{cases} \Delta \rho_2(\Phi_{\mathbb{D}}(z)) = - \sum_{i=1}^d e_i \otimes e_i, & \text{if } z \in \mathbb{D}; \\ \rho_2(\Phi_{\mathbb{D}}(z)) = 0, & \text{if } |z| = 1. \end{cases}$$

By Lemma 3.10,

$$\rho_2(\Phi_{\mathbb{D}}(z)) = \sum_{i=1}^d \frac{1}{2d} \left( 1 - \sum_{j=1}^d z_j^2 \right) e_i \otimes e_i$$

is the unique solution to (33). So our statement is true for  $n = 1, 2$ .

Suppose that the statement is true for every  $n < N$ . We are going to prove that it is true for  $n = N$ .

By Corollary 3.2, we obtain that for every  $n \geq 2$ ,

$$\Delta \rho_n(\Phi_{\mathbb{D}}(z)) = - \left( \sum_{i=1}^d e_i \otimes e_i \right) \otimes \rho_{n-2}(\Phi_{\mathbb{D}}(z)) - 2 \sum_{i=1}^d \left( e_i \otimes \frac{\partial \rho_{n-1}(\Phi_{\mathbb{D}}(z))}{\partial z_i} \right).$$

By the induction hypothesis, it is easy to show that the right-hand side should be in polynomial form of degree no more than  $n - 2$ . Using Lemma 3.10, each  $\pi^I \circ \Phi_{\mathbb{D}}$  with  $|I| = n$  should be a polynomial of degree no more than  $n$  with a factor of  $1 - \sum_{i=1}^d z_i^2$ , so  $\rho_n(\Phi_{\mathbb{D}})$  is in polynomial form of degree no more than  $n$  with a common factor  $1 - \sum_{i=1}^d z_i^2$ . Now our proof is complete.  $\square$

REMARK 3.5. It is natural to guess that  $\Phi_{\mathbb{D}}(z)$  has a common factor  $1 - |z|^2$ , since it would automatically satisfy the boundary condition that

$$\Phi(z) = \mathbf{1} \quad \forall |z| = 1.$$

In the last part of this section, we give the following truncated expected signature of two-dimensional Brownian motion upon the first exit time from the unit disk up to degree 4:

$$\begin{aligned} \rho_2(\Phi_{\mathbb{D}}(z)) &= \frac{1}{4} \left( 1 - \sum_{i=1}^2 z_i^2 \right) \left( \sum_{i=1}^2 e_i \otimes e_i \right), \\ \rho_3(\Phi_{\mathbb{D}}(z)) &= \left( 1 - \sum_{i=1}^2 z_i^2 \right) \left( - \sum_{i=1}^2 \frac{1}{8} z_i e_i \right) \otimes \left( \sum_{i=1}^2 e_i \otimes e_i \right) \quad \text{and} \\ \rho_4(\Phi_{\mathbb{D}}(z)) &= \left( 1 - \sum_{i=1}^2 z_i^2 \right) \\ &\quad \times \left( D_1(z) e_1 \otimes e_1 + \frac{z_1 z_2}{24} (e_1 \otimes e_2 + e_2 \otimes e_1) + D_2(z) e_2 \otimes e_2 \right) \\ &\quad \otimes \left( \sum_{i=1}^2 e_i \otimes e_i \right), \end{aligned}$$

where

$$\begin{aligned} D_1(z) &= \frac{7}{192} z_1^2 - \frac{1}{192} z_2^2 + \frac{1}{64}, \\ D_2(z) &= \frac{7}{192} z_2^2 - \frac{1}{192} z_1^2 + \frac{1}{64}. \end{aligned}$$

3.5. *The geometric bounds for  $\Phi_{\Gamma}$ .* In this subsection, we aim to show that under certain smoothness condition of a bounded domain  $\Gamma$ , each term of  $\Phi_{\Gamma}$  is geometrically bounded. In order to do so, we start with estimating the upper bounds for  $W^{m,2}$  norm of  $\rho_n \circ \Phi_{\Gamma}$ , using our PDE theorem and then show that  $\rho_n(\Phi_{\Gamma})(z)$  is geometrically bounded by the Sobolev theorem.

### 3.5.1. $W^{m,2}$ bounds for $\rho_n \circ \Phi_{\Gamma}$ .

LEMMA 3.11. *Let  $\Gamma$  be a bounded domain of class  $C^m$  in  $E = \mathbb{R}^d$ , where  $m = \lfloor \frac{d}{2} \rfloor + 1$ . Then there exists a constant  $C$  only depending on  $\Gamma$  and  $d$ , such that for every positive integer  $n \geq 2$ ,*

$$(34) \quad \|\rho_n(\Phi_{\Gamma})\|_{W^{m,2}(\Gamma)} \leq C(\|\rho_{n-1}(\Phi_{\Gamma})\|_{W^{m,2}(\Gamma)} + \|\rho_{n-2}(\Phi_{\Gamma})\|_{W^{m,2}(\Gamma)}).$$

PROOF. Since  $\Gamma$  is a bounded domain of class  $C^m$ , according to Theorem 2.3 there exists a constant  $C_1$  depending only on  $\Gamma$  and  $d$ , independent of  $\rho_n(\Phi_\Gamma)$ , such that

$$\|\rho_n(\Phi_\Gamma)\|_{W^{m,2}(\Gamma)} \leq C_1 \|\Delta \rho_n(\Phi_\Gamma)\|_{W^{m-2,2}(\Gamma)}.$$

Recall the PDE of  $\rho_n(\Phi_\Gamma)$ , that is,

$$(35) \quad \begin{aligned} & \Delta(\rho_n(\Phi_\Gamma(z))) \\ &= -2 \sum_{i=1}^d e_i \otimes \frac{\partial \rho_{n-1}(\Phi_\Gamma(z))}{\partial z_i} - \left( \sum_{i=1}^d e_i \otimes e_i \right) \otimes \rho_{n-2}(\Phi_\Gamma(z)). \end{aligned}$$

Then it follows immediately that

$$\begin{aligned} & \|\rho_n(\Phi_\Gamma)\|_{W^{m,2}(\Gamma)} \\ & \leq C_1 \left( \sum_{i=1}^d \left\| \frac{\partial \rho_{n-1}(\Phi_\Gamma)}{\partial z_i} \right\|_{W^{m-2,2}(\Gamma)} + d \|\rho_{n-2}(\Phi_\Gamma(z))\|_{W^{m-2,2}(\Gamma)} \right) \\ & \leq C_1 \left( \sum_{i=1}^d \|\rho_{n-1}(\Phi_\Gamma)\|_{W^{m-1,2}(\Gamma)} + d \|\rho_{n-2}(\Phi_\Gamma(z))\|_{W^{m-2,2}(\Gamma)} \right). \end{aligned}$$

Since for every  $f \in W^{k,2}(\Gamma)$ ,  $\|f\|_{W^{k-1,2}(\Gamma)} \leq \|f\|_{W^{k,2}(\Gamma)}$ , then we choose  $C = C_1 d$ , and (34) follows.  $\square$

LEMMA 3.12. *Let  $\Gamma$  be a bounded domain of class  $C^m$  in  $\mathbb{R}^d$ , where  $m = \lfloor \frac{d}{2} \rfloor + 1$ . Then there exists a constant  $C$  only depending on  $\Gamma$  and  $d$ , such that for every integer  $n \geq 0$ ,*

$$\|\rho_n(\Phi_\Gamma)\|_{W^{m,2}(\Gamma)} \leq |\Gamma|^{1/2} C^n.$$

PROOF. By Lemma 3.11, there exists a constant  $C_1 > 0$  such that

$$\|\rho_n(\Phi_\Gamma)\|_{W^{m,2}(\Gamma)} \leq C_1 (\|\rho_{n-1}(\Phi_\Gamma)\|_{W^{m,2}(\Gamma)} + \|\rho_{n-2}(\Phi_\Gamma)\|_{W^{m,2}(\Gamma)}).$$

We choose  $C = C_1 + 1$ . Let us prove this statement by induction on  $n$ . If  $n = 0$ ,  $\rho_0(\Phi_\Gamma(z)) = 1$  where  $z \in \Gamma$ , thus

$$\|\rho_0(\Phi_\Gamma)\|_{W^{m,2}(\Gamma)} = |\Gamma|^{1/2} \leq |\Gamma|^{1/2} C^0.$$

It is obvious that if  $n = 1$ ,  $\rho_1(\Phi_\Gamma(z)) = 0$  where  $z \in \Gamma$ , thus

$$\|\rho_1(\Phi_\Gamma)\|_{W^{m,2}(\Gamma)} = 0 \leq |\Gamma|^{1/2} C^1.$$

By the induction hypothesis, we have that

$$\begin{aligned} \|\rho_n \circ \Phi_\Gamma\|_{W^{m,2}(\Gamma)} & \leq C_1 (|\Gamma|^{1/2} C^{n-1} + |\Gamma|^{1/2} C^{n-2}) \\ & = |\Gamma|^{1/2} C^{n-2} C_1 (C + 1) \leq |\Gamma|^{1/2} C^n, \end{aligned}$$

since  $C = C_1 + 1$  and  $C_1(C + 1) = C^2 - 1 \leq C^2$ .  $\square$

3.5.2. *The geometric bounds for  $|\rho_n(\Phi_\Gamma(x))|$ .*

**THEOREM 3.6.** *Let  $\Gamma$  be of the class  $C^m$  and strong Lipschitz in  $\mathbb{R}^d$ . Then there exists a constant  $C$  depending on  $d$  and  $\Gamma$ , such that for every  $x \in \Gamma$ , for every positive integer  $n$ ,*

$$|\rho_n(\Phi_\Gamma(x))| \leq C^n.$$

**PROOF.** By Lemma 3.12, there exists a constant  $C_1$  such that for every  $n \in \mathbb{N}$ ,

$$\|\rho_n \circ \Phi_\Gamma\|_{W^{m,2}(\Gamma)} \leq C_1^n.$$

According to Theorem 2.2, there is a constant  $C_2(d, \Gamma)$  such that

$$|\rho_n \circ \Phi_\Gamma(x)| \leq C_2(d, \Gamma) \|\rho_n \circ \Phi_\Gamma\|_{W^{m,2}(\Gamma)}.$$

Let  $C = C_1 \max\{C_2, 1\}$ . Then it obvious that for every  $x \in \Gamma$ ,

$$|\rho_n \circ \Phi_\Gamma(x)| \leq C_2 d C_1^n \leq C^n. \quad \square$$

**REMARK 3.6.** According to Chevyrev [2], when the expected signature is compact-like, it determines the law of the signature. This result directs us to study the decay rates of  $\Phi_\Gamma$ . So far our best result is that  $\Phi_\Gamma$  is geometrically bounded, and it provides insufficient information for us to conclude whether  $\Phi_\Gamma$  is compact-like or not. Because the geometric boundedness of a tensor series does not imply that this tensor series is compact-like. It would be still unclear even in the simplest case that  $\Gamma = \mathbb{D}$  whether the expected signature of Brownian motion characterizes the law of the signature of stopped Brownian motion. One difficulty comes from the tail behavior of higher-order iterated integrals of Brownian motion; for example, the Lévy area of the Brownian motion stopped at the exit time from the disk is shown to have only exponential tail [14]. However, the story is not simply about tail behavior. The interaction between iterated integrals is clearly of great importance. We can see this by looking at the one-dimensional Gaussian random variable  $X$ . The law of  $X$  is certainly determined by its moments. On the other hand, it is known that the law of  $Y := X^3$  is not determined via its moments [1]. Nonetheless the joint distribution of  $(X, Y)$  is determined by its moments. This is because one can deduce from the moments of  $(X, Y)$  that the expected value of  $(X^3 - Y)^2$  is zero, hence recovering the algebraic relation  $X^3 = Y$  almost surely.

3.6. *A discrete analogy: The expected signature of a simple random walk up to an exit time.* Let  $X_1, X_2, \dots$  be independent, identically distributed random variables on a probability space  $(\Omega, \mathcal{F}, \mathbf{P})$  taking values in the integer lattice  $\mathbb{Z}^d$  with

$$\mathbf{P}\{X_j = e\} = \frac{1}{2d}, \quad |e| = 1.$$

A simple random walk starting at  $x \in \mathbb{Z}^d$  is a stochastic process  $S_n$  indexed by the nonnegative integers, with  $S_0 = x$ . We denote by  $\Gamma$  a regular domain of the integer lattice and denote by  $\tau_\Gamma$  the first exit time from  $\Gamma$ ; the definition of a regular domain can be found in [10]. The process  $S_n$  can be viewed as a random lattice path of step  $n$ , so that its iterated integrals are well defined. Moreover  $\Phi_\Gamma(x)$  is defined as the expected signature of a simple random walk starting at  $x$  and ending at the first exit time from  $\Gamma$  ( $\Gamma \subseteq \mathbb{Z}^d$  is a finite set). By the multiplicative property of the signature and the strong Markov property of the simple random walk, we have the following important equation: for every  $x \in \Gamma$ ,

$$(36) \quad \Phi_\Gamma(x) = \sum_{|e_j|=1} \frac{1}{2d} \exp(e_j) \otimes \Phi_\Gamma(x + e_j),$$

where for  $j \in \{1, \dots, d\}$ ,  $e_j$  is the unit vector in  $\mathbb{Z}^d$  with  $j$ th component 1, and for  $j \in \{d+1, \dots, 2d\}$ ,  $e_j$  is the unit vector in  $\mathbb{Z}^d$  with  $(j-d)$ th component  $-1$ .

Rewriting equation (36), we have

$$\rho_n(\Phi_\Gamma(x)) = \sum_{|e_j|=1} \frac{1}{2d} \sum_{i=0}^n \frac{(e_j)^{\otimes i}}{i!} \otimes \rho_{n-i}(\Phi_\Gamma(x + e_j)).$$

This is equivalent to

$$(37) \quad -\Delta \rho_n(\Phi_\Gamma(x)) = \sum_{|e_j|=1} \frac{1}{2d} \sum_{i=1}^n \frac{(e_j)^{\otimes i}}{i!} \otimes \rho_{n-i}(\Phi_\Gamma(x + e_j)),$$

where  $\Delta$  denotes the discrete Laplace operator; that is, for any function  $f: \Gamma \rightarrow E^{\otimes n}$ ,

$$\Delta f(x) = \frac{1}{2d} \sum_{i=1}^{2d} (f(x + e_i) - f(x)).$$

It is also easy to verify that for every  $x \in \Gamma^c$ , the following equation holds:

$$(38) \quad \Phi_\Gamma(x) = \mathbf{1} = (1, 0, 0, \dots),$$

where  $\Gamma^c$  is the complement of  $\Gamma$  in  $\mathbb{Z}^d$ .

We summarize our result in the following theorem.

**THEOREM 3.7.** *Let  $\Gamma \subseteq \mathbb{Z}^d$  be a finite set. Then  $\Phi_\Gamma: \mathbb{Z}^d \rightarrow T((E))$  satisfies the following conditions:*

- (1)  $\forall x \in \Gamma^c, \Phi_\Gamma(x) = \mathbf{1}$ ;
- (2)  $\forall x \in \Gamma, \rho_0(\Phi_\Gamma(x)) = 1, \rho_1(\Phi_\Gamma(x)) = 0$ ;
- (3)  $\forall n \geq 2, \forall x \in \Gamma,$

$$\Delta \rho_n(\Phi_\Gamma(x)) = - \sum_{|e_j|=1} \frac{1}{2d} \sum_{i=1}^n \frac{e_j^{\otimes i}}{i!} \otimes \rho_{n-i}(\Phi_\Gamma(x + e_j)).$$

**REMARK 3.7.** Notice that the right-hand side of (37) is determined totally by the truncated expected signature up to degree  $n - 1$ . This indicates that we can solve  $\Phi_\Gamma(x)$  recursively just as in the Brownian motion case. Classical potential theory guarantees that we can recursively solve a system of finite difference problems to obtain the whole expected signature of a simple random walk up to the first exit time.

**THEOREM 3.8.** *Let  $\Gamma \subseteq \mathbb{Z}^d$  be a finite set,  $F: \Gamma^c \rightarrow \mathbb{R}, g: \Gamma \rightarrow \mathbb{R}$ . Then the unique function  $f: \mathbb{Z}^d \rightarrow \mathbb{R}$  satisfying*

- (a)  $\Delta f(x) = -g(x), \quad x \in \Gamma,$
- (b)  $f(x) = F(x), \quad x \in \Gamma^c,$

is

$$f(x) = \mathbb{E}^x \left[ F(S_{\tau_\Gamma}) + \sum_{j=0}^{\tau_\Gamma-1} g(S_j) \right].$$

Immediately we have the following theorem in our setting.

**THEOREM 3.9.** *Let  $\Gamma \subseteq \mathbb{Z}^d$  be a finite set. Then  $\Phi_\Gamma: \mathbb{Z}^d \rightarrow T((E))$  is given as follows:*

- (1)  $\forall x \in \Gamma^c, \Phi_\Gamma(x) = \mathbf{1}$ ;
- (2)  $\forall x \in \Gamma, \rho_0(\Phi_\Gamma(x)) = 1, \rho_1(\Phi_\Gamma(x)) = 0$ ;
- (3)  $\forall n \geq 2, \forall x \in \Gamma,$

$$\rho_n(\Phi_\Gamma(x)) = \mathbb{E}^x \left[ \sum_{j=0}^{\tau_\Gamma-1} g_n(S_j) \right],$$

where the integration is understood in componentwise sense and

$$g_n(x) = \sum_{|e_j|=1} \frac{1}{2d} \sum_{i=1}^n \frac{(e_j)^{\otimes i}}{i!} \otimes \rho_{n-i}(\Phi_\Gamma(x + e_j)).$$

**Acknowledgments.** We thank the referees and the Associate Editor for their thorough review and highly appreciate their comments and suggestions, which significantly contributed to improve the quality of this paper. We would both like to thank Oxford-Man Institute for their funding.

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