

DETERMINATION OF THE SPACETIME FROM LOCAL TIME MEASUREMENTS

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ABSTRACT. We consider an inverse problem for a Lorentzian spacetime (M, g) , and show that time measurements, that is, the knowledge of the Lorentzian time separation function on a submanifold $\Sigma \subset M$ determine the C^∞ -jet of the metric in the Fermi coordinates associated to Σ . We use this result to study the global determination of the spacetime (M, g) when it has a real-analytic structure or is stationary and satisfies the Einstein-scalar field equations. In addition to this, we require that (M, g) is geodesically complete modulo scalar curvature singularities. The results are Lorentzian counterparts of extensively studied inverse problems in Riemannian geometry - the determination of the jet of the metric and the boundary rigidity problem. We give also counterexamples in cases when the assumptions are not valid, and discuss inverse problems in general relativity.

1. INTRODUCTION

Inverse problems for hyperbolic equations have been studied extensively using a geometric point of view, see e.g. [3, 5, 15, 16, 30, 32, 35]. This is due to the fact that for a hyperbolic equation with time-independent coefficients, the travel time of the waves between two points defines a natural Riemannian distance between these points. The corresponding Riemannian metric is called the travel time metric. A classical inverse problem is to determine the wave speed inside the object given the travel times between the boundary points, or equivalently, the distances between the boundary points. In this paper we study geometric inverse problems for Lorentzian manifolds, that are related to hyperbolic equations with time-depending coefficients and to general relativity.

Before formulating the geometric inverse problems for Lorentzian manifolds that we will study, let us recall earlier results for Riemannian manifolds. A paradigm problem is the boundary rigidity problem: does the restriction $\widehat{d}|_{\partial\widehat{M}\times\partial\widehat{M}}$ of the Riemannian distance function

Date: March 20, 2016.

$\widehat{d} : \widehat{M} \times \widehat{M} \rightarrow \mathbb{R}$ determine uniquely a Riemannian manifold with boundary $(\widehat{M}, \widehat{g})$. If this is possible, then $(\widehat{M}, \widehat{g})$ is said to be boundary rigid. Since the boundary distance function takes into account only the shortest paths, it is easy to construct counterexamples where $\widehat{d}|_{\partial\widehat{M} \times \partial\widehat{M}}$ does not carry information on an open subset of M . Thus some a-priori conditions on $(\widehat{M}, \widehat{g})$ are necessary for boundary rigidity.

Michel has conjectured that simple manifolds are boundary rigid [43]. We recall that a compact Riemannian manifold with boundary $(\widehat{M}, \widehat{g})$ is simple, if $\partial\widehat{M}$ is strictly convex and if for any $x \in \widehat{M}$ the exponential map \exp_x is a diffeomorphism. Pestov and Uhlmann proved the conjecture in the dimension two [53] but it is open in higher dimensions.

A related problem to determine the C^∞ -jet of the metric tensor \widehat{g} on the boundary from the Riemannian distance function $\widehat{d}|_{\partial\widehat{M} \times \partial\widehat{M}}$ was solved for simple Riemannian manifolds in [37]. Here we extend this result for Lorentzian manifolds. Our main motivation comes from the theory of relativity, whence we consider a Lorentzian manifold without boundary, and replace the restriction of the Riemannian distance function with the time separations between points on a timelike hypersurface.

Let us suppose that (M, g) is a Lorentzian manifold without boundary. The two main theorems of the paper concern determination of (M, g) given time separations between points on a timelike hypersurface Σ . Our first result is of local nature: we show that the time separations determine the C^∞ -jet of the metric tensor g at a point $x_0 \in \Sigma$ assuming that there are many timelike geodesics starting near x_0 and intersecting Σ again later. The result is obtained by adapting the method developed in [55] and [37] to the Lorentzian context. The method has been previously used only in the Riemannian setting. Our global result is that C^∞ -jet of the metric tensor at a point determines the universal Lorentzian covering space of (M, g) assuming that (M, g) is real-analytic and geodesically complete modulo scalar curvature singularities, see the definition before Theorem 2.

1.1. Previous literature. The boundary distance rigidity question in the Lorentzian context has been studied by Anderson, Dahl, and Howard [1] who have studied slab-like manifolds and show, in particular, that flat two dimensional product manifolds are boundary rigid, that is, the Lorentzian distances of boundary points determine the manifold uniquely under natural assumptions. In the Riemannian case, in addition to the above mentioned paper by Pestov and Uhlmann [53], boundary rigidity has been proved for subdomains of Euclidean space

[27], for subspaces of an open hemisphere in two dimension [43], for subspaces of symmetric spaces of constant negative curvature [6], for two dimensional spaces of negative curvature [9, 48]. It was shown in [57] that metrics a priori close to a metric in a generic set, which includes real-analytic metrics, are boundary rigid, and in [37] it was shown that two metrics with identical boundary distance functions differ by an isometry which fixes the boundary if one of the metrics is close to the Euclidean metric. For other results see [8, 10, 51, 56].

In [53], the Riemannian boundary rigidity problem is reduced to an inverse problem for the Laplace-Beltrami equation on a two-dimensional manifold. The solution of this problem is heavily based on the use of the underlying real-analytic conformal structure that the Riemannian surfaces have [38, 36, 39]. In the present paper we will use similar kind of underlying real-analytic structure to study inverse problems for the stationary spacetimes.

In addition to [37], the determination of the C^∞ -jet of the Riemannian metric tensor has been studied in [55], where the problem of this type was considered for a class of non-simple manifolds, and the authors showed that knowledge of the lens data in a neighborhood of a boundary point determines C^∞ -jet of the metric at this point. The boundary distances $\widehat{d}|_{\widehat{\partial M} \times \widehat{\partial M}}$ determine the lens data in the case of a simple Riemannian manifold.

2. STATEMENT OF THE RESULTS

Let (M, g) be a $(1 + n)$ -dimensional smooth manifold M with a Lorentzian metric g of signature $(-, +, \dots, +)$. We recall that a simply convex neighborhood is an open subset \mathcal{U} in M which is a normal neighborhood for every point inside. It is well-known that in a Lorentzian manifold, each point has a simply convex neighborhood, see e.g. [54].

We will begin by formulating a local result on a simply convex neighborhood $\mathcal{U} \subset M$. We regard (\mathcal{U}, g) as Lorentzian manifold and define the causality relation as usual, that is, for x and y in \mathcal{U} , we write $x \ll y$ if there is a future-pointing timelike curve $\mu([0, l])$, $l > 0$ in \mathcal{U} from x to y . We emphasize that a simply convex neighborhood (\mathcal{U}, g) is time-oriented and hence $x \ll y$ does not hold when $x = y$. Let us define two open subsets of \mathcal{U}

$$I_{\mathcal{U}}^+(x) := \{z \in \mathcal{U} : x \ll z\}, \quad I_{\mathcal{U}}^-(y) := \{z \in \mathcal{U} : z \ll y\}.$$

We call $I_{\mathcal{U}}^+(x)$ the chronological future of x in \mathcal{U} and $I_{\mathcal{U}}^-(y)$ the chronological past of y in \mathcal{U} .

On the Lorentzian manifold (\mathcal{U}, g) , we can define the Lorentzian distance function, also called the time separation function,

$$d : \mathcal{U} \times \mathcal{U} \rightarrow \mathbb{R}$$

as follows. For any $x, y \in \mathcal{U}$, by simple convexity of \mathcal{U} , there exists a unique geodesic $\gamma_{x,y} : [0, 1] \rightarrow \mathcal{U}$ connecting them, that is, $\gamma_{x,y}(0) = x$ and $\gamma_{x,y}(1) = y$. We define

$$L(\gamma_{x,y}) := \int_0^1 |\dot{\gamma}_{x,y}(t)|_g dt,$$

where $|v|_g = |(v, v)_g|^{\frac{1}{2}}$ and $(v, w)_g = g_{jk}(x)v^jw^k$ is the scalar product of vectors $v, w \in T_xM$ with respect to the metric tensor g . The Lorentzian distance function is defined as

$$(1) \quad d(x, y) = \begin{cases} L(\gamma_{x,y}), & \gamma_{x,y} \text{ is timelike and future-pointing,} \\ 0, & \text{otherwise.} \end{cases}$$

Note that this function encodes the causality information and thus is not symmetric.

We will assume that d is known on an oriented smooth timelike open submanifold $\Sigma \subset \mathcal{U}$ of codimension 1. Moreover, we assume that the topological closure of Σ is compact and satisfies $\bar{\Sigma} \subset \mathcal{U}$. Suppose $\hat{x}_0 \in \Sigma$ and $T_{\hat{x}_0}\Sigma$ is a timelike subspace of $T_{\hat{x}_0}M$, let ν be a unit normal vector field of Σ near \hat{x}_0 . We say that Σ is *timelike convex* near the point $\hat{x}_0 \in \Sigma$ and a timelike vector $\hat{\xi}_0 \in T_{\hat{x}_0}\Sigma$ in the direction $\nu(\hat{x}_0)$, if the following hypothesis **H** holds.

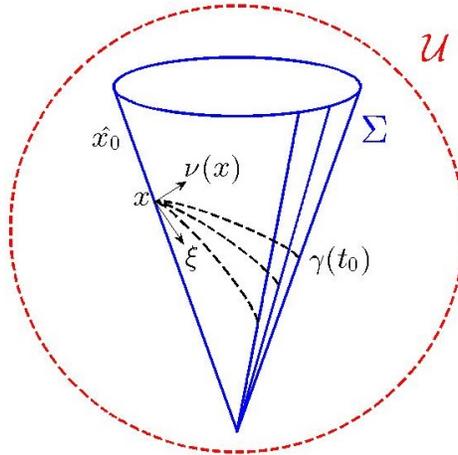


FIGURE 1. An example of Σ in \mathbb{R}^{2+1} . Here Σ is a cone minus the tip.

H: *There is an open neighborhood U of $(\widehat{x}_0, \widehat{\xi}_0)$ in $T\Sigma$ satisfying the following: for any $(x, \xi) \in U$, there is $\epsilon > 0$ such that if $r \in (0, \epsilon)$, then the geodesic $\gamma(t)$ with*

$$\gamma(0) = x, \quad \dot{\gamma}(0) = \xi + r\nu(x),$$

satisfies $\gamma(t_0) \in \overline{\Sigma}$ for some $t_0 > 0$, and $\gamma(t) \in \mathcal{U}$ for $t \in (0, t_0)$.

A physically motivated example of Σ can be found in Section 3.5, which consists of union of the world lines of freely falling material particles issued from a fixed point with identical Newtonian speeds. Our local result asserts that the knowledge of the Lorentzian distance function d on a timelike hypersurface Σ uniquely determines the C^∞ -jet of the Lorentzian metric g at a point $\widehat{x}_0 \in \Sigma$ assuming that Σ is timelike convex near $(\widehat{x}_0, \widehat{\xi}_0)$ for some timelike vector $\widehat{\xi}_0 \in T_{\widehat{x}_0}\Sigma$.

Theorem 1. *Let (M, g) and $(\widetilde{M}, \widetilde{g})$ be two smooth Lorentzian manifolds, let $\Sigma \subset M$ and $\widetilde{\Sigma} \subset \widetilde{M}$ be smooth timelike submanifolds of codimension 1 such that their closures are compact in simply convex neighborhoods \mathcal{U} and $\widetilde{\mathcal{U}}$ respectively. Let $\widehat{x}_0 \in \Sigma$ and let $\widehat{\xi}_0 \in T_{\widehat{x}_0}\Sigma$ be timelike. Suppose that there is a diffeomorphism $\Phi : \Sigma \rightarrow \widetilde{\Sigma}$, such that $\Phi_{*\widehat{x}_0}(\widehat{\xi}_0)$ is timelike, and such that Σ and $\widetilde{\Sigma}$ are timelike convex near $(\widehat{x}_0, \widehat{\xi}_0)$ in the direction of $\nu(\widehat{x}_0)$ and near $(\Phi(\widehat{x}_0), \Phi_{*\widehat{x}_0}(\widehat{\xi}_0))$ in the direction of $\widetilde{\nu}(\Phi(\widehat{x}_0))$ respectively. Suppose, furthermore, that the corresponding Lorentzian distance functions satisfy*

$$d(x, y) = \widetilde{d}(\Phi(x), \Phi(y)), \quad \text{for all } x, y \in \Sigma.$$

Then the C^∞ -jet of g at \widehat{x}_0 coincides with the C^∞ -jet of \widetilde{g} at $\Phi(\widehat{x}_0)$.

Here $\Phi_{*\widehat{x}_0}(\widehat{\xi}_0) \in T_{\Phi(\widehat{x}_0)}\widetilde{\Sigma}$ is the image of $\widehat{\xi}_0$ under the push-forward Φ_* at \widehat{x}_0 . A way to formulate the equality of the C^∞ -jets is to say that all the derivatives of the metric tensors are equal in suitable coordinates. In the proof we use semigeodesic coordinates (also called Fermi coordinates) associated to Σ and $\widetilde{\Sigma}$ when they are identified by using the diffeomorphism Φ , see the paragraphs before the proof of Theorem 1 and (14) for the details.

We say that (M, g) is real-analytic if the manifold M has an real-analytic structure with respect to which the metric tensor g is real-analytic. We note that when (M, g) is real-analytic, the determination of the C^∞ -jet of the metric tensor does not imply global uniqueness results without additional assumptions even if we assumed a priori that M is simply connected. The reason for this is that there can be multiple incompatible real-analytic extensions of a real-analytic manifold, as is

seen in Example 3.6 below. However, we will show that for a real-analytic manifold (M, g) the determination of the C^∞ -jet implies a global uniqueness result via analytic continuation under a topological completeness assumption that we will describe next.

We recall that a function $\iota : M \rightarrow \mathbb{R}$ is a scalar curvature invariant if it is of the form

$$\iota(x) = I(g(x), R(x), \nabla R(x), \dots, \nabla^k R(x)), \quad k = 1, 2, \dots,$$

where I is a smooth function, g is the metric tensor, R is the corresponding curvature tensor, and ∇ stands for covariant differentiation. For example, the Kretschmann scalar, written in local coordinates as $R_{abcd}R^{abcd}$, is a scalar valued curvature invariant of the form $\iota(x) = I(g(x), R(x))$. The Kretschmann scalar for the Schwarzschild black hole is a constant times r^{-6} where r is the radial coordinate.

We say that (M, g) is geodesically complete modulo scalar curvature singularities if every maximal geodesic $\gamma : (\ell_-, \ell_+) \rightarrow M$ satisfies $\ell_\pm = \pm\infty$ or there is a scalar curvature invariant ι such that $\iota(\gamma(t))$ is unbounded as $t \rightarrow \ell_\pm$. We will show the following global result.

Theorem 2. *Let (M, g) and $(\widetilde{M}, \widetilde{g})$ be two smooth Lorentzian manifolds satisfying the assumptions of Theorem 1. Suppose, furthermore, that (M, g) and $(\widetilde{M}, \widetilde{g})$ are connected, geodesically complete modulo scalar curvature singularities and real-analytic. Then the universal Lorentzian covering spaces of (M, g) and $(\widetilde{M}, \widetilde{g})$ are isometric.*

Let us emphasize that the result is sharp in the sense that only the universal covering space, and not the manifold itself, can be determined. For instance, the Minkowski space \mathbb{R}^{1+3} and the flat torus $\mathbb{T}^{1+3} := \mathbb{R}^{1+3}/\mathbb{Z}^4$ with the Minkowski metric contain isometric subsets $(0, 1)^4$, and the time measurements on a small submanifold $\Sigma \subset (0, 1)^4$ are identical in both cases.

3. EXAMPLES

3.1. Riemannian manifolds. Let us illustrate the relation between Theorem 2 and the earlier results for Riemannian manifolds discussed in the introduction.

Let $(\widehat{M}, \widehat{g})$ be a real-analytic complete Riemannian manifold of dimension n , and let \widehat{d} be the Riemannian distance function on \widehat{M} . The completeness could be replaced by an assumption similar to geodesically completeness modulo scalar curvature singularities. Let us consider the product manifold $M = \mathbb{R} \times \widehat{M}$ with the Lorentzian metric

$$(2) \quad g(t, x) = -dt^2 + \widehat{g}(x), \quad (t, x) \in \mathbb{R} \times \widehat{M}, \quad \widehat{g}(x) = \widehat{g}_{jk}(x)dx^j dx^k.$$

Let $\widehat{\mathcal{U}} \subset \widehat{M}$ be a simply convex open set and let $\widehat{\Sigma} \subset \widehat{\mathcal{U}}$ be an $n - 1$ dimensional submanifold. Let $\widehat{\nu}(y)$ be a Riemannian normal vector of $\widehat{\Sigma}$ at $y \in \widehat{\Sigma}$. For some fixed $(\widehat{y}_0, \widehat{\eta}_0) \in T\widehat{\Sigma}$, we assume the following is valid:

H2 : There is an open neighborhood $\widehat{U} \subset T\widehat{\Sigma}$ of $(\widehat{y}_0, \widehat{\eta}_0)$ in $T\widehat{\Sigma}$ satisfying the following: for any $(y, \eta) \in \widehat{U}$, there is $\epsilon > 0$ such that if $r \in (0, \epsilon)$, then the Riemannian geodesic $\widehat{\gamma}(s)$ with

$$\widehat{\gamma}(0) = y, \quad \partial_s \widehat{\gamma}(0) = \eta + r\widehat{\nu}(y),$$

satisfies $\widehat{\gamma}(s_0) \in \widehat{\Sigma}$ for some $s_0 > 0$, and $\widehat{\gamma}(s) \in \widehat{\mathcal{U}}$ for $s \in (0, s_0)$.

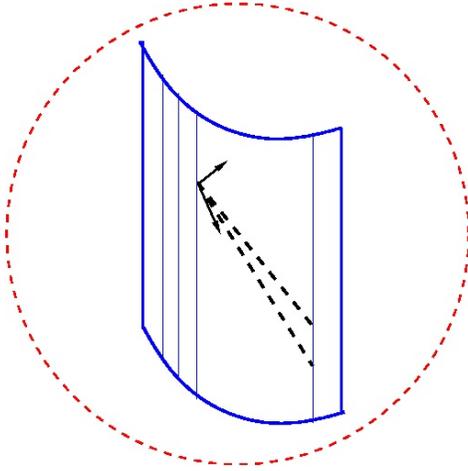


FIGURE 2. A product manifold $M = \mathbb{R} \times \widehat{M}$ with Lorentzian metric $g(t, x) = -dt^2 + \widehat{g}(x)$. Here $(\widehat{M}, \widehat{g})$ is a real-analytic complete Riemannian manifold.

In fact, the hypothesis **(H2)** on the Riemannian manifold \widehat{M} implies **(H)** on the Lorentzian manifold M . To see this, consider the point $(0, \widehat{y}_0) \in M$. We can choose $c_0 \in \mathbb{R}$ sufficiently large so that $c_0 \partial_t|_0 + \widehat{\eta}_0 \in T_{(0, \widehat{y}_0)} M$ is timelike. Let $\pi : M \rightarrow \widehat{M}$ be the canonical projection $\pi(t, y) = y$, then π maps a geodesic γ in M to the Riemannian geodesic $\pi \circ \gamma$ in \widehat{M} . In particular, π projects the geodesic γ in M with the initial data

$$\gamma(0) = (t_0, y), \quad \dot{\gamma}(0) = c_0 \partial_t|_{t_0} + \eta + r\pi^* \widehat{\nu}(t_0, y)$$

to the geodesic $\pi \circ \gamma$ in \widehat{M} with the initial data

$$\pi \circ \gamma(0) = y, \quad (\pi \circ \dot{\gamma})(0) = \eta + r\widehat{\nu}(y).$$

Here $\pi^*\widehat{\nu}$ is the lift to $\mathbb{R} \times \widehat{\Sigma}$ of the vector field ν . Thus if $\epsilon > 0$ is chosen as in **(H2)**, we have from **(H2)** that $\pi \circ \gamma(s_0) \in \widehat{\Sigma}$ for some $s_0 > 0$ and $\pi \circ \gamma(s) \in \widehat{\mathcal{U}}$ for $s \in (0, s_0)$. Consequently for large enough $\ell > 0$, we conclude $\gamma(s_0) \in (-\ell, \ell) \times \widehat{\Sigma}$ and $\gamma(s) \in (-\ell, \ell) \times \widehat{\mathcal{U}}$ for $s \in (0, s_0)$. Finally, we choose c to be sufficiently close to c_0 so that γ is still timelike, and choose t_0 to be sufficiently close to 0. Putting these together, we have shown that $(-\ell, \ell) \times \widehat{\Sigma}$ is timelike convex near $((0, \widehat{y}_0), c_0 \partial_t|_0 + \widehat{\eta}_0)$. (here we use $(-\ell, \ell) \times \widehat{\Sigma}$ instead of $\mathbb{R} \times \widehat{\Sigma}$ since the former has compact closure, see the condition before **(H)**). The simply convex neighborhood in **(H)** can be taken to be $\mathcal{U} := (-2\ell, 2\ell) \times \widehat{\mathcal{U}}$.

In particular, if $\widehat{N} \subset \widehat{M}$ is an open set that has a strictly convex smooth boundary, then any $\widehat{y}_0 \in \partial\widehat{N}$ has a simply convex neighborhood $\widehat{\mathcal{U}} \subset \widehat{M}$ and $\widehat{\Sigma} := \widehat{\mathcal{U}} \cap \partial\widehat{N}$ satisfies the assumption **(H2)**. That $\widehat{\Sigma}$ satisfies **(H2)** simply follows from the strict convexity of $\partial\widehat{N}$. As a result of the analysis in the previous paragraph, the submanifold $(-\ell, \ell) \times \widehat{\Sigma}$ is timelike convex near $((0, \widehat{y}_0), c_0 \partial_t|_0 + \widehat{\eta}_0)$, that is, **(H)** is satisfied.

The restriction of the Riemannian distance function $\widehat{d}|_{\widehat{\Sigma} \times \widehat{\Sigma}}$ determines the restriction of the Lorentzian distance function $d|_{\Sigma \times \Sigma}$ by

$$(3) \quad d((t_1, x_1), (t_2, x_2)) = \begin{cases} \sqrt{(t_2 - t_1)^2 - \widehat{d}(x_1, x_2)^2}, & \text{if } t_2 - t_1 > \widehat{d}(x_1, x_2), \\ 0, & \text{otherwise.} \end{cases}$$

Thus, if we are given $\widehat{\Sigma}$ and $\widehat{d}|_{\widehat{\Sigma} \times \widehat{\Sigma}}$, we can determine by Theorems 1 and 2, the universal covering space of the Lorentzian manifold (M, g) .

The manifold (M, g) is stationary (in fact, static) spacetime with the Killing field $Z = \frac{\partial}{\partial t}$ that corresponds to the ‘‘direction of time’’. Observe that there may be several Killing fields, as can be seen considering the standard Minkowski space \mathbb{R}^{1+3} . All elements in the Lorentz group $O(1, 3)$ define an isometry of \mathbb{R}^{1+3} that may change the time axis to any timelike line. In Section 3.2 we consider also determination of the Killing field in a stationary spacetime.

3.2. Stationary spacetimes satisfying Einstein-scalar field equations. We will apply Theorem 2 to the Einstein-scalar field model. For related inverse problems for the same model, see [33, 34].

Let M be a $(1 + 3)$ dimensional manifold. Let us recall the Einstein field equation $\text{Ein}(g) = T$, where the Einstein tensor $\text{Ein}(g)$ is defined

by

$$\text{Ein}_{jk}(g) = \text{Ric}_{jk}(g) - \frac{1}{2}S(g)g_{jk}, \quad j, k = 0, 1, \dots, 3,$$

Ric is the Ricci curvature tensor, S is the scalar curvature and T is a stress-energy tensor. If $T = 0$ and (M, g) is a solution to the Einstein field equation, then (M, g) is called a vacuum spacetime. We recall that a Lorentzian manifold (M, g) is stationary if there is a timelike vector field Z satisfying

$$(4) \quad \mathcal{L}_Z g = 0,$$

where \mathcal{L}_Z is the Lie derivative with respect to the vector field Z . The vector fields Z satisfying (4) are called Killing fields and when (4) is valid, we say that g is stationary with respect to the Killing field Z .

Below, we consider the Einstein field equations with scalar fields $\phi = (\phi_\ell)_{\ell=1}^L$,

$$(5) \quad \text{Ein}_{jk}(g) + \Lambda g_{jk} = T_{jk}(g, \phi),$$

$$(6) \quad T_{jk}(g, \phi) = \left(\sum_{\ell=1}^L \partial_j \phi_\ell \partial_k \phi_\ell - \frac{1}{2} g_{jk} g^{pq} \partial_p \phi_\ell \partial_q \phi_\ell \right) - \mathcal{V}(\phi) g_{jk},$$

$$(7) \quad \square_g \phi_\ell + \mathcal{V}'_\ell(\phi) = 0.$$

Here $\Lambda \in \mathbb{R}$ is the cosmological constant and $\mathcal{V} : \mathbb{R}^L \rightarrow \mathbb{R}$ is a real-analytic function that physically corresponds to the potential energy of the scalar fields, e.g., $\sum_{\ell=1}^L \frac{1}{2} m^2 \phi_\ell^2$ and $\mathcal{V}'_\ell(r) = \frac{\partial}{\partial r^\ell} \mathcal{V}(r_1, \dots, r_L)$. Another example of the real-analytic function \mathcal{V} is the Higgs-type potential $\mathcal{V}(\phi) = c(|\phi|^2 - m^2)^2$. Also,

$$\square_g u = \sum_{p,q=0}^3 |g|^{-1/2} \frac{\partial}{\partial x^p} \left(|g|^{1/2} g^{pq} \frac{\partial}{\partial x^q} u(x) \right),$$

where $|g| = -\det((g_{pq}(x))_{p,q=0}^3)$.

We will show that if (M, g) is a solution to the Einstein field equations with scalar fields ϕ , and if both (M, g) and ϕ are stationary, then (M, g) is real-analytic. We say that ϕ is stationary with respect to Z if $Z\phi_\ell = 0$ for all $\ell = 1, 2, \dots, L$. See e.g. [14, 44] for examples on non-trivial, real-analytic, spherically symmetric solutions to equations (5)-(7) with suitably chosen potentials $V(r)$.

Müller zum Hagen [45] has shown that a stationary vacuum spacetime is real-analytic. This result has been generalized to several systems of Einstein equations coupled with matter models, such as the Maxwell-Einstein equations [59]. Below we consider the Einstein field equations coupled with scalar fields, and show that stationary solutions

of such equations are real-analytic. Even though this result seems to be known in the folklore of the mathematical relativity, we include for the sake of completeness a proof in Section 5 as we want to apply Theorem 2 to this model.

Proposition 3. *Let M be a smooth manifold, g be a smooth metric tensor on M , ϕ_ℓ , $\ell = 1, 2, \dots, L$, be smooth functions on M , and suppose that (M, g) and $\phi = (\phi_\ell)_{\ell=1}^L$ satisfy the Einstein field equations (5)-(7). Suppose, furthermore, that there is a smooth timelike Killing field Z on M and that $Z\phi_\ell = 0$ for $\ell = 1, 2, \dots, L$. Then (M, g) is real-analytic.*

If the manifolds (M, g) and $(\widetilde{M}, \widetilde{g})$ in Theorem 2 are simply-connected, we have the following determination result.

Corollary 4. *Let (M, g) and $(\widetilde{M}, \widetilde{g})$ be simply connected and geodesically complete modulo scalar curvature singularities, let ϕ and $\widetilde{\phi}$ be \mathbb{R}^L -valued scalar fields on M and \widetilde{M} , respectively, and suppose that (M, g, ϕ) and $(\widetilde{M}, \widetilde{g}, \widetilde{\phi})$ satisfy the Einstein field equations with scalar fields (5)-(7). Also, assume that g , ϕ and \widetilde{g} , $\widetilde{\phi}$ are stationary with respect to timelike Killing fields Z and \widetilde{Z} , respectively. Moreover, let $\mathcal{U} \subset M$ and $\widetilde{\mathcal{U}} \subset \widetilde{M}$ be simply convex open sets. Let $\Sigma \subset \mathcal{U}$ and $\widetilde{\Sigma} \subset \widetilde{\mathcal{U}}$ be relatively compact smooth timelike submanifolds of codimension 1, and let $\Psi : \mathcal{U} \rightarrow \widetilde{\mathcal{U}}$ be a diffeomorphism such that $\Psi(\Sigma) = \widetilde{\Sigma}$ and such that Ψ_* maps the future-pointing unit normal vector field ν of Σ to the future-pointing unit normal vector field $\widetilde{\nu}$ of $\widetilde{\Sigma}$. Assume that Σ and $\widetilde{\Sigma}$ are timelike convex near $(\widehat{x}_0, \widehat{\xi}_0) \in T\Sigma$ and $(\Psi(\widehat{x}_0), \Psi_{\widehat{x}_0}(\widehat{\xi}_0)) \in T\widetilde{\Sigma}$, respectively, and that the Lorentzian distance functions d of \mathcal{U} and \widetilde{d} of $\widetilde{\mathcal{U}}$ satisfy*

$$(8) \quad d(x, y) = \widetilde{d}(\Psi(x), \Psi(y)), \quad \text{for all } x, y \in \Sigma.$$

Then there exists an isometry $F : (M, g) \rightarrow (\widetilde{M}, \widetilde{g})$.

Furthermore, assume that Z is transversal to Σ . Also, suppose $\widetilde{\phi} = \Psi_*\phi$ on $\widetilde{\Sigma}$, and $\widetilde{Z} = \Psi_*Z$ at $\Psi(\widehat{x}_0)$ and $\widetilde{\nabla}_{\Psi_*X}\widetilde{Z} = \Psi_*\nabla_X Z$ at $\Psi(\widehat{x}_0)$ for all vectors $X \in T_{\widehat{x}_0}M$. Here ∇ and $\widetilde{\nabla}$ are the covariant derivatives of (M, g) and $(\widetilde{M}, \widetilde{g})$, respectively. Then $\widetilde{\phi} = F_*\phi$ and $\widetilde{Z} = F_*Z$ on \widetilde{M} .

The proof will be given in Sec. 5.

Geodesic completeness is essential for the unique solvability of inverse problems for partial differential equations similar to (5)-(7). An important class of invisibility cloaking counterexamples is based on transformation optics [20, 21, 22, 19, 40, 49] and these examples are

not geodesically complete. By invisibility cloaking we mean the possibility, both theoretical and practical, of shielding a region or object from detection via electromagnetic or other physical fields.

A model for a spacetime cloak is suggested in [42]. There a point $p \in \mathbb{R}^{1+3}$ is removed from the Minkowski space to obtain the spacetime (N, g_0) with $N = \mathbb{R}^{1+3} \setminus \{p\}$ and $g_0 = \text{diag}(-1, 1, 1, 1)$. Then p is blown-up as follows: let $M = \mathbb{R}^{1+3} \setminus \overline{B}$, where B is the Euclidean unit ball in \mathbb{R}^4 , let $F : N \rightarrow M$ be a diffeomorphism, and define the metric $g_1 = F_*g_0$ on M . The manifold (M, g_1) can be considered as a spacetime with a hole \overline{B} , that can contain an object or an event, that is, a metric in \overline{B} can be chosen freely. When the manifolds \overline{B} and M are glued together, we obtain a spacetime \mathbb{R}^4 with a singular metric that contains the cloaked object in \overline{B} . When F is equal to the identity map outside a compact set, the metric g_1 can be considered as “spacetime cloaking device” around \overline{B} .

The theory of such models have inspired laboratory experiments [18] in optical systems analogous to the cloaking metric. The manifold (M, g_1) is Ricci flat and stationary but it is not complete, or even complete modulo scalar curvature singularities and therefore it does not satisfy the assumptions of Theorem 2 or Corollary 4. To the knowledge of the authors, rigorous cloaking theory for Einstein equations, in particular the question in what sense the non-linear Einstein field equations are valid in the cloaking examples, is still open.

3.3. Schwarzschild black hole. Let us recall the standard definition of the maximally extended Schwarzschild black hole in the Kruskal-Szekeres coordinates, see [48, Sec. 13], [47, Rem. 3.5.5]. Let us first consider the Schwarzschild coordinates (t, r, θ, ϕ) , where $t \in \mathbb{R}$ is the time coordinate (measured by a stationary clock located infinitely far from the massive body), $r \in \mathbb{R}_+$ is the radial coordinate and (θ, ϕ) are the spherical coordinates on the sphere S^2 . The non-extended Schwarzschild black hole is given on the chart

$$(t, r, \theta, \phi) \in M_0 = \mathbb{R} \times (\mathbb{R}_+ \setminus \{R\}) \times (-\pi/2, \pi/2) \times (-\pi, \pi)$$

by the metric

$$g = - \left(1 - \frac{R}{r}\right) dt^2 + \left(1 - \frac{R}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

where $R = 2GM$ is the Schwarzschild radius. Here G is the gravitational constant and M is the Schwarzschild mass parameter, and light speed $c = 1$.

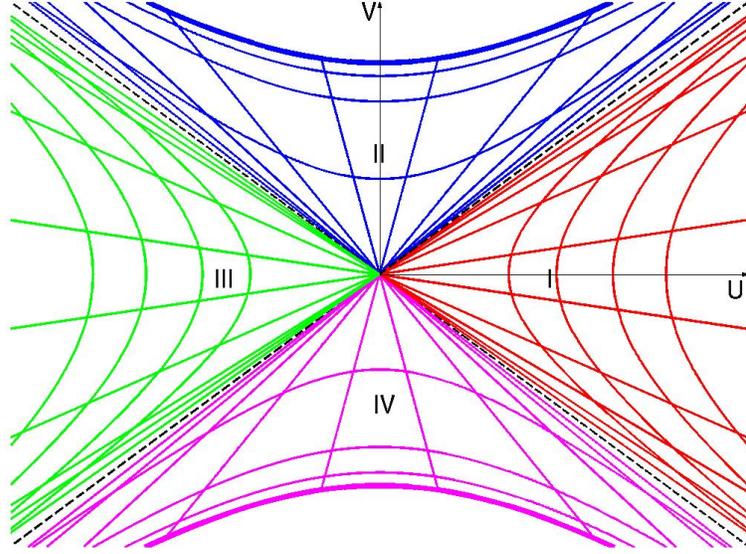


FIGURE 3. **Kruskal-Szekeres diagram.** The diagram shows the domain M in the (U, V) plane. The quadrants are the black hole interior (II), the white hole interior (IV) and the two exterior regions of the black hole (I and III). The diagonal lines $U = V$ and $U = -V$, which separate these four regions, are the event horizons. The hyperbolic curves which bound the top and bottom of the diagram are the physical singularities near which the Kretschmann scalar blow up. The hyperbolas represent contours of the Schwarzschild radial r coordinate, and the lines through the origin represent contours of the Schwarzschild time coordinate t . The region I is usually interpreted as the exterior of the black hole in one universe. The region III is the exterior of the black hole in an another, unreachable universe.

On the set M_0 the Kruskal-Szekeres coordinates are defined by replacing t and r by a new time and spatial coordinates V and U ,

$$\begin{aligned} V &= \left(\frac{r}{R} - 1\right)^{1/2} e^{r/(2R)} \sinh\left(\frac{t}{2R}\right), \\ U &= \left(\frac{r}{R} - 1\right)^{1/2} e^{r/(2R)} \cosh\left(\frac{t}{2R}\right), \end{aligned}$$

for the exterior region $r > R$, and

$$V = \left(1 - \frac{r}{R}\right)^{1/2} e^{r/(2R)} \cosh\left(\frac{t}{2R}\right),$$

$$U = \left(1 - \frac{r}{R}\right)^{1/2} e^{r/(2R)} \sinh\left(\frac{t}{2R}\right),$$

for the interior region $0 < r < R$.

In the Kruskal-Szekeres coordinates the manifold (M_0, g) can be extended real-analytically to a larger manifold. To consider maximal extension we denote

$$M = \{(U, V, w) \in \mathbb{R}^2 \times S^2 ; V^2 - U^2 < 1, w \in S^2\}.$$

The metric is given by

$$g = \frac{4R^3}{r} e^{-r/R} (-dV^2 + dU^2) + r^2 d\Omega^2,$$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2$, and the location of the event horizon, i.e., the surface $r = 2GM$, is given by $V = \pm U$. Here r is defined implicitly by the equation

$$V^2 - U^2 = \left(1 - \frac{r}{R}\right) e^{r/R}.$$

Note that the metric is well defined and smooth on M , even at the event horizon.

The manifold M with the above metric g is Ricci flat, real-analytic and geodesically complete modulo scalar curvature singularities. Indeed, the Kretschmann scalar for the Schwarzschild black hole goes to infinity as $V^2 - U^2$ goes to 1.

Let us consider a metric $\tilde{g} = g + h$, where the small perturbation h is real-analytic on M . We assume that also \tilde{g} is geodesically complete modulo scalar curvature singularities. When $p \in M_0$, $\mathcal{U} \subset M$ is a simply convex neighborhood of a p in (M, \tilde{g}) , and $\Sigma \subset \mathcal{U}$ is a 3-dimensional submanifold, the inverse problem considered in Theorem 2, can be interpreted as the question: Do the measurements in the exterior of the event horizon on “one side” of a black hole (region I in Fig. 1) determine the structure of the spacetime inside the event horizon (region II in Fig. 1), or even on “other side” of the black hole (region III in Fig. 1). By Theorem 2, the answer to this question is positive.

Roughly speaking, this means that if the black hole spacetime has formed so that it is real-analytic, any change of the metric in the exterior region III changes the metric close to the singularity, in the region II, and this further changes the metric and the results of the time separation measurements in the region I. On the other hand, the positive answer to the uniqueness of the inverse problem could be considered as

an argument that the assumption on the real-analyticity of the manifold and the metric is too strong assumption for physical black holes.

3.4. Other examples from the theory of relativity. There are several real-analytic solutions of vacuum Einstein equations for which Theorems 1 or 2 are applicable. These include e.g. the maximal analytic extension of the Kerr black holes, see [7, 60] that are real-analytic and geodesically complete modulo scalar curvature singularities. Similarly, the problem could be considered also for Kerr-Newman black holes, the solutions corresponding to several charged black holes (with the suitably chosen masses are charges), that is, the so-called Majumdar-Papapetrou and Hartle-Hawking solutions [23], and suitable gravitational wave solutions. Also, one can consider cosmological models, such as Friedmann-Lemaitre-Robertson-Walker metrics, see [54]. The detailed analysis of the inverse problem for these manifolds are outside the scope of this paper.

3.5. Material particles and clocks. In this example we construct a specific timelike hypersurface Σ satisfying the hypothesis **H**, and give a physical motivation behind this construction. We recall some facts from the theory of relativity, our main references are [48] and [54]. A point (p, u) on the tangent bundle TM is called an instantaneous observer if $u \in T_pM$ is a future-pointing timelike unit vector. Given such an instantaneous observer, the Lorentzian vector space T_pM admits a direct sum decomposition as

$$T_pM = \mathbb{R}u \oplus u^\perp,$$

where $\mathbb{R}u$ is the 1-dimensional subspace spanned by u , and u^\perp is an n -dimensional spacelike subspace which is orthogonal to $\mathbb{R}u$. $\mathbb{R}u$ is called the observer's time axis and u^\perp the observer's restspace.

A smooth curve $\alpha : I \rightarrow M$ is called a material particle if it is future-pointing timelike and $(\dot{\alpha}(\tau), \dot{\alpha}(\tau))_g = -1$ for all $\tau \in I$. A material particle is said to be freely falling if it is a (necessarily timelike) geodesic. We recall that if a material particle α passes through the point p , say $\alpha(0) = p$, then the energy E and momentum q of α as measured by the observer (p, u) are

$$E = -(\dot{\alpha}(0), u)_g, \quad q = \dot{\alpha}(0) - Eu.$$

Thus we have the decomposition $\dot{\alpha}(0) = Eu + q$. Moreover, the Newtonian velocity as measured by (p, u) is $v = q/E \in u^\perp$, see [54, p. 45].

Let \mathcal{U} be a simply convex neighborhood of p in M . We recall that $d : \mathcal{U} \times \mathcal{U} \rightarrow \mathbb{R}$ is defined in (1) and that $\gamma_{x,y}$ is the unique geodesic

in \mathcal{U} connecting points x and y in \mathcal{U} . Physically, if $\gamma_{x,y}$ is future-pointing and parametrized by arc length, one may think of $\gamma_{x,y}$ as a freely falling material particle, then $d(x,y)$ gives the elapsed proper time of the particle from the event x to the event y .

Let $c_0 \in (0,1)$ be a constant. We define \mathcal{G} to be the set of freely falling material particles α with $\alpha(0) = p$ and with the Newtonian velocities satisfying $|v|_g = c_0$. Choose $\epsilon > 0$ to be small so that the topological closure of the set

$$\Sigma_\epsilon := \{\alpha(\tau) : \alpha \in \mathcal{G}, \tau \in (0, \epsilon)\}$$

is contained in the simply convex neighborhood \mathcal{U} .

Notice that if $\alpha \in \mathcal{G}$, then by combining $(\dot{\alpha}(0), \dot{\alpha}(0))_g = -1$ and $|v|_g = |q/E|_g = c_0$ we get

$$\begin{aligned} -1 &= (\dot{\alpha}(0), \dot{\alpha}(0))_g = (Eu + q, Eu + q)_g = -E^2 + E^2(q/E, q/E)_g \\ &= -E^2 + c_0^2 E^2. \end{aligned}$$

Hence $E = (1 - c_0^2)^{-1/2}$ and $|q|_g = c_0 E$. Therefore, with the exponential map \exp_p , we can write $\alpha(\tau)$ as

$$\alpha(\tau) = \exp_p(\tau \dot{\alpha}(0)) = \exp_p\left(\frac{\tau}{\sqrt{1 - c_0^2}}(u + c_0 \xi)\right) \quad \text{for some } \xi \in S^{n-1}$$

where S^{n-1} denotes the unit sphere in the rest space u^\perp of the instantaneous observer (p, u) . This expression then provides a parametrization of Σ_ϵ . In fact, the map

$$\begin{aligned} \kappa : (0, \epsilon) \times S^{n-1} &\rightarrow \mathcal{U} \\ (\tau, \xi) &\mapsto \exp_p\left(\frac{\tau}{\sqrt{1 - c_0^2}}(u + c_0 \xi)\right) \end{aligned}$$

is a diffeomorphism of $(0, \epsilon) \times S^{n-1}$ onto Σ_ϵ .

Let $(\widetilde{M}, \widetilde{g})$ be another $(1+n)$ -dimensional smooth Lorentzian manifold, and $(\widetilde{p}, \widetilde{u})$ another instantaneous observer with $\widetilde{p} \in M$ and $\widetilde{u} \in T_{\widetilde{p}}\widetilde{M}$. Likewise, we can define the set $\widetilde{\Sigma}_\epsilon$ and assume $\epsilon > 0$ is so small that $\widetilde{\Sigma}_\epsilon$ is contained in a simply convex neighborhood $\widetilde{\mathcal{U}}$ of \widetilde{p} . We also have that

$$\begin{aligned} \widetilde{\kappa} : (0, \epsilon) \times S^{n-1} &\rightarrow \widetilde{\mathcal{U}} \\ (\tau, \xi) &\mapsto \widetilde{\exp}_{\widetilde{p}}\left(\frac{\tau}{\sqrt{1 - c_0^2}}(\widetilde{u} + c_0 \xi)\right) \end{aligned}$$

is a diffeomorphism of $(0, \epsilon) \times S^{n-1}$ onto $\tilde{\Sigma}_\epsilon$. Define the Lorentzian distance function \tilde{d} on $\tilde{\mathcal{U}}$ analogously to d . We can identify Σ_ϵ with $\tilde{\Sigma}_\epsilon$ via the diffeomorphism $\tilde{\kappa} \circ \kappa^{-1}$. Using the notation of Theorem 1, we can take $\Phi = \tilde{\kappa} \circ \kappa^{-1}$.

Now we relate the quantities appearing in Theorem 1 with the quantities in the present example. Clearly the closure of Σ_ϵ is compact in \mathcal{U} . As Σ_ϵ consists of future-pointing timelike geodesics parametrized by S^{n-1} , it is a timelike smooth submanifold of codimension 1. To show that the hypothesis **H** holds, simply notice that Σ_ϵ is diffeomorphic to a cone minus the tip in $T_p M$, under the exponential map \exp_p . Since the cone minus the tip in $T_p M$ satisfies **H**, so does its image Σ_ϵ .

Let us now give a physical interpretation of the above example. Imagine that a primary observer (p, u) shoots out numerous other observers with the same Newtonian speed c_0 in all the directions, and these secondary observers move under the influence of gravitation. Using the notation as above, each secondary observer can be denoted by a freely falling (i.e., under the influence of gravitation only) material particle α with $\alpha(0) = p$, thus the collection of the secondary observers is the set \mathcal{G} and the trajectories of the secondary observers in the space-time \mathbb{R}^{1+3} form the submanifold Σ_ϵ , provided $\epsilon > 0$ is small enough. Suppose each secondary observer carries many clocks, one of them is kept to read his own elapsed time, all the other clocks are to be emitted. Suppose when these secondary observers are shoot out from the primary observer, they start emitting continuously to each other the clocks which also move only under the influence of gravitation. Each emitted clock then records the elapsed time between the two secondary observers: the launcher and the receiver.

When a clock hits the receiver, he/she can read the clock to find out the elapsed time of this clock. After transmitting all the elapsed times collected this way to the primary observer, the primary observer standing at p then knows the elapsed time between any two secondary observers, which amounts to knowing the Lorentzian distance function d on Σ_ϵ . By the analysis in this section and Theorem 1, the primary observer is able to determine (the C^∞ -jet of) the metric structure of the universe at p at the beginning of the experiment. Furthermore, if the metric structure satisfies the assumptions of Theorem 2, this measurement determines the universal Lorentzian covering space of the universe.

3.6. Incompatible extensions. Let us now show that the assumption that the manifold is geodesically complete modulo scalar curvature singularities is essential in Theorem 2 and that the universal covering

space can not be determined without some kind of completeness assumption. We do this by constructing a counterexample of two manifolds one of which is not geodesically complete modulo scalar curvature singularities, and such that both the manifolds have the same time measurement data.

We recall, see [4, Def. 6.15] and [24, p. 58], that an extension of a Lorentzian manifold (M, g) is a Lorentzian manifold (M', g') together with a map $f : M \rightarrow M'$ onto a proper open subset $f(M)$ of M' such that $f : M \rightarrow f(M)$ is a diffeomorphism, and $f_*g = g'$. Also, if (M, g) has no extension, it is said to be inextendible or maximal Lorentzian manifold. If (M, g) , (M', g') and the map f are real-analytic, we say that (M', g') is a real-analytic extension of (M, g) .

Let us consider the product manifold $\mathbb{R} \times S^2$ endowed with the Lorentzian metric $\tilde{g} = -dt^2 + \widehat{g}_{S^2}$, where \widehat{g}_{S^2} is the standard Riemannian metric of S^2 . Let p_1 and p_2 be the South and the North pole. Also, let $M = \mathbb{R} \times (S^2 \setminus I(p_1, p_2))$ where $I(p_1, p_2)$ is one of the shortest Riemannian geodesics connecting p_1 to p_2 , that is, an arc of a great circle connecting the South pole to the North pole. Here, the arc $I(p_1, p_2)$ is closed and contains the points p_1 and p_2 . Let us endow M with the metric $g_M = \tilde{g}|_M$.

Next we construct two real-analytic extensions for manifold M , denoted by M_e and M_c . First, let $M_e = \mathbb{R} \times S^2$ be a Lorentzian manifold with metric $g_e = \tilde{g}$. Observe that manifold M_e is simply connected and geodesically complete. Second, let $N = \mathbb{R} \times (S^2 \setminus \{p_1, p_2\})$ be a Lorentzian manifold with the metric $g = \tilde{g}|_N$ and let (M_c, g_c) be the universal covering space of (N, g) . Using the spherical coordinates, we see that N is homeomorphic to $\mathbb{R} \times (0, \pi) \times S^1$ and the manifold M_c is homeomorphic to the simply connected manifold $\mathbb{R} \times (0, \pi) \times \mathbb{R}$. The obtained manifolds M_e and M_c are real-analytic extensions of the manifold M .

Let $p \in S^2 \setminus I(p_1, p_2)$ and $B_{S^2}(p, r)$ denote the open ball of radius r and center p in the Riemannian manifold S^2 . When $s > 0$ is small enough, let $\mathcal{U} = (-s, s) \times B_{S^2}(p, s)$. Then the manifolds M_e and M_c contain the set $\mathcal{U} \subset M$ that is a simply convex neighborhood of $x_0 = (0, p)$. More precisely, both M_e and M_c contain a simply convex open set that is isometric to $(\mathcal{U}, \tilde{g}|_{\mathcal{U}})$ that can be considered as the data given in Theorems 1 and 2. Note that the manifolds M_e and M_c are simply connected and therefore the both are their own universal covering spaces.

Let us next show that there are no 3-dimensional Lorentzian manifold (M_1, g_1) such that both M_e and M_c could be isometrically embedded in M_1 . To show this, let us assume the opposite, that such manifold M_1

exists. Since M_e is complete, it follows from [4, Prop. 6.16] that it is inextendible, i.e. maximal. Thus M_1 is isometric to M_e and there is an isometric embedding $F : M_c \rightarrow M_e$.

Let us recall, see [48, Sec. 1A], that when $\pi : \mathbb{R} \times S^2 \rightarrow \mathbb{R}$ and $\sigma : \mathbb{R} \times S^2 \rightarrow S^2$ are operators $\pi(s, q) = s$ and $\sigma(s, q) = q$, then the lift $L_\sigma X$, of a vector field X defined on S^2 , is the unique vector field $\tilde{X} = L_\sigma X$ defined on $\mathbb{R} \times S^2$ such that $d\sigma(\tilde{X}) = X$ and $d\pi(\tilde{X}) = 0$. Then, at the point (s, q) , we have $\tilde{X}|_{(s,q)} \in T_{(s,q)}(\{s\} \times S^2)$.

Using [48, Prop. 58 on page 89], we see that the curvature operator R of $\mathbb{R} \times S^2$ at $x = (s, q) \in \mathbb{R} \times S^2$ is such that for all $X, Y \in T_x(\mathbb{R} \times S^2)$,

$$R(\partial_s, X)Y = 0 \quad \text{and} \quad R(X, Y)\partial_s = 0.$$

Moreover, for all $U, V, W \in T_{(s,q)}(\{s\} \times S^2)$, we have at $(s, q) \in \mathbb{R} \times S^2$

$$R(U, V)W = L_\sigma(R_{S^2}(\sigma_*U, \sigma_*V)\sigma_*W),$$

that is, $R(U, V)W$ is equal to the lift of $R_{S^2}(\sigma_*U, \sigma_*V)\sigma_*W$, where R_{S^2} is the Riemannian curvature operator of S^2 . This implies that the linear space

$$\mathcal{V}_x = \{R(X, Y)Z \in T_{(s,q)}(\mathbb{R} \times S^2) \mid X, Y, Z \in T_{(s,q)}(\mathbb{R} \times S^2)\}, \quad x = (s, q),$$

is equal to the space $T_{(s,q)}(\{s\} \times S^2)$.

Since $F : M_c \rightarrow M_e$ is an isometry, we see that at $x \in M_c$ and $y = F(x) \in M_e$ the differential of F , $dF : T_x M_c \rightarrow T_y M_e$, maps \mathcal{V}_x onto \mathcal{V}_y . Let now $x = (s, q) \in M \subset N$ and $\xi \in \mathcal{V}_x \subset T_x M$ and so that the geodesic $\gamma_{x,\xi}^{M_e}$ on M_e is a great circle on $\{s\} \times S^2$. Moreover, let us assume that ξ is such that $\gamma_{x,\xi}^{M_e}$ does not intersect $\mathbb{R} \times \{p_1\}$ or $\mathbb{R} \times \{p_2\}$. Then, considering $(x, \xi) \in TN$ as an element of TM_c we see that the geodesic $\gamma_{x,\xi}^{M_c}$ on M_c is complete and it is homeomorphic to the real axis. However, its image of F , that is, $F(\gamma_{x,\xi}^{M_c}(\mathbb{R})) = \gamma_{F(x), F_*(\xi)}^{M_e}(\mathbb{R}) \subset M_e$ is a closed geodesic that is homeomorphic to S^1 . This is in contradiction with the assumption that F is an isometric embedding.

Summarizing, we have seen that the manifold M has two real-analytic extensions M_e and M_c that can not be isometrically embedded in any connected manifold M_1 of dimension 3 that would contain both of them and both these manifolds contain a subset \mathcal{U} with metric $\tilde{g}|_{\mathcal{U}}$. This shows that assumption that the manifold is geodesically complete modulo scalar curvature singularities is essential in Theorem 2.

4. PROOF ON THE C^∞ -JET DETERMINATION

In this section we investigate the local inverse problem of the C^∞ -jet determination. First, we establish some basic facts about the Lorentzian

distance function $d(x, y)$. Sometimes we would like to fix x and think of $d(x, y)$ as a function of y , in this case we may write $d(x, y)$ as $d_x^+(y)$; sometimes we would like to fix y and think of $d(x, y)$ as a function of x , in this case we may write $d(x, y)$ as $d_y^-(x)$. Notice that $d_x^+(y) > 0$ if and only if $y \in I_{\mathcal{U}}^+(x)$; $d_y^-(x) > 0$ if and only if $x \in I_{\mathcal{U}}^-(y)$. We start with the following simple results whose proof we include for the convenience of the reader.

Lemma 5. *Let \mathcal{U} be a simply convex neighborhood on a smooth Lorentzian manifold (M, g) . Then*

- (i). *d is continuous in $\mathcal{U} \times \mathcal{U}$.*
- (ii). *d is smooth in $\{(x, y) \in \mathcal{U} \times \mathcal{U} : x \ll y\}$.*
- (iii). *Let $c : [0, \epsilon) \rightarrow \mathcal{U}$ be a smooth curve with $c(0) = x$ and $x \ll c(s)$ for all $s \in (0, \epsilon)$, then*

$$\lim_{s \rightarrow 0^+} \frac{d_x^+(c(s))}{s} = \lim_{s \rightarrow 0^+} \frac{d(x, c(s))}{s} = |\dot{c}(0)|_g.$$

- (iv). *For $y \in I_{\mathcal{U}}^+(x)$, let $\gamma : [0, \ell] \rightarrow \mathcal{U}$ be a future pointing timelike radial geodesic with $\gamma(0) = x$, $\gamma(\ell) = y$, then*

$$\text{grad } d_x^+(y) = -\frac{\dot{\gamma}(\ell)}{|\dot{\gamma}(\ell)|_g}.$$

- (v). *The eikonal equation*

$$(\text{grad } d_x^+(y), \text{grad } d_x^+(y))_g = -1$$

holds for $y \in I_{\mathcal{U}}^+(x)$.

Proof. (i). As the exponential map is a radial isometry [48, Chapter 5 Lemma 13], we have

$$d(x, y) = |\exp_x^{-1} y|_g = \sqrt{|(\exp_x^{-1} y, \exp_x^{-1} y)_g|}.$$

It remains to show that $\exp_x^{-1} y$, as a function of (x, y) , is continuous. In fact we will prove a stronger result: $\exp_x^{-1} y$ is a smooth function of $(x, y) \in \mathcal{U} \times \mathcal{U}$.

To this end, introduce the notations

$$\mathcal{D} := \{(x, v) \in TM : (x, \exp_x v) \in \mathcal{U} \times \mathcal{U}\},$$

$$\mathcal{D}_x := \{v \in T_x M : (x, v) \in \mathcal{D}\}.$$

It is easy to see that \mathcal{D} and \mathcal{D}_x are open subsets of TM and $T_x M$, respectively. Consider the map

$$E : \mathcal{D} \rightarrow \mathcal{U} \times \mathcal{U}, \quad E(x, v) = (x, \exp_x v).$$

By simple convexity of \mathcal{U} , the exponential map $\exp_x : \mathcal{D}_x \rightarrow \mathcal{U}$ is non-singular at any $v \in \mathcal{D}_x$, thus by [48, Chapter 5 Lemma 6], E is non-singular at any $(x, v) \in \mathcal{D}$. Notice that E is a smooth map between manifolds of the same dimension, we conclude E is a local diffeomorphism. Notice further that E is bijective on \mathcal{D} , we come to the conclusion that E is in fact a diffeomorphism. By setting $y = \exp_x v$ it follows that $\exp_x^{-1} y$ is smooth for $(x, y) \in \mathcal{U} \times \mathcal{U}$. This completes the proof.

- (ii). If $x \ll y$, then the smooth function $\exp_x^{-1} y$ is non-vanishing. From the expression

$$(9) \quad d(x, y) = |\exp_x^{-1} y|_g = \sqrt{-(\exp_x^{-1} y, \exp_x^{-1} y)_g}$$

it is straightforward that d is smooth in $\{(x, y) \in \mathcal{U} \times \mathcal{U} : x \ll y\}$.

- (iii). Choose a normal coordinate chart (φ, ξ^i) where $\varphi = \exp_x^{-1}$. Let $\xi(s) = \varphi(c(s))$. The function d_x^+ composed with this normal coordinate chart reads

$$d_x^+ \circ \varphi^{-1} : \xi(s) \mapsto \sqrt{-(\xi(s), \xi(s))_g}.$$

Notice that $\xi(0) = \varphi(x) = 0$, so $\xi(s) = \dot{\xi}(0)s + \mathcal{O}(s^2)$. From this we conclude

$$\lim_{s \rightarrow 0^+} \frac{d_x^+(c(s))}{s} = \lim_{s \rightarrow 0^+} \frac{d_x^+ \circ \varphi^{-1}(\xi(s))}{s} = |\dot{\xi}(0)|_g = |\dot{c}(0)|_g.$$

- (iv). The exponential map is a radial isometry, thus

$$d_x^+(\gamma(t)) = d(x, \gamma(t)) = |\dot{\gamma}(0)|_g t.$$

Differentiate to get

$$(10) \quad (\text{grad } d_x^+(\gamma(t)), \dot{\gamma}(t))_g = |\dot{\gamma}(0)|_g.$$

Since $\dot{\gamma}(t)$ is orthogonal to the level sets of d_x^+ by Gauss Lemma [48, Chapter 5 Lemma 1], and since $\text{grad } d_x^+$ is also orthogonal to the level sets of d_x^+ , there exists a function $C(t)$ such that $\text{grad } d_x^+(\gamma(t)) = C(t)\dot{\gamma}(t)$. As γ is a geodesic, $|\dot{\gamma}(0)|_g = |\dot{\gamma}(t)|_g$ for all $0 \leq t \leq \ell$. Therefore, using (10) we derive that

$$C(t) = -\frac{1}{|\dot{\gamma}(t)|_g}.$$

Setting $t = \ell$ completes the proof.

- (v). Using the result in (iv) and that $(\dot{\gamma}(t), \dot{\gamma}(t))_g = -|\dot{\gamma}(t)|_g^2$ we have

$$(\text{grad } d_x^+(y), \text{grad } d_x^+(y))_g = \left(-\frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|_g}, -\frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|_g}\right)_g = -1.$$

□

Remark: Throughout this paper we only assume to know the Lorentzian distance between any two points on Σ . However, Lemma 5(i) says that by continuity we can further know the Lorentzian distance between any two points on the closure $\bar{\Sigma}$. In other words, we know not only $d|_{\Sigma \times \Sigma}$, but also $d|_{\bar{\Sigma} \times \bar{\Sigma}}$. This observation is useful in certain circumstances.

Without loss of generality we can suppose in the assumption of Theorem 1 that $\hat{\xi}_0$ is a past-pointing timelike vector, and we do this assumption below. From now on, we will systematically use \sim to denote the quantities which are related via the diffeomorphism Φ ; for instance, $\tilde{x} := \Phi(x)$.

As the first step towards proving Theorem 1, the following proposition says that the restriction of the Lorentzian distance function $d|_{\Sigma \times \Sigma}$ determines the metric g on the tangent bundle of Σ .

Proposition 6. *Under the assumption of Theorem 1, for any $x \in \Sigma$ and any $\xi \in T_x \Sigma$, we have*

$$(\xi, \xi)_g = (\tilde{\xi}, \tilde{\xi})_{\tilde{g}}$$

where $\tilde{\xi} := \Phi_{*x}(\xi)$ is the image of ξ under the push-forward Φ_* at x . Consequently, by polarization $g = \Phi^* \tilde{g}$ on $T_x \Sigma$.

Proof. For any fixed $x \in \Sigma$, since Σ is a timelike submanifold, we can find a future-pointing timelike vector $\xi_0 \in T_x \Sigma$. Let $c : [0, \epsilon) \rightarrow \Sigma \cap I_{\mathcal{U}}^+(x)$ be a smooth curve with $c(0) = x$ and $\dot{c}(0) = \xi_0$. By the assumption of Theorem 1, $\tilde{d}(\tilde{x}, \tilde{c}(s)) = d(x, c(s)) > 0$, we conclude $\tilde{c}(s) \in \tilde{I}_{\mathcal{U}}^+(\tilde{x})$. Hence $\tilde{c} : [0, \epsilon) \rightarrow \bar{\Sigma} \cap \tilde{I}_{\mathcal{U}}^+(\tilde{x})$ is a smooth curve with $\tilde{c}(0) = \tilde{x}$ and $\dot{\tilde{c}}(0) = \tilde{\xi}_0$. By Lemma 5(iii)

$$-(\tilde{\xi}_0, \tilde{\xi}_0)_{\tilde{g}} = \left(\lim_{s \rightarrow 0^+} \frac{\tilde{d}_{\tilde{x}}^+(\tilde{c}(s))}{s} \right)^2 = \left(\lim_{s \rightarrow 0^+} \frac{d_x^+(c(s))}{s} \right)^2 = -(\xi_0, \xi_0)_g.$$

Since this identity is true for all future-pointing timelike vectors in $T_x \Sigma$, we conclude that the two quadratic forms $(\Phi_* \cdot, \Phi_* \cdot)_{\tilde{g}}$ and $(\cdot, \cdot)_g$ coincide on the open set of timelike vectors, hence they must be equal everywhere. □

We introduce a local coordinate system which is an analogue of the semi-geodesic coordinates in Riemannian geometry. As the hypersurface Σ is an n -dimensional manifold, near the fixed point $\hat{x}_0 \in \Sigma$ we can find a coordinate chart $(W, (x^1, \dots, x^n))$ such that W is a neighborhood of \hat{x}_0 in Σ and the closure \bar{W} is compact in Σ . Let ν be the

unit normal vector field on Σ , chosen as in the hypothesis **H**. For small $\delta > 0$, we can define a diffeomorphism using W as follows:

$$(11) \quad \Psi(x, r) := \exp_x(r\nu(x)) \quad (x, r) \in W \times (-\delta, \delta).$$

Geometrically Ψ^{-1} parameterizes a tubular neighborhood of W in \mathcal{U} . Similarly we can define

$$\tilde{\Psi}(\tilde{x}, r) := \exp_{\tilde{x}}(r\tilde{\nu}(\tilde{x})) \quad (\tilde{x}, r) \in \tilde{W} \times (-\delta, \delta)$$

where $\tilde{\nu}$ is the normal vector field to $\tilde{\Sigma}$ chosen as in the hypothesis **H**, $\tilde{x} = \Phi(x)$, and $\tilde{W} := \Phi(W)$ is the image of W under the diffeomorphism Φ . Let $id : (-\delta, \delta) \rightarrow (-\delta, \delta)$ be the identity map. By identifying $x \in W$ with $(x, 0) \in W \times (-\delta, \delta)$, the diffeomorphism

$$(12) \quad \tilde{\Psi} \circ (\Phi \times id) \circ \Psi^{-1} : W \times (-\delta, \delta) \rightarrow \tilde{W} \times (-\delta, \delta),$$

is precisely $\Phi|_W$ when restricted to $W \times \{0\}$. In other words, the map (12) extends $\Phi|_W$ to a diffeomorphism which identifies the tubular neighborhood $\Psi(W \times (-\delta, \delta))$ with the tubular neighborhood $\tilde{\Psi}(\tilde{W} \times (-\delta, \delta))$. We will continue using a \sim to indicate that the quantities are related via this diffeomorphism.

Using the coordinates on W , (x^1, \dots, x^n, r) constitute coordinates in the tubular neighborhood $\Psi(W \times (-\delta, \delta))$. Similarly $(\tilde{x}^1, \dots, \tilde{x}^n, r)$ form local coordinates for the tubular neighborhood $\tilde{\Psi}(\tilde{W} \times (-\delta, \delta))$. In these coordinates, the metrics g and \tilde{g} can be expressed as

$$(13) \quad \begin{aligned} g &= \sum_{i,j=1}^n g_{ij}(x, r) dx^i dx^j + dr^2, \\ \tilde{g} &= \sum_{i,j=1}^n \tilde{g}_{ij}(\tilde{x}, r) d\tilde{x}^i d\tilde{x}^j + dr^2. \end{aligned}$$

The construction of such coordinates (x^1, \dots, x^n, r) and the special form of the metric tensors (13) are determined by P-Jacobi fields on geodesics, see [48, Definition 10.29, Lemma 8.7] or [58, Chapter 1 §8] for more details.

Now we are ready to prove our first main theorem. Roughly speaking, we shoot some timelike geodesics from near \hat{x}_0 , which by the timelike convexity assumption will intersect $\tilde{\Sigma}$. For those long geodesics, we adapt the proof of [55, Theorem 1] which is in the context of Riemannian geometry. For the short geodesics, we follow the idea of the proof of [37, Theorem 2.1] in the Riemannian setting to write distances as integrals over geodesics.

Proof of Theorem 1. Using the coordinates in (13), we only need to determine C^∞ -jet of each component g_{ij} at \widehat{x}_0 . As a conclusion of Proposition 6, the functions g_{ij} are uniquely determined on $\{(x, r) : r = 0\}$, from this we can find all tangential derivatives of g_{ij} at \widehat{x}_0 . Next we will show that $d|_{\Sigma \times \Sigma}$ uniquely determines the normal derivatives $\frac{\partial^k g_{ij}}{\partial r^k}(\widehat{x}_0)$ for $k = 1, 2, \dots$; that is, we will show

$$(14) \quad \frac{\partial^k g_{ij}}{\partial r^k}(\widehat{x}_0) = \frac{\partial^k \widetilde{g}_{ij}}{\partial r^k}(\Phi(\widehat{x}_0)) \quad k = 1, 2, \dots$$

We remark that in the following proof of (14), the key information of g that is used is the knowledge of its tangential derivatives on $\{(x, r) : r = 0\}$, thus the proof is also valid if g_{ij} in (14) is replaced by any of its tangential derivative $\frac{\partial^{s_1+\dots+s_n} g_{ij}}{(\partial x^1)^{s_1} \dots (\partial x^n)^{s_n}}$, and in this way we can determine any mixed derivative of the form $\frac{\partial^{s_1+\dots+s_n+s_{n+1}} g_{ij}}{(\partial x^1)^{s_1} \dots (\partial x^n)^{s_n} (\partial r)^{s_{n+1}}}$.

In order to make the proof of (14) clear, we divide it into two steps.

Step 1: Let us start with the case when $k = 1$. We will employ two types of argument alternately: one is constructive, that is, we give explicit procedures on how to recover quantities related to the metric g from the measurement function $d|_{\Sigma \times \Sigma}$; the other is non-constructive: we show that some quantities, which are related to g and \widetilde{g} respectively, are identical under the assumption that $d(x, y) = \widetilde{d}(\widetilde{x}, \widetilde{y})$ for all $x, y \in \Sigma$.

For the metric g , let $(x_0, \xi_0) \in \widehat{U}$ with ξ_0 a past-pointing timelike unit vector, and let ν be the unit normal vector field as in the hypothesis **H**. Define a sequence of vectors $\xi_l \in T_{x_0} \Sigma$ by

$$\xi_l := \xi_0 + \frac{1}{l} \nu(x_0) = \sum_{i=1}^n \xi_0^i \frac{\partial}{\partial x^i} \Big|_{x_0} + \frac{1}{l} \frac{\partial}{\partial r} \Big|_{x_0}.$$

Here the positive integer l is chosen to be sufficiently large so that ξ_l is also a timelike past-pointing vector. Let γ_l be the unique geodesic issued from x_0 in the direction ξ_l . By the hypothesis **H**, γ_l will intersect $\overline{\Sigma}$ for some $t_l > 0$, without loss of generality we may assume that t_l is the smallest parameter value such that the intersection happens. The corresponding encounter point $y_l := \exp_{x_0}(t_l \xi_l)$ will be different from x_0 since \mathcal{U} is simply convex (see pictures below). As $\overline{\Sigma}$ is compact, $\{t_l\}$ is a bounded sequence, hence has a convergent subsequence which we assume to be itself and write $t_l \rightarrow t_0 \in \mathbb{R}$ and $y_l \rightarrow y_0 \in \overline{\Sigma}$ as $l \rightarrow \infty$. Based on $\{y_l\}$ and y_0 , we can define functions $d_{y_l}^+, l = 1, 2, \dots$ and $d_{y_0}^+$. Notice that although $y_l \neq x_0$ for each l , it is possible that $y_0 = x_0$. In the following we consider two cases: $y_0 \neq x_0$ and $y_0 = x_0$. For the case

$y_0 \neq x_0$ we employ a constructive argument, while for the case $y_0 = x_0$ we prove the uniqueness in a non-constructive way. We write them as two claims.

Claim 1: If $y_0 \neq x_0$, we can uniquely determine $\sum_{i,j=1}^n \frac{\partial g^{ij}}{\partial r}(x_0) \xi_0^i \xi_0^j$.

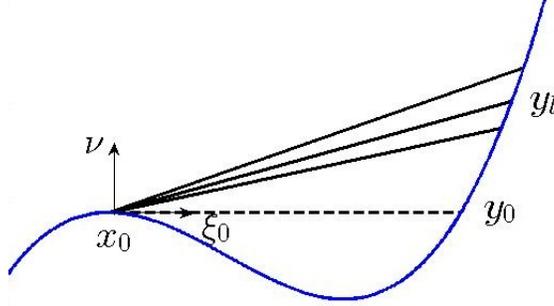


FIGURE 4. The case where $y_0 \neq x_0$.

To prove Claim 1, first notice that in this case $y_0 = \exp_{x_0}(t_0 \xi_0) \in I_{\mathcal{U}}^-(x_0)$, which means $y_0 \ll x_0$. By Lemma 5(ii), $d_{y_l}^+$ is smooth in a neighborhood, say $U_l \subset \mathcal{U}$, of x_0 when $l \geq l_0$ for some large integer $l_0 > 0$. For such an l , running the geodesic γ_l backwards from y_l to x_0 and applying Lemma 5(iv) yields

$$(15) \quad \text{grad } d_{y_l}^+(x_0) = \frac{\xi_l}{|\xi_l|_g}.$$

Also Lemma 5(v) states that the eikonal equation

$$(\text{grad } d_{y_l}^+(x), \text{grad } d_{y_l}^+(x))_g = -1$$

holds in U_l . We write this equation with the coordinates (x^1, \dots, x^n, r) :

$$(16) \quad \sum_{i,j=1}^n g^{ij} \frac{\partial d_{y_l}^+}{\partial x^i} \frac{\partial d_{y_l}^+}{\partial x^j} + \left(\frac{\partial d_{y_l}^+}{\partial r} \right)^2 = -1, \quad l \geq l_0.$$

In particular, this relation holds on $U_l \cap \Sigma$. Since the function $d|_{\Sigma \times \Sigma}$ is assumed to be known, we actually know the function $d_{y_l}^+$ on Σ (even if $y_l \in \bar{\Sigma}$ may not lie in Σ , see the remark after Lemma 5). This fact, together with Proposition 6, implies that all the tangential derivatives of g^{ij} and $d_{y_l}^+$ are known on $U_l \cap \Sigma$. Thus we can solve for $\frac{\partial d_{y_l}^+}{\partial r}$ in (16) by taking a square root. The sign of the square root can be determined in the following approach. Observing that in the coordinates

(x^1, \dots, x^n, r) the last component of $\text{grad } d_{y_l}^+(x_0)$ is $\frac{\partial d_{y_l}^+}{\partial r}(x_0)$, and the last component of $\frac{\xi_l}{|\xi_l|_g}$ is $\frac{1}{l|\xi_l|_g}$ by the definition of ξ_l , we derive from (15) that

$$(17) \quad \frac{\partial d_{y_l}^+}{\partial r}(x_0) = \frac{1}{l|\xi_l|_g},$$

which is positive. Therefore, from (16) we can recover $\frac{\partial d_{y_l}^+}{\partial r}$ in a neighborhood of x_0 in $U_l \cap \Sigma$ by taking the positive square root. Shrinking U_l if necessary, we may assume that this neighborhood is $U_l \cap \Sigma$ itself.

Differentiating (16) with respect to a tangential direction, say x^m , we get

$$(18) \quad \sum_{i,j=1}^n \left(\frac{\partial g^{ij}}{\partial x^m} \frac{\partial d_{y_l}^+}{\partial x^i} \frac{\partial d_{y_l}^+}{\partial x^j} + 2g^{ij} \frac{\partial^2 d_{y_l}^+}{\partial x^i \partial x^m} \frac{\partial d_{y_l}^+}{\partial x^j} \right) + 2 \frac{\partial d_{y_l}^+}{\partial r} \frac{\partial^2 d_{y_l}^+}{\partial x^m \partial r} = 0 \quad l \geq l_0.$$

From this identity we can recover $\frac{\partial^2 d_{y_l}^+}{\partial x^m \partial r}$ on $U_l \cap \Sigma$, and similarly up to all order the tangential derivatives of $\frac{\partial d_{y_l}^+}{\partial r}$ can be recovered on $U_l \cap \Sigma$.

On the other hand, differentiating (16) with respect to r we get

$$(19) \quad \sum_{i,j=1}^n \left(\frac{\partial g^{ij}}{\partial r} \frac{\partial d_{y_l}^+}{\partial x^i} \frac{\partial d_{y_l}^+}{\partial x^j} + 2g^{ij} \frac{\partial^2 d_{y_l}^+}{\partial x^i \partial r} \frac{\partial d_{y_l}^+}{\partial x^j} \right) + 2 \frac{\partial d_{y_l}^+}{\partial r} \frac{\partial^2 d_{y_l}^+}{\partial r^2} = 0 \quad l \geq l_0.$$

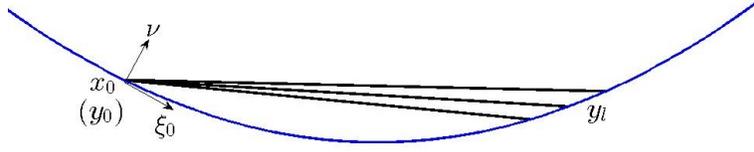
Evaluating this at $x = x_0$ and taking (17) into consideration we obtain

$$\sum_{i,j=1}^n \frac{\partial g_{ij}}{\partial r}(x_0) \frac{\xi_0^i}{|\xi_l|_g} \frac{\xi_0^j}{|\xi_l|_g} = - \left(2 \sum_{i,j=1}^n g^{ij} \frac{\partial^2 d_{y_l}^+}{\partial x^i \partial r} \frac{\partial d_{y_l}^+}{\partial x^j} + 2 \frac{\partial^2 d_{y_l}^+}{\partial r^2} \frac{1}{l|\xi_l|_g} \right) \Big|_{x=x_0}.$$

As $y_0 \neq x_0$, $\frac{\partial^2 d_{y_l}^+}{\partial r^2}(x_0)$ remains bounded as $l \rightarrow \infty$, so the second term on the right-hand side tends to zero as $l \rightarrow \infty$, and we recover $\sum_{i,j=1}^n \frac{\partial g_{ij}}{\partial r}(x_0) \xi_0^i \xi_0^j$. This completes the proof of Claim 1.

Claim 2: if $y_0 = x_0$, then $\sum_{i,j=1}^n \frac{\partial g_{ij}}{\partial r}(x_0) \xi_0^i \xi_0^j = \sum_{i,j=1}^n \frac{\partial \tilde{g}_{ij}}{\partial r}(\tilde{x}_0) \tilde{\xi}_0^i \tilde{\xi}_0^j$.

Notice that if $y_0 = x_0$, then $y_l \rightarrow x_0$ as $l \rightarrow \infty$, so for large l , γ_l will lie in $W \times (-\delta, \delta)$. We fix such an l and pull \tilde{g} back from the tubular neighborhood $\tilde{\Psi}(\tilde{W} \times (-\delta, \delta))$ onto the tubular neighborhood $\Psi(W \times (-\delta, \delta))$ via the diffeomorphism (12), the pullback metric, say

FIGURE 5. The case where $y_0 = x_0$.

g' , has the expression

$$g'(x, r) = \sum_{i,j=1}^n g'_{ij}(x, r) dx^i dx^j + dr^2.$$

Correspondingly let d' be the pullback of \tilde{d} via (12) onto $\Psi(W \times (-\delta, \delta))$, then we have $d(x, y) = d'(x, y)$ for all $x, y \in W$. Define $f_{ij} = g_{ij} - g'_{ij}$, in the following we will show

$$(20) \quad \sum_{i,j=1}^n \frac{\partial f_{ij}}{\partial r}(x_0, 0) \xi_0^i \xi_0^j = 0,$$

from which Claim 2 follows. Here we have identified $x_0 \in W$ with $(x_0, 0) \in W \times (-\delta, \delta)$.

Now we prove (20) by a contrapositive argument. Suppose it is not true, without loss of generality we may assume

$$\sum_{i,j=1}^n \frac{\partial f_{ij}}{\partial r}(x_0, 0) \xi_0^i \xi_0^j > 0.$$

By continuity, there exists a conic neighborhood V of $((x_0, 0); \xi_0)$ in the tangent bundle $T\mathcal{U}$ so that

$$\sum_{i,j=1}^n \frac{\partial f_{ij}}{\partial r}(x, r) \xi^i \xi^j > 0$$

for all $((x, r); \xi) \in V$. As $f_{ij}(x, 0) = 0$ by Proposition 6, developing f_{ij} in Taylor's expansion we obtain

$$f_{ij}(x, r) = \frac{\partial f_{ij}}{\partial r}(x, 0) r + \mathcal{O}(r^2),$$

thus

$$(21) \quad \sum_{i,j=1}^n f_{ij}(x, r) \xi^i \xi^j > 0$$

for all $((x, r); \xi) \in V$ with $r > 0$. By choosing l sufficiently large, we may assume $\{(\gamma_l(t); \dot{\gamma}_l(t)) : 0 \leq t \leq t_l\}$ is contained in V so that (21) is

valid. Since $\xi_0 \in T_{x_0}\Sigma$ is timelike with respect to g , $\xi_l \in T_{x_0}\Sigma$ is close to ξ_0 , and $g = g'$ on $T_{x_0}\Sigma$, we see that γ_l is a timelike curve with respect to g' for large l . (Notice that by definition γ_l is a timelike geodesic with respect to g . The argument here shows that γ_l is also a timelike curve with respect to g' , but not necessarily a timelike geodesic.) We assume the fixed l is chosen to be so large that γ_l is indeed a timelike curve with respect to g' . Therefore, for this timelike curve, we can find a strictly increasing smooth parametrization $\iota : [0, \ell] \rightarrow [0, t_l]$ such that the reparametrized curve

$$\Gamma := \gamma_l \circ \iota : [0, \ell] \rightarrow \mathcal{U}$$

satisfies $(\dot{\Gamma}(t), \dot{\Gamma}(t))_{g'} = -1$ for all t . It follows from (21) that

$$\begin{aligned} (22) \quad & \int_0^\ell (\dot{\Gamma}(t), \dot{\Gamma}(t))_g dt + \ell \\ &= \int_0^\ell \sum_{i,j=1}^n g_{ij}(\Gamma(t)) \dot{\Gamma}^i(t) \dot{\Gamma}^j(t) dt - \int_0^\ell \sum_{i,j=1}^n g'_{ij}(\Gamma(t)) \dot{\Gamma}^i(t) \dot{\Gamma}^j(t) dt \\ &= \int_0^\ell \sum_{i,j=1}^n f_{ij}(\Gamma(t)) \dot{\Gamma}^i(t) \dot{\Gamma}^j(t) dt > 0. \end{aligned}$$

On the other hand, let γ'_l be the pullback via (12) of the unique radial geodesic in $\tilde{\mathcal{U}}$ joining \tilde{y}_l and \tilde{x}_0 , hence γ'_l is, with respect to g' , the longest timelike curve joining y_l and x_0 in $\Psi(W \times (-\delta, \delta))$. Therefore, we conclude

$$\ell \leq d'(y_l, x_0) = d(y_l, x_0) = L(\Gamma).$$

By Cauchy-Schwarz inequality

$$(23) \quad \ell^2 \leq L^2(\Gamma) = \left(\int_0^\ell |\dot{\Gamma}(t)|_{\frac{1}{g}} dt \right)^2 \leq -\ell \int_0^\ell (\dot{\Gamma}(t), \dot{\Gamma}(t))_g dt$$

since $(\dot{\Gamma}(t), \dot{\Gamma}(t))_g < 0$ for all t . From (23) we derive

$$\ell + \int_0^\ell (\dot{\Gamma}(t), \dot{\Gamma}(t))_g dt \leq 0,$$

which contradicts (22). This completes the proof of identity (20), hence Claim 2.

Combining *Claim 1* and *Claim 2*, in either case $\sum_{i,j=1}^n \frac{\partial g_{ij}}{\partial r}(x_0) \xi_0^i \xi_0^j$ is uniquely determined. To recover $\frac{\partial g_{ij}}{\partial r}(x_0)$, we need to perturb ξ_0 : for any $\xi \in T_{x_0}\Sigma$ which is sufficiently close to ξ_0 , we run the above arguments to recover $\sum_{i,j=1}^n \frac{\partial g_{ij}}{\partial r}(x_0) \xi^i \xi^j$, these values are enough to

determine the matrix $(\frac{\partial g_{ij}}{\partial r}(x_0))_{1 \leq i, j \leq n}$, hence we obtain $\frac{\partial g_{ij}}{\partial r}(x_0)$. In particular, evaluating at $x_0 = \widehat{x}_0$ completes the proof of (14) for $k = 1$.

Step 2: In this step, we prove (14) for $k \geq 2$. This is an inductive argument. However, to make the idea clear, we state the proof only for $k = 2$. It should be obvious how this inductive process is done for any $k \geq 3$.

If $y_0 \neq x_0$, differentiate (19) with respect to r and evaluate at $x = x_0$. Since we have known $\frac{\partial g_{ij}}{\partial r}$ (note that in Step 1 we actually found $\frac{\partial g_{ij}}{\partial r}(x_0)$ for all x_0 near \widehat{x}_0 , not just $\frac{\partial g_{ij}}{\partial r}(\widehat{x}_0)$), from (19) we can recover $\frac{\partial^2 d_{y_l}^+}{\partial r^2}$ as well as all its tangential derivatives on $U_l \cap \Sigma$. The only unknown term will be $\frac{\partial^3 d_{y_l}^+}{\partial r^3}(x_0)$, but it will be multiplied by $\frac{\partial d_{y_l}^+}{\partial r}(x_0) = \frac{1}{l|\xi_l|_g}$. Taking the limit $l \rightarrow \infty$ will recover $\sum_{i,j=1}^n \frac{\partial^2 g_{ij}}{\partial r^2}(x_0) \xi_0^i \xi_0^j$ as above.

If $y_0 = x_0$, as in the proof of *Claim 2* we assume $\sum_{i,j=1}^n \frac{\partial^2 f_{ij}}{\partial r^2}(x_0) \xi_0^i \xi_0^j \neq 0$, without loss of generality we may assume it is positive. Writing the Taylor expansion of f_{ij} with respect to r at $r = 0$, by Proposition 6 and the case $k = 1$ in (14), we see that (21) holds for $((x, r); \xi)$ in a conic neighborhood of $((x_0, 0); \xi_0)$ with $r > 0$. Similarly we get a contradiction as above.

In either case, we can uniquely determine $\sum_{i,j=1}^n \frac{\partial^2 g_{ij}}{\partial r^2}(x_0) \xi_0^i \xi_0^j$. Finally, by first varying ξ_0 , and next varying x_0 , and then evaluating at $x_0 = \widehat{x}_0$, we prove (14) when $k = 2$. Applying this construction inductively completes the proof of the theorem. \square

5. GLOBAL DETERMINATION OF THE MANIFOLD

In this section we describe a procedure to obtain a maximal real-analytic extensions of a real-analytic manifolds that are geodesically complete modulo scalar curvature singularities.

Let (M, g) and $(\widetilde{M}, \widetilde{g})$ be a smooth Lorentzian manifolds, and let $\phi : U \rightarrow \widetilde{U}$ be an isometry of an open set $U \subset M$ onto an open set $\widetilde{U} \subset \widetilde{M}$. Let $\gamma : [0, \ell] \rightarrow M$ be a path starting from U , that is, $\gamma(0) \in U$. Let I be a connected neighborhood of zero in $[0, \ell]$. We say that a family $\phi_t : U_t \rightarrow \widetilde{U}_t$, $t \in I$, is a continuation of ϕ along γ if

- (i) $U_t \subset M$ and $\widetilde{U}_t \subset \widetilde{M}$ are open, $\gamma(t) \in U_t$ and ϕ_t is an isometry,
- (ii) for all $s \in I$ there is $\epsilon > 0$ such that $\phi_t = \phi_s$ in $U_t \cap U_s$ whenever $|t - s| < \epsilon$, and
- (iii) $\phi_0 = \phi$ in $U_0 \cap U$.

We say that ϕ is extendable along γ if there is a continuation of ϕ_t , $t \in [0, \ell]$, along γ .

We recall that a continuous path $\gamma : [0, \ell] \rightarrow M$ is a broken geodesic if there are $0 < l_0 < l_1 < \dots < l_N < \ell$ such that γ is a geodesic on $[l_{j-1}, l_j]$, $j = 1, 2, \dots, N$.

Theorem 7. *Suppose that (M, g) and $(\widetilde{M}, \widetilde{g})$ are connected. Let $\phi : U \rightarrow \widetilde{U}$ be an isometry of an open set $U \subset M$ onto an open set $\widetilde{U} \subset \widetilde{M}$, and suppose that ϕ is extendable along all broken geodesics $\gamma : [0, \ell] \rightarrow M$ starting from U . Suppose, furthermore, that all broken geodesics $\gamma : [0, \ell] \rightarrow M$, $\ell \in (0, \infty)$, starting from U satisfy the following:*

- (L) *If ϕ_t is a continuation of ϕ along γ and the limit $\lim_{t \rightarrow \ell} \phi_t(\gamma(t))$ exists, then the limit $\lim_{t \rightarrow \ell} \gamma(t)$ exists.*

Then (M, g) and $(\widetilde{M}, \widetilde{g})$ have the same universal Lorentzian covering space.

Although the assumptions in the theorem may seem unsymmetric with respect to M and \widetilde{M} , in fact, they are not. The extendability up to the end point ℓ implies the condition (L) with the roles of M and \widetilde{M} interchanged. We will also see below that if (M, g) and $(\widetilde{M}, \widetilde{g})$ are geodesically complete modulo scalar curvature singularities and real-analytic, and if there is a local isometry $\phi : U \rightarrow \widetilde{U}$ as above, then (M, g) and $(\widetilde{M}, \widetilde{g})$ satisfy the assumptions of the theorem.

Proof. We recall, see [48, p. 203], that (\mathcal{W}, \sim) is a mathced covering of M if $\mathcal{W} = \{W_j; j \in J\}$ is a covering of M by open sets W_j and \sim is a relation on the index set J such that for all $j, k, l \in J$

- (i) $j \sim j$,
- (ii) if $j \sim k$ then $k \sim j$,
- (iii) if $j \sim k$, $k \sim l$ and $W_j \cap W_k \cap W_l$ is nonempty, then $j \sim l$.

We begin by constructing a matched covering. Let $p \in U$. We denote by A the set of pairs (γ, t) where $\gamma : [0, \ell] \rightarrow M$ is a broken geodesic satisfying $\gamma(0) = p$ and $t \in [0, \ell]$. Let $(\gamma, t) \in A$. Let us choose a continuation $\phi_t^\gamma : V_t^\gamma \rightarrow \widetilde{M}$ of ϕ along γ . The sets $\{V_t^\gamma : (\gamma, t) \in A\}$, form an open covering of M since any two points of M can be joined by a broken geodesic, see e.g. [48, Lem. 3.32].

For each $(\gamma, t) \in A$ we may choose a fundamental system of neighborhoods $U_{t,j}^\gamma$, $j \in \mathbb{N}$, of $\gamma(t)$ such that $U_{t,j}^\gamma \subset V_t^\gamma$ and that $U_{t,j}^\gamma$ are strongly convex with respect to some smooth Riemannian metric tensor on M , see [11, Sec IX.6] for the definition of strong convexity. Then the intersections $U_{t,j}^\gamma \cap U_{s,k}^\mu$ are connected for all $(\gamma, t, j), (\mu, s, k) \in A \times \mathbb{N}$ whenever non-empty.

We define a relation $(\gamma, t, j) \sim (\mu, s, k)$ on $A \times \mathbb{N}$ by

$$U_{t,j}^\gamma \cap U_{s,k}^\mu \neq \emptyset \text{ and } \phi_t^\gamma = \phi_s^\mu \text{ in } U_{t,j}^\gamma \cap U_{s,k}^\mu,$$

and show next that (\mathcal{W}, \sim) is a matched covering where

$$\mathcal{W} = \{U_{t,j}^\gamma; (\gamma, t, j) \in A \times \mathbb{N}\}.$$

Clearly (\mathcal{W}, \sim) satisfies the properties (i) and (ii) of a matched covering. Let $(\gamma, t, j) \sim (\mu, s, k)$, $(\mu, s, k) \sim (\beta, r, l)$ and let us suppose that $W := U_{t,j}^\gamma \cap U_{s,k}^\mu \cap U_{r,l}^\beta \neq \emptyset$. Then there is $q \in W$ and $d\phi_t^\gamma = d\phi_r^\beta$ at q . Thus $\phi_t^\gamma = \phi_r^\beta$ in the connected set $U_{t,j}^\gamma \cap U_{r,l}^\beta$ by [48, Lem. 3.62]. That is $(\gamma, t, j) \sim (\beta, r, l)$, and therefore (\mathcal{W}, \sim) is a matched covering.

We define $\mathcal{X} = \{(y, \gamma, t, j) \in M \times A \times \mathbb{N} : y \in U_{t,j}^\gamma\}$ and an equivalence relation $(y, \gamma, t, j) \approx (z, \mu, s, k)$ on \mathcal{X} by

$$y = z \text{ and } (\gamma, t, j) \sim (\mu, s, k).$$

We let $X = \mathcal{X} / \approx$ and denote the equivalence classes by $[(y, \gamma, t, j)]$. Then the maps

$$\lambda_{t,j}^\gamma : U_{t,j}^\gamma \rightarrow X, \quad \lambda_{t,j}^\gamma(y) := [(y, \gamma, t, j)], \quad (\gamma, t) \in A, j \in \mathbb{N},$$

are bijective, the sets $X_{t,j}^\gamma := \lambda_{t,j}^\gamma(U_{t,j}^\gamma)$ cover X , and the functions $(\lambda_{s,k}^\mu)^{-1} \circ \lambda_{t,j}^\gamma$ coincide with the restrictions of the identity map of M on $U_{t,j}^\gamma \cap U_{s,k}^\mu$ whenever these intersections are non-empty.

Let us show the following separation property: if

$$[(y_0, \gamma_0, t_0, j_0)] \neq [(z_0, \mu_0, s_0, k_0)]$$

then there are $X_{t_1, j_1}^{\gamma_1}$ and $X_{s_1, k_1}^{\mu_1}$ such that $[(y_0, \gamma_0, t_0, j_0)]$ belongs to $X_{t_1, j_1}^{\gamma_1}$, $[(z_0, \mu_0, s_0, k_0)]$ belongs to $X_{s_1, k_1}^{\mu_1}$ and $X_{t_1, j_1}^{\gamma_1} \cap X_{s_1, k_1}^{\mu_1} = \emptyset$. Consider first the case that $(\gamma_0, t_0, j_0) \approx (\mu_0, s_0, k_0)$. Then $[(y, \gamma_0, t_0, j_0)]$ and $[(z, \mu_0, s_0, k_0)]$ are distinct for all $y \in U_{t_0, j_0}^{\gamma_0}$ and $z \in U_{s_0, k_0}^{\mu_0}$, and we may choose $X_{t_1, j_1}^{\gamma_1} = X_{t_0, j_0}^{\gamma_0}$ and $X_{s_1, k_1}^{\mu_1} = X_{s_0, k_0}^{\mu_0}$. Suppose now that $y_0 \neq z_0$. There are (γ_1, t_1) and (μ_1, s_1) in A such that $U_{t_1, j}^{\gamma_1}$, $j \in \mathbb{N}$, and $U_{s_1, k}^{\mu_1}$, $k \in \mathbb{N}$, are fundamental systems of neighborhoods of y and z , respectively. Moreover, there are j_1 and k_1 such that $U_{t_1, j_1}^{\gamma_1}$ and $U_{s_1, k_1}^{\mu_1}$ are disjoint. Also the sets $X_{t_1, j_1}^{\gamma_1}$ and $X_{s_1, k_1}^{\mu_1}$ are then disjoint, and we have shown that the separation property holds.

Now [48, Prop. 1.42] implies that there is a unique Hausdorff topology and complete atlas on X such that the functions $(\lambda_{t,j}^\gamma)^{-1}$, when composed with suitable coordinate systems of M , are coordinate systems in the resulting manifold.

To simplify the notation, we write for $a = (\gamma, t)$ and $(y, a, 1) \in \mathcal{X}$

$$\lambda_a := \lambda_{t,1}^\gamma, \quad \mathcal{U}_a := U_{t,1}^\gamma, \quad \Phi_a := \phi_t^\gamma, \quad [(y, a)] := [(y, a, 1)].$$

Moreover, we write $a \sim b$ when $(a, 1) \sim (b, 1)$, $a, b \in A$, and also $(y, a) \approx (z, b)$ when $(y, a, 1) \approx (z, b, 1)$, $(y, a, 1), (z, b, 1) \in \mathcal{X}$. We set $F([(y, a)]) = y$ and get a local diffeomorphism $F : X \rightarrow M$ such that

$$F = \lambda_a^{-1}, \quad \text{on } \lambda_a(\mathcal{U}_a).$$

We equip X with the pullback metric F^*g . Then $F : X \rightarrow M$ is a local isometry.

The map $\tilde{F}([(y, a)]) := \Phi_a(y)$ is well defined. Indeed, if $(y, a) \approx (z, b)$ then $y = z \in \mathcal{U}_a \cap \mathcal{U}_b$ and $\Phi_a(y) = \Phi_b(z)$. Moreover,

$$\tilde{F}([(y, a)]) = \Phi_a(y) = \Phi_a(F([(y, a)])),$$

whence $\tilde{F} = \Phi_a \circ F$ on $\lambda_a(\mathcal{U}_a)$, and $\tilde{F} : X \rightarrow \tilde{M}$ is a local isometry.

To finish the proof of the theorem we will need to invoke the following two lemmas several times. In the formulation of the lemmas we assume implicitly the facts that we have established so far in the proof.

Lemma 8. *Let $a = (\gamma, t) \in A$, $[(y, a)] \in X$, and let us consider a broken geodesic $\rho : [0, \ell] \rightarrow \mathcal{U}_a$ satisfying $\rho(0) = \gamma(t)$. We denote by μ the concatenation of γ and ρ . Then $(\gamma, t) \sim (\mu, s)$ for all $s \in [t, t + \ell]$.*

Proof. Notice that the set

$$I = \{s \in [t, t + \ell]; (\gamma, t) \sim (\mu, s)\}$$

is nonempty since $t \in I$. It is enough to show that I is closed and open. By [48, Lem. 3.62] we have

$$I = \{s \in [t, t + \ell]; \phi_t^\gamma(\mu(s)) = \phi_s^\mu(\mu(s)) \text{ and } d\phi_t^\gamma = d\phi_s^\mu \text{ at } \mu(s)\}$$

We have $\phi_s^\mu(x) = \phi_r^\mu(x)$ for s near r and x near $\mu(s)$. Thus the maps $s \mapsto \phi_s^\mu(\mu(s))$ and $s \mapsto d\phi_s^\mu|_{\mu(s)}$ are smooth and I is closed.

In order to show that I is open, let us suppose that $(\gamma, t) \sim (\mu, s)$. By the definition of a continuation $(\mu, s) \sim (\mu, r)$ for r close to s . The definition of a matched covering implies that $(\gamma, t) \sim (\mu, r)$ since $\mu(r)$ is in $U_{t,1}^\gamma \cap U_{s,1}^\mu \cap U_{r,1}^\mu$. Thus I is open. \square

Lemma 9. *Let $\mu : [0, \ell] \rightarrow M$ be a broken geodesic satisfying $\mu(0) = p$. Then there are*

$$0 = t_0 < t_1 < \dots < t_N = \ell$$

such that $\mu([t_{j-1}, t_j]) \subset \mathcal{U}_{a_j}$, $j = 1, \dots, N$, where $a_j = (\mu, t_{j-1})$, and we may define a continuous path $\hat{\mu} : [0, \ell] \rightarrow X$ by

$$\hat{\mu}(\tau) = \lambda_{a_j}(\mu(\tau)), \quad t \in [t_{j-1}, t_j].$$

Moreover, $F \circ \hat{\mu} = \mu$, that is, $\hat{\mu}$ is a lift of μ through F .

Proof. Compactness of $\mu([0, \ell])$ implies the existence of the intervals $[t_{j-1}, t_j]$, and Lemma 8 implies $(\mu, t_{j-1}) \sim (\mu, t_j)$. Hence $\lambda_{a_j}(\mu(t_j)) = \lambda_{a_{j+1}}(\mu(t_j))$ and $\widehat{\mu}$ is continuous. \square

Let us now return to the proof of the theorem. We will show next that F is a covering map. By [48, Th. 7.28] it is enough to show that all geodesics of M can be lifted through F . Let $\sigma : [0, \ell] \rightarrow M$ be a geodesic, let $[(y, a)] \in X$, $a = (\gamma, t)$, and suppose that $\sigma(0) = y$. There is a broken geodesic $\rho : [0, r] \rightarrow \mathcal{U}_a$ from $\gamma(t)$ to y . We denote by μ the concatenation of γ , ρ and σ , and by $\widehat{\mu}$ the lift of μ as in Lemma 9. Now $\widehat{\sigma}(\tau) = \widehat{\mu}(\tau + t + r)$ is a lift of σ . Let j be the index satisfying $t + r \in [t_{j-1}, t_j]$. Then

$$\widehat{\sigma}(0) = \widehat{\mu}(t + r) = [(y, (\mu, t_{j-1}))] = [(y, (\mu, t + r))] = [(y, (\gamma, t))],$$

where we have employed Lemma 8 twice. We have shown that F is a covering map.

Let us show next that \widetilde{F} is a covering map. Let $\sigma : [0, \ell] \rightarrow \widetilde{M}$ be a geodesic, let $[(y, a)] \in X$, $a = (\gamma, t)$, and suppose that $\sigma(0) = \Phi_a(y)$. There is a broken geodesic $\rho : [0, r] \rightarrow \mathcal{U}_a$ from $\gamma(t)$ to y . We denote by α the concatenation of γ and ρ and write $s = t + r$. Let $\beta : [0, L] \rightarrow M$ be the maximal geodesic satisfying $\beta(0) = y$ and $d\phi_s^\alpha \dot{\beta}(0) = \dot{\sigma}(0)$. Moreover, let μ be the concatenation of α and β . Then the geodesic $\widetilde{\sigma}(\tau) = \phi_{s+\tau}^\mu(\mu(s + \tau))$ coincides with σ on $[0, \ell] \cap [0, L]$ since both the geodesics have the same initial data. If $\ell \geq L$ then the limit $\lim_{t \rightarrow L} \beta(t)$ exists by (L) but this is a contradiction with the maximality of β . Thus $\ell < L$ and $\widetilde{\sigma} = \sigma$ on $[0, \ell]$.

Let $\widehat{\mu}$ be the lift of μ as in Lemma 9. Then for $t \in [t_{j-1}, t_j]$

$$\widetilde{F} \circ \widehat{\mu}(t) = \Phi_{a_j} \circ F \circ \widehat{\mu}(t) = \phi_{t_{j-1}}^\mu \circ \mu(t) = \phi_t^\mu \circ \mu(t).$$

Indeed, the first identity follows from the definition of \widetilde{F} , the second from the fact that $\widehat{\mu}$ is a lift of μ , and the last from Lemma 8. Hence $\widehat{\sigma}(\tau) = \widehat{\mu}(\tau + s)$ is a lift of σ through \widetilde{F} . As above we see that

$$\widehat{\sigma}(0) = [(y, a)].$$

We have shown that \widetilde{F} is a covering map.

Let us show that X is connected. It is enough to show that a point $[(y, a)] \in X$ can be connected to $[(p, b)] \in X$ where $a = (\gamma, t)$ and $b = (\beta, 0)$. There is a broken geodesic $\rho : [0, \ell] \rightarrow \mathcal{U}_a$ from $\gamma(t)$ to y . We denote by μ the concatenation of γ and ρ , and by $\widehat{\mu}$ the lift of μ as in Lemma 9. Now

$$\widehat{\mu}(t + \ell) = [(y, (\mu, t_{N-1}))] = [(y, (\mu, t + \ell))] = [(y, (\gamma, t))].$$

We have shown that X is connected.

As X is connected, it has the universal covering \tilde{X} . Moreover, as the composition of two covering maps is a covering map in the case of manifolds, the covering \tilde{X} is the universal covering of M and \tilde{M} . \square

Lemma 10. *Suppose that (M, g) and (\tilde{M}, \tilde{g}) are geodesically complete modulo scalar curvature singularities. Let $\phi : U \rightarrow \tilde{U}$ be an isometry of an open set $U \subset M$ onto an open set $\tilde{U} \subset \tilde{M}$. Let $\ell \in (0, \infty)$ and let $\gamma : [0, \ell) \rightarrow M$ be a broken geodesic such that $\gamma(0) \in U$. Suppose that $\phi_t, t \in [0, \ell)$, is a continuation of ϕ along γ , and define $\mu(t) = \phi_t(\gamma(t))$, $t \in [0, \ell)$. Then the limit $\lim_{t \rightarrow \ell} \mu(t)$ exists if and only if the limit $\lim_{t \rightarrow \ell} \gamma(t)$ exists.*

Proof. Local isometries map geodesics to geodesics and $\mu(t) = \phi_s(\gamma(t))$ for t near $s \in [0, r)$. Thus μ is a broken geodesic.

Notice that the limit $\lim_{t \rightarrow \ell} \gamma(t)$ exists if and only if $\kappa(\gamma(t))$ is bounded as $t \rightarrow \ell$ for all scalar curvature invariants κ of (M, g) . Indeed, if $\gamma(t) \rightarrow x$ in M as $t \rightarrow \ell$, then $\kappa(\gamma(t))$ is bounded as $t \rightarrow \ell$ for all scalar curvature invariants κ of (M, g) since κ is smooth near x . On the other hand if the limit $\lim_{t \rightarrow \ell} \gamma(t)$ does not exist, then γ can not be extended, and there is a scalar curvature invariant κ of (M, g) such that $\kappa(\gamma(t))$ is unbounded as $t \rightarrow \ell$. The analogous statement holds for the limit $\lim_{t \rightarrow \ell} \mu(t)$. The claim follows, since if $\tilde{\kappa}$ is a scalar curvature invariant of (\tilde{M}, \tilde{g}) then the corresponding scalar curvature invariant κ of (M, g) satisfies $\kappa(x) = \tilde{\kappa}(\phi_t(x))$ for all $x \in U_t$. \square

Lemma 11. *Suppose that (M, g) , (\tilde{M}, \tilde{g}) and $\phi : U \rightarrow \tilde{U}$ are as in Lemma 10, and let $\gamma : [0, \ell) \rightarrow M$ be a broken geodesic starting from U . Suppose, furthermore, that (M, g) and (\tilde{M}, \tilde{g}) are real-analytic. Then ϕ is extendable along γ .*

Proof. Let $S \geq 0$ be the supremum of $s \in [0, \ell]$ such that there is a continuation $\phi_t : U_t \rightarrow \tilde{U}_t$, $t \in [0, s]$, of ϕ . We define $\mu(t) := \phi_t(\gamma(t))$, $t \in [0, S)$. Lemma 10 implies that the limit $\lim_{t \rightarrow S} \mu(t)$ exists. We denote the limit by $\mu(S)$. Let \mathcal{U} and $\tilde{\mathcal{U}}$ be simply convex neighborhoods of $\gamma(S)$ and $\mu(S)$ respectively. Let $\epsilon > 0$ be such that $\gamma(t) \in \mathcal{U}$ and $\mu(t) \in \tilde{\mathcal{U}}$ for all $t \in [S - \epsilon, S]$. By decreasing ϵ we may also assume that both γ and μ are geodesics on $[S - \epsilon, S]$.

We write $s = S - \epsilon$, $p = \gamma(s)$ and $q = \mu(s)$. We will work in the normal coordinates around p and q . In the normal coordinates, the isometry ϕ_s coincides with the linear map $d\phi_s|_p$ in $U_s \cap \mathcal{U}$, see e.g. [48, p. 91]. In the normal coordinates, the simply convex neighborhoods \mathcal{U}

and $\tilde{\mathcal{U}}$ are neighborhoods of the origins in $T_p M$ and $T_q \tilde{M}$ respectively. We define W to be the connected component of $\mathcal{U} \cap d\phi_s|_p^{-1}(\tilde{\mathcal{U}})$ that contains the origin, and denote by ψ the linear map $d\phi_s|_p$ on W . Let X and Y be real-analytic vector fields on W . Then

$$(d\psi X, d\psi Y)_{\tilde{g} \circ \psi} = (X, Y)_g$$

in $U_s \cap W$ since ψ is an isometry there. The both sides of the above identity are real-analytic functions on the connected set W , whence the identity holds on W , see e.g. [25, Lem. VI.4.3]. Thus ψ is an isometry of W onto an open set in \tilde{M} .

We write $v = \dot{\gamma}(s)$. In the normal coordinates, the geodesic $\gamma|_{[s, S]}$ has the form $\gamma(t - s) = (t - s)v$, and the geodesic $\mu|_{[s, S]}$ has the form $\mu(t - s) = (t - s)d\phi_s|_p v$. Thus $[s, S]v \subset U$ and $d\phi_s([s, S]v) \subset \tilde{U}$. In particular, $\gamma(S) = (S - s)v \in W$. If $S < \ell$, then there is $\delta > 0$ such that $\gamma(t) \in W$ for $t \in [S, S + \delta]$ since W is open. Moreover, there is $r \in (s, S)$ such that $\gamma(t) \in U_s$ for $t \in [s, r]$. We define V to be the connected component of $W \cap U_s$ that contains $\gamma(s)$. Now

$$\psi_t = \begin{cases} \phi_t : U_t \rightarrow \tilde{U}_t, & t \in [0, s) \\ \psi : V \rightarrow \psi(V), & t \in [s, r), \\ \psi : W \rightarrow \psi(W), & t \in [r, S + \delta), \end{cases}$$

is a continuation of ϕ . Indeed, if $t < s$ is close to s , then $\phi_t = \phi_s$ in $U_t \cap U_s$. Hence $\phi_t = \psi$ in $U_t \cap V$. Moreover, $\gamma(r) \in V$ and $\psi_t = \psi_r$ in $V \cap W = V$ if $t < r$ is close to r . But this is a contradiction with maximality of S . Hence $S = \ell$. \square

Now we are ready to prove our second main theorem.

Proof of Theorem 2. The metric tensors g and \tilde{g} are real-analytic in geodesic normal coordinates, see e.g. [13, Th. 2.1]. Theorem 1 guarantees that there is a linear bijection $L : T_p M \rightarrow T_{\tilde{p}} \tilde{M}$ such that if V_0, \dots, V_n is a basis of $T_p M$ and we define $\tilde{V}_j = LV_j$, then the Taylor coefficients of the metric tensors g and \tilde{g} coincide in the normal coordinates

$$\psi(x^0, \dots, x^n) = \exp_p(x^j V_j), \quad \tilde{\psi}(x^0, \dots, x^n) = \widetilde{\exp}_{\tilde{p}}(x^j \tilde{V}_j)$$

defined on $\mathcal{B} = \{x \in \mathbb{R}^{1+n}; (x^0)^2 + \dots + (x^n)^2 < r\}$ where $r > 0$ is small enough. As g and \tilde{g} are real-analytic, they coincide in these coordinates. Hence $\phi = \tilde{\psi} \circ \psi^{-1}$ is an isometry of $U = \psi(\mathcal{B}) \subset M$ onto $\tilde{U} = \tilde{\psi}(\mathcal{B}) \subset \tilde{M}$. The claim follows from Theorem 7 together with Lemmas 10 and 11. \square

Next we prove Proposition 3. We show that stationary solutions of the Einstein field equations coupled with scalar fields are real-analytic.

Proof of Proposition 3. Given $p \in M$, first we show that there are local coordinates $y = (y^0, \dots, y^3)$ near p such that $Z = \partial_{y^0}$ and that $\widehat{g} = (g^{jk})_{j,k=1}^3$ is positive definite.

We start with the coordinates

$$(\widetilde{y}^0, \dots, \widetilde{y}^3) \mapsto \exp_p(\widetilde{y}^0 Z(p) + \widetilde{y}^1 V_1 + \widetilde{y}^2 V_2 + \widetilde{y}^3 V_3),$$

where $Z(p)/|Z(p)|_g, V_1, V_2, V_3$ form an orthonormal basis of $T_p M$. Here $Z(p) \neq 0$ since Z is timelike. In these coordinates Z can be written as

$$Z = Z^0 \partial_{\widetilde{y}^0} + Z^1 \partial_{\widetilde{y}^1} + Z^2 \partial_{\widetilde{y}^2} + Z^3 \partial_{\widetilde{y}^3}$$

with $Z^0(p) = 1$ and $Z^1(p) = Z^2(p) = Z^3(p) = 0$; and in these coordinates the metric \widetilde{g} at p is diagonal with diagonal elements $(-|Z(p)|_g, 1, 1, 1)$. In the following, we write $\widetilde{y}' = (\widetilde{y}^1, \widetilde{y}^2, \widetilde{y}^3)$ and use analogous notations also for other quantities. Denote the flow of Z by φ_t , and define a smooth map

$$(t, y') \mapsto \varphi_t(y') = (\widetilde{y}^0, \widetilde{y}').$$

This map is indeed a diffeomorphism near p . To see this, simply notice that

$$D\widetilde{y}/D(t, y') = \begin{pmatrix} Z^0 & D\widetilde{y}^0/Dy' \\ Z' & D\widetilde{y}'/Dy' \end{pmatrix},$$

which is the identity at the origin. Thus we can choose (t, y') as local coordinates near p . Renaming t as y^0 gives $\partial_{y^0} = Z$. Moreover, in the coordinates $y = (y^0, y')$ the metric g can be written as

$$g = \left(\frac{D\widetilde{y}}{Dy} \right)^T \widetilde{g} \left(\frac{D\widetilde{y}}{Dy} \right)$$

which, from our analysis above, is diagonal with diagonal elements $(-|Z(p)|_g, 1, 1, 1)$ at p . It follows that the matrix $\widehat{g} = (g^{jk})_{j,k=1}^3$ is positive definite at p , and hence by continuity is also positive definite in a neighborhood of p .

Now we choose the coordinates $y = (y^0, \dots, y^3)$ as above. As $Z = \partial_{y^0}$ is a Killing field, we have $\partial_{y^0} g^{pq} = 0$ in the coordinates y , and the wave operator has the form

$$\square_g u = \sum_{p,q=1}^3 |g|^{-1/2} \partial_{y^p} (|g|^{1/2} \widehat{g}^{pq} \partial_{y^q} u)$$

if u is a function of y' only. Note also that

$$\square_g y^0 = \sum_{p=1}^3 |g|^{-1/2} \partial_{y^p} (|g|^{1/2} \widehat{g}^{p0})$$

is a function of y' only. Let us choose functions $x^j(y')$, $j = 1, 2, 3$, solving the elliptic problem

$$\square_g x^j = 0, \quad \partial_{y^k} x^j(0) = \delta_k^j, \quad k = 1, 2, 3.$$

Moreover, let us choose a function $h(y')$ solving the problem

$$\square_g h = -\square_g y^0, \quad \partial_{y^k} h(0) = 0, \quad k = 1, 2, 3.$$

and define $x^0 = y^0 + h(y')$. Then Dx/Dy is the identity at the origin, and $x = (x^0, x')$ give local coordinates. Moreover, $Z = \partial_{x^0}$ and the coordinates x are harmonic, that is, $\square_g x^j = 0$, $j = 0, 1, 2, 3$.

Note that Einstein equations are equivalent to

$$\text{Ric}(g) = \rho, \quad \rho_{jk} = T_{jk} - \frac{1}{2}((g)^{nm} T_{nm}) g_{jk} + 2\Lambda g_{jk}, \quad \text{on } M,$$

see [12, p. 44] and we recall (see [17, 26]) that

$$(24) \quad \text{Ric}_{\mu\nu}(g) = \text{Ric}_{\mu\nu}^{(h)}(g) + \frac{1}{2}(g_{\mu q} \frac{\partial \Gamma^q}{\partial x^\nu} + g_{\nu q} \frac{\partial \Gamma^q}{\partial x^\mu})$$

where $\Gamma^q = g^{mn} \Gamma_{mn}^q$,

$$(25) \quad \text{Ric}_{\mu\nu}^{(h)}(g) = -\frac{1}{2} g^{pq} \frac{\partial^2 g_{\mu\nu}}{\partial x^p \partial x^q} + P_{\mu\nu},$$

$$P_{\mu\nu} = g^{ab} g_{ps} \Gamma_{\mu b}^p \Gamma_{\nu a}^s + \frac{1}{2} \left(\frac{\partial g_{\mu\nu}}{\partial x^a} \Gamma^a + g_{\nu l} \Gamma_{ab}^l g^{aq} g^{bd} \frac{\partial g_{qd}}{\partial x^\mu} + g_{\mu l} \Gamma_{ab}^l g^{aq} g^{bd} \frac{\partial g_{qd}}{\partial x^\nu} \right).$$

Note that $P_{\mu\nu}$ is a polynomial of g_{jk} , g^{jk} and the first derivatives of g_{jk} . In the harmonic coordinates x , we have $\Gamma^q = 0$, $q = 0, 1, 2, 3$, and thus $\text{Ric}_{\mu\nu}(g)$ coincides with $\text{Ric}_{\mu\nu}^{(h)}(g)$.

As $Z = \partial_{x^0}$, the stationarity of g and ϕ implies that the equation (5), (6), and (7), have the form

$$\begin{aligned} & - \sum_{p,q=1}^3 \frac{1}{2} \widehat{g}^{pq} \frac{\partial^2 g_{jk}}{\partial x^p \partial x^q} + P_{jk}(g, \partial g) = \rho_{jk}(g, T), \\ & T_{jk} = \left(\sum_{\ell=1}^L \partial_j \phi_\ell \partial_k \phi_\ell - \frac{1}{2} g_{jk} g^{pq} \partial_p \phi_\ell \partial_q \phi_\ell \right) - \mathcal{V}(\phi) g_{jk}, \\ & - \sum_{p,q=1}^3 \frac{1}{2} \widehat{g}^{pq}(x) \frac{\partial^2 \phi_\ell}{\partial x^p \partial x^q} - \mathcal{V}_\ell(\phi) = 0, \end{aligned}$$

in the coordinates x . This is an elliptic non-linear system of equations, where P_{jk} , ρ_{jk} and \mathcal{V} are real-analytic. By Morrey's theorem [46], a smooth solution of a real-analytic elliptic system is real-analytic. Thus g and ϕ are real analytic in the coordinates x , and also in the geodesic normal coordinates.

Let us now consider the differentiable structure of M given by the atlas of convex normal coordinates associated to g . The transition functions between such coordinates are real-analytic, and thus (M, g) can be considered as a real-analytic manifold. \square

Finally, we give the proof of Corollary 4.

Proof of Corollary 4. By Proposition 3, the manifolds (M, g) and $(\widetilde{M}, \widetilde{g})$ are real-analytic and ϕ and $\widetilde{\phi}$ are real-analytic functions on these manifold. By Theorem 2, the simply connected manifolds (M, g) and $(\widetilde{M}, \widetilde{g})$ are isometric and there is a real-analytic isometry $F : M \rightarrow \widetilde{M}$.

Next we consider the Killing fields. Let $U \subset T\Sigma$ be a neighborhood of $(\widehat{x}_0, \widehat{\xi}_0)$ and $\epsilon > 0$ for which the condition **H** is valid. We may assume that $\widehat{\xi}_0$ is a past-pointing timelike vector.

Pick past-pointing vectors $\eta_j \in T_{\widehat{x}_0}M$, $j = 1, \dots, n + 1$ so that they are linearly independent and so that $\gamma_{\widehat{x}_0, \eta_j}(\ell_j) \in \overline{\Sigma}$ for some $\ell_j > 0$. The latter condition can be achieved by the hypothesis **H** as long as η_j is sufficiently close to its projection on $T_{\widehat{x}_0}\Sigma$. Denote $\gamma_j := \gamma_{\widehat{x}_0, \eta_j}$ and let $x_j = \gamma_j(\ell_j) \in \overline{\Sigma}$ be the points where the geodesics intersect first time $\overline{\Sigma}$. Then by Lemma 5 (iv), we have

$$\text{grad } d_{x_j}^+(y) \Big|_{y=\widehat{x}_0} = \frac{\dot{\gamma}_j(0)}{|\dot{\gamma}_j(0)|_g} = \frac{1}{|\eta_j|_g} \eta_j.$$

As vectors η_j , $j = 1, 2, \dots, n + 1$ are linearly independent, the point \widehat{x}_0 has a neighborhood $V \subset M$ so that the map

$$(26) \quad \mathcal{D} : V \rightarrow \mathbb{R}^{n+1}, \quad \mathcal{D}(y) = (d_{x_j}^+(y))_{j=1}^{n+1}$$

defines regular coordinates on M near \widehat{x}_0 . By (9), \mathcal{D} can be written in terms of the inverse functions of the exponential functions and thus the coordinates (V, \mathcal{D}) are real-analytic coordinates of M . Let $\Sigma_0 = V \cap \Sigma$.

The above construction of coordinates (26) can be done also on \widetilde{M} . Thus we see that on $\widetilde{V} = F(V)$ we have coordinates $\widetilde{\mathcal{D}} : \widetilde{V} \rightarrow \mathbb{R}^{n+1}$, given by $\widetilde{\mathcal{D}}(y) = (\widetilde{d}_{\widetilde{x}_j}^+(y))_{j=1}^{n+1}$, $\widetilde{x}_j = F(x_j)$. As F is an isometry, we have $\mathcal{D} = \widetilde{\mathcal{D}} \circ F$ on V . Also, by (8), we have $\mathcal{D}|_{\Sigma_0} = \widetilde{\mathcal{D}} \circ \Psi|_{\Sigma_0}$. These yield

$$(27) \quad F|_{\Sigma_0} = \Psi|_{\Sigma_0}.$$

This in particular implies that $F|_{\Sigma_0} : \Sigma_0 \rightarrow \tilde{\Sigma}_0 = \Psi(\Sigma \cap V)$ is a diffeomorphism.

Recall that F is an isometry and a real-analytic map. Thus we see that the unit normal vectors satisfy $\tilde{\nu} = F_*(\nu)$. As $\tilde{\nu} = \Psi_*(\nu)$ by our assumptions, (27) yield $F_* = \Psi_*$ on $T_{\hat{x}_0}M$.

As Z is a Killing field on M , the field $\tilde{Z}_0 := F_*Z$ is a Killing field on \tilde{M} . Our next aim is to show that $\tilde{Z}_0 = \tilde{Z}$.

Now $\tilde{Z}_0 = F_*Z = \Psi_*Z = \tilde{Z}$ at \hat{x}_0 . As F is an isometry, we have, see e.g. [48, Prop. 3.59],

$$\tilde{\nabla}_{F_*X}\tilde{Z}_0 = F_*\nabla_X Z$$

for all vectors $X \in T_{\hat{x}_0}M$. As $F_* = \Psi_*$ at $T_{\hat{x}_0}M$, we have at $\Psi(\hat{x}_0)$,

$$\tilde{\nabla}_{\Psi_*X}\tilde{Z}_0 = \tilde{\nabla}_{F_*X}\tilde{Z}_0 = F_*\nabla_X Z = \Psi_*\nabla_X Z = \tilde{\nabla}_{\Psi_*X}\tilde{Z}.$$

Hence $\tilde{\nabla}\tilde{Z}_0 = \nabla\tilde{Z}$ at $\Psi(\hat{x}_0)$. By [48], see Lemma 9.27 and the text below it, the pair $(\tilde{Z}, A) := (\tilde{Z}, \nabla\tilde{Z})$ of the Killing field and its covariant derivative satisfy a first order differential equation over arbitrary smooth curve $\mu(s)$ on \tilde{M} (on the original Riemannian versions of this result, see [31, 50]),

$$\begin{aligned} \nabla_{\dot{\mu}(s)}\tilde{Z}(\mu(s)) &= -A(\mu(s))\dot{\mu}(s), \\ \nabla_{\dot{\mu}(s)}A(\mu(s)) &= R(\tilde{Z}(\mu(s)), \dot{\mu}(s)), \end{aligned}$$

where R is the curvature operator of \tilde{M} . Thus, as $\tilde{Z}_0 = \tilde{Z}$ and $\nabla\tilde{Z}_0 = \nabla\tilde{Z}$ at $\Psi(\hat{x}_0)$, we see that $\tilde{Z}_0 = \tilde{Z}$ on the whole manifold \tilde{M} .

Next we consider the scalar fields. As ϕ is stationary with respect to Z and Z is transversal to Σ_0 , we see that $\phi|_{\Sigma_0}$ determines ϕ in a neighborhood of Σ_0 . Indeed, we see that $\phi(\exp_x(sZ)) = \phi(x) = \tilde{\phi}(\exp_{\tilde{x}}(s\tilde{Z}))$ for $x \in \Sigma_0$, $\tilde{x} = F(x)$ and $s \in (-\epsilon(x), \epsilon(x))$, $\epsilon(x) > 0$. Thus see that $\tilde{\phi} \circ F = \phi$ in a neighborhood of Σ_0 . As F and the functions ϕ and $\tilde{\phi}$ are real-analytic, we have $\tilde{\phi} \circ F = \phi$ on the whole M . This proves the claim. \square

Acknowledgements. The authors express their gratitude to the Mittag-Leffler Institute, where parts of this work have been done. The authors would like to thank Prof. Gunther Uhlmann for his generous support related to this work, and for suggesting the method used in the proof of Theorem 1.

ML was partially supported by the Academy of Finland project 272312 and the Finnish Centre of Excellence in Inverse Problems Research 2012-2017. YY was partially supported by the NSF grants DMS

1265958 and DMS 1025372. LO was partially supported by the EPSRC grant EP/L026473/1.

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