REGULARIZATION STRATEGY FOR INVERSE PROBLEM FOR 1+1 DIMENSIONAL WAVE EQUATION

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Abstract. An inverse boundary value problem for a 1+1 dimensional wave equation with wave speed \( c(x) \) is considered. We give a regularisation strategy for inverting the map \( A : c \mapsto \Lambda \), where \( \Lambda \) is the hyperbolic Neumann-to-Dirichlet map corresponding to the wave speed \( c \). That is, we consider the case when we are given a perturbation of the Neumann-to-Dirichlet map \( \tilde{\Lambda} = \Lambda + \mathcal{E} \), where \( \mathcal{E} \) corresponds to the measurement errors, and reconstruct an approximative wave speed \( \tilde{c} \). We emphasize that \( \tilde{\Lambda} \) may not be in the range of the map \( A \). We show that the reconstructed wave speed \( \tilde{c} \) satisfies \( \| \tilde{c} - c \| \leq C \| \mathcal{E} \|^{1/54} \). Our regularisation strategy is based on a new formula to compute \( c \) from \( \Lambda \).

Keywords: Inverse problem, regularization theory, wave equation.

1. Introduction

We consider an inverse boundary value problem for the wave equation

\[
\left( \frac{\partial^2}{\partial t^2} - c(x)^2 \frac{\partial^2}{\partial x^2} \right) u(t, x) = 0,
\]

and introduce a regularization strategy to recover the sound speed \( c(x) \) by using the knowledge of perturbed Neumann-to-Dirichlet map \( \tilde{\Lambda} \). Our approach is based on the Boundary Control method \([2, 6, 54]\).

A variant of the Boundary Control method, called the iterative time-reversal control method, was introduced in \([9]\). The method was later modified in \([15]\) to focus the energy of a wave at a fixed time, and in \([47]\) to solve an inverse obstacle problem for the wave equation. Here we introduce yet another modification of the iterative time-reversal control method that is tailored for the 1+1 dimensional wave equation.

Classical regularization theory is explained in \([16]\). Iterative regularization of both linear and nonlinear inverse problems and convergence rates are discussed in Hilbert space setting in \([10, 18, 20, 41, 43]\) and in Banach space setting in \([19, 23, 24, 30, 48, 49, 50]\). Our new results...
give a direct regularization method for the nonlinear inverse problem for the wave equation. The result contains an explicit (but not necessarily optimal) convergence rate.

By direct methods for non-linear problems we mean explicit construction of a non-linear map solving the problem without resorting to a local optimisation method. In our case the map is given by (63) below. The advantage of direct approaches is that they do not suffer from the possibility that the algorithm converges to a local minimum. In particular, they do not require a priori knowledge that the solution is in a small neighbourhood of a given function. There are currently only few regularized direct methods for non-linear inverse problems. An example is a regularisation algorithm for the inverse problem for the conductivity equation in [31]. Also, a direct regularized inversion for blind deconvolution is presented in [21].

1.1. Statement of the results. We define

\[
\|c\|_{C^k(M)} = \sum_{p=0}^{k} \sup_{x \in (0, \infty)} |\partial_x^p c(x)|, \tag{1}
\]

where we denote by \( M \) the half axis \( M = [0, \infty) \subset \mathbb{R} \). We denote the set of bounded \( C^k(M) \)-functions by

\[
C^k_b(M) = \{ c \in C^k(M); \|c\|_{C^k(M)} < \infty \}. \tag{2}
\]

Let \( C_0, C_1, L_0, L_1, m > 0 \) and define the space of \( k \) times differentiable velocity functions

\[
\mathcal{V}^k = \{ c \in C^k(M); C_0 \leq c(x) \leq C_1, \|c\|_{C^k(M)} \leq m, c-1 \in C^k_0([L_0, L_1]) \}. \tag{3}
\]

Here \( C^k_0([L_0, L_1]) \) is the subspace of functions in \( C^k_b(M) \) that are supported on \([L_0, L_1]\). Let

\[
T > \frac{L_1}{C_0}. \tag{4}
\]

For \( c \in \mathcal{V}^2 \) and \( f \in L^2(0, 2T) \), the boundary value problem

\[
\left( \frac{\partial^2}{\partial t^2} - c(x)^2 \frac{\partial^2}{\partial x^2} \right) u(x,t) = 0 \quad \text{in} \ M \times (0, 2T),
\]

\[
\partial_x u(0,t) = f(t),
\]

\[
u|_{t=0} = 0, \quad \partial_t u|_{t=0} = 0,
\]
has a unique solution $u = u^f \in H^1(M \times (0, 2T))$. Using this solution we define the Neumann-to-Dirichlet operator $\Lambda = \Lambda_c$,

\begin{equation}
\Lambda : L^2(0, 2T) \rightarrow L^2(0, 2T), \quad \Lambda f = u^f|_{x=0}.
\end{equation}

We define for a Banach space $E$

\[ \mathcal{L}(E) := \{ A : E \rightarrow E; A \text{ is linear and continuous} \}. \]

Let $X = L^\infty(M)$, $Z = C^2_b(M)$ and $Y = \mathcal{L}(L^2(0, 2T))$. We define $\mathcal{D}(\mathcal{A}) = \mathcal{V}^2$ and the direct map

\begin{equation}
\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset Z \rightarrow \mathcal{R}(\mathcal{A}) \subset Y, \quad \mathcal{A}(c) = \Lambda.
\end{equation}

The notation in (6) means that the range $\mathcal{R}(\mathcal{A}) = \mathcal{A}(\mathcal{V}^2)$ and the domain $\mathcal{D}(\mathcal{A})$ are equipped with the topologies of $Y$ and $Z$, respectively. We show in Appendix A, Theorem 5 that the maps (5) and (6) are continuous.

We consider the inverse problem to recover the velocity function $c$ by using the boundary measurements $\Lambda$. In our case, it is well-known that $\mathcal{A}$ is invertible, see e.g. [12, 13]. Let us record the following:

**Theorem 1.** $\mathcal{A}$ is invertible, that is, there exist a map

\[ \mathcal{A}^{-1} : \mathcal{A}(\mathcal{V}^2) \subset Y \rightarrow \mathcal{V}^2 \subset Z, \quad \mathcal{A}^{-1}(\Lambda) = c. \]

For the convenience of the reader we give a proof of Theorem 1 in Section 2, where we also give a new formula to compute $c$ from $\Lambda$. When we restrict $\mathcal{A}$ to the set $\mathcal{V}^3 \subset \mathcal{V}^2$, the map $\mathcal{A}|_{\mathcal{V}^3} : \mathcal{V}^3 \subset Z \rightarrow \mathcal{A}(\mathcal{V}^3)$ has a continuous inverse operator in the following sense:

**Theorem 2.** The inverse map

\[ \mathcal{A}^{-1} : \mathcal{A}(\mathcal{V}^3) \subset Y \rightarrow \mathcal{V}^3 \subset Z, \quad \mathcal{A}^{-1}(\Lambda) = c, \]

is continuous.

Below, we will prove a result for the continuity modulus of $\mathcal{A}^{-1}$. The continuity of $\mathcal{A}^{-1}$ in Theorem 2 is abstract in the sense that it does not contain quantitative estimates. For the convenience of the reader we give a proof of Theorem 2 in Appendix B.

Our main result concerns perturbations of the Neumann-to-Dirichlet operator of the form

\begin{equation}
\tilde{\Lambda} = \Lambda + \mathcal{E},
\end{equation}

where $\mathcal{E} \in Y$ models the measurement error. We assume that $\|\mathcal{E}\|_Y \leq \epsilon$, where $\epsilon > 0$ is known. In this situation we can not use the map $\mathcal{A}^{-1}$ to calculate function $c$ since $\tilde{\Lambda}$ may not be in the range $\mathcal{R}(\mathcal{A})$. We recall the definition of a regularization strategy, see e.g. [16] and [30].
**Definition 1.** Let $Z, Y$ be Banach spaces and $\Omega \subset Z$. Let $A : \Omega \subset Z \to Y$ be a continuous mapping. Let $\alpha_0 \in (0, \infty]$. A family of continuous maps $R_\alpha : Y \to Z$ parametrized by $0 < \alpha < \alpha_0$ is called a regularization strategy for $A : \Omega \to Y$ if

$$\lim_{\alpha \to 0} R_\alpha(A(c)) = c$$

for every $c \in \Omega$. A regularization strategy is called admissible, if the parameter $\alpha$ is chosen as a function of $\epsilon > 0$ so that $\lim_{\epsilon \to 0} \alpha(\epsilon) = 0$ and for every $c \in \Omega$

$$\lim_{\epsilon \to 0} \sup \left\{ \left\| R_{\alpha(\epsilon)}(\tilde{\Lambda}) - c \right\|_Z : \tilde{\Lambda} \in Y, \left\| \tilde{\Lambda} - A(c) \right\|_Y \leq \epsilon \right\} = 0.$$

Below we will use Definition 1 for $A$ given in (6) with $\Omega = V^3$. Figure 1 gives a schematic illustration of regularization.

![Figure 1](image)

**Figure 1.** The idea of regularization is to construct a family $R_{\alpha(\epsilon)}$ of continuous maps from the data space $Y$ to the model space $Z$ in such a way that $c$ can be approximately recovered from noisy data $\tilde{\Lambda}$. For a smaller noise level $\epsilon$ the approximation $R_{\alpha(\epsilon)}(\tilde{\Lambda})$ is closer to $c$. More details can be found in [44, Fig. 11.5].

We are now ready to formulate our main result:

**Theorem 3.** Let $\beta = \frac{1}{54}$. For operator $A : V^3 \subset Z \to Y$, there exists an admissible regularization strategy $R_\alpha$ with the choice of parameter

$$\alpha(\epsilon) = 2\frac{13}{54}T^2 \epsilon^{\frac{2}{3}}$$

that satisfies the following: For every $c \in V^3$ there are $\epsilon_0$ and $C > 0$ such that

$$\sup \left\{ \left\| R_{\alpha(\epsilon)}(\tilde{\Lambda}) - c \right\|_Z : \tilde{\Lambda} \in Y, \left\| A(c) - \tilde{\Lambda} \right\|_Y \leq \epsilon \right\} \leq C\epsilon^{\beta},$$
for all $\epsilon \in (0, \epsilon_0)$.

We will give explicit choices of $R_\alpha$ and $\epsilon_0$, in formulas \((62), (63),\) and \((64)\) below. For the convenience of the reader we give a short summary on the regularization strategy. Assume that we are given $\tilde{\Lambda} \in Y$, that is, the Neumann-to-Dirichlet map for the unknown wave speed $c(x)$ with measurements errors. Then the regularization strategy is obtained by doing the following steps:

1. Using \((8)\) and \((24)\) we calculate the operator $\tilde{H}_r = P_r(\tilde{R}\tilde{A}R - J\tilde{A})P_r$ for $r \in [0, T]$. This operator determines approximately the inner products of the waves by $\langle u_{f1}(T), u_{f2}(T) \rangle_{L^2(M)} \approx \langle \tilde{H}_r f_1, f_2 \rangle_{L^2}$ for all boundary sources $f_1, f_2 \in L^2(T-r, T)$.

2. Using operator $\tilde{H}_r$ we construct in \((34)\) a source $\tilde{f}_{\alpha,r}$ that approximates the solution $f_{\alpha,r}$ of the minimization problem \((13)\). Here, the source $f_{\alpha,r}$ produces a wave such that $u_{f_{\alpha,r}}(t,x)|_{t=T}$ is close to the indicator function $1_{M(r)}(x)$ of the domain of influence $M(r)$, see \((11)\) and Figure 2.

3. Using sources $\tilde{f}_{\alpha,r}$ we compute approximately the volumes $V(r) = \text{Vol}_c(M(r))$ of the domains of influences, see \((20)\).

4. Using finite differences we compute approximate values of the derivative of the volume of the domain influences $\partial_r V(r)$, see \((44)\).

5. We interpolate the obtained values of $\partial_r V(r)$. This determines the approximate values of the wave speed $v(r)$ in the travel time coordinates, see \((21)\).

6. Finally, we change coordinates from the travel time coordinates to the Euclidean coordinates to obtain the approximate values of the wave speed $c(x)$ for $x \in M$.

1.2. Previous literature. From the point of view of uniqueness questions, the inverse problem for the 1+1 dimensional wave equation is equivalent with the one dimensional inverse boundary spectral problem. The latter problem was thoroughly studied in 1950s \([17, 32, 42]\) and we refer to \([22]\ pp. 65-67\) for a historical overview. In 1960s Blagovečenskii \([12, 13]\) developed an approach to solve the inverse problem for the 1+1 dimensional wave equation without reducing the problem to the inverse boundary spectral problem. This and later dynamical methods have the advantage over spectral methods that they require data only on a finite time interval. Applications of 1-dimensional inverse problems have been discussed widely in \([11, 22, 26]\).

The method in the present paper is a variant of the Boundary Control method that was pioneered by M. Belishev \([2]\) and developed by M.
Belishev and Y. Kurylev [5, 6] in late 80s and early 90s. Of crucial importance for the method was the result of D. Tataru [54] concerning a Holmgren-type uniqueness theorem for non-analytic coefficients. The Boundary Control method for multidimensional inverse problems has been summarized in [3, 26], and considered for 1+1 dimensional scalar problems in [4, 7] and for multidimensional scalar problems in [25, 28, 33, 36, 37]. For systems it has been considered in [34, 35]. Stability results for the method have been considered in [1] and [29].

The inverse problem for the wave equation can be solved also by using complex geometrical optics solutions. These solutions were developed in the context of elliptic inverse boundary value problems [53], and in [45] they were employed to solve an inverse boundary spectral problem. Local stability results can be proven using (real) geometrical optics solutions [8, 51, 52], and in [40] a stability result was proved by using ideas from the Boundary Control method together with complex geometrical optics solutions. Finally we mention the important method based on Carleman estimates [14] that can be used to show stability results when the initial data for the wave equation is non-vanishing.

2. Modification of the iterative time-reversal control method

In this section we prove Theorem 1 in such a way that we can utilize the proof to construct a regularization strategy as in Theorem 3. Let \( \Lambda \) be as defined in [5]. Let \( r \in [0, T] \). We define linear operators in \( Y \) by

\[
Jf(t) = \frac{1}{2} \int_0^{2T} 1_\Delta(t, s) f(s) ds,
\]

\[
Rf(t) = f(2T - t), \quad K = RARJ - JA,
\]

\[
Bf(t) = 1_{(0, T)}(t) \int_t^T f(s) ds, \quad Pf(t) = 1_{(T - r, T)}(t) f(t),
\]

where

\[
1_\Delta(t, s) = \begin{cases} 1, & t + s \leq 2T \text{ and } s > t > 0, \\ 0, & \text{otherwise} \end{cases}
\]

and

\[
1_{(T - r, T)}(t) = \begin{cases} 1, & t \in (T - r, T), \\ 0, & \text{otherwise}. \end{cases}
\]
Let \( f \in L^2(0, 2T) \). Using the solution \( u^f \in H^1(M \times (0, 2T)) \) of (4) we define
\[
U_T : L^2(0, 2T) \mapsto H^1(M), \quad U_T f = u^f|_{t=T}.
\]
We show in Appendix A, Theorem 5, that the map (9) is continuous. Let us denote \( dV = c^{-2}dx \) and \( u^f(T) = u^f|_{t=T} \). Let us recall the Blagovestchenskii identities
\[
\langle u^f(T), 1 \rangle_{L^2(M, dV)} = \langle f, B1 \rangle_{L^2(0, 2T)},
\]
\[
\langle u^f(T), u^h(T) \rangle_{L^2(M, dV)} = \langle f, Kh \rangle_{L^2(0, 2T)}.
\]
The identities (10) originate from [11], and their proofs can be found e.g. in [9]. We define the domain of influence
\[
M(r) = \{ x \in M; d(x, 0) \leq r \},
\]
where \( d(x, 0) = \int_0^x \frac{1}{\omega(t)} dt \) is the travel time of the waves from 0 to the point \( x \). See Figure 2 for a visualization of \( M(r) \).

**Figure 2.** When the boundary source \( f \) satisfies, \( \text{supp}(f) \subset [T - r, T] \), the solution \( u^f(t, x)|_{t=T} \) at time \( T \) is supported in the domain of influence \( M(r) \).

We use the following result that is closely related to [9, 46].
Theorem 4. Let \( r \in [0, T] \) and \( \alpha > 0 \). Let \( K, B, \) and \( P_r \) be as defined in (8). Let us define
\[
S_r = \{ f \in L^2(0, 2T) : \text{supp}(f) \subset [T - r, T] \}.
\]
Then the regularized minimization problem
\[
\min_{f \in S_r} \left( \langle f, Kf \rangle_{L^2(0,2T)} - 2\langle f, B1 \rangle_{L^2(0,2T)} + \alpha \| f \|^2_{L^2(0,2T)} \right).
\]
has unique minimizer
\[
f_{\alpha,r} = (P_rKP_r + \alpha)^{-1}P_rB1
\]
and the map \( r \mapsto f_{\alpha,r} \) is continuous \([0, T] \to L^2(0,2T)\). Moreover \( u^{f_{\alpha,r}}(T) \) converges to the indicator function of the domain of influence,
\[
\lim_{\alpha \to 0} \| u^{f_{\alpha,r}}(T) - 1_{M(r)} \|_{L^2(M;dV)} = 0.
\]
For the convenience of the reader we give a proof.

Proof of Theorem 4. Let \( \alpha > 0 \) and let \( f \in S_r \). We define the energy function
\[
E(f) : = \langle f, Kf \rangle_{L^2(0,2T)} - 2\langle f, B1 \rangle_{L^2(0,2T)} + \alpha \| f \|^2_{L^2(0,2T)}.
\]
The finite speed of wave propagation implies \( \text{supp}(u^f(T)) \subset M(r) \). Using (10) we can write
\[
E(f) = \| u^f(T) - 1_{M(r)} \|^2_{L^2(M;dV)} - \| 1_{M(r)} \|^2_{L^2(M;dV)} + \alpha \| f \|^2_{L^2(0,2T)}.
\]
Let \( (f_j)_{j=1}^{\infty} \subset S_r \) be such that
\[
\lim_{j \to \infty} E(f_j) = \inf_{f \in S_r} E(f).
\]
Then
\[
\alpha \| f_j \|^2_{L^2(0,2T)} \leq E(f_j) + \| 1_{M(r)} \|^2_{L^2(M;dV)},
\]
and we see that \( (f_j)_{j=1}^{\infty} \) is bounded in \( S_r \). As \( S_r \) is a Hilbert space, there is a subsequence of \( (f_j)_{j=1}^{\infty} \) converging weakly in \( S_r \). Let us denote the limit by \( f_\infty \in S_r \) and the subsequence still by \( (f_j)_{j=1}^{\infty} \).

The map \( U_T : L^2(0,2T) \to H^1(M) \), as defined in (9), is bounded. The embedding \( I : H^1(M) \to L^2(M) \) is compact and thus \( U_T : f \mapsto u^f(T) \) is a compact operator
\[
U_T : L^2(0,2T) \to L^2(M).
\]
Hence we have a subsequence \( (f_j)_{j=1}^{\infty} \) for which \( u^{f_j}(T) \to u^{f_\infty}(T) \) in \( L^2(M) \) as \( j \to \infty \). Moreover, the weak convergence implies
\[
\| f_\infty \|^2_{L^2(0,2T)} \leq \liminf_{j \to \infty} \| f_j \|^2_{L^2(0,2T)}.
\]
Thus

\[ E(f_\infty) = \lim_{j \to \infty} \| u^{f_j}(T) - 1_M(r) \|^2_{L^2(M;dV)} - \left\| 1_M(r) \right\|_{L^2(M;dV)}^2 + \alpha \left\| f_\infty \right\|_{L^2(0,2T)}^2 \]

\[ \leq \lim_{j \to \infty} \| u^{f_j}(T) - 1_M(r) \|^2_{L^2(M;dV)} - \left\| 1_M(r) \right\|_{L^2(M;dV)}^2 + \alpha \liminf_{j \to \infty} \left\| f_j \right\|_{L^2(0,2T)}^2 \]

\[ = \liminf_{j \to \infty} E(f_j) = \inf_{f \in S_r} E(f), \]

and thus \( f_\infty \in S_r \) is a minimizer for (16). We denote by \( D_h \) the Fréchet derivative to direction \( h \). Note that \( \inf_{f \in S_r} E(f) = \inf_{f \in L^2(0,2T)} E(P_r f) \).

If

\[ 0 = D_h E(f) = 2\langle h, P_r K P_r f \rangle_{L^2(0,2T)} - 2\langle h, P_r B1 \rangle_{L^2(0,2T)} + 2\alpha \langle h, f \rangle_{L^2(0,2T)}, \]

for all \( h \in S_r \subset L^2(0,2T) \), then

\[ (P_r K P_r + \alpha) f = P_r B1. \]

Using (10) we have

\[ \langle (P_r K P_r + \alpha) f, f \rangle_{L^2(0,2T)} = \langle u^{P_r f}(T), u^{P_r f}(T) \rangle_{L^2(M;dV)} + \langle \alpha f, f \rangle_{L^2(0,2T)}. \]

Operator \( P_r K P_r + \alpha \) is coercive when \( \alpha > 0 \). The Lax-Milgram Theorem implies that it is invertible, and we have an expression for minimizer

\[ f_{\alpha,r} := f_\infty = (P_r K P_r + \alpha)^{-1} P_r B1. \]

According to [54], see also [27], we know that

\[ \{ u^f(T) \in L^2(M(r)); \; f \in S_r \} \]

is dense in \( L^2(M(r)) \). Let \( \delta > 0 \). For \( \epsilon = \frac{\delta^2}{2} \), let us choose \( f_\epsilon \in S_r \), \( f_\epsilon \neq 0 \) such that

\[ \| u^{f_\epsilon}(T) - 1_M(r) \|^2_{L^2(M;dV)} \leq \epsilon. \]

Using (17) we have

\[ \| u^{f_{\alpha,r}}(T) - 1_M(r) \|^2_{L^2(M;dV)} \leq E(f_{\alpha,r}) + \left\| 1_M(r) \right\|_{L^2(M;dV)}^2. \]

Because \( E(f_{\alpha,r}) \leq E(f_\epsilon) \) we have

\[ \| u^{f_{\alpha,r}}(T) - 1_M(r) \|^2_{L^2(M;dV)} \leq \| u^{f_\epsilon}(T) - 1_M(r) \|^2_{L^2(M;dV)} + \alpha \| f_\epsilon \|^2. \]

\[ \leq \epsilon + \alpha \| f_\epsilon \|^2. \]

When \( 0 < \alpha < \alpha_r = \frac{\delta^2}{2\| f_\epsilon \|^2} \), we see that

\[ \| u^{f_{\alpha,r}}(T) - 1_M(r) \|_{L^2(M;dV)} \leq (\epsilon + \alpha \| f_\epsilon \|^2)^{\frac{1}{2}} = \delta. \]

Thus

\[ \lim_{\alpha \to 0} \| u^{f_{\alpha,r}}(T) - 1_M(r) \|_{L^2(M;dV)} = 0. \]
We define the travel time coordinates for $x \in M$ by
\[ \tau : [0, \infty) \to [0, \infty), \quad \tau(x) = d(x, 0). \]
The function $\tau$ is strictly increasing and we denote its inverse by
\[ \chi = \tau^{-1} : [0, \infty) \to [0, \infty). \]
We have
\[ \chi(0) = 0, \quad \chi'(t) = \frac{1}{\tau'(\chi(t))} = c(\chi(t)). \]
Thus denoting $v(t) = c(\chi(t))$ and using $V(r)$ to denote the volume of $M(r)$ with respect to the measure $dV$ we have
\[ V(r) = \|1_{M(r)}\|_{L^2(M; dV)}^2 = \int_0^{\chi(r)} \frac{dx}{c(x)^2} = \int_0^{\chi(r)} \frac{\chi'(t)dt}{v(t)^2} = \int_0^{\chi(r)} \frac{dt}{v(t)}. \]
Note that $M(r) = [0, \chi(r)]$. In particular, $V(r)$ determines the wave speed in the travel time coordinates,
\[ v(r) = \frac{1}{\partial_r V(r)}, \]
and also in the original coordinates since
\[ c(x) = v(\chi^{-1}(x)), \quad \chi(t) = \int_0^t v(t')dt'. \]
Using Theorem 4 and (10) we have a method to compute the volumes of the domains of influence
\[ V(r) = \|1_{M(r)}\|_{L^2(M; dV)}^2 = \lim_{\alpha \to 0} \langle f_{\alpha, r}, B1 \rangle_{L^2(0,2T)}, \]
where $r \in [0,T]$. We are ready to prove Theorem 1.

**Proof of Theorem 1.** For a given measurement $\Lambda$, Theorem 4 and equations (21), (22), (23) give us a way to calculate for all $x \in (0, L)$ the value of the velocity function
\[ c(x) = v(\chi^{-1}(x)) = A^{-1}(\Lambda)(x). \]
As we assumed that outside of the interval $(0, L)$ the function $c$ is identically one, the proof for the existence of inverse map $A^{-1}$ is complete. \[\square\]
3. Stability of regularized problem

In this section we prove Theorem 3. We will construct the operator $\mathcal{R}_{\alpha(c)}$ as a composition of several operators. The construction is motivated by the proof of Theorem 1. We define for a Banach space $E$

$$\mathcal{K}(E) = \{A \in \mathcal{L}(E); A \text{ is compact}\}.$$ 

Let $J, R$ be as defined in (8). Using (8) we see that $J \in \mathcal{K}(L^2(0, 2T))$. We define

$$K : Y \to \mathcal{K}(L^2(0, 2T)), \quad K\tilde{\Lambda} = R\tilde{\Lambda}RJ - J\tilde{\Lambda}.$$ 

$$H : Y \to C([0, T], Y), \quad H\tilde{\Lambda} = r \mapsto P_r(K\tilde{\Lambda})P_r.$$

**Proposition 1.** We have $\|H\|_{Y \to C([0, T], Y)} \leq T$.

**Proof.** Let $r \in [0, T]$. We have estimates $\|P_r\|_Y \leq 1$, $\|R\|_Y \leq 1$, $\|J\|_Y \leq \frac{T}{2}$, and

$$\|H\tilde{\Lambda}(r)\|_{\mathcal{L}(L^2(0, 2T))} \leq 2 \|J\|_{\mathcal{L}(L^2(0, 2T))} \|\tilde{\Lambda}\|_{\mathcal{L}(L^2(0, 2T))} \leq T \|\tilde{\Lambda}\|_{\mathcal{L}(L^2(0, 2T))}.$$ 

Thus

$$\|H\|_{Y \to L^\infty([0, T], Y)} \leq T.$$

It remains to show that $r \mapsto H\tilde{\Lambda}(r)$ is continuous. Let us denote $\tilde{K} = K\tilde{\Lambda}$. Let $r, s \in [0, T]$. We use the singular value decomposition for the compact operator $\tilde{K}$. There are orthonormal bases $\{\phi_n\}_{n=1}^\infty \in L^2(0, 2T)$ and $\{\psi_n\}_{n=1}^\infty \in L^2(0, 2T)$ such that

$$\tilde{K}f = \sum_{n=1}^\infty \mu_n \langle f, \phi_n \rangle_{L^2(0, 2T)} \psi_n,$$

for all $f \in L^2(0, 2T)$, where $\mu_n \in \mathbb{R}$ are the singular values of $\tilde{K}$. We define the family $\{\tilde{K}^m\}_{m=1}^\infty$ of finite rank operators by the formula

$$\tilde{K}^mf = \sum_{n=1}^m \mu_n \langle f, \phi_n \rangle_{L^2(0, 2T)} \psi_n.$$

Then

$$\|P_r\tilde{K}P_rf - P_s\tilde{K}P_sf\|_{L^2(0, 2T)} \leq \|P_r\tilde{K}P_rf - P_r\tilde{K}^mP_rf\|_{L^2(0, 2T)} + \|P_r\tilde{K}^mP_rf - P_s\tilde{K}^mP_rf\|_{L^2(0, 2T)} + \|P_s\tilde{K}^mP_rf - P_s\tilde{K}P_rf\|_{L^2(0, 2T)}.$$
Let $\epsilon > 0$ and let $\|f\|_{L^2([0, T])} \leq 1$. By choosing $m$ large enough we have
\[
\left\| P_r \tilde{K} P_r f - P_r \tilde{K}^m P_r f \right\|_{L^2([0, T])} + \left\| P_s \tilde{K}^m P_s f - P_s \tilde{K} P_s f \right\|_{L^2([0, T])} \leq \frac{\epsilon}{2}.
\]
Applying projections to (26) we see that
\[
P_s \tilde{K}^m P_r f = \sum_{n=1}^{m} \mu_n \langle f, P_r \phi_n \rangle P_s \psi_n.
\]
For the second term in the sum (27) we have an analogous estimate
\[
\left\| P_r \tilde{K}^m P_r f \right\|_{L^2([0, T])} = \left\| \sum_{n=1}^{m} \mu_n \langle f, P_r \phi_n \rangle (P_r - P_s) \psi_n \right\|_{L^2([0, T])}
\]
\[
\leq \sum_{n=1}^{m} |\mu_n| \left\| (P_r - P_s) \psi_n \right\|_{L^2([0, T])} \leq C(m) |r - s|^{\frac{1}{2}}.
\]
For the third term in the sum we have an analogous estimate
\[
\left\| P_s \tilde{K}^m P_r f - P_s \tilde{K} P_s f \right\|_{L^2([0, T])} \leq C(m) |r - s|^{\frac{1}{2}}.
\]
Putting these estimates together and choosing $|r - s| \leq \delta(\epsilon) = \frac{\epsilon^2}{4C(m)^2}$, we see that
\[
\left\| P_r \tilde{K} P_r - P_s \tilde{K} P_s \right\|_{Y} \leq \epsilon.
\]
Let us define
\[(28) \quad M_1 = \sup \{ \| \mathcal{A}(c) \|_{L(L^2([0, T]))} : c \in \mathcal{V}^2 \}.
\]
Using the continuity of $\mathcal{A}$, see Theorem 5 below, we see that $M_1 < \infty$. We define $M_2 = 2TM_1$. Let $c \in \mathcal{V}^2$ and denote $\Lambda = \mathcal{A}(c)$. We use again the notations $H = H^r \Lambda$, $\tilde{H} = H \tilde{\Lambda}$ and $\tilde{H}_r = H \tilde{\Lambda}(r)$. Using Proposition 1 we have
\[(29) \quad \| H \|_{C([0, T], Y)} \leq M_2.
\]
We define $M_3 = M_2 + 3$ and a family $\{ \Psi^Z_\alpha \}_{\alpha \in (0, 2]} \in C(\mathbb{R})$ by
\[
\Psi^Z_\alpha(s) = \begin{cases} 
0, & \text{if } s > M_3 - \frac{\alpha}{4}, \\
-\frac{4}{\alpha} s + \frac{4M_3}{\alpha} - 1, & \text{if } s \in (M_3 - \frac{\alpha}{2}, M_3 - \frac{\alpha}{4}], \\
1, & \text{if } s \leq M_3 - \frac{\alpha}{2}.
\end{cases}
\]
For $\alpha \in (0, 2]$ we define
\[(30) \quad Z_\alpha : C([0, T], Y) \to C([0, T], Y),
\]
\[
Z_\alpha(\tilde{H}) = r \mapsto \Psi^Z_\alpha \left( \left\| M_3 - (\tilde{H} + \alpha) \right\|_{C([0, T], Y)} \right) \left( \tilde{H}_r + \alpha \right)^{-1}.
\]
Let $E$ be a Banach space and let $H \in E$. Let $\epsilon > 0$. We denote

\[(31) \quad B_E(H, \epsilon) := \{ \tilde{H} \in E : \|H - \tilde{H}\|_E < \epsilon \}.\]

**Proposition 2.** Let $\epsilon \in (0, 1)$ and let $p \in (0, \frac{1}{2})$. Let $\alpha = 2\epsilon^p$ and let $\|H\|_{C([0,T],Y)} \leq M_2$. Let $H_r \in Y$ be positive semidefinite. Let us assume that $H \in B_{C([0,T],Y)}(H, \epsilon)$. Then

\[
\|Z_\alpha(H) - Z_\alpha(\tilde{H})\|_{C([0,T],Y)} \leq 2^{-1}\epsilon^{1-2p}.
\]

**Proof.** By the definition (30) of $\Psi^{Z}_\alpha$, we see that if

\[
\Psi^{Z}_\alpha\left(\|M_3 - (\tilde{H} + \alpha)\|_{C([0,T],Y)}\right) \neq 0,
\]

then

\[
\|M_3 - (\tilde{H} + \alpha)\|_{C([0,T],Y)} \leq M_3 - \frac{\alpha}{4} < M_3
\]

and $(\tilde{H}_r + \alpha)^{-1}$ is defined by the formula

\[
(\tilde{H}_r + \alpha)^{-1} = \frac{1}{M_3} \left(I - \frac{M_3 - (\tilde{H}_r + \alpha)}{M_3}\right)^{-1} = \frac{1}{M_3} \sum_{l=1}^{\infty} \left(\frac{M_3 - (\tilde{H}_r + \alpha)}{M_3}\right)^l.
\]

This gives that $Z_\alpha(\tilde{H})(r) \in Y$, when $r \in [0, T]$. Proposition 1 gives continuity for the map $r \mapsto \tilde{H}_r$. As $(\tilde{H}_r + \alpha) \mapsto (\tilde{H}_r + \alpha)^{-1}$ is continuous operation we see that $Z_\alpha(\tilde{H}) \in C([0,T],Y)$. It remains to show that the norm estimate holds. By assumption, $H_r = H(r)$ is positive semidefinite, that is, $H_r : L^2(0,2T) \rightarrow L^2(0,2T)$ is selfadjoint and $H_r \geq 0$. Also, $\|H_r\|_Y \leq M_2$. Thus $0 \leq H_r \leq M_2$ and as $M_3 = M_2 + 3$ and $0 \leq \alpha \leq 2$, we have $0 \leq M_2 - H_r \leq M_2$. Thus

\[
I \leq M_3 - \alpha I - H_r \leq M_3 - \alpha I.
\]

Hence $\|(M_3 - \alpha)I - H\|_{C([0,T],Y)} \leq M_3 - \alpha$. As $\|H - \tilde{H}\|_{C([0,T],Y)} \leq \epsilon \leq \frac{\alpha}{2}$, we have

\[
\|M_3 - (\tilde{H} + \alpha)\|_{C([0,T],Y)} \leq M_3 - \frac{\alpha}{2}.
\]

Thus $\Psi^{Z}_\alpha(\|M_3 - (\tilde{H} + \alpha)\|_{C([0,T],Y)}) = 1$ and $Z_\alpha(\tilde{H})$ is the map

\[
r \mapsto (\tilde{H}_r + \alpha)^{-1}.
\]

Let $r \in [0, T]$. We denote

\[
H_{\alpha,r} = (H_r + \alpha), \quad \tilde{H}_{\alpha,r} = (\tilde{H}_r + \alpha), \quad E = \tilde{H}_{\alpha,r} - H_{\alpha,r}.
\]
As $H_r$ is positive semidefinite we have
\[
\| H_{\alpha,r}^{-1} \|_Y \leq \alpha^{-1}. \tag{32} \]
Moreover
\[
\tilde{H}_{\alpha,r}^{-1} - H_{\alpha,r}^{-1} = \left( [I + H_{\alpha,r}^{-1}E]^{-1} - I \right) H_{\alpha,r}^{-1}. \]
Thus
\[
\| (\tilde{H}_{\alpha,r})^{-1} - (H_{\alpha,r})^{-1} \|_Y \leq \frac{\| (H_{\alpha,r}^{-1}E) \|_Y}{1 - \| (H_{\alpha,r}^{-1}E) \|_Y} \| (H_{\alpha,r})^{-1} \|_Y. \tag{33} \]
We have $\frac{1}{2} \geq \epsilon$. Using (32) and (33) we have
\[
\| (\tilde{H}_{\alpha,r})^{-1} - (H_{\alpha,r})^{-1} \|_Y \leq \frac{\epsilon \alpha^{-1}}{1 - \frac{1}{2}} \| (H_{\alpha,r})^{-1} \|_Y \leq 2 \frac{\epsilon}{\alpha^2} = 2^{-1} \epsilon^{-2p}. \tag{\Box} \]

Let $P_r$ and $B$ be as defined in (8). We define
\[
S : C([0,T], Y') \to C([0,T]),
\]
\[
S(\tilde{Z}_\alpha)(r) = \langle \tilde{Z}_\alpha(r)P_rB1, B1 \rangle_{L^2(0,2T)},
\]
\[
\tilde{f}_{\alpha,r} = \tilde{Z}_\alpha(r)P_rB1.
\]

Proposition 3. We have $\| S \|_{C([0,T])} \leq \frac{T^3}{3}$.

Proof. As the maps $r \mapsto P_rB1$ and $r \mapsto Z_\alpha(r)$ are continuous, we have that $S(\tilde{Z}_\alpha) \in C([0,T])$. Let $r \in [0,T]$. We have
\[
\| P_r \|_Y \leq 1, \quad \| B1 \|_{L^2(0,2T)}^2 = \frac{T^3}{3},
\]
and therefore
\[
|S(\tilde{Z}_\alpha)(r)| = |\langle \tilde{Z}_\alpha(r)P_rB1, B1 \rangle_{L^2(0,2T)}| \leq \frac{T^3}{3} \| \tilde{Z}_\alpha(r) \|_Y. \tag{\Box} \]

Lemma 1. Let $c \in \mathcal{Y}^2$. There is $C > 0$ such that for all $r > 0$ and $p \in H^1(M)$ satisfying $\text{supp}(p) \subset M(r)$ there is $f \in S_r$ such that $u^f(x,T) = p(x)$ and
\[
\| f \|_{L^2(0,2T)} \leq C \| p \|_{H^1(M)}. \tag{35} \]

We recall that $M(r)$ is defined by (11) and $S_r$ is defined by (12). We note that in the study of multidimensional inverse problem, estimate (35) need to be replaced by the Tataru inequality [54], [74] (see also [54], [74]), that is significantly weaker than (35). This is one of the key differences between one and multidimensional case.
By (38) and (39) the solution of (36) satisfies
\[
(36) \quad (\partial_x^2 - c(x)^{-2} \partial_t^2) \tilde{u}(x,t) = 0, \quad (x,t) \in (0, \chi(T)) \times (0, T),
\]
\[
\tilde{u}(x, T) = p(x), \quad x \in [0, \chi(T)],
\]
\[
\tilde{u}(\chi(T), t) = \partial_x \tilde{u}(\chi(T), t) = 0, \quad t \in (0, T).
\]

By (38) and (39) the solution of (36) satisfies
\[
(37) \quad \|\tilde{u}(0, \cdot)\|_{H^1(0,T)} \leq C \|p\|_{H^1(M(T))}.
\]

If \(\text{supp}(p) \subset M(r)\) then \(\text{supp}(\tilde{u}(0, \cdot)) \subset [T - r, T]\) and \(\tilde{u}(x, 0) = \partial_t \tilde{u}(x, 0) = 0\) when \(x \in [0, \chi(T)]\), by finite speed of propagation. We choose \(f(t) = \tilde{u}(0, t)\).

Let \(f_{\alpha, r}\) be as in (14) and define
\[
(38) \quad s_\alpha \in C([0, T]), \quad s_\alpha(r) := \langle f_{\alpha, r}, B1 \rangle_{L^2(0, 2T)}.
\]

**Lemma 2.** Let \(\alpha \in (0, \min(1, \frac{1}{\chi(T)})\). Let \(V\) be as defined in (20). Then there is \(C > 0\), independent \(\alpha\), such that
\[
\|s_\alpha - V\|_{C([0, T])} \leq C \alpha^{\frac{3}{2}}.
\]

**Proof.** Let \(r \in [0, T]\) and \(\delta > 0\). Let us define \(w_\delta \in H^1(M)\)
\[
w_\delta(x) = \begin{cases} 
1, & \text{if } x \in (0, \chi(r)), \\
1 - \frac{x - \chi(r)}{\delta}, & \text{if } x \in [\chi(r), \chi(r) + \delta], \\
0, & \text{if } x \in (\chi(r) + \delta, \infty).
\end{cases}
\]

Using \(c(x) > C_0\) we have
\[
(39) \quad \|w_\delta - 1_{M(r)}\|_{L^2(M, dV)}^2 \leq \frac{\delta}{3C_0^2}.
\]

When \(\delta \in (0, \min(1, \frac{1}{\chi(T)})\) we have
\[
(40) \quad \|w_\delta\|_{H^1(M)}^2 \leq \chi(T) + \frac{\delta}{3} + \frac{1}{\delta} \leq \frac{3}{\delta}.
\]

Below \(C > 0\) denotes a constant that may grow between inequalities, and that depends only on \(m, C_0, C_1, L_1\). Lemma 1 gives us \(f_\delta\) for which \(u_{f\delta}(x, T) = w_\delta(x)\). Thus (40) implies
\[
(41) \quad \|f_\delta\|_{L^2(0, 2T)} \leq C \|w_\delta\|_{H^1(M)} \leq \frac{C}{\delta^{\frac{3}{2}}}.
\]

Let \(f \in S_r\). We define
\[
(42) \quad G_{\alpha, r}(f) = \|u^{f}(T) - 1_{M(r)}\|_{L^2(M, dV)}^2 + \alpha \|f\|_{L^2(0, 2T)}^2.
\]
Using (43) and (44) we have
\[ G_{α,r}(f_δ) = \left\| w_δ - 1_{M(\alpha)} \right\|_{L^2(M,dV)}^2 + α \left\| f_δ \right\|_{L^2(M,dV)}^2 \leq \frac{δ}{C} + α \frac{C}{δ}. \]
Functional (42) and the functional defined in Theorem 4 have the same minimizer \( f_{α,r} \). Using (10), (23), and (38) we have
\[ \| s_α - V \|_{C([0,T])}^2 = \sup_{r \in [0,T]} |\langle f_{α,r}, B_1 \rangle_{L^2([0,2T])} - V(r) |^2. \]
Using (43) and choosing \( δ = α^{\frac{1}{2}} \) we have
\[ \| s_α - V \|_{C([0,T])}^2 \leq Cα^{\frac{1}{2}}. \]

**Lemma 3.** There is \( \bar{m} > 0 \) such that following holds: When \( c \in V^2 \), the functions \( v \) and \( V \), defined in (21) and (20), satisfy
\[ \| v \|_{C^2([0,T])} \leq \bar{m} \quad \text{and} \quad \| V \|_{C^2([0,T])} \leq \bar{m}. \]

**Proof.** Equations (19), (20), (21), and (22) with the chain rule and the formula for the derivatives of inverse functions give us the result. \qed

For small \( h > 0 \) we consider the partition
\[ (0, T) = (0, h) \cup [h, 2h) \cup [2h, 3h) \cup ... \cup [Nh - h, Nh) \cup [Nh, T), \]
where \( N \in \mathbb{N} \) satisfies \( T - h \leq Nh < T \). We define a discretized and regularized approximation of the derivative operator \( \partial_r \) by
\[ D_h : C([0,T]) \to L^∞(0,T), \]
\[ D_h(s_α)(r) = \begin{cases} \frac{\bar{s}_α(h)}{h}, & \text{if } r \in (0,h), \\ \frac{\bar{s}_α(jh+h) - \bar{s}_α(jh)}{h}, & \text{if } r \in [jh, jh+h), \\ \frac{\bar{s}_α(Nh) - \bar{s}_α(Nh)}{h}, & \text{if } r \in [Nh, T). \end{cases} \]

**Proposition 4.** Let \( β > 0 \) and \( ε \in (0, \text{min}(\frac{1}{β^4}, \frac{1}{β^4χ(T)^{\frac{1}{2}}})) \). Let \( α = β^ε, \)
\[ h = ε^{\frac{1}{2}}, \]
\[ V \]
be as defined in (20) and let \( s_α \) be as defined in (38). Let us assume that \( \bar{s}_α \in \mathcal{C}_{C([0,T])}(s_α, ε) \). Then
\[ \| D_h(\bar{s}_α) - ∂_r V \|_{L^∞(0,T)} \leq Cε^{\frac{5}{2}}, \]
where \( C \) is independent of \( α \) and \( \bar{s}_α \).
Proof. Let $r \in [jh, jh + h)$. Using the definition of $D_h(\tilde{s}_\alpha)$ (44) we have
\[
\left| D_h(\tilde{s}_\alpha)(r) - \partial_r V(r) \right| = \left| \frac{\tilde{s}_\alpha(jh + h) - \tilde{s}_\alpha(jh)}{h} - \partial_r V(r) \right|
\leq \left| \frac{\tilde{s}_\alpha(jh + h) - s_\alpha(jh + h)}{h} \right| + \left| \frac{s_\alpha(jh + h) - V(jh + h)}{h} \right|
+ \left| \frac{V(jh + h) - s_\alpha(jh)}{h} \right|
+ \left| \frac{V(jh + h) - V(jh)}{h} - \partial_r V(r) \right|.
\]
Lemma 3 gives us $\|V\|_{C^3([0,T])} \leq \tilde{m}$. When $r \in [jh, jh + h)$ there is $\xi \in (jh, jh + h)$ such that
\[
\left| V(jh + h) - V(jh) - \partial_r V(r) \right| = \left| \partial_r V(\xi) - \partial_r V(r) \right| \leq h\tilde{m}.
\]
Using (45) and Lemma 2 with assumption we get
\[
\left| D_h(\tilde{s}_\alpha)(r) - \partial_r V(r) \right| \leq \frac{2\epsilon}{h} + \frac{2C\alpha^{\frac{1}{2}}}{h} + h\tilde{m}.
\]
Let us choose $h = \epsilon^{\frac{1}{2}}$ and $\alpha = \beta\epsilon^4$. Then
\[
\left| D_h(\tilde{s}_\alpha)(r) - \partial_r V(r) \right| \leq C\epsilon^{\frac{3}{2}}.
\]
The proof is almost identical when $r \in (0, h)$ or $r \in [Nh, T)$. Note that the right hand side of (46) is independent of $r$.

Let $C_0$ and $C_1$ be as in (2). Let $\tilde{k}_\alpha \in L^\infty(0,T)$ and we define
\[
\Psi^W(\tilde{k}_\alpha)(r) = \begin{cases} 
\frac{1}{C_1}, & \text{if } \tilde{k}_\alpha(r) < C_1^{-1}, \\
\frac{1}{k_\alpha(r)}, & \text{if } C_1^{-1} \leq \tilde{k}_\alpha(r) \leq C_0^{-1}, \\
\frac{1}{C_0}, & \text{if } \tilde{k}_\alpha(r) > C_0^{-1}.
\end{cases}
\]

We define
(47)
\[
W : L^\infty(0,T) \to L^\infty(M), \quad W(\tilde{k}_\alpha)(r) = \begin{cases} 
\Psi^W(\tilde{k}_\alpha)(r), & \text{if } r \in (0,T), \\
1, & \text{if } r \in [T, \infty).
\end{cases}
\]

Proposition 5. Let $V$ be as defined in (20) and $v$ be as defined in (21). Let us assume that $\tilde{k}_\alpha \in B_{L^\infty(0,T)}(\partial_r V, \epsilon)$. Then
\[
\left\| W(\tilde{k}_\alpha) - v \right\|_{L^\infty(M)} \leq C_1^2 \epsilon.
\]
Proof. For all \( x \in M \), we have \( 0 < C_0 \leq c(x) \leq C_1 \). Let \( r \in (0, T) \) and assume that \( C^{-1}_1 \leq \tilde{k}_\alpha(r) \leq C^{-1}_0 \). Using (21) and (22) we have \( 0 < \frac{1}{C_1} \leq \partial_r V(r) \leq \frac{1}{C_0} \). Then
\[
|\frac{1}{\tilde{k}_\alpha(r)} - \frac{1}{\partial_r V(r)}| = \left| \frac{\tilde{k}_\alpha(r) - \partial_r V(r)}{\tilde{k}_\alpha(r)\partial_r V(r)} \right| \leq C_1^2 \epsilon.
\]
In the case when \( r \in (0, T) \) and \( \tilde{k}_\alpha(r) < C^{-1}_1 \) or \( \tilde{k}_\alpha(r) > C^{-1}_0 \) we obtain similar estimates. Note that the right hand side of (48) is independent of \( r \). When \( r \geq T \) the left hand side is identically zero. \( \square \)

For \( \tilde{w}_\alpha \in L^\infty(M) \) we define two operators
\[
\Psi^\Phi : L^\infty(M) \to L^\infty(M), \quad \Psi^\Phi(\tilde{w}_\alpha)(r) := \begin{cases} C_0, & \text{if } \tilde{w}_\alpha(r) < C_0, \\ \tilde{w}_\alpha(r), & \text{if } C_0 \leq \tilde{w}_\alpha(r) \leq C_1, \\ C_1, & \text{if } w_\alpha(r) > C_1. \end{cases}
\]
and
\[
\Upsilon : L^\infty(M) \to C(M), \quad \Upsilon(\tilde{w}_\alpha)(t) = \int_0^t \tilde{w}_\alpha(t') dt'.
\]
Using (49) and (50) we see that \( \Upsilon \circ \Psi^\Phi(\tilde{w}_\alpha) : M \to M \) is bijective as a function of \( t \). Let us denote \( \tilde{\chi} = \Upsilon \circ \Psi^\Phi(\tilde{w}_\alpha) \) and \( \tilde{\chi}^{-1} = (\Upsilon \circ \Psi^\Phi(\tilde{w}_\alpha))^{-1} \).

We define an operator
\[
\Phi : L^\infty(M) \to L^\infty(\mathbb{R}), \quad \Phi(\tilde{w}_\alpha) = \begin{cases} 1, & \text{if } x \in (-\infty, 0), \\ \tilde{w}_\alpha \circ \tilde{\chi}^{-1}, & \text{if } x \in [0, L_1), \\ 1, & \text{if } x \in [L_1, \infty). \end{cases}
\]
Let us define \( \eta \in C^\infty(\mathbb{R}) \) by
\[
\eta(x) = \begin{cases} C \exp \left( \frac{1}{x^2 - 1} \right), & \text{if } x \in (-1, 1), \\ 0, & \text{if } |x| \geq 1, \end{cases}
\]
where the constant \( C > 0 \) selected so that \( \int_\mathbb{R} \eta(x) = 1 \). For \( \nu > 0 \) we define
\[
\eta_\nu(x) = \frac{1}{\nu} \eta \left( \frac{x}{\nu} \right).
\]
By using convolution we define a smooth approximation to a given function \( \Phi(\tilde{w}_\alpha) \in L^\infty(\mathbb{R}) \) by setting
\[
\Gamma_\nu : L^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}), \quad \Gamma_\nu(\Phi(\tilde{w}_\alpha)) = \eta_\nu \ast \Phi(\tilde{w}_\alpha).
\]
Let us denote \( \tilde{\zeta}_\nu = (\Gamma_\nu \circ \Phi)(\tilde{w}_\alpha) = \eta_\nu \ast \Phi(\tilde{w}_\alpha) \).
Proposition 6. Let $\epsilon > 0$ and $\nu = \epsilon^{\frac{1}{3}}$. Let $m > 0$ as in (2). Let $v$ as in (27). Let $c \in V^3$. Let us assume that $\tilde{w}_\alpha \in B_{L^\infty(M)}(v, \epsilon)$. Thus we have

(i) $\|\Phi(\tilde{w}_\alpha) - c\|_{L^\infty(M)} \leq C\epsilon$,

(ii) $\|\tilde{c}_\nu - c\|_{C^2(M)} \leq C\epsilon^{\frac{2}{3}}$.

Proof. Let $x \in [0, L_1)$. Let us denote $t = \chi^{-1}(x)$ and $\tilde{t} = \tilde{\chi}^{-1}(x)$. Having $\chi$ as in (22) and $\tilde{\chi}$ as in (51) we see that

$|\Phi(\tilde{w}_\alpha)(x) - c(x)| = |\tilde{w}_\alpha(\tilde{t}) - v(t)| \leq |\tilde{w}_\alpha(\tilde{t}) - v(\tilde{t})| + |v(\tilde{t}) - v(t)|$.

Lemma 3 gives us $\|v\|_{C^2(0,T)} \leq \tilde{m}$ and we have

(55) $|v(\tilde{t}) - v(t)| \leq \tilde{m} |\tilde{t} - t|$.

Using (2) and (22) we see that $0 < C_0 \leq v(t) \leq C_1$ and hence

(56) $C_0 |\tilde{t} - t| \leq \int_t^\tilde{t} v(t') dt' = |\chi(\tilde{t}) - \chi(t)|$.

Having $\tilde{\chi}(\tilde{t}) = x = \chi(t)$ and using (22) and (50) we see that

(57) $|\chi(\tilde{t}) - \chi(t)| = |\chi(\tilde{t}) - \tilde{\chi}(\tilde{t})| = |\int_0^{\tilde{t}} (v(t') - \Psi^\Phi(\tilde{w}_\alpha)(t')) dt'|$.

Using (22) and (50) we see that $|v(t') - \Psi^\Phi(\tilde{w}_\alpha)(t')| \leq |v(t') - \tilde{w}_\alpha(t')|$ for all $t' \in M$. Hence

(58) $|\chi(\tilde{t}) - \chi(t)| = \int_0^{\tilde{t}} (v(t') - \tilde{w}_\alpha(t')) dt' \leq \tilde{\chi}^{-1}(L_1) \epsilon$.

Using (55), (56), and (58) we have

(59) $|\Phi(\tilde{w}_\alpha)(x) - c(x)| \leq \left( 1 + \frac{\tilde{m} \tilde{\chi}^{-1}(L_1)}{C_0} \right) \epsilon$.

Note that the right hand side in (59) does not depend on $x$. When $x \in [L_1, \infty)$ the left hand side in identically zero and we have inequality in case (i).

(ii) Let us define that $c(x) = 1$, for $x \in (-\infty, 0)$. Using (1) we have

$\|\eta_\nu * \Phi(\tilde{w}_\alpha) - \eta_\nu * c\|_{C^2(\mathbb{R})} \leq \|\eta_\nu\|_{W^{2,1}(\mathbb{R})} \|\Phi(\tilde{w}_\alpha) - c\|_{L^\infty(\mathbb{R})}$.

Let $\nu \in (0, 1)$. Using inequality (i) from Proposition 6 and definitions (51), (53) and (54) we have

(60) $\|\eta_\nu * \Phi(\tilde{w}_\alpha) - \eta_\nu * c\|_{C^2(\mathbb{R})} \leq C\nu^{-2} \epsilon$. 
Using (52) and (53) we see that \( \text{supp}(\eta _{\nu }) \subset [-\nu , \nu ] \). Combining that with assumption that \( c \in \mathcal{Y}^{3} \) we have
\[
(61) \quad \| \eta _{\nu } * c - c \|_{C^{2}(\mathbb{R})} \leq 2 \nu \| c \|_{C^{4}(\mathbb{R})} \leq 2 \nu m. 
\]
By \( \nu = \epsilon ^{\frac{3}{2}} \) and using (60) and (61) we have
\[
\| \tilde{c}_{\nu } - c \|_{C^{2}(\mathbb{R})} \leq C \nu ^{-\frac{3}{2}} \epsilon + \nu 2m \leq (C + 2m) \epsilon ^{\frac{3}{2}}.
\]

\[
\Box
\]

Proof of Theorem 3

Let
\[
(62) \quad \epsilon _{0} = \min \{ 1, \frac{1}{2T}, \frac{1}{34}, \frac{1}{24} T^{\chi (T)}, \frac{1}{2} C^{18} C^{36} T^{23} \}. 
\]
Suppose that \( \tilde{\Lambda} \in \mathcal{B}_{Y}(\Lambda , \epsilon ) \) and \( \epsilon \in (0, \epsilon _{0}) \). We denote \( H = H\tilde{\Lambda} \) and \( \tilde{H} = H\tilde{\Lambda} \). Using Proposition 1 we get
\[
\left\| H - \tilde{H} \right\|_{C([0,T],Y)} \leq 2T \epsilon . 
\]
We denote \( Z_{\alpha } = Z_{\alpha } (H) \) and \( \tilde{Z}_{\alpha } = Z_{\alpha } (\tilde{H}) \). We have \( \tilde{H} \in \mathcal{B}_{C([0,T],Y)}(H, 2T \epsilon ) \) and \( \epsilon \in (0, \min (1, \frac{1}{2T})) \). Proposition 2 with \( p = \frac{4}{3} \) gives us
\[
\left\| Z_{\alpha } - \tilde{Z}_{\alpha } \right\|_{C([0,T],Y)} \leq 2^{-\frac{\alpha }{2}} T^{\frac{\alpha }{2}} \epsilon ^{\frac{3}{2}} =: \epsilon _{1},
\]
since \( \alpha = 2^{p+1} T \epsilon ^{p} = 2^{\frac{12}{3}} T^{\frac{4}{3}} \epsilon ^{\frac{4}{3}} \).
We denote \( s_{\alpha } = S Z_{\alpha } \) and \( \tilde{s}_{\alpha } = S \tilde{Z}_{\alpha } \). We have \( \tilde{Z}_{\alpha } \in \mathcal{B}_{C([0,T],Y)}(Z_{\alpha }, \epsilon _{1}) \).
Proposition 3 gives us
\[
\left\| s_{\alpha } - \tilde{s}_{\alpha } \right\|_{C([0,T])} \leq 3^{-1} \cdot 2^{-\frac{\alpha }{2}} T^{\frac{3\alpha }{4}} \epsilon ^{\frac{1}{2}} =: \epsilon _{2}.
\]
We denote \( \tilde{k}_{\alpha } = D_{\alpha } (s_{\alpha }) \). We have \( \tilde{s}_{\alpha } \in \mathcal{B}_{C([0,T])}(s_{\alpha }, \epsilon _{2}) \) and \( \epsilon _{2} \in \left( 0, \min \left( \frac{1}{\beta ^{6} \chi (T)^{2}}, \frac{1}{\beta ^{3} \chi (T)^{2}} \right) \right) \). Proposition 4 with \( \beta = 3^{4} 2^{5} T^{-12} \) gives us
\[
\left\| \tilde{k}_{\alpha } - \partial _{x} V \right\|_{L^{\infty }([0,T])} \leq C 3^{-\frac{1}{2}} \cdot 2^{-\frac{\alpha }{2}} T^{\frac{14}{3}} \epsilon ^{\frac{1}{18}} =: \epsilon _{3},
\]
where \( \alpha = \beta (3^{-1} \cdot 2^{-\frac{\alpha }{2}} T^{\frac{3\alpha }{4}} \epsilon ^{\frac{1}{2}})^{4} = \beta (3^{-4} \cdot 2^{-\frac{\alpha }{2}} T^{112} \epsilon ^{\frac{1}{2}}) = 2^{34} T^{8} \epsilon ^{\frac{4}{3}} \).
We denote \( \tilde{w}_{\alpha } = W (\tilde{k}_{\alpha }) \). We have \( \tilde{k}_{\alpha } \in \mathcal{B}_{L^{\infty }([0,T])}(\partial _{x} V, \epsilon _{3}) \). Proposition 5 gives us
\[
\left\| \tilde{w}_{\alpha } - v \right\|_{L^{\infty }([M])} \leq C_{2} C 3^{-\frac{1}{2}} \cdot 2^{-\frac{\alpha }{2}} T^{\frac{14}{3}} \epsilon ^{\frac{1}{18}} =: \epsilon _{4}.
\]
Let \( \epsilon _{4} \in (0, 1) \) and \( \nu = \epsilon _{4}^{\frac{1}{2}} \). We denote \( \tilde{c}_{\nu } = \eta _{\nu } * \Phi (\tilde{w}_{\alpha }) \). We have \( \tilde{w}_{\alpha } \in \mathcal{B}_{L^{\infty }([M])}(v, \epsilon _{4}) \). Let \( c \in \mathcal{Y}^{3} \) and Proposition 6 gives us
\[
\left\| \tilde{c}_{\nu } - c \right\|_{C^{2}([M])} \leq C \nu ^{\frac{1}{2}}.
\]
where $\epsilon \in (0, \epsilon_0)$. Using (24), (30), (34), (44), (47), (51) and (54) we define

$$R_{\alpha(\epsilon)} : Y \to Z,$$

$$R_{\alpha(\epsilon)} = \Gamma_\nu \circ \Phi \circ W \circ D_h \circ S \circ Z_\alpha \circ H,$$

and we have an estimate

$$\|R_{\alpha(\epsilon)}(\tilde{\Lambda}) - c\|_Z \leq C\epsilon^{\frac{1}{4}}.$$

**Appendix A: The direct problem**

**Theorem 5.** Let $c \in \mathcal{V}^2$ and $f \in L^2(0, 2T)$. Then the boundary value problem (4) has a unique solution $u_f \in H^1((0, 2T) \times M)$. The operators $\Lambda$ and $U_\mathcal{V}$, defined in (5) and (9), are bounded, and the direct map $A : \mathcal{V}^2 \subset Z \to Y$, defined in (6), is continuous, and moreover

$$M_1 = \sup\{\|A(c)\|_{L^2(0, 2T)} ; c \in \mathcal{V}^2\} < \infty.$$

**Proof.** Let us consider the wave equation (4). When $c = 1$ on $M$ we denote the solution by $u^f_0$ and have

$$u^f_0(t, x) = h(t - x), \quad h(s) = \begin{cases} -\int_0^s f(t) dt, & t > 0, \\ 0, & t \leq 0. \end{cases}$$

Notice that $f \mapsto u^f_0$ is continuous from $L^2(0, 2T)$ to $C^1([0, 2T]; L^2(M)) \cap C([0, 2T]; H^1(M))$.

Let us consider the wave equation

$$\begin{align*}
(\partial^2_t - c(x)^2 \partial^2_x)w(t, x) &= F(t, x) \quad \text{in } (0, 2T) \times M, \\
\partial_x w(t, 0) &= 0, \\
w|_{t=0} &= \partial_x w|_{t=0} = 0.
\end{align*}$$

For the wave equation (65) the existence and uniqueness of the solutions and the continuity of the map $W : F \mapsto w$, given by

$$W : L^2((0, 2T) \times M) \to C([0, 2T]; H^1(M)) \cap C^1([0, 2T]; L^2(M)),$$

follow from the results of [38, Ch. 3, Theorems 8.1 and 8.2], and [39, p. 93].

Let $\psi \in C^\infty(M)$ be such that $\psi = 1$ near $x = 0$ and $\psi = 0$ when $x > \frac{L_0}{2}$. Note that $c = 1$ in the support of $\psi$. The commutator $A = [\partial^2_x, \psi]$ is a first order differential operator, whence $Au^f_0 \in L^2((0, 2T) \times M)$ for $f \in L^2(0, 2T)$. Let us choose $F(t, x) = Au^f_0(t, x)$ in (65) and define $u^f = \psi u^f_0 + w$. Then

$$(\partial^2_t - c^2 \partial^2_x)u^f = \psi(\partial^2_t - \partial^2_x)u^f_0 + Au^f_0 - Au^f_0 = 0,$$
where \( u^I := \psi u_0^I + w \in C^1([0, 2T]; L^2(M)) \cap C([0, 2T]; H^1(M)) \) is the solution of (4). As \( \psi = 1 \) near \( x = 0 \), we see that \( u \) satisfies also the boundary conditions in (4). In particular, \( u^I \in H^1((0, 2T) \times M) \). The above shows that \( f \mapsto u^I \) is continuous operator from \( L^2(0, 2T) \) to \( u^I \in C^1([0, 2T]; L^2(M)) \cap C([0, 2T]; H^1(M)) \), and hence \( U^I : L^2(0, 2T) \rightarrow H^1(M) \) is continuous. Using Trace theorem, we see that the map \( \Lambda \) is continuous from \( L^2(0, 2T) \) to \( H^\frac{1}{2}(0, 2T) \).

Let us now suppose that \( f \in C^0_c(0, 2T) \). Let \( u^I \) be solution for the boundary value problem in (4) and \( c(x) \) be as defined in (2). Let \( x \in M \) and we define
\[
k \in C^2(M), \quad k(x) = c(x)^{1/2}
\]
and
\[
G : C^2(M \times (0, 2T)) \rightarrow C(M \times (0, 2T)),
\]
\[
G(u) = k^{-1} \left( \partial_t^2 - c^2 \partial_x^2 \right) ku = \left( \partial_t^2 - c^2 \partial_x^2 - 2c^2 k^{-1} (\partial_x k) \partial_x - c^2 k^{-1} (\partial_x^2 k) \right) u.
\]
Let \( x \in M \) and define
\[
\phi(x) = \int_0^x c(x')^{-1} dx'.
\]
Let us denote \( \tilde{x} = \phi(x) \) and define
\[
\tilde{u}^I(\tilde{x}, t) = \tilde{u}^I(\phi(x), t) := \frac{u^I(x, t)}{k(x)}.
\]
Using (3), (66), (67), (68), (69) and the property of finite speed of propagation we see that \( \tilde{u}^I(\tilde{x}, t) \) is a solution of the boundary value problem
\[
(\partial_t^2 - \partial_x^2 + q(\tilde{x})) \tilde{u}^I(\tilde{x}, t) = 0, \quad (\tilde{x}, t) \in (0, 2T) \times (0, 2T),
\]
\[
\partial_t \tilde{u}^I(0, t) = f(t), \quad \partial_t \tilde{u}^I(2T, t) = 0, \quad t \in (0, 2T),
\]
\[
\tilde{u}^I(x, 0) = \partial_t \tilde{u}^I(x, 0) = 0, \quad \tilde{x} \in [0, 2T],
\]
where
\[
q(\tilde{x}) = -c^2 (\phi^{-1}(\tilde{x})) k^{-1} (\phi^{-1}(\tilde{x})) \partial_x^2 k(\phi^{-1}(\tilde{x})).
\]
Let \( \tilde{x} \in [0, T] \). Using (2), (3), (66), (68), (71) we see that for every \( q \) that corresponds to some \( c \in \mathbb{R}^2 \) via formula (71) there is a constant \( C_3 = C_3(C_0, C_1, L_1, m, T) \) for which
\[
|q(\tilde{x})| \leq C_3.
\]
Let \( \tilde{x} \geq T \). Using (2) we have \( c(x) = 1 \) and thus by using (71) we see that \( q(\tilde{x}) = 0. \)
We define \( \Lambda_q f = \tilde{u}_{|\tilde{x}=0} \). Let us consider two velocity functions \( c_1 \) and \( c_2 \), and let \( q_1 \) and \( q_2 \) be the potentials corresponding to \( c_1 \) and \( c_2 \) via formula (71). Using (69) and property that \( C^\infty_0((0,2T) \subset L^2((0,2T)) \) is dense we have

\[
(73) \quad \|A(c_1) - A(c_2)\|_{L(L^2((0,2T))} \leq \|\Lambda_{q_1} - \Lambda_{q_2}\|_{L(L^2((0,2T))}.
\]

Let us denote by \( u^{q_1}_f \) and \( u^{q_2}_f \) the two solutions with respect to potentials \( q_1 \) and \( q_2 \) for the problem (70). Let us define \( w(\tilde{x},t) = \tilde{u}^{q_1}_f(\tilde{x},t) - \tilde{u}^{q_2}_f(\tilde{x},t) \). Then \( w \) is the solution of

\[
(74) \quad (\partial_t^2 \partial_x^2 + q_1(\tilde{x}))w(\tilde{x},t) = F(\tilde{x},t), \quad (\tilde{x},t) \in (0,2T) \times (0,2T),
\]

\[
\partial_x w(0,t) = 0, \quad \partial_x w(2T,t) = 0, \quad t \in (0,2T),
\]

\[
w(\tilde{x},0) = \partial_t w(\tilde{x},0) = 0, \quad x \in [0,2T],
\]

where

\[
(75) \quad F(\tilde{x},t) = (q_1(\tilde{x}) - q_2(\tilde{x}))\tilde{u}^{q_1}_f(\tilde{x},t).
\]

Using results of [38, Ch. 3], or alternatively, the same proof that is in [27], Lemma 1.9 for initial boundary value problem with Dirichlet boundary condition, we see for (74), we see that there is a constant \( C_4 = C_4(C_0, C_1, C_3, L_1, m, T) \) such that for all potentials \( q \) satisfying (72) the solution of the wave equation satisfies

\[
(76) \quad \|w\|_{H^1((0,2T) \times (0,2T))} \leq C_4 \|F\|_{L^2((0,2T) \times (0,2T))}.
\]

Using (69) we see that \( u^{q_2}_f \in H^1((0,2T) \times (0,2T)) \). That with (72) and (75) imply

\[
(77) \quad \|F\|_{L^2((0,2T) \times (0,2T))} \leq \|q_1 - q_2\|_{L^\infty((0,2T))} \|u^{q_1}_f\|_{H^1((0,2T) \times (0,2T))}.
\]

When \( q = 0 \), we can construct an explicit solution of (70), see [27], formula (1.34). Similarly to the estimate (76), we see using results of [38, Ch. 3] or [27] that there is a constant \( C_5 \) that depends only on \( C_0, C_1, L_1, m, T \) such that

\[
(78) \quad \|\tilde{u}^{q_1}_f\|_{H^1((0,2T) \times (0,2T))} \leq C_5 \|f\|_{L^2((0,2T)).
\]

Using Trace Theorem we have

\[
(79) \quad \|\Lambda_{q_1} f - \Lambda_{q_2} f\|_{L^2((0,2T))} \leq C(T) \|u^{q_1}_f - u^{q_2}_f\|_{H^1((0,2T) \times (0,2T))}.
\]

Having \( q(\tilde{x}) = 0 \), when \( \tilde{x} \geq T \), and using (76), (77), (78), and (79) we have

\[
(80) \quad \|\Lambda_{q_1} - \Lambda_{q_2}\|_{L(L^2((0,2T)))} \leq C_7 \|q_1 - q_2\|_{L^\infty((0,2T))} = C_7 \|q_1 - q_2\|_{L^\infty((0,2T))},
\]

where \( C_7 \) depends only on \( C_0, C_1, L_1, m, \) and \( T \).
Let \( c_1, c_2 \in \mathcal{V}^2 \), where \( c_1 \in \mathcal{V}^2 \) is fixed. Let \( \|c_1 - c_2\|_{C^2(M)} \leq \epsilon \). Let \( \bar{x} \in (0, T) \). Using (68) and (71) we have
\[
q_1(\bar{x}) = -c_1^2(x)k_1^{-1}(x)\partial_x^2k_1(x)|_{x=\phi_1^{-1}(\bar{x})},
q_2(\bar{x}) = -c_2^2(x)k_2^{-1}(x)\partial_x^2k_2(x)|_{x=\phi_2^{-1}(\bar{x})}.
\]
Let us denote \( y = \phi_1^{-1}(\bar{x}) \) and \( x = \phi_2^{-1}(\bar{x}) \). Note that for all \( c_1, c_2 \in \mathcal{V}^2 \) and \( \bar{x} \in (0, T) \) we have \( x, y \in [0, TC_1] \). Let \( x \in M \) and define
\[
h_1(x) = -c_1^2(x)k_1^{-1}(x)\partial_x^2k_1(x), \quad h_2(x) = -c_2^2(x)k_2^{-1}(x)\partial_x^2k_2(x).
\]
We have
\[
(81)

For the second term on the right hand side of (81) we have
\[
|h_1(x) - h_2(x)| = |c_1^2(x)k_2^{-1}(x)\partial_x^2k_2(x) - c_2^2(x)k_1^{-1}(x)\partial_x^2k_1(x)|
\leq |c_1^2(x) - c_2^2(x)||k_1^{-1}(x)||\partial_x^2k_1(x)|
+ |k_1^{-1}(x) - k_2^{-1}(x)||c_2^2(x)||\partial_x^2k_1(x)|
+ |\partial_x^2k_1(x) - \partial_x^2k_2(x)||c_2^2(x)||k_1^{-1}(x)|,
\]
where \( x = \phi_2^{-1}(\bar{x}) \). Having \( x \in [0, TC_1] \) and using (2), (68), (66) we can bound each of these three terms and get
\[
(82)
|h_1(x) - h_2(x)| \leq C_8 \|c_1 - c_2\|_{C^2(M)},
\]
where \( C_8 \) depends only on \( C_0, C_1, L_1, m, T \). As \( h_1 \) is continuous on \( M \) and zero on \( [0, L_0) \cup (L_1, \infty) \), \( h_1 \) is uniformly continuous on \( M \). Moreover, we have a function \( \omega : M \to M \), the continuity modulus of \( h_1 \), for which
\[
(83)
|h_1(x) - h_1(y)| \leq \omega(\epsilon),
\]
for \( x, y \in M \) satisfying \( |x - y| \leq \epsilon \). Thus for the first term on the right hand side of (81) we have
\[
(84)
|h_1(\phi_1^{-1}(\bar{x})) - h_1(\phi_2^{-1}(\bar{x}))| \leq \omega(|\phi_1^{-1}(\bar{x}) - \phi_2^{-1}(\bar{x})|).
\]
Having \( x \in [0, TC_1] \) and using (2) and (68) we have
\[
(85)
|\phi_1(x) - \phi_2(x)| \leq \int_0^x \frac{|c_1(x') - c_2(x')|}{c_1(x')c_2(x')} \, dx \leq \frac{TC_0 \|c_1 - c_2\|_{C^2(M)}}{C_0^2}.
\]
As \( \frac{1}{c_1} \leq \frac{d\phi_1}{dx}(x) \leq \frac{1}{c_0} \), we have
\[
(86)
\frac{1}{C_1}|x - y| \leq |\phi_1(x) - \phi_1(y)| \leq \frac{1}{C_0}|x - y|.
\]
Using (85) and (86) we have
\begin{align}
|\phi_1^{-1}(\bar{x}) - \phi_2^{-1}(\bar{x})| & \leq C_1 |\phi_1(\phi_1^{-1}(\bar{x})) - \phi_1(\phi_2^{-1}(\bar{x}))| \\
& \leq C_1 \left( |\phi_1(\phi_1^{-1}(\bar{x})) - \phi_2(\phi_2^{-1}(\bar{x}))| + |\phi_2(\phi_2^{-1}(\bar{x})) - \phi_1(\phi_2^{-1}(\bar{x}))| \right) \\
& \leq C_1 \left( TC_0 \|c_1 - c_2\|_{C^2(M)}^2 \right) \leq C_0 C_1 T \|c_1 - c_2\|_{C^2(M)}^2.
\end{align}

Using (81), (82), (84), and (87) we have
\begin{equation}
|q_1(\bar{x}) - q_2(\bar{x})| \leq \omega(C) \|c_1 - c_2\|_{C^2(M)}^2 + C \|c_1 - c_2\|_{C^2(M)}^2,
\end{equation}
where $\bar{x} \in (0, T)$. As $\omega$ is continuous at zero, we see that when $c_2 \to c_1$ in $V^2 \subset C^2(M)$ we get by using (88) that $q_2 \to q_1$ in $L^\infty(0, T)$. Using this with (73) and (80) we obtain $A(c_1) \to A(c_2)$ in $L(L^2(0, 2T))$ when $c_2 \to c_1$ in $V^2 \subset C^2(M)$.

Choosing $c_2(x) = 1$ for all $x \in M$, we see that $A(c_2)f = 0$ for all $f \in L^2(0, 2T)$. Using this with (73) we have
\begin{equation}
\|A(c_1)\|_{L(L^2(0,2T))} = \|A(c_1) - A(c_2)\|_{L(L^2(0,2T))} \leq C_9 \|q_1 - q_2\|_{L^\infty(0,T)},
\end{equation}
where $C_9$ depends only on $C_0, C_1, L_1, m, T$. When $c_2(x) = 1$ for all $x \in M$, $q_2$ is zero on $M$. This with (72) and (80) imply $\|q_1 - q_2\|_{L^\infty(0,T)} \leq C_3$, for all $c_1 \in V^2$. Using this and (89) we see that
\[ M_1 = \sup\{\|A(c)\|_{L(L^2(0,2T))} : c \in V^2\} < \infty. \]

\section*{Appendix B: The proof of theorem (2)}

\textbf{Proof.} Let $V^k$ as defined in (2). By Theorem 5 the map
\[ A : V^2 \subset X \to Y, \quad A(c) = \Lambda, \]
is continuous. Also by Theorem 1 $A : V^2 \to A(V^2)$ is one-to-one. By Arzela-Ascoli Theorem $cl_{C^2(M)}(V_0^3) = \overline{V_0^3}$ is a compact subset of $C^2(M)$. Let $U \subset \overline{V_0^3}$ is open. Thus $\overline{V_0^3} \setminus U$ is closed and compact. Using continuity of $A$ we see that $A(\overline{V_0^3} \setminus U) = A(\overline{V_0^3}) \setminus A(U)$ is compact. As $Y$ is a Hausdorff space, $A(\overline{V_0^3}) \setminus A(U)$ is closed. Thus $A(U) \subset A(\overline{V_0^3})$ is open and
\[ A : \overline{V_0^3} \to A(\overline{V_0^3}), \quad A(c) = \Lambda, \]
is a homomorphism. Note that $\overline{V_0^3}$ has the relative topology determined by the norm $\|\cdot\|_{C^k(M)}$ and $A(\overline{V_0^3})$ has the relative topology induced from $Y$. \qed
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